Clifford-Steerable Convolutional Neural Networks

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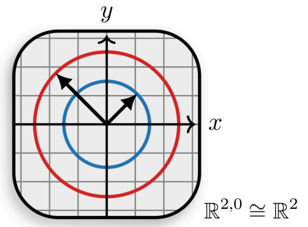
Preliminaries

Pseudo-Euclidean spaces $\mathbb{R}^{p,q}$: generalization of Euclidean spaces $\mathbb{R}^n$ to negative distances. Includes Euclidean space and Minkowski spacetime as special cases.

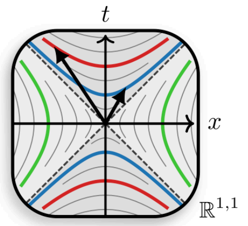
Euclidean space

$\Delta^2 = x^2 + y^2$



Minkowski spacetime

$\Delta^2 = (ct)^2 - x^2$



Pseudo-Euclidean group $\mathrm{E}(p,q)$: set of isometries, includes translations, spatial rotations, reflections, and also boosts.

Steerable CNNs

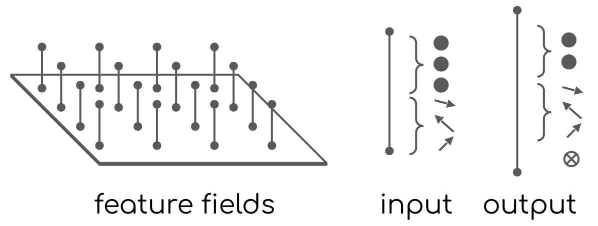
Known recipe to build E(n)-equivariant CNNs:

E(n)-equivariant CNNs = CNNs + O(n)-equivariant kernels

1. define input/output representations ρ_{in}, ρ_{out}

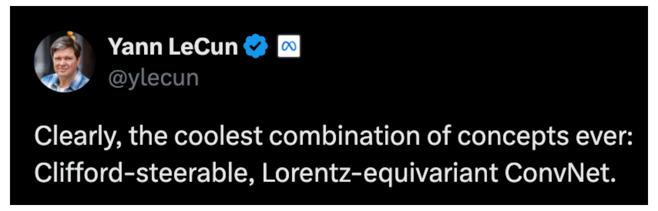
2. solve the kernel constraint for ρ_{in}, ρ_{out}

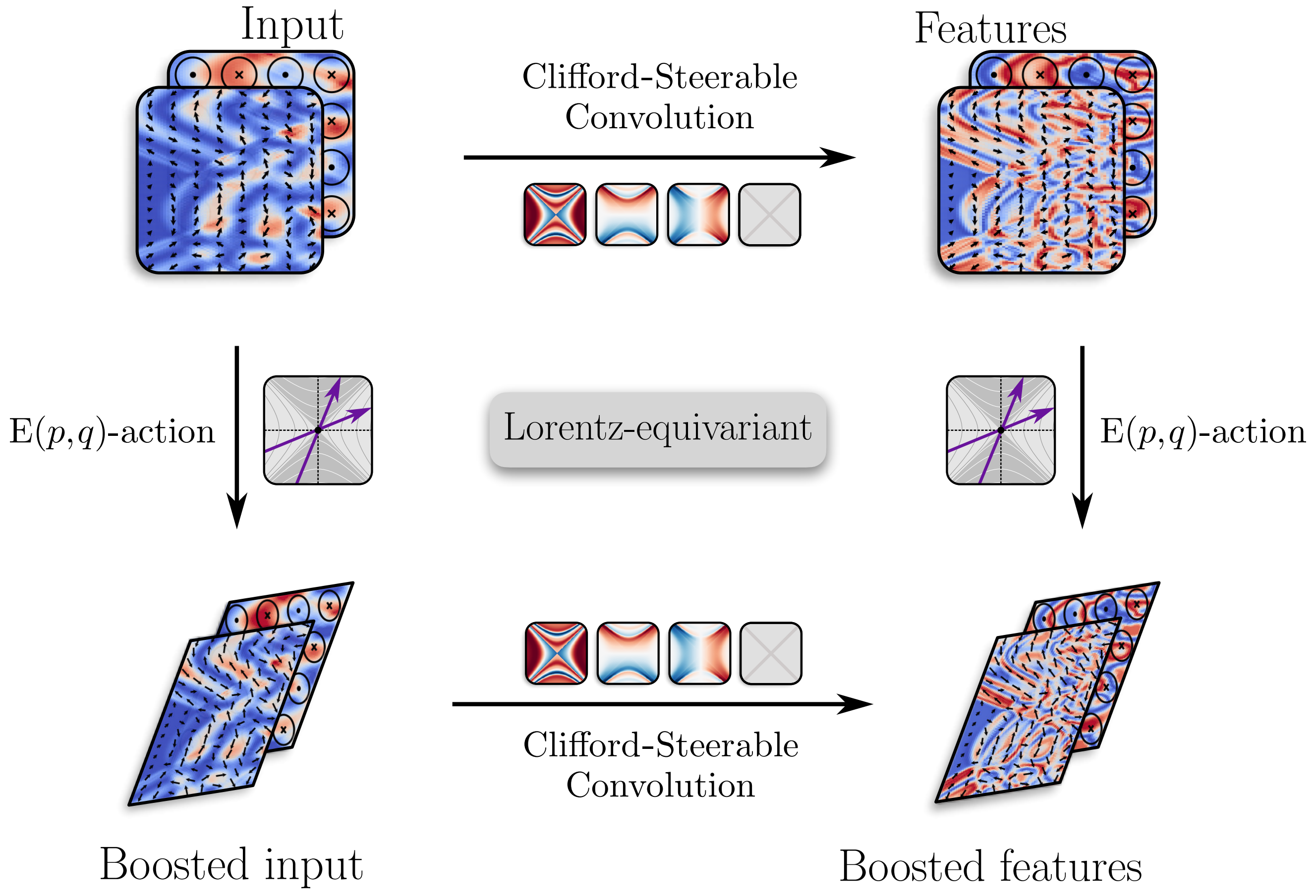
$$k(g.x) = \rho_{out}(g)k(x)\rho_{in}(g)^T \quad \forall g \in G$$



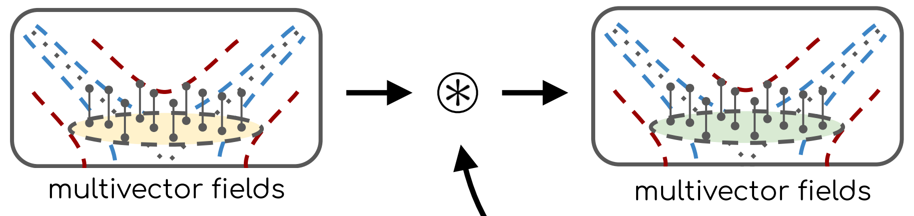
3. use in convolution

- → Don't want to solve the constraint analytically for each p,q.
- → Use Clifford group equivariant NNs [Ruhe et al. 2023] to parameterize $\mathrm{O}(p,q)$-equivariant convolutional kernels.
- → [Zhdanov et al. 2023] guarantees that the resulting CNN is $\mathrm{E}(p,q)$-equivariant.





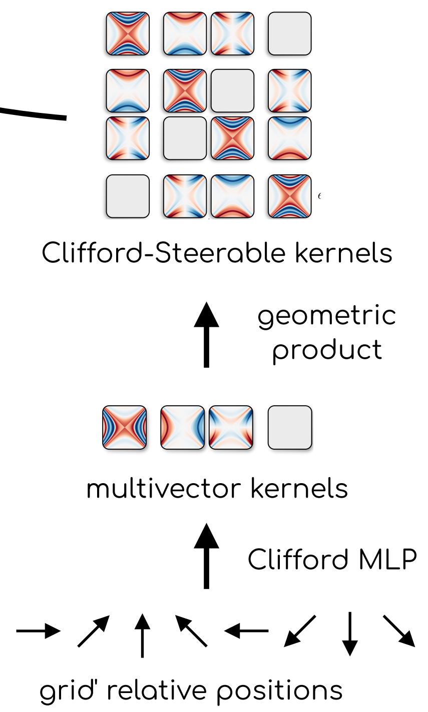
Clifford-Steerable CNNs



- → For each position on the grid, a multivector kernel matrix is computed.
- → For each multivector in the matrix, a weighted geometric product is partially evaluated, which yields Clifford-Steerable kernel.

$$\mathrm{K}: \mathbb{R}^{p,q} \xrightarrow{\mathrm{O}(p,q)-\mathrm{MLP}} \mathrm{Cl}(\mathbb{R}^{p,q})^{c_{out}\times c_{in}} \xrightarrow{\text{weighted GP}} \mathbb{R}^{c_{out}\cdot 2^{p+q}\times c_{in}\cdot 2^{p+q}}$$

- → The kernel can be used with nn.ConvNd → efficient and fast.



Experiments

Task: predict the next state of a system given previous states.

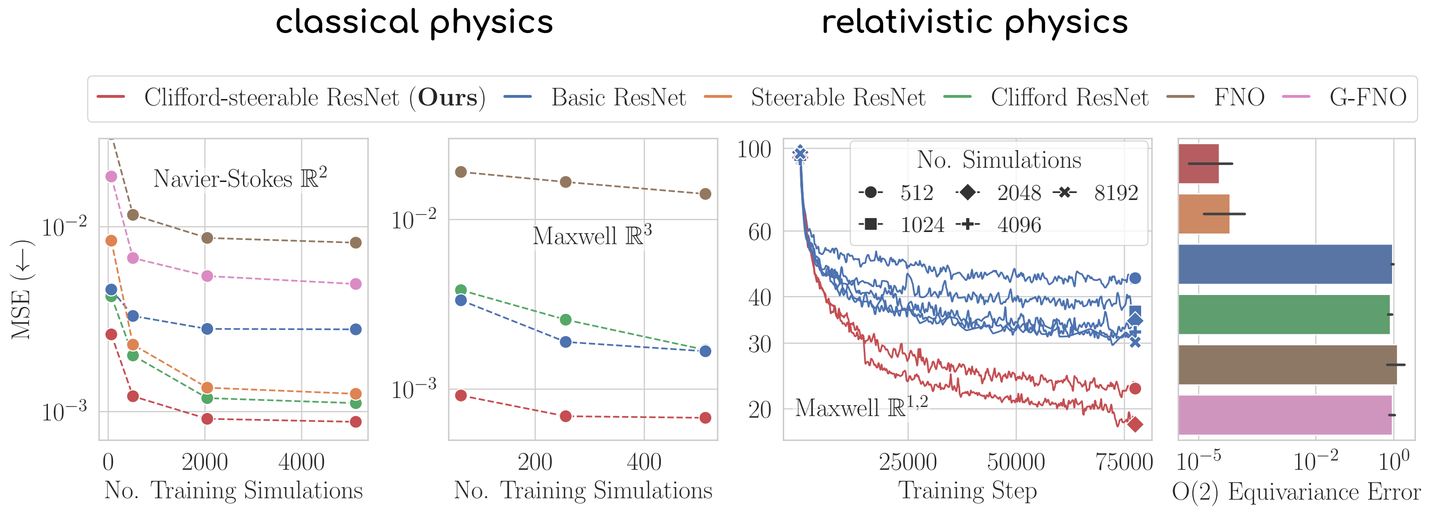
1) Fluid dynamics on $\mathbb{R}^2$ - incompressible Navier-Stokes eq. (PDEarena).

2) Electrodynamics on $\mathbb{R}^3$ - Maxwell eq. (PDEarena).

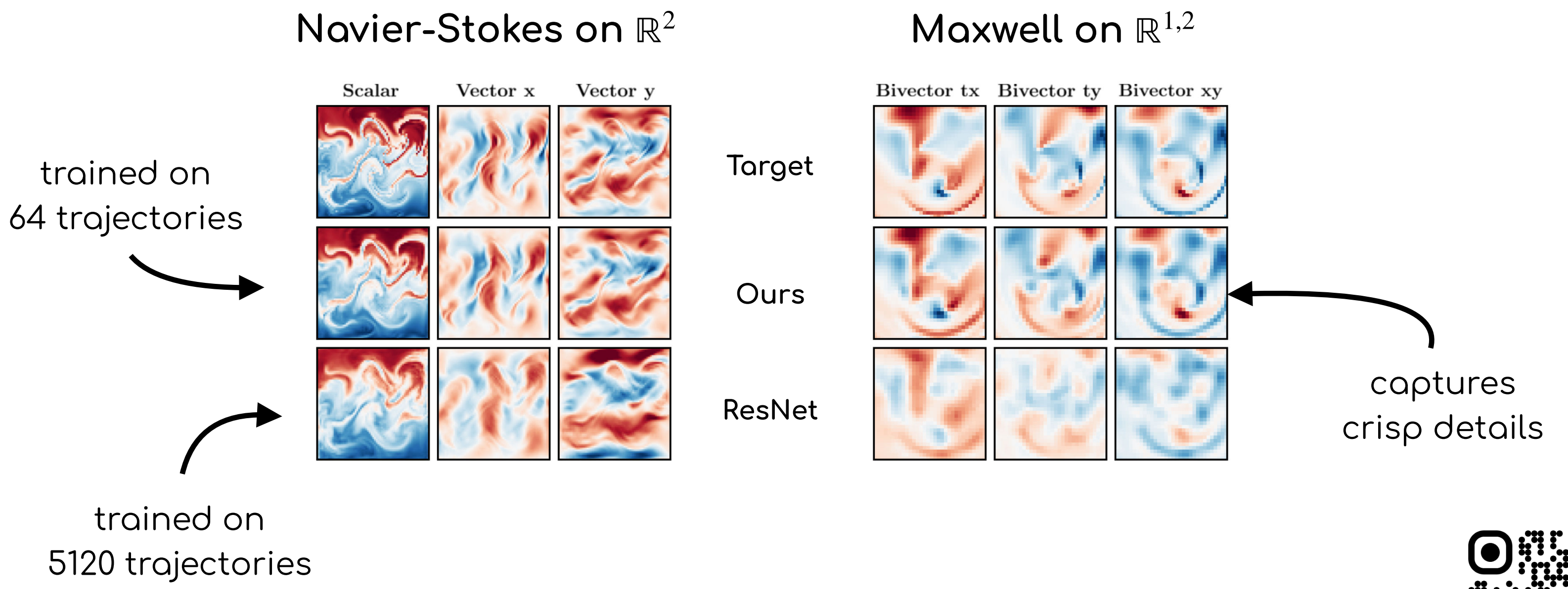
3) Electrodynamics on $\mathbb{R}^{1,2}$ - Maxwell eq., relativistic.

- EM field is generated by multiple charged particles moving with relativistic velocities.

→ In 1) and 2), time is given as channels, in 3), as a grid dimension.



- → CS-ResNets significantly and consistently outperform baselines.
- → CS-ResNets are 100x sample efficient than standard ResNets.
- → Allows for relativistic equivariant convolutions on spacetime.



for more details, check out the full paper →