PMATH 332

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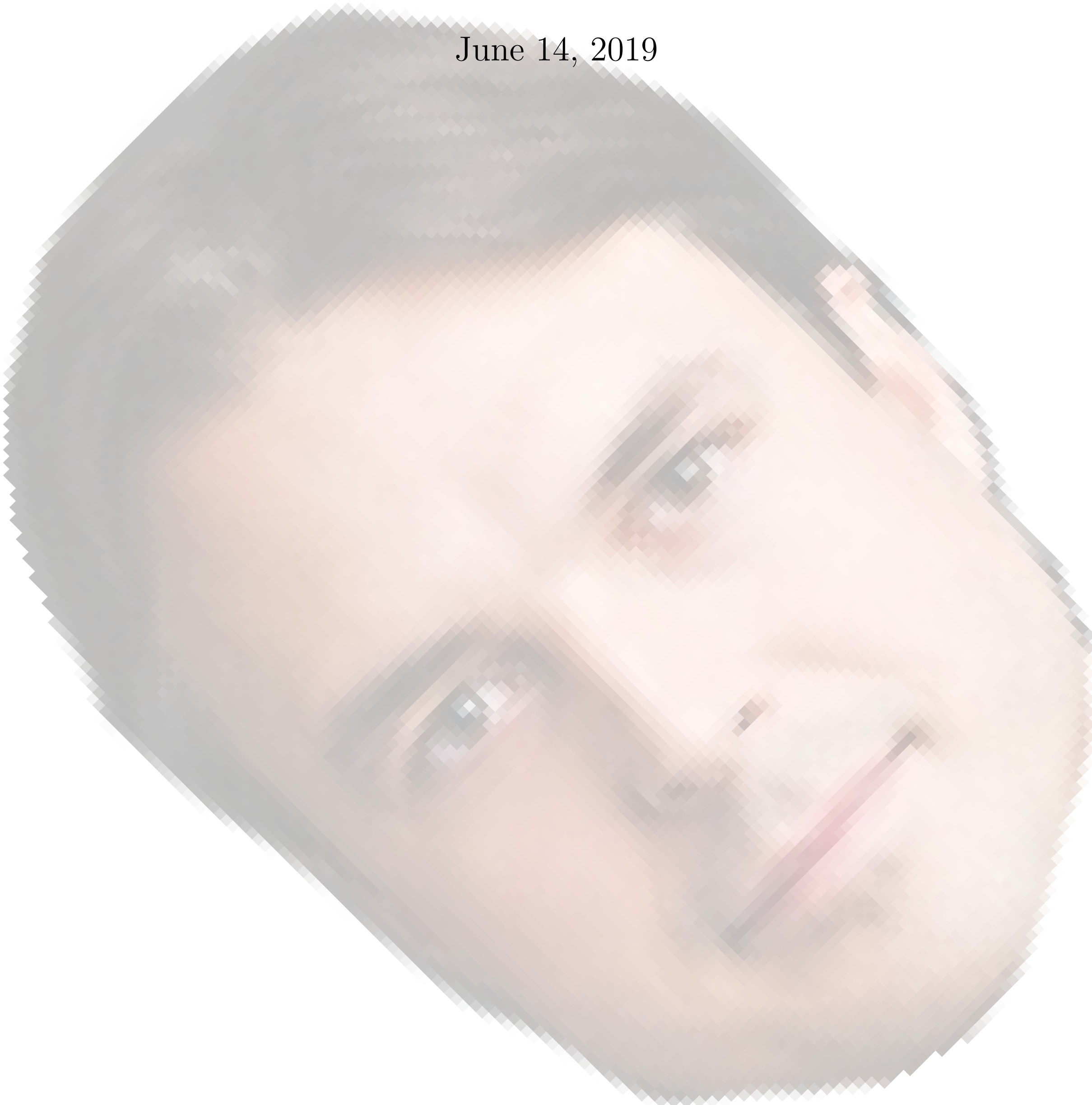

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Example 1.1: Most important example in this course

Evaluate $\int_C \frac{1}{z} dz$ where C is the unit circle, traversed once counter-clockwise.

The anti-derivative of $\frac{1}{z}$ is $\log(z)$, but this has a branch cut on C so we can't use it directly.

Method 1: Parametrize C as $z(t) = e^{it}$ for $t \in [0, 2\pi]$. Then,

$$\begin{aligned}\int_C \frac{1}{z} dz &= \int_0^{2\pi} \frac{1}{e^{it}} \cdot ie^{it} dt \\ &= 2\pi i\end{aligned}$$

Method 2: Separate the circle into two semicircles, C_1 and C_2. Define C_1 to go from $-i$ to i and C_2 to go from i to $-i$. Then,

$$\begin{aligned}\int_{C_1} \frac{1}{z} dz &= [\mathrm{Log}(z)]_{-i}^{i} \\ &= \mathrm{Log}(i) - \mathrm{Log}(-i) \\ &= i\frac{\pi}{2} - i\frac{-\pi}{2} \\ &= i\pi\end{aligned}$$

For C_2, use $\mathrm{Log}_0(z)$ instead to avoid the branch cut.

$$\begin{aligned}\int_{C_2} \frac{1}{z} dz &= [\mathrm{Log}_0(z)]_{i}^{-i} \\ &= \mathrm{Log}_0(-i) - \mathrm{Log}_0(i) \\ &= i\frac{3\pi}{2} - i\frac{\pi}{2} \\ &= i\pi\end{aligned}$$

So,

$$\begin{aligned}\int_C \frac{1}{z} dz &= \int_{C_1} \frac{1}{z} dz + \int_{C_2} \frac{1}{z} dz \\ &= 2\pi i\end{aligned}$$

Remark 1.2

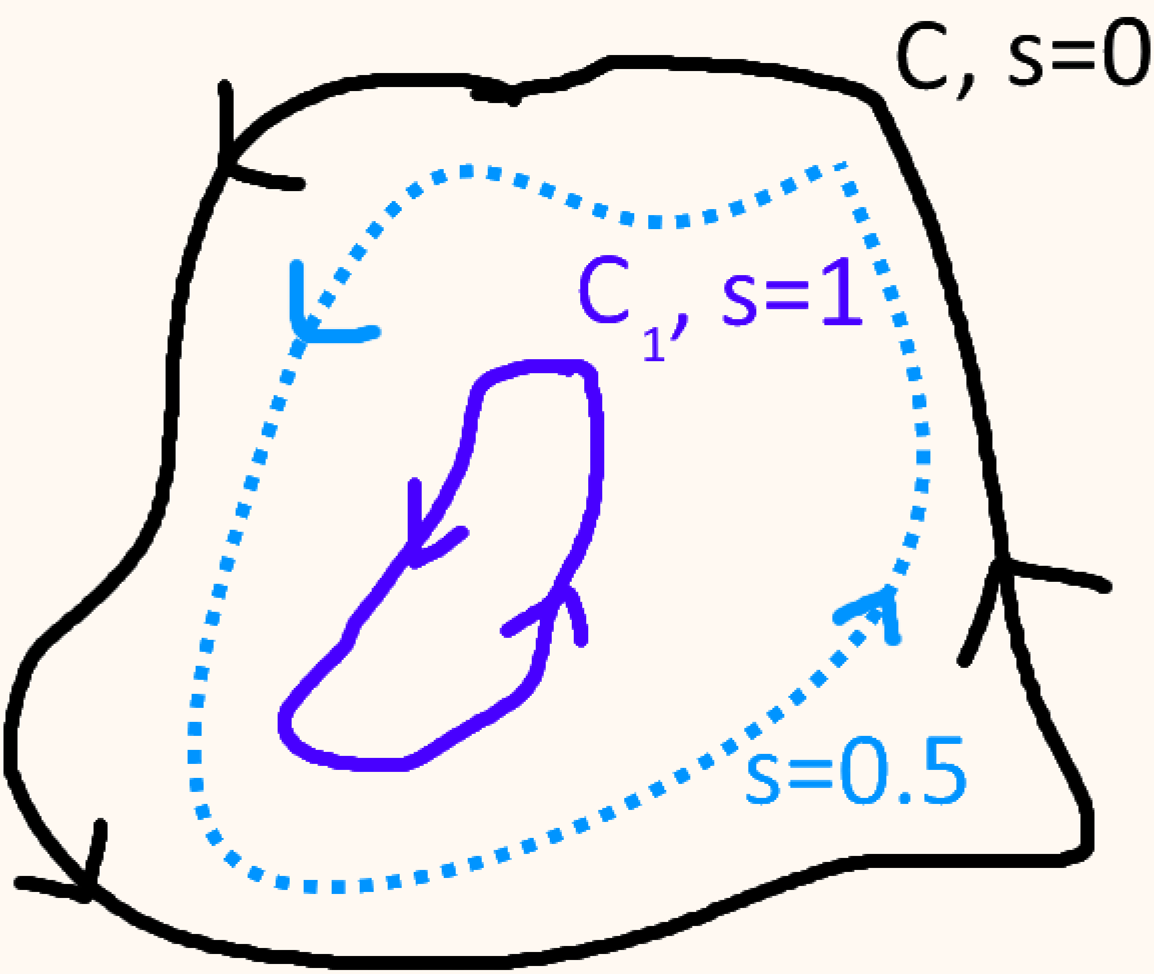
- For any circle of radius R, you will still get $2\pi i$.
- If you traverse the circle twice, you will get $4\pi i$.
- If you traverse backwards, you will get $-2\pi i$.

Definition 1.3: Continuously deformable

A closed contour C is said to be continuously deformable to a contour C_1 in a domain D if there exists a function $z(s,t)$ which is continuous on $s,t \in [0,1]$ such that:

- $z(s,t)$ is a closed contour in D for all $s \in [0,1]$.
- $z(0,t)$ parametrizes C.
- $z(1,t)$ parametrizes C_1.

Example 1.4



Theorem 1.5: Deformation invariance theorem

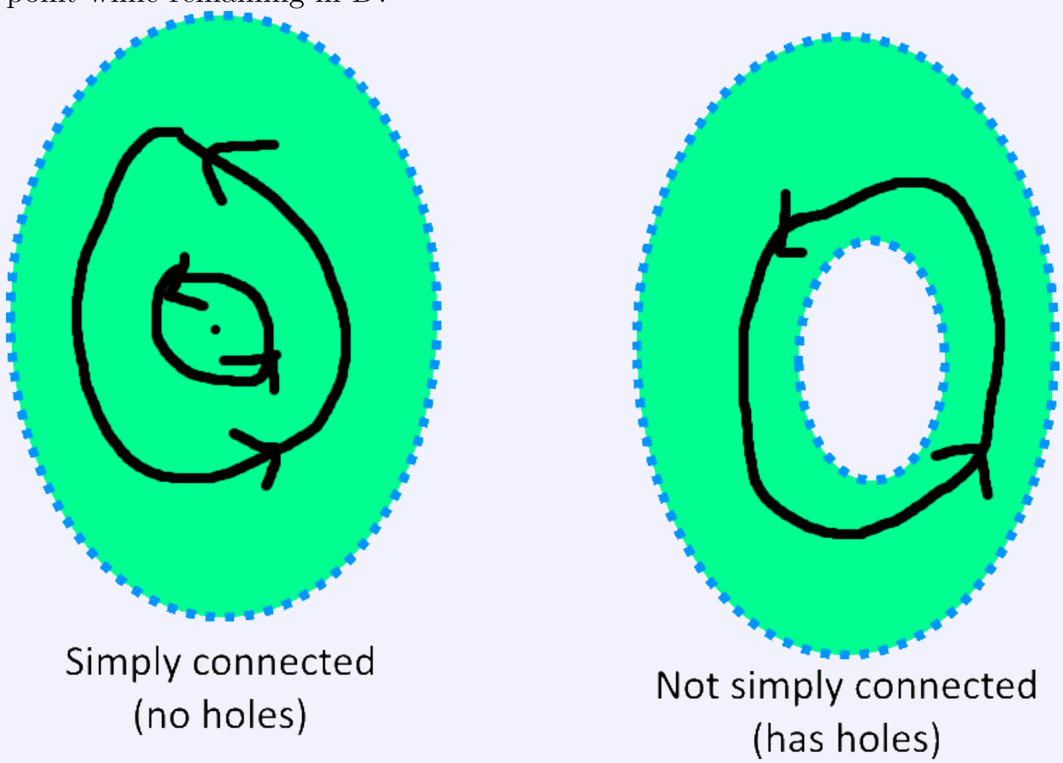
Let f be analytic in domain D containing contours C_1 and C_2. If C_1 can be continuously deformed into C_2, then

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$$

Proof. Too long to fit in these notes. □

Definition 1.6: Simply connected domain

A domain D is simply connected if every "loop" in D can be continuously deformed into a point while remaining in D.



Theorem 1.7: Cauchy's integral theorem (aka Cauchy-Coursat theorem)

If f is analytic in a simply connected domain D, and C is a closed contour in D, then

$$\oint_C f(z)dz = 0$$

Proof. Since D is simply connected, C can be continuously deformed into a point, so the result holds by deformation invariance theorem. □

Corollary 1.8

If f is analytic then f is infinitely anti-differentiable.

Proof. By Cauchy's integral theorem, $\oint_C f(z)dz = 0$ for any closed contour C, and so by theorem f is anti-differentiable. The anti-derivative of f must be analytic, so the anti-derivative itself must also be anti-differentiable, etc. $\square$

Example 1.9

We have

$$\oint_C \frac{1}{z} dz = 2\pi i$$

for any positively oriented contour C enclosing the origin. We could shift this:

$$\oint_C \frac{1}{z - z_0} dz = \begin{cases} 2\pi i & \text{if } z_0 \text{ is inside } C \\ 0 & \text{otherwise} \end{cases}$$