Numerical Analysis of Convection in the Inner Core (DRAFT)

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Abstract

Convection in the Earth's inner core has been a contentious topic in geoscience. Recently, it has been proposed through siesmic observations that Earths inner core is convecting. Here we numerically model convection in the Inner core, using the streamfunction-voriticty formulation in 2 dimensions and a three dimensional lattice boltzman approach. We find ...

Introduction

Plan: I want to also talk about why this is actually important, why is this something that is worth studying
Here I want to introduce the physics of what I want to talk about. I want to introduce the basic equations that I will use.
Throughout analysis we describe the fluid in the Eularian frame under a graviational acceleration $\vec{g}$ which may vary in space. We give each location in the fluid a velocity $\vec{u}$ and density ρ that vary in space $\vec{x}$ and time t. We assume the fluids viscosity μ, thermal diffusivity κ and specific heat capacity C_p are all constants. By conserving fluid momentum, we produce the navier stokes equation:

$$\rho\frac{D\vec{u}}{Dt} = \rho\vec{g} - \nabla p + \mu\nabla^2\vec{u}, \tag{1}$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{u}\cdot\nabla)$ is the material derivative, p the pressure of the fluid and ∇ the del operator. The dynamics of fluid temperature T are described the inhomoenous advection diffusion equation:

$$\frac{DT}{Dt} = \kappa\nabla^2 T + \frac{Q}{C_v\rho}, \tag{2}$$

where Q is the heat generated per unit volume per unit time, C_v the specific heat at constant volume and ρ the fluid density. The density of the fluid ρ is assumed to vary linearly in temeprature according to the equation of state:

$$\rho = \rho_0(1 - \alpha(T - T_0)), \tag{3}$$

where α is the volumetric expansion coeffecient and ρ_0 the density at a reference temperature T_0. Through siesmic imaging, variations in inner core density are $<< 1\%$. We also assume that the inner cores evolution occures over geologic timescales, and as such take $\vec{u}$ to be first order. These assumptions allow us to make the slow flow boussinesq approximation to equation 1:

$$\frac{\partial\vec{u}}{\partial t} = \frac{\rho'}{\rho}\vec{g} - \frac{\nabla p'}{\rho_0} + \nu\nabla^2\vec{u}, \tag{4}$$

where $\rho' = -\alpha(T - T_0)$, ν the kinomatic viscosity $\nu = \frac{\mu}{\rho_0}$ and p' a first order pertibation to the background pressure p_0.
The Earths

Numerical Methods

Two numerical methods are introduced for solving the thermal convection problem, the Lattice Boltzman Method and the streamfunction-vorticity formulation.

Here I want to introduce my two numerical schemes

Streamfunction-Vorticity formulation

Here I introduce the streamfunction-vorticity method and use it to eliminate pressure terms in 4 and 2 in cartesian and polar geometries giving a set of numerically solvable equations.

The streamfunction-vorticity formulation is a popular method for analytical and simple numerical analysis of incompressable fluids in two dimensions. Its key advantage is the elimination of all pressure terms, which would otherwise need to be iteratively accounted for or given in a constituative equation. We define the vorticity in the plane ω by:

$$\omega = (\nabla\times\vec{u})_z, \tag{5}$$

where the z subscript denotes the component out of the page. We also define a streamfunction ψ by:

$$\omega = -\nabla^2\psi. \tag{6}$$

Physically, the voriticty is the amount of spinning the fluid does about a point, while lines of constant streamfunction have the fluid flow perpendicular to them. Given a coordinate system, and a clever definition of $\vec{u}$ we can rewrite equations 2 and 4 in terms of ω and ψ rather than $\vec{u}$. In cartesian coordinates (x, y) we pick

$$u = \frac{\partial \psi}{\partial y}, v = -\frac{\partial \psi}{\partial x}, \tag{7}$$

allowing us to write equations 2 and 4 as:

$$\frac{\partial T}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial T}{\partial y} = \kappa \nabla^2 T + \frac{Q}{\rho_0 C_v} \tag{8}$$

$$\frac{\partial w}{\partial t} = -\frac{g_y}{\rho_0}\frac{\partial \rho'}{\partial x} + \nu \nabla^2 \omega \tag{9}$$

Similary, in polar coordinates (r, θ) we pick:

$$u = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, v = -\frac{\partial \psi}{\partial r}, \tag{10}$$

giving:

$$\frac{\partial T}{\partial t} + \frac{1}{r}\frac{\partial \psi}{\partial \theta}\frac{\partial T}{\partial r} - \frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial T}{\partial \theta} = \kappa \nabla^2 T + \frac{Q}{\rho_0 C_v} \tag{11}$$

and,

$$\frac{\partial \omega}{\partial t} = -\frac{g_r}{\rho_0 r}\frac{\partial \rho'}{\partial \theta} + \nu \nabla^2 \omega. \tag{12}$$

A full derivation of equations 8, 9, 11, 12 are given in appendix. Importantly, our definitions of u and v in equations 7 and 10 satisify equation 5 and 6. Together, equations 6 8, 9, 11 and 12 contain no unknown quantities like pressure p and so can be solved via finite difference methods.

Solving the streamfunction-vorticity equations

The finite difference method used for solving the streamfunction-vorticity-formulated governing equations is shown

We first discritize our domain $\mathcal{D}$. In the cartesian case, we use $(x_i, y_j) = (i\Delta x, j\Delta y)$ with integers i and j satisfying $0 \le i < N_x$ $0 \le j < N_y$. In the polar case, we use $(r_i, \theta_j) = (R_0 + i\Delta r, j\Delta\theta)$ again with $0 \le i < N_r$ and $0 \le j < N_\theta$. We impose $\Delta\theta = \frac{2\pi}{N_\theta - 1}$ for consistancy with θ-periodic boundary conditions and an inner radius R_0 in polar coordinates to avoid singularities generated by $r = 0$. We also discritize time t by $t_n = n\Delta t$. For a function f, we use $f^n_{i,j}$ to mean f evaluated at time n at position (x_i, y_j) in cartesian coorindates or (r_i, θ_j) in polar coordinates.

To approximate derivatives we use a finite difference approach. All time derivatives are approximated by forward difference:

$$\frac{\partial f}{\partial t} = \frac{f^{n+1} - f^n}{\Delta t} \tag{13}$$

Second order space derivatives are apprimxated by a central difference:

$$\frac{\partial^2 f_{i,j}}{\partial {x_1}^2} = \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{{\Delta x_1}^2}, \tag{14}$$

$$\frac{\partial^2 f_{i,j}}{\partial {x_2}^2} = \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{{\Delta x_2}^2}, \tag{15}$$

where x_1 is the first coordinate and x_2 is the second coordinate. For example, in cartesian coordinates (x, y), we would have $x_1 = x$ and $x_2 = y$. For non advection terms, we approximate first order spatial derivaitves by:

$$\frac{\partial f_{i,j}}{\partial x_1} = \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta x_1}, \tag{16}$$

and

$$\frac{\partial f_{i,j}}{\partial x_2} = \frac{f_{i,j+1} - f_{i,j+1}}{2\Delta x_2}. \tag{17}$$

For advection terms, of the form $a\frac{\partial f_{i,j}}{\partial x_1}$ we employ a first order godanov scheme:

$$a\frac{\partial f_{i,j}}{\partial x_1} = \frac{1}{\Delta x}(\mid a \mid (\frac{1}{2}f_{i+1,j} - \frac{1}{2}f_{i-1,j}) - a(\frac{1}{2}f_{i+1,j} - f_{i,j} - \frac{1}{2}f_{i-1,j})). \tag{18}$$

This scheme is always downstream, regardless of the direction of the advecting field a.

Other more accurate, but substantually more complex methods for solving these equations, particularlly the advection equation exists such as the semi-lagrange crank-nicolson scheme.

To solve the streamfuntion-vorticity equations we assume a starting vorticity ω on our domain $\mathcal{D}$. We then apply the Jacobi method to solve equation 6 for ψ on the interior of the domain which we call $\mathcal{D}'$. Using ψ we update T on $\mathcal{D}'$ using equation 8 (or 11 for polar). Finally, ω is updated on $\mathcal{D}'$ using equation 9 (12 for polar). This process is repeated.

The Jacobi Method

Here a basic numerical method for solving the Poisson equation, the Jacobi method is outlined. We cannot solve for ψ explicitly in equation 6. Instead we use an iterative jacobi method. In cartesian coordinates, equation 6 is:

$$\omega_{i,j} = \frac{\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j}}{\Delta x^2} + \frac{\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}}{\Delta y^2} \tag{19}$$

Rearanging for ψ

$$\psi_{i,j} = \frac{\Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)}\left(\frac{\psi_{i+1,j} + \psi_{i-1,j}}{\Delta x^2} + \frac{\psi_{i,j+1} + \psi_{i,j-1}}{\Delta y^2} + \omega_{i,j}\right). \tag{20}$$

We then use the result of $\psi_{i,j}$ back into equation 20 to solve for $\psi_{i,j}$ iterately as shown in equation 21:

$$\psi_{i,j}^{(k+1)} = \frac{\Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)}\left(\frac{\psi_{i+1,j}^{(k)} + \psi_{i-1,j}^{(k)}}{\Delta x^2} + \frac{\psi_{i,j+1}^{(k)} + \psi_{i,j-1}^{(k)}}{\Delta y^2} + \omega_{i,j}\right). \tag{21}$$

Where the superscript (k) means the result of the k^{th} iteration of the above equation and this operation us applied to all points in $\mathcal{D}'$. We terminate this iterative method once the error $\sum_{(i,j)\in\mathcal{D}'} \mid \psi_{i,j}^{(k+1)} - \psi_{i,j}^{(k)} \mid$ gets suffeciently small. A similar method is applied to the polar coordinate case.

Solution Stability and Accuracy

The solution stability is

Lattice Boltzman Method

The lattice boltzman method

I want to introduce the lattice boltzman method, go over its derivation, talk about why this is unique in the world of computational fluid dynamics. I pretty much just want to explain what it is and how I used it.

Boundary Conditions

In both cases

Advection and Heat Conservation

We next tested the accuracy of the advection schemes. Thermal diffusivity (κ) and thermal expansion (α) were set to zero to avoid convection and a segment of the fluid was set to 1 unit of temperature with all other regions set to 0 units of temperature. In the first test, the streamfunction was set so that the velocity field was purely horizontal and azimuthal in the cartesian and polar cases respectively.

The second set of tests moved the fluid in a diagonal motion over the domain bringing it back to its origin state.

I will also have a test of heat conservation. Here. This will be simple, just a few pictures of the codes running (show some nice plumes) and the energy vs time graph. It shows that neither of these conserve thermal energy.

Results

Figure 1: Polar azimuthal advection test. Temperature (color) is plotted in polar space. Red dot indicates temperature weighted mean position within the fluid and is taken as the location of the temperature 1 zone. (a) shows the initial condition, (b) shows the fluid being advected through the periodic boundry condition, (c) the analytic result for when the fluid would return to its initial state and (d) the final state of the fluid. Times are given above each pane.

Figure 2: Azimuthal advection test. Temperature (color) is plotted in polar space.

Figure 3: Cartesian diagonal advection test. Temperature (color) is plotted in polar space. Red dot indicates temperature weighted mean position within the fluid and is taken as the location of the temperature 1 zone.