Convection In A Box

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What is this?

A document where I can write down my working for modelling convection in a box as paper isn't the best at conveying ideas remotely. This is in no way polished document.

The Problem

We seek to model 2D convection for a slow moving fluid. We will format this problem using streamfunctions with the aim of solving it numerically. First, the solution in a box using cartesian cooridnates is deicussed. The problem is then formatted in polar coordinates and solved on a disk.

The bousinesque Approximation

In the bousinique approximation we assume density flucations are small. This leads us to considering only density flucations when they are multiplies by a gravity term.
We begin with the NSE, equation 6. We approximate the density by

$$\rho = \rho_0 + \rho', \tag{1}$$

where ρ_0 is a constant reference density and $\rho' << \rho_0$ is a pertibuation that depends on space. We similary split pressure up by equation 2.

$$p = p_0 + p' \tag{2}$$

Applying equation 1 and 2, the Navier-Stokes equation reads:

$$(\rho_0 + \rho')\frac{\partial \vec{u}}{\partial t} + (\rho_0 + \rho')(\vec{u} \cdot \nabla)\vec{u} = (\rho_0 + \rho')\vec{g} - \nabla(p_0 + p') + \mu\nabla^2\vec{u} \tag{3}$$

By assuming $\vec{u}$ is also first order, equation 3 to zeroth order produces:

$$\rho_0\vec{g} = \nabla p_0, \tag{4}$$

and so to first order:

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = \frac{\rho'}{\rho_0}\vec{g} - \frac{\nabla p'}{\rho_0} + \nu\nabla^2\vec{u}, \tag{5}$$

where $\mu = \rho\nu$.
And thats the bousinesque equation.

Equations

The navier stokes equation (equation 6) describes the conservation of momentum for an incompressable fluid.

$$\rho\frac{\partial \vec{u}}{\partial t} + \rho(\vec{u} \cdot \nabla)\vec{u} = \rho\vec{g} - \ \rho\nabla p + \mu\nabla^2\vec{u} \tag{6}$$

The density ρ is assumed to be a function of temperature T and governed by equation 7

$$\rho = \rho_0(1 - \alpha(T - T_0)), \tag{7}$$

where α is a coeffeicnet of thermal expansion and ρ_0 and T_0 are refernece densities and temperatures. The temperature is non constant, and modelled using equation **??**

$$\frac{\partial T}{\partial t} + (\vec{u} \cdot \nabla)T = \kappa\nabla^2 T + \frac{Q}{C_p}, \tag{8}$$

where κ is the diffusion constant, C_p the specific heat capacity per volume and Q a heat source.
We apply the creeping flow approximation first, giving:

$$\rho\frac{\partial \vec{u}}{\partial t} = \rho\vec{g} - \nabla p + \mu\nabla^2\vec{u} \tag{9}$$

Next we apply the bousinesque approximation giving:

$$\frac{\partial \vec{u}}{\partial t} = \frac{\rho'}{\rho}\vec{g} - \frac{\nabla p'}{\rho_0} + \nu\nabla^2\vec{u} \tag{10}$$

We now aim to numerically solve equations 10, 8 and 7 using a finite difference scheme.

Streamfunction-vorticity formulation in cartesian coorindates

We consider the problem in 2D cartesian co-ordinates (x,y) with $\vec{u} = (u, v)$, where u and v are the x and v velocities of the fluid respectively. We define a streamfunction ψ by equation **??**

$$u = -\frac{\partial \psi}{\partial y}, v = \frac{\partial \psi}{\partial x}. \tag{11}$$

We also define the vorticity in the $x - y$ plane ω by:

$$\omega = \nabla \times \vec{u} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{12}$$

By these definitions of the streamfunction and vorticity we get the incompressability condition (equation 13) and the vorticity-streamfunction relation, equation 14:

$$0 = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \nabla \cdot \vec{u} \tag{13}$$

$$\omega = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}. \tag{14}$$

Using the streamfunction we can immediately rewrite equation 8 as equation 15.

$$\frac{\partial T}{\partial t} - \frac{\partial \psi}{\partial y}\frac{\partial T}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial T}{\partial y} = \kappa\nabla^2 T + \frac{Q}{C_p} \tag{15}$$

The streamfunction-vorticity equation for the Navier Stokes equation is a little more involved. We first split equation 10 into its components:

$$\frac{\partial u}{\partial t} = \frac{\rho'}{\rho_0}g_x - \frac{1}{\rho_0}\frac{\partial p'}{\partial x} + \nu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2}) \tag{16}$$

and,

$$\frac{\partial v}{\partial t} = \frac{\rho'}{\rho_0}g_y - \frac{1}{\rho_0}\frac{\partial p'}{\partial y} + \nu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial x^2}). \tag{17}$$

Taking $\frac{\partial}{\partial x}$ of equation 17 and subtracting $\frac{\partial}{\partial y}$ of equation 16 we rearange to get equation 18:

$$\frac{\partial w}{\partial t} = \frac{g_y}{\rho_0}\frac{\partial \rho'}{\partial x} - \frac{g_x}{\rho_0}\frac{\partial \rho'}{\partial y} + \nu\nabla^2\omega \tag{18}$$

Equations 18 and 15 complete our streamfunction-vortivity description.

Boundry conditions

There are two classes of boundry conditions we must adress, fluid related boundy conditions and temperature related boundry conditions. We first consider the fluid related boundry conditions. We take the four boundries as stationary impermiable walls. (I cant draw latex graphics yet so I dont have a figure for this). We have four walls at $x = 0, x = L, y = 0, y = L$. At each wall, the normal fluid velocity is zero. For example, at the $y = 0$ and $y = L$ walls, we have:

$$u = \frac{\partial \psi}{\partial y} = 0 \tag{19}$$

and thus,

$$\psi = constant. \tag{20}$$

This is true for all four walls, which are connected. So on the walls, $\psi = constant$. From equation 12 on the left and right walls:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \frac{\partial^2\psi}{\partial y^2} = \omega_{wall} \tag{21}$$

and similary on the top and bottom wall:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \frac{\partial^2\psi}{\partial x^2} = \omega_{wall}. \tag{22}$$

Equations 20, 21 and 22 compose the boundry conditions used here.

Streamfunction-vorticity formulation in polar coorindates

Here we consider our domain $\mathcal{D}$ as a collection of discrized points (r, ϕ) where $r \in \{r_{inner}, R\}$ and $\phi \in \{0, 2\pi\}$. We define our velocity vector $\vec{u} = (u, v)$, where u points radially and v in the $\hat{\phi}$ direction. We define our streamfunction ϕ by equation 23:

$$u = \frac{1}{r}\frac{\partial\psi}{\partial\phi}, v = -\frac{\partial\psi}{\partial r}. \tag{23}$$

Similary to the cartesian case then, the vorticity is given by:

$$\omega = \nabla \times \vec{u} = \frac{\partial v}{\partial r} + \frac{1}{r}v - \frac{1}{r}\frac{\partial u}{\partial\phi} \tag{24}$$

By plugging the definitions of the velocities using the streamfunction into equation 24 we arrive at:

$$\omega = -\nabla^2\psi \tag{25}$$

Equation 10 can be seperated into equations 26 and 27:

$$\frac{\partial u}{\partial t} = \frac{\rho'}{\rho_0}g_r - \frac{1}{\rho_0}\frac{\partial p'}{\partial r} + \nu(\nabla^2 u - \frac{2u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial\phi}) \tag{26}$$

$$\frac{\partial v}{\partial t} = \frac{\rho'}{\rho_0}g_\phi - \frac{1}{r\rho_0}\frac{\partial p'}{\partial\phi} + \nu(\nabla^2 v + \frac{2}{r^2}\frac{\partial u}{\partial\phi} - \frac{v}{r^2}) \tag{27}$$

By multiplying equation 27 we get:

$$r\frac{\partial v}{\partial t} = r\frac{\rho'}{\rho_0}g_\phi - \frac{1}{\rho_0}\frac{\partial p'}{\partial\phi} + r\nu(\nabla^2 v + \frac{2}{r^2}\frac{\partial u}{\partial\phi} - \frac{v}{r^2}) \tag{28}$$

Now now cross differentiate similar to the cartesian case to give:

$$\frac{\partial r\omega}{\partial t} = \frac{-g_r}{\rho_0}\frac{\partial\rho'}{\partial\phi} + \nu(\frac{\partial}{\partial r}(r(\nabla^2 v + \frac{2}{r^2}\frac{\partial u}{\partial\phi} - \frac{v}{r^2})) - \frac{\partial}{\partial\phi}(\nabla^2 u - \frac{2u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial\phi})) \tag{29}$$

Note that we have assumed that $g_\phi = 0$ in equation 29.

$$\frac{\partial\omega}{\partial t} = -\frac{g_r}{\rho r}\frac{\partial\rho'}{\partial\phi} - \frac{3\psi^{(0,2)}}{r^4} - \frac{\psi^{(0,4)}}{r^4} - \frac{\psi^{(1,0)}}{r^3} + \frac{2\psi^{(1,2)}}{r^3} + \frac{\psi^{(2,0)}}{r^2} - \frac{2\psi^{(2,2)}}{r^2} - \frac{2\psi^{(3,0)}}{r} - \psi^{(4,0)}, \tag{30}$$

where $\psi^{(i,j)}$ means the derivative of ψ with respect to r i times and ϕ j times.
The advection-diffusion diffusion equation given in equation **??** follows a similar derivation to the cartesian case:

$$\frac{\partial T}{\partial t} + \frac{1}{r}\frac{\partial\psi}{\partial\phi}\frac{\partial T}{\partial r} - \frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial T}{\partial\phi} = \kappa\nabla^2 T + \frac{Q}{C_p}. \tag{31}$$