

# Numerical Analysis of Convection in the Inner Core (DRAFT)

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September 2021

## Abstract

Convection in the Earth's inner core has been a contentious topic in geoscience. Recently, it has been proposed through seismic observations that Earth's inner core is convecting. Here we numerically model convection in the inner core, using the streamfunction-vorticity formulation in 2 dimensions and a three dimensional lattice boltzman approach. We find ...

## Introduction

Plan: I want to also talk about why this is actually important, why is this something that is worth studying  
Here I want to introduce the physics of what I want to talk about. I want to introduce the basic equations that I will use.  
Throughout analysis we describe the fluid in the Eulerian frame under a gravitational acceleration  $\vec{g}$  which may vary in space. We give each location in the fluid a velocity  $\vec{u}$  and density  $\rho$  that vary in space  $\vec{x}$  and time  $t$ . We assume the fluid's viscosity  $\mu$ , thermal diffusivity  $\kappa$  and specific heat capacity  $C_p$  are all constants. By conserving fluid momentum, we produce the Navier-Stokes equation:

$$\rho \frac{D\vec{u}}{Dt} = \rho \vec{g} - \nabla p + \mu \nabla^2 \vec{u}, \quad (1)$$

where  $\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{u} \cdot \nabla)$  is the material derivative,  $p$  the pressure of the fluid and  $\nabla$  the del operator. The dynamics of fluid temperature  $T$  are described by the inhomogeneous advection-diffusion equation:

$$\frac{DT}{Dt} = \kappa \nabla^2 T + \frac{Q}{C_v \rho}, \quad (2)$$

where  $Q$  is the heat generated per unit volume per unit time,  $C_v$  the specific heat at constant volume and  $\rho$  the fluid density. The density of the fluid  $\rho$  is assumed to vary linearly in temperature according to the equation of state:

$$\rho = \rho_0(1 - \alpha(T - T_0)), \quad (3)$$

where  $\alpha$  is the volumetric expansion coefficient and  $\rho_0$  the density at a reference temperature  $T_0$ . Through seismic imaging, variations in inner core density are  $\ll 1\%$ . We also assume that the inner core's evolution occurs over geologic timescales, and as such take  $\vec{u}$  to be first order. These assumptions allow us to make the slow flow Boussinesq approximation to equation 1:

$$\frac{\partial \vec{u}}{\partial t} = \frac{\rho'}{\rho} \vec{g} - \frac{\nabla p'}{\rho_0} + \nu \nabla^2 \vec{u}, \quad (4)$$

where  $\rho' = -\alpha(T - T_0)$ ,  $\nu$  the kinematic viscosity  $\nu = \frac{\mu}{\rho_0}$  and  $p'$  a first order perturbation to the background pressure  $p_0$ . By introducing

## Numerical Methods

Here I want to introduce my two numerical schemes

### Streamfunction-Vorticity formulation

The streamfunction-vorticity formulation is a popular method for analytical and simple numerical analysis of incompressible fluids in two dimensions. Its key advantage is the elimination of all pressure terms, which would otherwise need to be iteratively accounted for or given in a constitutive equation. We define the vorticity of our fluid as:

$$\omega = \nabla \times \vec{u} \quad (5)$$

and the streamfunction  $\psi$  such that:

$$\omega = \nabla^2 \psi. \quad (6)$$

Physically, the vorticity is the amount of spinning the fluid does about a point, while lines of constant streamfunction have the fluid flow perpendicular to them.

We define a streamfunction  $\psi$  as the function such that:

$$\vec{u} = \nabla \times \psi \quad (7)$$

I want to introduce this in both cartesian AND polar coordinates

## **Finite Difference Schemes**

Here I want to look at how I numerically solved the streamfunction-vorticity equations, how did I discretize the domain, how did I approximate each derivative. I then want to look at the theoretical stability and accuracy of my equations.

## **Lattice Boltzman Method**

I want to introduce the lattice boltzman method, go over its derivation, talk about why this is unique in the world of computational fluid dynamics. I pretty much just want to explain what it is and how I used it.

## **Advection Tests**

## **Results**