Convection In A Box

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June 2021

The Problem

We will consider convection of a slow moving fluid in a box with a heat source on the bottom layer. We will format this problem using streamfunctions with the aim of solving it numerically,

The bousinesque Approximation

In the bousinique approximation we assume density flucations are small. This leads us to considering only density flucations when they are multiplies by a gravity term.
We begin with the NSE, equation 6. We approximate the density by

$$\rho = \rho_0 + \rho', \tag{1}$$

where ρ_0 is a constant reference density and $\rho' << \rho_0$ is a pertibuation that depends on space. We similary split pressure up by equation 2.

$$p = p_0 + p' \tag{2}$$

Applying equation 1 and 2, the Navier-Stokes equation reads:

$$(\rho_0 + \rho')\frac{\partial \vec{u}}{\partial t} + (\rho_0 + \rho')(\vec{u} \cdot \nabla)\vec{u} = (\rho_0 + \rho')\vec{g} - \nabla(p_0 + p') + \mu\nabla^2\vec{u} \tag{3}$$

By assuming $\vec{u}$ is also first order, equation 3 to zeroth order produces:

$$\rho_0\vec{g} = \nabla p_0, \tag{4}$$

and so to first order:

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = \frac{\rho'}{\rho_0}\vec{g} - \frac{\nabla p'}{\rho_0} + \nu\nabla^2\vec{u}, \tag{5}$$

where $\mu = \rho\nu$.
And thats the bousinesque equation.

Equations

The navier stokes equation (equation 6) describes the conservation of momentum for an incompressable fluid.

$$\rho\frac{\partial \vec{u}}{\partial t} + \rho(\vec{u} \cdot \nabla)\vec{u} = \rho\vec{g} - \ \rho\nabla p + \mu\nabla^2\vec{u} \tag{6}$$

The density ρ is assumed to be a function of temperature T and governed by equation **??**

$$\rho = \rho_0(1 - \alpha(T - T_0)), \tag{7}$$

where α is a coeffeicnet of thermal expansion and ρ_0 and T_0 are refernece densities and temperatures. The temperature is non constant, and modelled using equation **??**

$$\frac{\partial T}{\partial t} + (\vec{u} \cdot \nabla)T = \kappa\nabla^2 T + \frac{Q}{C_p}, \tag{8}$$

where κ is the diffusion constant, C_p the specific heat capacity per volume and Q a heat source.
We apply the creeping flow approximation first, giving:

$$\rho\frac{\partial \vec{u}}{\partial t} = \rho\vec{g} - \nabla p + \mu\nabla^2\vec{u} \tag{9}$$

Next we apply the bousinesque approximation giving:

$$\frac{\partial \vec{u}}{\partial t} = \frac{\rho'}{\rho}\vec{g} - \frac{\nabla p'}{\rho_0} + \nu\nabla^2\vec{u} \tag{10}$$

We now aim to numerically solve equations 10, **??** and **??** using a finite difference scheme.

Finite Difference Schemes