6.1 Logic Definition of Generalization:

a) Show empirically that the information limit of 2 prediction bits per parameter also holds for nearest neighbors.

We do this via recursion. We see the numbers tend around 2.

```
In [2]:
          1 def iteratively_remove_rows(X, y, original_X, original_y):
                if X.shape[0] <= 1:
                     return X, y
                for i in range(X.shape[0]):
                    X train = X.drop(X.index[i])
          6
                    y train = y.drop(y.index[i])
          9
                    clf = KNeighborsClassifier(n neighbors=1)
         10
                    clf.fit(X train, v train)
         11
         12
                    y_pred = clf.predict(original_X)
         13
                    accuracy = accuracy_score(original_y, y_pred)
         14
         15
                    if accuracy == 1:
                         return iteratively remove rows(X train, y train, original X, original y)
         16
         17
                return X, y
         18
```

```
In [3]:
          1 sizes and repeats = [(16, 2), (32, 4), (64, 8)]
          2 for size, repeat in sizes and repeats:
                y_sizes = []
                for in range(size):
                    X = np.array(list(product([0, 1], repeat=repeat)))
                    N_{\star} = X_{\star} shape
                    y = np.random.randint(2, size=(X.shape[0], 1)).ravel()
                    X = pd.DataFrame(X)
                    y = pd.Series(y)
                    X_fin, y_fin = iteratively_remove_rows(X, y, X, y)
         10
                    y_sizes.append(len(y_fin))
         11
                mean_size = np.mean(y_sizes)
         12
                 print(f"d={repeat}: n_full={2**repeat}, Avg. req. points for memorization n_avg={mean_size:.2f}, n_avg/d={2*
         13
        d=2: n_full=4, Avg. req. points for memorization n_avg=2.69, n_avg/d=1.49
        d=4: n_full=16, Avg. req. points for memorization n_avg=8.31, n_avg/d=1.92
```

b) Extend your experiments to multi-class classification.

Now they tend to 1.5.

```
In [*]:
          1 sizes and repeats = [(16, 2), (32, 4), (64, 8)]
            for size, repeat in sizes and repeats:
                v sizes = []
                for in range(size):
                    X = np.array(list(product([0, 1], repeat=repeat)))
                    N_{\star} = X_{\star} shape
                    y = np.random.randint(3, size=(X.shape[0], 1)).ravel()
                    X = pd.DataFrame(X)
         10
                    y = pd.Series(y)
                    X_fin, y_fin = iteratively_remove_rows(X, y, X, y)
                    y_sizes.append(len(y_fin))
         13
                mean size = np.mean(v sizes)
         14
                print(f"d={repeat}: n_full={2**repeat}, Avg. req. points for memorization n_avg={mean_size:.2f}, n_avg/d={2*
        d=2: n_full=4, Avg. req. points for memorization n_avg=2.81, n_avg/d=1.42
        d=4: n_full=16, Avg. req. points for memorization n_avg=11.03, n_avg/d=1.45
```

6.2 Finite State Machine Generalization

 a) Implement a program that automatically creates a set of if-then clauses from the training table of a binary dataset of your choice. Implement different strategies to minimize the number of if-then clauses. Document your strategies, the number of resulting conditional clauses, and the First I implement Algorithm 8 from the textbook in its original form.

Now use Algorithm 8 to create our basic set of if then clauses and test on AND.

```
n, d = data.shape
              df = pd.DataFrame(data).copy()
              df['sum'] = data.sum(axis=1)
df['label'] = labels
sorted_table = df.sort_values(by='sum').reset_index(drop=True)
        8
       10
             current class = sorted table.iloc[0, -1]
            16
       18
            return clauses
       1 # Check output using AND dataset
2 if_then_clauses(test_and[:, 0:-1], test_and[:, -1])
In [*]:
       Generate predictions and accuracy using if_then_clauses
```

```
In [*]:
             def predict_if_then(X, y):
    predictions = []
                      X = pd.DataFrame(X)
y = pd.Series(y)
                      X_train, X_test, y_train, y_test = train_test_split(X, y, test_size=0.40, random_state=42)
                      clauses = if_then_clauses(X_train, y_train)
            10
                      for row in range(X_test.shape[0]):
    row_sum = X_test.iloc[row].sum()
            13
14
15
                            predicted_class = y_train.iloc[0] #Copy from algo 8, set first class as default
                            for clause in clauses:
                                  parts = clause.split(" THEN ")
sum_threshold = float(parts[0].split(" >= ")[1])
clause_class = parts[1].replace("class=", "")
            16
17
            18
            19
                                  if row_sum >= sum_threshold:
    predicted class = int(clause class)
                            predictions.append(predicted_class)
            24
                       accuracy = accuracy_score(y_test, np.array(predictions))
return predictions, accuracy, clauses
            26
```

The above algorithm is optimal assuming the weights are zero. Now use gradient descent to optimise in case the weights are not 0.

```
def sigmoid(z):
    return 1 / (1 + np.exp(-z))
In [*]:
                        def update_weights(X, y, weights, learning_rate):
    predictions = sigmoid(np.dot(X, weights))
    gradient = np.dot(X.T, (predictions - y)) / len(y)
    weights -= learning_rate * gradient
    return weights
                  def predict(X, weights):
    probabilities = sigmoid(np.dot(X, weights))
    return [1 if prob >= 0.5 else 0 for prob in probabilities]
                   14
def evaluate(predictions, labels):
    acc = np.mean(predictions == labels)
                                   return acc
                  16
```

```
In [*]:
                    1
                         def if_then_clauses_weighted(data, labels, weights):
    clauses = []
                                   n, d = data.shape
                                   df = pd.DataFrame(data, columns=[f'X{i}' for i in range(d)])
df['weighted_sum'] = np.dot(data, weights)
df['label'] = labels
sorted_table = df.sort_values(by='weighted_sum').reset_index(drop=True)
                                    current_class = sorted_table.iloc[0, -1]
                    10
                                   for row in range(n):
    if sorted_table.iloc[row, -1] != current_class:
        current_class = sorted_table.iloc[row, -1]
        row_sum = sorted_table.loc[row, 'weighted_sum']
        conditions = ["X\fy]-\format\( (iax, value \) for idx, value in enumerate(sorted_table.iloc[row, :-2])\)
        clause = "IF weighted_sum >= {} THEN class={}".format\( (row_sum, current_class \)
                    12
13
14
                    16
                   18
                                                       clauses.append(clause)
```

```
20
                      return clauses
In [*]:
            1 def descent_opti(X, y):
2     X_train, X_test, y_train, y_test = train_test_split(X, y, test_size=0.4, random_state=39)
                      weights = np.ones(X_train.shape[1])
                      learning_rate = 0.01
                      for _ in range(100):
    weights = update_weights(X_train, y_train, weights, learning_rate)
                      predictions = predict(X_test, weights)
accuracy = evaluate(predictions, y_test)
clauses = if_then_clauses_weighted(X_train, y_train, weights)
            10
                      return predictions, accuracy, clauses, weights
```

b) Use the algorithms developed in (a) on different datasets. Again, observe how your choices make a difference.

Test on the iris dataset with only 2 outcomes (setosa and versicolor) and different logic gates. Note the optimised version works better for iris, where it makes sense to have different weights. However, the logic gate outcomes are not better with the optimised method, which make sense since weighting won't really work in these datasets, as there shouldn't be a difference in the weights of the features.

```
In [13]: 1 from sklearn.datasets import load_iris
          2 iris = load iris()
          3 X og = iris.data
          4 y_og = iris.target
          6 \text{ mask} = y_{og} < 2
          7 X = X og[mask]
          8 v = v oq[mask]
In [14]: 1 # Original Algo Results on binarised iris dataset
          predictions, accuracy, clauses = predict_if_then(X, y)
          3 print('Accuracy: ', accuracy, '\nClauses: ', clauses)
         Accuracy: 0.95
         Clauses: ['IF sum >= 11.5 THEN class=1']
In [15]:
          1 # Retest on iris
          predictions, accuracy, clauses, weights = descent_opti(X, y)
          4 print('Accuracy: ', accuracy, '\nClauses: ', clauses, '\nWeights: ', weights) #Success
         Accuracy: 1.0
         Clauses: ['IF weighted_sum >= 0.6636196471656209 THEN class=1']
         Weights: [-0.46199628 -0.03018867 0.67625226 0.9695596 ]
In [16]:
         1 xor = np.array([[0,1,1],[1,0,1], [1,1,0], [0,0,0]])
          2 predictions_xor, accuracy_xor, clauses_xor = predict_if_then(xor[:, 0:-1], xor[:, -1])
          3 predictions xor2, accuracy xor2, clauses xor2, weights xor2 = descent opti(xor[:, 0:-1], xor[:, -1])
          4 print('Orig Accuracy:', accuracy xor, '\nOptim Accuracy: ', accuracy xor2)
         Orig Accuracy: 0.5
         Optim Accuracy: 0.5
In [17]: 1 test_or = np.array([[0, 0, 0], [0, 1, 1], [1, 0, 1], [1, 1, 1]])
          2 predictions_or, accuracy_or, clauses_or = predict_if_then(test_or[:, 0:-1], test_or[:, -1])
          3 predictions_or2, accuracy_or2, clauses_or2, weights_or2 = descent_opti(test_or[:, 0:-1], test_or[:, -1])
          4 print('Orig Accuracy:', accuracy_or, '\nOptim Accuracy: ', accuracy_or2)
         Orig Accuracy: 1.0
         Optim Accuracy: 0.5
In [18]:
          1 xand = np.array([[0,1,0],[1,0,0], [1,1,1], [0,0,1]])
          2 | predictions_xand, accuracy_xand, clauses_xand = predict_if_then(xand[:, 0:-1], xand[:, -1])
          3 predictions xand2, accuracy xand2, clauses xand2, weights xand2 = descent opti(xand[:, 0:-1], xand[:, -1])
          4 print('Orig Accuracy:', accuracy_xand, '\nOptim Accuracy: ', accuracy_xand2)
         Orig Accuracy: 0.5
         Optim Accuracy: 0.5
```

c) Finally, use the programs developed in (a) on a completely random dataset, generated artificially. Vary your strategies but also the number of input columns as well as the number of instances. How many if-then clauses do you need?

This should not work on the random datasets because there are no correlations, so it doesn't matter what the algorithm used is. The accuracy should be about the same as random quessing, and this is indeed the case.

```
In [19]:
         1 feats = np.arrav(list(product([0. 1]. repeat=5)))
          2 N, _ = feats shape
          3 target = np.random.randint(2, size=(feats.shape[0], 1)).ravel()
          4 feats = pd.DataFrame(feats)
          5 target = pd.Series(target)
          8 predictions ft, accuracy ft, clauses ft = predict if then(feats, target)
          9 predictions ft2, accuracy ft2, clauses ft2, weights ft = descent opti(feats, target)
         10 print('Original Function: ', len(clauses_ft), '\nNew Function: ',
                   len(clauses ft2), '\nNew Function Accuracy: ', accuracy ft2)
         Original Function: 9
         New Function: 11
         New Function Accuracy: 0.38461538461538464
In [20]: 1 feats = np.array(list(product([0, 1], repeat=10)))
          2 N. = feats.shape
          3 target = np.random.randint(2, size=(feats.shape[0], 1)).ravel()
          4 feats = pd.DataFrame(feats)
          5 target = pd.Series(target)
          8 predictions ft, accuracy ft, clauses ft = predict if then(feats, target)
          9 predictions_ft2, accuracy_ft2, clauses_ft2, weights_ft = descent_opti(feats, target)
         10 print('Original Function: ', len(clauses_ft), '\nNew Function: ',
                   len(clauses_ft2), '\nNew Function Accuracy: ', accuracy_ft2)
         Original Function: 306
         New Function: 309
         New Function Accuracy: 0.43658536585365854
In [21]:
         1 feats = np.array(list(product([0, 1], repeat=12)))
          2 N, = feats.shape
          3 target = np.random.randint(2, size=(feats.shape[0], 1)).ravel()
          4 feats = pd.DataFrame(feats)
          5 target = pd.Series(target)
          8 predictions_ft, accuracy_ft, clauses_ft = predict_if_then(feats, target)
          9 predictions_ft2, accuracy_ft2, clauses_ft2, weights_ft = descent_opti(feats, target)
```

```
Original Function: 2455
New Function: 2458
New Function Accuracy: 0.5108330790357034
```

11

Original Function: 1262 New Function: 1213

10 print('Original Function: ', len(clauses_ft), '\nNew Function: ',

len(clauses ft2), '\nNew Function Accuracy: ', accuracy ft2)

len(clauses_ft2), '\nNew Function Accuracy: ', accuracy_ft2)

6.3 Compression:

a) Create a long random string using a Python program, and use a lossless compression algorithm of your choice to compress the string. Note the compression ratio.

```
In [23]:
          1 import zlib
          2 import random
          3 import sys
In [24]:
          1 N = [2**i for i in range(1, 13)]
          2 N
Out[24]: [2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096]
In [25]:
          1 string_size = []
          2 compressed sizes = []
          3 compression_ratios = []
             compression_ratios_bit = []
          6
          7
             for n in N:
          8
                 rndm_strng = ''.join(random.choices(string.ascii_lowercase + string.ascii_uppercase + string.digits
          9
                                                      , k=n)).encode('utf-8')
         10
                 compressed_rndm_strng = zlib.compress(rndm_strng)
         11
         12
                 # string_size.append(sys.getsizeof(rndm_strng))
         13
                 # compressed_sizes.append(sys.getsizeof(compressed_rndm_strng))
         14
                 # compression_ratios.append(sys.getsizeof(rndm_strng)/sys.getsizeof(compressed_rndm_strng))
         15
                 string_size.append(len(rndm_strng))
         16
         17
                 compressed sizes.append(len(compressed rndm strng))
         18
                 compression ratios.append(len(compressed rndm strng)/len(rndm strng))
         19
In [26]:
          1 df = pd.DataFrame({
                 'string_size': string_size,
          2
          3
                 'compressed_sizes': compressed_sizes,
                 'compression_ratios': compression_ratios
          5 })
          7 df
```

Out[26]:

	string_size	compressed_sizes	compression_ratios
0	2	10	5.000000
1	4	12	3.000000
2	8	16	2.000000
3	16	24	1.500000
4	32	40	1.250000
5	64	72	1.125000
6	128	129	1.007812
7	256	222	0.867188
8	512	415	0.810547
9	1024	797	0.778320
10	2048	1563	0.763184
11	4096	3097	0.756104

b) What is the expected compression ratio in (a)? Explain why?

The expected compression ratio if 1. This is because there should not be any patterns in the sting, meaning no patterns to use to reduce the size. However, we do not see this in practice. Rather, if the string is small then the compression has a high overhead, leading to a larger ratio than expected. On the other hand, longer strings changes the entropy. Basically, there are probably some patterns or redundancy that emerge with sufficiently large strings which can be used by the compression algorithm.

8.1 Maximum MEC of Neural Networks:

a) What is the maximum memory-equivalent capacity of the following neural networks

$$12 + min(12, 3) + min(3, 4) = 12 + 3 + 3 = 18$$

b) What is the maximum memory-equivalent capacity of the following neural networks

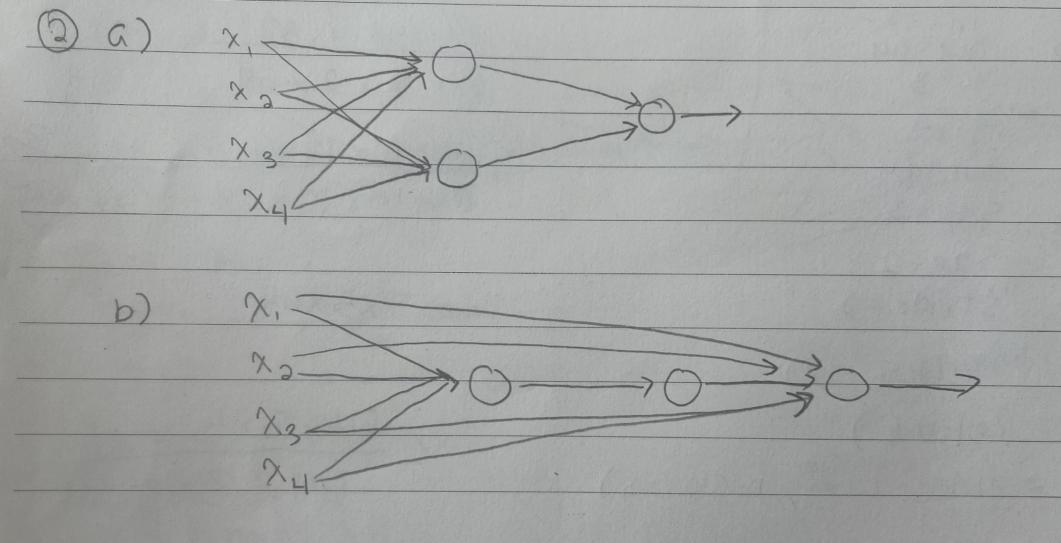
$$3 + 4 + 4 = 11$$

c) What is the maximum amount of rows that each network in (a) and (b) can memorize?

18 and 11 respectively

d) Answer (c) but for 4 classes instead of binary classification.

- · Logarithm calculation:
 - $\log_2(4) = 2$
- Calculations for networks:
 - Network A: $\frac{18}{2} = 9$
 - Network B: $\left\lfloor \frac{11}{2} \right\rfloor = 5$



8.4 Logic Definition of Generalization:

a) Do Exercise 40.8 in MacKay's book (MacKay 2003). It is cited here as follows:

Estimate in bits the total sensory experience that you have had in your life – visual information, auditory information, etc. Estimate how much information you have memorized. Estimate the information content of the works of Shakespeare. Compare these with the capacity of your brain assuming you have 1011 neurons each making 1000 synaptic connections and that the (information) capacity result for one neuron (two bits per connection) applies. Is your brain full yet?

Using the information found on this page for information transmission per second (https://www.britannica.com/science/information-theory/Physiology):

1. Visual: 10,000,000 per second:

Per Year = $10,000,000 \times 31,536,000 = 315.36$ TB/year

\$\$

2. Skin: 1,000,000 per second:

Per Year = $1,000,000 \times 31,536,000 = 31.536$ TB/year

Auditory: 100,000 per second:

Per Year = $100,000 \times 31,536,000 = 3.1536$ TB/year

4. Taste: 1,000 per second:

Per Year = $1,000 \times 31,536,000 = 0.031536$ TB/year

Total per year

350.081136 TB

I'm 27 years old, so multiplying that out gives me

 1.04×10^{16} bits experienced in my lifetime

Assuming (generously) about 20% is stored in memory, then we have:

 2.08×10^{15} bits memorized

Using this information for Shakespeare (https://nlp.stanford.edu/iR-book/html/htmledition/an-example-information-retrieval-problem-1.html). If the size of the collected works is about 1M words and we can assume 6 bytes per word, 6MB total.

Total for the brain:

$$10^{11} \times 10^3 \times 2 = 10^{11+3} \times 2 = 10^{14} \times 2 = 200 \times 10^{12}$$
 bits

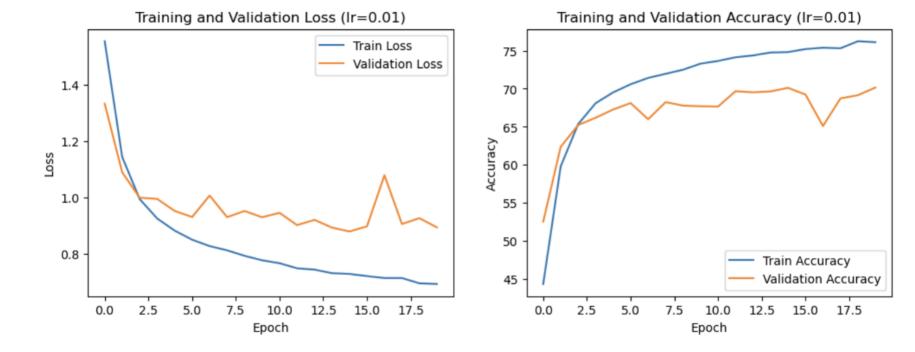
While my total sensory experience is far more than Shakespeare, my brain is far from full.

b) Expand Algorithm 8 to work with more than one binary classification.

The output of algorithm 8 with multi class classification is c/c-1 as stated in Law 5.1. We can confirm this with our outputs from section 6.2b, changing to a random dataset with c equi-distributed target classes. The clauses correspond to the thresholds, whereas the accuracy relates to the number of instances memorized. We divide the number of thresholds generated by the number of instances memorized.

c) Expand Algorithm 8 to work with regression.

As regression is essentially an infinite number of classes, c/c-1 tends to 1. Realistically, we expect more thresholds and less memorization the more points we add, hence the number of thresholds/memorization should increase. We can check using the same tactic as in part b, but with increasing numbers of classes.



b) Second, apply the measurements proposed in this book to reduce the hyperparameter search space and observe the model's performance.

In this part of the problem we add a layer to the network. This changes the compression ratio, so it now becomes

$$\left(\frac{64 \times 32 \times 32 \times 3}{64 \times 30 \times 30 \times 32} \right) \times \left(\frac{64 \times 30 \times 30 \times 32}{64 \times 28 \times 28 \times 32} \right) \times \left(\frac{64 \times 14 \times 14 \times 32}{64 \times 12 \times 12 \times 64} \right) \times \left(\frac{64 \times 12 \times 12 \times 64}{64 \times 10 \times 10 \times 64} \right) \times \left(\frac{64 \times 5 \times 5 \times 64}{64 \times 3 \times 3 \times 64} \right)$$

$$= \frac{32}{900} \times \frac{900}{784} \times \frac{98}{144} \times \frac{36}{25} \times \frac{25}{9} = 0.333 * 4 * 4 = 5.333$$

Each image is size 32x32x3 since there are 3 channels, and our ouput is now 576 into the linear layer. So

$$\frac{32 \times 32 \times 3}{576} = 5.333$$

which is the same

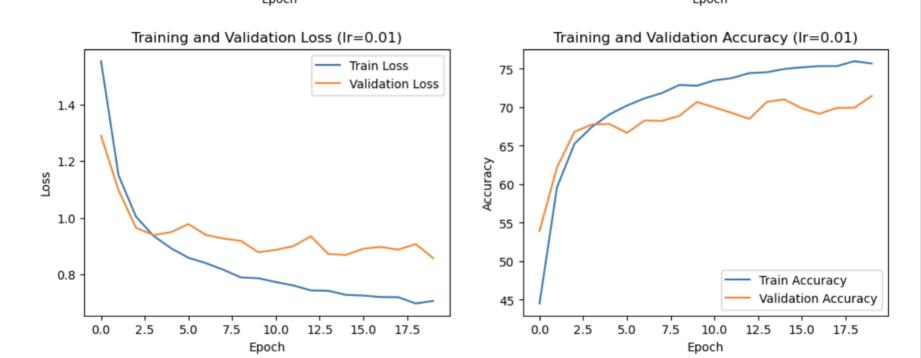
To compare this with the MEC, we multiply it by 8 bits, because in CIFAR-10, each pixel can maximally be represented with 8 bits.

$$\frac{32 \times 32 \times 3 \times 8}{5.33} = 4610.88$$

To find the MEC of the network, we calculate

$$(576 + 1) * 128 + min(128, 10) = 73866$$

```
[11]:
         class ConvNeuralNet(nn.Module):
       2
              # Determine what layers and their order in CNN object
       3
              def __init__(self, num_classes):
       4
                  super(ConvNeuralNet, self).__init__()
       5
                  self.conv_layer1 = nn.Conv2d(in_channels=3, out_channels=32, kernel_size=3)
       6
                  self.conv_layer2 = nn.Conv2d(in_channels=32, out_channels=32, kernel_size=3)
       7
                  self.max_pool1 = nn.MaxPool2d(kernel_size = 2, stride = 2)
       8
       9
                  self.conv_layer3 = nn.Conv2d(in_channels=32, out_channels=64, kernel_size=3)
                  self.conv_layer4 = nn.Conv2d(in_channels=64, out_channels=64, kernel_size=3)
      10
      11
                  self.max_pool2 = nn.MaxPool2d(kernel_size = 2, stride = 2)
      12
      13
                  self.conv layer5 = nn.Conv2d(in channels=64, out channels=64, kernel size=3)
      14
      15
                  self.fc1 = nn.Linear(576, 128)
                  self.relu1 = nn.ReLU()
      16
                  self.fc2 = nn.Linear(128, num_classes)
      17
      18
      19
              # Progresses data across layers
      20
              def forward(self, x):
      21
                  out = self.conv_layer1(x)
      22
                  #print(f"Shape of first conv : {out.shape}")
      23
                  out = self.conv_layer2(out)
                  # print(f"Shape of second conv : {out.shape}")
      24
      25
                  out = self.max_pool1(out)
      26
      27
                  out = self.conv_layer3(out)
      28
                  #print(f"Shape of third conv : {out.shape}")
      29
                  out = self.conv_layer4(out)
      30
                  #print(f"Shape of fourth conv : {out.shape}")
      31
                  out = self.max_pool2(out)
      32
      33
                  out = out.reshape(out.size(0), -1)
      34
      35
                  out = self.fc1(out)
      36
                  out = self.relu1(out)
      37
                  out = self.fc2(out)
      38
                  return out
```



a) First, experiment blindly with various hyperparameters and architectures and observe the model's performance.

We have experimented with multiple learning rates and kernel sizes. To save time, only the learning rates are included here. To calculate the compression ratio of this network we multiply

$$\left(\frac{64 \times 32 \times 32 \times 3}{64 \times 30 \times 30 \times 32}\right) \times \left(\frac{64 \times 30 \times 30 \times 32}{64 \times 28 \times 28 \times 32}\right) \times \left(\frac{64 \times 14 \times 14 \times 32}{64 \times 12 \times 12 \times 64}\right) \times \left(\frac{64 \times 12 \times 12 \times 64}{64 \times 10 \times 10 \times 64}\right)$$
$$= \frac{32}{900} \times \frac{900}{784} \times \frac{98}{144} \times \frac{36}{25} = 0.12 * 4 * 4 = 1.92$$

Each image is size 32x32x3 since there are 3 channels, and our ouput is 1600 in the linear layer. So

$$\frac{32 \times 32 \times 3}{1600} = 1.92$$

To compare this with the MEC, we multiply it by 8 bits, because in CIFAR-10, each pixel can maximally be represented with 8 bits.

$$\frac{32 \times 32 \times 3 \times 8}{1.92} = 12800$$

To find the MEC of the network, we calculate

$$(1600 + 1) * 128 + min(128, 10) = 204938$$

```
01:
        class ConvNeuralNet(nn.Module):
            # Determine what layers and their order in CNN object
     3
            def __init__(self, num_classes):
     4
                super(ConvNeuralNet, self).__init__()
     5
                self.conv_layer1 = nn.Conv2d(in_channels=3, out_channels=32, kernel_size=3)
     6
                self.conv_layer2 = nn.Conv2d(in_channels=32, out_channels=32, kernel_size=3)
     7
                self.max_pool1 = nn.MaxPool2d(kernel_size = 2, stride = 2)
     8
     9
                self.conv_layer3 = nn.Conv2d(in_channels=32, out_channels=64, kernel_size=3)
                self.conv_layer4 = nn.Conv2d(in_channels=64, out_channels=64, kernel_size=3)
    10
    11
                self.max_pool2 = nn.MaxPool2d(kernel_size = 2, stride = 2)
    12
    13
                self.fc1 = nn.Linear(1600, 128)
    14
                self.relu1 = nn.ReLU()
    15
                self.fc2 = nn.Linear(128, num classes)
    16
    17
            # Progresses data across layers
    18
            def forward(self, x):
    19
                out = self.conv_layer1(x)
    20
                #print(f"Shape of first conv : {out.shape}")
    21
                out = self.conv_layer2(out)
    22
                print(f"Shape of second conv : {out.shape}")
    23
                out = self.max pool1(out)
    24
    25
                out = self.conv_layer3(out)
    26
                #print(f"Shape of third conv : {out.shape}")
    27
                out = self.conv_layer4(out)
    28
                #print(f"Shape of fourth conv : {out.shape}")
    29
                out = self.max_pool2(out)
    30
    31
                out = out.reshape(out.size(0), -1)
    32
    33
                out = self.fc1(out)
    34
                out = self.relu1(out)
    35
                out = self.fc2(out)
    36
                return out
```