Question 1.

$m(X)=\frac{1}{N}\sum_{i=1}^{N}x_i \qquad cov(X,Y)=\frac{1}{N}\sum_{i=1}^{N}(x_i-m(X))(y_i-m(Y)) \qquad s^2=\frac{1}{N}\sum_{i=1}^{N}(x_i-m(X))^2$

1.1 $m(a+bX)=$

$$=\frac{1}{N}\sum_{i=1}^{N}(a+bx_i)$$

$$=\frac{1}{N}\left(\sum_{i=1}^{N}a+\sum_{i=1}^{N}bx_i\right)$$

$$=\frac{1}{N}\left(a\sum_{i=1}^{N}1+b\sum_{i=1}^{N}x_i\right)$$

$$=a\left(\cancel{\frac{1}{N}\sum_{i=1}^{N}1}\right)+b\cdot\frac{1}{N}\sum_{i=1}^{N}x_i$$

$$=a+b\cdot m(X)$$

1.2 $cov(X,a+bY)=$

$$=\frac{1}{N}\sum_{i=1}^{N}(x_i-m(X))(a+bY-m(a+bY))$$

$$=\frac{1}{N}\sum_{i=1}^{N}(x_i-m(X))(\cancel{a}+bY-\cancel{a}+b\cdot m(Y))$$

$$=\frac{1}{N}\sum_{i=1}^{N}(x_i-m(x))\,b\,(Y-m(Y))$$

$$=b\,cov(X,Y)$$

1.3 $cov(a+bX,a+bX)=$

$$=\frac{1}{N}\sum_{i=1}^{N}(a+bX-m(a+bX))(a+bX-m(a+bX))$$

$$=\frac{1}{N}\sum_{i=1}^{N}(\cancel{a}+bX-\cancel{a}+b\cdot m(X))(\cancel{a}+bX-\cancel{a}+b\cdot m(X))$$

$$=\frac{1}{N}\sum_{i=1}^{N}b(X-m(x))\,b(X-m(x))$$

$$=b^2\frac{1}{N}\sum_{i=1}^{N}(X-m(x))(X-m(x)) \rightarrow b^2s^2$$

$$=b^2\,cov(X,X)$$

1.4 $x\geq x' \Rightarrow g(x)\geq g(x')$ $\quad$ g is non-decreasing

let median(X) = m, so:

$P(X\leq m)\geq 0.5$ and $P(X\geq m)\geq 0.5$

$X\leq m\Rightarrow g(X)\leq g(m)$, $X\geq m\Longrightarrow g(X)\geq g(m)$

$P(g(X)\leq g(m))=P(X\leq m)\geq 0.5$, $P(g(X)\geq g(m))=P(X\geq m)\geq 0.5$

median$[g(X)]=g(\text{median}(X))$, for any non-decreasing transformation g.

Quartiles:

Q_1 : 25th percentile

Q_3 : 75th percentile

values $\leq Q_1$ become $\leq g(Q_1)$

values $\leq Q_3$ become $\leq g(Q_3)$

so

$Q_1[g(X)]=g(Q_1[X])$, $Q_3[g(X)]=g(Q_3[X])$

$IQR=Q_3-Q_1$, range = max − min

apply g

$IQR[g(X)]=g(Q_3)-g(Q_1)$

$Range[g(X)]=g(max)-g(min)$

1.5 Yes, for any non-decreasing transformation g, the median of g(X) equals g of the median of X because the transformation preserves the order of all values. Half the values remain below and above g(m(X)), so it satisfies the median definition. This only works if g is non-decreasing.