SLIIT ACADEMY

Mathematics II

Introduction to Differentiation

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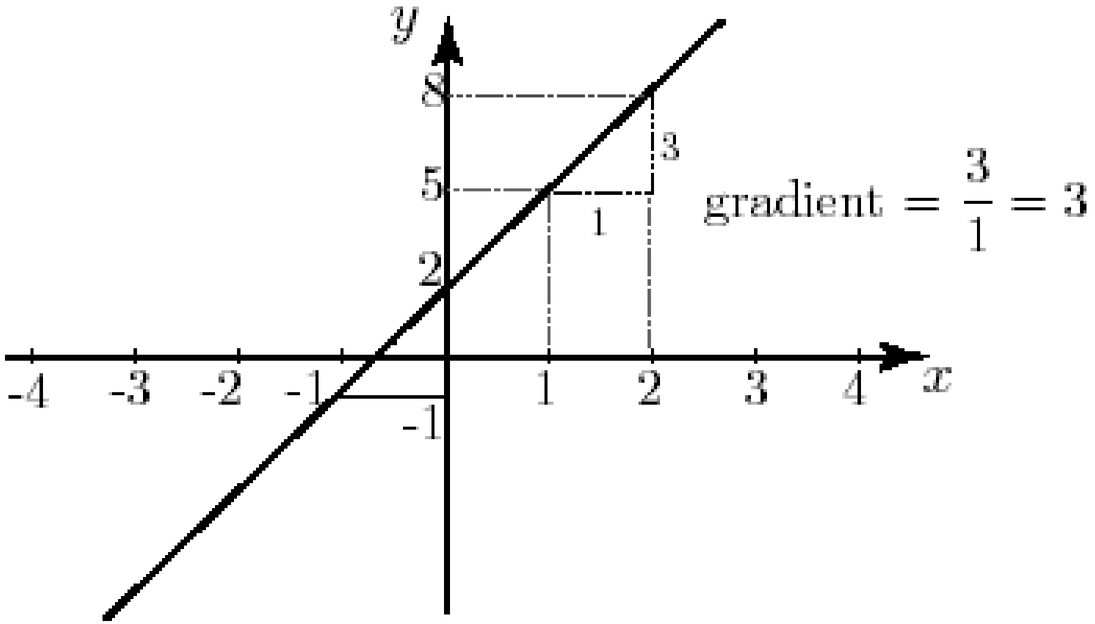
Differentiating a linear function

- A straight line has a constant gradient, or in other words, the rate of change of y with respect to x is a constant.

- Consider the straight line y = 3x + 2;



- We can find the gradient of this line by taking two points and calculating the change in y divided by the change in x.

- $$\text{The Gradient} = \frac{2-(-1)}{0-(-1)} = \frac{3}{1} = 3$$

- No matter which pair of points we choose the value of the gradient is always 3.

x	-3	-2	-1	0	1	2	3
$3x$	-9	-6	-3	0	3	6	9
2	2	2	2	2	2	2	2
$y = 3x + 2$	-7	-4	-1	2	5	8	11

Table 1: Table of values of $y = 3x + 2$

- Above note that y increases as a rate of 3 units, for every unit increase in x.

- We say that “the rate of change of y with respect to x is 3”.

- Observe that the gradient of the straight line is the same as the rate of change of y with respect to x.

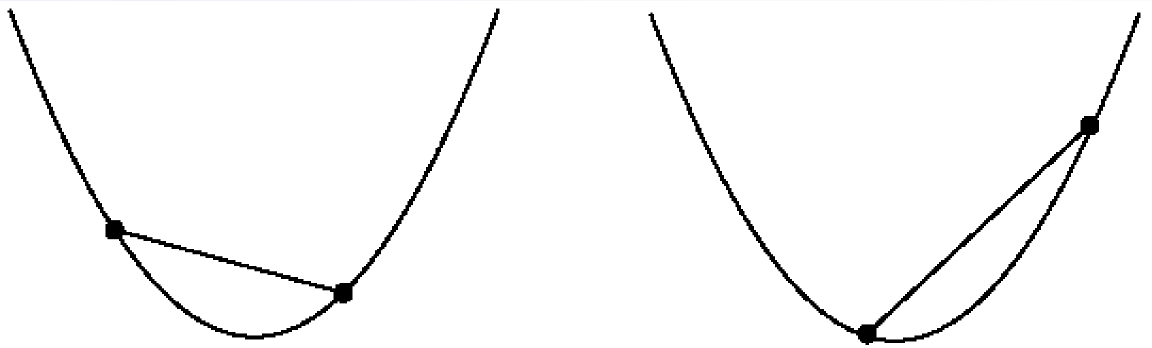
Key Point

- For a straight line:

"the rate of change of y with respect to x is the same as the gradient of the line."

Differentiation from first principles of some simple curves

- Consider the curve $y = x^2$



- Above for different pairs of points we will get different lines, with very different gradients.

x	-3	-2	-1	0	1	2	3
$y = x^2$	9	4	1	0	1	4	9

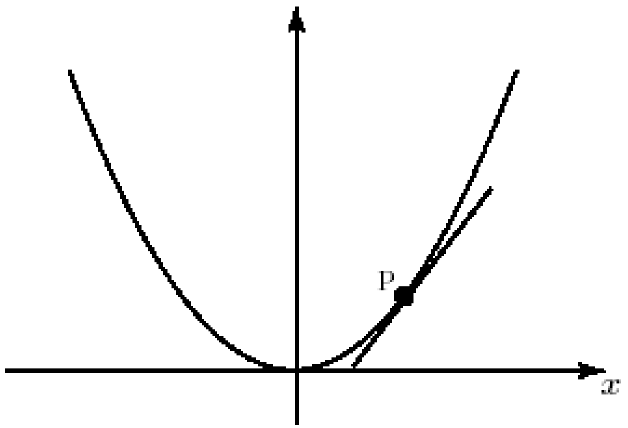
Table 2: values of $y = x^2$

- For a simple function like $y = x^2$ we see that y is not changing constantly with x.

- The rate of change of y with respect to x is not a constant.

Calculating the rate of change at a point

- Calculating the rate of change at any point on a curve $y = f(x)$ is defined to be the gradient of the tangent drawn at that point as shown below.

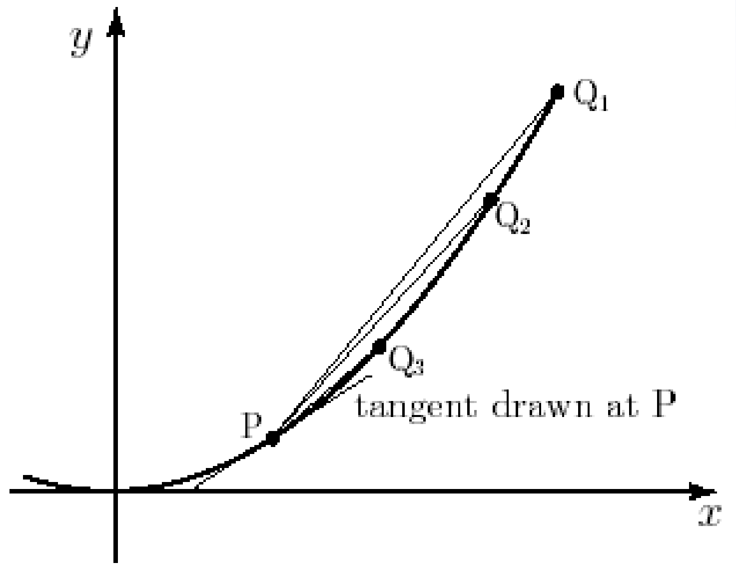


- The rate of change at a point ... dient of the tangent at P.

Key Point

- *The gradient of a curve $y = f(x)$ at a given point is defined to be the gradient of the tangent at that point.*

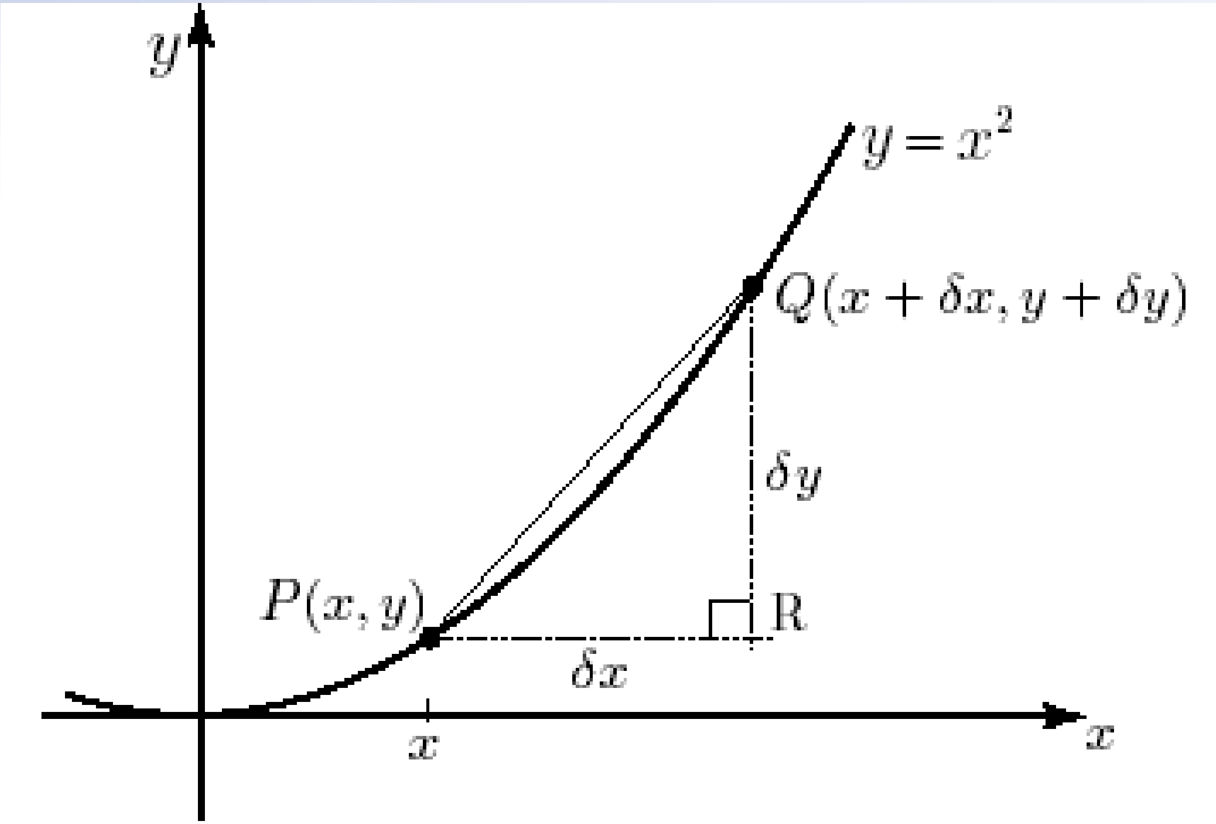
- Consider the figure below which shows a fixed point P on a curve $y=x^2$.



- The lines through P and Q approach the tangent at P when Q is very close to P.

- Calculate the gradient of one of these lines, and let the point Q approach the point P along the curve, then the gradient of the line should approach the gradient of the tangent at P, and **hence the gradient of the curve**.

- Consider a general point P which has coordinates (x, y).



- Choose point Q to be close to P on the curve.

- Because we are considering the graph of y = x^2;

$$y + \delta y = (x + \delta x)^2$$

$$y + \delta y = x^2 + 2x(\delta x) + (\delta x)^2$$

$$y = x^2;$$

$$\delta y = 2x(\delta x) + (\delta x)^2$$

- So the gradient of PQ is;

$$\frac{\delta y}{\delta x} = \frac{2x(\delta x) + (\delta x)^2}{\delta x}$$

$$\frac{\delta y}{\delta x} = \frac{\delta x(2x + \delta x)}{\delta x}$$

$$\frac{\delta y}{\delta x} = 2x + \delta x$$

- As we let δx become zero we are left with just 2x, and this is the formula for the gradient of the tangent at P.
- Gradient of tangent = $$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} (2x + \delta x) = 2x$$
- We can do this calculation in the same way for lots of curves. We have a special symbol for the phrase

$$\boxed{\lim_{\delta x \to 0} \frac{\delta y}{\delta x}}$$

- This is again written as “dy/dx” and referred to as “derivative of y with respect to x”.

Use of function notation

- We often use function notation y = f(x).

- Then, the point P has coordinates (x, f(x)). Point Q has coordinates $(x+\delta x,\ f(x+\delta x))$

- So, the change in y, that is δy is $f(x+\delta x) - f(x)$. Then,

$$\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

Key Point

- Given y = f(x), its derivative, or rate of change of y with respect to x is defined as;

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

Differentiation of $y = x^n$ when n is a positive integer

- Apply previous definition to the function $y = x^n$

- We have;

$$f(x) = x^n$$

$$f(x + \delta x) = (x + \delta x)^n$$

And so;

$$(x + \delta x)^n = x^n + nx^{n-1}\delta x + \ldots + (\delta x)^n$$

- Then, from the formula for the derivative;

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{[x^n + nx^{n-1}\delta x + \ldots + (\delta x)^n] - x^n}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{nx^{n-1}\delta x + \ldots + (\delta x)^n}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\delta x(nx^{n-1} + \ldots + (\delta x)^{n-1})}{\delta x}$$

$$= \lim_{\delta x \to 0} nx^{n-1} + \ldots + (\delta x)^{n-1}$$

- In the limit as x tends to zero, all the terms on the right, apart from the first become zero. We are left with the result that

$$\frac{dy}{dx} = nx^{n-1}$$

Key Point

- When n is a positive integer;

If $y = x^n$ *then* $$\frac{dy}{dx} = nx^{n-1}$$

- The result is true when n is a negative integer and when n is a fraction although we will not prove this here.

Linearity rules

- Also, if $y = k\,f(x)$ where k is a constant

then,
$$\frac{dy}{dx} = k\,\frac{df}{dx}$$

This means that we can differentiate a constant multiple of a function, simply by differentiating the function and multiplying by the constant.

Linearity rules

- If $y = f(x) \pm g(x)$

  then, $$\frac{dy}{dx} = \frac{df}{dx} \pm \frac{dg}{dx}$$

  This means that we can differentiate sums (and differences) of functions, term by term.

The Chain Rule

- The chain rule, exists for differentiating a function of another function.

- Consider the expression $(x^4+x^2-9)^{10}$. We can call such an expression a ‘function of a function’.

- Suppose, in general, that we have two functions, f(x) and g(x). Then y = f(g(x)) is a function of a function.

- $g(x) = x^4+x^2-9$ and $f(x) = x^{10}$

  $f(g(x)) = f(x^4+x^2-9) = (x^4+x^2-9)^{10}$

Key Point

- To differentiate y = f(g(x)), let u = g(x).

  Then y = f(u) and $$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Exercise

- Differentiate $y = (2x-5)^{10}$

Let u = 2*x*-5 so that y = u^{10}. It follows that

$$\frac{du}{dx} = 2$$

$$\frac{dy}{du} = 10u^9$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= 10u^9 \times 2$$

$$= 20(2x-5)^9$$

The Product Rule

- The product rule: if ***y = uv*** then

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Exercise

- Find the derivative of $y = (3x - 2x^2)\ (5 + 4x)$.

$$\frac{dy}{dx} = (3x - 2x^2)\frac{d}{dx}[5 + 4x] + (5 + 4x)\frac{d}{dx}[3x - 2x^2]$$

$$= (3x - 2x^2)(4) + (5 + 4x)(3 - 4x)$$

$$= (12x - 8x^2) + (15 - 8x - 16x^2)$$

$$= 15 + 4x - 24x^2$$

Key Point

$$\frac{d}{dx}(f(y)) = \frac{d}{dy}(f(y)) \times \frac{dy}{dx}$$

Remember, every time we want to differentiate a function of y with respect to x, we differentiate with respect to y and then multiply by (dy/dx).

The Quotient Rule

- The quotient rule: if $\boldsymbol{y = \frac{u}{v}}$ then

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

Exercise

- Find the derivative of

$$y = \frac{x-1}{2x+3}$$

$$\frac{dy}{dx} = \frac{(2x+3)\frac{d}{dx}[x-1] - (x-1)\frac{d}{dx}(2x+3)}{(2x+3)^2}$$

$$= \frac{(2\mathrm{x}+3)(1)-(\mathrm{x}-1)(2)}{(2\mathrm{x}+3)^2}$$

$$= \frac{5}{(2\mathrm{x}+3)^2}$$

Higher- order derivatives

- The derivative of ***f'*** is the second derivative of ***f*** and is denoted by ***f ''***.

$$\frac{d}{dx}[f'(x)] = f''(x)$$

- The derivative of ***f''*** is the third derivative of f and is denoted by ***f '''***.

Example

$$f(x) = 2x^4 - 3x^2$$

$$f'(x) = 8x^3 - 6x$$

$$f''(x) = 24x^2 - 6$$

$$f'''(x) = 48x$$

$$f^{(4)}(x) = 48$$

$$f^{(5)}(x) = 0$$

Higher- order derivatives

Find the 4th derivative of function $y = x^4 - 5x^2 + 2x$?

Higher- order derivatives

Find the 4th derivative of function $y = x^4 - 5x^2 + 2x$?

$$\frac{dy}{dx} = 4x^3 - 10x + 2$$

$$\frac{d^2y}{dx^2} = 12x^2 - 10x$$

$$\frac{d^3y}{dx^3} = 24x^1 - 10$$

$$\frac{d^4y}{dx^4} = 24$$

Implicit differentiation

- Sometimes functions are given not in the form $y = f(x)$ but in a more complicated form in which it is difficult or impossible to express y explicitly in terms of x.

- Such functions are called implicit functions.

- Now we look at how we might differentiate functions of y with respect to x.

- Consider an expression such as;

$$x^2 + y^2 - 4x + 5y - 8 = 0$$

- It would be quite difficult to re-arrange this so y was given explicitly as a function of x.

Example -

Suppose we want to differentiate the implicit function with respect to x.

$$y^2 + x^3 - y^3 + 6 = 3y$$

We differentiate each term with respect to x:

$$\frac{d}{dx}(y^2) + \frac{d}{dx}(x^3) - \frac{d}{dx}(y^3) + \frac{d}{dx}(6) = \frac{d}{dx}(3y)$$

$$\frac{d}{dy}(y^2) \times \frac{dy}{dx} + 3x^2 - \frac{d}{dy}(y^3) \times \frac{dy}{dx} + 0 = \frac{d}{dy}(3y) \times \frac{dy}{dx}$$

$$2y\frac{dy}{dx} + 3x^2 - 3y^2\frac{dy}{dx} = 3\frac{dy}{dx}$$

rearrange

$$3x^2 = 3y^2\frac{dy}{dx} - 2y\frac{dy}{dx} + 3\frac{dy}{dx}$$

$$3x^2 = (3y^2 - 2y + 3)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{3x^2}{(3y^2 - 2y + 3)}$$

Implicit differentiation

A function that defines the relationship between x and y but y is not the subject is called an Implicit function.

Example 1: $\boldsymbol{y^3 = x^2 + xy + xy^2}$

Derivative of Implicit functions can be found in terms of x and y by using the chain rule.

$$3y^2 \times \frac{dy}{dx} = 2x + x.\frac{dy}{dx} + y.1 + x.2y.\frac{dy}{dx} + y^2.1$$

$$\frac{dy}{dx}(3y^2 - x - 2xy) = 2x + y + y^2$$

$$\frac{dy}{dx} = \frac{2x + y + y^2}{3y^2 - x - 2xy}$$

Implicit differentiation

Example 2: $y^2 = x^2 y + \frac{x}{y} + x$

Derivative of Implicit functions can be found in terms of x and y by using the chain rule.

Differentiation of parametric functions

Chain rule can be used to find the rate of change in y with respective to x when x and y both are defined using a parameter (t).

- Example: If $x = f(t)$

  $y = g(t)$

Then, $$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

And, $$\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}$$

Therefore, $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}}$$

Differentiation of parametric functions

- Example: $x = 6t + 1$

  $y = 4t^3$

$$\frac{dx}{dt} = 6$$

$$\frac{dy}{dt} = 12t^2$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{12t^2}{6}$$

$$= 2t^2$$

Thank You