Hands-on PVS for mathematicians!

Case of Study:

The infinitude of primes by Fürstenberg's topological argument

PVS for Mathematicians - Tutorial CICM+LSFA 2025

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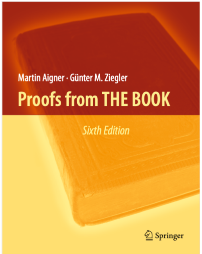
Plan

Infinitude of Primes:

Fürstenberg's Topological Argument [2], [1]

Goal: formalize Fürstenberg's topological proof.

Let's see the pen-and-paper proof.



Infinite of Primes:

Fürstenberg's Topological Argument

Topology

A **topology** over a set X is a collection τ of subsets of X satisfying the following properties:

i) $\emptyset$ and X belong to τ;

ii) The union of any sub-collection of τ belongs to τ;

iii) The intersection of any finite sub-collection of τ belongs to τ.

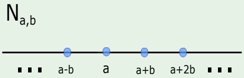
- A set X equipped with a topology τ is called a **topological space**.
- In a topological space (X, τ), the sets in τ are called **open sets of** X.

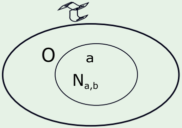
Fürstenberg's Topological Argument

Example

Consider the sets $N_{a,b} = \{a + nb : n \in \mathbb{Z}\}$, $a \in \mathbb{Z}, b \in \mathbb{Z}^+$.

A set $O \subseteq \mathbb{Z}$ is called **open** if and only if $O = \emptyset$ or for every $a \in O$, there is an integer $b \in \mathbb{Z}^+$ such that $N_{a,b} \subseteq O$.





The collection τ of open sets is a topology over $\mathbb{Z}$:

i) $\emptyset$ and $\mathbb{Z}$ belong to τ;

ii) Arbitrary unions of sets of τ belong to τ;

iii) If O_1 and O_2 belong to τ then $O_1 \cap O_2$ belongs to τ.

- In fact, if $a \in O_1 \cap O_2$, there exist b_1 and b_2 such that $N_{a,b_1} \subseteq O_1$ and $N_{a,b_2} \subseteq O_2$. Then, $N_{a,b_1 b_2} \subseteq O_1 \cap O_2$.

Fürstenberg's Topological Argument

- **Statement 1:** Any nonempty open set is infinite.
  - Proof: if $O \neq \emptyset$ then $N_{a,b} \subset O$, for some $a \in O$ and $b \in \mathbb{Z}^+$.
- **Statement 2:** The sets $N_{a,b}$, for $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$, are open sets.

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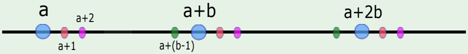
Closed sets

A subset A of a topological space X is called a **closed set** if and only if its complement A^c is an open set in X.

- **Statement 3:** For any $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$, $N_{a,b}$ is closed.

Indeed, $N_{a,b} = \mathbb{Z} \setminus \bigcup_{i=1}^{b-1} N_{a+i,b}$, and $\bigcup_{i=1}^{b-1} N_{a+i,b}$ is an open set.



Fürstenberg's Topological Argument

Some properties of closed sets

If X is a topological space then:

P1. $\emptyset$ and X are closed sets;

P2. The finite union of closed sets is a closed set:

- consider A_i, $1 \leq i \leq n$ closed sets. Thus,

$$X \setminus \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n} (X \setminus A_i) \text{ is an open set}$$

P3. The arbitrary intersection of closed sets is a closed set.

- Consider A_α, a family of closed sets. Then,

$$X \setminus \bigcap A_\alpha = \bigcup (X \setminus A_\alpha) \text{ is an open set}$$

Fürstenberg's topological argument

- **Statement 4:** Consider $k \in \mathbb{Z} \setminus \{-1, 1\}$. Then, k has a prime divisor p and, consequently, $k \in N_{0,p}$.

  Also,

  $$\mathbb{Z} \setminus \{-1, 1\} = \bigcup_{p \in \mathbb{P}} N_{0,p}, \text{ where } \mathbb{P} \text{ denotes the set of prime numbers.}$$

If we assume that $\mathbb{P}$ **is finite** then:

- $\bigcup_{p \in \mathbb{P}} N_{0,p}$ is a closed set (**Statement 3 + P2**).
- Thus, $\{-1, 1\}$ is an open set (**By the definition of a closed set**).
- Consequently, $\{-1, 1\}$ is an infinite set ⚡ (**Statement 1**)

Therefore, the set $\mathbb{P}$ of the prime numbers is infinite.

References

- Martin Aigner and Günter M. Ziegler, Proofs from THE BOOK. Springer (2018)
- Hillel Fürstenberg. On the Infinitude of Primes. Amer. Math, Monthly. **62**(5) (1955)
- Bruno B. de Oliveira Ribeiro, Mariano M. Moscato, Thaynara A. de Lima and Mauricio Ayala-Rincón, A PVS Library on the Infinitude of Primes. Proc. CICM 2025.

Hands-on!

Upload the exercise files:

https://github.com/mayalarincon/FormalProofsInPVS