Universidade de Brasília

A Strong Nominal Algebra for a Nominal Theory with Fixed-Point Constraints

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Nominal Syntax: atom permutations

- Variables (meta-level): $\mathbb{V} = \{X, Y, Z, \ldots\}$. (infinitely countable)

  Atoms (object-level): $\mathbb{A} = \{a, b, c \ldots\}$. (infinitely countable)

  Function symbols (term-formers): $\Sigma = \{\mathtt{f}, \mathtt{g}, \mathtt{h}, \ldots\}$. (finite)
- $\mathbb{A} \cap \mathbb{V} \cap \Sigma = \emptyset$ and $\mathtt{f} : n$ means that the arity of $\mathtt{f}$ is n.
- Swappings:

$$(a\ b)(c) = \begin{cases} a, & c = b, \\ b, & c = a, \\ c, & \text{otherwise.} \end{cases}$$

  (Finite) Permutations: bijections $\pi : \mathbb{A} \to \mathbb{A}$ with $\mathtt{dom}(\pi) = \{a \mid \pi(a) \neq a\}$, represented as lists of swappings ($\mathtt{id}$ is identity permutation, π^{-1} is the inverse of π, and $\pi \circ \pi'$ is the composition of π and π'). $\mathtt{Perm}(\mathbb{A})$ denotes the group of all (finite) permutations, $(a\ a) = \mathtt{id}$, and $\pi^{\pi'} = \pi'^{-1} \circ \pi \circ \pi'$.

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Nominal Syntax: nominal terms

Nominal terms are defined inductively by the following grammar:

$$t ::= a \mid \pi \cdot X \mid \mathtt{f}(t_1, \ldots, t_n) \mid [a]t$$

where

- a range over $\mathbb{A}$;
- $\pi \cdot X$ is a *suspension*, where π is an atom permutation. Suspensions of the form $\mathtt{id} \cdot X$ will be represented simply by X;
- $\mathtt{f}(t_1, \ldots, t_n)$ is the *application* of the term-former $\mathtt{f} : n$ to a tuple $(t_1, \ldots, t_n)$;
- $[a]t$ denotes the *abstraction* of the atom a over the term t.

Nominal Syntax: permutation action and substitutions

- Permutation action:

$$\begin{aligned} \pi \cdot a &= \pi(a) \\ \pi \cdot (\pi' \cdot X) &= (\pi \circ \pi') \cdot X \\ \pi \cdot ([a]t) &= [\pi(a)](\pi \cdot t) \\ \pi \cdot \mathtt{f}(t_1, \ldots, t_n) &= \mathtt{f}(\pi \cdot t_1, \ldots \pi \cdot t_n) \end{aligned}$$

- Substitutions:

$$\begin{aligned} a\sigma &= a \\ (\pi \cdot X)\sigma &= \pi \cdot (X\sigma) \\ ([a]t)\sigma &= [a](t\sigma) \\ \mathtt{f}(t_1, \ldots, t_n)\sigma &= \mathtt{f}(t_1\sigma, \ldots, t_n\sigma) \end{aligned}$$

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What are fixed-point constraints?

- A fixed-point constraint is an expression of the form $\pi \curlywedge t$ (reads: π fixes t). Intuitively, it indicates that the term t is fixed by the permutation up to a "safe" a renaming of the bound names:

$$\pi \curlywedge t \Leftrightarrow \pi \cdot t \overset{\curlywedge}{\approx}_\alpha t,$$

where $\overset{\curlywedge}{\approx}_\alpha$ is the α-equivalence defined using $\curlywedge$.

- freshness constraints $a\#t$ means that "a is doesn't occur free in t".
- The notion of fixed-point was inspired by a characterization of the freshness relation using the quantifier new (И), which quantifies over new names:

$$a\#x \Leftrightarrow \text{И}c.(a\ c) \cdot x = x,$$

that is, a is fresh for x iff for any new atom c, the permutation $(a\ c)$ fixes x.

- $\text{И}c.(a\ c) \cdot x = x$ means $((a\ c_1) \cdot x = x) \wedge ((a\ c_2) \cdot x = x) \wedge ((a\ c_3) \cdot x = x) \wedge \cdots$ for a cofinite amount of atoms c_i's.

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Why fixed-point constraints?

- Nominal unification and disunification was were first explored using freshness constraints.
- However, altought (pure) nominal unification via freshness constraints is unitary, when equational theories (such as A, C, and AC) are involved this property is lost.
- This issue was surpassed by the introduction of the fixed-point constraints.

Semantics of fixed-point constraints: an open question

- The soundness and completeness of the nominal theory via freshness is well-established, along with a nominal version of the HSP theorem (HSP stands for Homomophisms, Subalgebras, and Products).
- The intentional semantics of nominal theory via freshness was the class of nominal sets.
- The semantics of fixed-point constraints is a question yet to be investigated.

Judgments: fixed-point derivation rules

$$\frac{\pi(a) = a}{\Upsilon \vdash \pi \curlywedge a}(\curlywedge \mathbf{a}) \qquad \frac{\mathtt{dom}(\pi^{\pi'^{-1}}) \subseteq \mathtt{dom}(\mathtt{perm}(\Upsilon|_X))}{\Upsilon \vdash \pi \curlywedge \pi' \cdot X}(\curlywedge \mathbf{var})$$

$$\frac{\Upsilon \vdash \pi \curlywedge t_1, \ldots, \Upsilon \vdash \pi \curlywedge t_n}{\Upsilon \vdash \pi \curlywedge \mathtt{f}(t_1, \ldots, t_n)}(\curlywedge \mathtt{f}) \qquad \frac{\Upsilon, \overline{(c_1\ c_2) \curlywedge \mathtt{Var}(t)} \vdash \pi \curlywedge (a\ c_1) \cdot t}{\Upsilon \vdash \pi \curlywedge [a]t}(\curlywedge \mathbf{abs})$$

Figure: Fixed-point derivation rules; c_1, c_2 are fresh names.

- Fixed-point contexts, usually denoted by Υ, contain primitive constraints of the form $\pi \curlywedge X$.
- $\mathtt{perm}(\Upsilon|_X) = \{\pi \mid \pi \curlywedge X \in \Upsilon\}$.
- $\mathtt{dom}(\mathtt{perm}(\Upsilon|_X)) = \bigcup_{\pi \in \mathtt{perm}(\Upsilon|_X)} \mathtt{dom}(\pi)$.
- $\overline{\pi \curlywedge \mathtt{Var}(t)} := \{\pi \curlywedge X \mid X \in \mathtt{Var}(t)\}$ and $\pi^{\pi'^{-1}} = \pi' \circ \pi \circ \pi'^{-1}$.

Judgments: α-equality derivation rules

$$\frac{}{\Upsilon \vdash a \overset{\curlywedge}{\approx}_\alpha a}\,(\overset{\curlywedge}{\approx}_\alpha \textbf{a}) \qquad \frac{\mathtt{dom}((\pi')^{-1} \circ \pi) \subseteq \mathtt{dom}(\mathtt{perm}(\Upsilon|_X))}{\Upsilon \vdash \pi \cdot X \overset{\curlywedge}{\approx}_\alpha \pi' \cdot X}\,(\overset{\curlywedge}{\approx}_\alpha \textbf{var})$$

$$\frac{\Upsilon \vdash t_1 \overset{\curlywedge}{\approx}_\alpha t'_1, \ldots, \Upsilon \vdash t_n \overset{\curlywedge}{\approx}_\alpha t'_n}{\Upsilon \vdash \mathtt{f}(t_1, \ldots, t_n) \overset{\curlywedge}{\approx}_\alpha \mathtt{f}(t'_1, \ldots, t'_n)}\,(\overset{\curlywedge}{\approx}_\alpha \mathtt{f}) \qquad \frac{\Upsilon \vdash t \overset{\curlywedge}{\approx}_\alpha t'}{\Upsilon \vdash [a]t \overset{\curlywedge}{\approx}_\alpha [a]t'}\,(\overset{\curlywedge}{\approx}_\alpha [\textbf{a}])$$

$$\frac{\Upsilon \vdash s \overset{\curlywedge}{\approx}_\alpha (a\ b) \cdot t \qquad \Upsilon, \overline{(c_1\ c_2) \curlywedge \mathtt{Var}(t)} \vdash (a\ c_1) \curlywedge t}{\Upsilon \vdash [a]s \overset{\curlywedge}{\approx}_\alpha [b]t}\,(\overset{\curlywedge}{\approx}_\alpha \textbf{ab})$$

Figure: α-equality derivation rules; c_1 is fresh name.

Sets equipped with a permutation action

Definition

A $\mathtt{Perm}(\mathbb{A})$-set, denoted by $\mathscr{X}$, is a pair $(|\mathscr{X}|, \cdot)$ consisting of an underlying set $|\mathscr{X}|$ and a permutation action written $\pi \cdot_{\mathscr{X}} x$ or just $\pi \cdot x$ which is a group action on $|\mathscr{X}|$, that is, an operation $\cdot : \mathtt{Perm}(\mathbb{A}) \times |\mathscr{X}| \to |\mathscr{X}|$ such that

$$\begin{aligned} \mathtt{id} \cdot x &= x, \\ \pi \cdot (\pi' \cdot x) &= (\pi \circ \pi') \cdot x, \end{aligned}$$

for every $x \in |\mathscr{X}|$ and $\pi, \pi' \in \mathtt{Perm}(\mathbb{A})$.

(Strong) Support

Let $\mathscr{X} = (|\mathscr{X}|, \cdot)$ is a $\mathtt{Perm}(\mathbb{A})$-set.

Definition

- If $B \subseteq \mathbb{A}$ write $\mathtt{Fix}(B) = \{\pi \in \mathtt{Perm}(\mathbb{A}) \mid \forall a \in B.\pi(a) = a\}$.
- A set of atomic names $B \subseteq \mathbb{A}$ supports an element $x \in |\mathscr{X}|$ when for all permutations $\pi \in \mathtt{Perm}(\mathbb{A})$,
$$\pi \in \mathtt{Fix}(B) \Rightarrow \pi \cdot x = x.$$
- Additionally, we say that B strongly supports $x \in |\mathscr{X}|$ when for all permutations $\pi \in \mathtt{Perm}(\mathbb{A})$,
$$\pi \in \mathtt{Fix}(B) \Leftrightarrow \pi \cdot x = x.$$

(Strong) Nominal Sets

Definition

- We say that a $\mathtt{Perm}(\mathbb{A})$-set is a nominal set when all of whose elements are finitely supported.
- A nominal set is strong if every element is strongly supported by a finite set.
- Let $\mathscr{X}$ be a nominal set. We define the support of an element $x \in |\mathscr{X}|$ of a nominal set $\mathscr{X}$ by $\mathrm{supp}(x) = \bigcap\{B \mid B \text{ is finite and supports } x\}$.

Denote the class of nominal sets by $\mathbf{Nom}$ and the class of strong nominal sets by $\mathbf{SNom}$. Then $\mathbf{SNom} \subseteq \mathbf{Nom}$.

Example

1. The $\mathtt{Perm}(\mathbb{A})$-set $(\mathbb{A}, \cdot)$ with the action $\pi \cdot_{\mathbb{A}} a = \pi(a)$ is a nominal set and $\mathrm{supp}(a) = \{a\}$ for each $a \in \mathbb{A}$.

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(Strong) Nominal Sets: more examples

Example

2. The singleton set $\{\star\}$ equipped with the action $\pi \cdot \star = \star$ is a strong nominal set and $\mathtt{supp}(\star) = \emptyset$.
3. Consider the set $\mathcal{P}_{\mathtt{fin}}(\mathbb{A}) = \{B \subset \mathbb{A} \mid B \text{ is finite}\}$. Then the $\mathtt{Perm}(\mathbb{A})$-set $(\mathcal{P}_{\mathtt{fin}}(\mathbb{A}), \cdot)$ with the action $\pi \cdot_{\mathcal{P}_{\mathtt{fin}}(\mathbb{A})} B = \{\pi \cdot_{\mathbb{A}} a \mid a \in B\}$ is a nominal set and $\mathtt{supp}(B) = B$. Observe that $\mathcal{P}_{\mathtt{fin}}(\mathbb{A})$ is not strong because if we take $B = \{a, b\}$ and $\pi = (a\ b)$, then $\pi \cdot B = B$ but $\pi \notin \mathtt{Fix}(B)$.
4. The set $\mathbb{A}^* = \bigcup\{a_1 \cdots a_n \mid \forall i, j \in \{1, \ldots, n\}.a_i \in \mathbb{A} \wedge (j = i \Rightarrow a_j \neq a_i)\}$, that is, the set of finite words over distinct atoms, is a strong nominal set when equipped with the permutation action given by $\pi \cdot (a_1 \cdots a_n) = \pi(a_1) \cdots \pi(a_n)$.

Equivariant maps

Definition

For any nominal sets $\mathscr{X}, \mathscr{Y}$, call a map $f : |\mathscr{X}| \to |\mathscr{Y}|$ equivariant when

$$\pi \cdot f(x) = f(\pi \cdot x),$$

for all $\pi \in \mathtt{Perm}(\mathbb{A})$ and $x \in |\mathscr{X}|$.

For instance, any constant map is easily an equivariant map.

Semantics: freshness constraints vs fixed-point constrains

Definition

Let $\mathscr{X} = (|\mathscr{X}|, \cdot)$ be a nominal set, $x \in |\mathscr{X}|$, $a \in \mathbb{A}$, and $\pi \in \mathtt{Perm}(\mathbb{A})$.

- $a \#_{\mathtt{sem}}\, x$ means that $a \notin \mathrm{supp}(x)$. (Freshness)
- $\pi \curlywedge_{\mathtt{sem}}\, x$ means that $\pi \cdot x = x$. (Fixed-point)

Σ-algebras

Definition

Given a signature Σ, a Σ-**algebra** $\mathcal{A}$ consists of:

1. A domain nominal set $\mathscr{A} = (|\mathscr{A}|, \cdot)$.
2. An equivariant map $\mathtt{atom}\colon \mathbb{A} \to |\mathscr{A}|$ to interpret atoms; we write the interpretation $\mathtt{atom}(a)$ as $a^{\mathcal{A}} \in |\mathscr{A}|$.
3. An equivariant map $\mathtt{abs}\colon \mathbb{A} \times |\mathscr{A}| \to |\mathscr{A}|$ such that the set

$$\{c \in \mathbb{A} \mid (a\ c) \not\curlywedge_{\mathtt{sem}} \mathtt{abs}(a, x)\}$$

   is finite.
4. An equivariant map $f^{\mathcal{A}}\colon |\mathscr{A}|^n \to |\mathscr{A}|$ for each term-former $\mathtt{f} : n$ in Σ.

`Remark.` Condition of item 3 is based on the following equivalence:

$$a\#_{\mathtt{sem}}\mathtt{abs}(a, x) \Leftrightarrow \{c \in \mathbb{A} \mid (a\ c) \not\curlywedge_{\mathtt{sem}} \mathtt{abs}(a, x)\} \text{ is finite.}$$

How to interpret nominal terms?

In the following, $\mathcal{A}$ denotes a Σ-algebra.

Definition

- A valuation ς in $\mathcal{A}$ maps unknowns $X \in \mathbb{V}$ to elements $\varsigma(X) \in |\mathscr{A}|$.
- The interpretation of a nominal term t, denoted by $[\![t]\!]^{\mathcal{A}}_{\varsigma}$, or just $[\![t]\!]_{\varsigma}$, if $\mathcal{A}$ is understood, is defined inductively by:

$$[\![a]\!]^{\mathcal{A}}_{\varsigma} = a^{\mathcal{A}} \qquad [\![\pi \cdot X]\!]^{\mathcal{A}}_{\varsigma} = \pi \cdot \varsigma(X)$$

$$[\![\mathtt{f}(t_1, \ldots, t_n)]\!]^{\mathcal{A}}_{\varsigma} = f^{\mathcal{A}}([\![t_1]\!]^{\mathcal{A}}_{\varsigma}, \ldots, [\![t_n]\!]^{\mathcal{A}}_{\varsigma}) \qquad [\![[a]t]\!]^{\mathcal{A}}_{\varsigma} = \mathtt{abs}(a, [\![t]\!]^{\mathcal{A}}_{\varsigma}).$$

Validity and Soundness

Definition

Let $\mathcal{A}$ be a Σ-algebra and ς a valuation on $\mathcal{A}$.

- $[\![\Upsilon]\!]_\varsigma^\mathcal{A}$ is **valid** iff $\pi \curlywedge_{\mathtt{sem}} \varsigma(X)$ for each $\pi \curlywedge X \in \Upsilon$.
- $[\![\Upsilon \vdash \pi \curlywedge t]\!]_\varsigma^\mathcal{A}$ is **valid** iff $[\![\Upsilon]\!]_\varsigma^\mathcal{A}$ (valid) implies $\pi \curlywedge_{\mathtt{sem}} [\![t]\!]_\varsigma^\mathcal{A}$.
- $[\![\Upsilon \vdash s \stackrel{\curlywedge}{\approx}_\alpha t]\!]_\varsigma^\mathcal{A}$ is **valid** iff $[\![\Upsilon]\!]_\varsigma^\mathcal{A}$ (valid) implies $[\![s]\!]_\varsigma^\mathcal{A} = [\![t]\!]_\varsigma^\mathcal{A}$.

Theorem (Soundness)

For any $\mathcal{A}$ and any valuation ς:

- If $\Upsilon \vdash \pi \curlywedge t$ then $[\![\Upsilon \vdash \pi \curlywedge t]\!]_\varsigma^\mathcal{A}$ is valid.
- If $\Upsilon \vdash s \stackrel{\curlywedge}{\approx}_\alpha t$ then $[\![\Upsilon \vdash s \stackrel{\curlywedge}{\approx}_\alpha t]\!]_\varsigma^\mathcal{A}$ is valid.

A counter-example for soundness in Nom

The fixed-point derivation rule for suspended variables is given by:

$$\frac{\mathtt{dom}(\pi^{\pi'^{-1}}) \subseteq \mathtt{dom}(\mathtt{perm}(\Upsilon|_X))}{\Upsilon \vdash \pi \curlywedge \pi' \cdot X} \ (\curlywedge\mathbf{var})$$

We're going to present a Σ-algebra $\mathcal{A}$, a valuation ς (in $\mathcal{A}$) and a derivation using rule $(\curlywedge\mathbf{var})$ such that

$$\Upsilon \vdash \pi \curlywedge \pi' \cdot X \Rightarrow [\![\Upsilon \vdash \pi \curlywedge \pi' \cdot X]\!]_{\varsigma}^{\mathcal{A}} \text{ is not valid.}$$

A counter-example for soundness in $\mathbf{Nom}$

Take the domain of $\mathcal{A}$ as the nominal set $= (\mathcal{P}_{\mathtt{fin}}(\mathbb{A}), \cdot)$ that we presented a few slides ago.

We can give it a Σ-algebra structure as follows:

1. $\mathtt{atom}\colon \mathbb{A} \to \mathcal{P}_{\mathtt{fin}}(\mathbb{A})$ is given by $\mathtt{atom}(a) = \{a\}$;
2. $\mathsf{abs}\colon \mathbb{A} \times \mathcal{P}_{\mathtt{fin}}(\mathbb{A}) \to \mathcal{P}_{\mathtt{fin}}(\mathbb{A})$ is given by $\mathsf{abs}(a, B) = B \setminus \{a\}$;
3. $f^{\mathscr{A}} : \mathcal{P}_{\mathtt{fin}}(\mathbb{A})^n \to \mathcal{P}_{\mathtt{fin}}(\mathbb{A})$ is given by $f^{\mathscr{A}}(B_1, \ldots, B_n) = \bigcap_{i=1}^{n} B_i$, for each $\mathtt{f} : n$ in Σ.

The equivariance of these functions is easy to check. In addition,

$$\{c \in \mathbb{A} \mid (a\ c) \not\curlywedge_{\mathsf{sem}} \mathsf{abs}(a, B)\} \subseteq B \setminus \{a\}.$$

A counter-example for soundness in Nom

Now, fix an enumeration of $\mathbb{V} = \{X_1, X_2, \ldots\}$ and $\mathbb{A} = \{a_1, a_2, \ldots\}$, and consider the following valuation in $\mathcal{A} = (\mathcal{P}_{\mathtt{fin}}(\mathbb{A}), \cdot)$:

$$\varsigma(X_i) = \{a_i, a_{i+1}\}.$$

So $\varsigma(X_1) = \{a_1, a_2\}, \varsigma(X_2) = \{a_2, a_3\}, \ldots$

A counter-example for soundness in $\mathbb{N}\mathrm{om}$

Consider the context $\Upsilon = \{(a_1\ a_2) \circ (a_6\ a_4) \curlywedge X_1, (a_1\ a_2) \circ (a_3\ a_5) \curlywedge X_1\}$ and the following derivation:

$$\dfrac{\dfrac{\{a_1, a_3\} \subseteq \{a_1, a_2, a_3, a_4, a_5, a_6\}}{\mathtt{dom}((a_1\ a_2)^{(a_3\ a_1)}) \subseteq \mathtt{dom}(\mathtt{perm}(\Upsilon|_{X_1}))}}{\Upsilon \vdash (a_1\ a_2) \curlywedge (a_1\ a_3) \cdot X_1}\ (\curlywedge\mathbf{var})$$

We're going to show that $[\![\Upsilon \vdash (a_1\ a_2) \curlywedge (a_1\ a_3) \cdot X_1]\!]^{\mathcal{A}}_{\varsigma}$ is not valid, i.e. $[\![\Upsilon]\!]^{\mathcal{A}}_{\varsigma}$ (valid) doesn't imply $(a_1\ a_2) \curlywedge_{\mathsf{sem}} [\![(a_1\ a_3) \cdot X_1]\!]^{\mathcal{A}}_{\varsigma}$.

- $\varsigma(X_1) = \{a_1, a_2\}$.
- The context is valid since $(a_1\ a_2) \circ (a_6\ a_4) \curlywedge_{\mathsf{sem}} \varsigma(X_1)$ and $(a_1\ a_2) \circ (a_3\ a_5) \curlywedge_{\mathsf{sem}} \varsigma(X_1)$.
- However, $(a_1\ a_2) \not\curlywedge_{\mathsf{sem}} [\![(a_1\ a_3) \cdot X_1]\!]^{\mathcal{A}}_{\varsigma}$ since

$$\begin{aligned}(a_1\ a_2) \cdot [\![(a_1\ a_3) \cdot X_1]\!]^{\mathcal{A}}_{\varsigma} &= (a_1\ a_2) \cdot ((a_1\ a_3) \cdot \varsigma(X_1)) \\ &= \{a_3, a_1\} \\ &\neq [\![(a_1\ a_3) \cdot X_1]\!]^{\mathcal{A}}_{\varsigma}.\end{aligned}$$

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Why does this occur?

$$\frac{\mathtt{dom}(\pi^{\pi'^{-1}}) \subseteq \mathtt{dom}(\mathtt{perm}(\Upsilon|_X))}{\Upsilon \vdash \pi \curlywedge \pi' \cdot X}\ (\curlywedge \textbf{var})$$

The validity of the variable rule in $\mathbf{Nom}$ presupposes the following result:

Theorem A (?)

Let $\mathscr{X} = (|\mathscr{X}|, \cdot)$ be a nominal set, $x \in |\mathscr{X}|$, and $\pi, \gamma_1, \ldots, \gamma_n \in \mathtt{Perm}(\mathbb{A})$. If $\gamma_i \curlywedge_{\mathtt{sem}} x$ for every $i = 1, \ldots, n$, and $\mathtt{dom}(\pi) \subseteq \bigcup_{i=1}^{n} \mathtt{dom}(\gamma_i)$, then $\pi \curlywedge_{\mathtt{sem}} x$.

This is false in general: take $|\mathscr{X}| = \mathcal{P}_{\mathtt{fin}}(\mathbb{A})$, $x = \{a_1, a_2\}$, $\pi = (a_1\ a_3)$, $\gamma_1 = (a_1\ a_2) \circ (a_6\ a_4)$, and $\gamma_2 = (a_1\ a_2) \circ (a_3\ a_5)$.

- γ_1, γ_2 both fixes $\{a_1, a_2\}$,
- $\{a_1, a_3\} \subseteq \{a_1, a_2, a_3, a_4, a_5, a_6\}$,
- however, $(a_1\ a_3)$ doesn't fixes $\{a_1, a_2\}$.

How to solve this problem?

Theorem A

Let $\mathscr{X} = (|\mathscr{X}|, \cdot)$ be a strong nominal set, $x \in |\mathscr{X}|$, and $\pi, \gamma_1, \ldots, \gamma_n \in \mathtt{Perm}(\mathbb{A})$. If $\gamma_i \curlywedge_{\mathtt{sem}} x$ for every $i = 1, \ldots, n$, and $\mathtt{dom}(\pi) \subseteq \bigcup_{i=1}^{n} \mathtt{dom}(\gamma_i)$, then $\pi \curlywedge_{\mathtt{sem}} x$.

`Proof.` For each $i = 1, \ldots, n$, $\gamma_i \curlywedge_{\mathtt{sem}} x$ implies that $\gamma_i \in \mathtt{Fix}(\mathtt{supp}(x))$. Then by the inclusion $\mathtt{dom}(\pi) \subseteq \bigcup_{1 \leq i \leq n} \mathtt{dom}(\gamma_i)$, we have that $\pi \in \mathtt{Fix}(\mathtt{supp}(x))$ which gives us $\pi \curlywedge_{\mathtt{sem}} x$ by the definition of support. $\square$

As a consequence, considering strong nominal sets in the definition of Σ-algebras, now called strong Σ-algebras, we are able to prove soundness. Now we're investigating completeness (Ongoing Work).

An important property

Recall the definition of support: A set $B \subseteq \mathbb{A}$ supports an element $x \in |\mathscr{X}|$ if for all $\pi \in \mathtt{perm}(\mathbb{A})$, the following holds

$$\pi \in \mathtt{Fix}(B) \Rightarrow \pi \cdot x = x.$$

Next theorem follows directly from the definition of support:

Theorem B

Let $\mathscr{X} = (|\mathscr{X}|, \cdot)$ be a nominal set, $\pi \in \mathtt{Perm}(\mathbb{A})$, and $x \in |\mathscr{X}|$. Then

$$\mathtt{dom}(\pi) \cap \mathtt{supp}(x) = \emptyset \Rightarrow \pi \cdot x \curlywedge_{\mathtt{sem}} x.$$

The converse is not true in general. For instance, take $|\mathscr{X}| = \mathcal{P}_{\mathtt{fin}}(\mathbb{A})$, $\pi = (a\ b)$ and $x = \{a, b\}$. Then

- $(a\ b) \cdot \{a, b\} = \{a, b\}$,
- but $\mathtt{dom}((a\ b)) \cap \mathtt{supp}(\{a, b\}) = \{a, b\}$.

Freshness contexts vs Fixed-point contexts

- Freshness contexts Δ contain primitive constraints of the form $a\#X$.
- Fixed-point constraints Υ contain primitive constraints of the form $\pi \curlywedge X$.

Consider $\Delta = \{a\#X, b\#X\}$ and $\Upsilon = \{(a\ b) \curlywedge X\}$.

- We can make a translation from Δ to Υ because of the Theorem B, since

$$a\#X, b\#X \iff \mathtt{dom}((a\ b)) \cap \mathtt{supp}(X) = \emptyset \implies (a\ b) \curlywedge X.$$

- However, we cannot make a translation in the other direction, because it's not necessarily true that

$$(a\ b) \curlywedge X \implies a\#X, b\#X.$$

A more general solution

- Our contexts are of the form $\pi \curlywedge X$, and this is not sufficient to given information about fresh atoms in X.
- There is an incompatibility of fixed-point contexts vs freshness contexts.
- Pitts' characterization is

$$a\#x \Leftrightarrow \mathsf{N}c.(a\ c)\cdot x = x,$$

- To generalize our fixed-point context with the new quantifier consisting of primitive constraints of the form $\mathsf{N}c.(a\ c) \curlywedge X$ (Future work).
- Soundness and completeness should hold because of the Pitts' equivalence (Future work).

More future work

- To investigate the problems involving fixed-point constraints generalized with $\reflectbox{N}$ quantifier and equational theories.
- To extend the proof of HSP theorem for the fixed-point with $\reflectbox{N}$ and strong nominal algebras
- Apply the results to unification/disunification problems modulo equational theories.

References I

The principal references are (GABBAY, 2009),(GABBAY; MATHIJSSEN, 2009), and(AYALA-RINCÓN et al., 2019)

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Thanks for listening!