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Paradise of Georg Cantor

1 Cantor's Life

In this section we will present the life story of Georg Cantor based on the article "*The Nature of Infinity*" by Jorgen Veisdal [1] along with "Georg Cantor" by Britannica [2].

LIKE many mathematicians, Georg Cantor has had his fair share of criticism for his published theories. However, without his publications, many of today's mathematical applications would have remained obscure. Born on March 3rd, 1845, in Saint Petersburg, the German mathematician was raised by a family of cultural and philosophical interests. Being the eldest of six, Cantor's parents Marie Meyer and Georg Woldemar recognized his mathematical capabilities at an early age, where they encouraged him by private lessons.

In 1856, Cantor's family has moved to Frankfurt, Germany, where Georg attended Darmstadt school. Graduating with distinction from the Realschule, Darmstadt school in 1862, Georg then went to pursue his undergraduate engineering degree at Höheren Gewerbschule or two years before transferring to the Swiss Federal Polytechnic (ETH Zurich) to study mathematics. However, after receiving a substantial amount of money perceived by his inheritance in 1863, he traveled back to Berlin in order to continue his education. Upon his journey, he encountered many prestigious mathematics professors, including Ernst Kummer, Leopold Kronecker, and Karl Weierstrass that intrigued his interest in arithmetics. In 1868, He received his doctorate from the University of Berlin, where his dissertation was devoted to number theory, and moved to Saxony to teach at the University of Halle.

Thereafter, Cantor published two fundamental papers: "On a theorem concerning the trigonometric series" and "On the generalization of a theorem from the theory of trigonometric series." The results of both papers, along with his previous work, have then promoted him to associate professor at Halle University in 1872. That same year, after being acquainted with Richard Dedekind, a professor of mathematics at the Technische Hochschule at Brunswick, Cantor and Dedekind exchanged letters over the years discussing several mathematical ideas out of which the initiation of set theory was discovered. In 1974, Cantor married Vally Guttman, and in his honeymoon, he was able to finally meet Dedekind. Over the course of 11 years, Cantor proved his theories by publishing papers with proofs on "Countability of rational numbers, " "Infinite Sets, " "Cardinal Numbers," and "Infinite Cardinal Numbers," and set theory was established. In particular, the origins of set theory are traced back to one of Cantor's first papers published in 1874 entitled "On a Property of the Collection of All Real Algebraic Numbers."

While Cantor has successfully proved most of his findings, his last conjecture, The Continuum Hypothesis, was one of which perplexed him. Much of his work on the Continuum Hypothesis was published between 1879 and 1884 in a series of six papers: "On infinite, linear manifolds of points" in the journal Mathematische Annalenin. Cantor spent the remaining of his life trying to validate the hypothesis but failed eventually. In May 1884, Cantor had an argument with his then mentor Kronecker about the validity of the Continuum Hypothesis, which resulted in his first serious breakdown. However, this wasn't the last dispute between Kronecker and Cantor; at the time, Georg was constantly applying for positions at the University of Berlin, and Kronecker, head of the mathematical department, went out of his way to deny every request Cantor provided claiming that Cantor is "the corrupter of youth," and is challenging "the uniqueness of the absolute infinity in the nature of God."

Consequently, along with the death of the youngest of his five children Rudolph, Cantor's mental health deteriorated, and he has claimed to have "lost his passion for mathematics." Moreover, in 1903, another mathematician has published a paper refuting the set theory. There have been records of his admittance into and out of asylums ever since. The last 20 years of his life were a stage of chronic depression, validating his name and theories and trying to prove the continuum hypothesis. He repeatedly requested from his wife to be allowed back home but was eventually denied. Living in poverty, Cantor retired in 1913 during the beginning of the first world war, and in January 1918, he died of a heart attack while hospitalized in an asylum. In the end, although Cantor has spent the last of his days suffering, he was awarded the Sylvester Medal by the Royal Society. His work is highly recognized to this day and has been of great benefit to the field of mathematics in general.

2 Cantor's Theorem

To prove Cantor's Theorem, let us begin by defining a few terminologies that were raised by Georg Cantor in his *Set Theory* papers. We start by defining a set.

Definition 2.1. A *set* is a collection of objects.

Example 2.2. The following are examples of sets.

1. The set of natural numbers, i.e., $\mathbb{N} = \{0, 1, 2, 3, 4, ...\}$.
2. The set of the English Alphabet namely set $A = \{a, b, c, d, e, f, g, h, ..., x, y, z\}$.

Definition 2.3. Let A and B be sets. We say that A is a *subset* of B, and write $A \subseteq B$ if and only if every element of A is also and an element of B.

Using predicate logic, $A \subseteq B$ is expressed as follows:

$$\forall x(x \in A \rightarrow x \in B).$$

Example 2.4. The following are examples of subsets.

1. The set of natural numbers is a subset of the set of integers, i.e., $\mathbb{N} \subseteq \mathbb{Z}$.
2. The set {0,1} is a subset of the set real numbers, i.e. $\{0, 1\} \subseteq \mathbb{R}$.
3. The empty set ϕ is a subset of $\mathbb{N}$, i.e. $\phi \subseteq \mathbb{N}$.
4. The set of complex numbers is a subset of itself, i.e. $\mathbb{C} \subseteq \mathbb{C}$.

We next introduce the notion of a power set.

Definition 2.5. The *power set* $\mathcal{P}(S)$ of a set S is the set of all subsets of S.

Example 2.6. The following are examples of power sets:.

1. $\mathcal{P}(\{\phi, 1, 2\}) = \{\phi, \{\phi\}, \{1\}, \{2\}, \{\phi, 1\}, \{\phi, 2\}, \{1, 2\}, \{\phi, 1, 2\}\}$
2. $\mathcal{P}(\{1, 3, 5\} \cap \{2, 4\}) = \mathcal{P}(\phi) = \{\phi\}$
3. $\mathcal{P}(\{1, 2\} \times \{3\}) = \{\phi, \{(1, 3)\}, \{(2, 3)\}, \{(1, 3), (2, 3)\}\}$

Let us now introduce the concept of a function and some of its properties.

Definition 2.7. Let A and B be sets.

- A *function* $f : A \to B$ from A to B is an assignment which for each element $a \in A$, it assigns exactly one element $b \in B$.
- A function $f : A \to B$ is called *injective* if for all $a_1, a_2 \in A$, that if $a_1 \neq a_2$, then $f(a_1) \neq f(a_2)$.

  Using predicate logic, it is defined as follows:

  $$\forall x_1 \forall x_2 (x_1 \neq x_2 \to f(x_1) \neq f(x_1)).$$

- A function $f : A \to B$ is called *surjective* if for every $b \in B$ there exists $a \in A$ such that $f(a) = b$.

  $$\forall y \in B \exists x \in A(f(x) = y).$$

- A function $f : A \to B$ is called *bijective* if it is both injective and surjective.
  A bijective function is expressed using predicate logic as follows:

  $$\forall x \in A \forall y \in A(x \neq y \to f(x) \neq f(y) \land \forall z \in B \exists w \in A(f(w) = z)).$$

Next, we explain how functions are used to compare the sizes of sets.

Definition 2.8. Let A and B be any sets (finite or infinite).

- We say that the cardinality of A is equal to the cardinality of B, and write $|A| = |B|$, if there exists a bijection from A to B.
- We say that the cardinality of A is less than or equal to the cardinality of B, and write $|A| \leq |B|$, if there exists an injective function from A to B.
- We say that the cardinality of A is strictly less than the cardinality of B, and write $|A| < |B|$, if there is an injective function from A to B, but there is no bijection from A to B.

- A set S is *countably infinite* if there exists at least one bijection $f : \mathbb{N} \to S$.

Of course the set of natural numbers itself is a countably infinite set. We will show below another example of a countably infinite set.

Theorem 2.9. *Let* $B = \{n \in \mathbb{Z}^+ \mid n \equiv 2 \pmod 3\}$*. Then* B *is countably infinite.*

Proof. To prove set B is countably infinite, we can check if there $\exists$ a bijection from $\mathbb{N} \to B$, which can be represented as the following function:

$$f(n) = 3n + 2, \text{ where } f : \mathbb{N} \to B.$$

Now, we will prove $f(n) = 3n + 2$ is a bijection, meaning f is injective and surjective.
To prove injectivity using contraposition, choose any 2 elements x, y such that $x, y \in \mathbb{N}$. Assume $f(x) = f(y)$, or in other words,

$$3x + 2 = 3y + 2.$$

$$x = y.$$

Hence, f is injective.
Now, to prove surjectivity. Choose any element y such that $y \in \mathbb{Z}^+$ To search for a preimage of y, suppose the preimage of y is some x. Then, $f(x) = y$, implying that

$$3x + 2 = y \implies \tfrac{y-2}{3} = x \ .$$

Note that $y \geq 2$, meaning that $x \in \mathbb{N}$.
Now to check: $3(\frac{y-2}{3}) + 2 = y - 2 + 2 = y$, implying f is surjective.
Hence, there exists a bijection $f : \mathbb{N} \to B$, meaning f is countably infinite, or equivalently set B is countably infinite. $\blacksquare$

Observe that a countably infinite set is an infinite set which we can enumerate *all* of its elements in a sequence indexed by the natural numbers. We give another example of a countably infinite set.

Theorem 2.10. *The set* $\{1, 2, 3\} \times \mathbb{N}$ *is countably infinite.*

Proof. One method to prove the following set: $\{1, 2, 3\} \times \mathbb{N}$ is countably infinite is to list it in a sequence indexed by the natural numbers. For simplicity, name set $\{1, 2, 3\} \times \mathbb{N}$ as set S. Firstly, let's begin by constructing set S, where $S = \{(a, b) | a \in \{1, 2, 3\} \wedge b \in \mathbb{N}\}$. We obtain a set of some sequence:

$$
\begin{array}{cccccc}
(1,0) & (1,1) & (1,2) & (1,3) & (1,4) & \cdots \\
(2,0) & (2,1) & (2,2) & (2,3) & (2,4) & \cdots \\
(3,0) & (3,1) & (3,2) & (3,3) & (3,4) & \cdots
\end{array}
$$

Now, to list the set in a sequence, it is efficient to move across this set in a snake-like motion, where all pairs (a, b) are listed, indexed by the natural numbers in the following manner:

$$
\begin{tikzcd}
(1,0) \arrow[r] & (1,1) \arrow[dl] & (1,2) \arrow[r] & (1,3) \arrow[dl] & (1,4) & \cdots \\
(2,0) \arrow[d, bend right] & (2,1) \arrow[ur] & (2,2) \arrow[dl] & (2,3) \arrow[ur] & (2,4) \arrow[dl] & \cdots \\
(3,0) \arrow[ur] & (3,1) \arrow[r] & (3,2) \arrow[ur] & (3,3) \arrow[r] & (3,4) & \cdots
\end{tikzcd}
$$

Hence, the sequence produced is: $(1, 0), (1, 1), (2, 0), (3, 0), (2, 1), (1, 2), ...,$ showing the set $\{1, 2, 3\} \times \mathbb{N}$ is countably infinite.

∎

We will now present the main theorem of the article, Cantor's Theorem, which states that it is impossible to have a bijection between any set and its power set.

Theorem 2.11 (Cantor's Theorem)**.** *Let* S *be any (finite or infinite) set. Then* $|S| < |\mathcal{P}(S)|$*.*

Proof. Let S be any arbitrary set. By definition, to show that $|S| < |\mathcal{P}(S)|$ we need to prove that $|S| \leq |\mathcal{P}(S)|$ and $|S| \neq |\mathcal{P}(S)|$. In other words, we need to construct an injective function from S to $\mathcal{P}(S)$, and we need to show that it is impossible to have a bijection from S to $\mathcal{P}(S)$.

First, we will show that $|S| \leq |\mathcal{P}(S)|$ by constructing an injective function $g : S \rightarrow \mathcal{P}(S)$.

To do so, choose any two elements x, y such that $x, y \in S$. Now using contraposition, assume that $g(x) = g(y)$, where the function $g(n) = \{n\}$. More precisely, this means that

$$\{x\} = \{y\}.$$

Since both sets are equal, every element in the first set must also be in the second; hence, implying $x = y$, and proving that g is injective.
Therefore, $|S| \leq |\mathcal{P}(S)|$.

Note: Joe Roussos's proof has been used as a reference [3].

Now, we need to show that there is no bijection from S to $\mathcal{P}(S)$. Since there is an injective function from S to $\mathcal{P}(S)$, we need to show that there is no surjection from S to $\mathcal{P}(S)$.

Let $f : S \rightarrow \mathcal{P}(S)$ be any function. Suppose that f is surjective. Using contradiction, consider the following set:

$$B = \{x \in S : \neg(x \in f(x))\},$$

the set of all those elements x in S, whose image under f does not include x itself.

Clearly $B \subseteq S$, as all elements in B are in S. So $B \in \mathcal{P}(\mathcal{S})$, by definition of the power set. If f is surjective, then

$$\forall b \in B \exists s \in S(f(s) = b).$$

Now, suppose that $s \in B$. If $s \in B$, then by definition of B, we know

$$\neg(s \in f(s)).$$

But $f(s) = B$, so s has to be in $f(s)$ if $s \in B$. This is a contradiction. ‘
Therefore, our assumption must be wrong. f cannot be surjective. But f was just any arbitrary function. So there is no bijection between s and $\mathcal{P}(\mathcal{S})$.

Hence, $|S| < |\mathcal{P}(S)|$. ∎

Definition 2.12. A set is *uncountable* if it is not countable.

From Cantor's Theorem we can deduce the following consequences.

Corollary 2.13. *The power set of the natural numbers is uncountable.*

Proof. Clearly, $\mathcal{P}(\mathbb{N})$ is not finite since $\mathbb{N}$ is countably infinite.
Now, to prove $\mathcal{P}(\mathbb{N})$ is not countably infinite.
Using contradiction, assume $\mathcal{P}(\mathbb{N})$ is countably infinite, then there must exist bijection from $\mathbb{N} \to \mathcal{P}(\mathbb{N})$. Now, using Cantor's Theorem, we know that $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$, meaning that there is an injective function from $\mathbb{N} \to \mathcal{P}(\mathbb{N})$, but no bijection, which is a contradiction. Hence, $\mathcal{P}(\mathbb{N})$ is not finite or countably infinite, or in other words, is uncountable. ■

The following is another consequence of Cantor's Theorem.

Corollary 2.14. *There are infinitely many infinite sets $A_0, A_1, A_2, A_3, \ldots$ such that for each $i \in \mathbb{N}$ we have that $|A_i| < |A_{i+1}|$. That is,*

$$|A_0| < |A_1| < |A_2| < |A_3| < \cdots$$

In other words, there is an infinite hierarchy of infinities.

Proof. Choose the natural numbers to be your infinite set, and apply the power set on $\mathbb{N}$. Now, according to Cantor's Theorem, $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$. Now, apply the power set to $\mathcal{P}(\mathbb{N})$. Notice that once again according to Cantor's Theorem, $|\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))|$. Now, repeat this process infinitely many times and you obtain the following:

$$\begin{array}{ccccccccccc}
A_0 & & A_1 & & A_2 & & A_3 & & A_4 & & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
|\mathbb{N}| & < & |\mathcal{P}(\mathbb{N})| & < & |\mathcal{P}(\mathcal{P}(\mathbb{N}))| & < & |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))| & < & |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N}))))| & < & \cdots ,
\end{array}$$

an infinite hierarchy of infinities.

■

References

[1] Jorgen Veisdal. *The Nature of Infinity - and Beyond*. Medium, 2018.

[2] The Editors of Encyclopedia. *Georg Cantor*. Encyclopedia Britannica, 2021.

[3] Joe Roussos. *Cantor's Theorm*. 2017.