Anamorphic images

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Anamorphic images are images of objects which have been distorted in some way so that only by viewing them from some particular direction or in some particular optical surface do they become recognizable. Artists have been fascinated with these transformations since the 16th century, but there seems to be no modern explication of the mathematics of these transforms. In this paper we describe the most common of the anamorphic images found in art and derive the transform equations for plane, conical, and cylindrical cases. With these equations it is possible to analyze early anamorphs, and with computation to create modern ones with ease.

I. INTRODUCTION

The revolution in art in the 16th century was accompanied by a rediscovery of perspective. The artists, mathematicians, and philosophers of the Renaissance studied and exploited perspective, sometimes to the extent of obsession, as they saw it bearing on the nature of illusion, truth, and reality. Prodigious essays in perspective were executed resulting in startling *trompe-l'oeil* decorations such as the Colonna and Mitelli false architecture of 1616 in the Ducal Palace, Sassuole. A critical essay, with copious illustration of the development of perspective, can be found in the article by Clerici.[1] It is only a small further step to experiment with the rules of perspective and to carry them to extremes, producing distorted images which appear normal only when observed from a particular point of view, or via some optical device. Such images are generally known as "anamorphic images."

Some artist-mathematicians produced treatises on the methods for producing these images, the most notable being Jean-Francois Niceron[2] who provided geometrical constructions for making many types of anamorphic transforms. Some of these methods are exact, while others are clearly approximations.

The most extensive modern treatment of the history of anamorphic art is the monograph by Baltrušaitis.[3] An exhibition in Amsterdam in 1975 prompted a book containing several high-quality color reproductions by Leeman.[4] Earlier brief articles by Gardner[5] and Walker[6] cover much the same ground but from a more mathematical point of view. Walker deals explicitly with the geometrical optics of the anamorphic transformations and provides techniques devised by David G. Stork for producing anamorphic images photographically.

Familiar examples of anamorphic transformations are map projections and the images in "fun-house" mirrors.

In this paper we will present the transformations for the "plane," "conical," and "cylindrical" anamorphs which are the most common in anamorphic art.

II. PLANE ANAMORPHS

In these images a planar figure is distorted, mostly in one dimension, so that it appears normal (or a hidden image appears) only when viewed from a particular direction—usually very close to the plane of the drawing. By far the most famous of these is the painting "The Ambassadors" (1533) by Hans Holbein which hangs in the National Gallery in London.[7] This painting of two prosperous men standing on either side of a curio cabinet is a virtuoso exercise in perspective. The cabinet is filled with objects which were the standard subjects for artists to practice in their sketchbooks—spheres, polyhedra, a lute, etc., and the tessellated floor contains a variety of geometric shapes. However, spread across the lower portion of the painting is a gray smear which startlingly resolves itself into the image of a skull when the viewer stands on the right-hand side and glances down at an elevation of about 20° above the surface. This *memento mori* may also have been intended, by the artist, as a rubric on his name, as "hohle Bein" means "hollow bone."

A well-known *Vexierbild* (puzzle picture) by Erhard Schön (c. 1535) is shown in the left of Fig. 1.[8] The woodcut looks like a scene in which familiar objects are presented as perhaps in a nightmare landscape. When viewed from the left edge at an elevation of about 10° above the surface, the portraits of Charles V, Ferdinand I, Pope Paul III, and Francis I are seen, as shown in the reconstruction at the right in the figure. How was such a picture produced? Was it drawn by viewing the surface from that angle or was a formula of some kind employed? We know that both methods were used. What then are the transforms that produce these drawings?

The simplest transform is shown in the example of Fig. 2 by Samuel Marolois (1614).[9] In this case only one dimension is distorted and so the correct position from which to view the reconstruction is from infinity. The transformation equations are trivial in this case:

$$x' = x, \quad \text{and} \quad y' = y/\sin\alpha, \tag{1}$$

where the primes represent the transformed coordinates and α is the angle of elevation above the plane. Measurements on the drawing indicate that the angle is 13°.

Anamorphs of this type are seen by almost everyone every day in the symbols and drawings painted on roadways. The distorted figures appear undistorted to the car driver or bicycle rider as shown in Fig. 3.[10]

A more realistic situation for smaller scale images is one where the anamorph is viewed at an elevation angle α at a finite distance which we choose to be d, at height h from the center of the leading edge of the anamorph as shown in the top of Fig. 4. The transformation is found by mapping points x,y in a "target" grid to points x',y' on the anamorphic plane. We take the orientation of the target plane to be normal to the origin-line-of-sight, VO. From the figure,

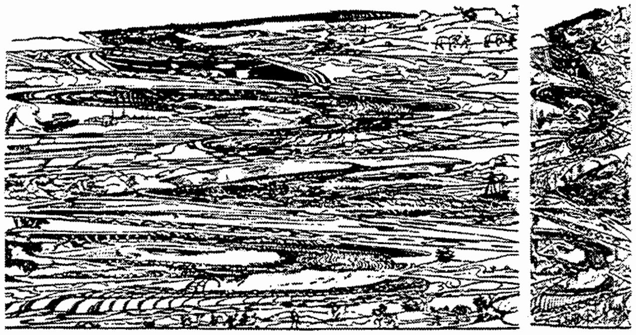

Fig. 1. Plane anamorph by Erhard Schön (c. 1535). At the right is the transformed image obtained by looking from the left at an elevation of 10° above the anamorphic plane.

$$h/(d+y')=\tan\gamma=\tan(\alpha-\beta)$$

and

$$\tan\beta=y/(d^2+h^2)^{1/2}.$$

Using the relation for the tangent of a difference, and writing the functions in terms of d, h, and α, we readily obtain for the y transformation and its inverse

$$y'=\frac{y/\sin\alpha}{1-(y/h)\cos\alpha}\quad\text{and}\quad y=\frac{y'\sin\alpha}{1+(y'/d)\cos^2\alpha}. \tag{2}$$

For each value of y, the x transformation follows from the lower part of Fig. 4 which is the (almost) horizontal plane including the line VAB. Writing $x/VA=x'/VB$ in terms of d, h, y, and y' gives

$$x'/\sqrt{h^2+(d+y')^2}=x/\sqrt{h^2+d^2+y^2}. \tag{3}$$

Equations (2) and (3) have Eqs. (1) as a limit as d and h go to infinity with a constant ratio $h/d=\tan\alpha$.

This transform is depicted in Fig. 5, originally drawn by Niceron.[2,11] The square grid (enlarged) is mapped onto the trapezoidal grid. The rest of the construction is Niceron's geometrical equivalent of Eqs. (2) and (3) and is given in detail by Gardner.[5] Analysis of the figure shows that it is drawn for $\alpha=19°$ and $a=26$ times the untransformed grid element.

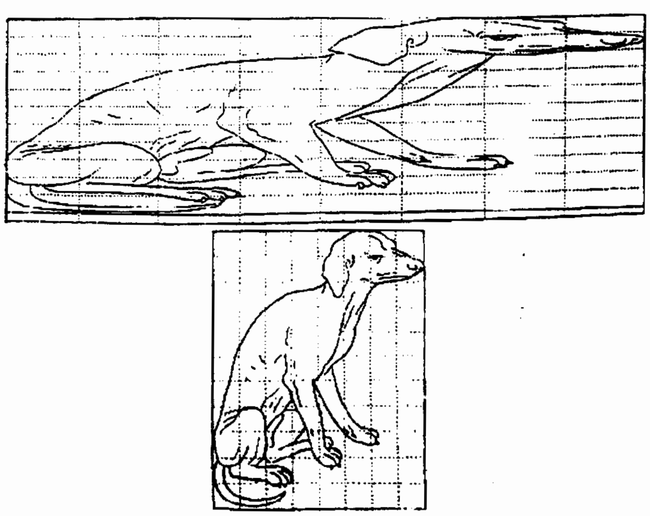

Fig. 2. Plane anamorph by Marolois (1614) constructed for viewing from infinity. The figure is to be viewed at an elevation angle of 13°.

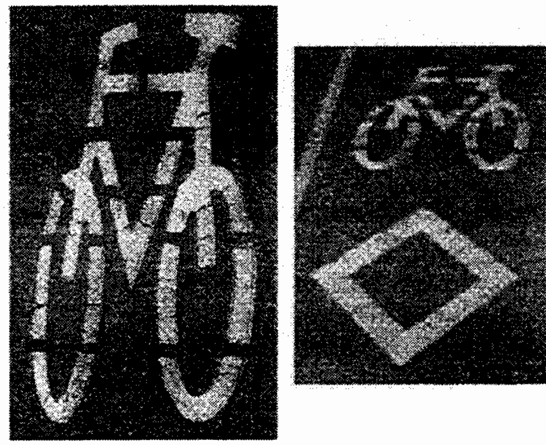

Fig. 3. Left: Anamorphic bicycle-lane symbol designed for viewing at about 20°. Right: View of the symbol at about the proper angle.

III. CONICAL ANAMORPHS

Anamorphic images which require an optical reflecting surface for reconstruction have, over the centuries, proven of more interest to both artist and viewer. The simplest is the conical reflector where the image is viewed from directly above the cone apex as shown in Fig. 6. The cone is characterized by an apex angle θ, and a base radius taken as unity so that the cone height is $h=\cot\frac{1}{2}\theta$. Clearly, the transformation has circular symmetry, so the target is taken to be a series of concentric rings of radius r which transform to rings of radius r' when reflected off the conical surface.

For the view from infinity (a not unreasonable assumption in practice) the transform is the strictly linear

$$r'=r+\frac{2(1-r)h^2}{h^2-1}. \tag{4}$$

More generally, the situation is as depicted in Fig. 6, where the anamorph is viewed from a point V at height d above the apex. From the figure,

$$r'=AR+(h-a)\tan\left(\frac{1}{2}\theta+\beta\right)=\frac{a}{h}+(h-a)\tan(\theta-\alpha)$$

and

$$\tan\alpha=\frac{r}{d+h}=\frac{a/h}{d+a}.$$

The last line can be used to eliminate both α and a and we find

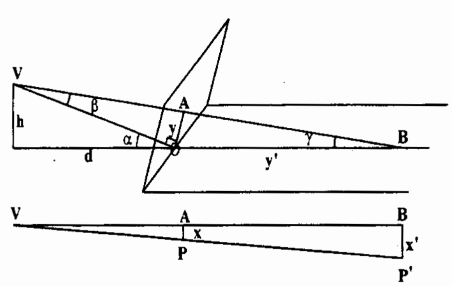

Fig. 4. The geometry of the plane anamorph viewed from a finite distance.

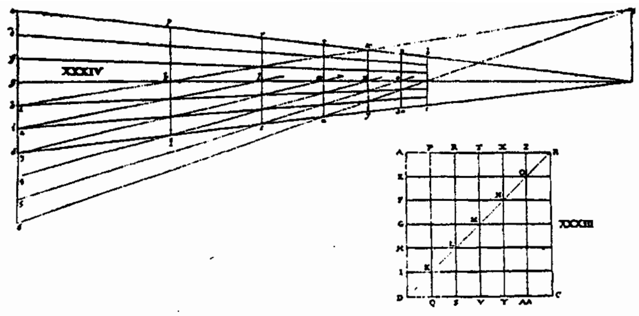

Fig. 5. Geometrical solution to the plane anamorph viewed from a finite distance by J.-F. Niceron (1638).

$$r' = \frac{dr}{d+h(1-r)} + \frac{(d+h)(1-r)h}{d+h(1-r)}\left[\frac{2h(d+h)-r(h^2-1)}{(d+h)(h^2-1)+2rh}\right]. \tag{5}$$

The limit of this as d goes to infinity is Eq. (4).

An example of this transformation is shown in Fig. 7,[12] which is again a construction by Niceron.[2] An analysis of the figure shows that the cone angle is 60°, a natural choice for a real situation. By fitting the measured values of r' it is further possible to show that the situation is not drawn for $d=\infty$, but rather that $d+h \approx 5$. The measured values of r' show some scatter and it is not unreasonable to speculate that the transformed rings were positioned experimentally by viewing a target of equally spaced circles in the cone about 5 radius units above its base.

Niceron also provided templates for making anamorphs by looking into a mirror from the side, but the templates, being circles in a sector, are certainly approximations.[13] Such anamorphs are rare.

IV. CYLINDRICAL ANAMORPHS

The anamorphs which have retained their interest for the longest time have been those which are reconstructed by observing them in a cylindrical mirror standing on the anamorphic plane. From objects of mystery in the 16th century to a rich man's possession in the 17th and 18th to children's toys in the 19th century,[14] the cylindrical anamorph has a long history.

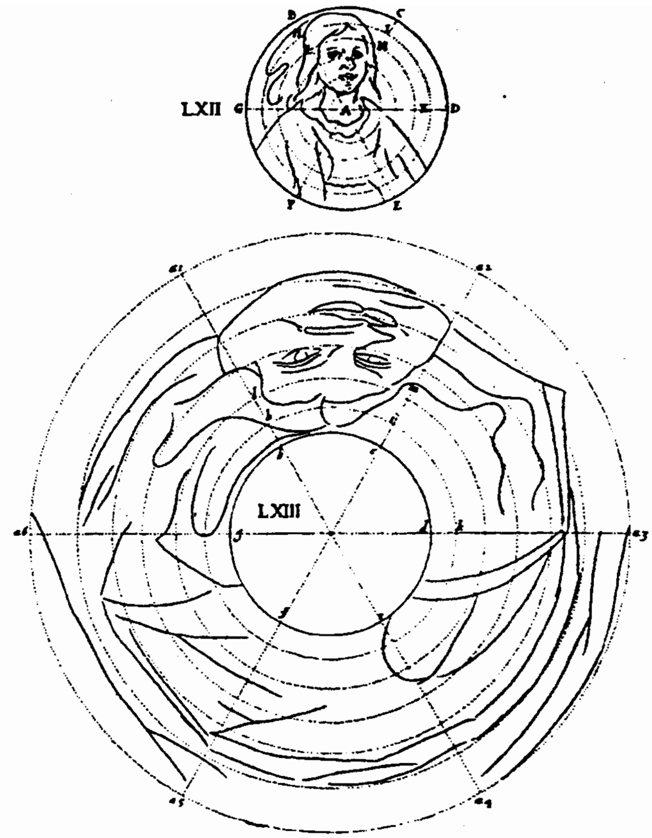

Fig. 7. Geometrical matrix for the construction of conical anamorphs by J.-F. Niceron (1638).

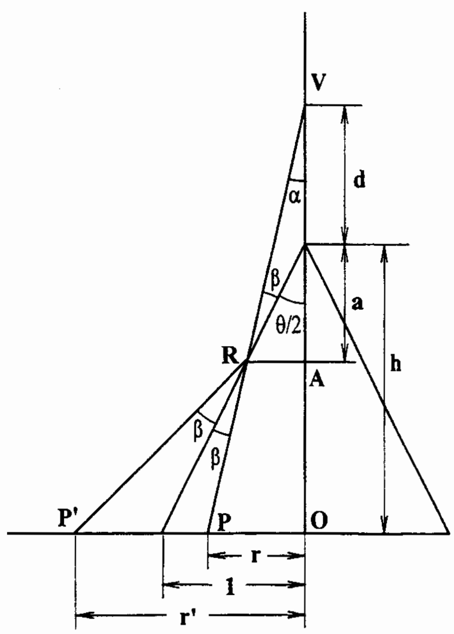

Fig. 6. The geometry of the conical anamorph viewed from a finite distance above the base.

The situation is depicted in Fig. 8. The cylinder with unit radius stands on the anamorphic plane and the transformation is determined by imagining rays from the point P'' in the anamorphic plane striking the mirror at R and being reflected to the point of observation V. The virtual extension of the ray VR passes through the anamorphic plane at P'. The left part of Fig. 8 shows the projection of the geometry onto the horizontal anamorphic plane, and on the right is a projection onto a vertical plane through the cylinder axis and V.

We take the origin of coordinates as the center of the base of the cylinder and V at an elevation h and distance d in

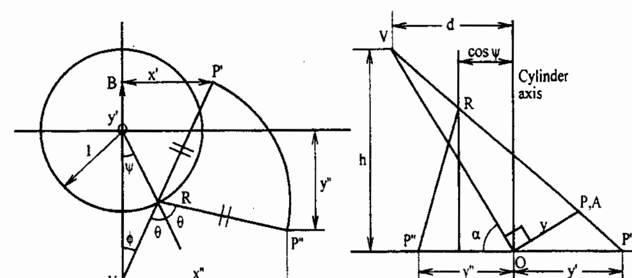

Fig. 8. The geometry of the cylindrical anamorph viewed from a finite distance at an elevation angle α.

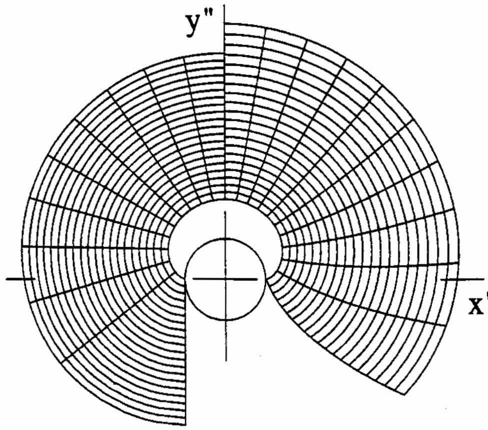

Fig. 9. Right: The cylindrically anamorphized contours of a 10×20 grid of squares viewed at $d=15$, $h=10$. Left: The same viewed from infinity at an elevation angle $\alpha=\tan^{-1}\frac{2}{3}$.

which case the geometry of the ray VP' is exactly that shown in Fig. 4. As a consequence we now can take the total anamorphic transform to P'' as the combination of the transformation Eqs. (2) and (3) plus that generated by the rotation of the ray segment RP' to RP'' by angle $\pi-2\theta$.

To obtain the latter transformation equations we start with

$$\tan\phi=\frac{x'}{d+y'}=\frac{\sin\psi}{d-\cos\psi}$$

as shown in Fig. 8. The second equality in this equation can be used to solve for ψ in terms of x',y', and d. We get, from the resultant quadratic in $\cos\psi$,

$$\cos\psi=\frac{dx'^2+(d+y')\sqrt{d^2(1-x'^2)+2dy'+x'^2+y'^2}}{(d+y')^2+x'^2}. \quad (6)$$

The point P'' coordinates are

$$x''=\sin\psi+RP''\sin(\psi+\theta),$$

$$y''=\cos\psi+RP''\cos(\psi+\theta),$$

where the horizontal projection $RP''=RP'=(\cos\psi+y')/\cos\phi$. On combining the above equations together with $\theta=\psi+\phi$ we obtain

$$x''=\sin\psi+(\cos\psi+y')(\sin 2\psi+\cos 2\psi\tan\phi)$$

$$=x'+\frac{2x'}{d+y'}(\cos\psi+y')(d\cos\psi-1). \quad (7)$$

and

$$y''=\cos\psi+(\cos\psi+y')(\cos 2\psi-\sin 2\psi\tan\phi)$$

$$=-y'+2\cos\psi(\cos\psi+y')\left(\frac{d\cos\psi-1}{d-\cos\psi}\right). \quad (8)$$

Since $\sin\psi=x'(d-\cos\psi)/(d+y')$, the second expressions in each of (7) and (8) make apparent the geometrical relationships $(x''-x')/(y''+y')=\tan\psi$, i.e., $P'P''$ is parallel to OR. The combined Eqs. (2), (3), and (6)–(8) solve the cylindrical anamorph problem of mapping (x,y) to (x'',y'').

The nature of these equations is illustrated in the right-hand section of Fig. 9 which displays the transformation of a 10×20 grid of squares on the interval $0\leq x\leq 1$, $0\leq y\leq 2$ with $h=10$ and $d=15$, i.e., an elevation angle of 33.7°. Note that the contours are not circles and the reflecting cylinder is displaced from what would appear to be the "center" of the array. The curved contours acquire an increasingly wider spacing further from the mirror as a consequence of the finite viewing distance. In many, if not most, situations the viewing distance can be taken to be infinity and a considerable simplification in the equations is realized. If $h=d\tan\alpha$, where α is the elevation angle of the line-of-sight above the plane, then Eqs. (2), (3), and (6)–(8) become

$$x''=x\left(3-2x^2+\frac{2y}{\sin\alpha}\sqrt{1-x^2}\right), \quad (9)$$

$$y''=2(1-x^2)^{3/2}+\frac{y}{\sin\alpha}(1-2x^2). \quad (10)$$

As an example of these equations, another 10×20 grid with $-1\leq x\leq 0$ viewed from infinity at $\alpha=33.7°$ is shown in the left-half of Fig. 9. The contours are not circles but they are equally spaced. The mirror is still not in the center of the array.

The contrast between the left and right sides of Fig. 9 illustrates that the edges of the cylinder are not lines of constant x except for viewing from infinity. At finite distance the image of the cylinder in the eye is a trapezium, wider at the top, which the brain insists on interpreting as a rectangle. This in turn means that, when viewing the anamorph as a "picture" framed by the cylinder, one is left with the impression that something is not quite right as the picture recedes from the cylinder edges with increasing y. To obtain an "artistically pleasing" transformation we must somehow compensate for the illusion that the edges of the cylinder appear as parallel lines. One possibility is to leave Eq. (2) and the corresponding target plane in Fig. 4 unchanged for specifying the y coordinate, but replace Eq. (3) with

$$x'/(d+y')=x/d, \quad (11)$$

which corresponds to specifying the x coordinate on a vertical target plane through the cylinder axis. By defining separate target planes we are clearly leaving the domain of perspective drawing in the strict sense, but it is precisely such apparent inconsistencies that prevents a unique solution.

Niceron[2] has provided both an exact geometrical method for the construction of such anamorphs and an approximation which is shown in Fig. 10.[15] In the approximation the contours are circles[16] with the spacing increasing with radial distance, indicating that it has been drawn for a finite viewing distance. In addition, the position of the cylinder is displaced from the center of the array. Finally, the radial lines in the correct transformation (see Fig. 9) do not converge to a common vertex whereas in the approximation they are taken to point back to the center of the displaced cylinder. The radial lines are drawn at equal angular separation, making the sectors in the forward direction larger than at the sides. From Fig. 9 it can be seen that this is exactly the opposite of the correct transformation. When the Niceron matrix is tested by transforming straight lines (a triangle, for example), the reconstructed image retains small curvatures. This failing would not be so noticeable or important for a drawing or painting which was probably to be finished by direct observation in the mirror.

The forward transformation Eqs. (2), (3), and (6)–(8) are appropriate if the target consists of simple elements. An ex-

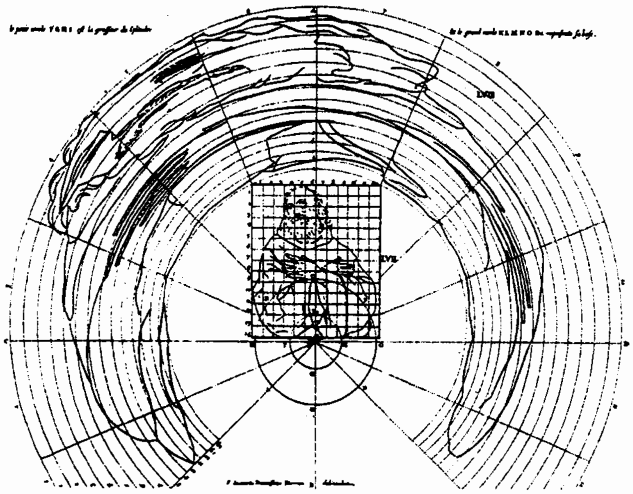

Fig. 10. Geometric matrix for the construction of cylindrical anamorphs by J.-F. Niceron (1638).

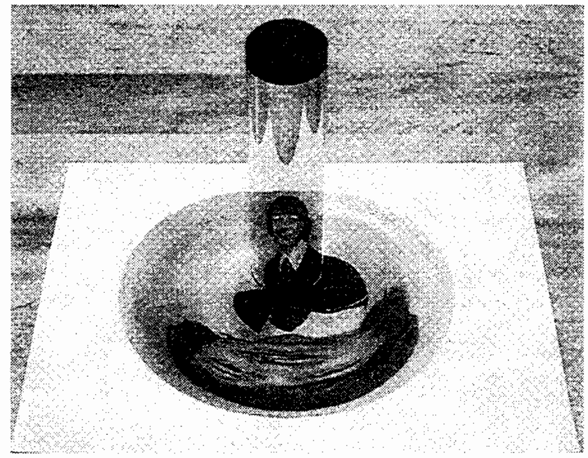

Fig. 12. An anamorphized photograph using Eqs. (A1)–(A5), (2), and (11).

ample of such a case and the resulting reconstruction is shown in Fig. 11. In more complicated cases such as a digitized photograph it is preferable to have the inverse equations. These are derived in the Appendix, and Fig. 12 is an example of an anamorphized photograph based on the inverse Eqs. (A1)–(A5) supplemented with (2) and (11).

V. VIEWING THE IMAGE

There is an ambiguity involved in describing where the virtual image in the cylindrical mirror appears to be located. The situation is described in Fig. 13. The anamorphic element at distance s is imaged at s' by the cylindrical mirror, i.e.,

$$\frac{1}{s'}=\frac{1}{s}+\frac{2}{R}.$$

Since $z=(s-s')\tan\alpha$, then,

$$z=\left(\frac{2s'^2}{R-2s'}\right)\tan\alpha. \qquad (12)$$

This image plane is shown in the figure and is responsible for some of the fascination of these objects in the past; the image seems to stand up in the mirror and have a certain three-dimensional air about it.[17]

There is, however, another possibility: Considered strictly as a plane mirror in the axial direction, then $s=s'$ and $z=0$. In other words, the image will be seen to lie in the anamorphic plane. Is such an image ever seen?

Almost all viewers see the cylindrical situation [Eq. (12)] when viewing the image normally, i.e., with the interocular axis perpendicular to the cylinder axis. It is, however, remarkable that when the head is turned to the side (interocular axis in the direction of the cylinder axis), then almost all viewers we have sampled see the image lying in the anamorphic plane. This illusion is very strong and viewing with one eye closed does not seem to affect it! We are not aware that this illusion has been previously reported.

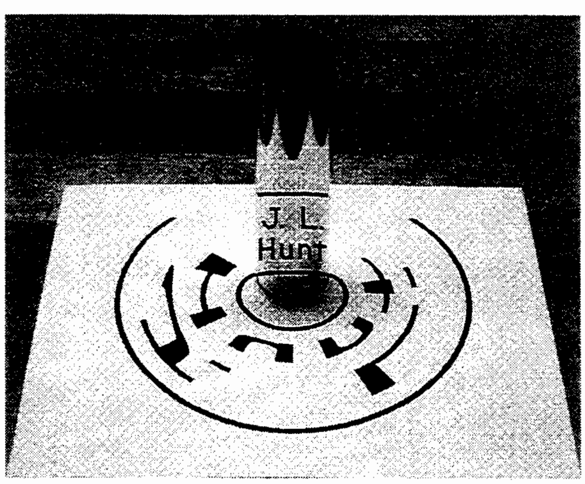

Fig. 11. A simple anamorph constructed with Eqs. (2), (3), and (6)–(8).

APPENDIX: THE INVERSE TRANSFORMS FOR THE CYLINDRICAL ANAMORPH

The inverse mapping for the cylindrical anamorph has been referred to by Dörrie[18] as *Alhazen's Billiard Problem.* We start by combining Eqs. (7) and (8) into the alternative pair,

$$(x''-\sin\psi)(\cos 2\psi-\sin 2\psi\tan\phi)$$
$$=(y''-\cos\psi)(\sin 2\psi+\cos 2\psi\tan\phi),$$
$$(x''-\sin\psi)^2+(y''-\cos\psi)^2=(\cos\psi+y')^2(1+\tan^2\phi).$$

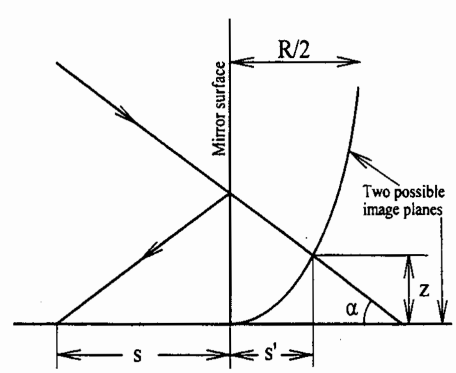

Fig. 13. The geometry of the image plane for the cylindrical anamorph.

The first of these equations, when used with $\tan \phi=\sin \psi/(d-\cos \psi)$, reduces to

$$(x''-\sin \psi)(d \cos 2\psi-\cos \psi) \\ =(y''-\cos \psi)(d \sin 2\psi-\sin \psi),$$

which is an equation for ψ in terms of x'',y'', and d. The remaining equation gives y' once ψ is known. Since $\cos \psi+y'$ is necessarily positive,

$$y'=-\cos \psi + \frac{d-\cos \psi}{\sqrt{d^2-2d \cos \psi+1}} \sqrt{(x''-\sin \psi)^2+(y''-\cos \psi)^2}, \tag{A1}$$

and, for completeness, from the discussion leading to Eq. (6),

$$x'=\frac{\sin \psi}{d-\cos \psi}(d+y'). \tag{A2}$$

The equation for ψ given above simplifies considerably if we introduce polar coordinates r, 2ϵ to replace x'',y'' and write ψ in terms of the deviation from the bisecting angle ϵ, i.e., $\psi=\epsilon+\delta$. With these changes the ψ-equation becomes equivalent to

$$\sin \delta \cos \delta=A \sin \delta+B \cos \delta,$$

where

$$A=\frac{1}{2}\left(\frac{1}{r}+\frac{1}{d}\right)\cos \epsilon, \quad B=\frac{1}{2}\left(\frac{1}{r}-\frac{1}{d}\right)\sin \epsilon, \\ x''=r \sin 2\epsilon, \quad y''=r \cos 2\epsilon. \tag{A3}$$

Since the equation for the coordinates x'',y'' define r and ϵ and hence A and B, we are left with the equation for δ as the only nontrivial relationship. This, in turn, is the quartic

$$\sin^4 \delta-2B \sin^3 \delta-(1-A^2-B^2)\sin^2 \delta+2B \sin \delta-B^2=0,$$

which yields, as the physical solution,

$$\sin \delta=\frac{b}{(a+\sqrt{a^2-2b})}, \tag{A4}$$

where

$$a=B+v, \quad b=u+B(2+u)/v, \quad v=\sqrt{1+u-A^2}, \\ u=4C \sin\left(\frac{t}{3}\right)\sin\left(\frac{\pi-t}{3}\right), \quad C=\frac{1}{3}(1-A^2-B^2), \tag{A5} \\ t=\sin^{-1}\left(\frac{AB}{C^{3/2}}\right)$$

with $|t|<\pi/2$. The solution of the inverse problem is now trivially completed with Eqs. (A1) and (A2) and $\sin \psi =\sin(\epsilon+\delta)$.

The solution given in Eq. (A4) is numerically stable with $|\delta|<\pi/2$ and all intermediate variables automatically real for any r, $d>1$ in the physically allowed region $|2\epsilon| \leqslant \cos^{-1}(1/d)+\cos^{-1}(1/r)$. In the extended or "shadow" region $\cos^{-1}(1/d)+\cos^{-1}(1/r)<|2\epsilon|<2\pi$, the solution predicted by Eq. (A4) corresponds to rays penetrating the cylinder and reflecting off the inner face. There is no dramatic simplification that occurs in the inverse Eqs. (A1)–(A5) when $d=\infty$ in contrast to the forward transformation case [cf. Eqs. (9) and (10)].

It is perhaps worth noting that none of the intermediate functions a, b, v, or u in Eqs. (A4) and (A5) have any particular symmetry under reflection $\epsilon\rightarrow-\epsilon$, and thus the symmetry requirement that $\sin \delta$ be an odd function of ϵ is not explicit. This does not mean Eq. (A4) is defective. On the contrary, Eq. (A4) is our "best" choice as we have found that other representations of the solution, which do satisfy the reflection symmetry explicitly, are numerically less stable and/or more complicated.

[1]Fabrizio Clarici, "The Grand Illusion: Some Considerations of Perspective, Illusionism and Trompe-l'oeil," Art News Annual **XXIII**, 96–180 (1954).

[2]Jean-Francois Niceron, *La perspective curieuse* (Paris, 1638).

[3]Jurgis Baltrušaitis (translated by W. J. Strachan), *Anamorphic Art* (Abrams, New York, 1977).

[4]F. Leeman, *Hidden Images: Games of Perception, Anamorphic Art, Illusion* (Abrams, New York, 1976).

[5]Martin Gardner, "Mathematical Games," Sci. Am. **232** (1), 110–116 (1975).

[6]Jearl Walker, "The Amateur Scientist," Sci. Am. **245** (1), 176–187 (1981).

[7]Figure p-125, 151(bw) of Ref. 1; Chap. 7 (bw) of Ref. 3; Plate 2, p-18 (color) of Ref. 4; Fig. p-112 (bw) of Ref. 5.

[8]Figure p-150 of Ref. 1; Fig. 7 of Ref. 3; Fig. 2, p-12 of Ref. 4.

[9]Figure 24 of Ref. 3.

[10]The image on the left in Fig. 3 could not be photographed from directly above but from the side at a steep angle; it has been corrected digitally for this anamorphic error. One-dimensional anamorphs are also the way in which wide-screen motion pictures are made. The camera imposes a compression in the horizontal direction to fit the wide image onto the film and the projector's anamorphic lens reverses the compression.

[11]Figure 28 of Ref. 3; Fig. 27, p-106 of Ref. 4: Fig. (redrawn) p-114 of Ref. 5.

[12]Figure 118 of Ref. 3; Fig. p-115 of Ref. 5.

[13]Figure 34 of Ref. 3.

[14]A child's toy consisting of 24 colored drawings and a cylindrical mirror was published about 1900 by the McLoughlin Bros. of New York. It was republished as *The Magic Mirror* (Dover, New York, 1979). A more recent child's book is *Anno's Magical ABC*, by Mitsumasa and Masaichiro Anno (Bodley Head, London, 1981).

[15]Figure 115 of Ref. 3; Fig. 35, p-132 of Ref. 4: Fig. p-115 of Ref. 5. The base of the mirror is the smallest circle centered at the convergence of the radial lines.

[16]When the curves are measured they are not strictly circles; the radii vary about 1%. The geometry of the drawing strongly suggests that they were originally circles. Various copying processes and the differential shrinking of almost four-century-old paper can well account for the deviations.

[17]Artists may have actually taken advantage of the shape of this image surface. A pretty example on p. 111 of Ref. 5 is an anamorph by an unknown artist of Nicolas Lancret's *Par une Tendre Chansonnette (With a tender little song)*. The reconstructed image of a man serenading a woman and her attendants in a pastoral scene has an almost horizontal lawn in the lower portion, and the figures erect further in the rear. The three-dimensional effect is very striking. It can be viewed using a cylindrical mirror 43 mm in diameter.

[18]Heinrich Dörrie, *A Hundred Great Problems of Elementary Mathematics: Their History and Solution* (translated by David Antin) (Dover, New York, 1965).