A Module-Theoretic Approach to Supersymmetric Gauge Theory

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Abstract

In particle physics, a field theory is a specification of how a field (such as functions or vector fields on spacetime) changes with time or with respect to other components of the field. A gauge theory is a field theory in terms of equivalence classes of covector fields.

Supersymmetry is a symmetry that relates bosons and fermions, the two fundamental classes of elementary particles. Together with rotations and translations, they form the super Poincare group. To study this group, we view the translations as generating a polynomial ring and the rotations as a standard (Lie or matrix) group. A representation maps elements of a group or ring to linear transformations or matrices on a vector space. We study a representation of the super Poincare group which maps the polynomial ring and the supersymmetry algebra to linear transformations on a module. Modules are basically vector spaces, except that their scalars come from polynomial rings. We show that in this that the equivalence classes of gauge fields are dual to a special submodule of a free module. This submodule is called the syzygy submodule.

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Dedication

To my friends and family.

Acknowledgments

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1
Foreword

This project was motivated by a problem in physics. An important problem in physics is to find mathematical representations for the group of symmetries of the universe. The group of symmetries of Euclidean space $\mathbb{R}^n$ have long been well understood. They form the so-called Euclidean group, which consist of translations and rotations. When a time dimension was added to a usual three-dimensional Euclidean space to form Minkowski space, it introduces extra symmetries. The group of symmetries of Minkowski space is therefore bigger. It is called *the Poincaré group.* Recently, additional symmetries between bozons and fermions have been postulated; hence the group of the symmetries of the universe was once more enlarged to something called *the super Poincaré group.* Attempts have been made to represent this group. However, these representations were only worked out case by case, and in many cases were arrived at computationally There has not been a general theoretical basis for understanding them. In this project, the two main theorems we proved are attempts to provide an algebraic perspective for some representation of the super Poincaré group (Theorem 3.3.4 and Theorem **??**).

The symmetries of the universe consist of translations, rotations and supersymmetries. Thus the super Poincaré group has three components, each corresponding to one type of symmetry: a ring (translations), a group (rotation) and an algebra (supersymmetry) that interacts with the ring. In this project, we study only representations of the ring that respect the super symmetry algebra (we do not study the rotation group).

So how do we represent an algebraic object in general? Let's take for example, electric force. We cannot see electric force, but we can learn about its existence and properties by observing how charges move under its influence. In other words, we learn about it via its action on other objects. Similarly, we can get information from an algebraic object by letting it act on other objects. A representation of an algebraic object is basically another object for it to act on together with the action itself.

We want to study these representations from a purely algebraic perspective, an approach that has not been fully exploited. But before introducing our translation of the problem into algebra terminologies, let us first familiarize the readers with the algebraic language.

2
Background

2.1 Modules

The definitions and theorems in this section are from Dummit [3] and Roman [6].

As mentioned above, we want to represent a ring. How are rings represented in general? We need to find an object for the ring to act on.

Readers are probably familiar with the concept of vector space over a field. Recall that if V is a vector space over a field F, then F acts on V via scalar multiplication. Thus V could be used to represent F. To represent a commutative ring R, we simply have a expand to concept of a vector space over a field to "a vector space over a ring", which is what is called a "module".

Definition 2.1.1. A *left R-module* M over a ring R consists of an abelian group $(M, +)$ and an operation $R \times M \longrightarrow M$ such that for all $r, s \in R$ and $x, y \in M$, we have

1. $r(x + y) = rx + ry$
2. $(r + s)x = rx + sx$
3. $(rs)x = r(sx)$

4. $1_R x = x$ if R has multiplicative identity $1R$.

The operation of the ring on M is called scalar multiplication, and is usually written by juxtaposition, i.e. as rx for $r \in R$ and $x \in M$.

From now on, we will assume all rings R are commutative, unless otherwise stated.

Examples

1. Any ring R is a module over it self.

2. In general, R^n is a module over R.

3. For all ideals I, we have I and R/I is a module over R.

4. All abelian groups are $\mathbb{Z}$-modules. Conversely, all $\mathbb{Z}$-modules are abelian groups.

Notice that the definition of module is almost exactly as that of vector space, the only difference is that scalar multiplication is from elements from a ring instead of a field. Many concepts in vector spaces carry over to module, like subspace, basis and linear independence.

Definition 2.1.2. Suppose M is a R-module and N is an additive subgroup of M. If N is also an R-module, then we call N a *submodule* of M.

Definition 2.1.3. A module M is *finitely generated* if there exist finitely many elements $x_1, \ldots, x_n \in M$ such that every element of M can be written as a linear combination of those elements with coefficients from the scalar ring R. The elements $x_1, \ldots, x_n$ are called *generators* of M.

Definition 2.1.4. Let M be an R-module. Let $x_1, \ldots, x_n \in M$. If there exist $a_1, \ldots a_n \in R$ such that $\sum_{i=1}^{n} a_i\, x_i = 0$ then $x_1, \ldots, x_n$ are said to be *linearly dependent.* If there do not exist such a_i's, then $x_1, \ldots, x_n$ are said to be *linealy independent.* A linearly independent set of generators of M is called a *basis* of M.

However, saying that a module is basically a vector space over a ring is very deceptive. In many ways, modules might not be as "nice" as vector spaces. It is the invertibility of the elements of the base fields that give vector spaces many nice properties. Since not all non-zero elements in a ring are invertible, many strange things can happen in a module. For example

1. Two linearly dependent elements might not be multiples of each other.

   e.g. Consider $\mathbb{Z}$ as a module over itself.

   $3 \cdot 2 - 2 \cdot 3 = 0$ but 3, 2 are not multiples of each other in $\mathbb{Z}$.

2. A module might contain no linearly independent set. (Every singleton set is linearly dependent.)

   e.g. Consider $\mathbb{Z}/n\mathbb{Z}$ as a module over $\mathbb{Z}$.

   $n \cdot x = 0$ for all $x \in \mathbb{Z}/n\mathbb{Z}$, but $n \neq 0$.

3. A spanning set might not contain a basis

   e.g. see previous example

There are, however, certain kinds of modules are that highly resemble vector spaces, and are therefore nice to work with. They are called *free modules*.

Definition 2.1.5. A *free module* is a module that has a basis.

Theorem 2.1.6. *Every basis of a free module has the same cardinality.*

This leads to the following definition.

Definition 2.1.7. The *rank* of a free module is the cardinality of its basis.

One convenience in working with a free module over a ring is that we can identify the module with copies of the ring itself. Let M be an R-module of rank n with basis

$\{b_1, \ldots, b_n\}$. Then the canonical map from R^n to M that sends every n tuple $(r_1, \ldots, r_n)$ to the element $r_1 b_1 + \cdots r_n b_n$ is an isomorphism.

Theorem 2.1.8. *Every R-module of rank n is isomorphic to R^n.*

The isomorphism mentioned above is isomorphism between modules. The definition is discussed in the next section.

2.2 R-homomorphisms

The concept of linear maps between vector spaces is generalized to R-homomorphisms between modules.

Definition 2.2.1 (R-homomorphism)**.** If M and N are R-modules, then a map $f : M \longrightarrow N$ is an *R-homomorphism* if, for any $m, n \in M$ and $r, s \in R$ we have

$$f(rm + sn) = rf(m) + sf(n).$$

A bijective module homomorphism is an isomorphism of modules, and the two modules are called *isomorphic*. An R-homomorphism from the module back to itself is called a *linear operator* or an *endormorphism* on the module. The endomorphisms of a module M form a ring called the ring of endomorphisms of M, which is denoted by $\mathrm{End}(M)$. Other concepts that are related to linear maps of vector spaces, such as kernel, linear functionals and annihilators, also exist for modules.

Definition 2.2.2. Let M, N be R-modules. Let $f : M \longrightarrow N$ be an R-homomorphism. Then the *kernel* of f is the submodule of M defined as $\ker f = \{x \in M \mid f(x) = 0\}$.

Definition 2.2.3. An R-homomorphism from an R-module M to R is called a *linear functional*. The set of all linear functionals on M is called the *dual space* of M and is denoted M^*.

Definition 2.2.4. Let M be a free module of finite rank with basis $\mathcal{B} = \{x_1, \ldots, x_n\}$. For $1 \le i \le n$, define a linear functional $x_i^* \in M^*$ by $x_i^*(x_j) = \delta_{ij}$. Then $\mathcal{B}^* = \{x_1^*, \ldots, x_n^*\}$ is a basis for M^*, called the *dual basis* of $\mathcal{B}$.

Definition 2.2.5. Let S be a nonempty subset of a module M. The *annihilator* M^0 of M is

$$M^0 = \{f \in M^* \mid f(S) = 0\}.$$

Theorem 2.2.6. *Let M be an R-module and N a submodule. Then*

1. $(M/N)^* \approx N^0 \subset M^*$
2. $N^* \approx M^*/N^0$.

Proof.

1. $(M/N)^* \approx N^0 \subset M^*$. Let ϕ be a homomorphism from $(M/N)^*$ to M^* such that for all $f \in (M/N)^*$ we have $\phi(f)(m) = f(m+N)$ for all $m \in M$. This map is well-defined. If $m - m' = \in N$ then $\phi(f)(m) - \phi(f)(m') = \phi(f)(m - m') = f(m - m' + N) = f(0) = 0$. It is also easy to see that ϕ is a homomorphism with trivial kernel.

   Since $\phi(f)(n) = 0$ for all $n \in N$, we have that $\operatorname{im}\phi \subset N^0$. Now let $g \in N^0$. We show that $g = \phi(f)$ for some $f \in (M/N)^*$. Simply, let f be defined as $f(m+N) = g(m)$. Then f is well defined because if $m - m' \in N$ then $f(m+N) - f(m'+N) = f(m - m' + N) = g(m - m') = 0$.

   Thus $(M/N)^* \approx N^0 \subset M^*$.

2. $N^* = M^*/N^0$.

□

2.2.1 Graded modules and homomorphisms

Grading is a powerful tool in Commutative Algebra. The general principle of using that tool is the following: in order to understand the properties of a graded object X, we consider X as a direct sum of vector spaces (its graded components) and we study the properties of each of these vector spaces.

Definition 2.2.7. A ring S is called a *graded ring* if it is the direct sum of additive subgroups: $S = S_0 \oplus S_1 \oplus S_2 \oplus \cdots$ such that $S_i S_j \subseteq S_{i+j}$ for all $i, j \geq 0$. The elements of S_k are said to be *homogeneous of degree* k, and S_k is called the *homogeneous component of* S *of degree* k.

Definition 2.2.8. An ideal I of the graded ring S is called a *graded ideal* if $I = \oplus_{k=0}^{\infty}(I \cap S_k)$.

Now we will define a grading on the polynomial rings. Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. Here $S_0 = k$ and the homogeneous components of degree i is the subgroup of all k-linear combinations of monomials of degree k.

Let I be a graded ideal in S. Note that $S_i I_j \subseteq I_{i+j}$ for all $i, j \in \mathbb{N}$.

Theorem 2.2.9. *The quotient ring $R = S/I$ inherits the grading from S by $R_i = S_i/I_i$ for every $i \in \mathbb{N}$.*

For the rest of the section, $R = S/I$, with standard grading described above.

Definition 2.2.10. An R-module M is called graded, if it has a direct sum decomposition $N = \oplus_{i \in \mathbb{Z}} N_i$ as a k-vector space and $R_i N_j \subseteq N_{i+j}$ for all $i, j \in \mathbb{Z}$. The k-spaces N_i are called the *homogeneous components* of N. An element $m \in N$ is called *homogeneous* if $m \in N_i$ for some i, and in this case we say that m has degree i and write $deg(m) = i$. Every element $m \in N$ can be written uniquely as a finite sum $m = \sum_i m_i$, where $m_i \in N_i$, and in this case m_i is called the *homogeneous component* of m of degree i.

Definition 2.2.11. Let M be a finitely generated graded R-module. Let $p \in \mathbb{Z}$. Denote by $M(-p)$ the graded R module such that $M(-p)_i = M_{i-p}$ for all i. We say that $M(-p)$ is the module M *shifted* p *degrees*, and call p the *shift*.

Definition 2.2.12. Let M and N be graded R-modules. We say that a homomorphism $\phi : M \longrightarrow N$ has *degree* i if $deg(\phi(m)) = i + deg(m)$ for each homogeneous element $m \in M$.

Recall that 0 has arbitrary degree, thus $deg(\phi(m)) = i + deg(m)$ is a condition only on the homogeneous elements outside $Ker(\phi)$.

2.3 The Exterior Algebra

The treatment of the exterior and tensor algebra in this section is from Dummit [3].

2.3.1 Tensor Product of Modules

Whereas the direct sum is a construction that allows "addition" of two modules, the tensor product is a way to define "product" of two modules. Given a right R-module M and a left R-module N, the tensor product between M and N is a "larger" module where we cab take "products" mn of elements $m \in M$ and $n \in N$. The tensor product also gives a way to "extend scalars". Suppose that R is a subring of a ring S and we want to to define an left S-module structure on N. Through this "extension of scalars" example, we will motivate the definition of tensor product.

Example 2.3.1 (Extending scalars)**.** We recall that an S-module consists of an abelian group N and a map from $S \times N$ to N (scalar multiplication). Thus it is natural to consider the free $\mathbb{Z}$-module $S \times N$. To satisfy all the module axioms and to ensure compatibility with the action of R on N, we need to quotient $S \times N$ by the subgroup generated by elements of the form

$$(s_1 + s_2, n) - (s_1, n) - (s_2, n)$$

$$(s, n_1 + n_2) - (s, n_1) - (s, n_2)$$

$$(sr, n) - (s, rn)$$

for $s, s_1, s_2 \in S, n, n_1, n_2 \in N$ and $r \in R$, where rn in the last element refers to the R-module structure already defined on N.

The resulting quotient group is denoted by $S \otimes_R N$ and is called the *tensor product of S and N over R*. Elements of $S \otimes_R N$ are called tensors and can be written as a sum of "simple tensors" of the form $s \otimes n$, which denotes the coset containing (s, n).

It is easy to see that $S \otimes_R N$ is a left S-module under the action defined by

$$s\left(\sum_{finite} s_i \otimes n_i\right) = \sum_{finite} (ss_i) \otimes n_i.$$

In general, we have the following definition.

Definition 2.3.2. Suppose that M is a right R-module and N is a left R-module. The quotient of the free $\mathbb{Z}$-module on the set $M \times N$ by the subgroup generated by all elements of the form

$$(m_1 + m_2, n) - (m_1, n) - (m_2, n)$$

$$(m, n_1 + n_2) - (m, n_1) - (m, n_2)$$

$$(mr, n) - (m, rn)$$

for $m, m_1, m_2 \in M, n, n_1, n_2 \in N$ and $r \in R$ is an abelian group, denoted by $M \otimes_R N$ and is called the *tensor product of M and N over R*. The elements of $M \otimes_R N$ are called *tensors* and the coset $m \otimes n$ of (m, n) in $M \otimes_R N$ is called a *simple tensor*.

2.3.2 The Tensor Algebra and Exterior Algebras

Definition 2.3.3. Let R be a commutative ring with identity. An *R-algebra* is a ring A with identity together with a ring homomorphism $f : R \longrightarrow A$ mapping 1_R to 1_A such that the subring $f(R)$ of A is contained in the center of A (i.e. $f(r)a = af(r)$ for all $r \in R, a \in A$).

If A is an R-algebra then it is easy to check that A has a natural left and right R-module structure defined by $r \cdot a = a \cdot r = f(r)a$ where $f(r)a$ is just the multiplication in the ring A.

For the rest of the section, let R be a commutative ring with 1. We assume the left and right actions of R on each R-module are the same.

Suppose M is an R-module.

Definition 2.3.4. For each interger $k \geq 1$, define

$$\mathcal{T}^k(M) = M \otimes_R M \otimes_R \cdots \otimes_R M \text{ k factors},$$

and set $\mathcal{T}^0(M) = R$. The elements of $\mathcal{T}^k(M)$ are called *k-tensors*.

Define

$$\mathcal{T}(M) = R \oplus \mathcal{T}^1(M) \oplus \mathcal{T}^2(M) \oplus \mathcal{T}^3(M) \cdots = \bigoplus_{k=0}^{\infty} \mathcal{T}^k(M).$$

Every element of $\mathcal{T}(M)$ is a finite linear combination of k-tensors for various $k \geq 0$. We identify M with $\mathcal{T}^1(M)$, so that M is an R-submodule of $\mathcal{T}(M)$.

Theorem 2.3.5. *If M is any R-module over the commutative ring R then $\mathcal{T}(M)$ is an R-algebra containing M with multiplication defined by mapping*

$$(m_1 \otimes \cdots m_i)(m'_1 \otimes \cdots \otimes m'_j) = m_1 \otimes \cdots m_i \otimes m'_1 \otimes \cdots \otimes m'_j.$$

and extended to sums via the distributive laws. With respect to this multiplication $\mathcal{T}^i(M)\mathcal{T}^j(M) \subseteq \mathcal{T}^{i+j}(M)$. Thus the tensor algebra has a natural grading structure.

Definition 2.3.6. The ring $\mathcal{T}(M)$ is called the *tensor algebra* of M.

Definition 2.3.7. The *exterior algebra* of an R-module M is the R-algebra obtained by taking the quotient of the tensor algebra $\mathcal{T}(M)$ by the ideal $\mathcal{A}(M)$ generated by all elements of the form $m \otimes m$, for $m \in M$. The exterior algebra $\mathcal{T}(M)/\mathcal{A}(M)$ is denoted by $\bigwedge(M)$ and the image of $m_1 \otimes m_2 \otimes \cdots \otimes m_k$ in $\bigwedge(M)$ is denoted by $m_1 \wedge m_2 \wedge \cdots \wedge m_k$.

The ideal $\mathcal{A}(M)$ is generated by homogeneous elements hence is a graded ideal. By Theorem 2.2.9, the exterior algebra is graded, with k-th homogeneous component $\bigwedge^k(M) = \mathcal{T}^k(M)/\mathcal{A}^k(M)$.

Definition 2.3.8. The R-module $\bigwedge^k(M)$ is called the *k-th exterior power* of M.

We can identify R with $\bigwedge^0(M)$ and M with $\bigwedge^1(M)$ and so consider M as an R-submodule of the R-algebra $\bigwedge(M)$. The multiplication

$$(m_1 \wedge \cdots \wedge m_i) \wedge (m'_1 \wedge \cdots \wedge m'_j) = m_1 \wedge \cdots m_i \wedge m'_1 \wedge \cdots \wedge m'_j$$

in the exterior algebra is called the *wedge product*. Note that $m_1 \wedge \cdots \wedge m_k = 0$ if $m_i = m_j$ for some $1 \leq i \neq j \leq k$ and if M is not of characteristic 2 then $m_1 \wedge m_2 = -m_2 \wedge m_1$.

Theorem 2.3.9. *Let M be an R-module over the commutative ring R and let $\bigwedge(M)$ be its exterior algebra. Then the k-th exterior power $\bigwedge^k(M)$ of M is equal to $M \otimes \cdots \otimes M$ (k factors) modulo the submodule generated by all the elements of the form*

$$m_1 \otimes m_2 \otimes \cdots \otimes m_k \text{ where } m_i = m_j \text{ for some } i \neq j.$$

In particular, $m_1 \wedge m_2 \wedge \cdots \wedge m_k = 0$ if $m_i = m_j$ for some $i \neq j$.

Corollary 2.3.10. *Let M be a free R-module of rank n with basis $m_1, \ldots, m_n$ then*

$$\bigwedge^n(M) = R(m_1 \wedge \cdots \wedge m_n)$$

is a free (rank 1) R-module with generator $m_1 \wedge \cdots \wedge m_n$ and

$$\bigwedge^{n+1}(M) = \bigwedge^{n+2}(M) = \cdots = 0.$$

In particular, $\dim_R(\bigwedge^k(M)) = \binom{n}{k}$.

2.4 Differential Forms

The elements of the modules we will use to represent to super Poincaré group and the relations between them will come from properties of differential forms. Thus, to motivate the construction of the module we will use to represent the our algebraic object, we first need to introduce differential forms.

These definitions and theorems are from Zorich [7].

2.4.1 Definitions

Definition 2.4.1. Let M be an R module. Then a k-linear form is an R-module homomorphism from M^k to R.

Definition 2.4.2. A k-linear form $\omega : M^k \longrightarrow R$ is *skew-symmetric* if the value of the form changes sign when any pair of its arguments are interchanged, that is

$$\omega(\xi_1, \ldots, \xi_i, \ldots, \xi_j, \ldots, \xi_k) = -\omega(\xi_1, \ldots, \xi_j, \ldots, \xi_i, \ldots, \xi_k).$$

In particular, if $\xi_i = \xi_j$ for some $i \neq j$ and $\mathrm{char} k \neq 2$ then the value of the form will be 0.

Definition 2.4.3. Let A^p be a skew-symmetric p-form, and B^q be a skew-symmetric q-form. Then the *exterior product* $\wedge$ of A^p and B^q is defined to be a $p+q$ skew-symmetric form satisfying

1. $(A^p \wedge B^q) \wedge C^r = A^p \wedge (B^q \wedge C^r)$

2. $(A^p + B^p) \wedge C^q = A^p \wedge C^q + B^p \wedge C^q$

3. $A^p \wedge B^q = (-1)^{pq} B^q \wedge A^p$

Definition 2.4.4. A *differential p-form* ω with domain $D \subset \mathbb{R}^n$ is a function sending every $x \in D$ to a skew symmetric form $\omega(x) : (TD_x)^p \longrightarrow \mathbb{R}$.

Thus for each x, the map $\omega(x)$ is an element of the dual space $((TD_x)^p)^*$.

For $1 \leq i \leq n$, let the linear functional dx_i from $\mathbb{R}^n$ to $\mathbb{R}$ be defined as

$$dx_i(\xi_1 \mathbf{e}_1 + \ldots + \xi_d \mathbf{e}_n) = \xi_i.$$

Then, every 1-form ω can be written uniquely as

$$\omega = \sum_{i=1}^{n} a_i dx_i$$

for some real-valued functions a_i.

In which case, for all $x \in (TD_x)^p$, we have $\omega(x)$ is the linear functional in $((TD_x)^p)^*$ given by $\sum_{i=1}^{n} a_i(x) dx_i$.

2.4.2 Coordinate Expressions

Every differential k-form is a combination of the elementary k-forms $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ formed from the differentials of the coordinates. Specifically, let ω be a differential k-form on $D \subset \mathbb{R}^n$ along with a curvilinear coordinate system $x_1, \ldots, x_n$. Then

$$\omega = \sum_{1 \leq i_1 \leq \ldots \leq i_k \leq n} a_{i_1 \ldots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

where $a_{i_1 \ldots i_k}$ are functions from D to $\mathbb{R}$.

In particular,

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n.$$

From now on let $\Omega^k(\mathbb{R}^n)$ denote the space of k-forms on $\mathbb{R}^n$ with **polynomial** coordinates, i.e. the space of all ω of the form $\sum_{1 \leq i_1 \leq \ldots \leq i_k \leq n} a_{i_1 \ldots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ where $a_{i_1 \ldots i_k} \in \mathbb{R}[x_1, \ldots, x_n]$ for all i_j.

2.4.3 Exterior Derivative

Definition 2.4.5 (Differential)**.** Let X and Y be normed space. A mapping $f : E \longrightarrow Y$ of a set $E \subset X$ into Y is differentiable at $x \in E$ if there exists a continuous linear transformation $L(x) : X \longrightarrow Y$ such that

$$f(x + h) - f(x) = L(x)h + \alpha(x; h)$$

where $\alpha(x; h) = o(h)$ as $h \longrightarrow 0, x + h \in E$.

The function $L(x)$ is called the *differential* of f at x

Definition 2.4.6 (Exterior derivative)**.** The *exterior derivative* of a 0-form f is the differential df of f.

In general, let $\omega = \sum a_{i_1 \dots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ be a differential k-form. Then its *exterior derivative* is the $k+1$-form

$$d\omega = \sum_{\alpha, i_j} \partial_{x_\alpha} a_{i_1 \dots i_k} dx_\alpha dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

A very important and well-known property of the exterior derivative is

Theorem 2.4.7. $d^2 = 0$.

2.5 Representations

In general, in mathematics, a representation refers to an establishment of similarities between two collection of objects. Roughly speaking, a collection of objects X is said to represent a collection of objects Y, if the properties and relationships among the objects in X conform in some consistent way to those of the objects in Y. We are particularly interested in the notion of representation in representation theory, branch of mathematics that studies abstract algebraic structures by representing their elements as linear transformations of vector spaces. In essence, representation theory reduces problems in abstract algebra to problems in linear algebra, which is better understood. A representation makes an abstract algebraic object more concrete by describing its elements by matrices and the algebraic operations in terms of matrix addition and matrix multiplication. The algebraic objects amenable to such a description include groups, associative algebras and Lie algebras.

A representation of a group G is a vector space V over a field F with a group homomorphism $\phi : G \longrightarrow GL(V, F)$.

A representation of an associative algebra A is a vector space V over a field F with an algebra homomorphism $\phi : A \longrightarrow \mathrm{End}_F(V)$.

A representation of a Lie algebra is a vector space V over a field F with a Lie algebra homomorphism $\phi : a \longrightarrow \mathfrak{gl}(V, F)$.

A representation of a ring R is a module M over R together with the homomorphism $\phi : R \longrightarrow \mathrm{End}(M)$ which takes r to multiplication by r.

3
The problem

3.1 Super Poincaré Group

The super Poincaré Group is the group of symmetries of the universe, which includes super symmetries. It is an algebraic object that has three main components.

$p^{d|N} = \mathrm{spin}(1, d-1) \oplus \mathbb{R}^{1,d-1} \oplus \prod \mathbb{S}$.

The spin group is the rotation group. The ring $\mathbb{R}[\partial_{x_1}, \ldots, \partial_{x_d}] \approx \mathbb{R}[x_1, \ldots, x_d]$ encodes infinitesimal translations. Finally, $\prod \mathbb{S} = \mathrm{span}\{Q_1, \ldots, Q_N\}$ is the supersymmetry group. The three components act on each other via the following relations. For all, $L \in \mathrm{spin}(1, d-1)$, $\partial \in \mathbb{R}^{1,d-1}$, $Q \in \prod \mathbb{S}$ we have

1. $[L_1, L_2] = L_1L_2 - L_2L_1$
2. $[\partial_1, \partial_2] = \partial_1\partial_2 - \partial_2\partial_1 = 0$ (derivatives commute)
3. $[L, \partial_v] = L\partial_v - \partial_v L = \partial_{Lv}$
4. $[\partial, Q] = 0$
5. $[L, Q] = L(Q)$

6. $\{Q_1, Q_2\} = \partial_{IJ}$

7. $\{Q_I, Q_J\} = Q_I Q_J + Q_J Q_I = 2\Gamma^k_{IJ} x_k$ where Γ is some given matrix.

As mentioned above, we are mainly interested in finding a representation for the ring $\mathbb{R}[x_1, x_2, \ldots, x_d]$ (the translations) which respects the supersymmetry algebra. A representation of a ring is a module over it. Thus in other words, we want to construct a module over $\mathbb{R}[x_1, \ldots, x_d]$ with linear operators Q_i's satisfying the supersymmetry relations $\{Q_I, Q_J\} = Q_I Q_J + Q_J Q_I = 2\Gamma^k_{IJ} x_k$.

3.2 Example of an approach: evaluation

Let M be a rank n free module over $\mathbb{R}[t, x, y, z]$. Let Q_i's be operators on M. One of the methods used to study M is to turn it into a finite dimensional vector space via evaluation map (Greg). In his paper [8], Landweber constructed an evaluation map $ev : \mathbb{R}[t, x, y, z] \to \mathbb{R}$ which sends t, x, y, z to some fixed t_0, x_0, y_0, z_0, correspondingly. This maps could be extend to a map from M to $\mathbb{R}^n$ by identifying M with $\mathbb{R}[t, x, y, z]^n$. Every element $(p_1(t, x, y, z), \ldots, p_n(t, x, y, z))$ of M is sent to the element$(p_1(t_0, x_0, y_0, z_0), \ldots, p_n(t_0, x_0, y_0, z_0))$ of $\mathbb{R}^n$. Thus the evaluation map identifies M with $\mathbb{R}^n$, a finite dimensional vector space over the reals.

The evaluation map also gives a natural way to identify each Q_i with a corresponding $\mathbb{R}$-linear transformation γ_i from $\mathbb{R}^n$ to $\mathbb{R}^n$. Fix the basis $(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, \ldots, 1)$ for $M \approx (\mathbb{R}[t, x, y, z])^n$, then each Q_i could be thought of as an n by n matrix with polynomial entries from $(\mathbb{R}[t, x, y, z])^n$. Evaluating these polynomials at t_0, x_0, y_0, z_0 gives a matrix γ_i with real entries.

Relations among Q_i's become

$$\gamma_i \gamma_j + \gamma_j \gamma_i = \delta_{ij}.$$

The γ_i's generate the algebra $Cl(m)$ called *Clifford Algebra.*

Some examples of Clifford Algebras are

1. $Cl(0) = \mathbb{R}$ (no γ)

2. $Cl(1) = \mathbb{C}$ ($\mathbb{R}^2$ with i, $i^2 = -1$)

3. $Cl(2) = \mathbb{H}$ quaternions ($\mathbb{R}^4$ with i, j, $i^2 = j^2 = -1, ij = -ji$).

3.3 The Module in 3-dimensional case

3.3.1 Original Picture

Recall that we are interested in studying modules over $\mathbb{R}[\partial_{x_1}, \ldots, \partial_{x_d}]$ some of whose operators Q_I satisfy the supersymmetry bracket relation

$$\{Q_I, Q_J\} = Q_I Q_J + Q_J Q_I = 2\Gamma^k_{IJ}\partial_{x_k}.$$

Let us first restrict the problem to the three dimensional case ($d = 3$). Physically, we are interested in A (Gauge fields, potential), λ (fermionic fields), $F = dA$ (field strength or curvature, 2-form). But what are these objects mathematically? A is the image of a 1-form with components A_{-2}, A_0, A_2 in the quotient space and F is the exterior derivative of A. We want a module over $R = \mathbb{R}[\partial_{x_2}, \partial_{x_0}, \partial_{x_{-2}}]$ with generators A_2, A_0, A_{-2}, λ_1, λ_{-1}. The reason we use the notations $\partial_{x_{-2}}, \partial_{x_0}, \partial_{x_2}$ instead of $\partial_{x_1}, \partial_{x_2}, \partial_{x_3}$ is because of the action of the rotation group $SO(1, 2)$ on Spin$(1, 2)$. Under our choice of notations, the actions will balance the indices in all equations involving A_i, F_i, λ_i and ∂_{x_i}.

Let us first recall the definition of 1-form. If we let $dx_i \in (\mathbb{R}^{1,2})^*$ be defined as $dx_i(\mathbf{e}_j) = \delta_{ij}$, where $\{\mathbf{e}_i\}$ is the standard basis of $\mathbb{R}^{1,2}$. Then $\{dx_i\}$ is a basis for $(\mathbb{R}^{1,2})^*$. A 1-form A from $\Omega^1(\mathbb{R}^{1,2})$ is a map from $\mathbb{R}^{1,2}$ to $(\mathbb{R}^{1,2})^*$ that is of the form $A = A_2 dx_2 + A_0 dx_0 + A_{-2} dx_{-2}$ where A_i are functions from $\mathbb{R}^{1,2}$ to $\mathbb{R}$. Now if we identify dx_i with the 1-form $1 \cdot dx_i$, then $\{dx_{-2}, dx_0, dx_2\}$ is a basis for $\Omega^1(\mathbb{R}^{1,2})$.

Definition 3.3.1 (Exact 1-forms)**.** We define $d\Omega^0$ to be the image of $\Omega^0(\mathbb{R}^{1,2})$, the space of smooth functions on $\mathbb{R}^{1,2}$, under the exterior derivative map $d : \Omega^0(\mathbb{R}^{1,2}) \longrightarrow \Omega^1(\mathbb{R}^{1,2})$ defined as

$$dg = (\partial_{x_{-2}} g)dx_{-2} + (\partial_{x_0} g)dx_0 + (\partial_{x_2} g)dx_2.$$

The elements in $d\Omega^0$ are called *exact 1-forms* Note that $g_{-2}dx_{-2}+g_0dx_0+g_2dx_2$ is an exact 1-form if and only if the vector field $V = (g_{-2}, g_0, g_2)$ is the gradient of some real-valued function.

The quotient $\Omega^1(\mathbb{R}^{1,2})/d\Omega^0(\mathbb{R}^{1,2})$ is a module over the polynomial ring $R = \mathbb{R}[\partial_{x_2}, \partial_{x_0}, \partial_{x_{-2}}]$, where scalar multiplication by ring element is defined to be differentiation in the corresponding direction. A is the image of a 1-form $A_{-2}dx_{-2} + A_0dx_0 + A_2dx_2$ in this quotient. On the other hand, $F = dA$ and since $d^2 = 0$ (Theorem 2.4.7), we have F is a 2-form.

The problem with this construction is that $\Omega^1(\mathbb{R}^{1,2})/d\Omega^0$ is not a free module. For example, we have $-x_0\, d_{x_{-2}} + x_{-2}\, dx_0$ is not zero in the quotient, since the vector field $(-x_0, x_{-2}, 0)$ has non zero curl and is therefore, not a gradient field. On the other hand $\partial_{x_2}(-x_0\, d_{x_{-2}} + x_{-2}\, dx_0) = 0$. Hence $\Omega^1(\mathbb{R}^{1,2})/d\Omega^0$ has a zero divisor and is therefore not free.

We note that since we are mainly interested in the field strength F_i's, and since $d^2 = 0$ (Theorem 2.4.7), it makes sense to think of A_i's as equivalence classes in $\Omega^1(\mathbb{R}^{1,2})/d\omega^0$. That is because if A and A' differ by $df \in d\Omega^0$, then $dA = dA' + ddf = dA'$, so they give rise to the same F.

3.3.2 Dual Picture

We will define a new concept of dual. But first, let us define ∂R to be the ring of power series in differential operators $\mathbb{R}[\partial_{x_{-2}}, \partial_{x_0}, \partial_{x_2}]$. Note that the rings $\Omega^1(\mathbb{R}^{1,2})$ and R both have a natural module structure over the ring ∂R, where scalar multiplication is defined to be differentiation.

Definition 3.3.2. Let $0 \leq n \leq 3$. We define $\mathrm{Hom}_{\partial R}\left(\Omega^n(\mathbb{R}^{1,2}), R\right)$ to be the space of all ∂R-homomorphisms from $\Omega^n(\mathbb{R}^{1,2})$ to R.

Definition 3.3.3. For $i = -2, 0, 2$, we define $\tilde{A}_i$ to be the R homomorphism from the space of 1-form $\Omega^1(\mathbb{R}^{1,2})$ to R that extract the i-th components. In other words, Let $f = f_{-2}\, dx_{-2} + f_0\, dx_0 + f_2\, dx_2$ be a 1-form, where f_{-2}, f_0, f_2 be smooth functions from $\mathbb{R}^{1,2}$ to $\mathbb{R}$. then $\tilde{A}_i(f) = f_i$. Note that $\tilde{A}_i(dx_j) = \delta_{ij}$.

Theorem 3.3.4. *[First Main Theorem] $Hom_{\partial R}\left(\Omega^1(\mathbb{R}^{1,2}), R\right) = \tilde{\Omega}^1 \otimes_{\partial R} \mathbb{R}[[x_{-2}, x_0, x_2]]$ where $\tilde{\Omega}^1$ is a free module over ∂R. The $\tilde{A}_i$'s form a basis for the free module $\tilde{\Omega}^1$. We call $\tilde{\Omega}^1$ the* dual *of* $\Omega^1(\mathbb{R}^{1,2})$.

Proof. Let $B \in \mathrm{Hom}_{\partial R}\left(\Omega^n(\mathbb{R}^{1,2}), R\right)$. We want to show that $B = D_{-2}\tilde{A}_{-2} + D_0\tilde{A}_0 + D_2\tilde{A}_2$ for some $D_i \in R[[x_{-2}, x_0, x_2]]$.

By definition, B is a ∂R- homomorphism from $\Omega^1(\mathbb{R}^{1,2})$ to R. Every element g in $\Omega^1(\mathbb{R}^{1,2})$ is of the form $g_{-2}d_{x_{-2}} + g_0 d_{x_0} + g_2 d_{x_2}$ for some polynomial g_i's. We have

$$\left(D_{-2}\tilde{A}_{-2} + D_0\tilde{A}_0 + D_2\tilde{A}_2\right)\left(g_{-2}d_{x_{-2}} + g_0 d_{x_0} + g_2 d_{x_2}\right) = D_{-2}g_{-2} + D_0 g_0 + D_2 g_2.$$

Thus, to show that $B(g) = \left(D_{-2}\tilde{A}_{-2} + D_0\tilde{A}_0 + D_2\tilde{A}_2\right)(g)$, it suffices to construct D_i such that $B(g_i dx_i) = \left(D_{-2}\tilde{A}_{-2} + D_0\tilde{A}_0 + D_2\tilde{A}_2\right)(g_i dx_i) = D_i g_i$ for all polynomial g_i and for $i = -2, 0, 2$. By the ∂R- linearity of D_i and B, it is enough to construct D_i such that $B(m\, dx_i) = D_i(m)$ for all monomials $m \in R$. Fix an i. Let us write dx and D instead of dx_i and D_i.

Suppose $D = \sum_{i,j,k} c(i,j,k)\ \partial^i_{x_{-2}}\, \partial^j_{x_0}\, \partial^k_{x_2}$ where the coefficients $c(i,j,k)$ are real-valued functions. Since $\frac{d^i}{dx^i}x^k = \binom{k}{i} i! x^{k-i}$, we have that for all m, n, l,

$$D\left(x_{-2}^m x_0^n x_2^l\right) = \sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{l} \binom{m}{i} i! \binom{n}{j} j! \binom{l}{k} k!\, c(i,j,k)\ x_{-2}^{m-i} x_0^{n-j} x_2^{l-k}.$$

Thus $c(i,j,k)$ could be constructed by equating the coefficients of $D\left(x_{-2}^m x_0^n x_2^l\right)$ with those of $B\left(x_{-2}^m x_0^n x_2^l\right)$ for all m,n,l. More precisely, let

$$B\left(x_{-2}^m x_0^n x_2^l\right) = \sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty}\sum_{\gamma=0}^{\infty} d(\alpha,\beta,\gamma)\ x_{-2}^{\alpha} x_0^{\beta} x_2^{\gamma}$$

where the coefficients $d(i,j,k)$ are real-valued functions and $d(\alpha,\beta,\gamma) = 0$ for α,β,γ sufficiently large so that this is a finite sum. (Actually, we can prove that $d(\alpha,\beta,\gamma) = 0$ for $\alpha > m, \beta > n, \gamma > l$, but we do not need this detail.) Then $c(i,j,k)$ is defined to be

$$c(i,j,k) = \frac{d(m-i,n-j,l-k)}{\binom{m}{i} i! \binom{n}{j} j! \binom{l}{k} k!}.$$

Now we simply need to check that $c(i,j,k)$ is well defined. Fix i,j,k and m,n,l. It suffices the check that the $c(i,j,k)$ we get from equating the coefficients of $D\left(x_{-2}^m x_0^n x_2^l\right) = B\left(x_{-2}^m x_0^n x_2^l\right)$ is the same that we get from equating the coefficients of $D\left(x_{-2}^{m+1} x_0^n x_2^l\right) = B\left(x_{-2}^{m+1} x_0^n x_2^l\right)$. That is because we can use the same argument on the value of $c(i,j,k)$ we get from equating $D\left(x_{-2}^m x_0^{n+1} x_2^l\right) = B\left(x_{-2}^m x_0^{n+1} x_2^l\right)$ or $D\left(x_{-2}^m x_0^n x_2^{l+1}\right) = B\left(x_{-2}^m x_0^n x_2^{l+1}\right)$.

Precisely, suppose

$$B\left(x_{-2}^{m+1} x_0^n x_2^l\right) = \sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty}\sum_{\gamma=0}^{\infty} e(\alpha,\beta,\gamma)\ x_{-2}^{\alpha} x_0^{\beta} x_2^{\gamma}$$

where the coefficients $e(i,j,k)$ are real-valued functions and $e(\alpha,\beta,\gamma) = 0$ for α,β,γ sufficiently large so that this is a finite sum. We want to show that

$$\frac{d(m-i,n-j,l-k)}{\binom{m}{i} i! \binom{n}{j} j! \binom{l}{k} k!} = \frac{e(m+1-i,n-j,l-k)}{\binom{m+1}{i} i! \binom{n}{j} j! \binom{l}{k} k!}$$

i.e.

$$d(m-i,n-j,l-k) = \frac{(m-i+1)}{m+1} e(m-i+1,n-j,l-k).$$

Since B is ∂R linear, we have $\partial_{x_{-2}} B\left(x_{-2}^{m+1} x_0^n x_2^l\right) = B\left(\partial_{x_{-2}}\, x_{-2}^{m+1} x_0^n x_2^l\right) = B\left((m+1))x_{-2}^m x_0^n x_2^l\right)$.

Thus

$$\partial_{x_{-2}}\left(\sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty}\sum_{\gamma=0}^{\infty} e(\alpha,\beta,\gamma)\ x_{-2}^{\alpha} x_0^{\beta} x_2^{\gamma}\right) = (m+1)\left(\sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty}\sum_{\gamma=0}^{\infty} d(\alpha,\beta,\gamma)\ x_{-2}^{\alpha} x_0^{\beta} x_2^{\gamma}\right).$$

On the other hand, we have

$$\partial_{x_{-2}}\left(\sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty}\sum_{\gamma=0}^{\infty} e(\alpha,\beta,\gamma)\ x_{-2}^{\alpha}x_0^{\beta}x_2^{\gamma}\right)=\sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty}\sum_{\gamma=0}^{\infty} \alpha\, e(\alpha,\beta,\gamma)\ x_{-2}^{\alpha-1}x_0^{\beta}x_2^{\gamma}.$$

Thus for all α,β,γ we have $(m+1)d(\alpha,\beta,\gamma)=(\alpha+1)e(\alpha,\beta,\gamma)$. In particular, $(m+1)d(m-i,n-j,l-k)=(m-i+1)e(m-i+1,n-j,l-k)$, as desired.

□

Recall that for N any submodule of a module M, we have $(M/N)^*=N^0$, the annihilator of N, which is a subset of M^*. Thus the "dual" of $\Omega^1(\mathbb{R}^{1,2})/d\Omega^0(\mathbb{R}^{1,2})$ is the annihilators of $d\Omega^0(\mathbb{R}^{1,2})$ in the dual space of $\Omega^1(\mathbb{R}^{1,2})=\tilde{\Omega}^1$.

Let M be a free module over $\partial R=\mathbb{R}[\partial_{x_1},\partial_{x_2},\partial_{x_3}]$ generated by $\tilde{A}_{-2},\tilde{A}_0,\tilde{A}_2,\tilde{\lambda}_{-1},\tilde{\lambda}_1$. We want to work with the submodule N of M generated by $\alpha\in\operatorname{span}\{\tilde{A}_{-2},\tilde{A}_0,\tilde{A}_2,\}$ such that $\alpha(dg)=0$ for all smooth functions g on $\mathbb{R}^{1,2}$ (i.e. α is an annihilator of $d\Omega^0(\mathbb{R}^{1,2})$).

Let $\alpha=f_2\tilde{A}_{-2}+f_0\tilde{A}_0+f_{-2}\tilde{A}_2$ be an element of M ($f_i\in\partial R$). Let $dg=(\partial_{x_{-2}}g)\,dx_{-2}+(\partial_{x_0}g)\,dx_0+(\partial_{x_2}g)\,dx_2$ be an element of $d\Omega^0(\mathbb{R}^{1,2})$).

Then

$$\begin{aligned}\alpha(dg)&=\left(f_2\tilde{A}_{-2}+f_0\tilde{A}_0+f_{-2}\tilde{A}_2\right)\left((\partial_{x_{-2}}g)\,dx_{-2}+(\partial_{x_0}g)\,dx_0+(\partial_{x_2}g)\,dx_2\right)\\&=f_2(\partial_{x_{-2}}g)+f_0(\partial_{x_0}g)+f_{-2}(\partial_{x_2}g)\ (\text{since }\tilde{A}_i(dx_j)=\delta_{ij})\\&=\left(f_2\partial_{x_{-2}}+f_0\partial_{x_0}+f_{-2}\partial_{x_2}\right)g\end{aligned}$$

So $\alpha(dg)=0$ for all g if and only if $f_2\partial_{x_{-2}}+f_0\partial_{x_0}+f_{-2}\partial_{x_2}=0$. Thus the submodule N is generated by λ's and the elements of the form $f_2\tilde{A}_{-2}+f_0\tilde{A}_0+f_{-2}\tilde{A}_2$ where $f_i\in\partial R$ satisfy $f_2\partial_{x_{-2}}+f_0\partial_{x_0}+f_{-2}\partial_{x_2}=0$. Later in Chapter 4.1 we will see that the elements (f_2,f_0,f_{-2}) in ∂R^3 that satisfy the equations above form a submodule of ∂R^3 called the syzygy of $(\partial_{x_{-2}},\partial_{x_0},\partial_{x_2})$. We will see that a basis for this submodule is $(0,\partial_{x_2},-\partial_{x_0}),(\partial_{x_2},0,-\partial_{x_{-2}}),(-\partial_{x_0},\partial_{x_{-2}},0)$. Thus a set of generators for N is λ_1, λ_{-1} together with $\partial_{x_2}\tilde{A}_0-\partial_{x_0}\tilde{A}_2,\partial_{x_2}\tilde{A}_{-2}-\partial_{x_{-2}}\tilde{A}_2,-\partial_{x_0}A_{-2}+\partial_{x_{-2}}\tilde{A}_0$.

Definition 3.3.5. For $i = -2, 0, 2$, we define $\tilde{F}_i$ to be the R homomorphism from the space of 1-form $\Omega^2(\mathbb{R}^{1,2})$ to R that extract the i-th components. In other words, Let $f = f_{-2}\, dx_0 \wedge dx_{-2} + f_0\, dx_0 dx_2 + f_2\, dx_{-2} dx_2$ be a 2-form, where f_{-2}, f_0, f_2 be smooth functions from $\mathbb{R}^{1,2}$ to $\mathbb{R}$. then $\tilde{F}_i(f) = f_i$. Note that $\tilde{F}_i(dx_j) = \delta_{ij}$.

Theorem 3.3.6. *$Hom_{\partial R}\left(\Omega^2(\mathbb{R}^{1,2}), R\right) = \tilde{\Omega}^2 \otimes_{\partial R} \mathbb{R}[[x_{-2}, x_0, x_2]]$ where $\tilde{\Omega}^2$ is a free module over ∂R. The $\tilde{F}_i$'s form a basis for the free module $\tilde{\Omega}^2$. We call $\tilde{\Omega}^2$ the* dual *of $\Omega^2(\mathbb{R}^{1,2})$.*

The proof is similar to that of 3.3.4

Later in Corollary 4.2.18 we will show that

$$\tilde{d}(\tilde{F}_{-2}) = -\partial_{x_0}\tilde{A}_{-2} + \partial_{x_{-2}}\tilde{A}_0$$

$$\tilde{d}(\tilde{F}_0) = \partial_{x_{-2}}\tilde{A}_2 - \partial_{x_2}\tilde{A}_{-2}$$

$$\tilde{d}(\tilde{F}_2) = \partial_{x_0}\tilde{A}_2 - \partial_{x_2}\tilde{A}_0.$$

where $\tilde{d}$ is the adjoint of d (see Definition 4.2.16). Thus $\tilde{d}(\tilde{F}_i)$ and $\tilde{\lambda}_i$ are a set of generators for N.

From now on, we will identify the ring ∂R with R by identifying ∂_{x_i} with x_i. Then M is a module over the polynomial ring $R = \mathbb{R}[x_{-2}, x_0, x_2]$. For convenience, we will write A_i, F_i, λ_i instead of $\tilde{A}_i, \tilde{d}(\tilde{F}_i), \tilde{\lambda}_i$.

3.4 The Operators Q_I in 3-dimensional case

The Γ matrix given in the three dimensional case is $\begin{pmatrix} \partial_{x_2} & \frac{1}{2}\partial x_0 \\ \frac{1}{2}\partial_{x_0} & \partial_{x_{-2}} \end{pmatrix}$. Thus we need two operators Q_I, Q_J on N satisfying

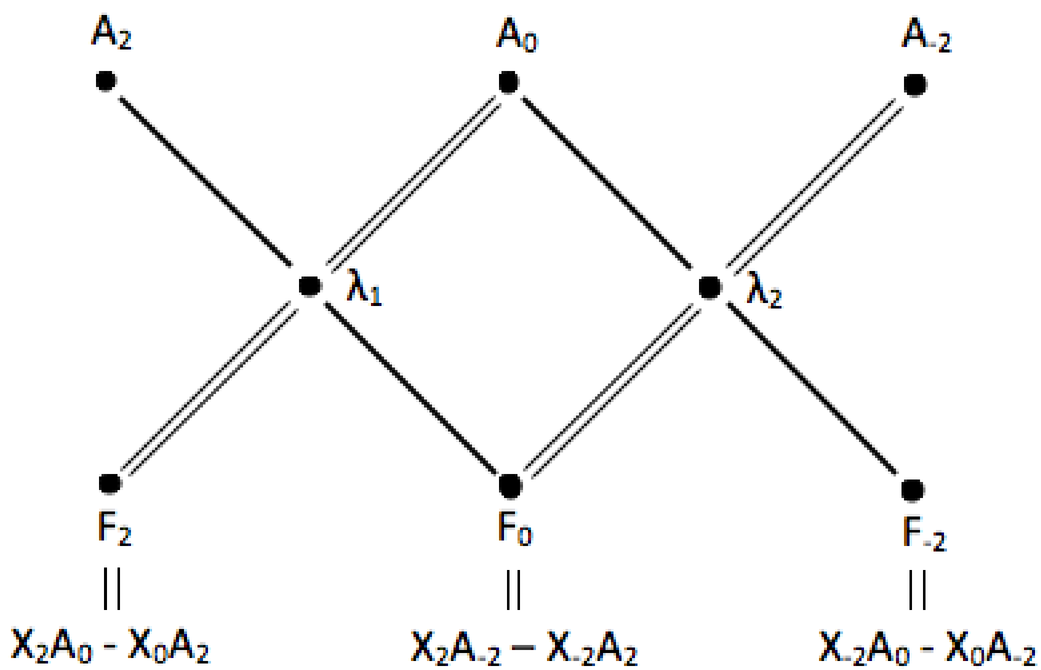
$$\{Q_I, Q_J\} = Q_I Q_J + Q_J Q_I = 2\Gamma_{IJ}$$

e.g. $\{Q_1, Q_2\} = 2\Gamma_{12} = \partial_{x_0}$.

For convenience, we will write Q_- instead of Q_1, Q_+ instead of Q_2. This notation would be make it easier to read off the action of Q's from *Adinkra diagrams.*

Adinkras diagrams are diagrams of the generators of a module with line segments connecting them. They often come together with formulas which describe how to read the action of operators Q's from the diagrams.

The Adinkra in the 3-dimensional case that we are working with is as in the following picture.



A way of seeing where A's and λ's go under Q's is by looking at the index. The F's are written in terms of the A's so we only need to look at the action on the A's. The Q's are operators only on the submodule generated by λ's and F's.)

Under every Q's, A's move to λ's and λ's move to F's.Under Q_+, every index go up by one. Under Q_-, every index go down by one. If the index gets over the range, then the element goes to 0. So for example, $Q_+\lambda_1 = \pm F_2$, whereas $Q_+A_2 = 0$.

We have carried out some computations to find out the appropriate signs in the Adinkra. (See Appendix)

4
A New Direction

In this chapter, we will provide an algebraic explanation for the construction of $\tilde{F}_i$'s and the submodule N in Section 3.3.2. The key point is the duality between the Koszul and the de Rham complex.

4.1 Syzygies

The definitions in this section come from Cox [2].

Recall that we want to restrict M to the submodule generated by elements of the forms $F = f_{-2}A_2 + f_0A_0 + f_2A_{-2}$ where $f_i \in R$ satisfy $x_2f_{-2} + x_0f_0 + x_{-2}f_2 = 0$.

Thus these tuples (f_{-2}, f_0, f_2)'s form something called a syzygy module of (x_2, x_0, x_{-2}).

Theorem 4.1.1. *Let M be a module over a ring R. Let $(f_1, \dots, f_t)$ be an ordered t-tuple of elements $f_i \in M$. The set of all $(a_1, \dots, a_t)^T \in R^t$ such that $a_1f_1 + \dots + a_tf_t = 0$ is an R-submodule of R^t, called the (first)* syzygy module *of $(f_1, \dots, f_t)$, and denoted $Syz(f_1, \dots, f_t)$.*

Thus the syzygy of a set of elements of M is like the set of all relations among those elements. The generators of the syzygy are the basic relations from which all other relations can be deduced. (We can connect this with reducing a system of equations).

4.1.1 Free Resolution

Definition 4.1.2. Consider a sequence of R-modules and homomorphisms $\ldots M_{i+1} \xrightarrow{\phi_{i+1}} M_i \xrightarrow{\phi_i} M_{i-1} \to \ldots$

1. We say that the sequence is *exact* at M, if $\mathrm{im}(\phi_{i+1}) = \ker(\phi_i)$.
2. The entire sequence is said to be *exact* if it is exact at each M_i, which is not at the beginning or the end of the sequence.

Definition 4.1.3. Let M be an R-module. A *presentation* for M is a set of generators $f_1, \ldots, f_t$, together with a set of generators for the syzygy module $\mathrm{Syz}(f_1, \ldots, f_t)$ of relations among $f_1, \ldots, f_t$.

A presentation of M is equivalent to an exact sequence formed in the following way. Suppose $f_1, \ldots, f_t$ are the generators of M. This gives a surjective homomorphism $\phi : R^t \to M$, which means the exact sequence

$$R^t \xrightarrow{\phi} M \to 0$$

where ϕ sends $(g_1, \ldots, g_t) \in R^t$ to $\sum_{i=1}^t g_i f_i \in M$. It follows that a syzygy on $f_1, \ldots, f_t$ is an element of the kernel of ϕ, i.e.,

$$Syz(f_1, \ldots, f_t) = ker(\phi : R^t \to M).$$

Now, choosing a set of generators for the syzygy module corresponds to choosing a homomorphism ξ of R^s onto $\ker(\phi) = \mathrm{Syz}(f_1, \ldots, f_t)$. Thus $\mathrm{im}(\xi) = \ker(\phi)$, which implies the sequence

$$R^s \to R^t \to M \to 0$$

is exact. We can keep going and look at the generators of the syzygy module of the syzygy module of $(f_1, \ldots, f_t)$. If we keep going, we will get something called a free resolution.

Definition 4.1.4. Let M be an R-module. A *free resolution* of M is an exact sequence of the form

$$\ldots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

where for all i, $F_i \approx R^{r_i}$ is a free R-module. If there is an l such that $F_{l+1} = F_{l+2} = \ldots = 0$, but $F_l \neq 0$, we say that the resolution is *finite* of *length* l. In a finite resolution of length l, we will actually write the resolution as

$$0 \longrightarrow F_l \longrightarrow F_{l-1} \longrightarrow \ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0.$$

Theorem 4.1.5. *In a finite free resolution*

$$0 \longrightarrow F_l \xrightarrow{\phi_l} F_{l-1} \xrightarrow{\phi_{l-1}} F_{l-2} \longrightarrow \ldots \longrightarrow F_0 \longrightarrow M \longrightarrow 0,$$

we have $\ker(\phi_{l-1})$ *is a free module. Conversely, if* M *has a free resolution in which* $\ker(\phi_{l-1})$ *is a free module for some* l*, then* M *has a finite free resolution of length* l*.*

Theorem 4.1.6 (Hilbert Syzygy Theorem)**.** *Let* $R = k[x_1, \ldots, x_n]$*. Then every finitely generated* R*-module has a finite free resolution of length at most* n*.*

4.2 Koszul Complex and De Rham Complex

4.2.1 Chain Complex and Homological Algebra

The definitions and theorem in this section are from Eisenbud [4].

Definition 4.2.1. A *chain complex* of modules F_i's over a ring R is a sequence of R-modules and homomorphisms

$$\mathcal{F} : \ldots \longrightarrow F_{i+1} \xrightarrow{\phi_{i+1}} F_i \xrightarrow{\phi_i} F_{i-1} \longrightarrow \ldots$$

such that $\phi_i \circ \phi_{i+1} = 0$ for all i.

Since $\phi_i \circ \phi_{i+1} = 0$, hence $\operatorname{im} \phi_{i+1}$ should be a submodule of $\ker \phi_i$. Recall the definition of exact sequence given in the previous section. To measure the extent by which a chain complex deviates from being an exact sequence, we have the following definition.

Definition 4.2.2 (Homology)**.** Given a chain complex $\mathcal{F}$ as above, we define the *homology* of $\mathcal{F}$ at F_i to be

$$H_i\mathcal{F} = \ker \phi_i / \operatorname{im} \phi_{i+1}.$$

Thus $\mathcal{F}$ is exact at F_i if and only if $H_i\mathcal{F} = 0$, and it is exact if and only if $H_i\mathcal{F} = 0$ for all i.

Definition 4.2.3. The complex is called *graded* if the modules F_i are graded and each ϕ_i is a homomorphism of degree 0.

Definition 4.2.4. For $p \in \mathbb{Z}$ denote by $F[-p]$ the homologically graded complex such that $F[-p]_i = F_{i-p}$ for all i. We say $F[-p]$ is the complex F *homologically shifted* p *degrees* and call p the *shift*.

4.2.2 Cochain Complex and Cohomology

The following definitions and theorems are from Peeva [5].

A cochain complex looks like a chain complex but with arrows pointing in the reverse direction.

Definition 4.2.5. A *cochain complex* of modules F^i's over a ring R is a sequence of R-modules and homomorphisms

$$\mathcal{F} : \ldots \leftarrow F_{i+1} \xleftarrow{\phi_{i+1}} F_i \xleftarrow{\phi_i} \ldots$$

such that $\phi_{i+1} \circ \phi_i = 0$ for all i.

Definition 4.2.6. Given a cochain complex $\mathcal{F}$ as above, we define the *cohomology* of $\mathcal{F}$ at F_i to be

$$H_i\mathcal{F} = \ker \phi_i / \operatorname{im} \phi_{i-1}.$$

4.2.3 The connecting homomorphism

Definition 4.2.7. A sequence of homomorphisms of complexes

$$0 \longrightarrow (\mathcal{F}, d) \xrightarrow{\phi} (\mathcal{F}', d') \xrightarrow{\psi} (\mathcal{F}'', d'') \longrightarrow 0$$

is *exact* if

$$0 \longrightarrow F_i \longrightarrow F_i' \longrightarrow F_i'' \longrightarrow 0$$

is exact for every i.

Given an exact sequence of complexes as above, we will define the *connecting homomorphism*

$$\tau = \{\tau_i : H_i(\mathcal{F}'') \longrightarrow H_{i-1}(\mathcal{F})\}$$

below, using the following diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F}_i & \xrightarrow{\phi} & y \in \mathcal{F}_i' & \xrightarrow{\psi} & x \in \mathcal{F}_i'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & z \in \mathcal{F}_{i-1} & \xrightarrow{\phi} & d'(y) \in \mathcal{F}_{i-1}' & \xrightarrow{\psi} & \mathcal{F}_{i-1}'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & d(z) \in \mathcal{F}_{i-2} & \xrightarrow{\phi} & \mathcal{F}_{i-2}' & \xrightarrow{\psi} & \mathcal{F}_{i-2}'' & \longrightarrow & 0
\end{array}$$

Let $\alpha \in H_i(\mathcal{F}'')$, so by definition it is an equivalence class. Its image $\tau(\alpha) \in H_{i-1}(\mathcal{F})$ is defined as follows. Choose a representative $x \in \mathcal{F}_i''$ of α. Note that $d''(x) = 0$, since $x \in \ker d_i''$. Since ψ_i is surjective, there exists $y \in \mathcal{F}_i'$ such that $\psi_i(y) = x$. Now we claim that there exists a $z \in \mathcal{F}_{i-1}$ such that $\phi_{i-1}(z) = d'(y)$. This is because $\psi_{i-1}(d'(y)) = d''(\psi_i(y)) = d''(x) = 0$, so $d'(y) \in \ker(\psi_{i-1}) = \mathrm{im}(\phi_{i-1})$ (this is where we use the exactness). We define $\tau(\alpha)$ to be the equivalence class of z in $H_{i-1}(\mathcal{F})$. To justify this definition, we have to verify two things: first, that $z \in \ker d$; second, that the map is well-defined.

That z is in the kernel of d is easy to see. We have $\phi_{i-2}(d(z)) = d'(\phi_{i-1}(z)) = d'(d'(y)) = 0$. Thus $d(z) \in \ker(\phi_{i-2}) = 0$.

Now, we will check that the map is well defined, using the following diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F}_{i+1} & \xrightarrow{\phi} & \bar{y} \in \mathcal{F}'_{i+1} & \xrightarrow{\psi} & \bar{x} \in \mathcal{F}''_{i+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bar{z} \in \mathcal{F}_i & \xrightarrow{\phi} & y - y' - d'(\bar{y}) \in \mathcal{F}'_i & \xrightarrow{\psi} & x - x'\mathcal{F}''_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & z - z' \in \mathcal{F}_{i-1} & \xrightarrow{\phi} & d'(y) - d'(y') \in \mathcal{F}'_{i-1} & \xrightarrow{\psi} & \mathcal{F}''_{i-1} & \longrightarrow & 0
\end{array}$$

Let x and x' be two representatives of the homology class α. Let y, z and y', z' be the elements constructed using x and x' respectively. We want to show that z and z' belong to the same homology class in $H_{i-1}(\mathcal{F}) = \ker(d_{i-1})/\operatorname{im}(d_i)$, i.e. $z - z' \in \operatorname{im}(d_i)$. Thus we want to find a $\bar{z} \in \mathcal{F}_i$ such that $z - z' = d(\bar{z})$.

First, since x and x' represent the same homology class in $H_i(\mathcal{F}'')$, we must have $x-x' = d''(\bar{x})$. Since ψ_{i+1} is surjective, we can choose $\bar{y} \in \mathcal{F}'_{i+1}$ such that $\psi_{i+1}(\bar{y}) = \bar{x}$. Now we choose $\bar{z} \in \mathcal{F}_i$ such that $\phi_i(\bar{z}) = y - y' - d'(\bar{y})$. This is possible because $\psi(d'(\bar{y})) = d''(\psi_{i+1}(\bar{y})) = x - x' = \psi_i(y - y')$. Thus $y - y' - d'(\bar{y}) \in \ker(\psi_i) = \operatorname{im}(\phi_i)$.

We claim $z - z' = d(\bar{z})$. We have $\phi_{i-1}(d(\bar{z}) = d'(\phi_i(\bar{z})) = d'(y - y' - d'(\bar{y})) = d'(y) - d'(y') = \phi_{i-1}(z - z')$. But ϕ_{i-1} is injective, hence $z - z' = d(\bar{z})$, as claimed.

Theorem 4.2.8. *A short exact sequence of complexes*

$$0 \longrightarrow (\mathcal{F}, d) \xrightarrow{\phi} (\mathcal{F}', d') \xrightarrow{\psi} (\mathcal{F}'', d'') \longrightarrow 0$$

gives the homology long exact sequence

$$\cdots \longrightarrow H_{i+1}(\mathcal{F}'') \xrightarrow{\tau} H_i(\mathcal{F}) \xrightarrow{\phi} H_i(\mathcal{F}') \xrightarrow{\psi} H_i(\mathcal{F}'') \xrightarrow{\tau} H_{i-1}(\mathcal{F}) \longrightarrow \cdots,$$

where τ is the connecting homomorphism.

Proof. See [5]. □

Corollary 4.2.9. *If any two of the complexes in the previous theorem are exact, then so is the third.*

4.2.4 Koszul Complex

Definition 4.2.10. Given a ring R and $x_1, \ldots, x_n \in R$, define the *Koszul Complex* $\mathbf{K}(x_1, \ldots, x_n)$ as the chain complex

$$\mathbf{K}(x_1, \ldots, x_n) : 0 \xrightarrow{d} K_q \xrightarrow{d} \ldots \xrightarrow{d} K_1 \xrightarrow{d} K_0 \longrightarrow 0$$

where the K_i's and the differential map d are defined as follows. Set $K_0 = R$ and $K_p = 0$ if $p > n$. For $1 \leq p \leq n$ define

$$K_p = \oplus R \cdot e_{i_1} \wedge \cdots \wedge e_{i_p}$$

to be the free module over R of rank $\binom{n}{p}$ with basis $\{e_{i_1} \wedge \cdots \wedge e_{i_p} : 1 \leq i_1 < \cdots < i_p \leq n\}$. The differential map $d : K_p \longrightarrow K_{p-1}$ is defined by

$$d(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \sum_{r=1}^{p} (-1)^{r-1} x_{i_r} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_r}} \wedge \cdots \wedge e_{i_p}.$$

($\widehat{e_{i_r}}$ means eliminating e_{i_r}. For $p = 1$, set $d(e_i) = x_i$.)

As before, it is easy to show that $d^2 = 0$, which ensures that the above definition of $\mathbf{K}(x_1, \ldots, x_n)$ indeed gives a chain complex.

Example 4.2.11. 1. Case $n = 1$: $\mathbf{K}(x)$

$$\mathbf{K}(x) : 0 \longrightarrow R \xrightarrow{x} R \longrightarrow 0.$$

2. Case $n = 2$: $\mathbf{K}(x, y)$

   We have K_0 has basis 1, K_1 has basis e_1, e_2 and K_2 has basis $e_1 \wedge e_2$. The differential acts on this basis as

$$d(e_1) = x, \text{ and } d(e_2) = y.$$

$$d(e_1 \wedge e_2) = -ye_1 + xe_2.$$

If we write the differential as matrices on the given basis, then the Koszul complex is

$$\mathbf{K}(x,y) : 0 \longrightarrow K_2 \xrightarrow{\begin{pmatrix} -y \\ x \end{pmatrix}} K_1 \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} K_0.$$

3. Case $n = 3$: $\mathbf{K}(x,y,z)$

Then the differential d acts as

$$d(e_1) = x, d(e_2) = y, d(e_3) = z.$$

$$d(e_1 \wedge e^2) = -ye_1 + xe_2, \;\; d(e_1 \wedge e_3) = -ze_1 + xe_3, \;\; d(e_2 \wedge e_3) = -ze_2 + ye_3.$$

$$d(e_1 \wedge e_2 \wedge e_3) = xe_2 \wedge e_3 - ye_1 \wedge e_3 + ze_1 \wedge e_2.$$

Therefore, the Koszul complex is

$$\mathbb{K}(x,y,z) : 0 \longrightarrow K_3 \xrightarrow{\begin{pmatrix} z \\ -y \\ x \end{pmatrix}} K_2 \xrightarrow{\begin{pmatrix} -y & -z & 0 \\ x & 0 & -z \\ 0 & x & y \end{pmatrix}} K_1 \xrightarrow{\begin{pmatrix} x & y & z \end{pmatrix}} K_0 \longrightarrow 0.$$

The Koszul complex has homology groups $H_p(\mathbf{K}(x_1,\ldots,x_n))$ which we abbreviate as $H_p(\bar{x})$. Recall that for $\mathbf{K}(\bar{x}) : 0 \longrightarrow K_q^{\phi_q} \longrightarrow \cdots \xrightarrow{\phi_1} K_1 \longrightarrow K_0 \xrightarrow{\phi_0} 0$, we define $H_p(\bar{x})$ to be $\ker(\phi_p)/\operatorname{im}(\phi_{p+1})$. Since $\ker(\phi_0) = K_0 = R$, whereas $\operatorname{im}(\phi_1) = \bar{x}R = x_1R + x_2R + \cdots + x_nR$, it is easy to see that $H_0(\bar{x}) = R/\bar{x}R$.

Theorem 4.2.12. *Let $\bar{x} = \{x_1,\ldots,x_n\}$ be a sequence of elements in R, and let $x_{n+1} \in R$. Let $\tilde{x} = \{x_1,\ldots,x_n,x_{n+1}$. Then we get an exact sequence of complexes*

$$0 \longrightarrow \boldsymbol{K}(\bar{x}) \longrightarrow \boldsymbol{K}(\tilde{x}) \longrightarrow \boldsymbol{K}(\bar{x})[-1] \longrightarrow 0$$

(recall that $\boldsymbol{K}(\bar{x})[-1]$ means that the complex $\boldsymbol{K}(\bar{x})$ is homologically shifted one degree). The homology long exact sequence obtained from this has the form

$$\cdots \longrightarrow H_i(\boldsymbol{K}(\bar{x})) \longrightarrow H_i(\boldsymbol{K}(\tilde{x})) \longrightarrow H_{i-1}(\boldsymbol{K}(\bar{x})) \xrightarrow{(-1)^{i+1}x_{n+1}} H_{i-1}(\boldsymbol{K}(\bar{x})) \longrightarrow \cdots.$$

Proof. We have $\mathbf{K}(\tilde{x})_p = \bigoplus_{1 \geq i_j \leq n+1} R \cdot e_{i_1} \wedge \ldots \wedge e_{i_p} = \left(\bigoplus_{1 \geq i_j \leq n} R \cdot e_{i_1} \wedge \ldots \wedge e_{i_p}\right) \oplus \left(\bigoplus_{1 \geq i_j \leq n} R \cdot e_{i_1} \wedge \ldots \wedge e_{i_{p-1}} \wedge e_{n+1}\right) = \mathbf{K}(\bar{x})_p \oplus \mathbf{K}(\bar{x})_{p-1} \wedge e_{n+1}$ for $1 \leq p \leq n+1$. So $\mathbf{K}(\bar{x})$ injects into $\mathbf{K}(\tilde{x})$ and $\mathbf{K}(\tilde{x})$ surjects onto $\mathbf{K}(\bar{x})[-1]$. Thus the sequence $0 \longrightarrow \mathbf{K}(\bar{x}) \longrightarrow \mathbf{K}(\tilde{x}) \longrightarrow \mathbf{K}(\bar{x})[-1] \longrightarrow 0$ is exact. By the Theorem 4.2.8, it yields a long exact sequence in homology. We will compute the connecting homomorphism following the construction in Section 4.2.3.

Let $\alpha \in H_i(\mathbf{K}(\bar{x})[-1]) = H_{i-1}(\mathbf{K}(\bar{x}))$. Choose a representative $x \in \mathbf{K}(\bar{x})_{i-1}$. Then $y = x \wedge e_{n+1} \in \mathbf{K}(\tilde{x})_i$. Now $d(y) = d(x) \wedge e_{n+1} + (-1)^{i-1} x_{n+1} x$. Since $(-1)^{i+1} x_{n+1} x \in \mathbf{K}(\bar{x})_{i-1}$ and $d(x) \wedge e_{n+1} \in \mathbf{K}(\bar{x})_{i-2} \wedge e_{n+1}$, from the construction, we have $z = (-1)^{i+1} x_{n+1} x$. Thus the connecting homomorphism is

$$H_{i-1}(\mathbf{K}(\bar{x})) \xrightarrow{(-1)^{i+1} x_{n+1}} H_{i-1}(\mathbf{K}(\bar{x})).$$

□

Definition 4.2.13. Let $R = k[x_1, \ldots, x_n]$ by a polynomial ring over some field k. Let M be a finitely generated R- module. If $rm \neq 0$ for every non-zero $m \in M$, then we say r is an M*-regular* element, or a *non-zero divisor* on M. A sequence $f_1, \ldots, f_q$ of elements in R is an M*-regular sequence* if the following two conditions are satisfied: $(f_1, \ldots, f_q)M \neq M$ and for every $1 \leq i \leq q$ we have that f_i is a non-zero divisor on the module $M/(f_1, \ldots, f_{i-1})M$.

Theorem 4.2.14. *Let R be a ring. $x_1, \ldots, x_n$ a regular sequence in R. Then $H_p(\bar{x}) = 0$ for $p > 0$.*

Proof. Proof by Induction.

For $n = 1$, we have $\mathbf{K}(x) : 0 \xrightarrow{\phi_2} R \xrightarrow{\phi_1 = x} R \xrightarrow{\phi_0} 0$.

So $H_1(x) = \ker(\phi_1)/\operatorname{im}(\phi_2) = ker(\phi_1) = \{r \in R \mid xr = 0\} = 0$.

When $n > 1$ and $p > 1$, Theorem 4.2.12 gives an exact sequence

$$0 = H_p(x_1, \ldots, x_{n-1}) \longrightarrow H_p(x_1, \ldots, x_n) \longrightarrow H_{p-1}(x_1, \ldots, x_{n-1}) = 0.$$

So $H_p(x_1, \ldots, x_n) = 0$. For $p = 1$, the corresponding exact sequence is

$$0 \longrightarrow H_1(x_1, \ldots, x_n) \longrightarrow H_0(x_1, \ldots, x_{n-1}) = R/(x_1, \ldots, x_{n-1})R \xrightarrow{\pm x_n} R/(x_1, \ldots, x_{n-1})R.$$

Since x_n is a regular element, the last map (multiplication by $\pm x_n$) has trivial kernel. Since the sequence is exact, H_1 must be trivial. $\square$

4.2.5 Koszul Complex and Syzygies

Apply Case 3 of Example 4.2.11 to the ring ∂R, we get the following Koszul complex

$$\mathbf{K} : 0 \longrightarrow K_3 \xrightarrow{M_3 = \begin{pmatrix} \partial_{x_2} \\ -\partial_{x_0} \\ \partial_{x_{-2}} \end{pmatrix}} K_2 \xrightarrow{M_2 = \begin{pmatrix} -\partial_{x_0} & -\partial_{x_2} & 0 \\ \partial_{x_{-2}} & 0 & -\partial_{x_2} \\ 0 & \partial_{x_{-2}} & \partial_{x_0} \end{pmatrix}} K_1 \xrightarrow{M_1 = \begin{pmatrix} \partial_{x_{-2}} & \partial_{x_0} & \partial_{x_2} \end{pmatrix}} K_0 \longrightarrow 0.$$

By Theorem 4.2.14, we have know that $\mathbf{K}$ is exact. Thus $\operatorname{im} M_2 = \ker M_1 = \operatorname{syz}(\partial_{x_{-2}}, \partial_{x_0}, \partial_{x_2})$. But $\operatorname{im} M_2$ are spanned by the columns of M_2. Hence the columns of M_2 are generators for $\operatorname{syz}(\partial_{x_{-2}}, \partial_{x_0}, \partial_{x_2})$. That a set of generators for the module N in Section 3.3.2 is $\tilde{\lambda}_i$'s together with $\tilde{d}(\tilde{F}_i)'s$ which are of the forms

$$\tilde{d}(\tilde{F}_{-2}) = -\partial_{x_0}\tilde{A}_{-2} + \partial_{x_{-2}}\tilde{A}_0$$
$$\tilde{d}(\tilde{F}_0) = \partial_{x_{-2}}\tilde{A}_2 - \partial_{x_2}\tilde{A}_{-2}$$
$$\tilde{d}(\tilde{F}_2) = \partial_{x_0}\tilde{A}_2 - \partial_{x_2}\tilde{A}_0.$$

The exactness of $\mathbf{K}$ also tells us that the second syzygy module of $(\partial_{x_{-2}}, \partial_{x_0}, \partial_{x_2})$ is free, with generator M_3.

4.2.6 De Rham Complex

Definition 4.2.15. [1] The *de Rham complex* is the cochain complex of exterior differential forms on $\mathbb{R}^n$, with the exterior derivative as the differential.

$$\cdots \xleftarrow{d} \Omega^i(\mathbb{R}^n) \xleftarrow{d} \cdots \xleftarrow{d} \Omega^1(\mathbb{R}^n) \xleftarrow{d} \Omega^0(\mathbb{R}^n) \leftarrow 0.$$

Recall that the dual $\tilde{\Omega}^n$ of $\Omega^n(\mathbb{R}^{1,2})$ is a free module over ∂R (see Section 3.3.2). We now define a "dual" version of the exterior derivative.

Definition 4.2.16. For $1 \leq n \leq 3$, let $\tilde{d}_n : \tilde{\Omega}^n$ be the adjoint of d defined as follows. For all $v \in \tilde{\Omega}^n$, we have $\tilde{d}_n(v)$ is the homomorphism in $\tilde{\Omega}^{n-1}$ such that $(\tilde{d}_n v)(w) = v(dw)$ for all $w \in \Omega^{n-1}(\mathbb{R}^{1,2})$.

Then it is easy to see that we will have the following chain complex

$$\tilde{\mathcal{D}} : 0 \xrightarrow{\tilde{d}_4} \tilde{\Omega}^3 \xrightarrow{\tilde{d}_3} \tilde{\Omega}^2 \xrightarrow{\tilde{d}_2} \tilde{\Omega}^1 \xrightarrow{\tilde{d}_1} \tilde{\Omega}^0 \xrightarrow{\tilde{d}_0} 0$$

which is dual to the de Rham complex

$$\mathcal{D} : \Omega^3(\mathbb{R}^{1,2}) \xleftarrow{d} \Omega^2(\mathbb{R}^{1,2}) \xleftarrow{d} \Omega^1(\mathbb{R}^{1,2}) \xleftarrow{d} \Omega^0(\mathbb{R}^{1,2}).$$

Theorem 4.2.17 (Second Main Theorem)**.** *main The dual complex $\mathcal{D}$ is the Koszul Complex $\boldsymbol{K}(\partial_{x_{-2}}, \partial_{x_0}, \partial_{x_2})$ over the ring $\partial R = \mathbb{R}[\partial_{x_{-2}}, \partial_{x_0}, \partial_{x_2}]$.*

Proof. We have

$$\tilde{\mathcal{D}} : 0 \xrightarrow{\tilde{d}_4} \tilde{\Omega}^3 \xrightarrow{\tilde{d}_3} \tilde{\Omega}^2 \xrightarrow{\tilde{d}_2} \tilde{\Omega}^1 \xrightarrow{\tilde{d}_1} \tilde{\Omega}^0 \xrightarrow{\tilde{d}_0} 0$$

By case 3 of Example 4.2.11, we simply need to prove that $\mathcal{D}$ is of the form

$$0 \longrightarrow K_3 \xrightarrow{M_3=\begin{pmatrix} \partial_{x_2} \\ -\partial_{x_0} \\ \partial_{x_{-2}} \end{pmatrix}} K_2 \xrightarrow{M_2=\begin{pmatrix} -\partial_{x_0} & -\partial_{x_2} & 0 \\ \partial_{x_{-2}} & 0 & -\partial_{x_2} \\ 0 & \partial_{x_{-2}} & \partial_{x_0} \end{pmatrix}} K_1 \xrightarrow{M_1=\begin{pmatrix} \partial_{x_{-2}} & \partial_{x_0} & \partial_{x_2} \end{pmatrix}} K_0 \longrightarrow 0$$

where K_i is a free module of rank $\binom{3}{i}$ for $i > 0$ and $K_0 = \partial R$. We need to show that given an appropriate choice of basis, M_i will be the matrix representation of $\tilde{d}_i$.

First, for each i, we show that the $\tilde{\Omega}^i$'s are free module of rank i and fix a basis for it. By Theorem 3.3.4 we have that $\tilde{\Omega}^1$ is a free module of rank 3 with generators $\tilde{A}_{-2}, \tilde{A}_0, \tilde{A}_2$. By Theorem 3.3.6, we also know that $\tilde{\Omega}^2$ is a free module of rank 3 with generators $\tilde{F}_{-2}, \tilde{F}_0, \tilde{F}_2$. Using the same method as in the proof of Theorem 3.3.4, we can show that $\tilde{\Omega}^0$ and $\tilde{\Omega}^3$

are free modules of rank 1. $\tilde{\Omega}^0$ is generated by the function $\mathbf{1}$ which sends everything to 1. $\tilde{\Omega}^3$ is generated by $\tilde{K}$ which sends a 3-form $k\ dx_{-2} \wedge dx_0 \wedge dx_2$ to k. Thus both of these module is isomorphic to $\partial R = K_0$.

Now, we will show that under the bases chosen above, M_i will be the matrix representation for $\tilde{d}_i$. Abusing notation, we will write this statement as $M_i = \tilde{d}_i$. But first of all, let us recall the definition of $\tilde{A}_i$ and $\tilde{F}_i$. Let $g = g_{-2}dx_{-2} + g_0 dx_0 + g_2 dx_2 \in \Omega^1(\mathbb{R}^{1,2})$ and $h = h_{-2}\, dx_{-2} \wedge dx_0 + h_0\, dx_{-2} \wedge dx_2 + h_2\, dx_0 \wedge dx_2$. Then $\tilde{A}_i(g) = g_i$ and $\tilde{F}_i(h) = h_i$.

First, we show $\tilde{d}_1 = M_1$. Let $f \in \Omega^0(\mathbb{R}^{1,2})$. Then $df = \partial_{x_{-2}} f\, dx_{-2} + \partial_{x_0} f\, dx_0 + \partial_{x_2} f\, dx_2$. Hence $\tilde{d}_1(\tilde{A}_{-2})(f) = \tilde{A}_{-2}(df) = \partial_{x_{-2}} f$. Thus $\tilde{d}_1(\tilde{A}_{-2}) = \partial_{x_{-2}}$. Similarly, $\tilde{d}_1(\tilde{A}_0) = \partial_{x_0}$ and $\tilde{d}_1(\tilde{A}_2) = \partial_{x_2}$. This implies $\tilde{d}_1 = M_1$.

Now we prove that $\tilde{d}_2 = M_2$. Let $g = A_{-2}dx_{-2} + A_0 dx_0 + A_2 dx_2 \in \Omega^1(\mathbb{R}^{1,2})$. Then

$$
\begin{aligned}
dg &= \sum_{i,j} (\partial_{x_i} A_i)\, dx_i \wedge dx_j \\
&= (\partial_{x_0} A_{-2})\, dx_0 \wedge dx_{-2} + (\partial_{x_2} A_{-2}) dx_2 \wedge dx_{-2} + (\partial_{x_{-2}} A_0) dx_{-2} \wedge dx_0 + (\partial_{x_2} A_0) dx_2 \wedge dx_0 \\
&\qquad + (\partial_{x_{-2}} A_2) dx_{-2} \wedge dx_2 + (\partial_{x_0} A_2) dx_0 \wedge dx_2 \\
&= \left(-\partial_{x_0} A_{-2} + \partial_{x_{-2}} A_0\right) dx_{-2} \wedge dx_0 + \left(-\partial_{x_2} A_{-2} + \partial_{x_{-2}} A_2\right) dx_{-2} \wedge dx_2 \\
&\qquad + (-\partial_{x_2} A_0 + \partial_{x_0} A_2)\, dx_0 \wedge dx_2.
\end{aligned}
$$

Thus $\tilde{d}_2(\tilde{F}_{-2})(g) = -\partial_{x_0} A_{-2} + \partial_{x_{-2}} A_0 = \left(-\partial_{x_0} \tilde{A}_{-2} + \partial_{x_{-2}} \tilde{A}_0\right)(g)$. This is true for all 1-form g so $\tilde{d}_2(\tilde{F}_{-2}) = -\partial_{x_0} \tilde{A}_{-2} + \partial_{x_{-2}} \tilde{A}_0$. Similarly $\tilde{d}_2(\tilde{F}_0) = \partial_{x_{-2}} \tilde{A}_2 - \partial_{x_2} \tilde{A}_{-2}$ and $\tilde{d}_2(\tilde{F}_2) = \partial_{x_0} \tilde{A}_2 - \partial_{x_2} \tilde{A}_0$. This implies that $\tilde{d}_2 = M_2$.

Finally, we show that $\tilde{d}_3 = M_3$. Let $h = h_{-2}\, dx_{-2} \wedge dx_0 + h_0\, dx_{-2} \wedge dh_2 + h_2\, dx_0 \wedge dx_2$ be a 2-form. Then

$$\begin{aligned}
dh &= d\left(h_{-2}\, dx_{-2} \wedge dx_0 + h_0\, dx_{-2} \wedge dx_2 + h_2\, dx_0 \wedge dx_2)\right) \\
&= \sum_{i,j,k} (\partial_{x_i} h_j) dx_i \wedge dx_j \wedge dx_k \\
&= (\partial_{x_2} h_{-2})\, dx_2 \wedge dx_{-2} \wedge dx_0 + (\partial_{x_0} h_0)\, dx_0 \wedge dx_{-2} \wedge dx_2 \\
&\quad + \left(\partial_{x_{-2}} h_2\right) dx_{-2} \wedge dx_0 \wedge dx_2 \\
&= \left(\partial_{x_2} h_{-2} - \partial_{x_0} h_0 + \partial_{x_{-2}} h_2\right) dx_{-2} \wedge dx_0 \wedge dx_2 \\
&= \left(\partial_{x_2} \tilde{F}_{-2}(h) - \partial_{x_0} \tilde{F}_0(h) + \partial_{x_{-2}} \tilde{F}_2(h)\right) dx_{-2} \wedge dx_0 \wedge dx_2
\end{aligned}$$

Thus we have $\tilde{d}_3 \tilde{K}(h) = \tilde{K}(dh) = \partial_{x_2} \tilde{F}_{-2}(h) - \partial_{x_0} \tilde{F}_0(h) + \partial_{x_{-2}} \tilde{F}_2(h)$. This is true for all h so $\tilde{d}_3 \tilde{K} = \partial_{x_2} \tilde{F}_{-2} - \partial_{x_0} \tilde{F}_0 + \partial_{x_{-2}} \tilde{F}_2$. This implies $\tilde{d}_3 = M_3$.

Thus we have shown that $\mathcal{D}$ is a Koszul complex. $\square$

Corollary 4.2.18.

$$\begin{aligned}
\tilde{d}(\tilde{F}_{-2}) &= -\partial_{x_0} \tilde{A}_{-2} + \partial_{x_{-2}} \tilde{A}_0 \\
\tilde{d}(\tilde{F}_0) &= \partial_{x_{-2}} \tilde{A}_2 - \partial_{x_2} \tilde{A}_{-2} \\
\tilde{d}(\tilde{F}_2) &= \partial_{x_0} \tilde{A}_2 - \partial_{x_2} \tilde{A}_0
\end{aligned}$$

5
Appendix

For convenience, in this section. we will identify the ring ∂R with R by identifying ∂_{x_i} with x_i. Then M is a module over the polynomial ring $R = \mathbb{R}[x_{-2}, x_0, x_2]$. For convenience, we will write A_i, F_i, λ_i instead of $\tilde{A}_i, \tilde{F}_i, \tilde{\lambda}_i$. We will let $F_0 = x_2 A_{-2} - x_{-2} A_2$, the negative of what it actually is, so that we have $\sum x_{-i} F_i = 0$.

The Adinkra given in the last section of the previous chapter is without sign. We carried out some computations to figure out the signs. The computations are documented below. The problem is that for the computations, I have mistakenly assume that one of the bracket relations was $Q_{+-} + Q_{-+} = 0$ instead of x_0. Thus, to make cancellation works out, I had to introduce extra λ_3, λ_{-3}, which serve as "bridge". λ_3, for example, save A_2 from going to 0 under Q_+. Thus under Q_{+-}, it will safely go back to $\pm F_2$. Under Q_{-+}, by the index rule, A_2 should also goes back to $\pm F_2$. Thus after twisting index, we will get $Q_{+-} + Q_{-+}$ applying on A_2 gives 0. This also works for A_0 and A_{-2} and hence also for the F's.

The problem is that λ_3, λ_{-3} now interfere with Q_{++} and Q_{--} and as we can see from the index, appear in, say $Q_{++}(\lambda_1)$ and $Q_{--}(\lambda_{-1})$. To get rid of these new λ's, I thought maybe we could introduce complex weight. The result of Q applying on each A and λ is not simply $\pm\lambda, \pm F$, but can also be $\pm i\lambda$ and $\pm iF$. Then, we can twist the weight in some way such that when we take the real parts of the image of Q_{++} and Q_{--}, λ_3 and λ_{-3} will get kicked out (because they will have imaginary weight). We do not necessarily have to use complex weight, nor do we have to take the real parts. The idea is just to use some kind of extra factors to pollute the weight, and then use some kind of map to filter the unwanted part out.

So from the mistake I made, I saw some possible ways of handling the Adinkra:

1. Add bridges
   (a) Horizontal bridges: adding more λ's, A's, F's
   (b) Vertical bridges: adding more layers of symbols
2. Add weights and use them to filter out unwanted components
3. Compose the Q's with a map, like projection to the "reals".

Greg commented that in the four dimension case $\{Q_+, Q_-\}$ does give 0, and the weights are complex, but I have not looked into how that is related yet, since the rule for reading Adinkra in four dimension is too complicated.

Below is the incomplete computation:

3D Computations

Let $F =$ Then $x_{-2}F_2 + x_0F_0 + x_2F_{-2} = 0$. For $\alpha \in \{\pm 1\}$ we have

$$Q_\alpha(A_i) = \epsilon_{i,\alpha+i}\lambda_{\alpha+i}$$

and

$$Q_\alpha(\lambda_j) = \epsilon_{j,j+\lambda}F_{j+\lambda}.$$

Note: if the index of a variable is out of range, we define the variable to be 0. $\epsilon_{i,j} \in \{\pm 1\}$.

Under the action of Q_{++}:

$$\lambda_{-1} \xrightarrow{Q_+} \epsilon_{-1,0} F_0 = \epsilon_{-1,0}(x_2 A_{-2} - x_{-2} A_2) \xrightarrow{Q_+} \epsilon_{-1,0}\epsilon_{-2,-1} x_2 \lambda_{-1}$$

$$\lambda_1 \xrightarrow{Q_+} \epsilon_{1,2} F_2 = \epsilon_{1,2}(x_0 A_2 - x_2 A_0) \xrightarrow{Q_+} -\epsilon_{1,2}\epsilon_{0,1} x_2 \lambda_1$$

$$F_2 = x_0 A_2 - x_2 A_0 \xrightarrow{Q_+} -\epsilon_{0,1} x_2 \lambda_1 \xrightarrow{Q_+} -\epsilon_{0,1}\epsilon_{1,2} x_2 F_2$$

$$F_0 = x_2 A_{-2} - x_{-2} A_2 \xrightarrow{Q_+} \epsilon_{-2,-1} x_2 \lambda_{-1} \xrightarrow{Q_+} \epsilon_{-2,-1}\epsilon_{-1,0} x_2 F_0$$

$$F_{-2} = x_{-2} A_0 - x_0 A_{-2} \xrightarrow{Q_+} \epsilon_{0,1} x_{-2} \lambda_1 - \epsilon_{-2,-1} x_0 \lambda_{-1} \xrightarrow{Q_+} \epsilon_{0,1}\epsilon_{1,2} x_{-2} F_2 - \epsilon_{-2,-1}\epsilon_{-1,0} x_0 F_0$$

To get $Q_{++} = x_2$, it is sufficient and necessary that

$$\epsilon_{-1,0}\epsilon_{-2,-1} = 1 \text{ and}$$

$$\epsilon_{1,2}\epsilon_{0,1} = -1$$

By symmetry, to get $Q_{--} = x_{-2}$, it is sufficient and necessary that

$$\epsilon_{1,0}\epsilon_{2,1} = 1 \text{ and}$$

$$\epsilon_{-1,-2}\epsilon_{0,-1} = -1$$

Under the action of Q_{+-}

$$\lambda_{-1} \xrightarrow{Q_-} \epsilon_{-1,-2} F_{-2} \xrightarrow{Q_+} \epsilon_{-1,-2}(\epsilon_{0,1} x_{-2} \lambda_1 - \epsilon_{-2,-1} x_0 \lambda_{-1})$$

$$\lambda_1 \xrightarrow{Q_-} \epsilon_{1,0} F_0 \xrightarrow{Q_+} \epsilon_{1,0}\epsilon_{-2,-1} x_2 \lambda_{-1}$$

$$F_2 \xrightarrow{Q_-} \epsilon_{2,1} x_0 \lambda_1 - \epsilon_{0,-1} x_2 \lambda_{-1} \xrightarrow{Q_+} \epsilon_{2,1}\epsilon_{1,2} x_0 F_2 - \epsilon_{0,-1}\epsilon_{-1,0} x_2 F_0$$

$$F_0 \xrightarrow{Q_-} -\epsilon_{2,1} x_{-2} \lambda_1 \xrightarrow{Q_+} -\epsilon_{2,1} x_{-2} \epsilon_{1,2} F_2$$

$$F_{-2} \xrightarrow{Q_-} \epsilon_{0,-1} x_{-2} \lambda_{-1} \xrightarrow{Q_+} \epsilon_{0,-1} x_{-2} \epsilon_{-1,0} F_0$$

Under the action of Q_{-+}

$$\lambda_{-1} \xrightarrow{Q_+} \epsilon_{-1,0}F_0 \xrightarrow{Q_-} -\epsilon_{-1,0}\epsilon_{2,1}x_{-2}\lambda_1$$

$$\lambda_1 \xrightarrow{Q_+} \epsilon_{1,2}F_2 \xrightarrow{Q_-} \epsilon_{1,2}(\epsilon_{2,1}x_0\lambda_1 - \epsilon_{0,-1}x_2\lambda_{-1})$$

$$F_2 \xrightarrow{Q_+} -\epsilon_{0,1}x_2\lambda_1 \xrightarrow{Q_-} -\epsilon_{0,1}x_2\epsilon_{1,0}F_0$$

$$F_0 \xrightarrow{Q_+} \epsilon_{-2,-1}x_2\lambda_{-1} \xrightarrow{Q_-} \epsilon_{-2,-1}x_2\epsilon_{-1,-2}F_{-2}$$

$$F_{-2} \xrightarrow{Q_+} \epsilon_{0,1}x_{-2}\lambda_1 - \epsilon_{-2,-1}x_0\lambda_{-1} \xrightarrow{Q_-} \epsilon_{0,1}x_{-2}\epsilon_{1,0}F_0 - \epsilon_{-2,-1}x_0\epsilon_{-1,-2}F_{-2}$$

$Q_{+-} + Q_{-+} = 0$ is equivalent to the following set of equations

$$\epsilon_{-1,-2}(\epsilon_{0,1}x_{-2}\lambda_1 - \epsilon_{-2,-1}x_0\lambda_{-1}) - \epsilon_{-1,0}\epsilon_{2,1}x_{-2}\lambda_1 = 0$$

$$\epsilon_{1,0}\epsilon_{-2,-1}x_2\lambda_{-1} + \epsilon_{1,2}(\epsilon_{2,1}x_0\lambda_1 - \epsilon_{0,-1}x_2\lambda_{-1}) = 0$$

$$\epsilon_{2,1}\epsilon_{1,2}x_0F_2 - \epsilon_{0,-1}\epsilon_{-1,0}x_2F_0 - \epsilon_{0,1}x_2\epsilon_{1,0}F_0 = 0$$

$$-\epsilon_{2,1}x_{-2}\epsilon_{1,2}F_2 + \epsilon_{-2,-1}x_2\epsilon_{-1,-2}F_{-2} = 0$$

$$\epsilon_{0,-1}x_{-2}\epsilon_{-1,0}F_0 + \epsilon_{0,1}x_{-2}\epsilon_{1,0}F_0 - \epsilon_{-2,-1}x_0\epsilon_{-1,-2}F_{-2} = 0.$$

As it is now, this set of equalities do not hold. If, on the other hand, we introduce λ_3 and λ_{-3}, the problem can be fixed.

Under the action of Q_{++}:

$$\lambda_{-1} \xrightarrow{Q_+} \epsilon_{-1,0}F_0 = \epsilon_{-1,0}(x_2A_{-2} - x_{-2}A_2) \xrightarrow{Q+} \epsilon_{-1,0}(\epsilon_{-2,-1}x_2\lambda_{-1} - \epsilon_{2,3}x_{-2}\lambda_3)$$

$$\lambda_1 \xrightarrow{Q_+} \epsilon_{1,2}F_2 = \epsilon_{1,2}(x_0A_2 - x_2A_0) \xrightarrow{Q_+} \epsilon_{1,2}(\epsilon_{2,3}x_0\lambda_3 - \epsilon_{0,1}x_2\lambda_1)$$

$$F_2 = x_0A_2 - x_2A_0 \xrightarrow{Q_+} \epsilon_{2,3}x_0\lambda_3 - \epsilon_{0,1}x_2\lambda_1 \xrightarrow{Q_+} ? - \epsilon_{0,1}\epsilon_{1,2}x_2F_2$$

$$F_0 = x_2A_{-2} - x_{-2}A_2 \xrightarrow{Q_+} \epsilon_{-2,-1}x_2\lambda_{-1} - \epsilon_{2,3}x_{-2}\lambda_3 \xrightarrow{Q_+} \epsilon_{-2,-1}\epsilon_{-1,0}x_2F_0$$

$$F_{-2} = x_{-2}A_0 - x_0A_{-2} \xrightarrow{Q_+} \epsilon_{0,1}x_{-2}\lambda_1 - \epsilon_{-2,-1}x_0\lambda_{-1} \xrightarrow{Q_+} \epsilon_{0,1}\epsilon_{1,2}x_{-2}F_2 - \epsilon_{-2,-1}\epsilon_{-1,0}x_0F_0$$

Suppose we can find some good way to define $Re(Q)$. To get $Re(Q_{++}) = x_2$, it is necessary that

$$\epsilon_{-1,0}\epsilon_{-2,-1} = 1$$
$$\epsilon_{1,2}\epsilon_{0,1} = -1$$
$$\epsilon_{-1,0}\epsilon_{2,3} = \epsilon_{1,2}\epsilon_{2,3} = \pm i$$

Under the action of Q_{--}:

$$\lambda_{-1} \xrightarrow{Q_-} \epsilon_{-1,-2}F_{-2} = \epsilon_{-1,-2}(x_{-2}A_0 - x_0A_{-2}) \xrightarrow{Q_-} \epsilon_{-1,-2}(x_{-2}\epsilon_{0,-1}\lambda_{-1} - x_0\epsilon_{-2,-3}\lambda_{-3})$$
$$\lambda_1 \xrightarrow{Q_-} \epsilon_{1,0}F_0 = \epsilon_{1,0}(x_{-2}A_2 - x_2A_{-2}) = \epsilon_{1,0}(\epsilon_{2,1}x_{-2}\lambda_1 - \epsilon_{-2,-3}x_2\lambda_{-3})$$

Thus, to get $Re(Q_{--}) = x_{-2}$, it is necessary that

$$\epsilon_{0,-1}\epsilon_{0,-1} = 1$$
$$\epsilon_{2,1}\epsilon_{1,0} = -1$$
$$\epsilon_{-1,-2}\epsilon_{-2,-3} = \epsilon_{1,0}\epsilon_{-2,-3} = \pm i$$

Under the action of Q_{+-}

$$\lambda_{-1} \xrightarrow{Q_-} \epsilon_{-1,-2}F_{-2} \xrightarrow{Q_+} \epsilon_{-1,-2}(\epsilon_{0,1}x_{-2}\lambda_1 - \epsilon_{-2,-1}x_0\lambda_{-1})$$
$$\lambda_1 \xrightarrow{Q_-} \epsilon_{1,0}F_0 \xrightarrow{Q_+} \epsilon_{1,0}(\epsilon_{-2,-1}x_2\lambda_{-1} + \epsilon_{2,3}x_{-2}\lambda_3)$$
$$F_2 \xrightarrow{Q_-} \epsilon_{2,1}x_0\lambda_1 - \epsilon_{0,-1}x_2\lambda_{-1} \xrightarrow{Q_+} \epsilon_{2,1}\epsilon_{1,2}x_0F_2 - \epsilon_{0,-1}\epsilon_{-1,0}x_2F_0$$
$$F_0 \xrightarrow{Q_-} \epsilon_{-2,-3}x_2\lambda_{-3} - \epsilon_{2,1}x_{-2}\lambda_1 \xrightarrow{Q_+} \epsilon_{-2,-3}x_2\epsilon_{-3,-2}F_{-2} - \epsilon_{2,1}x_{-2}\epsilon_{1,2}F_2$$
$$F_{-2} \xrightarrow{Q_-} \epsilon_{0,-1}x_{-2}\lambda_{-1} - \epsilon_{-2,-3}x_0\lambda_{-3} \xrightarrow{Q_+} \epsilon_{0,-1}x_{-2}\epsilon_{-1,0}F_0 - \epsilon_{-2,-3}x_0\epsilon_{-3,-2}F_{-2}$$

Under the action of Q_{-+}

$$\lambda_{-1} \xrightarrow{Q_+} \epsilon_{-1,0} F_0 \xrightarrow{Q_-} \epsilon_{-1,0} x_2 \epsilon_{-2,-3} \lambda_{-3} - \epsilon_{-1,0} \epsilon_{2,1} x_{-2} \lambda_1$$

$$\lambda_1 \xrightarrow{Q_+} \epsilon_{1,2} F_2 \xrightarrow{Q_-} \epsilon_{1,2}(\epsilon_{2,1} x_0 \lambda_1 - \epsilon_{0,-1} x_2 \lambda_{-1})$$

$$F_2 \xrightarrow{Q_+} \epsilon_{2,3} x_0 \lambda_3 - \epsilon_{0,1} x_2 \lambda_1 \xrightarrow{Q_-} \epsilon_{2,3} \epsilon_{3,2} x_0 F_2 - \epsilon_{0,1} x_2 \epsilon_{1,0} F_0$$

$$F_0 \xrightarrow{Q_+} \epsilon_{-2,-1} x_2 \lambda_{-1} - \epsilon_{2,3} x_{-2} \lambda_3 \xrightarrow{Q_-} \epsilon_{-2,-1} x_2 \epsilon_{-1,-2} F_{-2} - \epsilon_{2,3} \epsilon_{3,2} x_{-2} F_2$$

$$F_{-2} \xrightarrow{Q_+} \epsilon_{0,1} x_{-2} \lambda_1 - \epsilon_{-2,-1} x_0 \lambda_{-1} \xrightarrow{Q_-} \epsilon_{0,1} x_{-2} \epsilon_{1,0} F_0 - \epsilon_{-2,-1} x_0 \epsilon_{-1,-2} F_{-2}$$

$Re(Q_{+-} + Q_{-+}) = 0$ is equivalent to the following set of equations

$$\epsilon_{-1,-2}(\epsilon_{0,1} x_{-2} \lambda_1 - \epsilon_{-2,-1} x_0 \lambda_{-1}) + \epsilon_{-1,0} x_2 \epsilon_{-2,-3} \lambda_{-3} - \epsilon_{-1,0} \epsilon_{2,1} x_{-2} \lambda_1 = 0$$

$$\epsilon_{1,0}(\epsilon_{-2,-1} x_2 \lambda_{-1} + \epsilon_{2,3} x_{-2} \lambda_3) + \epsilon_{1,2}(\epsilon_{2,1} x_0 \lambda_1 - \epsilon_{0,-1} x_2 \lambda_{-1}) = 0$$

$$\epsilon_{2,1} \epsilon_{1,2} x_0 F_2 - \epsilon_{0,-1} \epsilon_{-1,0} x_2 F_0 + \epsilon_{2,3} \epsilon_{3,2} x_0 F_2 - \epsilon_{0,1} x_2 \epsilon_{1,0} F_0 = 0$$

$$\epsilon_{-2,-3} x_2 \epsilon_{-3,-2} F_{-2} - \epsilon_{2,1} x_{-2} \epsilon_{1,2} F_2 + \epsilon_{-2,-1} x_2 \epsilon_{-1,-2} F_{-2} - \epsilon_{2,3} \epsilon_{3,2} x_{-2} F_2 = 0$$

$$\epsilon_{0,-1} x_{-2} \epsilon_{-1,0} F_0 - \epsilon_{-2,-3} x_0 \epsilon_{-3,-2} F_{-2} + \epsilon_{0,1} x_{-2} \epsilon_{1,0} F_0 - \epsilon_{-2,-1} x_0 \epsilon_{-1,-2} F_{-2} = 0.$$

This set of equations is equivalent to

$$\epsilon_{-1,-2} \epsilon_{0,1} - \epsilon_{-1,0} \epsilon_{2,1} = 0$$

$$\epsilon_{-1,-2} \epsilon_{-2,-1} = \pm \epsilon_{-1,0} \epsilon_{-2,-3} = \pm i$$

$$\epsilon_{1,0} \epsilon_{-2,-1} - \epsilon_{1,2} \epsilon_{0,-1} = 0$$

$$\epsilon_{1,0} \epsilon_{2,3} = \pm \epsilon_{1,2} \epsilon_{2,1} = \pm i$$

$$\epsilon_{1,2} \epsilon_{2,1} + \epsilon_{2,3} \epsilon_{3,2} = 0$$

$$\epsilon_{0,-1} \epsilon_{-1,0} + \epsilon_{0,1} \epsilon_{1,0} = 0$$

$$\epsilon_{-2,-3} \epsilon_{-3,-2} + \epsilon_{-2,-1} \epsilon_{-1,-2} = 0$$

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