Partial Differential Equations

Brief Introduction.

Historically, partial differential equations (PDE) originated from the study of surfaces in geometry and a wide variety of problems in mechanics. Almost all physical phenomena obey mathematical laws that can be formulated by differential equations. This was first observed by Newton when he formulated the laws of mechanics and applied them to describe the motion of the planets. Many partial differential equations that govern physical, chemical and biological phenomena have been found and successfully solved by numerous methods. These equations include Euler's equation for the dynamics of rigid body.

The key defining property of a PDE is that there is more than one independent variable $x, y, \ldots$. There is a dependent variable that is an unknown function of these variables $u(x, y, \ldots)$. We will often denote its derivatives by subscripts thus $\frac{\partial u}{\partial x} = u_x$ and so on..

A PDE describes a relation between the unknown function and its partial derivatives. PDEs appear frequently in all areas of physics & engineering. Moreover, in recent years there have been use of PDEs in biology, chemistry, C.S. Infact in each area where there is an interaction between a number of independent variables, we attempt to define functions in these variables and to model a variety of processes by constructing equations for these functions.

The general form of a PDE for a function $u(x_1, x_2, \ldots, x_n)$ is

$$F(x_1, x_2, \ldots, x_n, u, u_{x_1}, u_{x_2}, \ldots u_{x_{11}}, \ldots) = 0 \quad (1)$$

where x_i's are independent variables, u is dependent variable or the unknown

Eqn (1) is, in general supplemented by additional conditions such as initial conditions & boundary conditions.

The most general PDE in two independent variables of first order is.

$$F(x, y, u, u_x, u_y) = 0. \quad \text{--- (2)}$$

and the most general PDE in two independent variables of 2nd order is

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0. \quad \text{--- (3)}$$

we will study the PDE mostly in two dimensions.

A solution of a PDE is a function $u(x_1, x_2, --, x_n)$ that satisfies the equation identically, at least in some region of the $x_1, x_2, -, x_n$ variables.

There exist very complex equations that cannot be solved with the aid of computers. All we can do is to find qualitative information on the solutions.

The fundamental theoretical questions is whether the problem consisting of the equation and its associated side conditions is well posed.

Q.1's What is a well posed problem?

A problem is called well posed if it satisfies all of the following criteria.

1). Existence : The problem has a soln.

2) Uniqueness : There is no more than one solution.

3) Stability : A small change in the equation or in the side conditions gives rise to a small change in the solution.

Some of the examples of PDEs (all of which occur in physical theory) are,

1) $u_x + u_y = 0$ (transport)

2) $u_x + y u_y = 0$ (transport)

3) $u_x + u u_y = 0$ (shock wave)

4) $u_{xx} + u_{yy} = 0$ (Laplace equation)

5). $u_t + u u_x + u_{xxx} = 0$ (dispersive wave)

Classification of PDE :

Order of an equation

The order is defined to be the ~~highest~~ order of the highest

derivative in the equation.
①, ② & ③ are first order PDE.
④ is 2nd order & ⑤ is 3rd order
PDE.

Another classification is into two groups: linear versus non linear equations. An equation is called linear if in ①, F is a linear fn. of the unknown function u & its derivatives. Thus ①, ②, & ④ are linear.

An equation is non. linear if it is not linear. The non linear equations are classified into subclass.

a) Quasilinear PDE): A PDE is called as a quasilinear if all the terms with highest order derivatives of dependent variables, occur linearly. that is the coefficients of such terms are functions of only lower order derivatives of the dependent variable. However terms with lower order derivatives can occur in any manner.

Ex. (1) $u_t + u u_x = 0$.

$u_t + a(u) u_x = 0$.

A PDE is said to be semilinear in which the coefficients of the terms involving the highest order derivatives of u depend only on independent variables not on u or its derivatives.

Ex. $u_t + u_x + u^2 = 0$.

$u_t + u_{xxx} + u u_x = 0$.

We shall be primarily concerned with linear 2nd order partial differential equations, which frequently arise in problems of mathematical physics. The most general 2nd order PDE in n independent variables has the form.

$$\sum_{i,j=1}^{n} A_{ij} u_{x_i x_j} + \sum_{i=1}^{n} B_i u_{x_i} + F u = G \quad -(2).$$

Where we assume without loss of generality that $A_{ij} = A_{ji}$, we also assume that B_i, F and G are functions of the n independent variables x_i.

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If G is identically zero; the equation is said to be homogeneous otherwise it is non homogeneous.

The general solution of a linear ODE of nth order is a family of functions depending on n independent arbitrary constants. In the case of PDE, the general solution depends on arbitrary functions rather than arbitrary constants. To illustrate this,

Consider. $u_{xy} = 0$.

Integrate w.r.t y, we get.

$$u_x = f(x)$$

A 2nd integration w.r.t. x yields.

$$\boxed{u(x,y) = g(x) + h(y)}$$

Ex $u_{yy} = 0$

$\Rightarrow$ $u_y = f(x)$

$$\boxed{u(x,y) = y f(x) + g(x)}.$$

Ex $u_{xx} + u = 0$.

$$\boxed{u(x,y) = f(y) \cos x + g(y) \sin x.}$$

Ex. $u_{xy} + u_x = 0$.

Let $u_x = w$.

$\Rightarrow \quad w_y + w = 0$.

(Can be considered as Linear ODE in w)

$$W(x,y) = f(x)\,e^{-y}.$$

$$u_x = f(x)\,e^{-y}.$$

$$u(x,y) = \int f(x)\,dx\; e^{-y} + g(y)$$

$$= h(x,y) + g(y).$$

Linear operators:

Another important concept pertaining to a PDE is ~~line~~ that of linearity.

differential operator to be applied to a function u are. $\mathcal{L}u = \frac{\partial u}{\partial x}$,

$\mathcal{L}u = 3u + \sin y \frac{\partial u}{\partial x}$ & $\mathcal{L}u = u \frac{\partial^2 u}{\partial x^2}$

The operator is said to be linear if for any two functions u & v and any constant c,

$$\mathcal{L}(u+v) = \mathcal{L}u + \mathcal{L}v$$

$$\mathcal{L}(cu) = c\,\mathcal{L}u$$

A PDE is said to be linear if it can be written in the form

$$\mathcal{L}u = g. \quad -①$$

where $\mathcal{L}$ is a linear operator and g is a given function.. In case $g=0$ ① is said to be homogeneous.

The most general 2nd order linear PDE in two variables is of the form.

$$A(x,y)\,u_{xx} + B(x,y)\,u_{xy} + C(x,y)\,u_{yy} + D(x,y)\,u_x + E(x,y)\,u_y + F(x,y)\,u = G(x,y).$$

where A, B, C, D, E, F & G are known.

<u>Superposition principle for homogeneous linear PDE.</u>

If $u_1, u_2, \ldots, u_N$ are solutions of the same linear homogeneous PDE. $\mathcal{L}u = 0$. and $C_1, C_2, \ldots, C_N$ are constants., then $C_1u_1 + C_2u_2 + \ldots + C_Nu_N$ is also a solution of the same PDE.. For example. For any constant K, $u(x,y) = e^{Kx}\cos Ky$ is a soln of $u_{xx} + u_{yy} = 0$. by superposition

$$u(x,y) = e^{-x}\cos y + 2e^{-3x}\cos 3y - 5e^{-\pi x}\cos \pi y$$

is also a solution of $u_{xx} + u_{yy} = 0$.

Note:- The superposition principle does not apply to non-homogeneous equation.

Subtraction principle for non-homogeneous equation:- If u_1 & u_2 are solutions of the same linear non-homogeneous equation $Lu = g$ then the function $u_1 - u_2$ is a solⁿ of the associated homogeneous eqⁿ $Lu = 0$.

Note:- The general solⁿ of the linear PDE $Lu = g$ can be written as $u = U + v$ where U is a particular solution of the equation $LU = g$ and v is the general solⁿ of $Lv = 0$

Ex Find the general solⁿ of

$$u_{xx} = 2.$$

It can be verified that $u = x^2$ is a solⁿ of the given eqⁿ. The general solⁿ of the associated homogeneous eqⁿ. $u_{xx} = 0$ is.

$$u(x,y) = x\,g(y) + h(y)$$

Therefore the general solⁿ of

$u_{xx} = 2$ is.

$u(x,y) = x^2 + x\,g(y) + h(y)$.

Mathematical models.

PDEs arise frequently in formulating fundamental laws of nature and in the study of a wide variety of physical, chemical & biological models.

We will start with 2nd order linear PDE for the following reasons.

1) 2nd order linear PDE arise more frequently in a wide variety of applications.

2). Their mathematical treatment is simpler & easier to understand than that of 1st order PDE in general.

Usually in all physical processes the dependent variable $u = u(x,y,z,t)$ is a function of three space variables x, y, z and time variable t.

The three basic types of 2nd order linear PDE are

a) The wave equation

$$u_{tt} - c^2(u_{xx} + u_{yy} + u_{zz}) = 0$$

b) The heat equation.

$$u_t - K(u_{xx} + u_{yy} + u_{zz}) = 0.$$

c) The Laplace equation

$$u_{xx} + u_{yy} + u_{zz} = 0.$$

Following are some more 2nd order PDE.

d) The Poisson equation.

$$\nabla^2 u = f(x, y, z)$$

e) The Helmholtz equation

$$\nabla^2 u + \lambda u = 0$$

f) the biharmonic equation

$$\nabla^4 u = \nabla^2(\nabla^2 u) = 0.$$

Classification of 2nd. order linear PDE

Classification in 2-D.

Let u be the dependent variable & x, y be the independent variables. The most general 2nd order linear PDE is of the form

$$A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u = G \quad \text{---} (*)$$

where the coefficients are the f^n of x & y and do not vanish simultaneously. We assume that u & the twice continuously differentiable. In some domain in $\mathbb{R}^2$.

The classification of 2nd order PDE is suggested by the classification of the quadratic equation of conic sections in analytical geometry. The equation.

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

represents hyperbola, parabola or ellipse accordingly as $B^2 - 4AC$ is +ve, zero or -ve.

The classification of (*) is based upon the possibility of reducing (*) by co-ordinate transformation to canonical or standard form at a point. An equation is said to be hyperbolic parabolic or elliptic at a point (x_0, y_0) accordingly as.

$$B^2(x_0, y_0) - 4A(x_0, y_0)\,C(x_0, y_0)$$

is +ve, zero or negative. If this is true at all points, then the equation is said to hyperbolic, parabolic, or elliptic in a domain.

Ex// Classify the following equation

$$y^2 u_{xx} - x^2 u_{yy} = 0.$$

Here, $A = y^2$, $B = 0$, $C = -x^2$.

Thus, $B^2 - 4AC = 4x^2y^2 > 0$.

The equation is hyperbolic every where except on the coordinate axes $x = 0$ & $y = 0$.

Ex// Classify the following equ^n

$$y^2 u_{xx} - 2xy\, u_{xy} + x^2 u_{yy} - \frac{y^2}{x} u_x + \frac{x^2}{y} u_y = 0$$

Here, $A = y^2$, $B = -2xy$, $C = x^2$

$D = -\frac{y^2}{x}$, $E = -\frac{x^2}{y}$, $F = 0$, $G = 0$

Thus, $B^2 - 4AC = 4x^2y^2 - 4x^2y^2 = 0$

The equ^n is parabolic everywhere.

Ex: classify the following equation.

$$(1+x^2) u_{xx} + (1+y^2) u_{yy} + x u_x + y u_y = 0.$$

Here $A = (1+x^2)$, $B = 0$, $C = (1+y^2)$.

thus $B^2 - 4AC = -4 (1+x^2)(1+y^2) < 0$.

The equation is elliptic everywhere.

classification in higher dimension.

The most general form of the eq^n. (2nd order linear)

$$\sum_{i,j=1}^{n} A_{ij} u_{x_i x_j} + \sum_{i=1}^{n} A_i u_{x_i} + Bu = 0.$$ (*)

Assuming that the mixed partial derivatives are equal, we may as well assume that $A_{ij} = A_{ji}$

For example, the equation.

$$u_{xx} + 2u_{xy} - 3u_{yz} + 5u_{zz} = 0$$

Can be written as

$$u_{xx} + u_{xy} + u_{yx} - \frac{3}{2} u_{yz} - \frac{3}{2} u_{zy} + 5u_{zz} = 0.$$

By making an appropriate change of variables. we can write the top order term $\sum_{i,j=1}^{n} A_{ij} u_{x_i x_j}$ as.

(**) $$\sum_{K=1}^{n} d_K u_{x_K x_K}.$$

where the coefficients d_k are the eigenvalues of the $n \times n$ matrix $B = [A_{ij}]$.

Let $B = [A_{ij}]$ be the coefficient matrix ~~with~~ associated with (**). By the the assumption $A_{ij} = A_{ji}$, we know that B is symmetric.

Extending the ~~definition~~ of classification of (**) we saw previously. we say that (**) is hyperbolic if the ~~of~~ none of the eigenvalues of $B = [A_{ij}]$ is zero and one of them has the opposite sign of the $(n-1)$ others.

(**) is parabolic if exactly one of the eigenvalues is zero. and all the others have the same sign.

(**) is elliptic if the eigenvalues are all +ve or all -ve.

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Ex: Consider the

$$u_{xx} + 2u_{xy} - 3u_{yz} + 5u_{zz} = 0.$$

This can be written as.

$$u_{xx} + u_{xy} + u_{yx} - \frac{3}{2}u_{yz} - \frac{3}{2}u_{zy} + 5u_{zz} = 0$$

the matrix B.

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & -\frac{3}{2} \\ 0 & -\frac{3}{2} & 5 \end{bmatrix}$$

characteristic eqn of B.

$$|\lambda I - B| = 0.$$

$$\Rightarrow \begin{vmatrix} \lambda-1 & -1 & 0 \\ 0 & \lambda & \frac{3}{2} \\ 0 & \frac{3}{2} & \lambda-5 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda-1)\left(\lambda^2 - 5\lambda - \frac{9}{4}\right) = 0.$$

$$\Rightarrow \lambda = 1, \quad \lambda^2 - 5\lambda - \frac{9}{4} = 0.$$

$$\Rightarrow \lambda = \frac{5 \pm \sqrt{25+9}}{2}$$

$$= \frac{5 \pm \sqrt{34}}{2}.$$

Note that none of the $\lambda_i = 0$ and, one of the eigenvalues has opposite sign of the remaining 2. So the given equation is hyperbolic

Ex: Classify the following eqn

$3u_{xx} + u_{yy} + 4u_{yz} + 4u_{zz} = 0$

The coefficient matrix is.

$$B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

The characteristic eqn is.

$$|\lambda I - B| = 0$$

$$\Rightarrow \begin{vmatrix} \lambda-3 & 0 & 0 \\ 0 & \lambda-1 & -2 \\ 0 & -2 & \lambda-4 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda-3)\{(\lambda-1)(\lambda-4)-4\} = 0$$

$$\Rightarrow (\lambda-3)(\lambda^2-5\lambda+\cancel{4}-\cancel{4}) = 0$$

$$\Rightarrow \lambda(\lambda-3)(\lambda-5) = 0$$

$$\Rightarrow \lambda = 0, 3, 5$$

Here exactly one of the eigen values is zero. and, the rest are of the same sign. therefore the eqnation is parabolic.

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Canonical Forms :-

In case of two independent variables, a transformation can always be found to reduce the equation

$$A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u = G \quad \text{--- (1)}$$

to canonical form in a given domain. However, in the case of several independent variables it is not, in general possible to find such a transformation.

To transform (1) to a canonical form we make a change of independent variables. Let the new variables be

$$\xi = \xi(x, y), \quad \eta = \eta(x, y) \quad \text{--- (2)}$$

Assuming that ξ & η are twice continuously differentiable and that the Jacobian

$$J = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0$$

in the region under consideration, then x & y can be determined uniquely from (2). Let x & y be twice continuously differentiable functions of ξ & η. Then, we have

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial x}$$

$$= u_\xi \xi_x + u_\eta \eta_x \quad \text{(chain rule)}$$

Similarly,

$$u_y = u_\xi \xi_y + u_\eta \eta_y$$

$$u_{xx} = \frac{\partial}{\partial x}(u_\xi \xi_x) + \frac{\partial}{\partial x}(u_\eta \eta_x)$$

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$$u_{xx} = \frac{\partial}{\partial x} u_\xi \cdot \xi_x + u_\xi \cdot \frac{\partial}{\partial x}\xi_x + \frac{\partial}{\partial x} u_\eta \cdot \eta_x + u_\eta \cdot \frac{\partial}{\partial x}\eta_x.$$

$$= \frac{\partial}{\partial \xi} u_\xi \cdot \frac{\partial \xi}{\partial x} \cdot \xi_x + u_\xi \xi_{xx} + \left(\frac{\partial}{\partial \xi} u_\eta \cdot \frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta} u_\eta \cdot \frac{\partial \eta}{\partial x} \right) \eta_x + \frac{\partial}{\partial \eta} u_\xi \cdot \frac{\partial \eta}{\partial x} \cdot \xi_x + u_\eta \eta_{xx}$$

$$= u_{\xi\xi} \cdot \xi_x^2 + u_{\eta\xi} \xi_x \eta_x + u_{\eta\eta} \eta_x^2 + u_{\xi\eta} \xi_x \eta_x .$$

(Assuming continuity of mixed partial derivative).

$$u_{xx} = u_{\xi\xi} \xi_x^2 + 2u_{\xi\eta} \xi_x \eta_x + u_{\eta\eta} \eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}.$$

Similarly, it can be obtained,

$$u_{xy} = u_{\xi\xi} \xi_x \xi_y + u_{\xi\eta} (\xi_x \eta_y + \xi_y \eta_x) + u_{\eta\eta} \eta_x \eta_y + u_\xi \xi_{xy} + u_\eta \eta_{xy}.$$

$$u_{yy} = u_{\xi\xi} \xi_y^2 + 2u_{\xi\eta} \xi_y \eta_y + u_{\eta\eta} \eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}.$$

Substituting these values in (1), we get

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} + D^* u_\xi + E^* u_\eta + F^* u = G^* \quad \text{(3)}$$

where.

$$A^* = A \xi_x^2 + B \xi_x \xi_y + C \xi_y^2$$

$$B^* = 2A \xi_x \eta_x + B (\xi_x \eta_y + \xi_y \eta_x) + 2C \xi_y \eta_y$$

$$C^* = A \eta_x^2 + B \eta_x \eta_y + C \eta_y^2$$

$$D^* = A \xi_{xx} + B \xi_{xy} + C \xi_{yy} + D \xi_x + E \xi_y$$

$E^* = A h_{xx} + B h_{xy} + C h_{yy} + D h_x + E h_y$.

$F^* = F$, $G^* = G$.

The eqn (3) is in the same form as the original equation (1) under the general transformation. The nature of the equation remains invariant under such a transformation if the Jacobian does not vanish.

This can be seen from the fact that the sign of the discriminant does not alter.

$$B^{*2} - 4A^*C^* = J^2 (B^2 - 4AC), \quad \text{—(4)}$$

which can be easily verified.

Note :- The equation can be of a different type at different points of the domain, but for our purpose we shall assume that the equation under consideration is of the single type in a given domain.

The classification of eqn (1) depends on the coefficients $A(x,y)$, $B(x,y)$ & $C(x,y)$ at a given point (x,y), we shall therefore rewrite (1) as.

$$A u_{xx} + B u_{xy} + C u_{yy} = H(x, y, u, u_x, u_y) \quad \text{—(5)}$$

and (3) as.

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} = H^*(\xi, \eta, u, u_\xi, u_\eta) \quad \text{—(6)}$$

Here we will consider the problem of reducing eqn (5) to canonical form.

We suppose first that none of A, B, C is zero. Let ξ & η be new variables such that the coefficients A^* and C^* in eqn. (6) vanish. Thus from the expressions of A^* & C^*, we have.

$$A^* = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0,$$

$$C^* = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0.$$

These two equations are of the same type hence, we may write them in the form,

$$A\mu_x^2 + B\mu_x\mu_y + C\mu_y^2 = 0 \quad \text{--- (7)}$$

in which μ stands for either of the fn ξ or η. Dividing through by μ_y^2, (7) becomes

$$A\left(\frac{\mu_x}{\mu_y}\right)^2 + B\left(\frac{\mu_x}{\mu_y}\right) + C = 0, \quad \text{--- (8)}$$

Along the curve $\mu =$ constant, we have.

$$d\mu = \mu_x dx + \mu_y dy = 0.$$

Thus, $$\frac{dy}{dx} = -\frac{\mu_x}{\mu_y}. \quad \text{--- (9)}$$

and therefore eqn (8) may be written in the form

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0 \quad \text{--- (10)}$$

and roots of which are.

$$\frac{dy}{dx} = \left(B + \sqrt{B^2 - 4AC}\right) / 2A \quad — (11)$$

$$\frac{dy}{dx} = \left(B - \sqrt{B^2 - 4AC}\right) / 2A \quad — (12)$$

These equations are known as characteristic equations and these are ODE for families of curves in the x-y plane along which $\xi =$ Constant & $\eta =$ Constant. The integral of (11) & (12) are called the characteristic curves.

The solutions of (11) & (12) may be written as $\phi_1(x, y) = c_1$, $c_1 =$ Constant.

$\phi_2(x, y) = c_2$, $c_2 =$ Constant

Hence the transformation $\xi = \phi_1(x, y)$ & $\eta = \phi_2(x, y)$ will transform. eqn (5) to a Canonical form.

Hyperbolic Type :- If $B^2 - 4AC > 0$, then integration of (11) & (12) yield two real & distinct families of characteristics. Eqn (6) reduces to

$$\boxed{u_{\xi\eta} = H_1 \quad \text{where } H_1 = H^*/B^*} \quad — (13)$$

It can be easily shown that $B^* \neq 0$. This form is called the first Canonical form of the hyperbolic equation.

Now, if new independent variables

$\alpha = \xi + \eta$ & $\beta = \xi - \eta$ — (14)

are introduced. Then eqn (13) becomes.

$$u_\xi = \frac{\partial u}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial \xi} + \frac{\partial u}{\partial \beta}\cdot\frac{\partial \beta}{\partial \xi}$$

$$= u_\alpha + u_\beta \quad \left(\because \frac{\partial \alpha}{\partial \xi} = 1, \frac{\partial \beta}{\partial \xi} = 1\right)$$

$$u_{\xi\eta} = \frac{\partial}{\partial \eta}(u_\alpha + u_\beta)$$

$$= \frac{\partial}{\partial \alpha}u_\alpha \cdot \frac{\partial \alpha}{\partial \eta} + \frac{\partial}{\partial \beta}u_\alpha \cdot \frac{\partial \beta}{\partial \eta} + \frac{\partial u_\beta}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial \eta} + \frac{\partial u_\beta}{\partial \beta}\cdot\frac{\partial \beta}{\partial \eta}$$

$$= u_{\alpha\alpha} - u_{\alpha\beta} + u_{\beta\alpha} - u_{\beta\beta} \quad \left(\frac{\partial \alpha}{\partial \eta} = 1, \frac{\partial \beta}{\partial \eta} = -1\right)$$

$$= u_{\alpha\alpha} - u_{\beta\beta} \quad (\because u_{\alpha\beta} = u_{\beta\alpha}).$$

The eqn (13) becomes.

$$\boxed{u_{\alpha\alpha} - u_{\beta\beta} = H_2(\alpha, \beta, u, u_\alpha, u_\beta)} \quad \text{— (15)}$$

This form is called the 2nd Canonical form.

Parabolic Type: In this case, we have $B^2 - 4AC = 0$ and equations (11) & (12) coincide. There there exists one real family of characteristics and we obtain only a single integral either $\xi =$ Constant (or $\eta =$ Constant)

Since $B^2 = 4AC$ & $A^* = 0$, we find that

$$A^* = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 =$$

$$A^* = (\sqrt{A}\,\xi_x + \sqrt{C}\,\xi_y)^2 = 0.$$

From this, it follows that

$$B^* = \cancel{2A\xi_x \eta_x}\; 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y.$$

$$= 2(\sqrt{A}\,\xi_x + \sqrt{C}\,\xi_y)(\sqrt{A}\,\eta_x + \sqrt{C}\,\eta_y)$$

$$= 0.$$

for arbitrary values of $\eta(x,y)$ which is independent of $\xi(x,y)$; for instance if $\eta = y$, the Jacobian does not vanish in the domain of parabolicity.

Division of Eqn 6, by C^* yields.

$$\boxed{u_{\eta\eta} = H_3(\xi, \eta, u, u_\xi, u_\eta) \qquad C^* \neq 0.} \quad 16$$

This is called the Canonical form of parabolic eqn.

Eqn 6 may also have the form.

$$\boxed{u_{\xi\xi} = H_3^*(\xi, \eta, u, u_\xi, u_\eta)} \quad (17)$$

If we choose $\eta =$ Constant as the integral of eqn. (11).

Elliptic type. :- For elliptic type we have $B^2 - 4AC < 0$, consequently the eqn (10) has no real solutions but it has two complex conjugate solⁿ.

Since ξ & η are complex, we introduce new variables.

$$\alpha = \frac{1}{2}(\xi + \eta), \quad \beta = \frac{1}{2i}(\xi - \eta) \quad (18)$$

So that $\xi = \alpha + i\beta$, $\eta = \alpha - i\beta$ — (19)

First, we transform the eqn (5) as

$$A^{**} u_{\alpha\alpha} + B^{**} u_{\alpha\beta} + C^{**} u_{\beta\beta} = H_4(\alpha, \beta, u, u_\alpha, u_\beta)$$ — (20)

in which the coefficients assume the same form as the coefficients in eqn (6)

with the use of (19) the equations

$A^* = C^* = 0$, yield.

$$(A^{**} - C^{**}) + iB^{**} = 0, \quad (A^{**} - C^{**}) - iB^{**} = 0$$

These are satisfied iff

$$A^{**} = C^{**} \text{ \& } B^{**} = 0.$$

Hence, (20) transform into.

$$A^{**} u_{\alpha\alpha} + A^{**} u_{\beta\beta} = H_4(\alpha, \beta, u, u_\alpha, u_\beta)$$

or $$\boxed{u_{\alpha\alpha} + u_{\beta\beta} = H_5(\alpha, \beta, u, u_\alpha, u_\beta)}$$ — (21)

where $H_5 = H_4 / A^{**}$.

Ex. Consider the equation.

$$y^2 u_{xx} - x^2 u_{yy} = 0.$$

Here, $A = y^2$, $B = 0$, $C = -x^2$.

Thus $B^2 - 4AC = 4x^2y^2 > 0$.

The eqn is hyperbolic everywhere except on the coordinate axes $x = 0$ and $y = 0$.

From the characteristic equations (11) & (12), we have

$$\frac{dy}{dx} = \frac{0 + \sqrt{4x^2y^2}}{2y^2}, \qquad \frac{dy}{dx} = \frac{0 - \sqrt{4x^2y^2}}{2y^2}.$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{y}, \qquad \frac{dy}{dx} = -\frac{x}{y}.$$

After integration of these equations we obtain,

$$\boxed{\frac{1}{2}y^2 - \frac{1}{2}x^2 = c_1} \quad \& \quad \boxed{\frac{1}{2}y^2 + \frac{1}{2}x^2 = c_2.}$$

Family of hyperbolas. Family of circles.

To transform the given eqn to canonical form, we consider

$$\xi = \frac{1}{2}y^2 - \frac{1}{2}x^2, \qquad \eta = \frac{1}{2}y^2 + \frac{1}{2}x^2$$
$$2\xi = y^2 - x^2, \qquad \Rightarrow x^2 + y^2 = 2\eta$$

Now, we compute,

$$u_x = u_\xi \xi_x + u_\eta \cdot \eta_x = -u_\xi \cdot x + u_\eta \cdot x$$
$$(\because \xi_x = -x, \ \eta_x = x).$$

$$u_{xx} = \frac{\partial}{\partial x}(-u_\xi x + u_\eta \cdot x).$$
$$= -\frac{\partial}{\partial x}u_\xi \cdot x - u_\xi + \frac{\partial}{\partial x}u_\eta \cdot x + u_\eta$$
$$= -\left(\frac{\partial}{\partial \xi}u_\xi \cdot \frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}u_\xi \cdot \frac{\partial \eta}{\partial x}\right)x - u_\xi.$$
$$+ \left(\frac{\partial}{\partial \xi}u_\eta \cdot \frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}u_\eta \cdot \frac{\partial \eta}{\partial x}\right) \cdot x + u_\eta$$
$$= (u_{\xi\xi} - u_{\xi\eta}) \cdot x^2 - u_\xi - (u_{\eta\xi} - u_{\eta\eta})x^2$$
$$+ u_\eta \cdot (u_{\xi\eta} = u_{\eta\xi})$$
$$= u_{\xi\xi}x^2 - 2u_{\xi\eta}x^2 + u_{\eta\eta}x^2 - u_\xi + u_\eta$$

Similarly it can be obtained that

$$u_{yy} = y^2 u_{\xi\xi} + 2y^2 u_{\xi\eta} + y^2 u_{\eta\eta} + u_\xi + u_\eta$$

Thus the given eqn.

$$y^2 u_{xx} - x^2 u_{yy} = 0 \quad \text{becomes}$$

$$\cancel{u_{\xi\xi} x^2 y^2} - 2u_{\xi\eta} x^2 y^2 + \cancel{u_{\eta\eta} x^2 y^2} - y^2 u_\xi + y^2 u_\eta$$

$$- \cancel{u_{\xi\xi} x^2 y^2} - 2u_{\xi\eta} x^2 y^2 - \cancel{u_{\eta\eta} x^2 y^2} - x^2 u_\xi - x^2 u_\eta$$

$$= 0$$

$$-4x^2 y^2 u_{\xi\eta} = (x^2+y^2) u_\xi + (x^2 - y^2) u_\eta$$

$$y^2 + x^2 = 2\eta, \quad y^2 - x^2 = 2\xi.$$

$$(y^2 - x^2)^2 - (y^2 + x^2)^2 = -4x^2 y^2$$

$$\Rightarrow \quad -4x^2 y^2 = 4\xi^2 - 4\eta^2.$$

therefore the eqn. becomes.

$$u_{\xi\eta} = \frac{2\eta}{4(\xi^2 - \eta^2)} u_\xi - \frac{2\xi}{4(\xi^2 - \eta^2)} u_\eta.$$

$$\boxed{u_{\xi\eta} = \frac{\eta}{2(\xi^2 - \eta^2)} u_\xi - \frac{\xi}{2(\xi^2 - \eta^2)} u_\eta}$$

Ex. Consider the PDE.

$$x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = 0$$

Here $A = x^2$, $B = 2xy$, $C = y^2$,

$$B^2 - 4AC = 4x^2y^2 - 4x^2y^2 = 0$$

$\Rightarrow$ the equation is parabolic.

Therefore this eqn has only one real characteristic.

From eqn (11) we get

$$\frac{dy}{dx} = \frac{2xy f'}{2x^2} = \frac{y}{x}.$$

integration yields.

$\frac{y}{x} = c_1$, which is a family of straight lines.

let $\xi = \frac{y}{x}$. choose η independent of ξ.

let $\eta = x$.

we compute.

$$u_x = u_\xi \xi_x + u_\eta \eta_x.$$

$$= -\frac{y}{x^2} u_\xi + u_\eta.$$

$$u_{xx} = \frac{2y}{x^3} u_\xi - \frac{y}{x^2}\left(\frac{\partial}{\partial \xi} u_\xi \cdot \frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta} u_\xi \cdot \frac{\partial \eta}{\partial x}\right)$$

$$+ \frac{\partial}{\partial \xi} u_\eta \cdot \frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta} u_\eta \cdot \frac{\partial \eta}{\partial x}$$

$$= \frac{2y}{x^3} u_\xi - \frac{y}{x^2}\left(-\frac{y}{x^2} u_{\xi\xi} + u_{\xi\eta}\right).$$

$$-\frac{y}{x^2} u_{\xi\eta} + u_{\eta\eta}$$

$$= \frac{y^2}{x^4} u_{\xi\xi} - \frac{2y}{x^2} u_{\xi\eta} + u_{\eta\eta} + \frac{2y}{x^3} u_\xi.$$

$$u_{xy} = -\frac{1}{x^2} u_\xi - \frac{y}{x^2}\left(\frac{\partial}{\partial \xi} u_\xi \cdot \frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta} u_\xi \cdot \frac{\partial \eta}{\partial y}\right)$$

$$+ \frac{\partial}{\partial \xi} u_\eta \cdot \frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta} u_\eta \cdot \frac{\partial \eta}{\partial y}.$$

$$= -\frac{1}{x^2} u_\xi - \frac{y}{x^3} u_{\xi\xi} + \frac{1}{x} u_{\xi\eta}$$

$$u_y = u_\xi \cdot \xi_y + u_\eta \cdot \eta_y.$$

$$= \frac{1}{x} u_\xi + 0 \qquad (\eta_y = 0)$$

$$u_{yy} = \frac{1}{x}\left(\frac{\partial}{\partial \xi} u_\xi \cdot \frac{\partial \xi}{\partial y} + \cancel{\frac{\partial}{\partial \eta} u_\xi \cdot \frac{\partial \eta}{\partial y}}^{\,0}\right)$$

$u_{yy} = \frac{1}{x^2} u_{\xi\xi}$.

Therefore the eqn becomes.

$$x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = 0$$

$$\cancel{\frac{y^2}{x^2} u_{\xi\xi}} - \cancel{2y\, u_{\xi\eta}} + x^2 u_{\eta\eta} + \cancel{\frac{2y}{x} u_\xi} - \cancel{\frac{2y}{x} u_\xi}$$

$$- \cancel{\frac{2y^2}{x^2} u_{\xi\xi}} + \cancel{2y\, u_{\xi\eta}} + \cancel{\frac{y^2}{x^2} u_{\xi\xi}} = 0$$

$$\Rightarrow \boxed{u_{\eta\eta} = 0} \quad \text{for } x \neq 0$$

Ex Consider the eqn. $u_{xx} + x^2 u_{yy} = 0$.

Here $A = 1$, $B = 0$, $C = x^2$.

Thus. $B^2 - 4AC = -4x^2 < 0$.

eqn is elliptic except on the coordinate axis $x = 0$.

The characteristic equations are.

$$\frac{dy}{dx} = \frac{2xi}{2} = xi, \quad \& \quad \frac{dy}{dx} = -xi$$

Integration yields.

$$y = \frac{ix^2}{2} + c_1 \quad \& \quad y = -\frac{ix^2}{2} + c_2.$$

or $2y - ix^2 = c_1$ & $2y + ix^2 = c_2$.

Thus, if we write.

$$\xi = 2y - ix^2 \quad \& \quad \eta = 2y + ix^2.$$

and hence,

$$\alpha = \frac{1}{2}(\xi + \eta) = 2y, \quad \beta = \frac{1}{2i}(\xi - \eta)$$
$$= -x^2.$$

$$u_x = u_\alpha \cdot \alpha_x + u_\beta \cdot \beta_x$$
$$= -2x\, u_\beta.$$

$$u_{xx} = -2u_\beta + x\left(\frac{\partial u_\beta}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial x} + \frac{\partial u_\beta}{\partial \beta}\cdot\frac{\partial \beta}{\partial x}\right)$$
$$= -2u_\beta - x\left(0 - 2x\,u_{\beta\beta}\right)$$
$$= -2u_\beta + 2x^2 u_{\beta\beta}.$$

$$u_y = u_\alpha \alpha_y + u_\beta \beta_y.$$
$$= 2u_\alpha.$$
$$u_{yy} = 2.\left(\frac{\partial u_\alpha}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial y} + \frac{\partial u_\alpha}{\partial \beta}\cdot\frac{\partial \beta}{\partial y}\right)$$
$$= 2\left(2u_{\alpha\alpha} + 0\right) = 4u_{\alpha\alpha}.$$

Therefore, the eqn. $u_{xx} + x^2 u_{yy} = 0$ reduces to $-2u_\beta + 2x^2 u_{\beta\beta} + 4x^2 u_{\alpha\alpha} = 0$.

or
$$\boxed{2u_{\alpha\alpha} + u_{\beta\beta} = -\frac{2}{\beta}u_\beta.}$$

Ex Consider the eqn.
$$u_{xx} - 4u_{xy} + 4u_{yy} = e^y.$$

Here $A = 1$, $B = -4$, $C = 4$.

$B^2 - 4AC = 16 - 16 = 0$.

Therefore the eqn is parabolic.

The characteristic equations are

$$\frac{dy}{dx} = \frac{4}{2} = 2, \quad \frac{dy}{dx} = 2.$$

$$y = 2x + C. \Rightarrow y - 2x = C.$$

let $\xi = y - 2x$. we chose $\eta = y$, that is independent of ξ.

$\xi = c_1$, $\eta = c_2$ are the characteristic curves.

$$u_x = u_\xi \xi_x + u_\eta \cdot \eta_x$$
$$= -2u_\xi + 0 = -2u_\xi. \quad (\eta_x = 0)$$
$$u_{xx} = -2\left(\frac{\partial u_\xi}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial u_\xi}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}\right)$$
$$= -2(-2)u_{\xi\xi} + 0 = 4u_{\xi\xi}.$$
$$u_{xy} = -2\cdot\left(\frac{\partial u_\xi}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial u_\xi}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}\right)$$
$$= -2\left(u_{\xi\xi} + u_{\xi\eta}\right)$$
$$u_y = u_\xi \xi_y + u_\eta \eta_y.$$
$$= u_\xi + u_\eta \quad (\because \xi_y = 1,\ \eta_y = 1)$$
$$u_{yy} = \left(\frac{\partial u_\xi}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial u_\xi}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}\right) + \left(\frac{\partial u_\eta}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial u_\eta}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}\right)$$
$$= \left(u_{\xi\xi} + u_{\xi\eta}\right) + \left(u_{\eta\xi} + u_{\eta\eta}\right)$$
$$= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}.$$

Therefore, the eqn.

$u_{xx} - 4u_{xy} + 4u_{yy} = e^y$ becomes.

$$\cancel{4u_{\xi\xi}} - \cancel{8u_{\xi\xi}} - \cancel{8u_{\xi\eta}} + \cancel{4u_{\xi\xi}} + \cancel{8u_{\xi\eta}} + 4u_{\eta\eta} = e^\eta$$
$$\Rightarrow \boxed{u_{\eta\eta} = \frac{1}{4}e^\eta.}$$

Canonical form.

General Soln: In general it is not so simple to determine the general solution of a given equation

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sometimes, further simplification of the canonical form of an eqn may yield the general solution.

Ex Consider the problem.

$$u_{xx} - 4u_{xy} + 4u_{yy} = e^y.$$

This eqn is parabolic and its canonical form is. $u_{\eta\eta} = \frac{1}{4} e^{\eta}$.

Integrating w.r.t. η. yields.

$$u_{\eta} = \frac{1}{4} e^{\eta} + f(\xi)$$

Further integrating w.r.t. η, we get,

$$u = \frac{1}{4} e^{\eta} + f(\xi)\eta + g(\xi).$$

Therefore, the general soln is,

$$\boxed{u(x,y) = \frac{1}{4} e^{y} + y\, f(y-2x) + g(y-2x)}$$

where, f and g are arbitrary functions.

Ex Find the general solution of the following equations.

(a) $y\, u_{xx} + 3y\, u_{xy} + 3u_x = 0, \quad y \neq 0.$

(b) $u_{xx} + 2u_{xy} + u_{yy} = 0$

(c) $u_{xx} + 2u_{xy} + 5u_{yy} + u_x = 0.$

(a) Note that, $A = y,\ B = 3y,\ C = 0.$

$B^2 - 4AC = 9y^2 > 0$

therefore, (a) is hyperbolic everywhere.

The characteristic equations are

$$\frac{dy}{dx} = \frac{3y + 3y}{2y} = 3, \qquad \frac{dy}{dx} = 0.$$

Integrating gives.

$y = 3x + C_1$ & $y = C_2$.

Let $\xi = y - 3x$ & $\eta = y$. be the characteristic curves.

Now, we reduce the given equation to canonical form.

$u_x = u_\xi \xi_x + u_\eta \eta_x = -3u_\xi + 0$

$= -3u_\xi$.

$$u_{xx} = -3\left(\frac{\partial u_\xi}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial u_\xi}{\partial \eta} \cdot \cancel{\frac{\partial \eta}{\partial x}}^{0}\right)$$

$$= 9u_{\xi\xi}.$$

$$u_{xy} = -3\left(\frac{\partial u_\xi}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial u_\xi}{\partial \eta} \cdot \frac{\partial \eta}{\partial y}\right)$$

$$= -3(u_{\xi\xi} + u_{\xi\eta})$$

Therefore (a) reduces to.

$y u_{xx} + 3y u_{xy} + 3u_x = 0$

$\Rightarrow$ $\cancel{9\eta u_{\xi\xi}} - \cancel{9\eta u_{\xi\xi}} - 9\eta u_{\xi\eta} - 9u_\xi = 0$

$\Rightarrow$ $u_{\xi\eta} + \frac{1}{\eta} u_\xi = 0$.

Let $u_\xi = v$. then we get,

$v_\eta + \frac{1}{\eta} v = 0$.

using variable separable; we get.

$v = \frac{1}{\eta} f(\xi)$

$u_\xi = \frac{1}{\eta} f(\xi)$

$u = \frac{1}{\eta}\int f(\xi)\,d\xi + g(\eta)$

$= \frac{1}{\eta} h(\xi) + g(\eta)$

$$\boxed{u(x,y) = \frac{1}{y} f(y-3x) + g(y).}$$

(b) $u_{xx} + 2u_{xy} + u_{yy} = 0.$

Here $A = 1$, $B = 2$, $C = 1$

$B^2 - 4AC = 4 - 4 = 0.$

The eqn is parabolic. The characteristic eqn is.

$$\frac{dy}{dx} = \frac{2}{2} = 1$$

integration yields. $y = x + c$

The characteristic curves $\xi = y - x$.

Let $\eta = x$. we transform the equation into canonical form.

$$u_x = u_\xi \xi_x + u_\eta \eta_x = -u_\xi + u_\eta.$$

$$u_{xx} = -\left(\frac{\partial u_\xi}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial u_\xi}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}\right) + \left(\frac{\partial u_\eta}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial u_\eta}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}\right)$$

$$= -\left(-u_{\xi\xi} + u_{\xi\eta}\right) + \left(-u_{\eta\xi} + u_{\eta\eta}\right)$$

$$= u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}.$$

$$u_{xy} = -\left(\frac{\partial u_\xi}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial u_\xi}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}\right) + \left(\frac{\partial u_\eta}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial u_\eta}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}\right)$$

$$= -\left(u_{\xi\xi} + 0\right) + \left(u_{\eta\xi} + 0\right)$$

$$= -u_{\xi\xi} + u_{\xi\eta}$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y$$

$$= u_\xi + 0 = u_\xi,$$

$$u_{yy} = \frac{\partial u_\xi}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial u_\xi}{\partial \eta}\cdot\cancel{\frac{\partial \eta}{\partial y}}^{0} = u_{\xi\xi}.$$

Therefore the eqn. $u_{xx} + 2u_{xy} + u_{yy} = 0$

becomes $\cancel{u_{\xi\xi}} - \cancel{2u_{\xi\eta}} + u_{\eta\eta} - 2u_{\xi\xi} + 2u_{\xi\eta}$
$+ \cancel{u_{\xi\xi}} = 0$

$\Rightarrow \quad u_{\eta\eta} = 0.$

integrating w.r.t. η, twice, we get,

$$u(\xi,\eta) = \eta f(\xi) + g(\xi).$$

In terms of original variables.

$$\boxed{u(x,y) = x f(y-x) + g(y-x)}$$

Ex Consider the wave equation,

$$u_{tt} - c^2 u_{xx} = 0, \quad c \text{ is constant}.$$

Since $A = -c^2$, $B = 0$, $C = 1$

$B^2 - 4AC = 0 + 4c^2 > 0$. the equation is hyperbolic everywhere. The characteristic equation.

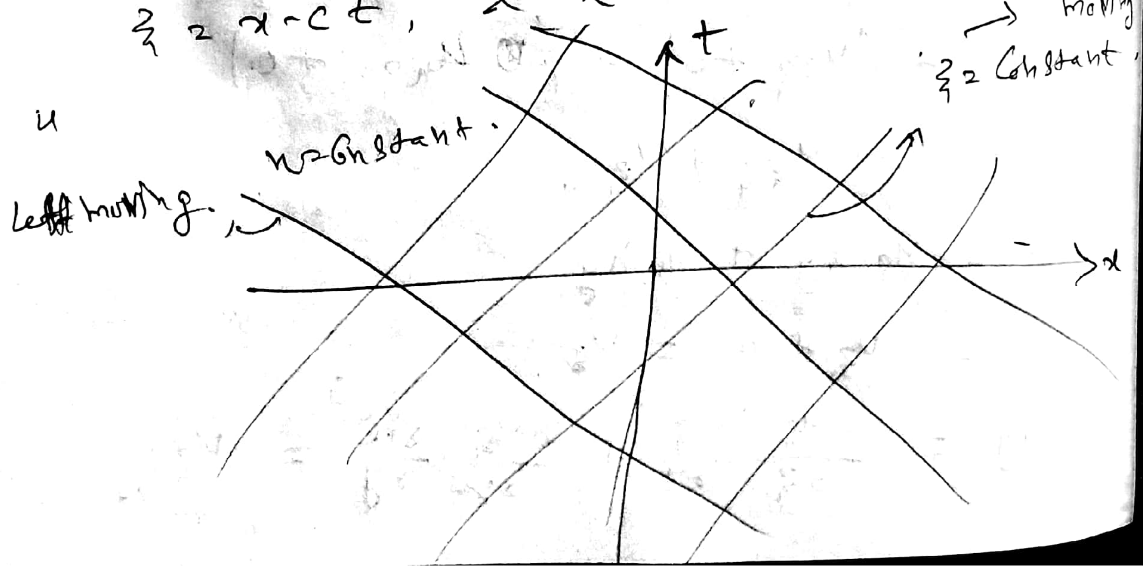
$$\frac{dx}{dt} = \frac{2c}{2} = c, \quad \& \quad \frac{dx}{dt} = -c.$$

integration yields.

$x = ct + c_1 \quad \& \quad x = -ct + c_2$.

The characteristic curves. are.

$\xi = x - ct, \quad \& \quad \eta = x + ct$.



$$u_x = u_\xi \cdot \xi_x + u_\eta \cdot \eta_x$$
$$= u_\xi + u_\eta$$
$$u_{xx} = \left(\frac{\partial u_\xi}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial u_\xi}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}\right) + \left(\frac{\partial u_\eta}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial u_\eta}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}\right)$$
$$= (u_{\xi\xi} + u_{\xi\eta}) + (u_{\eta\xi} + u_{\eta\eta})$$
$$= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}.$$

$$u_t = u_\xi \cdot \xi_t + u_\eta \cdot \eta_t$$
$$= -cu_\xi + cu_\eta.$$

$$u_{tt} = -c\left(\frac{\partial u_\xi}{\partial \xi}\cdot\frac{\partial \xi}{\partial t} + \frac{\partial u_\xi}{\partial \eta}\cdot\frac{\partial \eta}{\partial t}\right) + c\left(\frac{\partial u_\eta}{\partial \xi}\cdot\frac{\partial \xi}{\partial t} + \frac{\partial u_\eta}{\partial \eta}\cdot\frac{\partial \eta}{\partial t}\right)$$

$$= -c(-cu_{\xi\xi} + cu_{\xi\eta}) + c(-cu_{\eta\xi} + cu_{\eta\eta})$$
$$= c^2u_{\xi\xi} - 2c^2u_{\xi\eta} + c^2u_{\eta\eta}.$$

therefore the eqn. $u_{tt} - c^2u_{xx} = 0$ reduces to.

$$\cancel{c^2u_{\xi\xi}} - 2c^2u_{\xi\eta} + \cancel{c^2u_{\eta\eta}} - \cancel{c^2u_{\xi\xi}} - 2c^2u_{\xi\eta} - \cancel{c^2u_{\eta\eta}} = 0$$

$$\Rightarrow -4c^2u_{\xi\eta} = 0 \Rightarrow u_{\xi\eta} = 0.$$

Integrating w.r.t. η yields.

$$u_\xi = f(\xi),$$

integrating w.r.t. ξ, we get,

$$u(\xi,\eta) = \int f(\xi)\,d\xi + G(\eta)$$
$$= F(\xi) + G(\eta)$$

In terms of original variables,

$$\boxed{u(x,t) = F(x-ct) + G(x+ct)}$$

provided F & G are arbitrary but twice differentiable functions.

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Note that F is Constant on "wave fronts" $x = ct + \xi$, that travel towards increasing x as t increases, whereas G is Constant on wavefront $x = -ct + \eta$ that travel toward decreasing x as t increases. Thus any general solution can be expressed as the sum of two waves, one travelling to the right with constant velocity 'c' & the other travelling to the left with the same velocity.

<u>Ex</u> - Find the characteristics and characteristic coordinates, and reduce the following equation to Canonical form.

a). $u_{xx} + 5u_{xy} + 4u_{yy} + 7u_y = \sin x$.

b). $6u_{xx} - u_{xy} + u = y^2$.

a) Here $A = 1$, $B = 5$, $C = 4$.

$B^2 - 4AC = 25 - 16 = 9 > 0$

Therefore, the equation is hyperbolic everywhere. The characteristic equations are $\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A}$, $\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A}$

$\Rightarrow \frac{dy}{dx} = \frac{5+3}{2} = 4, \quad \frac{dy}{dx} = \frac{5-3}{2} = 1$

integration yields

$y = 4x + c_1 \quad \& \quad y = x + c_2$

The characteristic curves are

$\xi = y - 4x \quad \& \quad \eta = y - x.$

Now, we transform the eqn to canonical form.

First, we compute

$u_x = u_\xi \xi_x + u_\eta \cdot \eta_x$

$= -4u_\xi - u_\eta.$

$$u_{xx} = -4\left(\frac{\partial u_\xi}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial u_\xi}{\partial \eta} \cdot \frac{\partial \eta}{\partial x}\right) - \left(\frac{\partial u_\eta}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial u_\eta}{\partial \eta} \cdot \frac{\partial \eta}{\partial x}\right)$$

$$= -4(-4u_{\xi\xi} - u_{\xi\eta}) - (-4u_{\eta\xi} - u_{\eta\eta})$$

$$= 16u_{\xi\xi} + 8u_{\xi\eta} + u_{\eta\eta}$$

$$u_{xy} = -4\left(\frac{\partial u_\xi}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial u_\xi}{\partial \eta} \cdot \frac{\partial \eta}{\partial y}\right) - \left(\frac{\partial u_\eta}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial u_\eta}{\partial \eta} \cdot \frac{\partial \eta}{\partial y}\right)$$

$$= -4(u_{\xi\xi} + u_{\xi\eta}) - (u_{\eta\xi} + u_{\eta\eta})$$

$$= -4u_{\xi\xi} - 5u_{\xi\eta} - u_{\eta\eta}.$$

$u_y = u_\xi \xi_y + u_\eta \eta_y.$

$= u_\xi + u_\eta.$

$$u_{yy} = \left(\frac{\partial u_\xi}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial u_\xi}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}\right) + \left(\frac{\partial u_\eta}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial u_\eta}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}\right)$$

$$= (u_{\xi\xi} + u_{\xi\eta}) + (u_{\eta\xi} + u_{\eta\eta})$$

$$= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

Therefore, the eqn

$u_{xx} + 5u_{xy} + 4u_{yy} + 7u_y = \sin x$, reduces to.

$$\cancel{16u_{\xi\xi}} + 8u_{\xi\eta} + \cancel{u_{\eta\eta}} - \cancel{20u_{\xi\xi}} - 25u_{\xi\eta} - \cancel{5u_{\eta\eta}} + \cancel{4u_{\xi\xi}} + 8u_{\xi\eta} + \cancel{4u_{\eta\eta}} + 7u_\xi + 7u_\eta = \sin\left(\frac{\eta-\xi}{3}\right)$$

$$\Rightarrow \boxed{-9u_{\xi\eta} + 7u_\xi + 7u_\eta = \sin\left(\frac{\eta-\xi}{3}\right)}$$

2

(b) Here $A=6$, $B=-1$, $C=0$

$B^2-4AC = 1 > 0$, The eqn is hyperbolic everywhere.

The characteristic equations are

$$\frac{dy}{dx} = \frac{-1+1}{2\times 6} = 0 \qquad \frac{dy}{dx} = \frac{-1-1}{2\times 6} = -\frac{1}{6}$$

Integration yields.

$$y = c_1 \quad \& \quad y = -\frac{1}{6}x + c_2$$

The characteristic curves are

$$\xi = y \quad \& \quad \eta = y + \frac{1}{6}x.$$

we reduce the equation to canonical form.

$$u_x = u_\xi \xi_x + u_\eta \eta_x = \frac{1}{6} u_\eta .$$

$$u_{xx} = \frac{1}{6}\left(\frac{\partial u_\eta}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial u_\eta}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} \right)$$

$$= \frac{1}{36} u_{\eta\eta}$$

$$u_{xy} = \frac{1}{6}\left(\frac{\partial u_\eta}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial u_\eta}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} \right)$$

$$= \frac{1}{6}\left(u_{\eta\xi} + u_{\eta\eta} \right).$$

the eqn. $6u_{xx} - u_{xy} + u = y^2$ becomes

$$\cancel{\frac{1}{6} u_{\eta\eta}} - \frac{1}{6} u_{\xi\eta} - \cancel{\frac{1}{6} u_{\eta\eta}} + u = \xi^2 .$$

$$\Rightarrow \boxed{u_{\xi\eta} - 6u = -6\xi^2}.$$