Chapter 1: Part II: Asymptotic Notations

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Design and Analysis of Algorithms
IIT Jammu, Jammu

The Big-Oh Notation

- def: for a given function g(n), we say that O(g(n)) = f(n) — if there exists positive constants c and n_0 such that, $0 \leq f(n) \leq cg(n)$, for all $n \geq n_0$

$f(n) = O(g(n)) \Rightarrow$

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- The Big-oh defines an upper bound for a function within a constant factor i.e. except for a constant factor and a finite number of exceptions, f is bounded above by g.

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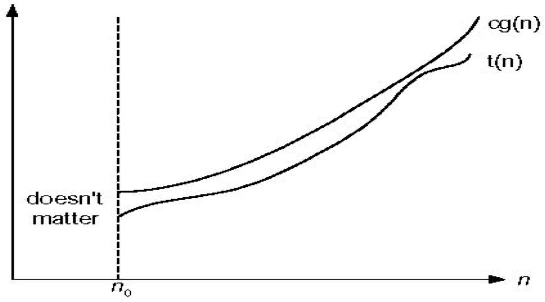

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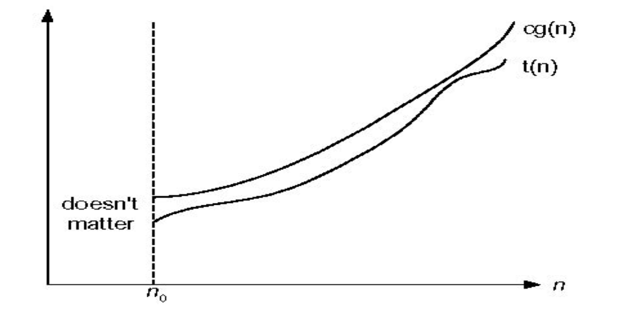

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- What does the growth rate imply ?

The Big-Oh Notation Illustrations

Function	notation in O
$f(n) = 5n + 8$	$f(n) = O(?)$
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$F(n) = 12n^2 - 11$	$f(n) = O(?)$
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- The symbol = is not proper truly it is ϵ which should be used i.e. f(n) ϵ O(g(n)

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- When O notation bounds the worst case running time of an algorithm, by implication we also bound the running time of an algorithm on EVERY input.

Abuse

Technically, it is abuse to say that the running time of insertion sort is $O(n^2)$. Why?

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- When O notation bounds the worst case running time of an algorithm, by implication we also bound the running time of an algorithm on EVERY input.
- this is not so when using other notations i.e. the worst case $\theta(n^2)$ or θ(n) does not apply to every input.

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f(n) always dominates the growth of g(n) i.e. f(n) is of the order at least g(n) i.e. g(n) grows at the most as fast as f(n).

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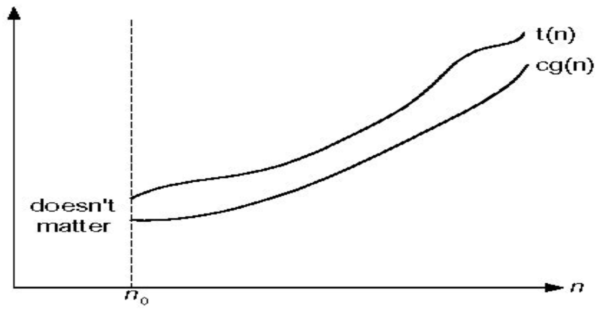

Fig. 2.2 Big-omega notation: $t(n) \in \Omega(g(n))$

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- Neither the big-O notation nor the big-Ω notation describe the asymptotically tight bounds.
- Θ-notation to express tighter bounds - used to specify the exact order of growth of functions.
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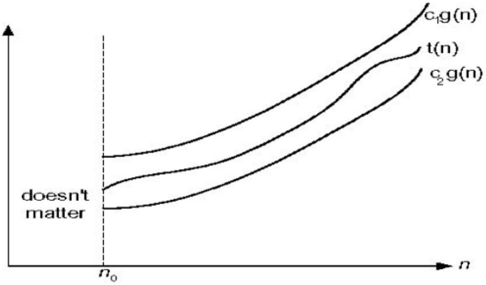

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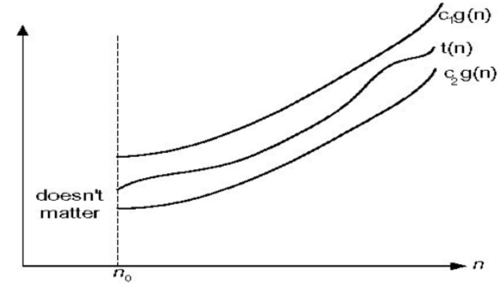

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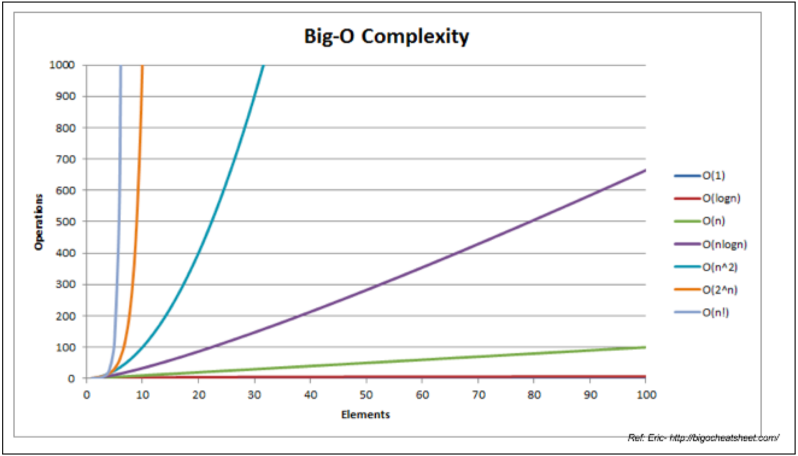
The Asymptotic Classes



Complexity of Data Structures

Ref: Eric- http://bigocheatsheet.co

Data Structure	Time Complexity								Space Complexity
	Average				Worst				Worst
	Access	Search	Insertion	Deletion	Access	Search	Insertion	Deletion	
Array	O(1)	O(n)	O(n)	O(n)	O(1)	O(n)	O(n)	O(n)	O(n)
Stack	O(n)	O(n)	O(1)	O(1)	O(n)	O(n)	O(1)	O(1)	O(n)
Singly-Linked List	O(n)	O(n)	O(1)	O(1)	O(n)	O(n)	O(1)	O(1)	O(n)
Doubly-Linked List	O(n)	O(n)	O(1)	O(1)	O(n)	O(n)	O(1)	O(1)	O(n)
Skip List	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(n)	O(n)	O(n)	O(n)	O(n log(n))
Hash Table	-	O(1)	O(1)	O(1)	-	O(n)	O(n)	O(n)	O(n)
Binary Search Tree	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(n)	O(n)	O(n)	O(n)	O(n)
Cartesian Tree	-	O(log(n))	O(log(n))	O(log(n))	-	O(n)	O(n)	O(n)	O(n)
B-Tree	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(n)
Red-Black Tree	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(n)
Splay Tree	-	O(log(n))	O(log(n))	O(log(n))	-	O(log(n))	O(log(n))	O(log(n))	O(n)
AVL Tree	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(log(n))	O(n)

Complexities of Sorting Algorithms

Algorithm	Time Complexity			Space Complexity
	Best	Average	Worst	Worst
Quicksort	O(n log(n))	O(n log(n))	O(n^2)	O(log(n))
Mergesort	O(n log(n))	O(n log(n))	O(n log(n))	O(n)
Timsort	O(n)	O(n log(n))	O(n log(n))	O(n)
Heapsort	O(n log(n))	O(n log(n))	O(n log(n))	O(1)
Bubble Sort	O(n)	O(n^2)	O(n^2)	O(1)
Insertion Sort	O(n)	O(n^2)	O(n^2)	O(1)
Selection Sort	O(n^2)	O(n^2)	O(n^2)	O(1)
Shell Sort	O(n)	O((nlog(n))^2)	O((nlog(n))^2)	O(1)
Bucket Sort	O(n+k)	O(n+k)	O(n^2)	O(n)
Radix Sort	O(nk)	O(nk)	O(nk)	O(n+k)

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  - represent each internal node by a_i:a_j in the range $1 \leq i \leq n$
  - denote each leaf by permutation ($\pi(1)$, $\pi(2)$, $\pi(3)$, $\pi(4)$.. $\pi(n)$)

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- Can we find out a lower bound on the height of the decision tree under consideration ?
- How does it relate to the running time of any comparison based sort ?

Lower Bound on Sorting...

Theorem

Any decision tree that sorts n-elements has height $\Omega(nlgn)$

Corollary

Heapsort and Mergesort are asymptotically optimal comparison sorts.

The Counting Sort

- Let input instance A=<7 1 3 1 2 4 5 7 2 4 3>

A

1	2	3	4	5	6	7	8	9	10	11
7	1	3	1	2	4	5	7	2	4	3

- `C[A[j]] = C[A[j]] + 1`

C

1	2	3	4	5	6	7
2	2	2	2	1	0	2

- `C[i] = C[i] + C[i-1]`

1	2	3	4	5	6	7
2	4	6	8	9	0	11

- `B[C[A[j]]] = A[j]`

The Counting Sort ...

Algorithm CountingSort(A[i],n)

```
for i=1 to k
        do C[j] =0
for j = 1 to length[A]
        do C[A[j]] = C[A[j]] + 1
for i=2 to k
        do C[i] = C[i] + C[i-1]
for j=length[A] downto 1
        do B[C[A[j]]] = A[j]
                C[A[j]] = C[A[j]] - 1
```

What is the time complexity of this sort ?

The Interger Sorting Algorithms

- Integer sorting
  - sorting a collection of data values by numeric keys, each of which is an integer.
  - the ability to perform integer arithmetic on the keys allows integer sorting algorithms to be faster than comparison sorting algorithms in many cases, depending on the details of which operations are allowed in the model of computing and how large the integers to be sorted are.
  - the classical integer sorting algorithms of bucket sort, counting sort, and radix sort

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