

Chapter 3: Greedy Algorithm Design Technique - II

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Design and Analysis of Algorithms
IIT Jammu, Jammu

1 Review of Essential Graph Basics

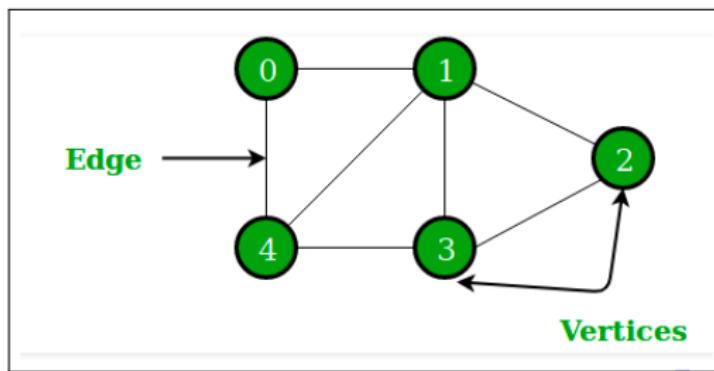
2 Minimum Spanning Trees

3 Applications of MST approach

Revisiting a Graph

Definition

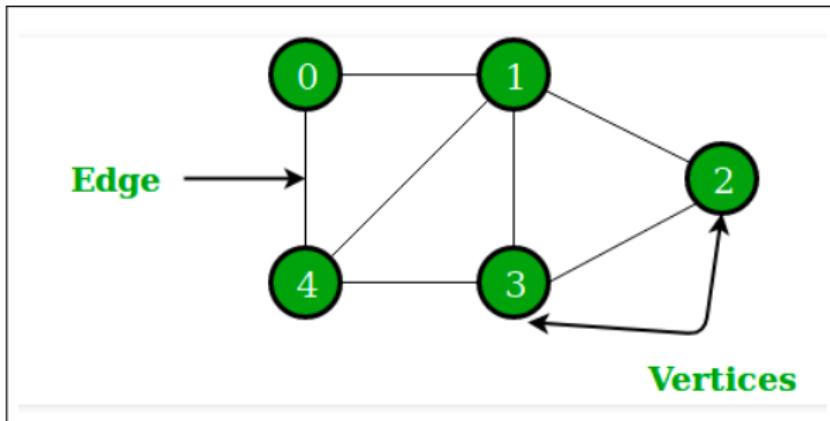
- informally, a set of vertices (nodes) and edges connecting them.
- a graph $G = (V, E)$ where, V is a set of vertices i.e. $V = v_i$ with an edge $e = v_i, v_j$ connecting two vertices with each $e \in E$ to a set of all such edges between two vertices of the graph viz. $E = (v_i, v_j)$. An example graph $G=(V,E)$ shown here



Graph attributes

A Path

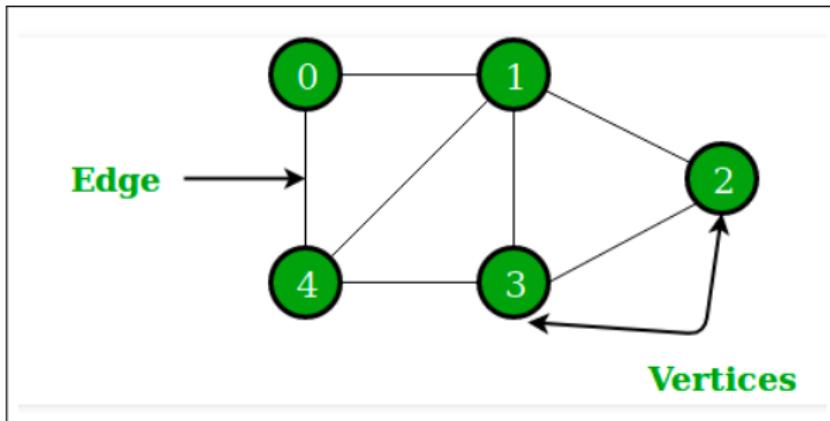
- A Path p of length k , is a sequence of connected vertices $p = \langle v_0, v_1, \dots, v_k \rangle$ where $(v_i, v_{i+1}) \in E$



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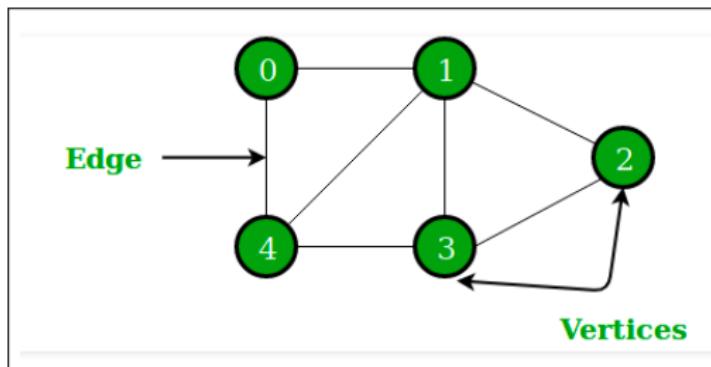
- A Path p of length k , is a sequence of connected vertices $p = < v_0, v_1, \dots, v_k >$ where $(v_i, v_{i+1}) \in E$
- in the figure given here various paths are :



Graph attributes : Path, Path-length

A Path and path-length

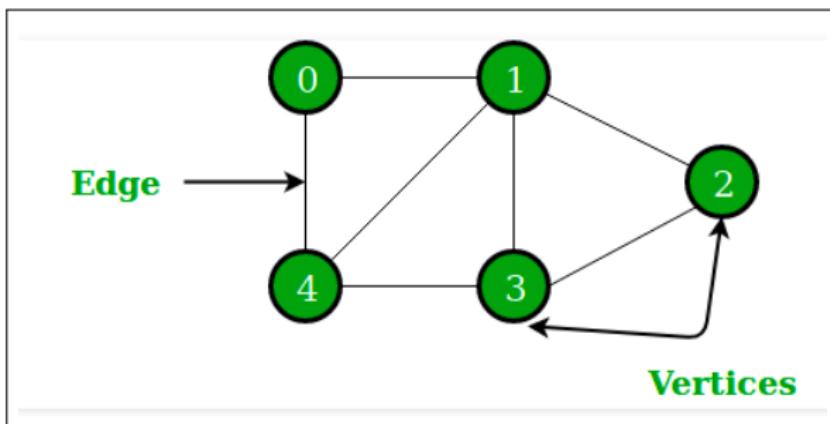
- Path-length is the number of vertices in a graph e.g. the path lengths in this graph are:



Graph attributes: Cycle

A Cycle

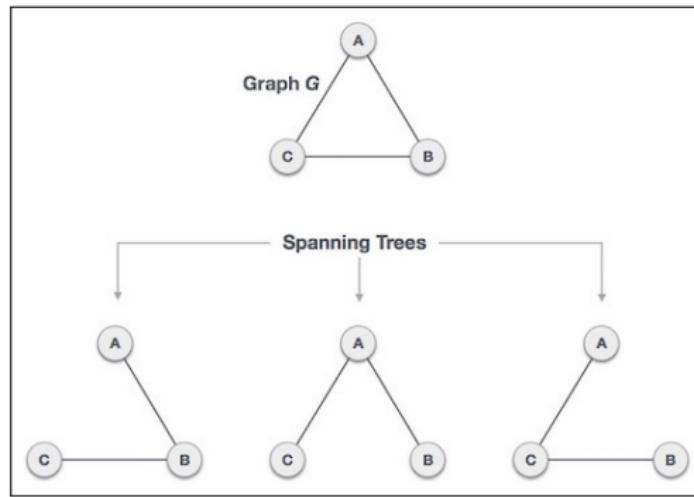
- Cycle: A graph contains no cycles if there is no path $p = \langle v_0, v_1, \dots, v_k \rangle$ such that $v_0 = v_k$ e.g. in the graph shown one of the cycles is :



Graph attributes : A Spanning Tree

A Spanning Tree

- A Spanning Tree is a set of $|V| - 1$ edges that connect all the vertices of a graph.



Graph attributes : A Spanning Tree...

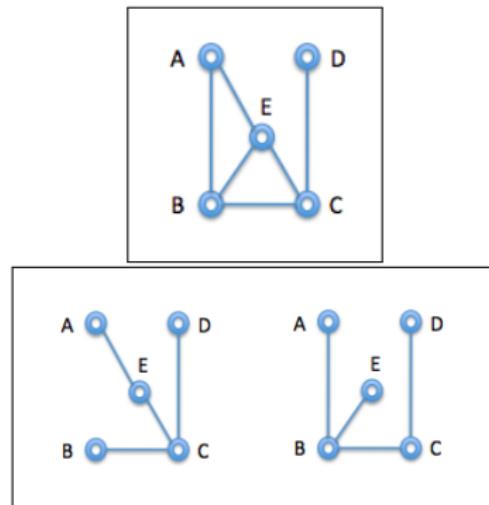


Figure: Another illustration of Spanning Tree

Graph attributes : A Spanning Tree...

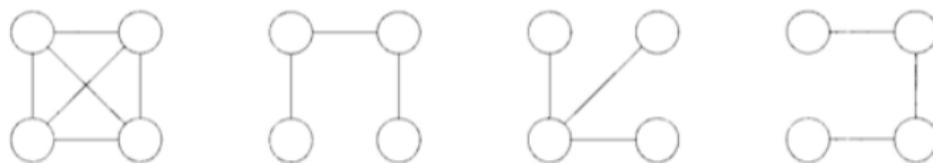


Figure 4.5 An undirected graph and three of its spanning trees

Figure: Another illustration of Spanning Tree

Graph attributes : A Spanning Tree...



A spanning tree of a [graph](#) on n vertices is a subset of $n - 1$ edges that form a [tree](#) (Skiena 1990, p. 227). For example, the spanning trees of the [cycle graph](#) C_4 , [diamond graph](#), and [complete graph](#) K_4 are illustrated above.

Figure: Another illustration of Spanning Tree

Properties of a Spanning Tree

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- Spanning trees are a subset of connected Graph G and disconnected graphs do not have spanning tree.



Sparse and Dense Graphs

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By comparison, a tree is sparse because $|E| = n - 1 = O(n)$.

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- That is $|E| = n^2$

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Constructing of a Spanning Tree

Algorithm1: Node-centric ST Algorithm

- Pick an arbitrary node and mark it as being in the tree.
- We iterate (n_1) times in Step 2, because there are (n_1) vertices that have to be added to the tree.
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 - Add e to the spanning tree and mark w as in the tree.
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Constructing of a Spanning Tree

Algorithm2: Edge-centric ST Algorithm

- 1 Start with the collection of singleton trees, each with exactly one node.

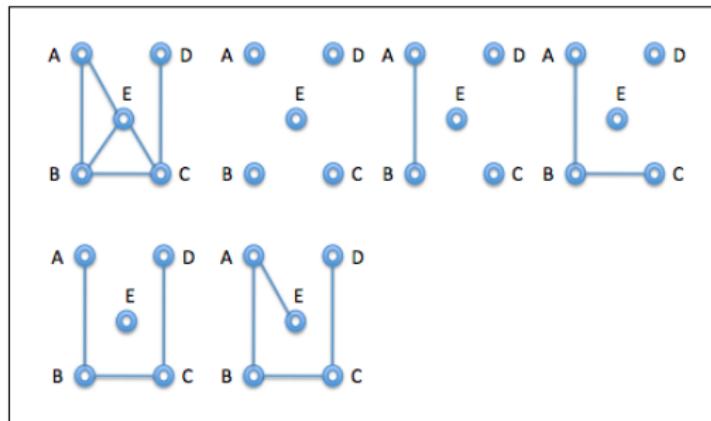


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Constructing of a Spanning Tree

Algorithm2: Edge-centric ST Algorithm

- ① Start with the collection of singleton trees, each with exactly one node.
- ② As long as we have more than one tree, connect two trees together with an edge in the graph.

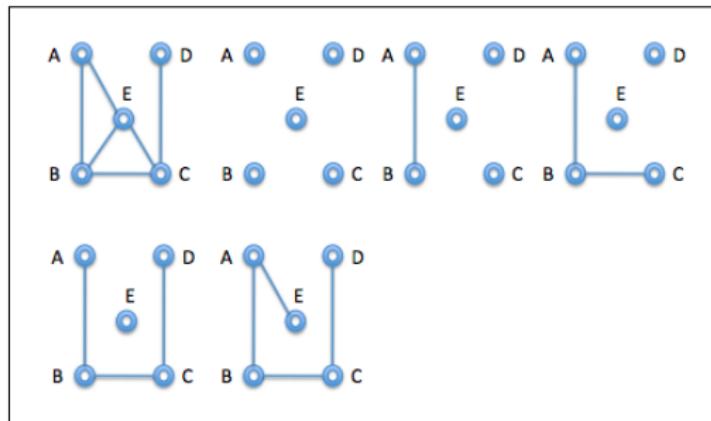


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Minimum Spanning Tree of a Graph

Definition

If a cost c_{ij} is associated with an edge $e_{ij} = (v_i, v_j)$ then the minimum spanning tree is the set of edges E_{span} such that
 $C = \sum_{\text{for all } e_{ij} \in E_{span}} c_{ij}$ is a minimum

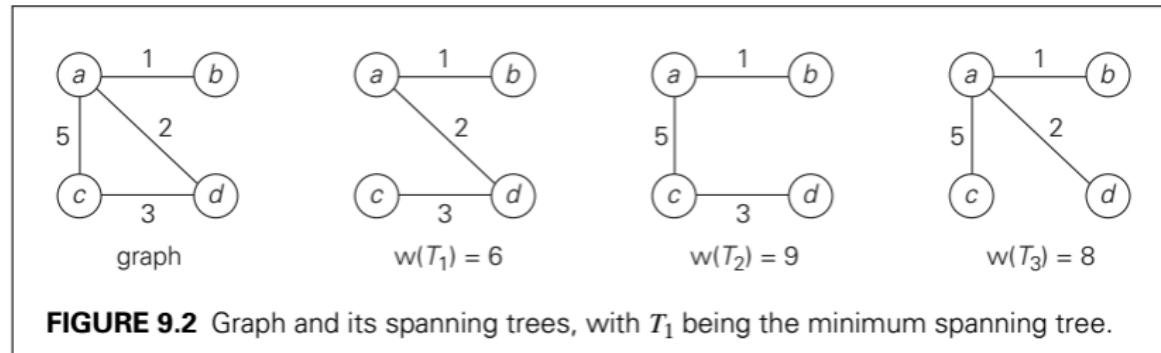


FIGURE 9.2 Graph and its spanning trees, with T_1 being the minimum spanning tree.

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- The approach uses the concepts of **safe edges**, **light edges** and **a cut** of a graph
- Let us investigate these concepts.

Safe edge and Generic MST approach

Safe edge

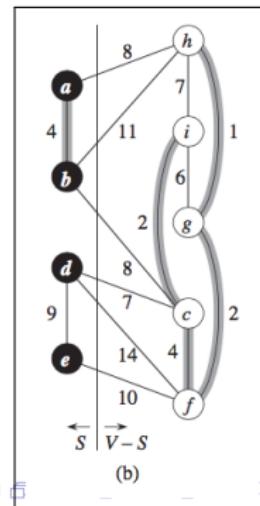
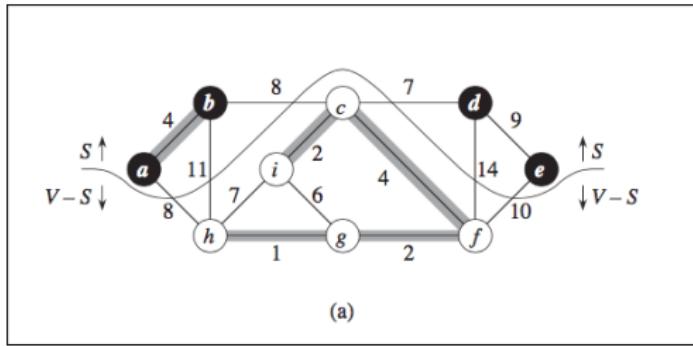
Given the loop invariant that *Prior to each iteration, A is a subset of some minimum spanning tree at each step*, we determine an edge (u, v) that can be safely added to the set A - we call such an edge a **safe edge for A**, since we can add it safely to A while maintaining the invariant.

Algorithm GENERIC-MST(G, w)

- 1 $A = \emptyset$
- 2 while A does not form the spanning tree
- 3 find an edge (u, v) that is safe for A
- 4 $A = A \cup \{(u, v)\}$
- 5 return A

Definitions

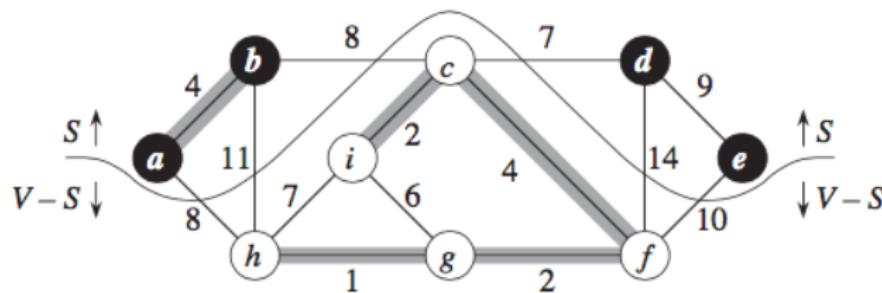
Cut of a graph : A cut $(S, V-S)$ of an undirected graph $G = (V, E)$ is a partition of the nodes of the graph V in such a way that an edge $(u,v) \in E$ crosses the cut $(S, V-S)$ if one of its endpoints is in S and the other endpoint is in $V-S$. We say, a cut respects a set A of edges if no edge in A crosses the cut.



Theorem: Determining Safe edge

Theorem

Let $G=(V,E)$ be a connected, undirected graph with a real-valued weight function w defined on E . Let A be a subset of E that is included in some minimum spanning tree for G , let $(S, V-S)$ be any cut of G that respects A , and let (u,v) be a light edge crossing $(S, V-S)$. Then, edge (u,v) is safe for A .



(a)

How to find a safe edge ?

Determining a safe edge

- Calculate the minimum spanning tree
- Put all the vertices into single node trees by themselves
- Put all the edges in a priority queue
- Repeat until we have constructed a spanning tree
- Extract the cheapest edge
- If it forms a cycle, ignore it, else add it to the forest of trees -
(it will join two trees into a larger tree)
- Return the spanning tree

Pseudocode: How to find a safe edge ?

```
Algorithm MinimumSpanningTree( Graph G(V,E) ,  
                               Weight_function w)  
1. for each vertex v ∈ V[G]  
2.     Construct a single-vertex forest from G  
3. Sort the edges of E into  
               nondecreasing order by weight w ∈ E  
4. for each edge (u,v)  
5.         do extract the cheapest edge  
6.                 add it to A  
7.                 while (it does not form a Cycle)  
8.             return A;
```

How to find detect a cycle ?

Cycle detection

- Uses a Union-find structure
- Union-Find data structure is based on partitioning of a set
- How do we view Partition formally ?
- Partition is a set of subsets of elements of a set where
 - every element belongs to one of the sub-sets
 - no element belongs to more than one sub-set
 - that is, given that Set, $S = s_i$,
 - $\text{Partition}(S) = P_i$, where $P_i = s_i$
 - $\forall s_i \in S_i, s_i \in P_i$
 - $\forall j, k, P_j \cap P_k = \emptyset$
 - $S = \cup P_j$

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Equivalence Class

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 - symmetric i.e. if $x \sim y$ then $y \sim x$
- Each element of the set is related to every other element

Applying Partitions and Union Find in MST

In a typical MST algorithm

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- Initially, each vertex is in a class by itself
- As edges are added, more vertices become related and the equivalence classes grow, until finally all the vertices are in a single equivalence class

Applying Partitions and Union Find in MST

Representatives of Equivalence Classes

- One vertex in each class may be chosen as the representative of that class

Now we are ready to apply these concepts for cycle determination

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- This is the union-find structure

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 - if the two representatives are equal, then the edge will form a cycle

Applying Union-find structure to find MST

Let us apply the concept of equivalence classes to determine MST in the given graph

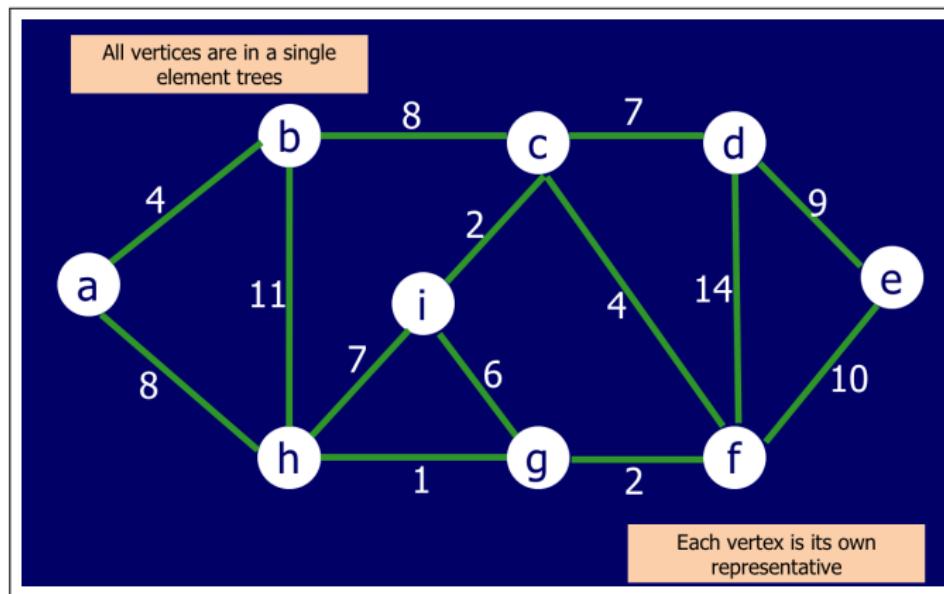


Figure: MST determination (a)

Applying Union-find structure to find MST...

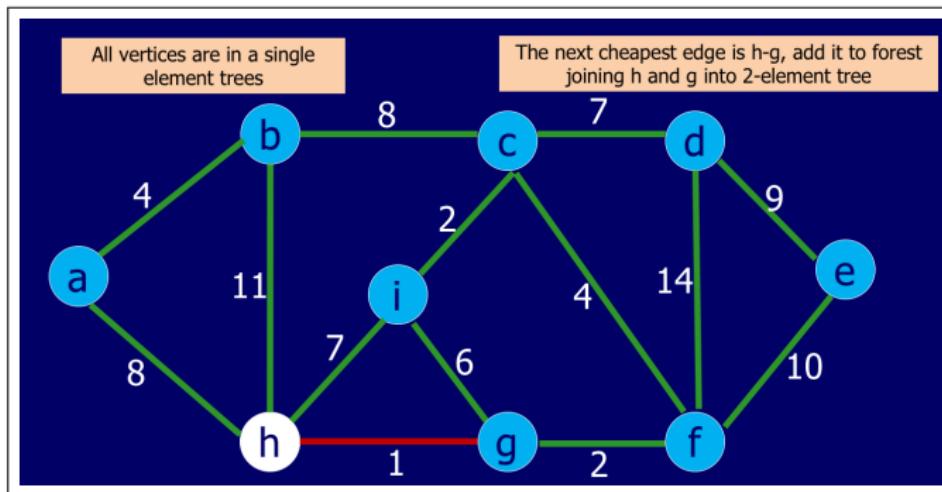


Figure: MST determination (b)

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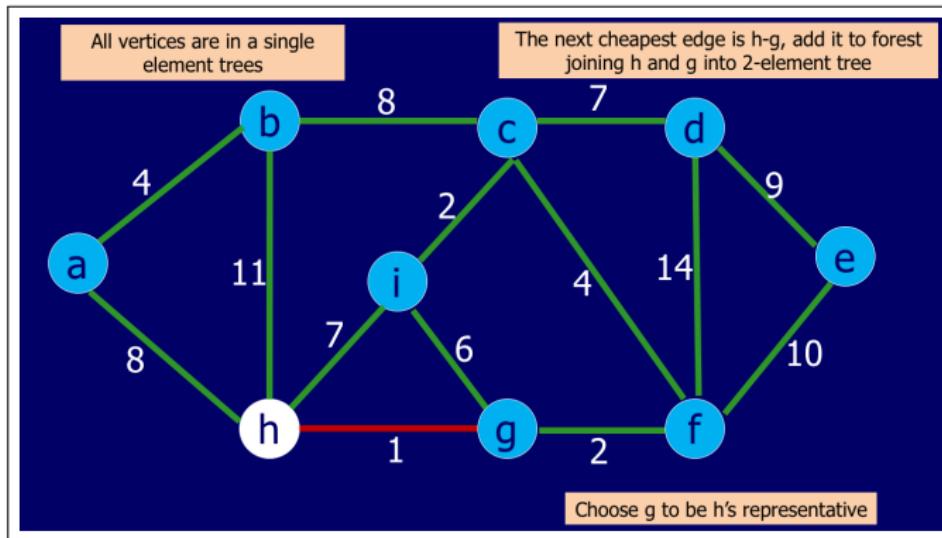


Figure: MST determination (c)

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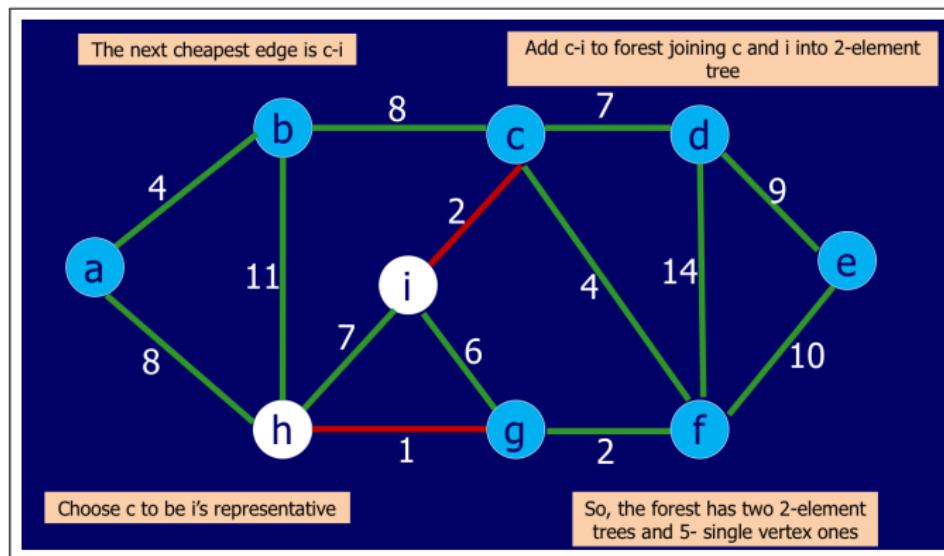


Figure: MST determination (d)

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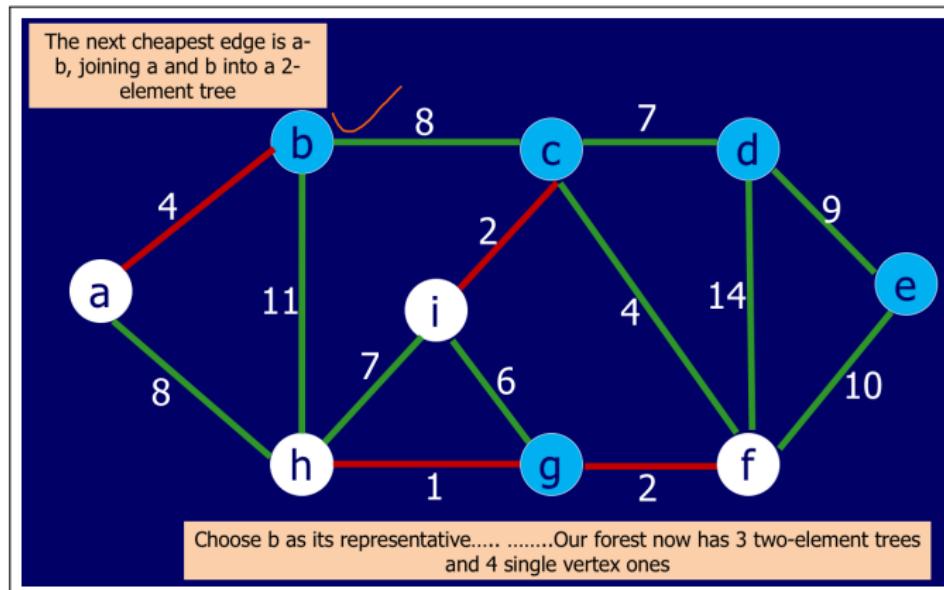


Figure: MST determination (e)

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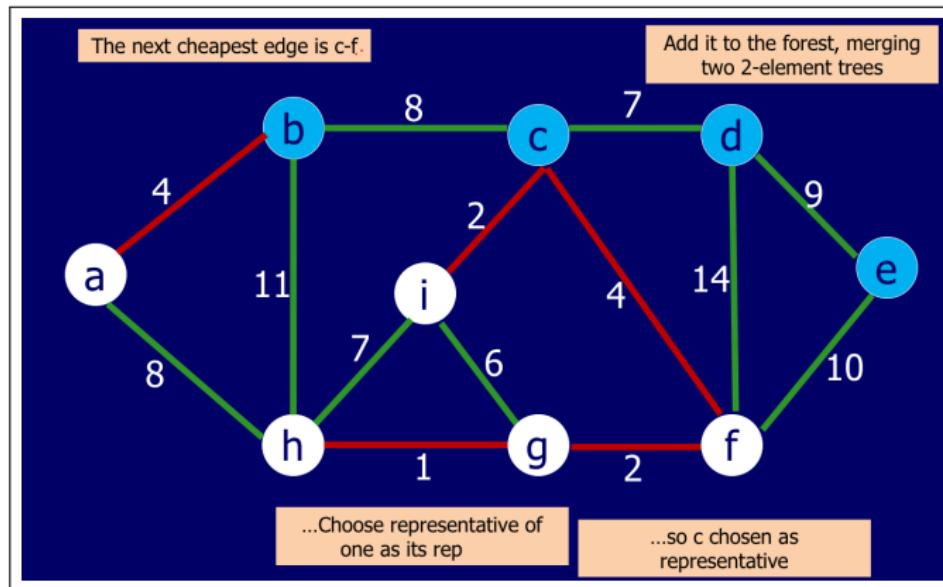


Figure: MST determination (f)

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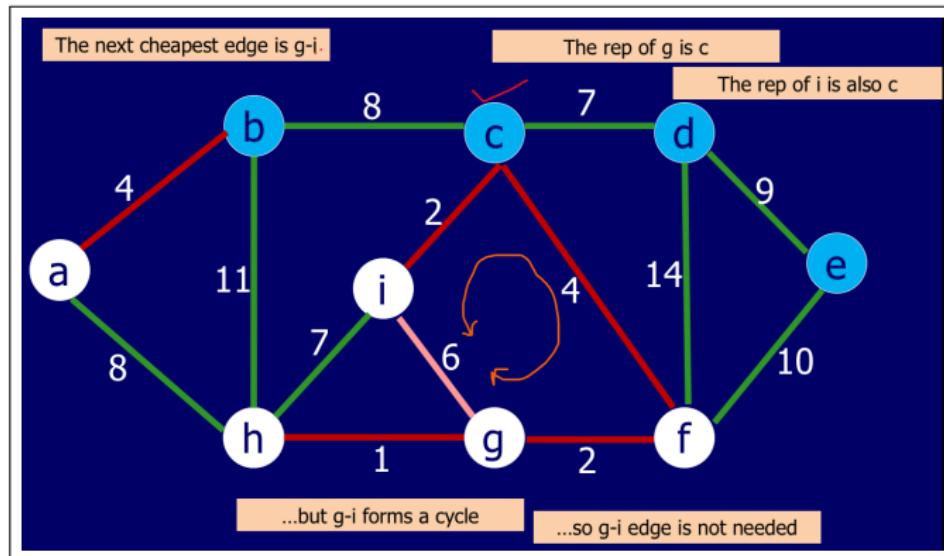


Figure: MST determination (g)

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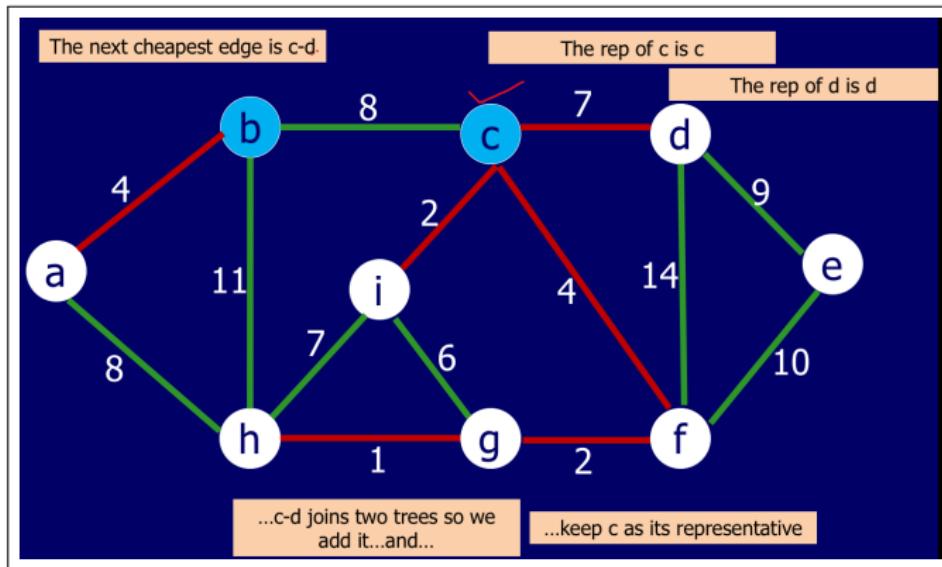


Figure: MST determination (h)

Applying Union-find structure to find MST...

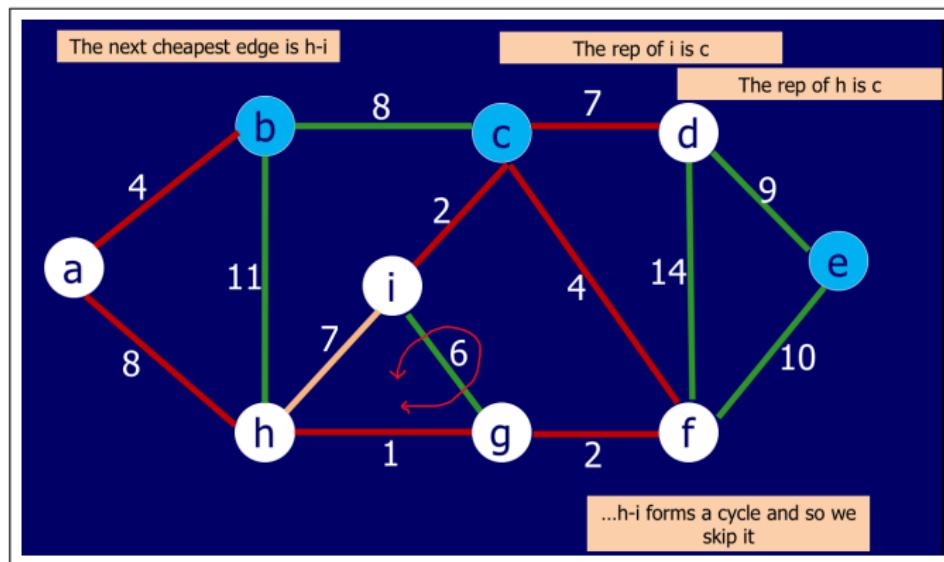


Figure: MST determination (i)

Applying Union-find structure to find MST...

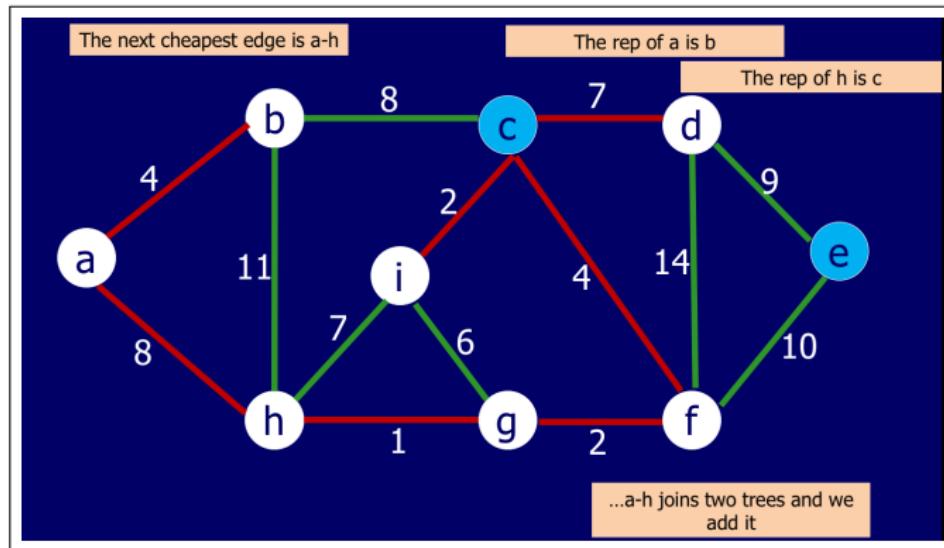


Figure: MST determination (j)

Applying Union-find structure to find MST...

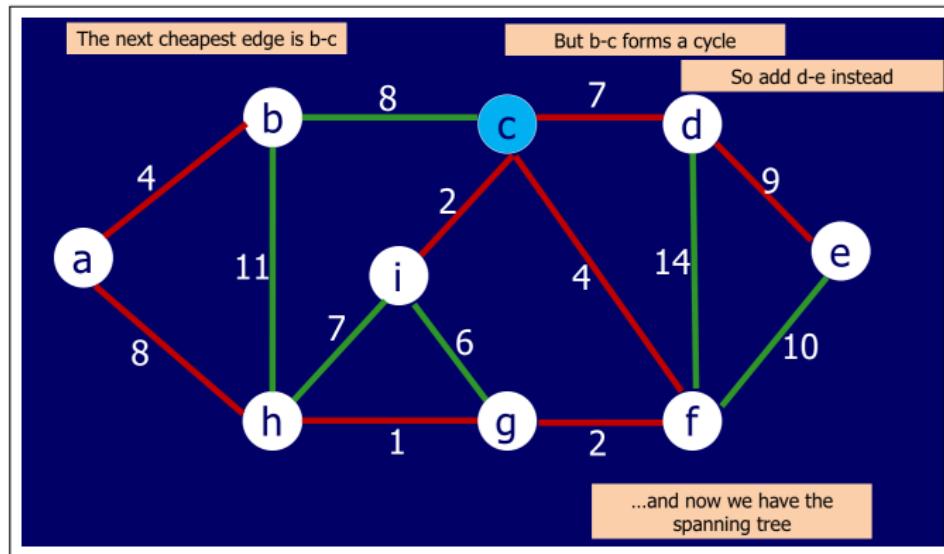


Figure: MST determination (k)

Applying Union-find structure to find MST...

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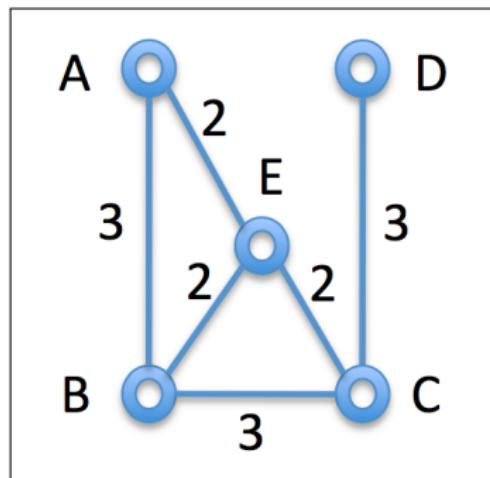


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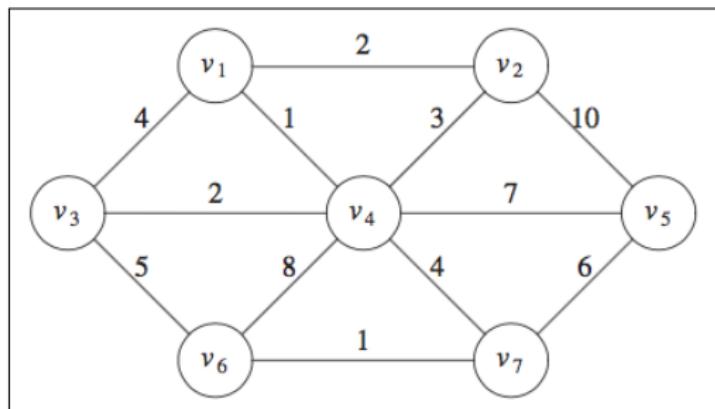
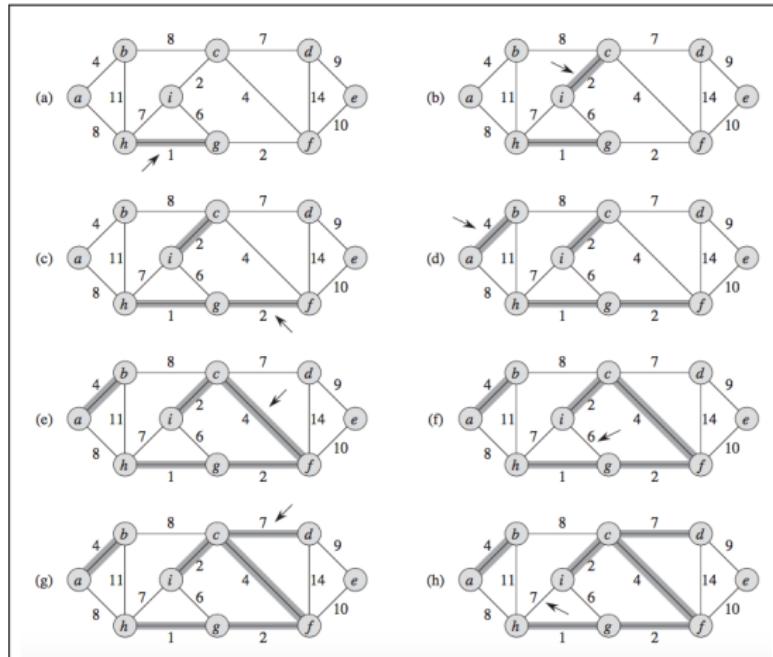


Figure: MST determination (k)

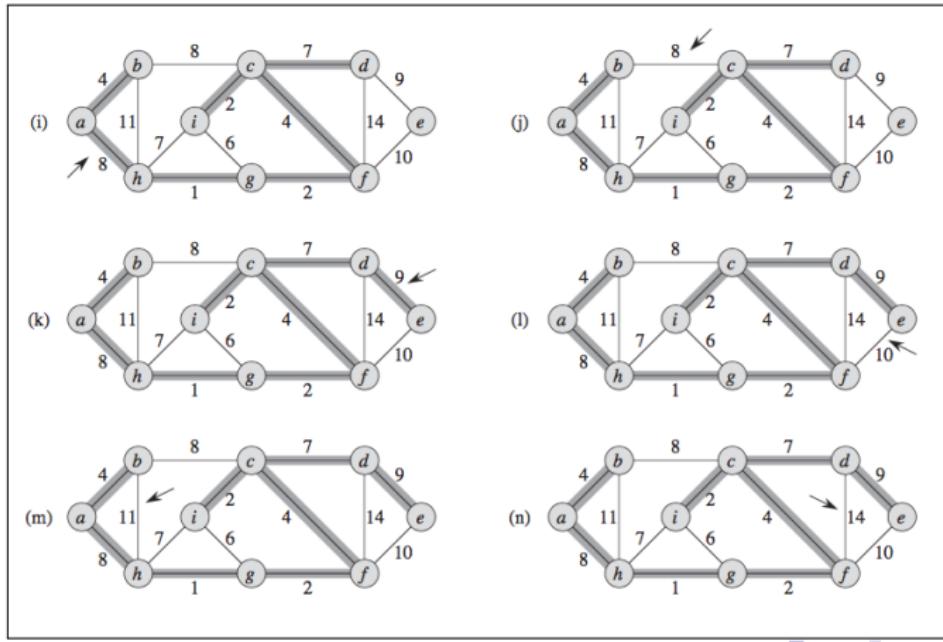
Applying Union-find structure to find MST...

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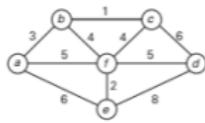
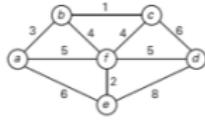
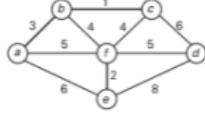
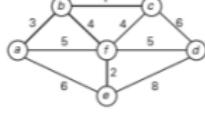
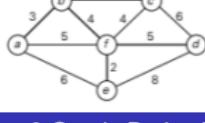


Applying Union-find structure to find MST...

Illustrate the algorithm discussed on the following graph to compute its MST



Applying Union-find structure to find MST...

Tree edges	Sorted list of edges	Illustration
bc 1	bc ef ab bf cf af df ae cd de 1 2 3 4 4 5 5 6 6 8	
ef 2	bc ef ab bf cf af df ae cd de 1 2 3 4 4 5 5 6 6 8	
ab 3	bc ef ab bf cf af df ae cd de 1 2 3 4 4 5 5 6 6 8	
bf 4	bc ef ab bf cf af df ae cd de 1 2 3 4 4 5 5 6 6 8	
af 5	bc ef ab bf cf af df ae cd de 1 2 3 4 4 5 5 6 6 8	

Kruskal's Algorithm

Algorithm Kruskal(Graph G(V,E), Weight_function w)

1. $A = \emptyset$
2. for each vertex $v \in V[G]$
3. do MAKE-SET(v)
4. Sort the edges of E into nondecreasing order
 by weight w
5. for each edge $(u,v) \in E$,
6. if FIND-SET(u) \neq FIND-SET(v)
7. $A = A \cup \{(u,v)\}$
8. UNION(u,v)
9. return A

Proving Kruskal's Algorithm

Proof by contradiction

- Note that any edge creating a cycle is not needed

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- The cheapest way to do this is to add e_x - because e_x is the cheapest edge
- So we should have added e_x instead of e_z
- This proves that the greedy approach is correct for MST

Theorems Associated

Lemma

Let F be a forest, that is any undirected acyclic graph. Let $e = (v, w)$ be an edge that is NOT in F . Then, there is a cycle in F , consisting of edges of F and edge e , iff v and w are in the same connected component of F .

Theorem

Let $G = (V, E)$ be a connected, undirected graph with a real-valued weight function w defined on the set of edges E . Let A be a subset of E that is included in some MST for G . Let $(S, V-S)$ be a cut of G that respects A and let (u, v) be a light edge crossing $(S, V-S)$. Then, edge (u, v) is safe for A .

Theorems Associated

Theorem

Let $G=(V,E)$ be a connected, undirected graph and $G'=(V',E')$ be a partial graph formed by the nodes of G and the edges in E' . Let there be n nodes in V . Then, prove that the set E' with n or more edges cannot be optimal. Also, prove that E' must have exactly $(n-1)$ edges and so E' must be a tree.

Theorem

Prove that Kruskal's algorithm is correct and it finds its MST.

Tutorial Assignment Proofs

Theorem

- Let T be a MST for a graph G , let e be an edge in T and let T' be T with e removed. Show that e is a minimum weight edge between components of T' .
- Comment on the validity of the statement. If all the weights in G are distinct, distinct spanning trees of G have distinct weights. Give a counterexample.
- Let T and T' be two STs of a connected graph G . Suppose that an edge e is in T but not in T' . Show that there is an edge e' in T' , but not in T , such that $(T - e \cup e')$ and $(T' - e' \cup e)$ are STs of G .

Time Complexity

Steps

- Initialise forest $O(|V|)$
 - Sort edges $O(|E|\log|E|)$
 - Check edge for cycles $O(|V|) \times$
 - Number of edges $O(|V|)$ i.e. $O(|V|^2)$
 - Total $O(|V| + |E|\log|E| + |V|^2)$
 - Since $|E| = O(|V|^2)$
Total = $O(|V|^2 \log|V|)$
-
- Thus, Kruskal's algorithm is tagged as $O(n^2 \log n)$ algorithm for a graph of n vertices
 - This is an upper bound, some improvements on this are known...
 - Prim's algorithm can be $O(|E| + |V|\log|V|)$ using Fibonacci heaps.
 - Even better variants are known for restricted cases, such as sparse graphs ($|E| \approx |V|$)

Prim's Algorithm

PRim's Algorithm

Approach

- Follows the natural greedy approach starting with the source vertex to create the spanning tree,
- add an edge to the tree that is attached at exactly one end to the tree & has minimum weight among all such edges.
- Prim's algorithm starts from one vertex and grows the rest of the tree an edge at a time.
- As a greedy algorithm, which edge should we pick?

Prim's Algorithm

Points to note

- In Kruskal, the selection function uses a greedy approach, i.e. chooses an edge that is minimum weighted edge, then in increasing order.

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 - the result is a sort of forest of trees that grow haphazardly and later merge into a tree.
- As compared in Prim's the MST grows in a natural manner, starting from an arbitrary root.
- At each, a new edge is added to a tree already constructed.

PRim's Algorithm

Approach

- Consider B - a set of nodes, T - a set of edges
- initially B contains a single arbitrary node
- at each step, Prim's algorithm looks for the shortest edge $\{u,v\}$ such that $u \in B$ and $v \in N - B$.
- then it adds v to B and $\{u, v\}$ to T .
- in this way, in T at any instant an MST for the nodes in B is formed.

Prim's Algorithm

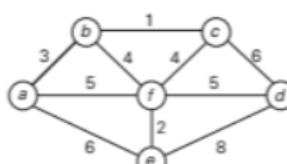
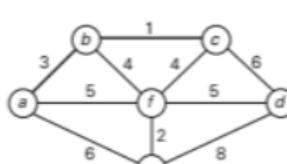
```
Algorithm Prim(Graph G(V,E), Weight_function w)
{initialization}
1. T  $\in \phi$ 
2. B  $\leftarrow$  {an arbitrary member of V}
3. while B  $\neq$  N do
4.     find E={u, v} of minimum weight such that
        u  $\in$  B and v  $\in$  V-B
5.     T  $\leftarrow$  T  $\cup$  {e}
6.     B  $\leftarrow$  B  $\cup$  {v}
7. return T
```

How to determine the edge with the minimum weight ?

Prim's Algorithm Refined

```
Algorithm Prim(Graph G(V,E), Weight-function w, r)
/* the connected graph G and the root r of the minimum
spanning tree to be grown are inputs to the algorithm.*/
1. for each  $u \in G.V$ 
2.      $u.key = \infty$ 
3.      $u.\pi = NIL$ 
4.  $r.key = 0$ 
/* During execution of the algorithm, all vertices that
are not in the tree reside in a min-priority queue Q
based on a key attribute.*/
5.  $Q = G.V$ 
6. while  $Q \neq \emptyset$ 
7.      $u = \text{EXTRACT-MIN}(Q)$ 
8.     for each  $v \in G.\text{Adj}[u]$ 
9.         if  $v \in Q$  and  $w(u,v) < v.key$ 
10.             $v.\pi = u$ 
11.             $v.key = w(u,v)$ 
```

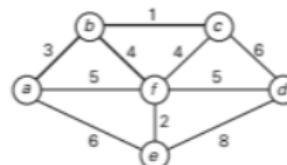
Applying Prim's Algorithm...

Tree vertices	Remaining vertices	Illustration
a(-, -)	b(a, 3) c(-, ∞) d(-, ∞) e(a, 6) f(a, 5)	 A graph with six vertices labeled a through f. Vertex a is the root. Edges and weights: a-b (3), a-e (6), b-f (1), b-c (4), f-e (2), f-d (5), c-d (6), c-e (4), d-e (8). Vertices a and b are in the tree.
b(a, 3)	c(b, 1) d(-, ∞) e(a, 6) f(b, 4)	 The same graph as above, but now vertex b is shaded, indicating it is the current vertex being considered for the next step in the algorithm.

Applying Prim's Algorithm...

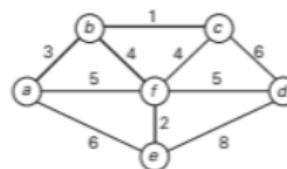
c(b, 1)

d(c, 6) e(a, 6) f(b, 4)



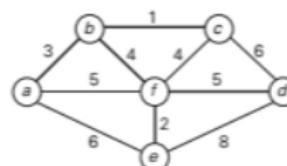
f(b, 4)

d(f, 5) e(f, 2)



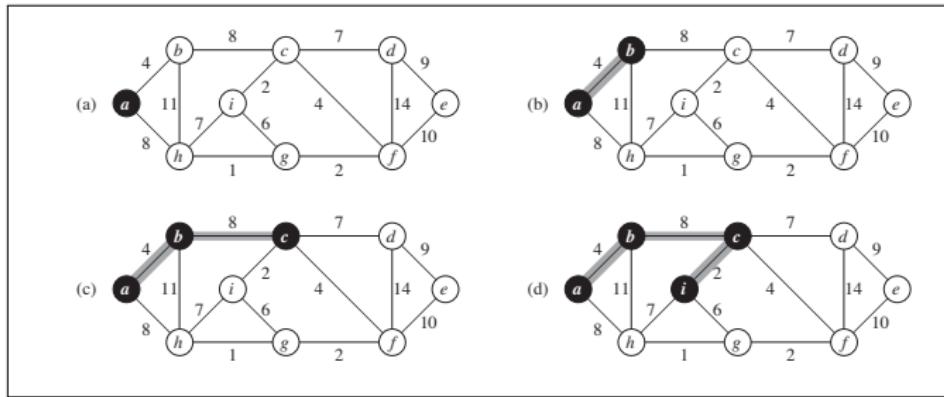
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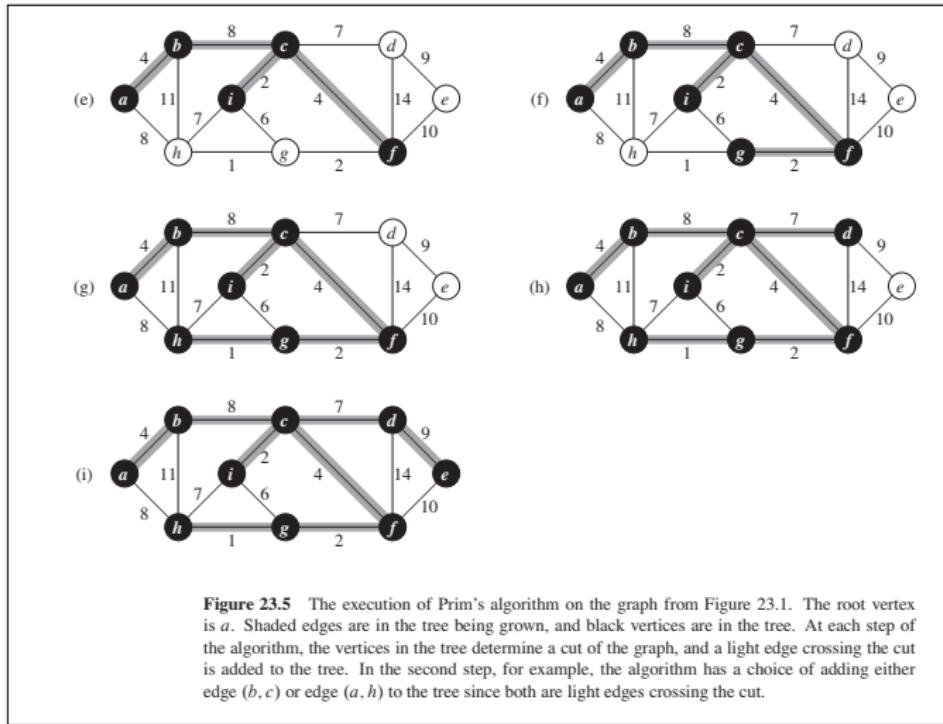


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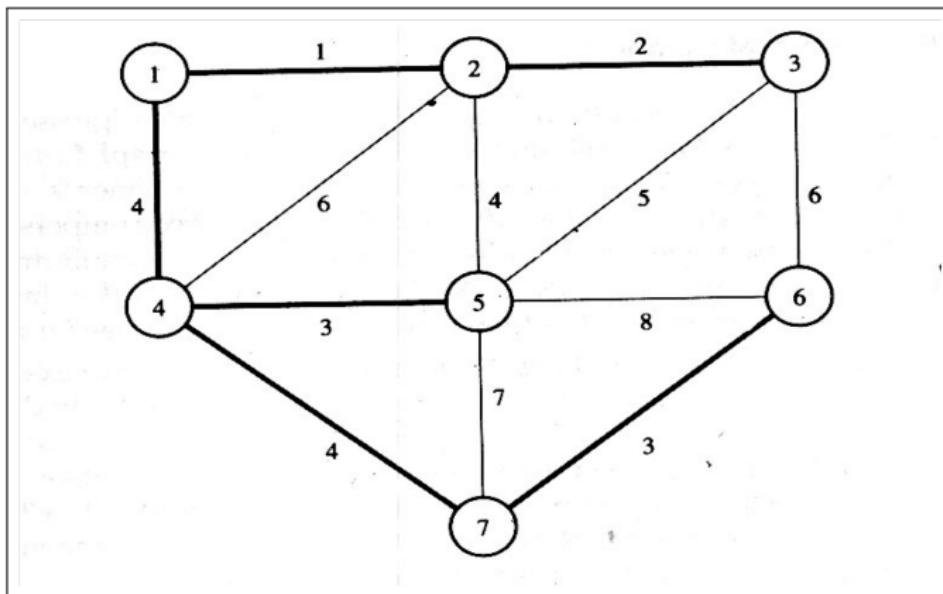
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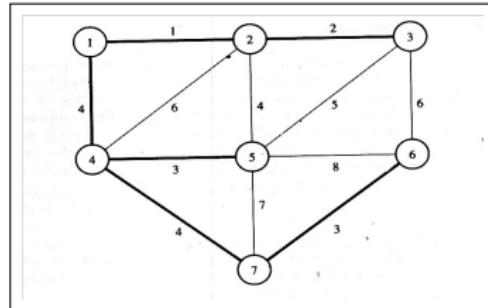
Applying Prim's Algorithm...



Comparing Kruskal and Prim by applying them

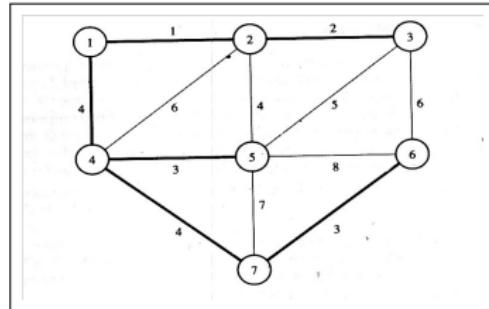


Running Prim's on the given graph



Step	Edge Considered	Connected Components
Init	-	{1}
1	{1 2}	{1 2}
2	{2 3}	{1 2 3}
3	{1 4}	{1 2 3 4 }
4	{4 5}	{1 2 3 4 5 }
5	{4 7}	{1 2 3 4 5 7}
6	{5}	Rejected
7	{4 7}	{1 2 3 4 5 6 7}

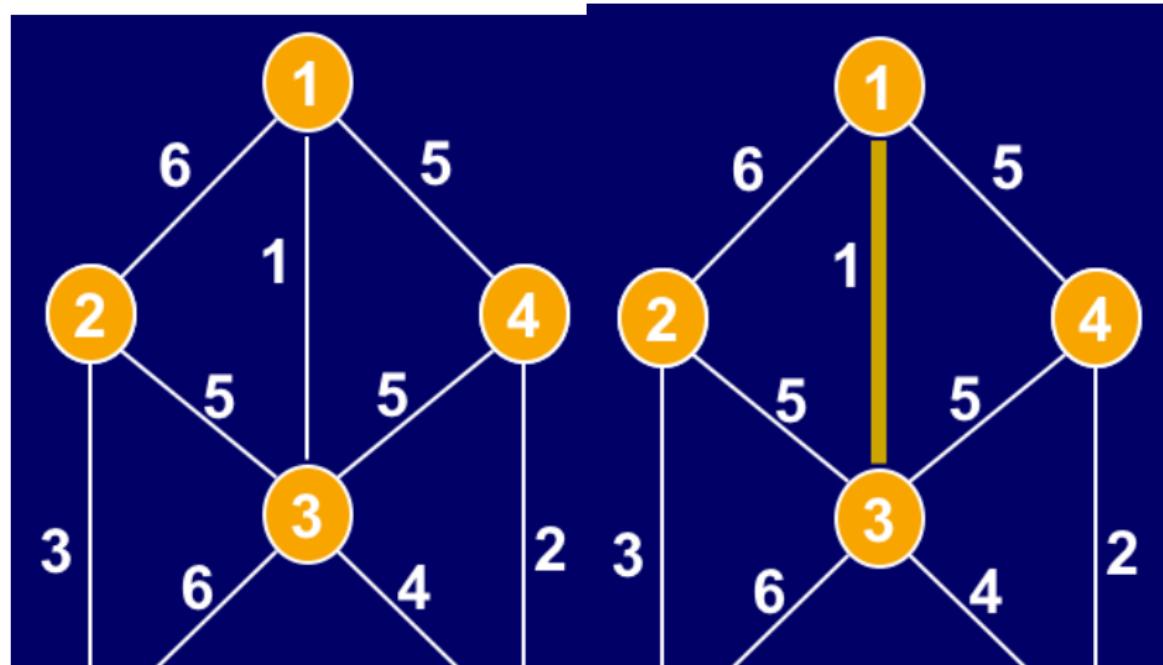
Running Kruskal's on the given graph



Step	EdgeConsidered	ConnectedComponents
Init	-	{1} {2} {3} {4} {5} {6} {7}
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3	{4 5}	{1 2 3} {4 5} {6} {7}
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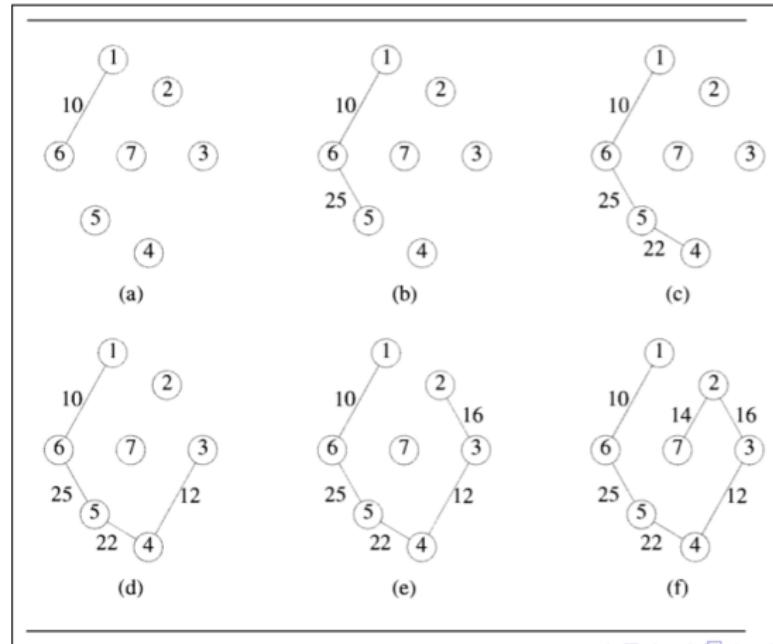
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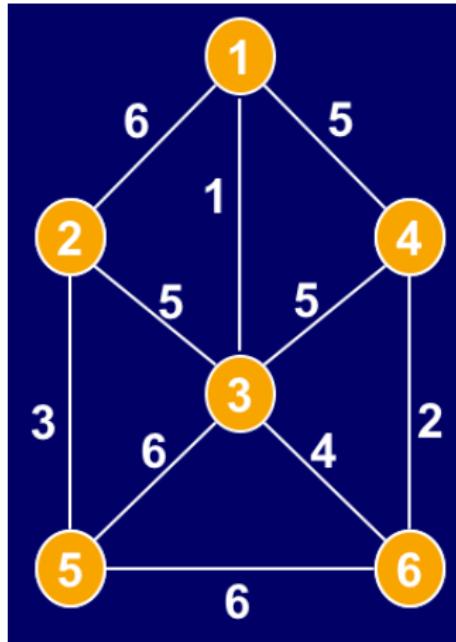


Figure: Various iterations of Prim

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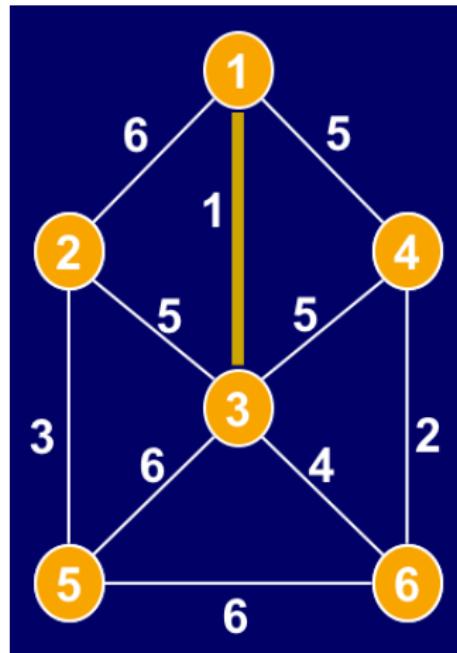


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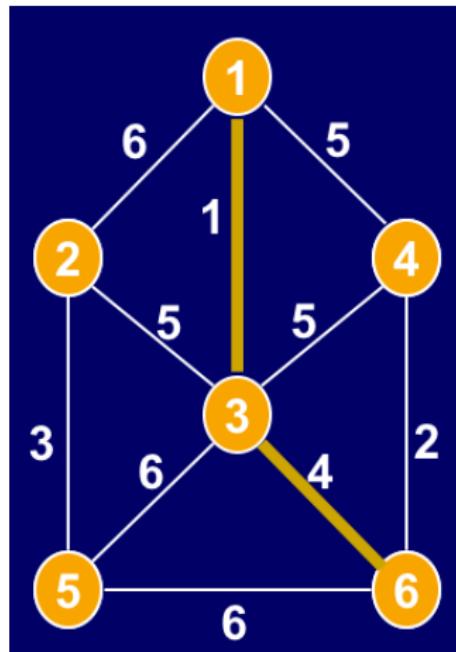


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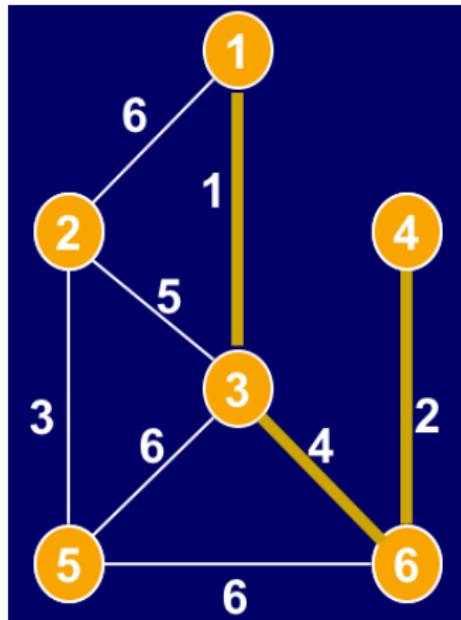


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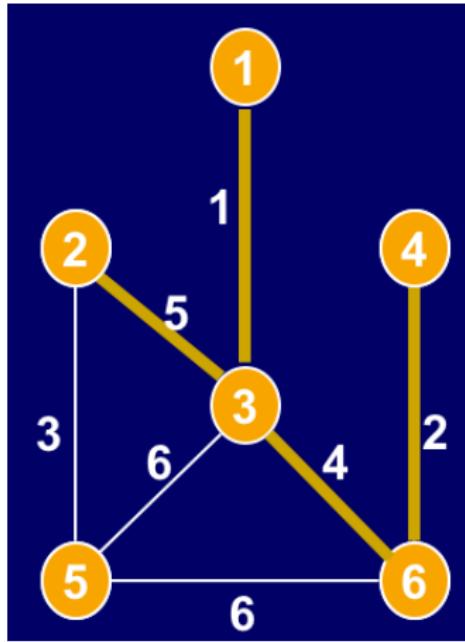


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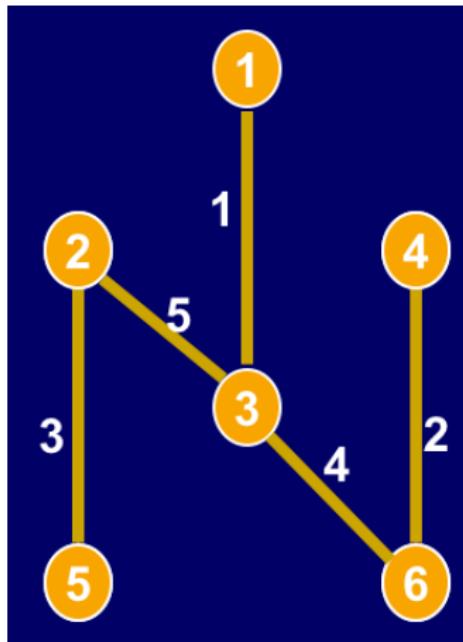
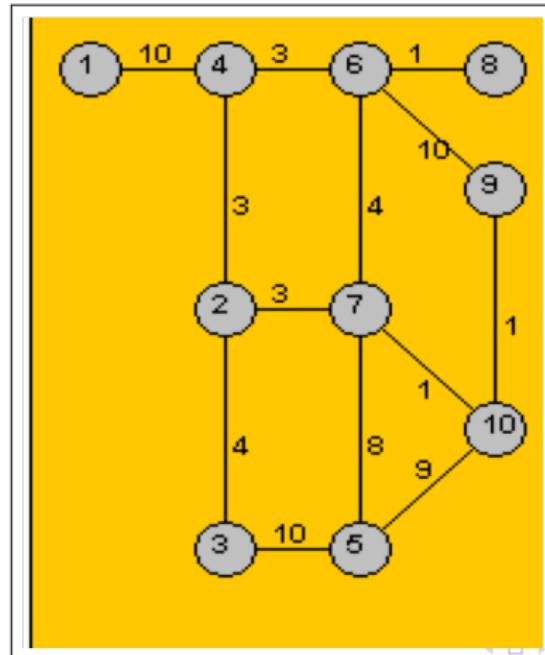


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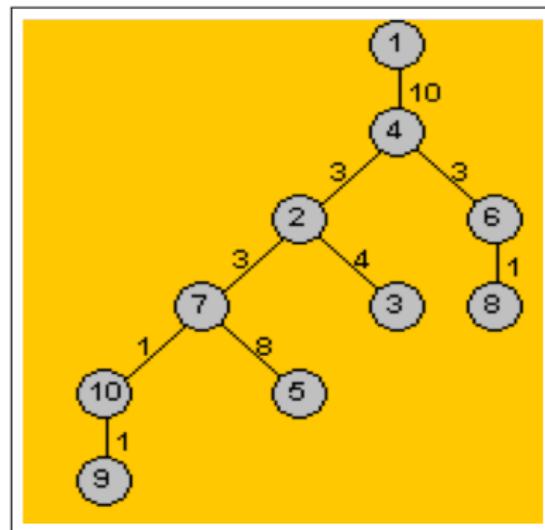
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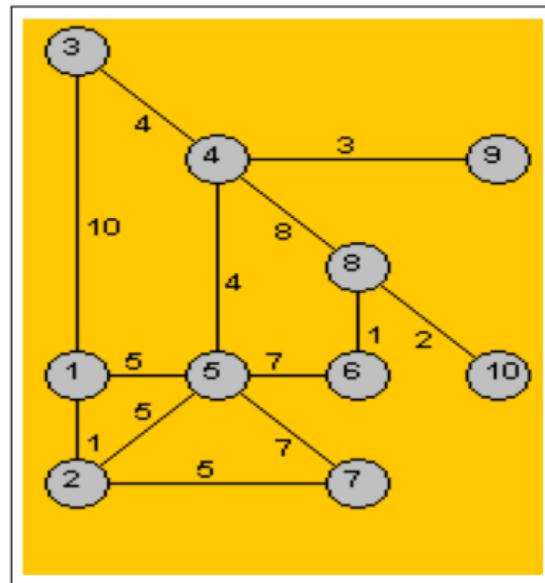
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Answer...



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- Therefore, Prim takes $O(V \lg V + E \lg V)$ time.

Application of MST approach : The K-clustering Problem

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 - point-to-point distance between the edges it has, is also the minimum ?
 - sometimes, one is willing to pay more for the total cost but reduce the cost between specific nodes

Issues with the MST algorithms...

- Alternately, does it ensure that
 - the congestion on the edges is minimized i.e. given traffic that needs to be routed between pairs of nodes
 - one desires a ST in which no single edge carries more than a certain amount of traffic
- the network design problem is mitigated wholly?
 - a tree has the property that destroying any edge disconnects it
 - is not robust against any failures
 - what if network resilience is the explicit goal at the same time, seeking the cheapest connected network on the set of sites that remain connected even after deletion of one site ?
- All these lead to various variations

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- So, here the objective function for clustering is *maximizing spacing*.
- the spacing of a set of clusters as the distance between the closest pair of points in different clusters.
- For a good clustering, one expects the spacing to be large; hence, maximizing the spacing seems like a reasonable objective function for clustering.

The K-Clustering Problem...

- The distance may not be in terms of physical distance
- distance may take more abstract meaning e.g.
 - distance between two species - to be the no of years in the course of the evolution
 - distance between two images number of corresponding pixels at which their intensity values differ by at least some threshold
- Thus, Given a distance function on the objects
 - divide the objects into groups so that intuitively objects within the same group are close whereas
 - objects within the different groups are far apart.

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- Also, $d(p_i, p_j) = d(p_j, p_i)$
- Given a parameter k , a k -clustering of U is a partition of U into k nonempty sets $C_1, C_2, C_3, C_4, \dots, C_k$
- Spacing of a k -clustering is defined as the minimum distance between any two points lying in different clusters.

The K-Clustering Problem

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- How many k-clustering of a set u can be there ?
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 - finding out then one with maximum spacing. How ?

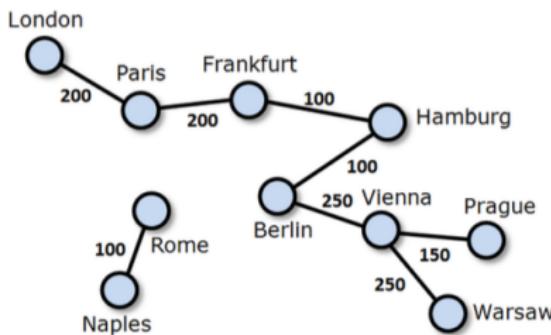
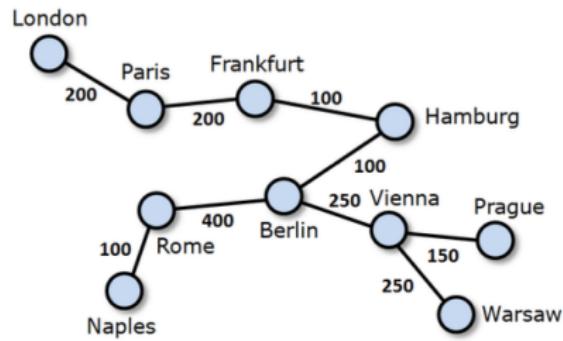
The Algorithm Approach

- consider growing a graph on the vertex set U .
- connected components become the clusters.
 - bring the nearby points into the same cluster.
- draw an edge between the closest pair of points
 - add edges between pairs of points in order of increasing distance $d(p_i, p_j)$.
 - if p_i and p_j belong to the same cluster, do not add that edge.
- How is this related to Kruskal's MST ?

The Algorithm Approach...

- Follow the same method as Kruskal's algorithm, but stop when obtaining k-clustering.
- How to stop at having k connected components ?
- stop just before it adds the last $k - 1$ edges.
- Run Kruskal's MST on the given graph fully and
 - then delete the $k-1$ most expensive edges
 - define the resulting structure to be k connected components.
 - an example of 3-clustering

Examples of 2-clustering, 3-clustering



Proving the algorithm

- Prove that the algorithm as outlined before actually works.