COUNTING-2

Combination with Repetition

$V(n, r)$ = r Combination of n Objects with repetition

Ex: 10 Combination of 3 fruits [A, M, O]

How many ways we can form a basket of 10 fruits from Apple, Mangoes & oranges

A		M		O		
6	+	3	+	1	=	10
7	+	0	+	3	=	10
1	+	8	+	1	=	10

$x_1 + x_2 + x_3 = 10$

$3 + 5 + 2 \longrightarrow \underbrace{000100000100}_{10\ 0\text{'s}\ \&\ 2\ 1\text{'s}}$

no of binary strings with

$\boxed{1}$ $\boxed{2}$ $\boxed{3}$ . . . . $\boxed{n}$

$x_1 + x_2 + x_3 + \dots\dots + x_n = r$

(iii) no of non-negative integral soln $x_i \geqslant 0 \ \forall i = \quad i = 1, 2, \dots n$

We can ~~also~~ also interpret this problem as

i) $\hookrightarrow$ No of ways of distributing r ~~similar~~ identical balls into n boxes

ii) $\hookrightarrow$ No of binary strings with $(n-1)$ 1's and r 0's.

or BS of length $(n-1)+r$ with exactly r 0's

$$^{n-1+r}C_r = {}^{(n-1)+r}C_{n-1}$$

Ex i) No of ways 20 ~~similar~~ identical balls can be placed into 5 ~~boxes~~ boxes ?

$\square\ \square\ \square\ \square\ \square$

$x_1 \ x_2 \ x_3 \ x_4 \ x_5$

Sol. $x_1 + x_2 + \dots + x_5 = 20$

$x_i \geqslant 0 \ \forall i = 1 \text{ to } 5$

$r = 20$

$n = 5$

~~$(20-1)+5 C_{5-1}$~~

$^{(5-1)+20}C_{5-1} \longrightarrow {}^{24}C_4$

(ii) No of ways 20 ~~similar~~ identical balls can be placed into 5 boxes so that each box is ~~non~~ non-empty

Sol: $\square\ \square\ \square\ \square\ \square$

$x_1 \ x_2 \ x_3 \ x_4 \ x_5$

$x_1 \geqslant 1 \quad x_2 \geqslant 1 \ \dots\dots \ x_5 \geqslant 1$

put 1 ball in each box to satisfy the condition,

$\boxed{1}$ $\boxed{1}$ $\boxed{1}$ $\boxed{1}$ $\boxed{1}$ left $= 20 - 5 = 15$

$x_1 + x_2 + \dots + x_5 = 15 \qquad x_i \geqslant 0 \ \forall i$

$n = 5$

$r = 15 \qquad {}^{n-1+r}C_{n-1}$

$^{5-1+15}C_{5-1}$

$^{19}C_4$

iii) first box contain at least 5 balls
second box contain at least 3 balls
~~3rd box " " " 3 balls~~
3rd box " " " 3 balls
4th & 5th are non-empty

$\boxed{5}$ $\boxed{3}$ $\boxed{3}$ $\boxed{1}$ $\boxed{1}$ $11+ = \cancel{8}\,20-13 = 7$

$x_1+x_2+x_3+x_4+x_5 = 7 \quad x_i \geq 0 \; \forall i$

$n=5 \quad r=7 \quad {}^{(n-1)+r}C_{n-1}$

${}^{5-1+7}C_{5-1} = {}^{11}C_4$

WB
Q2
Sol:

$(F_1, F_2, F_3, F_4, F_5)$

$x_1+x_2+x_3+x_4+x_5 = 6 \quad x_i \geq 0 \; \forall i = 1 \text{ to } 5$

$r=6 \quad n=5$

${}^{(n-1)+r}C_{n-1} \quad {}^{5-1+6}C_{5-1} = {}^{10}C_4$

$= {}^{10}C_4$

$= \dfrac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2} = 210$

Ex
Q) No of outcome possible when 2 of
i) 2 ~~similar~~ identical coins are tossed
ii) 10 ~~similar~~ identical coins are tossed

it can appear 0 times, 1 times or both the times

Sol: i) H T

$x_1 + x_2 = 2 \quad x_i \geq 0 \; \forall i$

$n=2 \quad r=2$

${}^{n-1+r}C_r \quad {}^{2-1+2}C_2 = {}^3C_2 = 3$

(ii) H T

$x_1 + x_2 = 10$

$r=10 \quad n=2 \quad x_i \geq 0 \; \forall i$

${}^{n-1+r}C_{n-1} \Rightarrow {}^{2-1+10}C_{2-1} = {}^{11}C_1 = 11$

Ex
Q) No of outcomes possible when
i) 2 ~~similar~~ identical dice are ~~tos~~ rolled
ii) 10 ~~similar~~ identical dices are rolled

outcome

Sol. i) 1 2 3 4 5 6

$x_1+x_2+x_3+x_4+x_5+x_6 = 2 \quad x_i \geq 0 \; \forall i$

$r=2 \quad n=6 \quad {}^{n-1+r}C_r \quad {}^{6-1+2}C_2 = {}^7C_2 = 21$

ii) $x_1+x_2+\dots+x_6 = 10 \quad x_i \geq 0 \; \forall i$

$n=6 \quad r=10 \quad {}^{n-1+r}C_{n-1} \Rightarrow {}^{6-1+10}C_{6-1} = {}^{15}C_5$

Permutations with Constrained Repetitions

AAABB $\longrightarrow x$ ← possible arrangements

(considering same) $\underline{A_1A_2A_3}\ \underline{B_1B_2} \longrightarrow x \cdot 3!\,2!$ → these many we get extra

if all are different $\longrightarrow 5!$

$x \cdot 3! \cdot 2! = 5!$

$x = \dfrac{5!}{3!\,2!}$

HELLO

Total $\longrightarrow x$ $\quad$ $HE\underline{L_1L_2}O \longrightarrow x \cdot 2!$

Let 2 L's are different (5 diff letters)

$HEL_1L_2O \longrightarrow 5!$

$5! = x \cdot 2!$

$x = \dfrac{5!}{2!}$

Result

1) No of permutations of n objects in which

q_1 are same

q_2 are same

q_t are same

Where $q_1 + q_2 + q_3 + \dots + q_t = n$

$C(n; q_1, q_2, \dots q_t) = \dfrac{n!}{q_1!\,q_2! \cdots q_t!}$

$= {}^{n}C_{q_1} \cdot {}^{n-q_1}C_{q_2} \cdot {}^{n-q_1-q_2}C_{q_3} \cdots {}^{n-(q_1+q_2 \dots q_{t-1})}C_{q_t}$

Ex: No of arrangements of letters in the word

MISSISSIPPI $= \dfrac{11!}{4!\,4!\,2!}$

$C(11; 4, 4, 2, 1)$ (I S P M)

ii) 4 S's are together

iii) 4 S's and 4 I's are together

Sol: ii) MIIIPPI $\boxed{SSSS}$ $= \dfrac{8!}{4!\,2!}$ ✓ for each of these arrangement in which 4 S's are together they can arrange in 1 way

iii) MPP $\boxed{IIII}$ $\boxed{SSSS}$ $= \dfrac{5!}{2!}$

WB T1

Sol: | G | G | G | G | G | A | A | A | E | C | S | S |

$\dfrac{12!}{5!\,3!\,2!} \times \dfrac{{}^{13}C_3 \times 3!}{3!} = \dfrac{12!}{5!\,3!\,2!} \times \dfrac{{}^{13}P_3}{3!}$

Arrange these letters in which there is no condition then from 13 gaps pick 3 to place 3 T's.

Multinomial Theorem

The coefficient of $x_1^{q_1}, x_2^{q_2}, \ldots x_t^{q_t}$ in the expansion of $(x_1+x_2+\ldots+x_t)^n$ where $q_1+q_2+\ldots+q_t = n$ is given by

$$\frac{n!}{q_1!\,q_2!\,q_3!\ldots q_t!}$$

Ex- What is the coefficient of

i) $x^3y^2z^5$ in the expansion of $(x+y+z)^{10}$

$$= \frac{10!}{3!\,2!\,5!} = \frac{10\times9\times\overset{4}{\cancel{8}}\times7\times\cancel{6}}{\cancel{6}\times\cancel{2}} = 90\times28 = 2520$$

ii) coefficient of $x^2y^3z^2$ in the expansion of $(2x-y+3z)^7$

$$= \frac{7!}{2!\,3!\,2!}(2)^2(-1)^3(3)^2$$

WB
Q42
उत्तर -
$$\frac{6!}{3!\,2!\,1!}(2)^3(-1)^2(3)^1$$

Result

No of ordered partitions of a set S, $|S| = n$ of type $(q_1, q_2, q_3, \ldots, q_t)$ = is given by $\dfrac{n!}{q_1!\,q_2!\ldots q_t!}$

Ex: $S = \{a, b, c, d\}$

No of ordered partitions of S of type $(1, 1, 2)$

2nd set → 1 element
3rd set → 2 elem
1st set → 1 element

$$\frac{4!}{1!\,1!\,2!} = 12$$

$\bar{a}, \bar{b}, \overline{cd}$	$\bar{b}, \bar{a}, \overline{cd}$	$\bar{c}, \bar{a}, \overline{bd}$
$\bar{a}, \bar{c}, \overline{bd}$	$\bar{b}, \bar{c}, \overline{ad}$	$c, \bar{b}, \overline{ad}$
$\bar{a}, d, \overline{bc}$	$\bar{b}, \bar{d}, \overline{ac}$	$\bar{c}, \bar{d}, \overline{ab}$

$\bar{d}, \bar{b}, \overline{ac}$
$\bar{d}, \bar{a}, \overline{bc}$
$\bar{d}, \bar{c}, \overline{ab}$

Ex No of ways 14 men can be partitioned into 3 teams, where

1st team has 6 members
2nd team has 5 members
3rd team has 3 members

Sol: Ordered partition = {6, 5, 3}

$$= \frac{14!}{6!\,5!\,3!}$$

Ex: No of ways 12 of 14 people can be partitioned into 3 teams where

i) 1st team has 5 members

ii) 2nd team has 4 members

iii) 3rd team has 3 members

Sol: $= {}^{14}C_{12} \cdot \frac{12!}{5!\,4!\,3!}$ ← Ordered partition = {5, 4, 3}

Result

Let S be a set with n elements where $n = q \cdot t$ then → q is repeated t times

No of unordered partitions of S of type $\underbrace{\{q, q, \dots q\}}_{t\ \text{times}}$

is given by $\frac{n!}{(q!)^t} \times \frac{1}{t!}$

Ex: No of ways 12 of 14 people can be partitioned into 3 teams of 4 each.

Sol: $${}^{14}C_{12} \times \frac{12!}{(4!)^3} \times \frac{1}{3!} \qquad \left\{ \frac{12!}{4!\,4!\,4!} \times \frac{1}{3!} \right\}$$ ← bcz unordered

Two dice are Rolled

Sum	2	3	4	5	6	$\boxed{7}$	8	9	10	11	12
fav	1	2	3	4	5	$\boxed{6}$	5	4	3	2	1

Three dices are rolled

Sum	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
fav	1	3	6	10	15	21	25	27										

Sum	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
fav	1	3	6	10	15	21	25	27	27	25	21	15	10	6	3	1

☆ Sum for n dice case → we need previous 6 of (n−1) dice

Sum = 14

8	9	10	11	12	13
5	4	3	2	1	0

15

Ex: 3 dice are rolled

No of cases favourable for sum = 16

①

1	2	3	4	5	6	Sum = 16
0	0	0	1	0	2	$\rightarrow \frac{3!}{2!1!} = 3$
0	0	0	0	2	1	$\rightarrow \frac{3!}{2!1!} = 3$
						$\overline{6}$

Q 4 dice are rolled. No of cases favourable for sum = 22

Sol: We consider

1	2	3	4	5	6	Sum = 22
0	0	0	1	0	3	$= \frac{4!}{3!1!} = 4$
0	0	0	0	2	2	$= \frac{4!}{2!2!} = 6$
						$\overline{10}$

Generating function

G.F of a sequence $\{a_n\}_{n=0}^{\infty}$ is given by

$$A(x) = \sum_{n=0}^{\infty} a_n x^n$$

Ex GF for sequence 1, 1

Sol: $= 1 + x$

②

$$A(x) = 1 \cdot x^0 + 1 \cdot x^1 = 1 + x$$

Ex:- G.F for the sequence 1, 1, 1......

Sol: $A(x) = 1 + x + x^2 + x^3 + \ldots\ldots$

$$\boxed{A(x) = \left(\frac{1}{1-x}\right)}$$

$$\boxed{A(x) = \frac{1}{1-x}} \quad \text{——} (1)$$

Q G.F for the sequence 1, 2, 3, 4, 5, ...

Sol: $\cancel{A(x) = 1 + 2x + 3x^2 + 5x^3 + 6x^4 + \ldots\ldots}$

$A(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + \ldots\ldots$

$x A(x) = x + 2x^2 + 3x^3 + 4x^4 + \ldots$

$A(x)[(1-x)] = 1 + x + x^2 + \ldots$

$(1-x) A(x) = \frac{1}{(1-x)}$

$$\boxed{A(x) = \frac{1}{(1-x)^2}}$$

Differentiate (1) we get result

$A'(x) = \frac{-1}{(1-x)^2} \times -1 = \frac{1}{(1-x)^2}$

$$\boxed{A(x) = (1-x)^{-2}}$$

(3)

Ex: GF for sequence $^nC_0, ^nC_1, \ldots\ldots ^nC_n$

Sol $A(x) = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \ldots\ldots {}^nC_n x^n$

$\boxed{A(x) = (1+x)^n}$

alternating +ve & -ve no's. imp

Remmamber (Important)

$(1-x)^{-1} = \sum_{n=0}^{\infty} x^n$

$\boxed{(1+x)^{-n} = \sum_{n=0}^{\infty} (-1)^n \, {}^{n+n-1}C_{n-1} \, x^n}$

1) $\boxed{\frac{1}{(1-x)} = \sum_{n=0}^{\infty} x^n} = 1+x+x^2+\ldots\ldots$

2) $\boxed{\frac{1}{(1-x)^n} = \sum_{n=0}^{\infty} {}^{n-1+n}C_n \, x^n}$

$\boxed{(1-x)^{-n} = \sum_{n=0}^{\infty} {}^{n+n-1}C_{n-1} \, x^n}$

${}^{n+n-1}C_{n-1} = {}^{n+n-1}C_n$

3) $\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} {}^{n+1}C_n x^n = \sum_{n=0}^{\infty} (n+1) x^n$

$\boxed{\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1) x^n}$

$\boxed{(1-x)^{-2} = \sum_{n=0}^{\infty} (n+1) x^n}$

$\sum_{n=0}^{\infty} (n+1) x^n = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots\ldots\ldots$

Ex

Write in simplified form

$x + 2x^2 + 3x^3 + 4x^4 + \ldots$

$x(1 + 2x + 3x^2 + \ldots) = \frac{x}{(1-x)^2}$

(4)

Ex Write

$\sum_{n=0}^{\infty} n x^n = x\left[\sum_{n=0}^{\infty} n x^{n-1}\right] = x\left(\frac{1}{(1-x)^2}\right)$

$1 \cdot x^1 + 2x^2 + 3x^3 + 4x^4 + \ldots\ldots\ldots$

$x(1 + 2x + 3x^2 + 4x^3 + \ldots\ldots)$

$= \frac{x}{(1-x)^2}$

$\boxed{(1-x)^{-3} = \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2} x^n}$

4) $\frac{1}{(1-x)^3} = \sum_{n=0}^{\infty} {}^{n+2}C_n \, x^n$

$= \sum_{n=0}^{\infty} \left[\frac{(n+2)(n+1)}{2} x^n\right]$

$\boxed{\frac{1}{(1-x)^3} = 1 + 3x + 6x^2 + 10x^3 + \ldots\ldots\ldots}$

5) $\boxed{1 + x + x^2 + \ldots\ldots + x^{n-1} = \cancel{\frac{x^n-1}{x-1}} \left(\frac{1-x^n}{1-x}\right) \quad 0 < x < 1}$

So $\boxed{1 + x + x^2 + \ldots\ldots + x^n = \left(\frac{1-x^{n+1}}{1-x}\right)}$

DB T10

Sol: $(x^3 + x^4 + x^5 + x^6 + \ldots)^3$

$x^9 (1 + x + x^2 + \ldots\ldots)^3$

$x^9 \left(\frac{1}{(1-x)}\right)^3 \Rightarrow \frac{x^9}{(1-x)^3}$

$= x^9 (1-x)^{-3}$

5

$x^9(1-x)^{-3}$

$= x^9(1 + 3x + 6x^2 + 10x^3 + \dots)$

$= x^9 + 3x^{10} + 6x^{11} + 10x^{12} + \dots$

$= 10$ (OR)

Coefficient of x^{12} in $(x^3 + x^4 + x^5 + \dots)^3$

Coefficient of x^{12} in $x^9(1 + x + x^2 + \dots)^{+3}$

= Coefficient of x^3 in $(1 + x + x^2 + x^3 + \dots)^3$

= Coefficient of x^3 in $(1-x)^{-3}$

$= 1 + 3x + 6x^2 + 10x^3 + \dots$

$= 10$

WB
Q51

Sol: $\{a_n\}_{n=0}^{\infty} = \frac{1+z}{(1-z)^3}$

$(1+z)(1-z)^{-3}$

$(1+z)(1 + 3z + 6z^2 + 10z^3 + \dots)$

$\cancel{(1+z)}(1 + 4z + 9z^2 + 16z^3 + \dots)$

$a_3 = 16 \quad a_0 = 1$

$a_3 - a_1 = 16 - 1 = 15$

6

$\cancel{A(z) = (1+z)\sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2} z^n}$

$A(z) = (1+z)\sum_{n=0}^{\infty} {}^{n+2}C_n z^n$

$\sum_{n=0}^{\infty} {}^{n+2}C_n z^n + z\sum_{n=0}^{\infty} {}^{n+2}C_n z^n$

put $z = 3$ put $n = 2$

$a_3 = {}^{3+2}C_3 + {}^{2+2}C_2$

$= {}^5C_3 + {}^4C_2$

$= 10 + 6$

$= 16$

$a_0 = {}^2C_0 = 1$ [put $n = 0$ in]

$a_3 - a_0 = 16 - 1 = 15$

Gate 2018

Q Which one of the following is closed form expression for the GF of the sequence $\{a_n\}$ where $a_n = 2n + 3$ for all $n = 0, 1, 2, \dots$

Sol: $A(x) = 3 + 5x + 7x^2 + 9x^3 + \dots$

$G.F = \sum_{n=0}^{\infty} (2n+3)x^n$

$$G.F = \sum_{n=0}^{\infty} (2n+3)\, x^n$$

$$= 2\sum_{n=0}^{\infty} n x^n + 3\sum_{n=0}^{\infty} x^n$$

$$= 2\left[x + 2x^2 + 3x^3 + 4x^4 + \dots\right] + 3\left(\frac{1}{(1-x)}\right)$$

$$= 2x\left(1 + 2x + 3x^2 + 4x^3 + \dots\right] + \frac{3}{(1-x)}$$

$$= 2x\left[\frac{1}{(1-x)^2}\right] + \frac{3}{(1-x)}$$

$$= \frac{2x}{(1-x)^2} + \frac{3}{(1-x)}$$

$$\Rightarrow = \frac{2x + 3 - 3x}{(1-x)^2}$$

$$\Rightarrow \frac{3-x}{(1-x)^2}$$

NOTE: Converting G.F problem to Combination with Repetition Problem

Ex: What is the coefficient of x^8 in

(i) $(1 + x^3 + x^5 + x^8)(1 + x^4 + x^5 + x^8)$

Sol:

$$x^{e_1} \cdot x^{e_2} = x^8$$

$e_1 + e_2 = 8$

e_1 = powers of x in first ~~bracket~~ fun

e_2 = powers of x in 2nd ~~bracket~~ fun

$e_1 = 0, 3, 5, 8$

$e_2 = 0, 4, 5, 8$

$$\left.\begin{array}{l} 0 + 8 = 8 \\ 3 + 5 = 8 \\ 8 + 0 = 8 \end{array}\right\} \rightarrow 3$$

Ans = 3

Ex: $e_1 + e_2 + e_3 = 8$

e_i represents the power of x in $A_i(x)$

$e_1 = 0, 3, 8$

$e_2 = 0, 5, 8$

$e_3 = 0, 1, 6, 8$

Coefficient of x^8 in $\underset{A_1(x)}{(1 + x^3 + x^8)}\ \underset{A_2(x)}{(1 + x^5 + x^8)}\ \underset{A_3(x)}{(1 + x + x^6 + x^8)}$

Ex: $e_1 + e_2 + e_3 + \dots + e_n = r$

$0 \le e_1 \le 1$; $0 \le e_2 \le 1$; ...... $0 \le e_n \le 1$

at most case

We can convert this ~~problem~~ problem to

Coefficient of x^r in $\underbrace{(1+x)(1+x)(1+x)\dots(1+x)}_{n \text{ times}}$

Coefficient of x^r in $(1+x)^n$

$= {}^nC_r$

(9)

Ex What is coefficient of x^n in $(1+x+x^2+\ldots)^n$

Sol: $(1+x+x^2+\ldots)^n = \frac{1}{(1-x)^n}$

Coefficient of x^n in $\underbrace{(1+x+x^2+\ldots)(1+x+x^2+\ldots)(1+x+x^2+\ldots)\ldots}_{n \text{ times}}$

$= c_1 + c_2 + c_3 + \ldots + c_n = n$

$c_1 \geq 0;\ c_2 \geq 0;\ \ldots\ c_n \geq 0$

$= {}^{n-1+n}C_n$ (or) ${}^{n-1+n}C_{n-1}$

WB Q48

Sol: — — — — — — —

$x_1 + x_2 + \ldots + x_7 = 11$

~~$x_i \in (0, (1,2,3)$~~

$\boxed{1 \leq x_i \leq 3}$

to handle upper conditions we use GF

~~$n = 11$ $r = 7$~~

~~${}^{n-1+n}C_{n-1} \Rightarrow {}^{7-1+11}C_{7-1}$~~

Convert to GF problem

(10)

Coefficient of x^{11} in $(x+x^2+x^3)(x+x^2+x^3)\ldots(x+x^2+x^3)$

Coefficient of x^{11} in $(x+x^2+x^3)^7$

Coefficient of x^{11} in $x^7(1+x+x^2)^7$

Coefficient of x^4 in $(1+x+x^2)^7$

= Coefficient of x^4 in $\left(\frac{1-x^3}{1-x}\right)^7$

Coeffi of x^4 in $(1-x^3)^7 \times \frac{1}{(1-x)^7}$

$\downarrow$ ignoring higher powers (power > 4)

" " x^4 in $[{}^7C_0 - {}^7C_1 x^3] \sum_{n=0}^{\infty} {}^{7-1+n}C_n x^n$

$[1 - 7x^3] \sum_{n=0}^{\infty} {}^{n+6}C_n x^n$

" " x^4 in $\left[\sum_{n=0}^{\infty} {}^{n+6}C_n x^n - 7x^3 \sum_{n=0}^{\infty} {}^{n+6}C_n x^n\right]$

$n=4$ $\qquad$ $n=1$

$= {}^{10}C_4 - 7 \cdot {}^7C_1$

II - Method

$x_1 + x_2 + \dots + x_7 = 11$

$1 \le x_i \le 3$

we put satisfy minimum condition

$x_1 + x_2 + \dots + x_7 = 4$

$0 \le x_i \le 2$

first solve for

$x_i \ge 0$

$7-1+4C_4 = {}^{10}C_4$

we have remove those ~~cases~~ cases which violet the $x_i \le 2$

ie we have to remove $x_i > 2$ cases

$x_1 + x_2 + \dots + x_7 = 4$

$\left.\begin{array}{l} 4+0+0 \dots +0 \\ 0+4+ \dots \end{array}\right\} 7$

$\left.\begin{array}{l} \overbrace{\boxed{1 \quad\quad}}^{6} + 3 \\ \boxed{\quad\quad} 3 + \boxed{\ } \end{array}\right\} 7.6 = 42$

x_1 can have values 1, 2, 3, 4, फि

We need to remove 3, & 4 cases

3 & 4 से sum 4 बनता है।

∴ total cases which violet = 42 + 7 = 49

∴ Ans: ${}^{10}C_4 - 49$

Recurrence Relation

Def: Expressing a problem in terms of sub problem

Mathematically: Expressing n^{th} term of a sequence in terms of 1 or more previous terms.

Ex: $a_n = 4a_{n-1}$ $\qquad n! = n(n-1)!$

$a_n = 3a_{n-1} + 2a_{n-2}$ $\qquad a_n = na_{n-1}$

WB

Q 36

Sol: $a_0 = 37$ ants $a_5 = ?$, a_n = no of ants after n years.

a_n = no of ~~crazy~~ ants after n years

a_{n-1} = no of ants after $(n-1)$ years

a_{n-1} = 1st order

a_{n-2} = 2nd order

a_{n-k} = k^{th} order

$a_n = 2(2a_{n-1})$

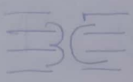
$\boxed{a_n = 4a_{n-1}}$ 1st order R.R

$a_0 = 37 \leftarrow$ stopping condition

~~$a_1 = 4a_0 = 4 \times 37 = 148$~~

$a_5 = 4^5 \cdot a_0 = 4^5 \cdot 37$

Ex: Find recurrence relation for no of ways a person can climb up flight of n steps if person can skip at most 1 step at time



Sol: ~~$a_n = a_{n-2} + a_{n-1}$~~

Let a_n = no of ways a person can climb n steps such that (he can skip at most 1 step.)

only 1 way — still n-2 to climb

$a_n \xrightarrow{\text{skip}} 1 \cdot a_{n-2}$

$a_n \xrightarrow{\text{not skip}} 1 \cdot a_{n-1}$ — still n-1 to climb

$\therefore a_n = a_{n-1} + a_{n-2}$ — 2nd Order $\therefore$ we need 2 stopping Condition

$a_1 = 1$

$a_2 = 2$

Gate 2016

a_n = no of n-bit strings that do not Contain 2 ~~Consecy~~ ~~Consequitive~~ Consecutive 1's.

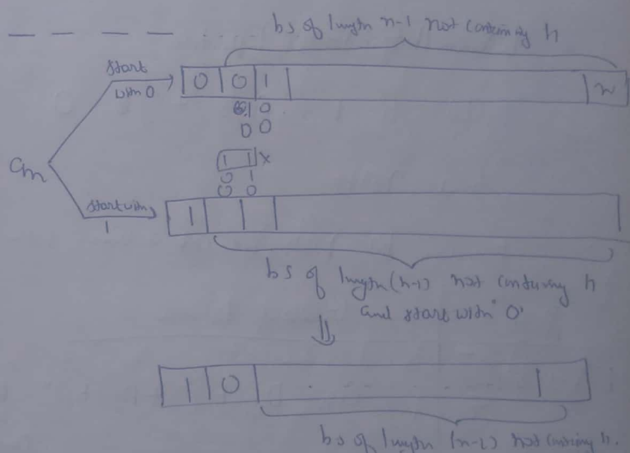
Write recurrence relation for a_n?

(14)

(a) $a_n = a_{n-1} + 2a_{n-2}$ (b) $a_n = 2a_{n-1} + a_{n-2}$

(c) $a_n = a_{n-1} + a_{n-2}$ (d) $a_n = 2a_{n-1} + 2a_{n-2}$

Sol: — — — — ......



$a_n = a_{n-1} + a_{n-2}$

Verification Method

$a_1 = 2$

$a_2 = 3$

$a_3 = 5$

(a) $a_3 = a_2 + 2a_1$
$= 3 + 2(2)$
$= 7 \rightarrow$ but a_3 is 5 so wrong

(c) $a_3 = a_2 + a_1$
$= 3 + 2$
$= 5$ this is write

Second Order Homogeneous Linear Recursive Relation

$$a_n + C_1 a_{n-1} + C_2 a_{n-2} = 0$$

Write characteristic Equation

$C(t): t^2 \cdot 1 + C_1 \cdot t + C_2 = 0$

find roots

based on the roots we can determine soln

Roots	General Solution
1) Real & distinct b_1, b_2	$a_n = D_1 (b_1)^n + D_2 (b_2)^n$ D_1, D_2 are arbitrary constants
2) Real & Equal b, b	$a_n = D_1 (b)^n + D_2 (b)^n (n)$ $= [D_1 + D_2 n](b)^n$

WB Q36

Ex: $a_n - 4a_{n-1} = 0$

$C(t): t - 4 = 0$

$t = 4$

$a_n = D(4)^n$ $a_0 = D = 37$ given

$a_n = 37(4)^n$

$a_5 = 37(4)^5$

Ex: 1 $a_n - 5a_{n-1} + 6a_{n-2} = 0$

Ex. 2 $a_n - 4a_{n-1} + 4a_{n-2} = 0$

Sol: 1 $a_n - 5a_{n-1} + 6a_{n-2} = 0$

$C(t): t^2 - 5t + 6 = 0$

$(t-3)(t-2) = 0$

$t = 3, 2$

$a_n = C_1 (3)^n + C_2 (2)^n$

Sol: 2 $a_n - 4a_{n-1} + 4a_{n-2} = 0$

$C(t): t^2 - 4t + 4 = 0$

$t = 2, 2$ $a_n = [C_1 + C_2 n](2)^n$

WB
P-13
Q32

Sol- $S(k) - 10S(k-1) + 9S(k-2) = 0$

$S(0) = 3 \quad S(1) = 11$

$S_k - 10S_{k-1} + 9S_{k-2} = 0$

$C(t): t^2 - 10t + 9 = 0$

$t = 9, 1$

$S_k = C_1(1)^k + C_2(9)^k$

Ex: $T_n - 7T_{n-1} + 12T_{n-2} = 0$

$T_0 = 1 \qquad T_1 = 0$

a) $3(2)^n - 2(3)^n$

b) $7(6)^n - 6(7)^n$

c) $4(3)^n - 3(4)^n$

d) $5(4)^n - 4(5)^n$

Sol: $T_n - 7T_{n-1} + 12T_{n-2} = 0$

$C(t): t^2 - 7t + 12 = 0$

$t = 3, 4$

$T_n = C_1(3)^n + C_2(4)^n$

Non-Linear Reccurrence Relation

WB
Q25

Sol: $T(2^k) = 3T(2^{k-1}) + 1$

$T(1) = 1$

$2^k = n \quad k\log_2 2 = \log n \quad k = \log n$

$T(n) = 3T\left(\frac{n}{2}\right) + 1$

$T_n = 3T_{n/2} + 1$

$T(2^k) = S_k$

$T(2^{k-1}) = S_{k-1}$

$T(1) = S_0 = 1$

$S_k - 3S_{k-1} - 1 = 0 \qquad S_0 = 1$

$C(t): t - 3 = 0$

$t = 3$

$S_1 = \frac{3^{1+1} - 1}{2} = 4$

(b) $S_k = \frac{3^{k+1} - 1}{2}$

$S_0 = \frac{3^1 - 1}{2} = 1$

$S_1 = 3S_0 + 1$

$= 3(1) + 1$

$= 4$

a) $S_k = 2^k$

$S_0 = 2^0 = 1$

$S_1 = 2^1 = 2 \neq 4$

(8) ~~Test~~ Gate 2016

$a_1 = 8 \qquad a_n = 6n^2 + 2n + a_{n-1} \qquad a_{99} = k \times 10^4$

$k = ?$

$a_n - a_{n-1} = 6n^2 + 2n$

$a_{n-1} - a_{n-2} = 6(n-1)^2 + 2(n-1)$

$a_{n-2} - a_{n-3} = 6(n-2)^2 + 2(n-2)$

$\vdots$

$a_2 - a_1 = 6(2)^2 + 2(2)^2$

$a_n - a_1 = 6(n^2 + (n-1)^2 + \dots 2) + 2(n + (n-1) + \dots + 2)$

$a_n - a_1 = \cancel{6\left(\frac{n(n+1)(2n+1)}{6}\right)}\ 6\left[\frac{n(n+1)(2n+1)}{6} - 1\right] + 2\left[\frac{n(n+1)}{2} - 1\right]$

$a_n - a_1 = n(n+1)(2n+1) - 6 + n(n+1) - 2$

$a_n - \cancel{8} = n(n+1)(2n+1+1) - \cancel{8}$

$a_n = n(n+1)(2n+2)$

$a_{99} = 99(99+1)(2 \times 99 + 2)$

$= 198 \cdot 10^4$

$k = 198$

Pigeon hole Principle

If there are n pigeon holes and

1) $n+1$ … $2n$ pigeons then same pigeon holes contain at least 2 pigeons.

2) $2n+1$ … $3n$ Pigeon then same pigeonhole contain atleast 3 pigeons.

3) $kn+1$ pigeons then same pigeonhole contains at least $k+1$ pigeons.

Ex: In a class there are 36 students, at least how many of them are born on same day of a week.

Sol: 6 $\qquad \left\lceil \frac{36}{7} \right\rceil = 6$

7 = no of pigeon holes
36 = pigeons

If there are n pigeon holes and k pigeons then same pigeon hole contain at least $\left\lceil \frac{k}{n} \right\rceil$ pigeons born on same day of a week. What is the no of students which would guarantee this result a) 35 b) 36 (c) 42 (d) 49

Sol: $\left\lceil \frac{n}{7} \right\rceil = 5$ $n = 35$

$$\underbrace{\left\lceil \frac{35}{7} \right\rceil = 5 \quad \left\lceil \frac{36}{7} \right\rceil = 6 \quad \left\lceil \frac{42}{7} \right\rceil = 6 \quad \left\lceil \frac{49}{7} \right\rceil = 7}$$

We have to find minimum

Numerical Answer type

min: $kn+1$ at least $k+1$

$k+1 = 5$
$n = 7$ $k = 4$

min $= 7 \times 4 + 1 = 29$

Q What is min no of students to guarantee at least 7 students born on same month of a year

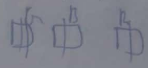
Sol: at least $\rightarrow k+1 = 7$ $n = 12$ $12 \times 6 + 1 = 73$
$k = 6$

min $= kn+1$

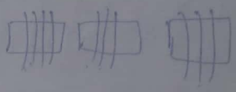
Ex: A box contains lot of green, blue and red socks. What is the min no of socks one to pick to be sure of getting

1) at least one pair of same color
2) at 2 pairs of same color

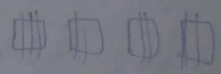
Sol:
1) $k+1 = 2$ min $= kn+1 = 1 \times 3 + 1 = 4$
$k = 1$
$n = 3$



2) $k+1 = 4$ min: $kn+1 = 9+1 = 10$
$k = 3$
$n = 3$



WB
Q 23
Sol: at least: $k+1 = 3$ $n = 4$ suits: 4
$k = 2$



min: $kn+1 = 2 \times 4 + 1 = 9$

Graph Theory (HO)

Q7 1 2 3

123 0

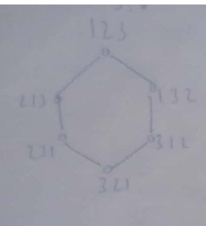
0 132

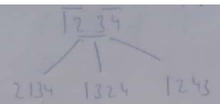
3 12 0

0 213

321 0

0 231





3!V   4!V   5!V ...... 100!V

d(V) = 2   d(V) = 3   d(V) = 4 ...... d(V) = 99

y = 99

z = 1

y + 10z = 99 + 10(1) = 109

Logic

1) $\boxed{\forall x\,[P \to Q(x)] \equiv P \to \forall x\, Q(x)}$

$\forall x\,[\sim P \vee Q(x)]$

$\sim P \vee \forall x\, Q(x)$

$P \to \forall x\, Q(x)$

(2) $\boxed{\exists x\,[P \to Q(x)] \equiv P \to \exists x\, Q(x)}$

(3) $\boxed{\forall x\,[P(x) \to Q] \equiv \exists x\, P(x) \to Q}$

$\forall x\,[\sim P(x) \vee Q]$

$\sim(\exists x\, P(x)) \vee Q$, $\exists x\, P(x) \to Q$

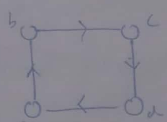
4) $\boxed{\exists x\,[P(x) \to Q] \equiv \forall x\, P(x) \to Q}$

Graph Theory

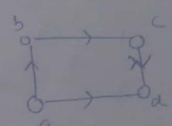
Strongly Connected

A directed graph is strongly connected if b/w every pair of vertices [U & V / U and V] there exists a directed path. from U to V & V to U

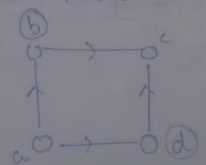
A directed graph is unilaterally connected if ~~there exist a directed~~ path b/w every pair of vertices U & V there exist a directed path from U to V or V to U .



strongly connected



unilaterally connected



no path b/w b & d

not connected