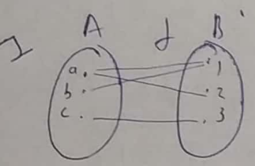
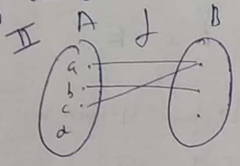
Function

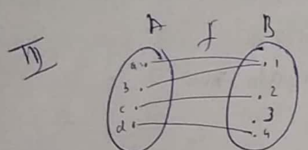
A function f from set A to B is a rule which assigns every element of set A to a unique element of set B.



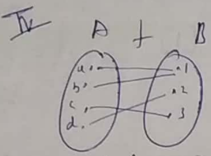
Not fn



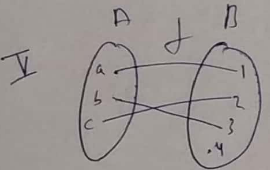
Not fun



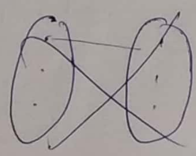
fun (Into)
every element of B not used

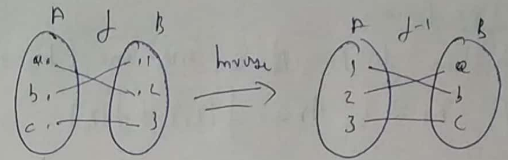


fun (onto)

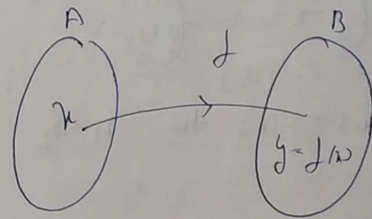


fun
(one-one)





onto
1-1 } fun



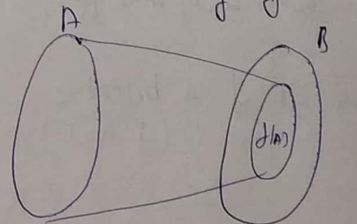
$f : A \longrightarrow B$ is a fun

A : Domain

B : Codomain

$y = f(x)$: image of x under f

x : Pre-image y

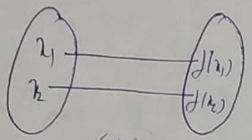


$f(A) \subset B$ (Into fun)

$f(A) = B$ (Onto to fun)

One-One function

Def: A func $f: A \to B$ is one-one (Injective) if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$.

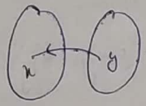


(or)

used for solving question → if $f(x_1) = f(x_2)$ then $x_1 = x_2$

Onto function

Def: A fun $f: A \to B$ is onto (surjection) if $\forall y \in B \ \exists x \in A \ [f(x) = y]$



Bijection : one-one and onto func

Result: f^{-1} exists iff f is bijective (f is 1-1 & onto)

Ex. $f: Z \to Z$

$f(x) = \frac{2x+6}{3}$ then f is

(a) 1-1 only (b) onto only (c) both d) none

Sol: $f(x_1) = \frac{2x_1+6}{3}$ $\quad f(x_2) = \frac{2x_2+6}{3}$

$\frac{2x_1+6}{3} = \frac{2x_2+6}{3}$

$x_1 = x_2$

$y = \frac{2x+5}{3}$

$3y = 2x+5$

$\frac{3y-5}{2} = x$

$f(1) = \frac{7}{3} \notin Z$ Not a fun

2) $f: R \to R$

$f(x) = \frac{2x+5}{3}$

Sol: To check 1-1

$f(x_1) = f(x_2)$

$\frac{2x_1+5}{3} = \frac{2x_2+5}{3}$

$x_1 = x_2$ → Hence 1-1 ✓

to check Onto

$y = f(x)$

y in terms of x

for onto →

we find x in terms of y

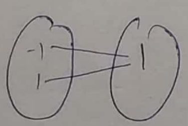
$y = f(x) = \frac{2x+5}{3}$

$x = \frac{3y-5}{2} \in R$ (domain)

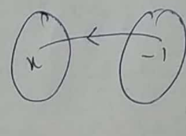
onto fun

3) $f: R \to R$

$f(x) = x^2$ fun



not 1-1



not onto

4) $f: R^+ \to R$

$f(x) = x^2$ fun

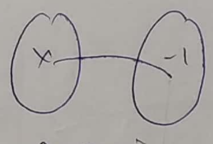
1-1

$f(x_1) = f(x_2)$

$x_1^2 = x_2^2$ $x_1, x_2 \in R^+$

$x_1 = x_2$

onto



no pre image

not onto

no pre image for -1

$f(x) = x^2$

$y = x^2$

$x = \sqrt{y}$

$y \in R^+$ & $x \in R^+$

5) $f: R^+ \to R^+$

$f(x) = x^2$ 1-1 & onto

Find $f^{-1}(x)$

(1) $f(x) = 2x+3$

$y = 2x+3$ y in terms of x

write x in terms of y

$f^{-1}(y) = x = \frac{y-3}{2}$

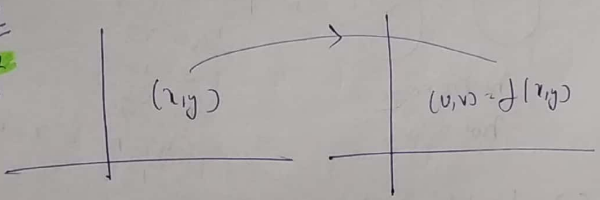
$f^{-1}(y) = \frac{y-3}{2}$

$\boxed{f^{-1}(x) = \frac{x-3}{2}}$

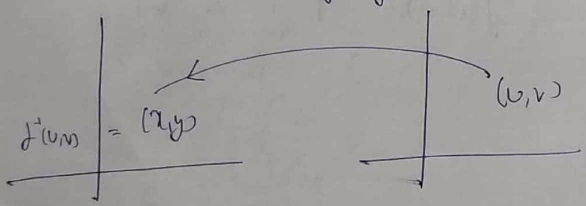
HQ

Q2

Sol



u, v in terms of x, y



x, y in terms of u, v

$f(x,y) = (x+y, x-y)$

$(u,v) = (x+y, x-y)$

$u = x+y$ $\quad$ $v = x-y$

$2x = u+v$ $\quad$ $\frac{u-v}{2} = y$

$x = \frac{u+v}{2}$

$f^{-1}(u,v) = \left(\frac{u+v}{2}, \frac{u-v}{2}\right)$

$f^{-1}(x,y) = \left(\frac{x+y}{2}, \frac{x-y}{2}\right)$

Q2

Sol: $f(x,y) = (2x+y, x-2y)$

$(u,v) = (2x+y, x-2y)$

$u = 2x+y$

$v = x-2y$

$2v = 2x-4y$

$u-2v = 5y$

$\frac{u-2v}{5} = y$

$v+2y = x$

$v + 2\left(\frac{u-2v}{5}\right) = x$

$v + \frac{2u-4v}{5} = x$

$\frac{2u+v}{5} = x$

Ans: $f^{-1}(x,y) = \left(\frac{x-2y}{5}, \frac{2x+y}{5}\right)$

Ans $f^{-1}(x,y) = \left(\frac{2x+y}{5}, \frac{x-4y}{5}\right)$

Ex: $f(x) = \frac{x+3}{x+4}$ find f^{-1}

$y = \frac{x+3}{x+4}$

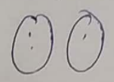
$xy + 4y = x+3$

$\left(\frac{4y-3}{1-y}\right) = x$

$f^{-1}(x) = \frac{4x-3}{1-x}$

Results

1-1 not possible

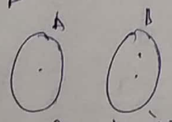


(1) $f: A \to B$ is 1-1 then

~~a) $|A| \le |B|$~~ (b) $|A| \geq |B|$ c) $|A| = |B|$

d) none

(2) $f: A \to B$ is onto then $|A| \geq |B|$

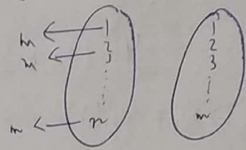


onto not possible

3) $f: A \to B$ is 1-1 and onto then $|A| = |B|$

4) $|A| = n$ $|B| = m$

No of fun's from A to B = $|B|^{|A|}$

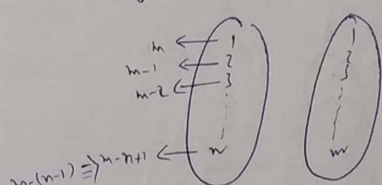


m m m m .. n times

$= m^n$

5) $|A| = n$ $|B| = m$

no of 1-1 fun from A to B = ^mP_n



$m \cdot (m-1) \cdot (m-2) \cdot (m-3) \ldots (m-n+1)$

$= {}^mP_n$

6) $|A| = n$ $|B| = m$

No of onto funs from A to B

$= \boxed{\sum_{i=0}^{m} {}^mC_i \, (-1)^i (m-i)^n}$

$$= \sum_{i=0}^{m} {}^{m}C_i (-1)^i (m-i)^n$$

$${}^{m}C_0\, m^n - {}^{m}C_1 (m-1)^n + {}^{m}C_2 (m-2)^n - {}^{m}C_3 (m-3)^n + \dots$$

$$\dots + {}^{m}C_m (-1)^m (m-m)^n$$

HO

Q4

Sol: $X = \{1,2,3,4\}$ $\quad Y = \{a,b,c\}$

$${}^3C_0 (3^4) - {}^3C_1 (2)^4 + {}^3C_2 (1)^4$$

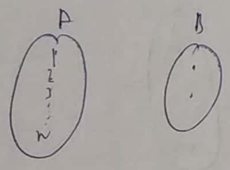
$$= (1)\,81 - 3\times 16 + 3$$

$$= 81 - 48 + 3$$

$$= 36$$

Q5

Sol:



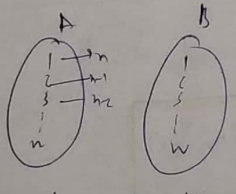
$$= {}^2C_0 (2)^n - {}^2C_1 (1)^n$$

$$= 1(2)^n - 2$$

$$= (2^n - 2)$$

Q6

Sol:



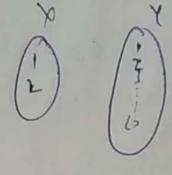
✓ 1-1 fun

$${}^nP_n = n!$$

Result: $f : A \to A$ $\quad |A| = n$ $\quad$ no. of 1-1 fun = $n!$

Q7

Sol:



$F = \{(20)^2\}$

$400 - (20)(19)$

$\frac{400 - (20)(19)}{400}$

1-1 fun $= \frac{(20)(19)}{400} = 380$

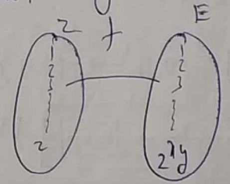
$$\text{Prob} = \frac{380}{400} = \frac{38}{40} = \frac{19}{20}$$

Q 8/11

Sol: $|X| = x$ $\quad |Y| = y$ $\quad |Z| = z$

$W = X \times Y$

$|W| = xy$ $\quad E = P(W) = 2^{xy}$



$n(f) : Z \to E$

$$f : (2^{xy})^z = 2^{xyz}$$

NOTE: $\{0,1\} \times \{0,1\} = \{0,1\}^2 \to$ binary ordered pair

$\cancel{\{0,1\}^2 \times \{0,1\} = \{0,1\}^3}$

$\{(0,0), (0,1), (1,0), (1,1)\}$

triple → $\{0,1\}^3 = \{0,1\}^2 \times \{0,1\}$

ordered pair

$= \{(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1)\}$

$\{0,1\}^4 = \{0,1\}^3 \times \{0,1\}$

$A = \{0,1\}$

$|A| = 2$

$A^4 = \{0,1\}^4$

$|A^4| = 2^4 = 16$

HW

Q12

Sol:

~~$S = \{f : \{0,1\}$~~

$S = \{f \mid f : \{0,1\}^4 \to \{0,1\}\}$

$\boxed{\cancel{N = |f : \{0,1\}^4 \to \{0,1\}|}}$ $|S| = 2^{16}$

$N = |f : \underset{2^{16}}{S} \to \underset{2}{\{0,1\}}|$

$\boxed{\cancel{N = \quad = 2^{16}}}$

$N = 2^{2^{16}}$

$\log_2 \log_2 N = 16$

Q13 (Imp)

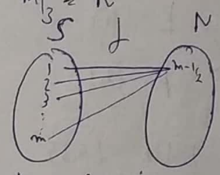
Sol: $S = \{1, 2, 3, \ldots, m\}$ $m > 3$

$(X_1, X_2, X_3; \ldots X_n) \subseteq S$ $|X_i| = 3 \ \forall i$

~~$f : S \to N$~~ N = set of natural nos.

No of subsets of size 3 = ${}^mC_3 = n$

$f : S \longrightarrow N$



$f(i)$ = no of sets X_j that contain i

$\{i, -, -\}$

$= {}^{m-1}C_2$

$$\sum_{i=1}^{m} f(i) = f(1) + f(2) + f(3) + \ldots + f(m)$$

$$= \underbrace{{}^{m-1}C_2 + {}^{m-1}C_2 + \ldots + {}^{m-1}C_2}_{m \text{ times}}$$

$$= m \times {}^{m-1}C_2$$

$$= m \frac{(m-1)!}{2!\,(m-3)!}$$

$$= \frac{3\, m!}{3 \cdot 2!\,(m-3)!}$$

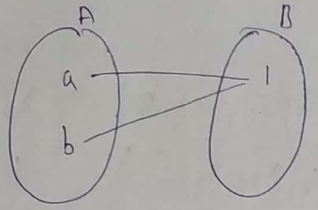
$$= 3 \cdot {}^mC_3 = 3n$$

Imp Ex:

Q16-HO

Concept

(Gate)



$A = \{a, b\}$ $B = \{1\}$

$f(A) = \{1\}$ ← set A of elements of image

$\subseteq A$ < $E = \{a\}$, $F = \{b\}$

$f(E) = \{1\}$

$f(F) = \{1\}$

$E \cup F = \{a, b\}$ $f(E \cup F) = \{1\}$

$E \cap F = \{\}$ $f(E \cap F) = \{\}$

$\therefore f(E \cup F) = (f(E) \cup f(F)) = \{1\}$

$f(E \cap F) \subseteq (f(E) \cap f(F)) = \{1\}$

Result Let $f: A \to B$ be a fun^n

Let $E \subseteq A$ & $F \subseteq A$

1) $f(E \cup F) = f(E) \cup f(F)$

2) $f(E \cap F) \subseteq f(E) \cap f(F)$

* 3) $f(E \cap F) = f(E) \cap f(F)$ iff f is 1-1

Q16 HO

Sol^n- a) False

bcz intersection may contain common elements

$|f(A \cup B)| = |f(A)| + |f(B)| - |f(A \cap B)|$

d) $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$ ✓

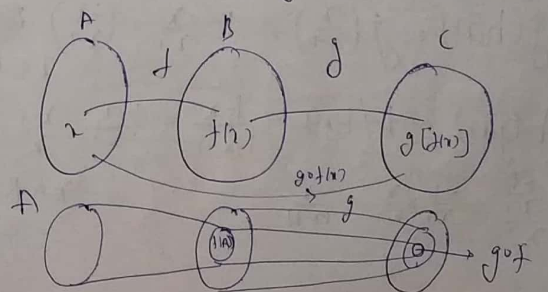
f^{-1} is one-one & onto

we can use Result (3)

Def: Composite function

Def: Let $f: A \to B$ and $g: B \to C$ then Composite fun^n $g \circ f: A \to C$ defined as

$$g \circ f(x) = g[f(x)]$$



NOTE:

1) If $g \circ f$ is defined then $f \circ g$ need not be defined

2) When both $f \circ g$ and $g \circ f$ are defined then $f \circ g$ need not be equal to $g \circ f$.

Ex: $f(x) = x^2$ $\quad g(x) = x + 2$

$g \circ f = g(f(x)) = g(x^2) = x^2 + 2$

$f \circ g = f(g(x)) = f(x+2) = (x+2)^2$

$g \circ f \neq f \circ g$

HO

Q8

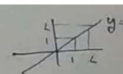
Sol:

$g(x) = 1 - x \qquad h(x) = \frac{x}{x-1} \qquad x \in R - \{1\}$

$g(h(x)) = g\left(\frac{x}{x-1}\right) = 1 - \frac{x}{x-1} = \left(\frac{-1}{x-1}\right) \qquad x \in R - \{1\}$

$h(g(x)) = h(1-x) = \frac{1-x}{1-x-1} = \frac{1-x}{-x} = \frac{x-1}{x} \qquad x \in R - \{0\}$

$\frac{g(h(x))}{h(g(x))} = \frac{\frac{-1}{x-1}}{\frac{x-1}{x}} = \frac{-1}{x-1} \times \frac{x}{x-1} = \frac{-x}{(x-1)^2}$



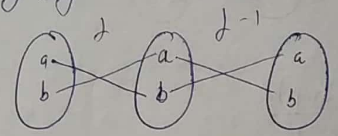
Def: **Identity fun**

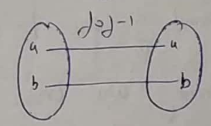
$I(x) = x \longrightarrow$ **1-1 & onto**

Result

1) $f \circ I(x) = f(x)$

2) $f \circ f^{-1}(x) = I(x)$





3) $f \circ f(i) = i$

$f = I$ $\qquad$ **$f = f^{-1}$**

HO

Q14

Sol:

$\underbrace{f(f(i)) = i}$ $\rightarrow$ such fun are identity fun $\begin{cases} f(i) = k \\ f(k) = i \end{cases}$

$f(i) = i$

identity

I case

P

Q

R

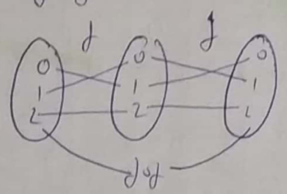
$f : \{0, 1, 2\} \rightarrow \{0, 1, 2\}$

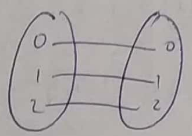
$f(f(i)) = i$

↑ true only if domain and codomain intersect. It might not be true for all i.

Hence P is false

II case $f = f^{-1}$



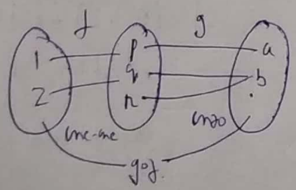


Results (Imp)

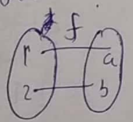
1) If f and g are 1-1 then $g \circ f$ is 1-1
2) If f and g are onto then $g \circ f$ is onto
3) If f and g are 1-1 and onto then $g \circ f$ is 1-1 and onto.

But converse need not be True

Ex.



$g \circ f(1) = a$
$g \circ f(2) = b$



II
1) If $g \circ f$ is 1-1 then f is 1-1
2) If $g \circ f$ is onto then g is onto
3) If $g \circ f$ is 1-1 & onto then f is 1-1 and g is onto

Graph Theory

Graph $G = (V, E)$

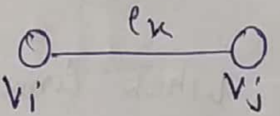
V = Vertex set $\{v_1, v_2, v_3 \dots v_n\}$

E = edge set $\{e_1, e_2, e_3 \dots e_n\}$

Order of $G = |V|$

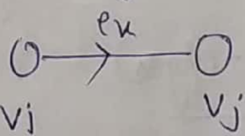
unordered pair of vertices

If each edge $e_k = \{v_i, v_j\}$ then G is called Undirected graph (or) Graph

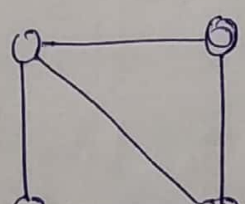


ordered pair of vertices

If each edge $e_k = \{v_i, v_j\}$ then G is called directed graph (or) digraph

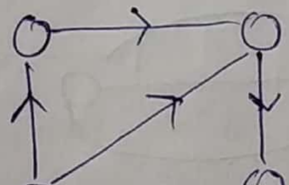


Ex:



G_1

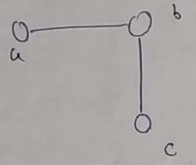
Graph



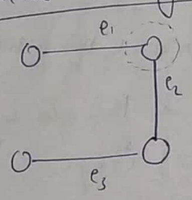
G_2

digraph

Adjacent Vertices :- they have common edge

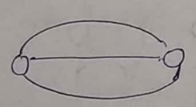


Adjacent ~~Ad~~ Edges : they have common vertex



Self loops : edge which connect vertex to itself

Parallel edges (multi -edges)

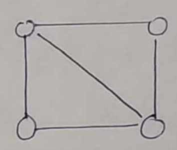


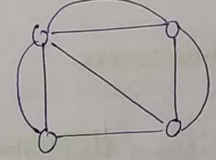
~~Definit~~

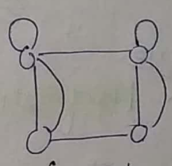
Definition	self loops	multi edges
Simple graph	X	X
multigraph	X	✓
General (or) Pseudograph	✓	✓

✓ → allowed
X → not allowed

Simple can be called → multi & general but vice versa not true



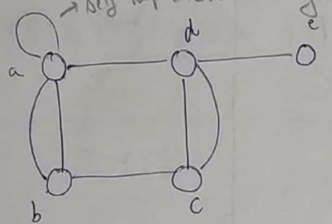




Degree of a Vertex

No of edges ~~inco~~ incident on a vertex.
(counting loop twice)

→ self loop contribute 2 degree



dg(e) = 1
dg(d) = 4
dg(c) = 3
dg(b) = 3
dg(a) = 5

V	d(V)
a	5
b	3
c	3
d	4
e	1

$|V| = 5$
$|E| = 8$

$\sum d(v) = 16 = 2|E|$

NOTE
★ No of odd degree vertices must be even

First Theorem of Graph Theorem (Handshaking Lemma)

In any graph $G = (V, E)$
The sum of degrees of vertices is twice the number of edges

i.e. $\boxed{\sum \deg(V) = 2|E|}$

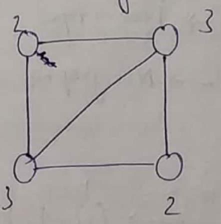
Proof:- Since each edge contribute 2 degrees

1 edge ⟶ 2 degree
∴ $|E|$ edges ⟶ $2|E|$

Result: In any graph $G = (V, E)$
The no. of odd degree vertices must be even.

Degree Sequence

It is the sequence of degrees in non-increasing order



(3, 3, 2, 2)

Def: A degree sequence is graphic if there exists a simple graph corresponding to it.

Havel-Hakimi Algorithm

Tell given degree sequence is graphic (or) not.

1) Arrange degrees in non-increasing order
2) Let "k" be the highest degree vertex then delete k and subtract 1 from next k degrees.
3) Continue step 1 and step 2 till stop condition is reached

Stop Condition

1) All zero entries ~~degrees~~ → graphic
2) Any -ve entry → Not graphic
3) Not enough degrees → Not graphic

Ex: (3, 3, 2, 2)

Sol: $(\cancel{3}, 3, 2, 2) \longrightarrow (2, 1, 1)$

$$\begin{array}{ccc} 3 & 2 & 2 \\ -1 & -1 & -1 \\ \hline 2 & 1 & 1 \end{array}$$

$(\cancel{2}, \underline{1, 1}) \longrightarrow 0, 0 \Rightarrow$ graphic

Ex: (4, 3, 3, 2, 1, 1)

Sol: $\cancel{4}, \underline{3, 3, 2, 1}, 1 \longrightarrow (2, 2, 1, 0, 1)$

$(\cancel{2}, \underline{2, 1}, 1, 0) \longrightarrow (1, 0, 1, 0)$

$(\cancel{1}, 1, 0, 0) \longrightarrow (0, 0, 0) \Rightarrow$ graphic

Ex: (4, 2, 2, 2, 1, 1)

Sol: $(\cancel{4}, \underline{2, 2, 2, 1}, 1) \longrightarrow (1, 1, 1, 0, 1)$

$(\cancel{1}, \underline{1}, 1, 1, 0) \longrightarrow (0, 1, 1, 0)$

$(\cancel{1}, \underline{1}, 0, 0) \longrightarrow (0, 0, 0) \rightarrow$ graphic

Ex: (3, 2, 1, 1)

Sol: $(\cancel{3}, 2, 1, 1)$

$(\cancel{1}, 0, 0) \longrightarrow (-1, 0) \rightarrow$ not graphic

HP Q2 Sol: Ex: $\cancel{7}, \underline{6, 5, 4, 3, 2, 1} \rightarrow$
↳ not enough degrees

not graphic

Ho

Q 2

Sol I) ~~7~~, 6, 5, 4, 4, 3, 2, 1

~~5~~, 4, 3, 3, 2, 1, 0

~~3~~, 2, 2, 1, 0, 0

~~1~~, 1, 0, 0, 0

0, 0, 0, 0 graphic

II > 6, 6, 6, 6, 3, 3, 2, 2

~~5~~, 5, 5, 2, 2, 2, 1

~~4~~, 4, 1, 1, 1, 1

3, 0, 0, 0, 1 → ~~3~~, 1, 0, 0, 0

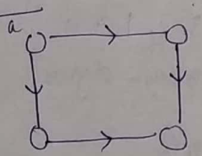
0, -1, -1, 0 not graphic

III 7, 6, 6, 4, 4, 3, 2, 2 → graphic

(IV) ~~8~~, 7, 7, 6, 4, 2, 1, 1 → not enough degree

10/7/2018

Digraph



indegree : No of edges incident in.

outdegree : No of edges incident out.

V	in(v)	O(v)
a	0	2
b	1	2
c	1	1
d	3	0

$$\boxed{\sum in(v) = \sum O(v) = 5 = |E|}$$

Result

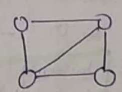
In a digraph ~~no of edges inc~~ $G = (V, E)$

$\boxed{|E| = \text{Sum of } in(v) \text{ degree} = \text{the sum of out degree}}$

Notation In graphs

$\delta \to$ min degree

$\Delta \to$ max degree



$\delta = 2$

$\Delta = 3$

Result

In any graph G with v vertices and e edges

$$\boxed{\delta \le \frac{2e}{|V|} \le \Delta}$$

Proof: I) $\sum d(v) = 2e$

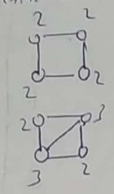
replace every deg with δ

$\underbrace{\delta + \delta + \delta \cdots \delta}_{V \text{ times}} \leq 2e$

$V \cdot \delta \leq 2e$

$\boxed{\delta \leq \frac{2e}{|V|}}$

2+2+2+2 = 8   4(2)



2+2+2+2 ≤ 2e

II) replace every deg with Δ

$V \cdot \Delta \geq 2e$

$\boxed{\Delta \geq \frac{2e}{|V|}}$

Ex. Let $G = (V, E)$ be a graph with 11 edges and $\delta = 3$. max no of vertices in G = ~~8~~ 7

NOTE: In case of $\leq$ less than equal to we take floor

$\delta \leq \frac{2e}{|V|}$

$|V| \leq \frac{2 \times 11}{3}$

$|V| \leq \lfloor 7.33 \rfloor = 7$

Ex: Let $G = (V, E)$ be a graph with 7 vertices and $\delta = 5$, Min no of edges G = 18

NOTE: In case of greater than equal to we take ceil

$\delta \leq \frac{2e}{|V|}$

$\frac{5 \times 7}{2} \leq e$

$\lceil 17.5 \rceil \leq e$

$18 \leq e$

Ex: Let G be a graph with 13 edges and $\Delta = 5$ then min no of vertices in G = 6

Sol: $\frac{2|e|}{|V|} \leq \Delta$

$\frac{2 \times 13}{5} \leq |V|$

$\frac{26}{5} \leq |V|$

$\lceil 5.2 \rceil \leq |V|$

$\min(|V|) = 6$

Ex. let G be a graph with 11 vertices and $\Delta = 7$. max no of edges in G = 38

$\frac{2e}{|V|} \leq \Delta$

$2 \cdot e \leq 7 \times 11$

$e \leq \frac{77}{2}$

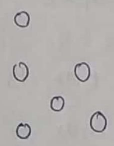
$e \leq \lfloor 38.5 \rfloor$

$e \leq 38$

Special Graphs

1) Null graph (N_n) : Graph with no edges

N_1 N_2 N_3

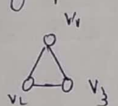


NOTE:
1) $d(v) = 0$ isolated vertex

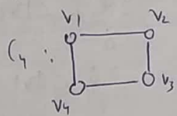
wit $d(v) = 1$, we call it as Pendent vertex

2) Cycle graph (C_n ; $n \geq 3$)

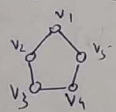
C_3 :



C_4 :



C_5 :

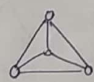


A cycle graph C_n is a simple graph with n vertices $v_1, v_2, v_3, \ldots v_n$ such that i) each v_i is adjacent to v_{i+1} $i = 1, 2, 3, \ldots n-1$
& ii) v_n is adjacent to v_1

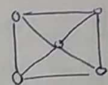
~~degr~~

Wheel graph (W_n ; $n \geq 3$)

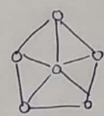
W_3 :



W_4 :



W_5 :

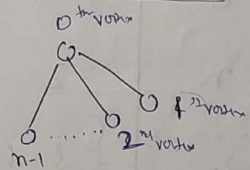


Wheel graph W_n is a cycle graph with additional vertex (hub) which is adjacent to all the vertices of C_n.

Results

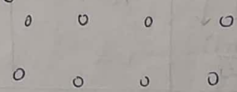
1) Maximum degree of a vertex in a simple graph with n vertices = $n-1$

$d(v) = n-1$



2) Max ~~degree of~~ no of edges in a simple graph with n vertices = $^nC_2 \Rightarrow \frac{n(n-1)}{2}$

reason



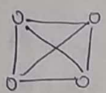
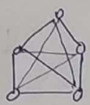
Every pair of vertices must be adjacent. max no of edges = no of 2-C of n vertices $= {}^nC_2$

2-C => 2 combinations

ie for max no of edges, every vertex must be adjacent to every other vertex (ie it must form a complete graph)

4) Complete graph (K_n)

K_1 o K_2 :  K_3 : 

K_4 :  K_5 : 

Simple graph in which every pair of vertices are adjacent.

(Imp)

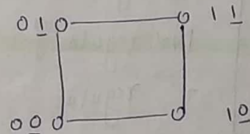
G	V	e	d(v)	
1) Null (N_n)	n	0	0	
2) Cycle graph C_n	n	n	2	
3) Wheel graph (W_n)	$n+1$	$2n$	3 n	when v is non-hub when v is hub
Complete graph (K_n)	n	$\binom{n}{2}$ ie $\frac{n(n-1)}{2}$	$n-1$	
Ex: K_4	4	6	3	
K_5	5	10	4	

5) Q_n (n-cube)

Q_n is a simple graph with 2^n vertices whose vertices are labelled with 2^n binary strings of length n such that two vertices in Q_n are adjacent iff they differ in exactly one bit position

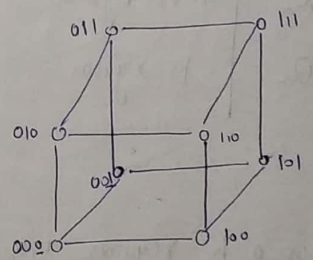
Q_1 

$V = 2^1$

Q_2 

take Q_1 and join corresponding vertices

$V = 2^2 = 4$

Q_3 

$V = 2^3 = 8$

G	V	e	d(v)
Q_n	2^n	$n \cdot 2^{n-1}$	n

Scanned by CamScanner

Regular Graph

It is a graph in which every vertex has same degree

$\forall v \quad d(v) = k \longrightarrow k\text{-regular}$

G	Type
N_n	0 - regular
C_n	2 - regular
W_n	Not regular for $n > 3$
W_3	3 - regular
K_n	$(n-1)$ regular
Q_n	n - regular

Result

If G is a k-regular graph with n vertices then no of edges in G = ________

Sol: $\sum d(v) = 2e \Rightarrow \underbrace{k+k+k+\dots}_{n \text{ times}} = 2e$

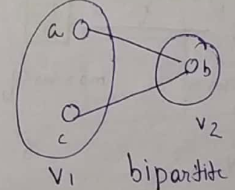
$k \cdot n = 2e$

$\frac{k \cdot n}{2} = e$

7) Bipartite graph

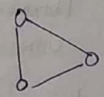
A bipartite graph $G = (V, E)$ is a simple graph in which vertex set V is ~~partitioned~~ partitioned into 2 sets V_1 and V_2 $(V_1 \cup V_2 = V)$ such that every edge in G is b/w a vertex of V_1 to a vertex of V_2 only

i)



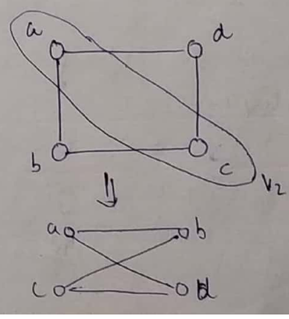
bipartite

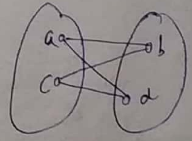
ii)



Not bipartite

iii)



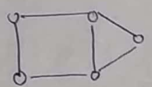


bipartite

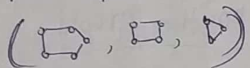
Result

$G = (V, E)$

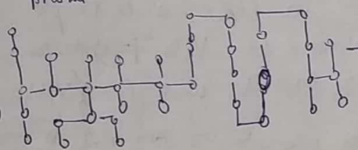
1) A simple graph is bipartite iff every cycle in G is even cycle



Not bipartite



Cycles present — (C_5, C_4, C_3)

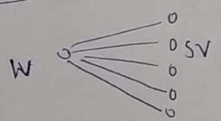


(Tree) → Every cycle is 0-cycle. Hence it is bipartite.

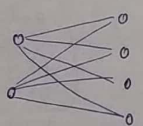
Result

Maximum no of edges in a bipartite graph with n vertices $= \left\lfloor \frac{n^2}{4} \right\rfloor$ (Greatest)

Bipartite G — 6 vertices — max no of edges



1V, 5V — 5



2V, 4V — 8

$n/2$, $n/2$ — 9 — $\frac{n^2}{4}$

max edges we get when we divided half-half

Complete Bipartite Graph ($K_{m,n}$)

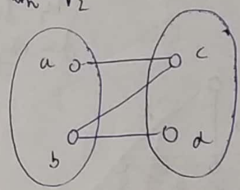
$K_{m,n}$ is a bipartite graph $G = (V_1 \cup V_2, E)$

$|V_1| = m$, $|V_2| = n$

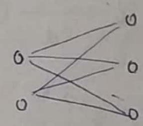
such that,

every vertex in V_1 is adjacent to every vertex in V_2 (all vertices of V_2)

Ex.



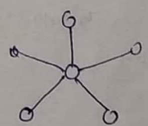
Bipartite, but no complete bipartite



$K_{2,3}$

G	V	e	d(v)
$K_{m,n}$	$(m+n)$	mn	n, $v \in V_1$; m, $v \in V_2$

NOTE: $K_{1,n}$ is called star graph

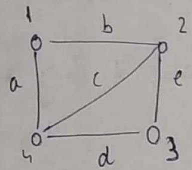


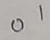
$K_{1,5}$

Subgraph

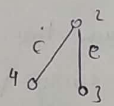
$H_1 = (V_1, E_1)$ is subgraph of $G = (V, E)$

if $V_1 \subseteq V$ and $E_1 \subseteq E$

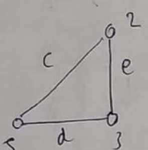


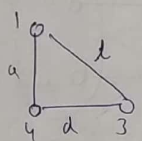


H_1 Subgraph



H_2 Subgraph formed by {2,3,4}

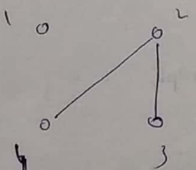




not a subgraph

H_3 Subgraph ~~indirect~~ induced → obtained by deleting set of vertices.

Induced by {2,3,4}



~~Subgraph~~ Subgraph containing all the vertices of a graph is called spanning subgraph

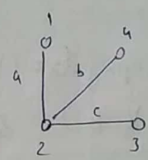
✱ Operations

$G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$

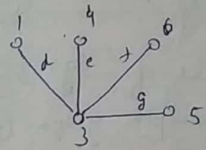
Union: $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$

Intersection $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$

Ex.



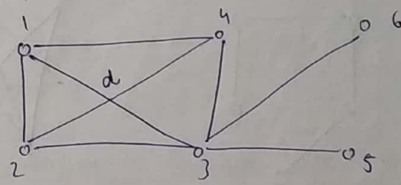
G_1



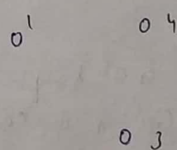
~~$G_1 \cup G_2 = (V, E)$~~

~~$V = \{1, 2, 3, 4,$~~

$G_1 \cup G_2 =$



$G_1 \cap G_2$

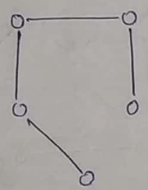


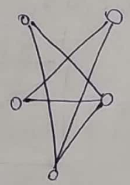
Com

Complement of a graph

Complement of a given simple graph $G = (V, E)$ is a simple graph $G' = (V, E')$ such that two vertices in G^c are adjacent iff they are not adjacent in G.

Ex:





$G \cup G' = K_5$

~~Result~~ Ex: G is a simple graph with 13 edges and G' is its complement with 15 edges then no of vertices in G

$13 + 15 = {}^nC_2$

$28 = \frac{n(n-1)}{2} \Rightarrow 2 \times 28 = n(n-1)$

$8 \times 7 = n(n-1)$

$n = 8$

Result: G is a simple graph with n vertices & e edges and G' is its complement with e' edges

$G \cup G' = K_n$

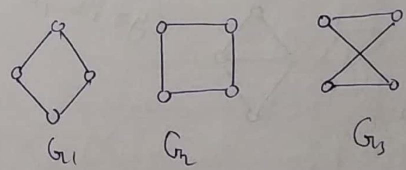
$$\boxed{e + e' = {}^nC_2}$$

${}^nC_2 = \frac{n(n-1)}{2}$

Isomorphism

→ Isomorphic graphs are same graphs drawn differently

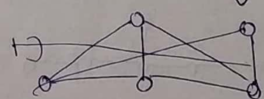
→ G_1 is isomorphic to G_2 we write $G_1 \cong G_2$

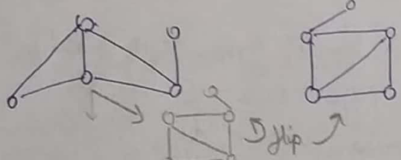


$G_1 \cong G_2 \cong G_3$

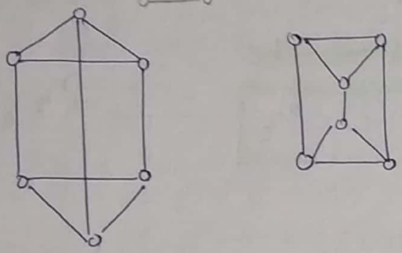
Q Check which of the following pair are ~~isomorphic~~ not isomorphic

1)

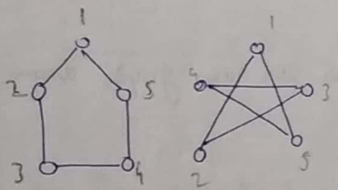


(1) 

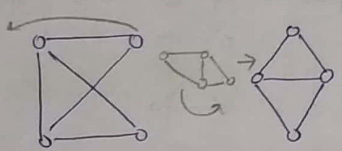
not isomorphic

2) 

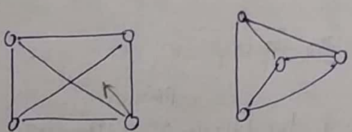
yes isomorphic

3) 

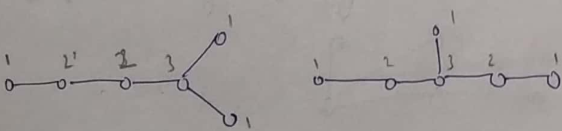
yes isomorphic

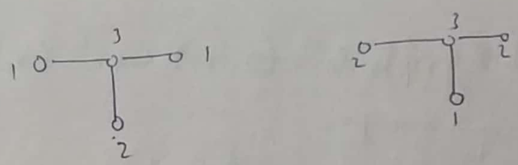
4) 

yes isomorphic

5) 

yes isomorphic

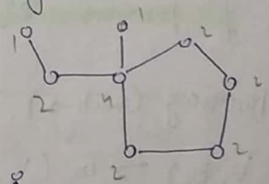
6) 

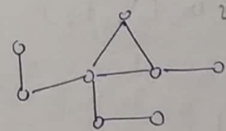


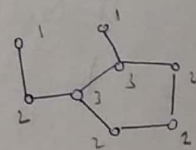
∴ adjacency not preserved. Not Isomorphic

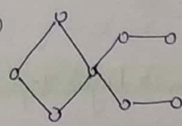
Gate

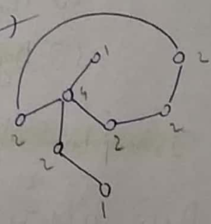
Q Which of the following is isomorphic to

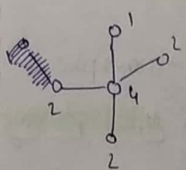


a) 

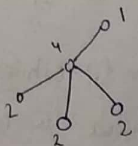
c) 

b) 

d) 



given



Here adjacency also preserved

Def: Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic

if there exist a fun $f : V_1 \longrightarrow V_2$

Such that

(1) f is 1-1

(2) f is onto

(3) f preserves adjacency

Result : (Necessary conditions)

If $G_1(V, E_1) \cong G_2(V_2, E_2)$

then

1) $|V_1| = |V_2|$

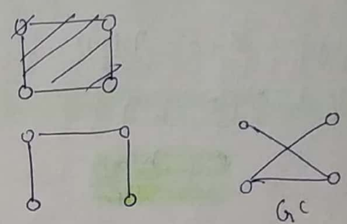
2) $|E_1| = |E_2|$

3) Degree sequence of G_1 = Degree sequence of G_2

→ Isomorphism algorithm complexity $= O(n!)$

Def: A simple graph G which is isomorphic to its complement is called self-complementary graph

ie G is complementary means $G \cong G^c$



G

G^c

$G \cong G^c$ (self-complementary)

Q which of the following can not be no of vertices in a self-complementary graph.

a) 4 b) 5 c) 9 d) 10

no of edges with

a) $K_4 \to 6$ → 3 (G), 3 (G^c)

b) K_5 → 10 → 5, 5

c) K_9 → 36 → 18, 18

d) $K_{10} \to 45$ → X, X — Can not be equal distributed b/w G & G^c

Result (Imp)

G is a self complementary graph with n vertices and e edges and G^c is its ~~Complementary~~ complement with e' edges

(1) $G \cup G^c = K_n$

(2) $e = e'$

3) $e = \frac{n(n-1)}{4}$

Proof :- $e + e' = \frac{n(n-1)}{2}$

$\because (e = e')$

$e = \frac{n(n-1)}{4}$

4) n is of the form $4k$ (or) $4k+1$

Proof $e = \frac{n(n-1)}{4}$

$\therefore 4 | n$ (or) $4 | n-1$

$\frac{n}{4} = k$ $\qquad$ $\frac{n-1}{4} = k$

$n = 4k$ (or) $n = 4k+1$

Ex: A cycle on n-vertices is isomorphic to its complement. What is the value of $n =$ ______

G self-complementary	G is cycle
$e = \frac{n(n-1)}{4}$	$e = n$

both conditions need to be satisfied simultaneously

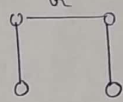
$n = \frac{n(n-1)}{4} \Rightarrow 4 = n-1 \Rightarrow n = 5$

$n = 5$

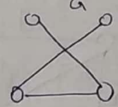
Note: C_5 is the only self-complementary cycle

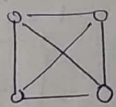
(Imp) Ex: $(d_1, d_2 \ldots d_n)$ is degree sequence of G then degree sequence of $G' =$ ______

G



G^c





K_4

Ans: $\boxed{(n-1-d_1,\ n-1-d_2,\ n-1-d_3, \ldots\ldots\ n-1-d_n)}$

K_4 max deg $= 3$

K_n max deg $= n-1$

d_1	$n-1-d_1$
d_2	$n-1-d_2$
$\vdots$	$\vdots$

Walk, path, cycle

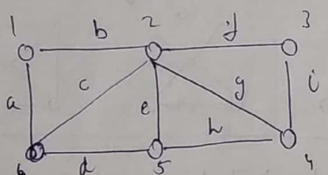
An alternating sequence of vertices & edges is called walk

A walk which starts and end at the same vertex is called closed walk

→ A walk in which no edge is repeated is called trail.

→ A walk in which no vertex is repeated is called path.

→ A closed walk with 3 or more vertices in which no vertex, except starting vertex is repeated is called cycle.



1 b 2 b 1 b 2 j 3 · walk

1 b 2 b 1 closed walk

1 b 2 e 5 d 6 c 2 j 3 trail

Representation

Adjacency Matrix

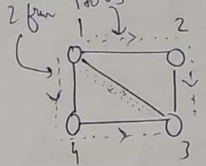
Let G be a graph with n vertices and no parallel edges

Thus,

$$A_G = [a_{ij}]_{n \times n}$$

where, $a_{ij} = \begin{cases} 0 & (i,j) \notin E \\ 1 & (i,j) \in E \end{cases}$

2 walks of length 2 from 1 to 3



$$A_G = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 1 \\ 4 & 1 & 0 & 1 & 0 \end{array}$$

$$A_G^2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{array}{l} \deg(V) \\ \to 3 \\ \to 2 \\ \to 3 \\ \to 2 \end{array}$$

means no self loop

$$A_G^2 = \begin{bmatrix} 3 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

11/7/18

Length of walk : no of edges in the walk

Result : The $(i,j)^{th}$ element of A_G^k denote the no of walks of length k from vertex i to vertex j

WB
Q33
Sol.

$$\begin{array}{c|c} & \begin{array}{cccccc}1&2&3&4&5&6\end{array} \\ \begin{array}{c}1\\2\\3\\4\\5\\6\end{array} & \begin{bmatrix} 0&1&0&0&1&1\\ 1&0&1&0&1&0\\ 0&1&0&1&1&1\\ 0&0&1&0&1&0\\ 1&1&1&1&0&1\\ 1&0&1&0&1&0 \end{bmatrix} \end{array} \begin{array}{c} \begin{array}{cccccc}1&2&3&4&5&\end{array} \\ \begin{bmatrix} 0&1&0&0&1&1\\ 1&0&1&0&1&0\\ 0&1&0&1&1&1\\ 0&0&1&0&1&0\\ 1&1&1&1&0&1\\ 1&0&1&0&1&0 \end{bmatrix} \end{array}$$

$= (3,5)^{th}$

$= 3$

(or)

5th column

$$3^{rd}\text{ row } \begin{bmatrix} 0&1&0&1&1&1 \end{bmatrix} \times \begin{bmatrix} 1\\1\\1\\1\\0\\1 \end{bmatrix} = 3$$

WB
Q37
Sol:

$$A^2 = \begin{array}{c|c} & \begin{array}{cccc}1&2&3&4\end{array} \\ \begin{array}{c}1\\2\\3\\4\end{array} & \begin{bmatrix} 0&0&0&1\\ 1&0&1&1\\ 1&0&0&1\\ 0&0&1&0 \end{bmatrix} \end{array} \begin{bmatrix} 0&0&0&1\\ 1&0&1&1\\ 1&0&0&1\\ 0&0&1&0 \end{bmatrix}$$

$$\begin{bmatrix} 0&0\\ &\times\ 0\\ &0\\ &0 \end{bmatrix}$$

We have to find A^3

if $A^T \neq A$ → then this Matrix A ~~is~~ does not represents Digraph

if $A^T = A$, then A represents a simple graph (ie undirected graph)

✱ 2 vertices U & V are said to be connected if there exists at least 1 path b/w them.

Def: A graph in which every pair of vertices are connected is called ~~connected is called~~ connected graph.

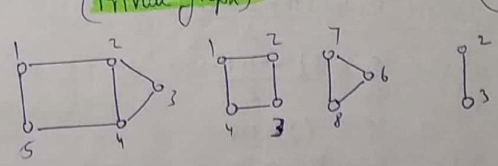
Otherwise, graph is disconnected

Def: The maximal connected subgraph of a graph is called Component. (we can't add anything more to it)

(minimal - means we can't remove anything from it)

o → single vertex is always connected
6
(Trivial graph)

Ex



Results

If G is a simple graph with n vertices and $\delta \geq \frac{n-1}{2}$ then G is connected

$p \to q$

(Sufficient Condition)

Condition is in P therefore sufficient

But not Conversly

Ex: C_7 $\delta = 2 \not> \frac{7-1}{2}$

but G is connected

2) If G is a simple graph with n vertices and k components then max no of edges in G is $\frac{(n-k)(n-k+1)}{2}$ (Necessary condition)

ie $e \le \frac{(n-k)(n-k+1)}{2}$ Here condition is in q

Ex: G is a simple graph with 13 vertices and 4 components. find max no of edges in G.

Sol:-

$$e \le \frac{(n-k)(n-k+1)}{2}$$

$$e \le \frac{(13-4)(13-4+1)}{2}$$

$$e \le \frac{(9)(10)}{2}$$

$$e \le 45$$

3) If G is a simple graph with "n" vertices and no of edges $e > \frac{(n-1)(n-2)}{2}$ then G is connected

(Sufficient condition)

Proof: G is disconnected ← our assumption

$k = 2$

$$e \le \frac{(n-2)(n-2+1)}{2}$$

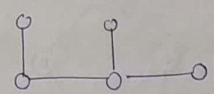
$$e \le \frac{(n-2)(n-1)}{2}$$

∴ Our assumption is wrong.

∴ G is connected

Note: Converse need not be true

Ex



$n = 5$

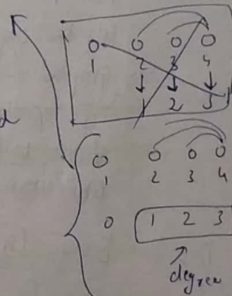
$e = 4$

$e > \frac{(n-1)(n-2)}{2}$

$e \not> 6$, but G is connected

Result (Crate)

In a connected simple graph there are atleast 2 vertices of same degree.



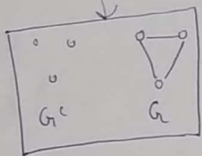
Result: G or G' must be connected

Ex.

Ex: Which of the following always TRUE?

(Imp) I : G is connected then G^c is disconnected.

II : G^c is disconnected then G is connected. ✓ T



a) only I true

~~b)~~ only II true

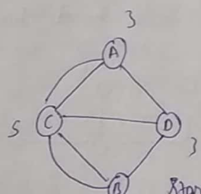
c) both true

d) none

G G^c

I need not be true

Euler graph



Def: Euler ~~walk~~ path

every edge covered exactly once starting & ending at different vertices

An open walk containing all the edges of a connected graph in which no edge is repeated is called Euler walk or ~~Unico~~ unicursal ~~Unicursal~~ line

Start at any vertex cover every edge exactly once and come back to starting position.

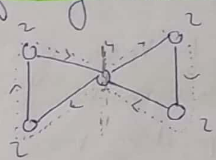
Def: Euler Circuit every edge covered exactly once, starting ending at same vertex.

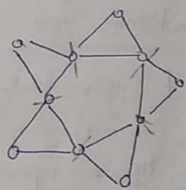
A closed walk containing all the edges of a connected graph in which no edge is repeated is called Euler Circuit.

Def: A connected graph containing Euler circuit is called Euler graph

· A connected graph containing open Euler walk is called Unicursal graph

Result: A connected graph is Euler graph iff degree of every vertex is even (p2→79)



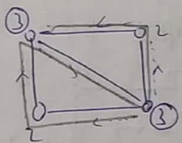


(Imp) (Gate) Ex: C_{25}^{c} every vertex degree $= 24 - 2 \ \forall v$

$= 22$

∴ it is Euler graph

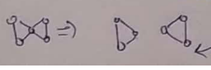
Ex.



← Unicursal graph

Result: A connected graph is unicursal graph iff there are exactly 2 vertices of odd degree

Result: A connected graph is Euler iff it can be decomposed into edge disjoint cycle



edge disjoint cycle

Hamiltonian Graph

Hamiltonian Path

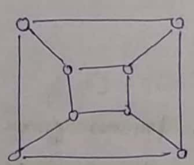
An open path containing all the vertices of a connected graph in which no vertex is repeated

Hamiltonian Cycle

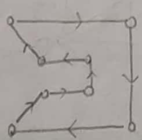
A cycle containing all the vertices of a connected graph in which no vertex is ~~repeated~~, repeated. (except starting vertex) is called Hamiltonian cycle or Spanning cycle

A connected graph containing Hamiltonian cycle is called Hamiltonian graph.

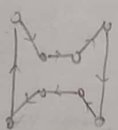
$Q_3 \Rightarrow$



H. Graph

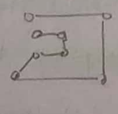


H. cycle



H cycle

H. cycle $\xrightarrow[\text{get}]{\text{we can}}$ H. path   vice versa not true



Result Graph G is H. graph with n vertices then

(1) number of

i) vertices in H. cycle = n

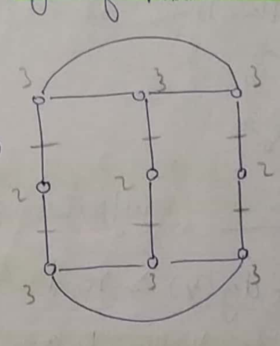
ii) edges in H. cycle = n

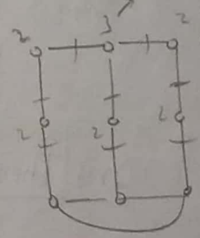
iii) vertices in H. path = n

edges in H. path = $n-1$

(2) Degree of vertex in H. cycle = 2

H. cycle not possible



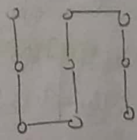


to get H. cycle

1) fix edges corresponding to even degree vertex

2) Try to reduce degree of odd degree vertices to even. (by removing the edges).

H. path possible →



Results

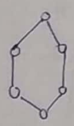
1) Dirac's Theorem

If G is a simple G with n vertices and $\delta \geq \frac{n}{2}$ then G is Hamiltonian

[Sufficient Condition]

NOTE: Converse need not be true

Ex: C_6 is Hamiltonian but $\delta = 2 \not\geq 3$



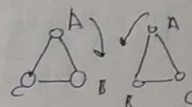
2) Ore's Theorem [Sufficient Condition]

If $\deg(u) + \deg(v) \geq n$ for every pair of non-adjacent vertices u and v in a simple graph with n vertices then G is Hamiltonian

If sum of degrees of every pair of non-adjacent vertices $\geq n$

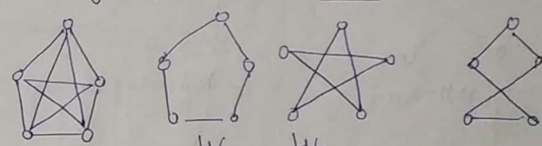
3) No of H-cycles in K_n $(n \geq 3)$ = $\frac{(n-1)!}{2}$

Since clock wise & anti clock wise circular permutation represent same graph



Diff CP but same graph

Ex: No of H-cycle in K_5 = 12



HC1 HC2

Edge Disjoint cycles

4) No of edge disjoint H-cycle in K_n $(n \geq 3)$ with odd no of vertices = $\frac{(n-1)}{2}$

Ex: no of edge disjoint cycles in K_5 & K_9

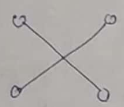
Sol: $K_5 = 2$

$K_9 = 4$

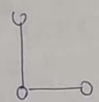
NOTE: this problem is analogous to no of ways 5 people can be seated around a circular table such that every person has different neighbour in every arrangement

Planar Graph

Def: drawing a graph in a plane without cross-overs is called planar representation or planar embedding



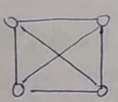
Cross-over



not cross over

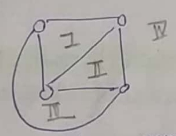
Def: A graph having planar representation is called planar graph.

Ex.



planar graph

$\Rightarrow$



planar representation

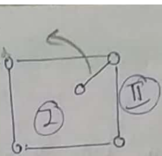
No of regions = 4

I, II, III → interior region — also called bounded faces

IV → exterior region

Def: The planar representation of planar graph divides the entire plane into regions (or) faces.

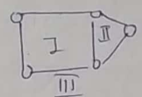
Def: No of edges in the boundary of a region is called degree of a region



interior edge is not considering as boundary edge

r	d(r)
I	4
II	4

$\sum d(r) = 8$



r	d(r)
I	4
II	3
III	5

bounded faces → 2
I, II

$\sum d(r) = 12$

2|E| when we don't have interior edge

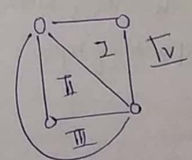
Result:

1) In a planar graph ~~the sum of degrees of region~~ $G = (V, E)$

$$\boxed{\text{Sum of degrees of regions} \leq \text{twice the no of edges}}$$

ie $\boxed{\sum d(r) \leq 2|E|}$

Ex)



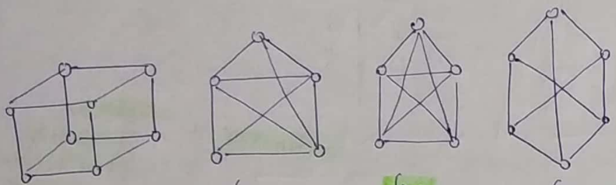
$V - e + r = \underline{\quad}$

$4 - 6 + 4 = 2$

Euler's formula

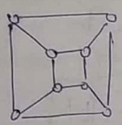
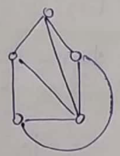
In a connected planar graph with v-vertices, e-edges, and r-regions we have

$$\boxed{v - e + r = 2}$$



G_1 G_2 G_3 G_4

G_3 & G_4 are non-planar

$G_1 \longrightarrow$  G_2 

~~$G_3 \rightarrow$ we assume K_5~~ : ~~Here~~ G_4 is $K_{3,3}$ which is a bipartite graph

Note: minimum degree of region ≥ 3.

Result In a connected planar graph with min degree of region $= 3$

1) $3r \leq 2e$

2) $e \leq 3v - 6$

Proof: 1) $\sum d(r) \leq 2e$

replace every degree with min degree

$$\underbrace{3+3+3+3 \dots +3}_{r \text{ times}} \leq 2e$$

$$\boxed{3r \leq 2e}$$

$$r \leq \frac{2e}{3}$$

2) $2 = v - e + r$ $\qquad 3r \leq 2e$

$2 \leq v - e + \frac{2e}{3}$

$2 \leq v - \frac{e}{3}$

$6 \leq 3v - e$

$$\boxed{e \leq 3v - 6}$$

Scanned by CamScanner

Result

In a connected planar graph with min degree of region = k

we have

1) $kr \le 2e$

2) $e \le \frac{k(v-2)}{k-2}$

In case of complete bipartite graph $K_{m,n}$ min deg $(r) = 4$

Note: min degree of region = 4

$\boxed{4r \le 2e} \rightarrow 2r \le e$

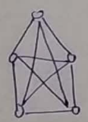
$e \le \frac{4(v-2)}{2}$

$e \le 2(v-2)$

$\boxed{e \le 2v-4}$

Result

Ex K_5



K_5 is non-planar

$v = 5$ $e = 10$

We assume K_5 as planar

1) $v - e + r = 2$

$5 - 10 + r = 2$

$r = 7$

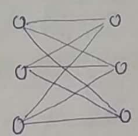
2) $3r \le 2e$

$3(7) \le 2(10)$

$21 \nleq 20$

Our assumption is wrong

Result $K_{3,3}$



$v = 6$ $e = 9$

We assume $K_{3,3}$ to be planar

min deg of region = 4

1) $v - e + r = 2$

$6 - 9 + r = 2$

$r = 5$

~~2) $3r \le 2e$~~

~~$3(5) \le 2(9)$~~

~~$15 \le 18$~~

~~4) $e \le 3v - 6$~~

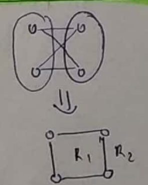
~~$9 \le 3(6) - 6$~~

~~$9 \le 18 - 6$~~

~~$9 \le 12$~~

Note: In a $K_{m,n}$ min deg = 4

In bipartite deg of region is even or ≥ 4 greater to



$d(R_1) = 4$

$d(R_2) = 4$

$4r \le 2e$

$4(5) \le 2(9)$

$20 \nleq 18$

Hence our assumption is wrong

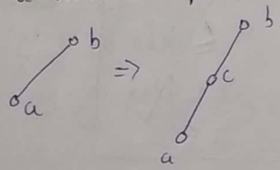
$K_{3,3}$ is non-planar

Kuratowski's graph

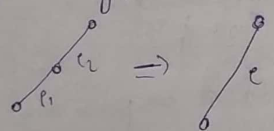
1) K_5 & $K_{3,3}$
2) Both are non-planar graph
3) K_5 is non-planar graph with min no of vertices
4) $K_{3,3}$ is non-planar graph with min no of edges

~~Homo~~ Homeomorphic Operations

1) Adding a vertex in sequence



2) Merging the edges in the sequence

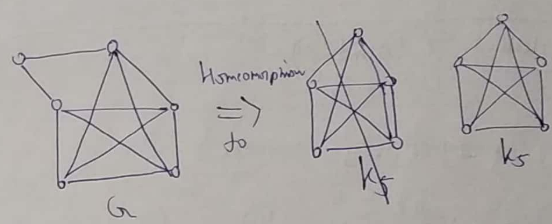


Kuratowski's Theorem

A Graph G is planar iff it does not contain any subgraph homeomorphic to K_5 (or) $K_{3,3}$

$p \leftrightarrow q$

$\sim p \leftrightarrow \sim q$



G is non-planar

HW (Imp)

ex

sol:

$V = 10$

$d(R) = 3$

no interior edge

$\sum d(R) = 2e$

$3R = 2e$

$v - e + R = 2$

$10 - e + \frac{2e}{3} = 2$

$10 - \frac{e}{3} = 2 \Rightarrow e = 24$

HO

Q1.

Sol:

$v = 13$

$e = 19$

$v - e + r = 2$

$13 - 19 + r = 2$

$r = 8$

Imp

Q11

bounded region = (regions − 1)

$10 - 15 + r = 2$

$r = 7$

bounded region $= 7 - 1$

$= 6$

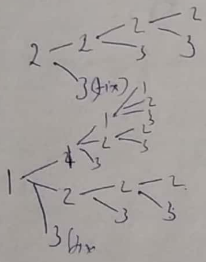
12/8/18

HO solution (Counting)

Q1

Sol:

$3 - 3 - 3 - 3$ (1)



(4)

(10)

Total = 15

Q7

Sol:

$k + {}^{k}C_{2} \geq 26$

means selecting but not arranging

if (R, G) used then (G, R) ~~not used~~ can not be used

$$A = \frac{(2n)!}{(n!)^2 2}$$

$$B = \frac{(2n)!}{2^n n!}$$

(no of perfect matching = always integer.

$${}^{2n}C_n = \frac{2n!}{n!\,n!}$$

$${}^{2n}C_n = \frac{2\cancel{n}(2n-1)!}{\cancel{n}(n-1)!\,n!}$$

$$\frac{1}{2}\,{}^{2n}C_n = \frac{(2n-1)!}{(n-1)!\,n!}$$

$$\frac{1}{2}\frac{(2n)!}{(n!)^2} = \underbrace{{}^{2n-1}C_{n-1}}_{\text{integer}}$$

Graph Theory Continued

HO

Q12

Sol:

$G = (V, E)$

$e \leq 3v - 6$

Let min degree of G be $\delta = k$ $\left(\because \delta \leq \frac{2e}{|V|} \leq \Delta\right)$

$kv \leq 2e$ — (1)

$2e \leq 6v - 12$ — (2) (Multiply by 2)

$kv \leq 6v - 12$ (from (1) and (2))

put $k = 3, 4, 5, 6$

put $k = 6$

$6v \leq 6v - 12$

$0 \nleq -12$ $\therefore$ min degree can not be 6

HP
Q13
Sol.

$\delta \geq 3$

$v - e + r = 2$

$3v \leq 2e \quad \left(\because \ \delta \leq \frac{2e}{v}\right)$

$3v \leq 2(v + r - 2)$

$3v \leq 2v + 2r - 4$

$v + 4 \leq 2r$

$2r \geq v + 4$

$r \geq \frac{v}{2} + 2$

WB
Q23
Sol.

Cubic $\cdot \rightarrow$ 3-regular

$\Sigma d(v) = 2e$

$\boxed{3v = 2e}$ — ①

planar with no bridge (ie no minor edge)

$\Sigma d(r) = 2e$

$5r = 2e$

Euler formula $v - e + r = 2$

$\frac{2e}{3} - e + \frac{2e}{5} = 2$

$10e - 15e + 6e = 30$

$e = 30$ Ans

Result : In a planar graph with

v - vertices
e - edges
r - regions
k - components

we have $\boxed{v - e + r = k + 1}$

1	2	3	. . . .	k
v_1	v_2	v_3	. . . .	v_k
e_1	e_2	e_3		e_k
r_1	r_2	r_3		r_k

$v_1 - e_1 + r_1 = 2 + (v_2 - e_2 + r_2 = 2) + \ldots \ldots + (v_k - e_k + r_k = 2)$

$v_1 + v_2 + \ldots + v_k = V$

$e_1 + e_2 + \ldots + e_k = E$ external region is counted only once

$r_1 + r_2 + \ldots r_k - (k-1) = r \Rightarrow r_1 + r_2 + \ldots + r_k = r + k - 1$

$(v_1 + v_2 + \ldots + v_k) - (e_1 + e_2 + \ldots + e_k) + (r_1 + r_2 + \ldots + r_k) = 2k$

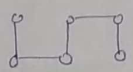
$\boxed{\cancel{V - E + r_1 + r_2 + \ldots + k - 1 = 2k}}$

$V - E + r + k - 1 = 2k$

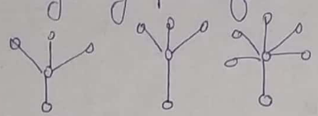
$\boxed{V - E + r = k + 1}$

Trees

Def: Connected and acyclic graph is Tree.



An acyclic graph is forest.



Every tree is forest but every forest need not be tree

Result The following properties are equivalent for any graph T with n vertices (means if 1 stmt True others are also true)

(1) T is a tree

(2) T is connected and has (n-1) edges

(3) T is Acyclic and has (n-1) edges

(4) T is minimally connected

(5) There exist exactly one path b/w every pair of vertices in T.

Result: T is a tree with n vertices then no of edges in T = n-1

Ex: Let T be a tree with ~~5 vertices of degree~~

5 vertices of degree 2

3 vertices of degree 3

2 vertices of degree 4

and remaining vertices of degree 1

No of vertices in T = ___

Sol:-

~~5×2 + 3×3 + 2×4 + 1(n-10)×1 = 2~~

Degree of Vertex	no of Vertices
1	x
2	5
3	3
4	2

$\sum d(V) = x \cdot 1 + (5)2 + (3)3 + (4)2$

$= 27 + x$

$\sum d(v) = 2e$

$27 + x = 2e$ —(1)

$n = x + 5 + 3 + 2$

$n = 10 + x$

$e = n - 1$

$e = x + 9$ —(2)

$$27 + x = 2(9 + x)$$
$$27 + x = 18 + 2x$$
$$9 = x$$

$\therefore$ total vertices $= 9 + 10 = 19$ <- Ans

WB
Q17

Sol:

v	deg
n_2	2
n_3	3
.	
.	
n_k	k
?	x

$$(n_2 + n_3 + n_4 + \dots n_k + x) - 1 = e$$
$$(\because e = n - 1)$$

$$\sum d(v) = 2e$$

$$1 \cdot x + 2n_2 + 3n_3 + 4n_3 + \dots kn_k = 2(e)$$

$$1 \cdot x + 2n_2 + 3n_3 + \dots + kn_k = 2(x + n_2 + n_3 + \dots + n_k - 1)$$

$\cancel{n_3}$:

$$x + 2n_2 + 3n_3 + \dots + kn_k = 2x + 2n_2 + 2n_3 + \dots + 2n_k - 2$$

$$n_3 + 2n_4 + \dots + (k-2)n_k = x - 2$$

$$\boxed{n_3 + 2n_4 + \dots + (k-2)n_k + 2 = x}$$

Imp

Gate: G is a forest with n vertices and ~~k-com~~ k-components.

No of edges in G = ________ ?

a) $\left\lfloor \frac{n}{k} \right\rfloor$ b) $\left\lceil \frac{n}{k} \right\rceil$ c) $n-k$ d) $n-k+1$

Sol: 1st component

1 . 2 . 3 . . . . k (ie k trees)

Vertices : n_1 n_2 n_3 ... n_k

edges : $n_1 - 1$ $n_2 - 1$ $n_3 - 1$ ... $n_k - 1$

$$n_1 + n_2 + n_3 + \dots + n_k = n$$

$$(n_1 - 1) + (n_2 - 1) + (n_3 - 1) + \dots + (n_k - 1) = e$$

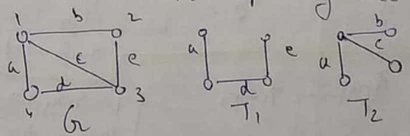
$$\underbrace{(n_1 + n_2 + \dots + n_k)}_{n} - k = e$$

$$\boxed{n - k = e}$$

Spanning Tree

(connected)

Spanning subgraph T of a graph G which is a tree is called Spanning Tree



$T_1 \rightarrow$ a, d, e $\rightarrow$ branches (edge of a spanning tree)

c, b $\rightarrow$ chords (not an edge of spanning tree)

T_2 - a, c, b $\rightarrow$ branches

d, e $\rightarrow$ chords

Result

G is a connected graph with n vertices & e edges.

No of vertices in any spanning tree $= n$

no of edges in any spanning tree $= n-1$

no of branches in any " " $= n-1$

no of chords in any " " $= e-(n-1)$
$= e-n+1$

Result

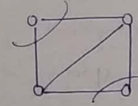
G is a connected graph with n vertices and e edges.

(v.v.Imp)

$\boxed{\text{No of spanning trees} = {}^{e}C_{e-n+1} - (\text{no of } e-n+1 \text{ selection of edges which disconnect the graph})}$

Ex: Find no of spanning trees

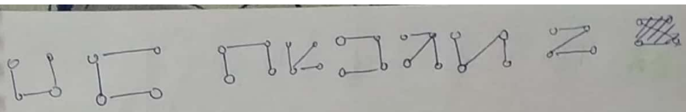
1)



$n = 4$
$e = 5$
branches $= 3$
chords $= 2$

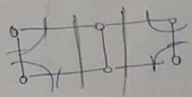
Sol: ${}^{5}C_{5-4+1} \Rightarrow {}^{5}C_{2} = 10 - 2 = 8$

We may not get spanning tree by removal of ~~(e-n+1) chords~~ every $(e-n+1)$ chords, some removal might disconnect the graph. Hence we need to subtract those cases.



WB
Q 24

Sol: $e = 7$
$n = 6$



chords $= e-n+1$
$= 7-6+1$
$= 2$

no of spanning tree $= {}^{7}C_{2} - (6)$
$= 21 - 6$
$= 15$

Result: No of spanning trees in a complete graph K_n

$K_n = n^{n-2}$

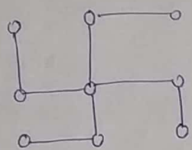
Removal of Vertex

Delete all the edges incident on the vertex and delete the vertex.

Cut Edge (Bridge)

A single edge whose removal disconnect the connected graph.

Ex: Find no of cut edges (bridges)



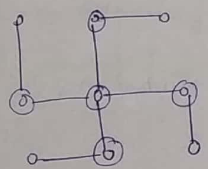
← this is a tree

8 → cut edges

Result: In a tree every edge is bridge (cut edge)

Cut Vertex (Articulation Point)

A single vertex whose removal disconnects the of connected graph is called cut vertex (or) articulation point.

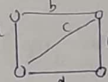


Articulation Point = 5

Def: A graph with no articulation points is called biconnected graph.

Cut ~~Set~~ Set

A set of edges whose removal disconnects the connected graph is cut set.



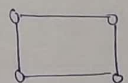
$C_1 = \{a, b, c, d, e\}$

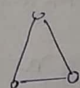
$C_2 = \{a, c, d\}$
$C_3 = \{b, c, e\}$
$C_4 = \{b, c, d\}$
} minimal cut set → any proper subset will not be a cut set.

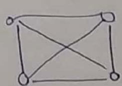
$C_5 = \{a, b\}$
$C_6 = \{d, e\}$
} smallest min cut set

Edge Connectivity (λ)

Minimum no of edges whose removal disconnects the connected graph

 $\lambda = 2$

 $\lambda = 2$

 K_4 $\lambda = 3$

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