14/7/18

Engineering Mathematics

1) Calculus — 1
2) Probability — 2
3) Linear Algebra — 1

Calculus

Limit

$\lim_{x \to a} f(x) = L$ means

$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$

ie LHL = RHL

Ex $f(x) = \dfrac{x^2-4}{x-2}$ $f(2) =$ not defined

$$\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{(x-2)} = 4$$

$x \to 2^-$	$x \to 2^+$
$x = 1.9$ $f(x) = 3.9$	$x = 2.1$ $f(x) = 4.1$
$x = 1.99$ $f(x) = 3.99$	$x = 2.01$ $f(x) = 4.01$
$x = 1.999$ $f(x) = 3.999$	$x = 2.001$ $f(x) = 4.001$

2

NOTE: $f(a)$ need not be defined

Ex: $f(x) = 3x+2$; $f(2)$

Sol: $f(2) = 3(2)+2 = 8$

$\lim_{x \to 2} f(x) = \lim_{x \to 2} 3x+2 = 8$

Def: $f(x)$ is Continuous at $x = a$ means

$$\boxed{\lim_{x \to a} f(x) = f(a)}$$

ie $\boxed{LHL = RHL = f(a)}$ Otherwise, $f(x)$ is discontinuous

NOTE if $f(x)$ continuous then definitely limit exists but vice versa not true (ie converse not true)

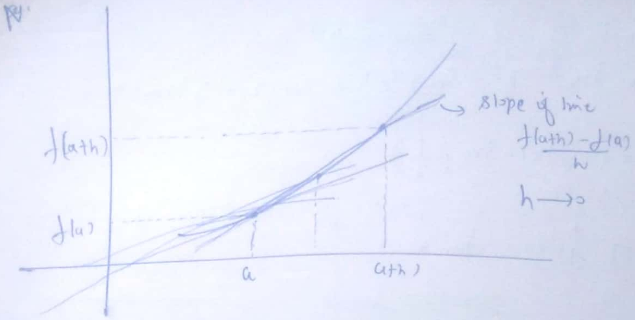
Def: Derivative of $f(x)$ is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided limit exists

Note: $f'(a)$ means slope of the tangent to the curve $y = f(x)$ at $x = a$

Pf:



$\therefore$ as $h \longrightarrow 0$

$f'(x) =$ slope of tangent at $x = a$

Results:

1) Polynomial functions, e^x, $\sin x$, and $\cos x$ are continuous (Differentiable) for all x

2) If $f(x)$ and $g(x)$ are continuous (differentiable) then

$f+g$

$f-g$

$f \times g$

$\frac{f}{g}$ $(g \neq 0)$

$f \circ g$

are also continuous (differentiable)

3) Diff $\Rightarrow$ Conti But converse not true.

Ex 1

$f(x) = x^3 + \sin x \cdot e^x + \cos 3x + x^2 \sin e^x$

Ex 2 $f(x) = \sin\left(\frac{1}{x}\right)$

Ex 3 $f(x) = \tan x$

Sol:-1) $f(x) = x^3 + \sin x \cdot e^x + \cos 3x + x^2 \sin e^x$

Continuous & Differentiable for all

Sol:2) $f(x) = \sin\left(\frac{1}{x}\right)$

$g(x) = \sin x \quad h(x) = \frac{1}{x}$

$f(x) = g \circ h = g(h(x))$

$= g\left(\frac{1}{x}\right) = \sin\left(\frac{1}{x}\right)$

Not conti & not diff at $x = 0$

Sol 3) $f(x) = \tan x$

$\tan x = \frac{\sin x}{\cos x}$

Conti & Diff when $\cos x \neq 0$

$x \neq (2n+1)\frac{\pi}{2}$ n is integer

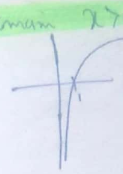
Ex $f(x) = \frac{\sin x}{e^x}$

$e^x \neq 0 \quad \therefore f$ is Cont & Diff for all x

5) $f(x) = \log x$ Domain $x > 0$

$\log_e x = \ln x$

$\log_{10} x = \lg x$



$\log x$ is Cont & diff when $x > 0$

6) $f(x) = \sqrt{x}$ $x \geq 0$ ← Domain

$\sqrt{x}$ is conti for $x \geq 0$

$\sqrt{x}$ is diff for $x > 0$ $\left(\frac{d(\sqrt{x})}{dx} = \frac{1}{2\sqrt{x}} \quad x \neq 0\right)$

Ex. $f(x) = (1+x)\log(1+x)$

$\log(1+x)$ is cont. & diff when $1+x > 0$

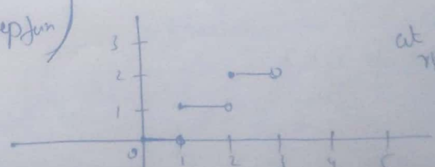
ie $x > -1$

$\therefore (1+x)\log(1+x)$ is conti & diff when $x > -1$

Ex: $f(x) = [x]$ = GIF $\leq x$

(floor fn (or) step fun)

$f(x) = [x] = \lfloor x \rfloor$



at $x = a$ right continuous

$0 \leq x < 1$

$1 \leq x < 2$

At $x = a$ Right Continuous $RHL = f(a)$

Cont & Diff when $x \neq$ integer

9) $|x|$ is

a) Differm for all x

b) Diff only at $x = 0$

c) not Diff only at $x = 0$

d) not diff for all x

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$x < 0$ Conti (bcz poly)

$x > 0$ Conti (bcz poly)

At $x = 0$

$$\left.\begin{array}{l} LHL = 0 \\ RHL = 0 \\ f(0) = 0 \end{array}\right\} \text{Cont at } x = 0$$

Cont for all x

$$f'(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$$

~~LHD~~ ~~at~~ at $x = 0$

$LHD = -1$

$RHD = +1$

$LHD \neq RHD$

Not differentiable ~~at~~ only at $x = 0$

2) $$f(x) = \begin{cases} -x^2 & x < 0 \\ x^2 & x \geq 0 \end{cases}$$

Conti for all x

$$f'(x) = \begin{cases} -2x & x < 0 \\ 2x & x > 0 \end{cases}$$

at $x = 0$ ~~LHL~~ $LHD = RHD$

$$f''(x) = \begin{cases} -2 & x < 0 \\ 2 & x > 0 \end{cases}$$

Not twice differentiable at $x = 0$

ie first derivative is not diff at $x = 0$ only

WB

Q1

Sol.
$$f(x) = \begin{cases} 2x+1 & x \le 1 \\ ax^2+b & 1 < x < 3 \\ 5x+2a & x \ge 3 \end{cases}$$

Sol.
LHL = RHL = f(x)

at $x = 1$

$2(1)+1 = 3$

$a(1)^2 + b = 3$ (RHL = f(1)) at $x=1$

$a + b = 3$

at $x = 3$

LHL = f(3)

$9a + b = 15 + 2a$

$7a + b = 15$

$6a = 12$

$a = 2$

$b = 1$

WB

Q3

Sol.
$$f(x) = \begin{cases} x^2+3x+a & x \le 1 \\ bx+2 & x > 1 \end{cases}$$

Sol.

LHL = RHL = f(x)

$$f'(x) = \begin{cases} 2x+3+0 & x \le 1 \\ b & x > 1 \end{cases}$$

at $x = 1$

LHD = RHD

$2(1)+3 = b$

$5 = b$

Using Continuity

LHL = f(x) at $x = 1$

$1 + 3(1) + a = b(1) + 2$

$1 + 3 + a = 5(1) + 2$

$4 + a = 7$

$a = 3$

Indeterminate forms

$$\left\{ \left(\frac{0}{0}\right), \left(\frac{\infty}{\infty}\right), (0\cdot\infty), (\infty - \infty), (1^\infty), (\infty^0), (0^0) \right\}$$

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L' Hospital's Rule

If $\lim_{x\to a} \frac{f(x)}{g(x)}$ is of the form $\left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$

then $\boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}}$

Ex. $\lim_{x\to 0} \left(\frac{e^x-1}{x}\right) \Rightarrow \lim_{x\to 0} \frac{e^x}{1} = 1$

$\left(\frac{0}{0}\right)$ form

apply L' Hospital's Rule

2) $\lim_{x\to 0} x\log x = \lim_{x\to 0} \frac{\log x}{(1/x)} = \frac{\frac{1}{x}}{-\frac{1}{x^2}} = -x = 0$

3) $\lim_{x\to \pi/2} [\sec x - \tan x] \Rightarrow \left(\frac{1-\sin x}{\cos x}\right) =$

$\lim_{x\to \pi/2} \left(\frac{1-\sin x}{\cos x}\right) \to \left(\frac{0}{0}\right)$ form

$\lim_{x\to \pi/2} \left(\frac{+\cos x}{+\sin x}\right) \Rightarrow \lim_{x\to \pi/2} \left(\frac{0}{1}\right) = 0$

4) $\lim_{x\to 0} (1+x)^{1/x} = 1^{\infty}$ form

$\boxed{e^{\lim_{x\to 0} \frac{1}{x}(1+x)} \quad \times \quad \Rightarrow e}$

Note: $y = f(x)^{g(x)} \Rightarrow e^{\lim_{x\to a} g(x)(f(x)-1)}$

$\log y = g(x) \log f(x)$

Method-1 (apply formula directly)

$e^{\lim_{x\to 0} \frac{1}{x} (1+x-1)}$

$= e^1$

$= e$

$\lim_{x\to 0} \frac{1}{x}(1+x) \leftarrow \left(\frac{0}{0}\right)$ form

$\lim_{x\to 0} \left(\frac{1}{1}\right) = 1$

(OR)

Method-2

Apply log $\quad \log y = \lim_{x\to 0} \frac{\log(1+x)}{x}$

$= \lim_{x\to 0} \left(\frac{1}{1}\right) = 1$

$\log y = 1$

$y = e^1$

Ex. $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = 1^{\infty}$ form

Sol:
$$e^{\lim_{x\to\infty} x\left(1+\frac{1}{x}\right)}$$

$$e^{\lim_{x\to\infty}(x+1)}$$

$$e^{\lim_{x\to\infty}}$$

put $x = \frac{1}{t}$

$x \to \infty \quad t \to 0$

$$\lim_{t\to 0} (1+t)^{1/t} = e$$

Results (Important)

1) $\lim_{x\to 0} \frac{\sin x}{x} = 1$

2) $\lim_{x\to 0} \frac{\tan x}{x} = 1$

3) $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$ $\qquad \lim_{x\to 0} \frac{a^x - 1}{x} = \log a$

4) $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$

5) $\lim_{x\to a} \frac{x^n - a^n}{x-a} = na^{n-1}$

6) $\lim_{x\to\infty} \frac{\sin x}{x} = 0$

7) $\lim_{x\to 0} (1+x)^{1/x} = e$

8) $\lim_{x\to 0} (1+ax)^{1/x} = e^a$

9) $\lim_{x\to 0} \left(1+\frac{1}{x}\right)^x = e$

10) $\lim_{x\to\infty} \left(1+\frac{a}{x}\right)^x = e^a$

11) $\lim_{x\to\infty} x^{1/x} = 1$

12) $\lim_{x\to 0} x^x = 1$

Ex: $\lim_{x\to\infty} \frac{\sin x}{x}$

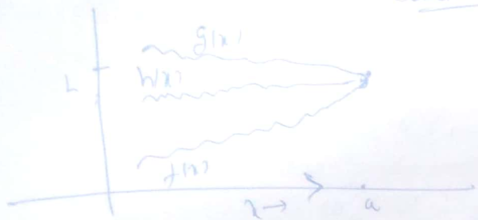
$-1 \le \sin x \le 1$ $\quad$ ~~$\forall x \in \mathbb{R}$~~ $\quad x > 0$

b/c we are considering +ve x

$$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$$

$$\lim_{x\to\infty} -\frac{1}{x} = \lim_{x\to\infty} \frac{1}{x} = 0$$

Sandwith Theorm



$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = L$$

$$f(x) \le h(x) \le g(x)$$

$$\lim_{x\to a} h(x) = L$$

$$\therefore \lim_{x\to \infty} \frac{\sin x}{x} = 0$$

Problems

1) $\lim_{x\to\infty} \frac{\cos x}{x}$

2) $\lim_{x\to\infty} \frac{x - \sin x}{x + \cos x}$

3) $\lim_{x\to 0} \frac{\sin 3x\,(2^x - 1)}{x^2}$

4) $\lim_{x\to\infty} \frac{3^{x+1} + a}{3^{x+2} + 4}$

5) $\lim_{x\to\infty} \left(\frac{x-5}{x-2}\right)^{x+4}$

WB
Q47

$$\lim_{x\to\infty} \left(e^{\frac{1}{5x}} - 1\right)\left(5x + \frac{x}{5}\sin\frac{1}{x}\right)$$

Sol:- 1) $\lim_{x\to\infty} \frac{\cos x}{x}$

As $x \to \infty$ $\cos x$ always oscillatin finitely b/w -1 to $+1$

$\frac{1}{x} \to$ approach to 0

$$\therefore \lim_{x\to\infty} \frac{\cos x}{x} = 0 \text{ (finite)}$$
$$= 0$$

2) $\lim_{x\to\infty} \frac{x - \sin x}{x + \cos x}$

Sol: $\lim_{x\to\infty} \frac{x\left(1 - \frac{\sin x}{x}\right)}{x\left(1 + \frac{\cos x}{x}\right)} = \lim_{x\to\infty} \frac{\left(1 - \frac{\sin x}{x}\right)}{\left(1 + \frac{\cos x}{x}\right)} = \frac{1-0}{1+0} = 1$

3) $\lim_{x\to 0} \frac{\sin 3x\,(2^x - 1)}{x^2}$

Sol:- $\lim_{x\to 0} \left(\frac{\sin 3x}{x} \cdot \frac{(2^x - 1)}{x}\right)$

$$\lim_{x\to 0} \left(3\left(\frac{\sin 3x}{3x}\right) \cdot \frac{(2^x-1)}{x}\right) = 3 \cdot \cancel{x}^{0}$$

$\downarrow$ 0/0 form

$$\cancel{\lim_{x\to 0} 3\left(\frac{\sin 3x}{3x}\right) \cdot \lim_{x\to 0}\left(\frac{2^x \log 2}{1}\right) = 3 \cdot 1 = 3}$$

$\lim_{x\to 0} 3\left(\frac{\sin 3x}{3x}\right) \cdot \lim_{x\to 0}\left(\frac{2^x \log 2}{1}\right) = 3\log 2$

5) $\lim_{x\to\infty}\left(\frac{x-5}{x-2}\right)^{x+4}$

Sol: $\lim_{x\to\infty} \frac{x}{x}\left(\frac{1-5/x}{1-2/x}\right)^{x+4}$

$\lim_{x\to\infty}\left(\frac{1-5/x}{1-2/x}\right)^{x+4} = (1^\infty \text{ form})$

4) $\lim_{x\to\infty} \frac{3^{x+1}+4}{3^{x+2}+4}$

Sol: $\cancel{\lim_{x\to\infty} \frac{3^x(3+4/3^x)}{3^x(3^2+4/3^x)}}$ $\lim_{x\to\infty} \frac{3^x(3+4/3^x)}{3^x(3^2+4/3^x)}$

$\lim_{x\to\infty}\left(\frac{3+4/3^x}{9+4/3^x}\right) = \frac{3}{9} = \frac{1}{3}$

5) $\lim_{x\to\infty}\left(\frac{x-5}{x-2}\right)^{x+4}$

Sol: $\cancel{\lim_{x\to\infty} \frac{x}{x}\left[\frac{1-5/x}{1-2/x}\right]^{x+4}}$

$\lim_{x\to\infty}\left[\frac{x(1-5/x)}{x(1-2/x)}\right]^{x+4}$

$\lim_{x\to\infty}\left(\frac{(1-5/x)}{(1-2/x)}\right)^x + \lim_{x\to\infty}\left(\frac{1-5/x}{1-2/x}\right)^4$

$x\to\infty \quad \frac{1}{x}\to 0$

$= \frac{e^{-5}}{e^{-2}}\left(\frac{1-0}{1-0}\right)^4$

$= e^{-3} = \frac{1}{e^3}$

WB-47

6) $\lim_{x\to\infty}\left(e^{1/5x}-1\right)\left[5x+\frac{x}{5}\sin\left(\frac{1}{x}\right)\right]$

Sol: $\lim_{x\to\infty}\left(e^{1/5x}-1\right)5x\left[1+\frac{1}{25}\sin\left(\frac{1}{x}\right)\right]$

$\lim_{x\to\infty}\left(\frac{(e^{1/5x}-1)}{1/5x}\right)\left(1+\frac{1}{25}\sin\left(\frac{1}{x}\right)\right)$

as $x\to\infty$ put $5x = \frac{1}{t}$ $\quad x\to\infty$ $\frac{1}{x}\to 0$ $\sin\left(\frac{1}{x}\right)\to 0$

$t\to 0$

$\lim_{t\to 0}\left(\frac{e^t-1}{t}\right)\left(1+\frac{1}{25}\sin(5t)\right)$

$[1] \quad (1+0)$

$= 1$

Grate

$$\lim_{x\to 1} \frac{x^7-2x^5+1}{x^3-3x^2+2} =$$

Sol: $\lim_{x\to 1} \left(\frac{x^7-2x^5+1}{x^3-3x^2+2}\right) = \left(0/0\right)$ form

Apply L Hospital

$$\lim_{x\to 1}\left(\frac{7x^6-10x^4}{3x^2-6x}\right) \Rightarrow \lim_{x\to 1} \frac{(7x^5-10x^4)}{(3x^2-6x)}$$

$$\lim_{x\to 1}\left(\frac{7(1)^5-10(1)^4}{3(1)-6(1)}\right) = \left(\frac{7-10}{3-6}\right) = \frac{-3}{-3} = 1$$

Grate

Q $\lim_{x\to 0} \frac{3x^2-6x+5}{4x^2+5x+3}$

Sol:- $\boxed{\frac{x^2\left(3-\frac{6}{x}+\frac{5}{x^2}\right)}{x^2\left(\quad \right)}} = 5/3$ Ans

Q $\lim_{x\to \infty} \frac{3x^2-6x+5}{4x^2+5x+3}$

Sol: $\lim_{x\to\infty} \frac{\cancel{x^2}\left(3-\frac{6x}{x}+\frac{5}{x^2}\right)}{\cancel{x^2}\left(4+\frac{5x}{x}+\frac{3}{x^2}\right)} = 3/4$

Q) $\lim_{x\to\infty} \frac{6x+5}{4x^2+5x+3} =$ $\left(\deg(D) > \deg(N)\right)$

$$\lim_{x\to\infty}\left(\frac{6}{8x+5}\right) = 0$$

Q) $\lim_{x\to\infty} \frac{3x^2-6x+5}{(5x+3)} =$ $\left(\deg(N) > \deg(D)\right)$

$$\lim_{x\to\infty}\left(\frac{6x-6}{5}\right) = \infty$$

WB

Q5 $\lim_{x\to 0} e^x(\cos x)^{1/\sin^2 x}$

Sol. $\lim_{x\to 0} e^x(\cos x)^{1/\sin^2 x} = 1^\infty$ form

Sol: $\log y = \frac{1}{\sin^2 x} e^x(\cos x)$

$\lim_{x\to 0} \frac{1}{\sin^2 x} e^x(\cos x)$

$\lim_{x\to 0}\left(\frac{e^x \cos x}{\frac{1-\cos 2x}{2}}\right) = \lim_{x\to 0} \frac{e^x \cos x}{\quad}$

$$y = \lim_{x\to 0} e^{x} \lim_{x\to 0} (\cos x)^{1/\sin^2 x}$$

$$\lim_{x\to 0} (\cos x)^{1/\sin^2 x} \to 1^{\infty} \text{ form}$$

Apply log on both the sides

$$\log y = \lim_{x\to 0} \frac{\log(\cos x)}{\sin^2 x} \quad (0/0) \text{ form}$$

$$= \lim_{x\to 0} \frac{\frac{1}{\cos x}(-\sin x)}{2\sin x \cos x}$$

$$= \lim_{x\to 0} \left(-\frac{1}{2\cos^2 x}\right)$$

$$\log y = \left(-\frac{1}{2}\right) \Rightarrow y = e^{-1/2}$$

$$\boxed{\begin{aligned} &\lim_{x\to 0} \left((\cos x)^{1/\sin^2 x}\right) \\ &e^{\lim_{x\to 0} \frac{1}{\sin^2 x}(\cos x - 1)} \\ &e^{\lim_{x\to 0} \frac{(\cos x - 1)}{\sin^2 x}\cdot\frac{x^2}{x^2}} \\ &e^{\lim_{x\to 0} -\frac{(1-\cos x)}{x^2}/\frac{\sin^2 x}{x^2}} \\ &= e^{-1/2} \end{aligned}}$$

WB Q7 Sol

$$\lim_{x\to 0} \frac{a\sin^2 x + b\log(\cos x)}{x^2} = \frac{1}{2} \quad \text{find } a, b$$

a) $-1, -2$
b) $1, 2$
c) $-1, 2$
d) $1, -2$

Sol: $\lim_{x\to 0} \left(a\frac{\sin^2 x}{x^2} \cdot \frac{b\log(\cos x)}{x^2}\right)$

$\cancel{\frac{a\sin^2 x}{x^2} \cdot \frac{b\log\left(1 - \sin^2 x/2\right)}{x^2}}$

$$\lim_{x\to 0} \left(a\frac{\sin^2 x}{x^2} \cdot \frac{b\log(1-\sin^2 x/2)}{x^2}\right) = \frac{1}{2}$$

$$\downarrow (0/0)$$

$$\frac{b\,\frac{1}{1-\sin^2 x/2} \times 2\sin x/2 \cos x/2 \times \frac{1}{2}}{2x}$$

$$\lim_{x\to 0} \left(\frac{a\sin^2 x}{x^2}\right)\left(\frac{b \cdot \sin x}{(1-\sin^2 x/2)2x} \times \frac{1}{2}\right) = \frac{1}{2}$$

$$\lim_{x\to 0} a\left(\frac{\sin^2 x}{x^2}\right) \cdot \frac{b}{2}\left(\frac{\sin x}{(1-\sin^2 x/2) \times 2x}\right)$$

$$\lim_{x\to 0} a\left(\frac{\sin x}{x}\right)^2 \cdot \frac{b}{4}\left[\frac{\left(\frac{\sin x}{x}\right)}{(1-\sin^2 x/2)}\right] = \frac{1}{2}$$

$$\lim_{x\to 0} a\left(\frac{\sin x}{x}\right)^2 \cdot \frac{b}{4}\left[\frac{\left(\frac{\sin x}{x}\right)}{\cos x}\right] = \frac{1}{2}$$

$a = 1 \quad b = 2$

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Mean Value Theorem

Rolle's MVT

If $f(x)$ is defined in $[a,b]$ such that

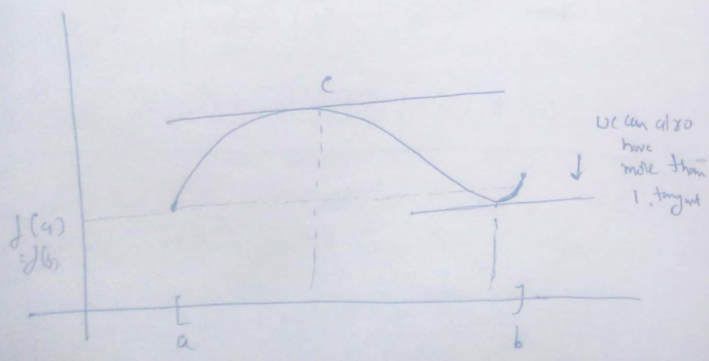
1) f is continuous in $[a,b]$
2) f is differentiable in (a,b)
3) $f(a) = f(b)$

Then there exists at least one $c \in (a,b)$ such that

$$\boxed{f'(c) = 0}$$

i.e. slope of the tangent at $c = 0$

$[a,b]$ means $a \le x \le b$



Lagrange's MVT (First MVT)

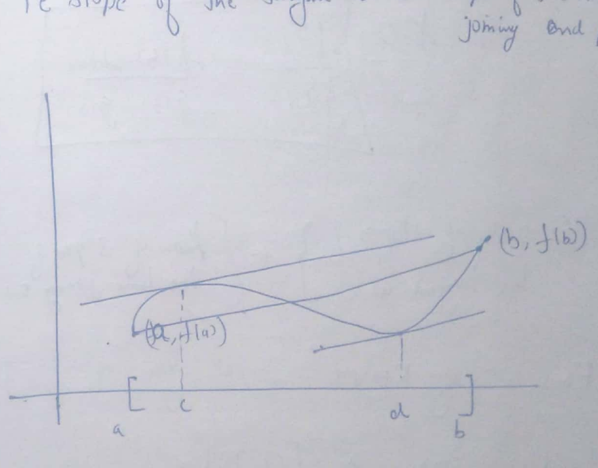
If $f(x)$ is defined in $[a,b]$ such that

i) f is continuous in $[a,b]$
ii) f is differentiable in (a,b)

then there exists at least one $c \in (a,b)$ such that

$$\boxed{f'(c) = \frac{f(b) - f(a)}{b-a}}$$

i.e. slope of the tangent at c = slope of the line joining end points.



Note: if $f(a) = f(b)$ then LMVT reduces to RMVT

3) Cauchy's MVT

If $f(x)$ and $g(x)$ are defined in $[a,b]$ such that

i) both f and g are continuous in $[a,b]$

ii) both f and g are differentiable in (a,b)

iii) $g' \neq 0$ in (a,b)

then there exists at least one $c \in (a,b)$ such that

$$\boxed{\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}}$$

i.e. {ratio of slopes of the tangent at c} = {Ratio of slopes of the lines joining end points}

(1) Find c of RMVT

$f(x) = \frac{\sin x}{e^x}$ in $[0, \pi]$

Sol:- f is cont. & Diff for all x

f is diff $f(0) = f(\pi) = 0$

$\therefore$ Rolls MVT applicable

$f(x) = e^{-x}\sin x$

$f'(x) = -e^{-x}\sin x + e^{-x}\cos x$

$f'(x) = e^{-x}[\cos x - \sin x]$

find c such that

$f'(c) = 0$

$f'(c) = e^{-c}[\cos c - \sin c] = 0$

$\cos c = \sin c \qquad e^{-c} \neq 0$

$c = \frac{\pi}{4} \qquad \pi/4 \in (0, \pi)$

2) Find c of MVT

$f(x) = \frac{\sin x}{e^x}$ in $[\pi, 2\pi]$

Sol. f According to RMVT

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$f'(x) = e^{-x} \sin x$

$f'(x) = e^{-x} \cos x - e^{-x} \sin x$

$= e^{-x}(\cos x - \sin x)$

$$e^{-x}(\cos x - \sin x) = \frac{f(2\pi) - f(\pi)}{2\pi - \pi}$$

$f(\pi) = \frac{\sin \pi}{e^{\pi}} = 0$

$f(2\pi) = \frac{\sin(2\pi)}{e^{2\pi}} = 0$

$\therefore f(\pi) = f(2\pi)$

$f'(c) = e^{-c}(\cos c - \sin c) = 0$

$e^{-c} \neq 0 \quad \cos c = \sin c$

$c = \pi + \pi/4$

$c = 5\pi/4 \in (\pi, 2\pi)$

3) find c of MVT

$f(x) = (1+x)\log(1+x)$ in $[0,1]$

Sol: $f(x) = (1+x)\log(1+x)$

f is cont & Diff in

$f(0) = 0 \quad f(1) = 2\log 2$

$f(0) \neq f(1)$

$\therefore$ Using LMVT

$f(x) = (1+x)\log(1+x)$

$f'(x) = 1 + \log(1+x)$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$f'(c) = \frac{f(1) - f(0)}{b - a}$$

$$1 + \log(1+c) = \frac{2\log 2 - 0}{\cancel{2\log 2}\ 1 - 0}$$

$1 + \log(1+c) = 2\log 2$

$\log\left(\frac{1+c}{4}\right) = -1$

Scanned by CamScanner

$\frac{1+c}{4} = e^{-1}$

$\boxed{c = \frac{4}{e} - 1} \in (0,1)$

H.W

Ex find c of CMVT

$f(x) = x^2 \quad g(x) = x^3$ in $[1,2]$

Sol: f & g are cont & diff.

$g'(x) = 3x^2 \neq 0$

then CMVT

$f'(x) = 2x \quad g'(x) = 3x^2$

$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

$\frac{2c}{3c^2} = \frac{(2)^2 - (1)^2}{(2)^3 - (1)^3}$

$\frac{2}{3c} = \frac{4-1}{8-1}$

$\frac{2}{3c} = \frac{3}{7} \Rightarrow c = \frac{9}{14}$

$c = \frac{14}{9}$

Taylor's Series

The expansion of $f(x)$ in powers of $(x-a)$. [T.S of $f(x)$ about $x=a$]

$$\boxed{f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots}$$

If $a=0$, then it is called ~~Maclor~~ Maclaurin's Series.

$$\boxed{f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots}$$

$f(x) = \cos x \qquad f(0) = 1$

$f'(x) = -\sin x \qquad f'(0) = 0$

$f''(x) = -\cos x \qquad f''(0) = -1$

$f'''(x) = \sin x \qquad f'''(0) = 0$

$f^{(4)}(x) = \cos x \qquad f^{(4)}(0) = 1$

$\cos(x) = 1 + x \cdot 0 + \frac{x^2}{2!} \cdot (-1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(1) + \ldots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$

Ex: What is the coefficient of $(x-2)^4$ in Taylor series expansion of e^x about $x=2$

$$f(x) = f(a) + (x-a)f'(x) + \frac{(x-a)^2}{2!}f''(x) + \frac{(x-a)^3}{3!}f'''(x) + \cdots$$

Put $x-2=t$
$x=t+2$

We have to express in terms of $(x-2)$

$$e^x = e^{t+2}$$
$$= e^2 . e^t$$
$$= e^2\left[1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\cdots\right]$$

put $t = x-2$

$$= e^2\left[1+(x-2)+\frac{(x-2)^2}{2!}+\frac{(x-2)^3}{3!}+\frac{(x-2)^4}{4!}+\cdots\right]$$

Coefficient of $(x-2)^4$ is $\frac{e^2}{4!}$

Ex. Expand $f(x) = \frac{\sin x}{x-\pi}$ in powers of $(x-\pi)$

$$\boxed{f'(x) = \frac{(x-\pi)\cos x - \sin x}{(x-\pi)^2} \qquad f'} \quad \times$$

Sol:- Put $x-\pi = t$
$x = (\pi + t)$

Sin	All
tan	cos

$$f(x) = \frac{\sin(\pi+t)}{\pi+t-\pi} = \frac{-\sin t}{t}$$

$$f(x) = \frac{-\sin t}{t}$$

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots$$

$$f(x) = \frac{-1}{t}\left[t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\right]$$

$$= -1 + \frac{t^2}{3!} - \frac{t^4}{5!} + \frac{t^6}{7!} + \cdots$$

$$= -1 + \frac{(x-\pi)^2}{3!} - \frac{(x-\pi)^4}{5!} + \frac{(x-\pi)^6}{7!} - \cdots$$

Maxima & Minima

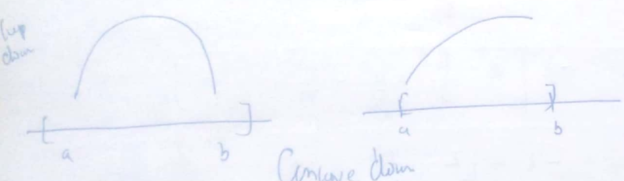
Note: Let $y = f(x)$ be the fun

1) $f'(x) > 0$ then $f(x)$ is increasing
2) $f'(x) < 0$ then $f(x)$ is decreasing
3) $f'(x) = 0$ then $f(x)$ is stationary

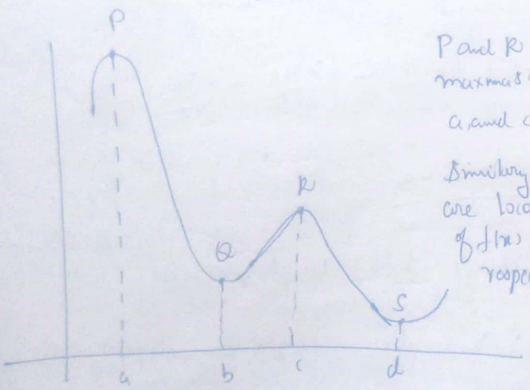
4) $f''(x) > 0$ then $f(x)$ is concave up

5) $f''(x) < 0$ then $f(x)$ is concave down

Concave Up



Concave down



P and R are local maximas of $f(x)$ at a and c respectively

Similarly Q and S are local minimas of $f(x)$ at b and d respectively

Procedure for finding local maxima & local minima

Let $y = f(x)$

1) Find $f'(x)$
2) Solve $f'(x) = 0$ to find stationary points
3) Find $f''(x)$

Let a be the stationary point

i) If $f''(a) > 0$ then $f(x)$ has local minima at $x = a$

ii) If $f''(a) < 0$ then $f(x)$ has local maxima at $x = a$

iii) If $f''(a) = 0$ then this test fails

5) If $f''(a) = 0$ but $f'''(a) \neq 0$ then $f(x)$ has point of inflection at $x = a$

Note: The point where $f(x)$ changes from concave up to concave down (or) vice-versa is called point of inflection

~~Concave up to~~

Concave down to concave up

Concave ~~down~~ up to concave ~~up~~ down

1) Find maxima, minima

$f(x) = x^3 - 6x^2 + 9x + 25$

Sol: $f(x) = x^3 - 6x^2 + 9x + 25$

$f'(x) = 3x^2 - 12x + 9$

~~Put~~ $f'(x) = 0$

$3x^2 - 12x + 9 = 0$

$x^2 - 4x + 3 = 0$

$x^2 - 3x - x + 3 = 0$

$(x-3)(x-1) = 0$

$x = 1, 3 \rightarrow$ stationary points

$f''(x) = 6x - 12$

at $x = 1$

$f''(1) = 6 - 12 = -6 < 0$ ~~f(1) = point of local maxima~~

at $x = 3$ $f''(3) = 18 - 12 = 6 > 0$ ~~f(3) = point of local minima~~

$f(1) = (1)^3 - 6(1)^2 + 9(1) + 25$

$= 1 - 6 + 9 + 25$

$= 29 \longrightarrow$ local maxima

$f(3) = (3)^3 - 6(3)^2 + 9(3) + 25$

$= 27 - 54 + 27 + 25$

$= 54 - 54 + 25$

$= 25 \longrightarrow$ ~~toto~~ local minima

2) $f(x) = x^3 - 9x^2 + 24x + 5$

$f'(x) = 3x^2 - 18x + 24$

$f'(x) = 0$

$3x^2 - 18x + 24 = 0$

$x^2 - 6x + 8 = 0$

$x^2 - 4x - 2x + 8 = 0$

$(x-4)(x-2) = 0$

$x = 2, 4 \rightarrow$ stationary points

$f''(x) = 6x - 18$

at $x = 2$ $f''(2) = 12 - 18 = -6 < 0$ point of maxima

at $x = 4$ $f''(4) = 24 - 18 = 6 > 0$ point of minima

at $x = 2$, $f(2) = 8 - 36 + 48 + 5$

$= 13 + 12$

$= 25 \rightarrow$ local maxima

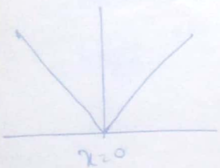
at $x=4$ $f(4) = 64 - 144 + 56 + 5$

$= 69 + 96 - 144$

$= 165 - 144$

$= 21$ → local minima

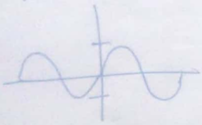
Ex: Min value of $|x|$ = ______



$|x| \geq 0$

min value $= 0$ at $x = 0$

Ex: max value of $\sin x$ = $\underline{+1}$



Ex: min value of $f(x) = (x-1)^{2/3}$

Sol: $\cancel{\sqrt[3]{(x-1)^2}}$ $\quad$ $\cancel{(x-1)^2} \geq 0$

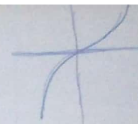
$\cancel{x^2}$

$\left[(x-1)^{1/3}\right]^2 \geq 0$ ← always +ve

min value at $x = 1$

min value $= 0$

6) $f(x) = x^3$



Sol. $f'(x) = 3x^2$

$f'(x) = 0$

$3x^2 = 0$

$x^2 = 0$

$x = 0$ → stationary point

$f''(x) = 6x$

at $x = 0$ $f''(0) = 0$ <u>Test fails</u>

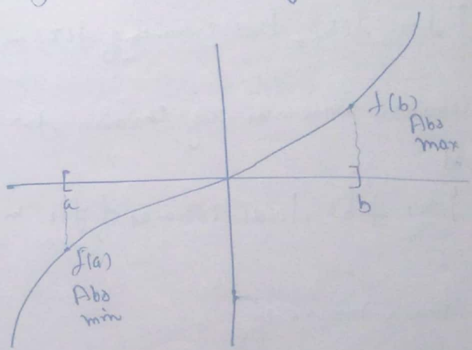
$f'''(x) = 6 \Rightarrow f'''(0) = 6 \neq 0$

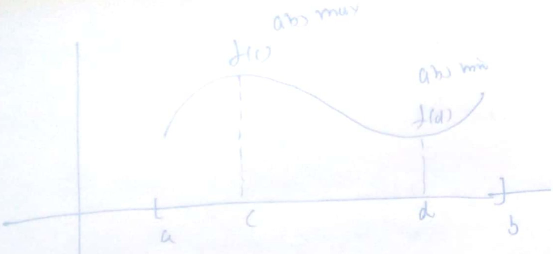
∴ $f(x)$ has point of inflection at $x = 0$

Abs max & Abs min

~~Global maxima~~

= Does not exists





Extreme Value Theorem

if $f(x)$ is conti in $[a,b]$ then

Result: The absolute maximum of continuous fn $f(x)$ in $[a,b]$

$= \max \{ f(a), f(b), \text{local extremes of } f(x) \text{ in } (a,b) \}$

The absolute minimum of continuous fn $f(x)$ in $[a,b]$

$= \min \{ f(a), f(b), \text{local extremas of } f(x) \text{ in } (a,b) \}$

Ex. Find maxima, minima of

$f(x) = x^3 - 6x^2 + 9x + 25$ in $[0,5]$

Sol: $f(x) = x^3 - 6x^2 + 9x + 25$ in $[0,5]$

$f'(x) = 3x^2 - 12x + 9$

$f'(x) = 0$

$\Rightarrow 3x^2 - 12x + 9 = 0$

$x^2 - 4x + 3 = 0$

$x^2 - 3x - x + 3 = 0$

$(x-3)(x-1) = 0$

$x = 3, 1 \leftarrow$ stationary points

end points $\rightarrow 0, 5$

stationary points $\rightarrow 3, 1$

$f(3) = 27 - 54 + 27 + 25 = 54 - 54 + 25 = 25$

$f(1) = (1) - 6 + 9 + 25 = 29$

$f(0) = 25$

$f(5) = 125 - 150 + 45 + 25$

$= 150 - 150 + 45$

$= 45$

Abs max $= 45$ at $x = 5$

Abs min $= 25$ at $x = 0$ & 3

(2) $f(x) = x^3 - 3x^2 - 24x + 100$ in $[-3,3]$

Sol:- $f(x) = x^3 - 3x^2 - 24x + 100$

$f'(x) = 3x^2 - 6x - 24$

$f'(x) = 0$

$3x^2 - 6x - 24 = 0$

$x^2 - 2x - 8 = 0$

$x^2 - 4x + 2x - 8 = 0$

$(x-4)(x+2) = 0$

$x = -2, 4$

End points $= -3, 3$ Stationary points $= -2, 4$

$f(-2) = -8 - 12 + 48 + 100 \Rightarrow 128$

$f(-3) = -27 - 27 + 72 + 100 = 118$

$f(3) = 27 - 27 - 72 + 100 = 28$

$f(4) = 64 - 48 - 96 + 100 = 20$

Abs max of $f(x) = 128$ at $x = -2$

Abs min of $f(x) = 28$ at $x = 3$

WB
Q60, 62 (H.W)

NOTE:

1) The maximum value of $a\sin x + b\cos x = \sqrt{a^2+b^2}$

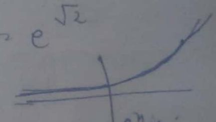
2) The minimum value of $a\sin x + b\cos x = -\sqrt{a^2+b^2}$

WB
Q23

Q Max value of $f(x) = \frac{e^{\sin x}}{e^{\cos x}}$

Sol: $f(x) = e^{\sin x - \cos x} = e^{\sqrt{(1)^2+(-1)^2}} = e^{\sqrt{2}}$

[e^x is max when x is max]



e^x is increasing fn

Ex: Max value of $e^{2\sin x + 3\cos x}$ = $e^{\sqrt{13}}$

Ex: Min value of $e^{4\sin x - 3\cos x}$ = e^{-5}

Ex: $x^{1/x}$ is max when $x =$ ______

Sol: $y = x^{1/x}$

apply log on both sides

$\log y = \frac{1}{x} \log x$

$\log y = \frac{\log x}{x}$

$y = e^{\frac{\log x}{x}}$

but $y = x^{1/x}$

$x^{1/x} = e^{\frac{\log x}{x}}$

[$e^{\frac{\log x}{x}}$ is maximum when $f(x) = \frac{\log x}{x}$ is maximum]

$f(x) = \frac{\log x}{x}$

$f'(x) = \left(x \cdot \frac{1}{x} - \log x \cdot (1)\right) / x^2$

$= \frac{1 - \log x}{x^2}$

$f'(x) = 0$

$\frac{1 - \log x}{x^2} = 0$

$1 - \log x = 0$

$\log x = 1$

$x = e$

$f''(x) = \frac{x^2\left(-\frac{1}{x}\right) - (1 - \log x)(2x)}{x^4}$

$= \frac{-x - 2x + 2x\log x}{x^4}$

at $x = e$

$f''(e) = -e - 2e + 2e\log e$

$= -3e + 2e$

$= -e < 0$

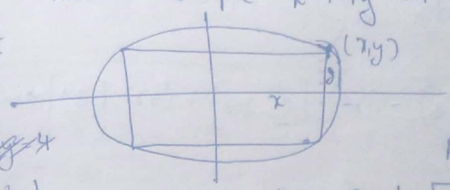
$\therefore$ maxima at $x = e$

$f(x) = \frac{\log x}{x}$ is max. at $x = e$

$\therefore x^{1/x}$ is max at $x = e$

Ex maximum area of rectangle whose vertices lies on ellipse $x^2 + 4y^2 = 1$

Sol:-



$l = 2x$

$b = 2y$

$A = 4xy$

$x^2 + 4y^2 = 1$

$y^2 = \frac{1 - x^2}{4}$

$y = \frac{+\sqrt{1 - x^2}}{2}$

MAX $A = 4x\frac{\sqrt{1-x^2}}{2}$

$A = 2x\sqrt{1 - x^2}$

A is max when A^2 is max

$A^2 = 4x^2(1 - x^2) = f(x)$

$f(x) = 4x^2 - 4x^4$

$f'(x) = 8x - 16x^3$, Now $f'(x) = 0$

$x = 0, \pm\frac{1}{\sqrt{2}}$

$f''(x) = 8 - 48x^2$

$f''\left(\frac{1}{\sqrt{2}}\right) = 8 - 48\frac{1}{2} < 0$

max when $x = \frac{1}{\sqrt{2}}$

Max $A = 2\left(\frac{1}{\sqrt{2}}\right)\sqrt{1 - \left(\frac{1}{\sqrt{2}}\right)^2}$

$2 \times \frac{1}{\sqrt{2}} \cdot \sqrt{1 - \frac{1}{2}}$

$= \sqrt{2} \times \frac{1}{\sqrt{2}}$

$= 1$

Integration

Leibnitz rule (DI rule) $U = x^n$

$V = e^{ax}$ (or) $\sin ax$ (or) $\cos ax$

$$\int UV = UV_1 - U'V_2 + U''V_3 - U'''V_4 + \cdots$$

Where $U', U'', U''', \ldots$ represents Derivative

$V_1, V_2, V_3 \ldots \longrightarrow$ Integration

Ex. $\int x^3 e^{2x}\,dx$

$$= x^3\left[\frac{e^{2x}}{2}\right] - (3x^2)\cdot\left[\frac{e^{2x}}{4}\right] + (6x)\cdot\left[\frac{e^{2x}}{8}\right] - 6\left[\frac{e^{2x}}{16}\right] + 0$$

Ex. $\int x^3 \sin x\,dx$

$$= (x^3)(-\cos x) - (3x^2)(-\sin x) + (6x)(\cos x) - 6(\sin x) + 0$$

$$= -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x$$

Ex: $\int x^2 e^{-3x}\,dx$

$$= x^2\left[\frac{-e^{-3x}}{3}\right] - 2x\left[\frac{e^{-3x}}{9}\right] + x\left[\frac{-e^{-3x}}{27}\right]$$

4) $\int x^2 \cos 3x\,dx$

$$= x^2\left(\frac{+\sin 3x}{3}\right) - 2x\left[\frac{-\cos 3x}{9}\right] + 2\left(\frac{-\sin 3x}{27}\right)$$

$$= +\frac{x^2\sin 3x}{3} + \frac{2x\cos 3x}{9} - \frac{2\sin 3x}{27}$$

Integration by Parts (I L A T E)

I → inv, L → log, A → algebric, T → Trigno, E → Exponential

$$\boxed{\int UV = U\int V - \int\left(U'\int V\right)}$$

Ex. $\int \underset{\text{II}}{\sqrt{x}}\ \underset{\text{I}}{\log x}\,dx$

$$= \log x\left[\frac{x^{3/2}}{3/2}\right] - \int \frac{+1}{x}\left(\frac{x^{3/2}}{3/2}\right)dx$$

$$= (\log x)\left(\frac{x^{3/2}}{3/2}\right) - \frac{2}{3}\int x^{1/2}\,dx$$

$$= \frac{2}{3}(\log x)\,x^{3/2} - \frac{2}{3}\,\frac{x^{3/2}}{3/2}$$

$$= \frac{2}{3}x^{3/2}(\log x) - \frac{4}{9}x^{3/2} + C$$

$$= \frac{2}{3}x^{3/2}\left[\log x - \frac{2}{3}\right] + C$$

Results

1) $\int \frac{f'(x)}{f(x)}\, dx = \log[f(x)] + C$

2) $\int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} + C$

3) $\int e^{f(x)} f'(x)\, dx = e^{f(x)} + C$

$\boxed{\begin{array}{l}\text{Concept: } f(x) = t \\ f'(x)\, dx = dt\end{array}}$

~~Ex.~~ 4) $\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + C$

Ex) $\int \frac{1}{x \log x}\, dx = \int \frac{1/x}{\log x}\, dx = \log(\log x) + C$

Ex $\int \frac{\cot x}{\log(\sin x)}\, dx = \log(\log(\sin x)) + C$

Ex. $\int \cot^2 x \operatorname{cosec}^2 x\, dx$

Sol. $-\int [\cot x]^2 (-\operatorname{cosec}^2 x)\, dx \quad \left(\because \frac{d}{dx} \cot x = -\operatorname{cosec}^2 x\right)$

$= -\frac{\cot^3 x}{3} + C$

Ex: $\int \frac{1}{1+e^{-x}}\, dx$

Sol: $\int \frac{e^x}{1+e^x}\, dx = \cancel{\log(e^x) + C}\ \log(1+e^x) + C$

Ex: $\int e^{x^2} x\, dx$

$= \frac{1}{2}\int e^{x^2}(2x)\, dx = \frac{1}{2} e^{x^2} + C$

Ex. $\int e^x [\sec x + \sec x \tan x]\, dx$

$= e^x \sec x + C$

Ex: $\int e^x x^2 dx + \int e^x 2x\, dx$

$= \int e^x [x^2 + 2x]\, dx$

$= e^x x^2 + C$

Ex 1 $\int_0^{2\pi} \sin 3x \cos 5x\, dx = 0$

Ex 2 $\int_{-\pi}^{\pi} \cos 6x \cdot \cos 7x\, dx = 0$

Ex 3 $\int_0^{3\pi} \sin 5x\, dx = 0$

Ex 4 $\int_0^{2\pi} \cos^2 7x\, dx = \pi$

Trigonometric System

Set of funs^n

$1, \sin x, \cos x, \sin 2x, \cos 2x, \sin 3x, \cos 3x, \ldots$

Results The following results are valid over any interval of length 2π

(1) $\int_{-\pi}^{\pi} \sin nx\, dx = 0 \quad \int_{-\pi}^{\pi} \cos nx\, dx = 0$

(2) $\int_{-\pi}^{\pi} \sin mx \cdot \cos nx\, dx = 0$

(3) $\int_{-\pi}^{\pi} \sin mx \cdot \sin nx\, dx = 0 \quad (m \neq n)$

(4) $\int_{-\pi}^{\pi} \cos mx \cos nx\, dx = 0 \quad (m \neq n)$

5) $\int_{-\pi}^{\pi} \sin^2 nx\, dx = \pi$

6) $\int_{-\pi}^{\pi} \cos^2 nx\, dx = \pi$

Reduction formulae

1) $\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x$ Where $(n>1)$

$= \frac{(n-1)(n-3)(n-5)(n-7)\ldots\ldots}{(n)(n-2)(n-4)(n-6)\ldots\ldots} \left\{ \cdot \frac{\pi}{2} \right.$ only when n is even

2) $\int_0^{\pi/2} \sin^m x \cos^n x\, dx =$ $(m, n > 1)$

$= \cancel{\frac{(m-1)(m-3)(m-5)\ldots\ldots}{(n)(n-2)(n-4)\ldots\ldots}}$

$\frac{(m-1)(m-3)(m-5)\ldots\ldots(n-1)(n-3)(n-5)\ldots\ldots}{(m+n)(m+n-2)(m+n-4)\ldots\ldots} \left\{ \cdot \frac{\pi}{2} \right.$ only when both m, n are even

Ex: $\int_0^{\pi/2} \sin^7 x\, dx = \frac{6\cdot 4\cdot 2}{7\cdot 5\cdot 3\cdot 1}$

2) $\int_0^{\pi/2} \cos^6 x\, dx = \frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2\cdot}\left(\frac{\pi}{2}\right)$

3) $\int_0^{\pi/2} \sin^6 x \cos^7 x\, dx = \frac{(5\cdot 3\cdot 1)(6\cdot 4\cdot 2)}{(13)\cdot 11\cdot 9\cdot 7\cdot 5\cdot 3\cdot 1}$

4) $\int_0^{\pi/2} \sin^7 x \cos^5 x\, dx = \frac{(6\cdot 4\cdot 2)(4\cdot 2)}{12\cdot 10\cdot 8\cdot 6\cdot 4\cdot 2}$

5) $\int_0^{\pi/2} \sin^6 x \cos^4 x\, dx = \left(\frac{5\cdot 3\cdot 1\; 3\cdot 1}{10\cdot 8\cdot 6\cdot 4\cdot 2} \times \frac{\pi}{2}\right)$

Ex: $\int_0^1 x^6\sqrt{1-x^2}\, dx$ (Standard Problem)

Sol: put $x = \sin\theta$ $x=0$ $\theta=0$

$dx = \cos\theta\, d\theta$ $x=1$ $\theta=\pi/2$

$\int_0^{\pi/2} \sin^6\theta \sqrt{1-\sin^2\theta}\; d(\cos\theta)\, d\theta$

$\int_0^{\pi/2} \sin^6\theta \cos^2\theta\, d\theta = \left(\frac{5\cdot 3\cdot 1\cdot 1}{8\cdot 6\cdot 4\cdot 2\cdot} \times \frac{\pi}{2}\right)$

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Ex $\displaystyle\int_0^1 \frac{x^7}{\sqrt{1-x^2}}\,dx$

put $x = \sin\theta$ at $x=0$ $\theta=0$

$dx = \cos\theta\, d\theta$ $x=1$ $\theta=\pi/2$

$$\int_0^{\pi/2} \frac{\sin^7\theta}{\sqrt{1-\sin^2\theta}} \cos\theta\, d\theta$$

$$\int_0^{\pi/2} \frac{\sin^7\theta}{\cos\theta} \times \cos\theta\, d\theta$$

$$\int_0^{\pi/2} \sin^7\theta\, d\theta = \frac{6\cdot 4\cdot 2}{7\cdot 5\cdot 3\cdot 1}$$

Results

(1) $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(2) $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

3) $\displaystyle\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

4) $\displaystyle\int_{-a}^a f(x)\,dx = \begin{cases} 2\displaystyle\int_0^a f(x)\,dx & f(-x)=f(x) \text{ even fun} \\ 0 & f(-x)=-f(x) \text{ odd fun} \end{cases}$

5) $\displaystyle\int_0^a f(x)\,dx = \begin{cases} 2\displaystyle\int_0^{a/2} f(x)\,dx & f(a-x)=f(x) \\ 0 & f(a-x)=-f(x) \end{cases}$

6) $\displaystyle\int_0^a x f(x)\,dx = \frac{a}{2}\int_0^a f(x)\,dx \quad f(a-x)=f(x)$

Ex $\displaystyle\int_{-1}^1 x^2\,dx$

Sol: $f(x)=x^2$ $f(-x)=(-x)^2=x^2=f(x)$

$\therefore f(x)$ is even

$$2\int_0^1 x^2\,dx = 2\left[\frac{x^3}{3}\right]_0^1 = \frac{2}{3}$$

Ex $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$

Sol. $f(x) = \frac{1}{1+x^2}$ $\quad f(-x) = \frac{1}{1+(-x)^2} = \frac{1}{1+x^2} = f(x)$

$\therefore f(x)$ is even fun

$= 2\int_{0}^{\infty} \frac{1}{1+x^2}\, dx = 2\left[\tan^{-1}x\right]_0^{\infty}$

$= 2 \times \left[\frac{\pi}{2} - 0\right] = 2\left[\frac{\pi}{2}\right] = \pi$

Ex $\int_{-1}^{1} \frac{x^7}{x^4+x^2+1}\, dx = 0$ (bcz odd fun)

Ex: $\int_0^{\pi} \sin^6 x \cos^4 x\, dx$

Sol: $f(x) = \sin^6 x \cos^4 x$

$f(\pi - x) = \sin^6(\pi - x)\cos^4(\pi - x)$

$= \sin^6 x \cos^4 x = f(x)$

$\because f(a-x) = f(x)$

$\int_0^{a} f(x)\,dx = 2\int_0^{a} f(x)\,dx \quad \because f(a-x) = f(x)$

$2\int_0^{\pi/2} \sin^6 x \cos^4 x\, dx$

$= \cancel{2}\left[\frac{5\cdot 3\cdot 1\cdot 3\cdot 1}{\underset{2}{\cancel{10}}\cdot 8\cdot \underset{2}{\cancel{6}}\cdot 4\cdot 2} \times \frac{\pi}{\cancel{2}}\right]$

$= \frac{3\pi}{256}$

Ex: $\int_0^{\pi} \sin^6 x \cos^5 x\, dx = 0$ $\quad \because f(x) =$

Sol. $\because f(\pi - x) = \sin^6(\pi - x)\cdot\cos^5(\pi - x)$

$= \sin^6 x\,(-\cos^5 x)$

$= -\sin^6 x \cos^5 x$

$= -f(x)$

$f(\pi - x) = -f(x)$

$\therefore \int_0^{\pi} \sin^6 x \cos^5 x\, dx = 0$

Q6

$$\int_0^{\pi} x \sin^6 x \cos^4 x \, dx$$

Sol. $f(\pi - x) = \sin^6 x \cos^4 x = f(x)$

Rule 6 → $\int_0^{\pi} x f(x) = \frac{a}{2}\int_0^{a} f(x)\,dx \quad f(a-x) = f(x)$

$$= \frac{\pi}{2}\int_0^{\pi} \sin^6 x \cos^4 x \, dx$$

$$\frac{\pi}{2}\int_0^{\pi} \sin^6 x \cos^4 x \, dx$$

$f(\pi - x) = f(x)$

Rule-5 $\int_0^{a} f(x)\,dx = 2\int_0^{a/2} f(x)\,dx \quad f(a-x) = f(x)$

$$\frac{\pi}{2}\cdot(2)\int_0^{\pi/2} \sin^6 x \cos^4 x \, dx$$

$$\pi \int_0^{\pi/2} \sin^6 x \cos^4 x \, dx$$

$$\pi \left[\frac{5\cdot 3\cdot 1\cdot 3\cdot 1}{10\cdot 8\cdot 6\cdot 4\cdot 2}\right] \times \frac{\pi}{2} = \frac{3\pi^2}{256}$$

Problems

1) $\int_0^{\pi/4} \log(1 + \tan x)\,dx$

2) $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$

3) $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx$

4) $\int_0^{\pi} \frac{1}{a^2\cos^2 x + b^2 \sin^2 x}\,dx$

Gate 2018

5) $\int_0^{\pi/4} x \cos(x^2)\,dx$ take value in radians in Calculator

(Imp)

6) $\int_{0.25}^{1.25} x - [x]\,dx$

$$= \int_{0.25}^{1}(x-0)\,dx + \int_{1}^{1.25}(x-1)\,dx$$

$$= \left[\frac{x^2}{2}\right]_{0.25}^{1} + \left[\frac{x^2}{2} - x\right]_{1}^{1.25}$$

7) $\int_1^3 \{x\}^{[x]}\,dx$

$$\int_1^3 (x - [x])^{[x]}\,dx$$

$$\int_1^2 (x-1)^1\,dx + \int_2^3 (x-2)^2\,dx$$

$$= \left[\frac{x^2}{2} - x\right]_1^2 + \left[\frac{x^3}{3} + 4x - 4\frac{x^2}{2}\right]_2^3$$

Q1

Sol $\int_0^{\pi/4} \log(1+\tan x)\,dx$

$\cancel{\int_a^b f(x)\,dx} = \quad I = \int_0^{\pi/4} \log(1+\tan x)\,dx \quad \text{—— (1)}$

Rule-2 $\left[\int_0^b f(x) = \int_0^a f(a-x)\,dx\right]$

$I = \int_0^{\pi/4} \log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right]dx$

$I = \int_0^{\pi/4} \log\left(1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\cdot\tan x}\right)dx$

$I = \int_0^{\pi/4} \log\left(1+\frac{1-\tan x}{1+\tan x}\right)dx$

$I = \int_0^{\pi/4} \log\left(\frac{2}{1+\tan x}\right)dx \quad \text{—— (2)}$

or $\left[I = \int_0^{\pi/4}\left[\log 2 - \log(1+\tan x)\right]dx\right]$

Add (1) & (2)

$2I = \int_0^{\pi/4} \log(1+\tan x)\,dx + \int_0^{\pi/4} \log\left(\frac{2}{1+\tan x}\right)dx$

$2I = \int_0^{\pi/4} \log\left((1+\tan x)\times\frac{2}{(1+\tan x)}\right)dx$

$2I = \int_0^{\pi/4} \cancel{\log}\ \log(2)\,dx$

$2I = \log 2 \int_0^{\pi/4} (1)\,dx$

$2I = \log 2 \times \pi/4$

$I = \frac{\pi}{8} \times \log 2$

(2) $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$

$I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}} \quad \text{—— (1)}$

Rule 3 $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

$\frac{\pi}{6} + \frac{\pi}{3} = \frac{3\pi}{6} = \frac{\pi}{2}$

$$I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan\left(\frac{\pi}{2}-x\right)}}$$

$$I = \int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\cot x}}\,dx$$

$$I = \int_{\pi/6}^{\pi/3} \frac{1}{1+\frac{1}{\sqrt{\tan x}}}\,dx$$

$$I = \int_{\pi/6}^{\pi/3} \left(\frac{\sqrt{\tan x}}{1+\sqrt{\tan x}}\right) dx \quad \text{—(2)}$$

Add (1) + (2)

$$2I = \int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\tan x}}\,dx + \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}}{1+\sqrt{\tan x}}\,dx$$

$$2I = \int_{\pi/6}^{\pi/3} dx$$

$$I = \frac{1}{2}\left[\frac{\pi}{3} - \frac{\pi}{6}\right]$$

$$= \frac{\pi}{12}$$

3) $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$

$f(x) = \frac{\sin x}{1+\cos^2 x}$ $\quad f(\pi - x) = \frac{\sin x}{1+\cos^2 x} = f(x)$

Rule- $\int_0^a x f(x) = \frac{a}{2}\int_0^a f(x)\,dx$

$$= \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$$

$$= \frac{\pi}{2}\int_1^{-1} \frac{1}{1+t^2}\,dt$$

$\cos x = t$ $\quad x = 0 \; t = 1$

$-\sin x\,dx = dt$ $\quad x = \pi \; t = -1$

$$= +\frac{\pi}{2}\int_{-1}^{1} \frac{+1}{1+t^2}\,dt$$

even

$\cancel{= -\frac{\pi}{2}\left[\log(1+\cos x)\right]_0^{\pi}}$

$\cancel{= -\frac{\pi}{2}\left[\log(0) - \log(1+1)\right]}$

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$= \frac{\pi}{2} \log 2 + C$

(4) $\displaystyle\int_0^{\pi} \frac{1}{a^2\cos^2 x + b^2\sin^2 x}\, dx$

$\displaystyle\frac{\pi}{2} \cdot 2 \int_0^1 \frac{1}{1+t^2}\, dt$

$\pi \left[\tan^{-1} t\right]_0^1$

$\pi \left[\tan^{-1}(1) - \tan^{-1}(0)\right]$

$\pi \left[\frac{\pi}{4} - 0\right]$

$= \frac{\pi^2}{4}$

(4) $\displaystyle\int_0^{\pi} \frac{1}{a^2\cos^2 x + b^2\sin^2 x}\, dx$

$\displaystyle I = \int_0^{\pi} \frac{1}{a^2\cos^2 x + b^2\sin^2 x}\, dx$

$\displaystyle\int_0^{\pi} \frac{1}{\cos^2 x\left[a^2 + b^2\tan^2 x\right]}\, dx$

$\displaystyle\int_0^{\pi} \frac{\sec^2 x}{\left[a^2 + b^2\tan^2 x\right]}\, dx$

$\boxed{\text{put } \tan x = t \quad \sec^2 x\, dx = dt}$

we can not put directly bcz $\tan x$ is not continuous in $[0, \pi]$ ~~because~~

Using Rule 5

$f(x) = \dfrac{\sec^2 x}{a^2 + b^2\tan x}$

$f(\pi - x) = \dfrac{\sec^2(\pi - x)}{a^2 + b^2\tan^2(\pi - x)} = \dfrac{\sec^2 x}{a^2 + b^2\tan^2 x} = f(x)$

$\displaystyle\int_0^a f(x)\,dx = 2\int_0^{a/2} f(x)\,dx \quad f(a-x) = f(x)$

$\displaystyle = 2\int_0^{\pi/2} \frac{\sec^2 x}{a^2 + b^2\tan^2 x}\, dx$

put ~~b~~ $b\tan x = t$ at $x = 0 \quad t = 0$

$\sec^2 x\, dx = \dfrac{dt}{b}$ $\quad x = \pi/2 \quad t = \infty$

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$$= \frac{2}{b}\int_0^\infty \frac{dt}{a^2+t^2}$$

$$= \frac{2}{b}\int_0^\infty \frac{1}{a^2+t^2}\,dt = \frac{2}{b}\left[\frac{1}{a}\tan^{-1}\frac{t}{a}\right]_0^\infty$$

$$= \frac{2}{ab}\left[\frac{\pi}{2} - 0\right]$$

Ans $= \dfrac{\pi}{ab}$

6) $\displaystyle\int_0^{\pi/4} x\cos(x^2)\,dx$ (Gate -2018)

~~$f(x) = \cos(x^2)$ $f(t)$~~ put $x^2 = t$

$2x\,dx = dt$

$x\,dx = \dfrac{dt}{2}$

$$= \frac{1}{2}\int_0^{\frac{\pi^2}{16}} \cos t\,dt$$

$$\frac{1}{2}\int_0^{\frac{\pi^2}{16}} \cos t\,dt$$

$$= \frac{1}{2}\left[\sin t\right]_0^{\pi^2/16}$$

$$= \frac{1}{2}\left[\sin\frac{\pi^2}{16}\right] = \left[\frac{1}{2}\sin\left(\frac{\pi^2}{16}\right)\right] = \cancel{0.288}$$

0.289

7) $\displaystyle\int_{0.25}^{61.25} x\,\cancel{}$

Limiting Case Problems

~~Imp~~

8) $\displaystyle\lim_{n\to\infty}\left[\frac{n}{n^2} + \frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \dots\dots + \frac{n}{n^2+(n-1)^2}\right]$

Sol. $\displaystyle\lim_{n\to\infty}\left[\sum_{k=0}^{n-1}\frac{n}{n^2+k^2}\right]$ limiting case of summation is integration

$\displaystyle\lim_{n\to\infty}\int_0^{n-1}\frac{n}{n^2+x^2}\,dx$ replace summation variable by integration variable

$$\lim_{n\to\infty}\left[n\,\frac{1}{n}\tan^{-1}\left(\frac{x}{n}\right)\right]_0^{n-1}$$

$$\lim_{n\to\infty}\left[\tan^{-1}\left(\frac{n-1}{n}\right) - \tan^{-1}0\right]$$

$$\lim_{n\to\infty}\left[\tan^{-1}\left(1-\frac{1}{n}\right)\right]$$

$$= \cancel{\lim_{n\to\infty}}\tan^{-1}1$$

$$= \frac{\pi}{4}$$

Result

1) Area under the curve $y = f(x)$ from $x = a$ to $x = b$ is

$$\boxed{A = \int_a^b f(x)\,dx}$$

2) Area under the polar curve $r = f(\theta)$ from $\theta = \theta_1$ to $\theta = \theta_2$

$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\,d\theta$$

HW

1) Find area under curve $y = x^3$ from $x = 1$ to $x = 2$

2) Find area under the curve $r = a(1 - \cos\theta)$ from $\theta = 0$ to $\theta = \pi$

15/7/18

WB (Imp)

Q7) $$\lim_{x\to 0} \frac{a\sin^2 x + b\log(\cos x)}{x^4} = \frac{1}{2}$$

LHS: $(0/0)$ form

apply L'H

$$\lim_{x\to 0} \frac{2a\sin x\cos x + b\left(\frac{-\sin x}{\cos x}\right)}{4x^3}$$

$$= \left[2a\left(\frac{\sin x}{x}\right)\cos x + b(-\,)\right]x$$

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)\left[\frac{2a\cos x - b\sec x}{4x^2}\right]$$

$$\lim_{x\to 0}\frac{2a\cos x - b\sec x}{4x^2} \rightarrow \frac{2a-b}{0} \text{ form}$$

Concept: if $2a - b \neq 0$ Then $\lim \rightarrow \infty$ which contradict our result ie $\frac{1}{2}$

$\therefore 2a - b = 0$

$2a = b$

$$\lim_{x\to 0}\frac{2a\cos x - 2a\sec x}{4x^2} \quad (0/0) \text{ form}$$

$$\lim_{x\to 0}\frac{-2a\sin x - 2a\sec x\tan x}{8x}$$

$$\frac{-2a}{8}\left[\lim_{x\to 0}\frac{\sin x}{x} + \lim_{x\to 0}\sin x\cdot\frac{\tan x}{x}\right]$$

$$= \frac{-2a}{8}[1+1]$$

$$= \frac{-4a}{8}$$

$$\frac{-4a}{8} = \frac{1}{2}$$

$$\boxed{\begin{array}{l} a = -1 \\ b = -2 \end{array}}$$

Gamma function

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\,dx$$

Result

1) $\Gamma(1) = 1$
2) $\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$
3) $\Gamma(n+1) = n\Gamma n$
4) $\Gamma(n+1) = n!$ (n is positive integer)

Ex: $\Gamma 5 = 4!$

2) $$\int_0^\infty e^{-x} x^4\,dx = \Gamma 5 = 4!$$

3) $$\Gamma\tfrac{3}{2} = \Gamma(\tfrac{1}{2}+1) = \tfrac{1}{2}\Gamma\tfrac{1}{2} = \frac{\sqrt{\pi}}{2}$$

4) $$\Gamma\tfrac{5}{2} = \Gamma(\tfrac{3}{2}+1) = \boxed{\cancel{\tfrac{3}{2}\Gamma\tfrac{3}{2} = \tfrac{3}{2}\sqrt{\pi}}}\ \tfrac{3}{2}\cdot\tfrac{1}{2}\cdot\sqrt{\pi}$$

5) $$\Gamma\tfrac{7}{2} = \tfrac{5}{2}\cdot\tfrac{3}{2}\cdot\tfrac{1}{2}\sqrt{\pi}$$

CWB

Q34

Sol.

$$\int_0^\infty e^{-y^3} y^{1/2}\,dy$$

puts $y^3 = t$ $y = 0,\ t = 0$ $y = \infty,\ t = \infty$

$y = t^{1/3}$

$dy = \frac{1}{3} t^{-2/3}\,dt$

$$= \int_0^\infty e^{-t}\, t^{1/6}\, \frac{1}{3} t^{-2/3}\,dt$$

$$= \frac{1}{3}\int_0^\infty e^{-t}\, t^{-1/2}\,dt$$

$n-1 = -\frac{1}{2}$

$n = \frac{1}{2}$

$$= \frac{1}{3}\Gamma\tfrac{1}{2} = \frac{1}{3}\sqrt{\pi}$$

$n-1=0$
$n>1$

Ex $\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi}$

put $x^2 = t$ $\quad x = \sqrt{t}$ $\quad x = t^{1/2}$ $\quad x=0 \;\; t=0$
$x=\infty \;\; t=\infty$

$dx = \frac{1}{2} t^{-1/2}\,dt$

$= \int_0^\infty e^{-t}\frac{1}{2}t^{-1/2}\,dt$

$= \frac{1}{2}\int_0^\infty e^{-t}t^{-1/2}\,dt = \frac{1}{2}\Gamma\frac{1}{2} = \frac{1}{2}\sqrt{\pi}$

Partial Derivatives

$f(x,y) = x^3 + y^3 + 3x^2y$

$p = \frac{\partial f}{\partial x} = f_x = 3x^2 + 6xy$

differentiate partially with respect to x treating other constant

$q = \frac{\partial f}{\partial y} = f_y = 3y^2 + 3x^2$

$r = \frac{\partial^2 f}{\partial x^2} = f_{xx} = 6x + 6y$

$\frac{\partial^2 f}{\partial y \partial x} = f_{yx} = 6x$

$\frac{\partial^2 f}{\partial x \partial y} = f_{xy} = 6x$ $\qquad s = \boxed{f_{xy} = f_{yx}}$

$t = \frac{\partial^2 f}{\partial y^2} = 6y$

Derivative of Implicit form

$f(x,y) = c$

$$\boxed{\frac{dy}{dx} = \frac{-f_x}{f_y}}$$

Ex:- $f(x,y) = x^3 + y^3 + 3x^2y = 0$

$f_x = 3x^2 + 6xy$

$f_y = 3y^2 + 3x^2$

$\frac{dy}{dx} = \frac{-f_x}{f_y} = \frac{-(3x^2+6xy)}{3y^2+3x^2}$

Ex:

$y = \sin x$ (explicit)

$x^3 + y^3 + 3x^2y = 0$ (implicit)

$y - \sin x = 0$ (every explicit can be converted to implicit but converse not true)

Ex: $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \cdots}}}$

$y = \sqrt{\tan x + y}$

$y^2 = \tan x + y$

$f(x,y) = y^2 - y - \tan x = 0$

$f_x = -\sec^2 x \qquad f_y = 2y - 1$

$$\frac{dy}{dx} = \frac{-f_x}{f_y} = \frac{+\sec^2 x}{2y-1}$$

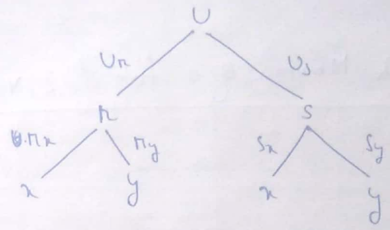
Chain rule

Let $U = f(r,s)$

$r = f_1(x,y)$

$s = f_2(x,y)$

Tree Diagram



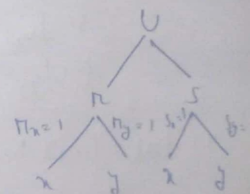
$$\frac{\partial U}{\partial x} = U_r \cdot r_x + U_s \cdot s_x$$

$$\frac{\partial U}{\partial y} = U_r \cdot r_y + U_s \cdot s_y$$

Ex: $U = f(r,s)$

$r = x + y \qquad s = x - y$

$U_x + U_y = ?$



$$U_x = \frac{\partial U}{\partial x} = U_r \cdot r_x + U_s \cdot s_x$$

$= U_r \cdot 1 + U_s \cdot (1)$

$= U_r + U_s$

$U_y = U_r \cdot r_y + U_s \cdot s_y$

$= U_r(1) + U_s(-1)$

$= U_r - U_s$

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$\therefore U_x + U_y = (U_r + U_s) + (U_r - U_s) = 2U_r$

Maxima & Minima of a fun of 2 variables

Procedure

Let $f(x,y)$ be the fun of 2 variables

1) Find f_x and f_y
2) Solve the eqn $f_x = 0$ & $f_y = 0$ to find critical points
3) find
$r = f_{xx}$
$s = f_{xy}$
$t = f_{yy}$
4) Let (a,b) be the critical point
At (a,b)
If $rt - s^2 > 0$ and
(1) $r > 0$ then $f(x,y)$ has local minima at (a,b)
(2) $r < 0$ then $f(x,y)$ has local maxima at (a,b)

2) If $rt - s^2 < 0$ then $f(x,y)$ has neither max nor min and (a,b) is called saddle point
3) If $rt - s^2 = 0$ then the test fails

Find maxima, minima for the fun

$f(x,y) = 2x^4 + y^2 - x^2 - 2y$

$f_x = 8x^3 + 0 - 2x - 0$

$f_y = 2y - 2$

$f_x = 0$ & $f_y = 0$

$8x^3 - 2x = 0$ $\quad$ $f_y = 2y - 2 = 0$
$x(8x^2 - 2) = 0$ $\quad$ $y = 1$
$x = 0$ or $8x^2 - 2 = 0$
$x = \pm \frac{1}{2}$
$x = \pm \frac{1}{2}$

$x = 0, \pm\frac{1}{2}$ $\quad$ $y = 1$ are critical points $(0,1)(\frac{1}{2},1)$ & $(-\frac{1}{2},1)$

$r = f_{xx} = 24x^2 - 2$

$s = f_{xy} = 0$

$t = f_{yy} = 2$

$\cancel{rt - s^2 = (24x^2-2)2 - 0}$

$= \cancel{2^{48}}$

Critical point	$r = 24x^2-2$	$t = 2$	$s = 0$	$rt - s^2$	Nature
$(0,1)$	-2	2	0	$-4 < 0$	Saddle point
$(\frac{1}{2}, 1)$	4	2	0	$8 > 0$	local min
$(-\frac{1}{2}, 1)$	4	2	0	$8 > 0$	

$$f(\tfrac{1}{2},1) = 2\times\frac{1}{16} + 1 - \frac{1}{4} - 2(1)$$
$$= \frac{1}{8} + 1 - \frac{1}{4} - 2$$
$$= \frac{1-2}{8} - 1$$
$$= -\frac{1}{8} - 1$$
$$= -9/8$$

$$f(-\tfrac{1}{2},1) = 2\times\frac{1}{16} + 1 - \frac{1}{4} - 2(1)$$
$$= -9/8$$

local mini of $f(x,y) = -9/8$ at $(\frac{1}{2},1)$ & $(-\frac{1}{2},1)$

Linear Algebra

Matrix : $A = [a_{ij}]_{m\times n}$

a_{ij} = i^{th} row j^{th} column element

$m\times n$ = order of the matrix

Ex: $A = \begin{bmatrix} 2 & 3 & 5 & -1 \\ 7 & 6 & 2 & 5 \\ 8 & 3 & -4 & 2 \end{bmatrix}_{3\times 4}$

$a_{23} = 2$ $\quad a_{13} = 5$

Matrix Multiplication

$A = [a_{ij}]_{m\times n}$ $\quad B = [b_{ij}]_{n\times n}$

$C = AB = [c_{ij}]_{m\times n}$

$c_{ij} = [a_{i1}, a_{i2}, \ldots\ldots a_{in}] \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$

$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots\ldots a_{in}b_{nj}$

Result

$A = [a_{ij}]_{m \times n}$ $\quad B = [b_{ij}]_{n \times n}$

1) No of multiplications required to find $AB = \underline{m \times n \times n}$

2) No of additions required to find $AB = \underline{m \times (n-1) \times n}$

Concept 1) 1 element $\longrightarrow$ n multiplication

$m \times n$ elements $\longrightarrow$ $(m \times n \times n)$ multiplication

Concept 2) 1 element $\longrightarrow$ $(n-1)$ additions

$m \times n$ elements $\longrightarrow$ $m \times n \times (n-1)$ addition

Ex $A_{4 \times 5}$ $B_{5 \times 3}$

No of multiplication required to find $AB = 4 \times 5 \times 3 = 60$

Ex: $A_{4 \times 5}$, $B_{5 \times 3}$ and $C_{3 \times 2}$

No of multiplications required to find ABC

$(A_{4 \times 5} \times B_{5 \times 3})\ C_{3 \times 2}$

$D_{4 \times 3} \times C_{3 \times 2}$

$E_{4 \times 2}$

60

24

84

$A_{4 \times 5} \times (B_{5 \times 3} \times C_{3 \times 2})$

$D_{5 \times 2}$

$E_{4 \times 2}$

30

40

70

Ex: No of square matrices of order 2 whose elements can be either 0 or 1

✓ 2 ways for every elements

Sol: $\begin{bmatrix} 0/1 & 0/1 \\ 0/1 & 0/1 \end{bmatrix} \Rightarrow 2^4 = 16$

Result

No of square matrices of order n whose elements can be either 0 or $1 = \underline{2^{n^2}}$

Invertible (Non-Singular)

$A_{n \times n}$ is invertible, $|A| \neq 0$

if $AB = BA = I_n$ for some $B_{n \times n}$

we write $B = A^{-1}$

Otherwise, A is non-invertible (singular)

Results

(1) $(A^{-1})^{-1} = A$

(2) $(kA)^{-1} = \frac{1}{k} A^{-1}$

(3) $(AB)^{-1} = B^{-1} \cdot A^{-1}$

(4) $(ABC)^{-1} = C^{-1} \cdot B^{-1} \cdot A^{-1}$

(5) $(I_n)^{-1} = I_n$

NOTE

(6) $(A+B)^{-1} \neq A^{-1} + B^{-1}$

WB

Q1

Sol: $XY = Y$ & $YX = X$ (given)

$|X| = 0$ $|Y| = 0$ (given)

find $X^2 + Y^2$

~~$(X+Y)^2 = X^2 + Y^2 + 2XY$~~

$$(X+Y)(X+Y) = X\,X + X \cdot Y + Y \cdot X + Y \cdot Y$$
$$= X^2 + Y + X + Y^2$$
$$= X$$

X

$$X^2 + Y^2 = X \cdot X + Y \cdot Y$$
$$= \underline{X} \cdot YX + Y \cdot X \cdot Y$$
$$= YX + X \cdot Y$$
$$= X + Y$$

Q2

Sol: $DABEC = I$ $A, B, C, D, E \rightarrow$ non-singular

$B^{-1} = \underline{ECDA}$

$DABEC$ $B^{-1} = ECDA$

$A^{-1} = BECD$

Q7

Sol: $M^4 = I$ $M \neq I$, $M^2 \neq I$ & $M^3 \neq I$

for any k $M^{-1} = ?$

$M^4 = I$

Multiply by M^{-1}

$M^3 = M^{-1}$

$M^3 . I = M^{-1} . I$

$M^3 . M^{4k} = M^{-1}$

$M^{4k+3} = M^{-1}$

$\because M^4 = I$

$(M^4)^2 = I$

$\vdots$

$(M^4)^k = I$

$M^{4k} = I$

Vectors

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ are 2 vectors

1) Length or norm of X

$$||X|| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

2) Normalizing Vector X

$$\frac{X}{||X||}$$

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3) Dot Product

$$X \cdot Y = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

Note: $X \cdot Y = X^T Y$

4) If $X \cdot Y = 0$ then X and Y are orthogonal vectors.

5) If $X \cdot Y = 0$ $\|X\| = 1$ & $\|Y\| = 1$ then X & Y are orthonormal vectors.

Ex: $X = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$ $Y = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$

length

$$\|X\| = \sqrt{(2)^2 + (1)^2 + (-3)^2} = \sqrt{14}$$

$$\|Y\| = \sqrt{(1)^2 + (-2)^2 + (0)^2} = \sqrt{5}$$

Normalizing X (means getting unit vector in the direction of vector)

$$\frac{X}{\|X\|} = \begin{bmatrix} 2/\sqrt{14} \\ 1/\sqrt{14} \\ -3/\sqrt{14} \end{bmatrix}$$

Dot Product ((corresponding elements are multiplied & added)

$$X \cdot Y = 2(1) + (1)(-2) + (-3)(0)$$
$$= 0$$

$$X \cdot Y = X^T \cdot Y$$
$$= \begin{bmatrix} 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$X \cdot Y = 0$ then X, Y are orthogonal vectors.

Ex: $X = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -1/\sqrt{2} \end{bmatrix}$ $Y = \begin{bmatrix} 1/\sqrt{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

Length $\|X\| = \sqrt{(1/\sqrt{2})^2 + (-1/\sqrt{2})^2} = 1$

$\|Y\| = \sqrt{(1/\sqrt{2})^2 + (1/\sqrt{2})^2} = 1$

$X \cdot Y = \frac{1}{2} - \frac{1}{2} = 0$

$\therefore X \cdot Y = 0$ && ($\|X\| = 1$ & $\|Y\| = 1$)

$\therefore X$ & Y are orthonormal vectors

Def: A square matrix $A = [a_{ij}]_{n \times n}$ is

1) symmetric if $\boxed{A^T = A}$

Note: (1) $a_{ij} = a_{ji}$

2) skew symmetric if $\boxed{A^T = -A}$

Note: 1) ~~a_{ii}~~ main diagonal elements are zero

2) $a_{ij} = -a_{ji}$

3) Orthogonal if $\boxed{A^T = A^{-1}}$

(or)

$\boxed{A \cdot A^T = I}$

~~NOTE~~: Results

The rows (columns) of an orthogonal matrix form an orthonormal set.

Ex: $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ $AA^T = I$

X Y

$\|X\| = 1$ & $\|Y\| = 1$ $X \cdot Y = 0$

A is orthogonal.

Ex: $A = \begin{bmatrix} 4 & 3 & 2 \\ a & 5 & 6 \\ b & c & 1 \end{bmatrix}$ is symmetric

mirror image

$A = A^T \Rightarrow \begin{bmatrix} 4 & 3 & 2 \\ a & 5 & 6 \\ b & c & 1 \end{bmatrix} \begin{bmatrix} 4 & a & b \\ 3 & 5 & c \\ 2 & 6 & 1 \end{bmatrix}$

$a = 3$, $b = 2$ $c = 6$

Ex: mirror

$A = \begin{bmatrix} a & -2 & 3 \\ b & 0 & -5 \\ c & d & e \end{bmatrix}$ is skew-symmetric

-ve value mirror of each other

Sol: $A^T = -A \Rightarrow \begin{bmatrix} -a & 2 & -3 \\ -b & 0 & 5 \\ -c & -d & -e \end{bmatrix} \Rightarrow \begin{bmatrix} a & b & c \\ -2 & 0 & d \\ 3 & -5 & e \end{bmatrix}$

~~a = a~~ $b = 2$ $c = -3$ $d = 5$

$a = 0$ $e = 0$

Complex Matrix

$$A = \begin{bmatrix} 2+i & 3-i \\ 1+i & 4 \end{bmatrix}$$

Conjugate $(\bar{A})$

$$\bar{A} = \begin{bmatrix} 2-i & 3+i \\ 1-i & 4 \end{bmatrix}$$

Conjugate Transpose $(\bar{A}^T)$

$$\bar{A}^T = \begin{bmatrix} 2-i & 1-i \\ 3+i & 4 \end{bmatrix}$$

Def: A complex square matrix $A = [a_{ij}]_{n \times n}$ is

1) Hermitian if $\boxed{\bar{A}^T = A}$

Note: (1) main diagonals are real no

(2) $a_{ij} = \overline{a_{ji}}$

2) Skew - Hermitian if $\boxed{\bar{A}^T = -A}$

Note: (1) Main diagonal elements are pure-imaginary

(2) $a_{ij} = -\overline{a_{ji}}$

Unitary if $\boxed{\bar{A}^T = A^{-1}}$

(or)

$\boxed{A\bar{A}^T = I}$

16/7/18

Hermitian Matrix

mirror

$$A = \begin{bmatrix} 3 & 2+i & 1-i \\ 2-i & 0 & -5i \\ 1+i & 5i & 5 \end{bmatrix}$$

images are conjugate of each other

$\bar{A}^T = A$

Skew Hermitian Matrix

mirror

$$A = \begin{bmatrix} 2i & 2+i & 3+i \\ -2+i & -3i & -1-i \\ -3+i & 1-i & 0 \end{bmatrix}$$

images are conjugate w.r.t to real no part

$\bar{A}^T = -A$

$$A = \begin{bmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Unitary Matrix

$$\bar{A}^T = A^{-1}$$

(or)

$$A \cdot \bar{A}^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Ex:

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 2 \\ 4 & 1 & 2 \end{bmatrix}$$

$R_1 \longleftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 4 & 1 & 2 \end{bmatrix}$$

$R_1 \longrightarrow 3R_1$

$$\sim \begin{bmatrix} 3 & 3 & 6 \\ 3 & 2 & 1 \\ 4 & 1 & 2 \end{bmatrix}$$

$R_2 \longrightarrow R_2 - R_1$

$$\sim \begin{bmatrix} 3 & 3 & 6 \\ 0 & -1 & -5 \\ 4 & 1 & 2 \end{bmatrix}$$

$R_3 \rightarrow 3R_3 - 4R_1$

$$\sim \begin{bmatrix} 3 & 3 & 6 \\ 0 & -1 & -5 \\ 0 & -9 & -18 \end{bmatrix}$$

Elementary row operation

1) Interchanging two rows : $R_{ij} : R_i \longleftrightarrow R_j$

2) Multiplying a row by a ~~ro~~ nonzero constant

$$R_i^{(k)} : R_i \longrightarrow (k) R_i$$

3) Adding constant multiple of a row to another row

$$R_{ij}^{(k)} : R_i \rightarrow R_i + (k) R_j$$

Determinant

2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \det(A) = (ad - bc) = |A|$$

Ex: $A = \begin{bmatrix} 4 & 3 \\ 5 & 2 \end{bmatrix}$

$|A| = 8 - 15 = -7$

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Ex

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 4 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

order $(A) = 3 \times 3$

Minor of 2nd row 1st col element = M_{21}

$$M_{21} = \begin{bmatrix} 1 & 2 \\ 1 & 5 \end{bmatrix} = 5 - 2 = 3$$

Cofactor

$$C_{21} = (-1)^{2+1} \begin{bmatrix} 1 & 2 \\ 1 & 5 \end{bmatrix} = (-1)^3 (5-2) = -3$$

$M_{32} = 1$ $\quad C_{32} = -1$

$M_{13} = -3$ $\quad C_{13} = -3$

Cofactor Matrix of A

$$= \begin{bmatrix} 19 & -4 & -3 \\ -3 & +13 & -2 \\ +(-7) & -1 & +11 \end{bmatrix}$$

$$\text{adj}(A) = [\text{Cofactor}(A)]^T$$

$$= \begin{bmatrix} 19 & -3 & -7 \\ -4 & +13 & -1 \\ -3 & -2 & 11 \end{bmatrix}$$

$|A|$ = multiply every element in a row with its corresponding cofactors.

these are Laplace Expansion

$$|A| = 3(19) + 1(-4) + 2(-3) = 47$$
$$1(-4) + 4(+13) + 1(-1) = 47$$
} row expansion

$$3(19) + 1(-3) + 1(-7) = 47$$
$$1(-3) + 4(+13) + 2(-2) = 47$$
} column expansion

We can expand across any row or any column

$$A^{-1} = \frac{\text{adj}(A)}{|A|} = \frac{1}{47} \begin{bmatrix} 19 & -3 & -7 \\ -4 & +13 & -1 \\ -3 & -2 & 11 \end{bmatrix}$$

Short Method (Market Method)

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 4 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

$$\begin{array}{ccccc} 3 & 1 & 2 & 3 & 1 \\ 1 & 4 & 1 & 1 & 4 \\ 1 & 1 & 5 & 1 & 1 \\ 3 & 1 & 2 & 3 & 1 \\ 1 & 4 & 1 & 1 & 4 \end{array}$$

Column wise multiply row wise लिखना है।

$$\begin{bmatrix} 10 & -3 & -7 \\ -4 & 11 & -1 \\ -3 & -2 & 11 \end{bmatrix} = adj(A)$$

LOB Q5

Sol $A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 3 & 1 \\ 0 & 5 & -2 \end{bmatrix}$

$adj\, A = \begin{bmatrix} -11 & -9 & 1 \\ 4 & -2 & -3 \\ 10 & k & 7 \end{bmatrix}$ $\quad k = \_\_\_$

$k \to 32$ $\quad k = C_{23}$

Cofactor (A) में k must be 23 ie $C_{23} = ?$

$C_{23} = -5$

Ex: $A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 0 & 4 \\ 2 & 0 & 6 \end{bmatrix}$

$|A| = -3(6-8) = +6$

Ex: $A = \begin{bmatrix} 2 & 4 & 6 & 8 \\ 0 & 3 & 0 & 0 \\ 0 & 5 & 4 & 6 \\ 0 & 8 & 0 & 5 \end{bmatrix}$

$|A| = 2 \begin{vmatrix} 3 & 0 & 0 \\ 5 & 4 & 6 \\ 8 & 0 & 5 \end{vmatrix}$

$= 2 \times 3(20) = 120$

Properties of Determinant

1) $A = \begin{bmatrix} 2 & 3 & 5 \\ 6 & 7 & 8 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad |A| = 0$

2) $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 5 \\ 1 & 2 & 3 \end{bmatrix}$ identical rows $\quad |A| = 0$

3) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 5 \\ 2 & 4 & 6 \end{bmatrix}$ (dependent) proportional rows $\quad |A| = 0$

$\boxed{bz \quad R_3 = 2R_1}$

4) $A = \begin{bmatrix} 30 & -20 & -10 \\ -15 & 40 & -25 \\ -10 & -30 & 50 \end{bmatrix} \begin{array}{l} \text{Sum} \\ \to 0 \\ \to 0 \\ \to 0 \end{array}$ $\quad |A| = 0$

*

5> $A = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & 5 \\ -3 & -5 & 0 \end{bmatrix}_{3\times 3}$ skew-sym matrix of odd order (3)

$|A| = 0$

6> $A = [a_{ij}]_{5\times 5}$

$a_{ij} = i^2 - j^2 \rightarrow$ this is skew-sym

$\begin{bmatrix} 0 & -3 & & & \\ 3 & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix}_{5\times 5}$ skew-sym matrix of odd order

$|A| = 0$

7> $A = \begin{bmatrix} 3 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ upper Δ matrix

$a_{ij} = 0 \;\forall\; i > j$

$|A| = 3 \cdot 2 \cdot 5 = 30$

8> $A = \begin{bmatrix} 4 & 0 & 0 \\ 6 & 3 & 0 \\ 0 & 2 & 2 \end{bmatrix} \rightarrow$ lower Δ matrix

$a_{ij} = 0 \;\forall\; i < j$

$|A| = 4 \cdot 3 \cdot 2 = 24$

9> $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 5 \end{bmatrix} \rightarrow$ diagonal matrix

$|A| = 2 \cdot 6 \cdot 5 = 60$

Ex: $A = [a_{ij}]_{n\times n}$

$A \xrightarrow{R_i \leftrightarrow R_j} B \qquad |B| = \underline{-|A|}$

$A \xrightarrow{R_i \rightarrow kR_j} B \qquad |B| = \underline{k|A|}$

$A \xrightarrow{R_i \rightarrow R_i + kR_j} B \qquad |B| = \underline{|A|}$

Ex: $|A_{5\times 5}| = 15$

$A \xrightarrow{R_1 \leftrightarrow R_5} B, \quad |B| = \underline{-15}$

$B \xrightarrow{R_2 \rightarrow 3R_2} C, \quad |C| = \underline{-45}$

$C \xrightarrow{R_1 \rightarrow R_1 + 5R_3} D \quad |D| = \underline{-45}$

Ex: $|A_{5\times 5}| = 4$

$|2A| = \underline{128}$ $\qquad |kA| = k^n|A|$

$\boxed{|kA_{n\times n}| = k^n|A|}$

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Ex. $|A| = 4 \qquad |B| = 5$

$$|AB| = \underline{|A|\cdot|B|}$$
$$= 4\cdot 5$$
$$= 20$$

Ex
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad |A| = 1$$

Ex.
$$A = \begin{bmatrix} 3 & 2 & 7 \\ 1 & 2 & 5 \\ 2 & 2 & 6 \end{bmatrix} \qquad C_3 = C_1 + 2C_2$$

$$|A| = 3(2) - 2(6-10) + 7(2-4)$$
$$= 6 + 8 - 14$$
$$= 0$$

Method 1 : Direct Method

$$|A| = 3(12-10) - 2(6-10) + 7(2-4) = 0$$

Method 2 : (only for 3×3 matrix

$$\begin{array}{ccc|cc} & & - & - & - \\ 3 & 2 & 7 & 3 & 2 \\ 1 & 2 & 5 & 1 & 2 \\ 2 & 2 & 6 & 2 & 2 \\ & & + & + & + \end{array}$$

$$|A| = 36 + 20 + 14 - 28 - 30 - 12 = 0$$

Method 3 (Imp)

$$A = \begin{bmatrix} 3 & 2 & 7 \\ 1 & 2 & 5 \\ 2 & 2 & 6 \end{bmatrix}$$

$$R_1 \longleftrightarrow R_2$$

$$A = \begin{bmatrix} +1 & +2 & +5 \\ 3 & 2 & 7 \\ 2 & 2 & 6 \end{bmatrix}$$

$$R_2 \to R_2 + 3R_1$$
$$R_3 \to R_3 + 2R_1$$

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & -4 & -8 \\ 0 & -2 & -4 \end{bmatrix}$$

$$|A| = \cancel{86} - \cancel{18} \cancel{40}\ 16 - 16 = 0$$

$$R_1 \to R_1 - 3R_1$$
$$R_3 \to R_3 - 2R_2$$

$$\begin{bmatrix} 0 & -4 & -8 \\ 1 & 2 & 5 \\ 0 & -2 & -4 \end{bmatrix}$$ proportional

$$= 0$$

Method 4 (only when

$$|A| = \begin{vmatrix} 3 & 2 & 7 \\ 1 & 2 & 5 \\ 2 & 2 & 6 \end{vmatrix} \qquad C_1 + 2C_2 = C_3$$

$$|A| = 0$$

$$A = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 4 \\ 4 & 3 & 7 \end{vmatrix} \qquad C_1 + C_2 = C_3 \qquad R_1 + R_2 = R_3$$

$\therefore |A| = 0$

Results

1) $A^{-1} = \dfrac{\text{adj}A}{|A|} \qquad |A| \neq 0$

2) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Ex $A = \begin{bmatrix} 3+i & 2 \\ 4 & 3-i \end{bmatrix}$

$$A^{-1} = \frac{1}{10-8}\begin{bmatrix} 3-i & -2 \\ -4 & 3+i \end{bmatrix}$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 3-i & -2 \\ -4 & 3+i \end{bmatrix}$$

Results

(1) A is invertible (non singular) iff $|A| \neq 0$

(2) A is non-invertible (singular) iff $|A| = 0$

(3) A is invertible $|A^{-1}| = \dfrac{1}{|A|}$

Concept:
$AA^{-1} = I$

$|AA^{-1}| = |I|$

$|A||A^{-1}| = 1$

$|A^{-1}| = \dfrac{1}{|A|}$

(4) $\boxed{\text{adj} A = |A| A^{-1}}$

5) $\boxed{A \cdot \text{adj} A = |A| \cdot I}$

$A \cdot \text{Adj}A = \cancel{|A|A^{-1}}\ |A| A A^{-1}$

$A \cdot \text{Adj}A = |A| I$

6) $(\text{adj} A)^{-1} = \dfrac{A}{|A|} \quad ; |A| \neq 0$

$\text{adj } A = |A| A^{-1}$

$(\text{adj} A)^{-1} = (|A| A^{-1})^{-1}$

$= \dfrac{1}{|A|} (A^{-1})^{-1} \qquad \left(\because (kA)^{-1} = \dfrac{1}{k} A^{-1}\right)$

$= \dfrac{1}{|A|} \cdot A$

7) $|adj\,A| = |A|^{n-1}$

$|adj\,A| = ||A|A^{-1}|$ $\quad |kA| = k^n|A|$

$= |A|^n |A^{-1}|$

$= |A|^n \times \frac{1}{|A|}$

$= |A|^{n-1}$

$adj(AB) = |AB|(AB)^{-1}$
$= |A||B| B^{-1}A^{-1}$
$= |A|A^{-1} |B|B^{-1}$
$= (adj\,B).(adj\,A)$

8) $adj(adj\,A) = |A|^{n-2}A$

$\because (adj\,A) = |A|.A^{-1}$

$adj(adj\,A) = |adj\,A|(adj\,A)^{-1}$

$= |A|^{n-1} \frac{A}{|A|}$

$= |A|^{n-2}.A$

Proof: 9

$adj\,A = |A|A^{-1}$

$adj\,kA = |kA|(kA)^{-1}$

$= k^n|A| \cdot \frac{1}{k}(A)^{-1}$

$= k^{n-1}|A|(A)^{-1}$

$= k^{n-1}.adj\,A$

9) $adj(kA) = k^{n-1}adj\,A$

10) $adj(AB) = (adj\,B)(adj\,A)$

Ex $A_{3\times3}$, $|A| = 4$

1) $|2A| = 2^n|A| = 2^3|A| = 8\times4 = 32$

2) $|3A^{-1}| = \cancel{\frac{1}{3}|A^{-1}| = \frac{1}{3}\times\frac{1}{|A|} = \frac{1}{3}\times\frac{1}{4} = \frac{1}{12}}$

$3^3|A^{-1}| = 3^3\frac{1}{|A|} = \frac{27}{4}$

3) $|(3A)^{-1}| = \frac{1}{(3)^3}|A^{-1}| = \frac{1}{27}\times\frac{1}{|A|} = \frac{1}{27}\times\frac{1}{4} = \frac{1}{108}$

4) $|adj\,A| = \cancel{|A|^{n-1} = |A|^{3-1} = (4)^2 = 16}$

$|A|^{n-1} = (4)^{3-1} = 16$

5) $adj(adj\,A) = |A|^{n-2}.A = 4^{3-2}.A = 4A$

Ex $\begin{bmatrix} 2 & 3 & 7 \\ 1 & 2 & 4 \\ 3 & 1 & x \end{bmatrix}$ is singular

$x =$ ______

Sol. $|A| = 0$ $\quad 2C_1 + C_2 = C_3$ $\quad x = 6+1 = 7$

Properties of Determinant

1) $|A| = |A^T|$

2) No of terms in Expansion of a $\det(A_{n\times n}) = n!$

3) A = skew sym matrix of odd order, $|A| = 0$

4) $|kA| = k^n|A|$

5) $|AB| = |A|\cdot|B|$.

6) $|A+B| \neq |A| + |B|$

7) $|I_n| = 1$

8) If sum of all elements in rows (columns) of A equal to zero then $|A| = 0$

Row-echelon form

A matrix is said to be in echelon form if

1) The zero rows (if any) occurs at the bottom of the matrix

2) The leading non-zero entry in any row is to the right side of leading non-zero entry in the preceding row.

Reduced Row-echelon form

A matrix is said to be in reduced row-echelon form if

(1-2) it is in echelon form

(3) The leading non-zero entries (if any) must be 1 (leading 1)

(4) The leading 1 is the only non-zero entry in its column.

Note: Every matrix can be reduced to echelon form and reduced row echelon form using finite number of elementary row operations

Ex:

$$A = \begin{bmatrix} 4 & 3 & 5 & 6 \\ 0 & 0 & 6 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ ech

leading non-zero means first non zero element.

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$ ech

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ → trivial Example ech

$$A = \begin{bmatrix} 4 & 2 & 5 & 6 \\ 0 & 3 & 7 & 8 \\ 0 & 2 & 1 & 2 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$ not in ech

Reduced Row Echelon form

Ex: $$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ Reduce row ech

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ Ech
R.R.ech

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ Ech
R.R.ech

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ Ech
R.R.ech

Elimination Method

$$A = \begin{bmatrix} 1 & 1 & 1 & 4 \\ \boxed{1} & 2 & 3 & 6 \\ \boxed{2} & 3 & 6 & 14 \end{bmatrix}$$

Forward elimination

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 2 \\ 0 & \boxed{1} & 4 & 6 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix} \to \text{Echelon form}$$

$\rho(A) = 3$

Backward elimination

$R_3 \to \frac{R_3}{2}$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$R_1 \to R_1 - R_3$; $R_2 \to R_2 - 2R_3$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$R_1 \to R_1 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad \text{R.R ech}$$

Result

The no. of non-zero rows in echelon form of a matrix A is rank of A. [rank(A) (or) $\rho(A)$]

Ex. Find rank

(1) $$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 3$

Whenever we find 0's row pull it down

Arrange the remaining rows in such a way that a row having more zeros before leading non-zero element is below a row having less zeros before leading non-zero element.

Find rank

Q $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\rho(A) = 1$

Q $A = [a_{ij}]_{m \times n}$ $a_{ij} = 3 \quad \forall i, j$

$\rho(A) = 1$

$$\begin{bmatrix} 3 & 3 & \cdots & 3 \\ 3 & 3 & \cdots & 3 \\ \vdots & \vdots & & \vdots \\ 3 & 3 & \cdots & 3 \end{bmatrix} \xrightarrow{\text{reduced}} \begin{bmatrix} 3 & 3 & \cdots & 3 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Q

$$A = \begin{bmatrix} 1 & a & a^2 & \cdots & a^n \\ 1 & a & a^2 & \cdots & a^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a & a^2 & \cdots & a^n \end{bmatrix}$$ $\rho(A) = 1$

identical rows

$$\sim \begin{bmatrix} 1 & a & a^2 & \cdots & a^n \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

5) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ proportional $\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\rho(A) = 1$

6) $A = [a_{ij}]_{m \times n}$

$a_{ij} = i \cdot j \quad \forall i, j$

$$A = \begin{bmatrix} 1\cdot1 & 1\cdot2 & 1\cdot3 & \cdots & 1\cdot n \\ 2\cdot1 & 2\cdot2 & 2\cdot3 & \cdots & 2\cdot n \\ \vdots & & & & \\ m\cdot1 & m\cdot2 & m\cdot3 & \cdots & m\cdot n \end{bmatrix}$$ proportional rows

$\rho(A) = 1$

Sub Matrix

Sub matrix of a matrix is obtained by deleting zero or more rows and zero or more columns.

Minor : determinant of square sub matrix is called minor of the matrix.

Ex. $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 1 & 2 \\ 2 & 4 & 6 & 8 \end{bmatrix}_{3 \times 4}$

1x1 Submatrices [1] [2] [3].

Determinants

minor of order 1 → 1 2 3 . . . . .

2x2 Submatrices $\begin{bmatrix}1&2\\1&3\end{bmatrix}$ $\begin{bmatrix}1&3\\1&1\end{bmatrix}$ $\begin{bmatrix}1&2\\2&4\end{bmatrix}$ . . . . .

Determinants

minor of order 2 → 1 −2 0 . . . . .

3x3 Submatrices $\begin{bmatrix}1&2&3\\1&3&1\\2&4&6\end{bmatrix}$ $\begin{bmatrix}2&3&4\\3&1&2\\4&6&8\end{bmatrix}$ $\begin{bmatrix}1&2&4\\1&3&2\\2&4&8\end{bmatrix}$ $\begin{bmatrix}1&3&4\\1&1&2\\2&6&8\end{bmatrix}$

Determinant

minors of order 3 0 0 0 0

Def: The order of highest order non-zero minor of a matrix A is rank of A [rank(A) (or) $\rho(A)$]

prev. Ex : $\rho(A) = 2$

Rank Properties

1) $\rho(A) = r$ means

i) There is a non-zero minor of order r

and

ii) every minor of order greater than r must be zero

2) $\rho(A) = 0$ iff $A = 0$ (ie null matrix)

Ex: $A_{4\times6}$ rank ≤ 4

$A_{6\times3}$ → highest order non-zero minor possible is of size 3x3. rank ≤ 3

We have to look for sub matrix possible.

3) $\rho(A_{m\times n}) \leq \min\{m,n\}$

Ex: $A = \begin{bmatrix}1&0\\0&0\end{bmatrix}$ $B = \begin{bmatrix}0&0\\0&1\end{bmatrix}$

$\rho(A) = \underline{1}$ $\rho(B) = \underline{1}$

$A \cdot B = \begin{bmatrix}0&0\\0&0\end{bmatrix}$ $\rho(AB) = 0$

Ex: $A = \begin{bmatrix}1\\2\\3\end{bmatrix}_{3\times1}$ $B = \begin{bmatrix}1&3&5\end{bmatrix}_{1\times3}$

$\rho(A) = \underline{1}$ $\rho(B) = \underline{1}$

$$(AB) = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 3 & 9 & 15 \end{bmatrix}_{3\times 3}$$ Proportional $\rho(AB) = 1$

4) $\rho(AB) \le \min\{\rho(A), \rho(B)\}$

5) $A_{m\times 1}$ $B_{1\times n}$ $\rho(AB) = 1$

(6) $X_{n\times 1}$ $\cancel{\rho(X\,X^T)}$ $\rho(X.X^T) = 1$

Imp *

(7) $\rho(A) = \rho(A^T)$

*

(8) $\rho(A.A^T) = \rho(A)$

Important

9) $A_{n\times n}$, $\rho(A) = n$ iff $|A| \neq 0$

$A_{n\times n}$, $\rho(A) < n$ iff $|A| = 0$

Ex: $A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 4 & 6 \\ 5 & 10 & x \end{bmatrix}$ $\rho(A) = 2$

$x =$ ______

Sol: $\rho(A) = 2$ hence $|A|$ must be 0.

$2(4x-60) - 3(x-30) + 5(10-20) = 0$

$8x - 120 - 3x + 90 - 50 = 0$ (or) $R_3 \to 2R_1 + R_2$

$5x = 80$

$x = 16$

Ex: Let $c_1 X_1 + c_2 X_2 = 0$

iff $c_1 = 0$ and $c_2 = 0$

$c_1 X_1 = -c_2 X_2$

$X_1 \neq -\frac{c_2}{c_1} X_2$ $X_2 \neq \frac{c_1}{c_2} X_1$

$\therefore$ X_1 and X_2 are Linear Independent (L.I)

We can not express one vector in terms of another vector

Ex: $X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $X_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$X_2 = 2X_1$ (Linear dependent) (L.D)

Results

Let $X_1, X_2, \ldots\ldots X_n$ are n column vectors

Construct the matrix R

$R = [X_1\ X_2 \ldots\ X_n]$

1) $\rho(R) = n$ then $X_1, X_2, \ldots X_n$ are L.I

$[|R| \neq 0]$

2) $\rho(R) < n$ then $X_1, X_2 \ldots X_n$ are L.D

$[|R| = 0]$

Ex: $X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $X_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$R = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$|R| = 4 - 4 = 0$ then $\Rightarrow$ X_1, X_2 are L.D

Ex: $X_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $X_2 = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$ $X_3 = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$

$R = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$ $R_3 = R_1 + R_2$ $\therefore |R| = 0$

$\therefore X_1, X_2$ & X_3 are L.D

$|R| = \cancel{(9+5) - 2(6+3) + 4(10-9)}\ 0$

$\cancel{= 14 - 18 + 12 =}$

Ex: $X_1 = [1\ -1\ 2]$ $X_2 = [2\ 1\ 0]$ $X_3 = [3\ 1\ 4]$

$R = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ 3 & 1 & 4 \end{bmatrix}$

$|R| = 1(4) + 1(8) + 2(2-3) = 4 + 8 - 2 = 10$

X_1 & X_2 are L.I.

System of linear Equations

Linear System (LS): A Linear system of m linear equations and n unknown is:

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n = b_m$

Where $x_1, x_2 \ldots x_n$ are variables or unknowns and a's and b's are constants.

Note:- The LS ~~of~~ is called

1) Under determined LS if $m < n$
2) Determined LS if $m == n$
3) Over determined LS if $m > n$

Matrix form of System of linear Equations

$AX = b$

Where A = coefficient matrix
X = Solution Vector

$b = 0$ Homogeneous LS (HLS)
$b \neq 0$ Non-homogeneous (NHLS)

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System of linear Equations

$AX = b$

$b = 0$ Homogeneous LS (HLS)

$b \neq 0$ Non-homogeneous LS (NHLS)

Consistent $\longrightarrow$ unique (or) infinite

Inconsistent $\longrightarrow$ no solutions

Augmented Matrix

$[A:b]$

NHLS

$AX = b$ (n = no of unknowns)

$A_{m\times n}$

$X_{n\times 1}$

$b_{m\times 1}$

- $\rho(A) \neq \rho(A:b)$ — No Solution (Inconsistent)
- (Consistent) $\rho(A) = \rho(A:b)$
  - $r = n$ (Unique soln)
  - $r < n$ Infinite soln

2) If $AX = b$ has infinite soln

then no of linearly independent solutions (or) (L.I Vectors)

= No of free variables

= Dimensions of solution space

$= n - r$

Where n : no of unknowns $r = \rho(A) = \rho(A:b)$

Ex:
$$x + y + z = 4$$
$$x + 2y + 3z = 6$$
$$2x + 3y + 4z = 12$$

a) No soln b) Infinite soln c) Exactly 1 soln d) Exactly 2 soln

never possible eliminate this

Sol. $AX = b$
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 12 \end{bmatrix}$$

Augmented Matrix
$$[A:b] = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 1 & 2 & 3 & 6 \\ 2 & 3 & 4 & 12 \end{bmatrix}$$

$R_2 \rightarrow R_2 - R_1$; $R_3 \rightarrow R_3 - 2R_1$
$$\sim \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 4 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$\rho(A) = 2 \qquad \rho(A:b) = 3$

$\rho(A) \neq \rho(A:b)$

Hence no soln

System is Inconsistent

Ex
$$x + y + z = 4$$
$$x + 2y + 3z = 6$$
$$2x + 3y + 6z = 14$$

Sol: $Ax = b$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 14 \end{bmatrix}$$

Augmented Matrix

$$(A:b) = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 1 & 2 & 3 & 6 \\ 2 & 3 & 6 & 14 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 1 & 2 & 3 & 6 \\ 2 & 3 & 6 & 14 \end{bmatrix}$$

$R_2 \to R_2 - R_1$; $R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 4 & 6 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$\rho(A) = 3 \qquad \rho(A:b) = 3$

n = no of unknowns

$r = 3 \qquad n = 3 \qquad \therefore r = n$

Hence Unique soln

System is Consistent

$$x + y + z = 4$$
$$y + 2z = 2$$
$$2z = 4$$

$z = 2 \qquad y = -2 \qquad x = 4$

$$x = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}$$

WB
Q13)

$x+y+z=6$

$x+4y+6z=20$

$x+4y+\lambda z=\mu$

For what values of λ & μ the system has
(i) No solutions
(ii) Unique soln
(iii) Infinite solution

a) $\lambda=6$ $\mu=20$
b) $\lambda\neq 6$
c) $\lambda=6$ $\mu\neq 20$
d) None

Sol. $AX=B$

$$\begin{bmatrix}1&1&1\\1&4&6\\1&4&\lambda\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\20\\\mu\end{bmatrix}$$

Augmented Matrix $(A:B)$

$$\begin{bmatrix}1&1&1&6\\1&4&6&20\\1&4&\lambda&\mu\end{bmatrix}$$

$R_2\to R_2-R_1$; $R_3\to R_3-R_1$

$$\sim\begin{bmatrix}1&1&1&6\\0&3&5&14\\0&3&\lambda-1&\mu-6\end{bmatrix}$$

$R_3\to R_3-R_2$

$$\sim\begin{bmatrix}1&1&1&6\\0&3&5&14\\0&0&\lambda-6&\mu-20\end{bmatrix}$$

(c) i) No soln $\rho(A)\neq\rho(A:B)$ ie $\rho(A)<\rho(A:B)$

$\therefore \lambda-6=0$ $\mu-20\neq 0$ $\therefore \rho(A)=2$ & $\rho(A:B)=3$

$\lambda=6$ $\mu\neq 20$

(b) ii) Unique soln $\rho(A)=\rho(A:B)$ & $r=n$

$r=n$ $n=3$ for $r=3$ $\lambda-6\neq 0$ $\lambda\neq 6$

$\therefore \rho(A)=3$ for any value of "$(\mu-20)$"

(a) iii) Infinite soln

$\rho(A)=\rho(A:B)$ & $r<n$

$r<n$ $n=3$

$\therefore r<3$ → Hence $\lambda-6=0 \Rightarrow \lambda=6$; $\mu-20=0 \Rightarrow \mu=20$

$\therefore \rho(A)=\rho(A:B)=2$

Q11)
Sol.

$x+2y-3z=a$

$2x+3y+3z=b$

$5x+9y-6z=c$

For System to be Consistent

a) $7a-b-c=0$
b) $3a+b-c=0$
c) $3a-b+c=0$
d) $7a-b+c=0$

Sol:

$AX = b$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 3 \\ 5 & 9 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Augmented Matrix $(A:b)$

$$\begin{bmatrix} 1 & 2 & -3 & a \\ 2 & 3 & 3 & b \\ 5 & 9 & -6 & c \end{bmatrix}$$

$R_2 \to R_2 - 2R_1 \;;\; R_3 \to R_3 - 5R_1$

$$\sim \begin{bmatrix} 1 & 2 & -3 & a \\ 0 & -1 & 9 & b-2a \\ 0 & -1 & 9 & c-5a \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & -3 & a \\ 0 & -1 & 9 & b-2a \\ 0 & 0 & 0 & \cancel{c-b} \end{bmatrix} \to c-b-3a$$

$c-5a$
$-b+2a$
$c-b-3a$

for consistent $\rho(A) = \rho(A:b)$

$$\boxed{c - b - 3a = 0}$$

HLS

$AX = 0$

1) A HLS $AX = 0$ is always consistent
[because $X = 0$ is trivial solution]

2) $A_{m \times n} \cdot X_{n \times 1} = 0$
n: no of unknown

- $\rho(A) = n$
  Unique soln
  (or) trivial solution (ie $X = 0$)
- $\rho(A) < n$
  Infinite soln
  (or) Non - trivial soln
  $X \neq 0$
  {zero होगा और x zero के अलावा और भी values होंगे}

3) $A_{n \times n} \cdot X_{n \times 1} = 0$
n: no of unknown

- $|A| \neq 0$
  Unique soln
- $|A| = 0$
  Infinite soln

(4) If $AX = 0$ has infinite solutions

= No of ~~free~~ L.I. solutions
= no of free variables
= Dimension of Null space / Dimension of space of solutions
= Nullity
= $n - r$

where n: no of unknown
r: rank

Ex

$x + y - z = 0$
$x - y + 2z = 0$
$x + y + z = 0$

Sol

$AX = 0$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\cancel{\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}}$$

$R_2 \to R_2 - R_1 ; \; R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

$\rho(A) = \pi = 2$

Sol

$A_{3\times3} \cdot X_{3\times1} = 0$

$$\begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = -4 \neq 0$$

Unique soln

$\boxed{X = 0}$

WB
Q14

$ax + y + z = 0$
$x + ay + z = 0$
$x + y + az = 0$

has infinite soln, set of values of a = ?

Sol.

$A_{3\times3} \cdot X_{3\times1} = 0$

$$\begin{bmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

for infinite many soln $|A| = 0$

$$\begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} = 0$$

$R_1 \to R_1 + R_2 + R_3$

$$(a+2) \begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} = 0$$

$R_2 \to R_2 - R_1 \quad R_3 \to R_3 - R_1$

$$(a+2) \begin{vmatrix} 1 & 1 & 1 \\ 0 & a-1 & 0 \\ 0 & 0 & a-1 \end{vmatrix} = 0$$

$\therefore (a+2)(a-1)^2 = 0$

$a = 1, 1, -2$

$a \in \{1, -2\}$

WB Q13

Sol: $A = (a_{ij})_{n\times n}$ $\quad a_{ij} = 3 \ \forall i, j$

Nullity $= n - r$ $\quad r = \rho(A) = 1$

$= n - 1$

Q15

Sol: $A_{m\times n}$ $\quad \rho(A) = r$

$AX = 0$

Dimension of space of soln $= n - r$

Q14

Sol: $A_{3\times 4} \cdot X_{4\times 1} = 0$ $\quad n = 4$

$r = \rho(A) \leq 3 \Rightarrow$

$\therefore r < n$

Infinite soln

(2 mark) NOTE

> An under determined HLS always has infinite soln
> An under determined NHLS will never have unique soln

Ex: $x + y + z = 0$

$x + 2y + 3z = 0$

$2x + 3y + 4z = 0$

Sol: $AX = 0$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$$

$R_2 \to R_2 - R_1 \ ; \ R_3 \to R_3 - (R_1 + R_2)$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$n = 3$ $\quad \rho(A) = r = 2$

$\because r < n$

Hence infinite solutions

→ No of L.I solutions (or) L.I soln vectors

→ No of free variables

→ Dimension of null space

Nullity

$= n - r$

$= 3 - 2$

$= 1$

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Now convert to Reduced Row Ech form

$R_1 \rightarrow R_2 - R_1$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{ RRE}$$

Basic Variables (Dependent variables)

Variables corresponding to leading 1's in RRE.

$b.V \longrightarrow x, y$

other variables are called <u>free variables</u>

$f.V \longrightarrow z$

$x - z = 0$

$y + 2z = 0$

express every variable in terms of free variable

let $z = \alpha$

$x = \alpha$

$y = -2\alpha$

$$X = \begin{bmatrix} \alpha \\ -2\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

1 L.I. solution vector which create infinite solutions

Basis

It is a set of L.I. ~~solution~~ vectors which can span (create) the entire solution space (infinite solution vectors)

$$\text{Basis} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$$

Dimension of solution space

No of L.I. vectors which can span ~~(create)~~ (create) the entire solution space.

Ex: find Basis & Dimension

1) $x + y + z = 0$

$n = 3 \quad r = 1$

$[\,1 \quad 1 \quad 1\,]$

x is b.V

y & z both are L.I. vectors

Nullity $= n - r$
$= 3 - 1$
$= 2$

$$\text{Basis} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

x in terms of y

x in terms of z

2) $p + q + r + s + t = 0$

$n = 5 \quad r = 1 \quad [\,1 \quad 1 \quad 1 \quad 1 \quad 1\,]$

Dimension of null space $= n - r = 5 - 1 = 4$

$$\text{Basis} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Ex $x + y + z + u + v = 5$

$n = 5 \quad r = 1$

Nullity $= n - r \Rightarrow 5 - 1 = 4$

here square 5 in RHS

Basis $= \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Ex: $x - 2y + 3z = 7$

$n = 3 \quad r = 1$

Dimension of null space $= n - r = 3 - 1 = 2$

basis $= \cancel{\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}} \quad \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

Eigen Values & Eigen Vectors

a scalar λ is called eigen value of a square matrix A if there exists a non-zero vector X ($X \neq 0$) such that

$$\boxed{AX = \lambda X}$$

X is eigen vector for eigen value λ

eigen vector ↑ Characteristic vector

eigen value ↑ Characteristic root

Characteristic Equation

$$AX = \lambda X$$
$$AX - \lambda X = 0$$
$$A - \lambda I X = 0$$
$$\boxed{(A - \lambda I)\underset{n \times n}{X} = 0} \quad \text{HLS} \quad \text{(eigen vector)}$$

$X \neq 0$

$$\boxed{|A - \lambda I| = 0} \quad \text{Characteristic Eqn (eigen values)}$$

Ex. $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

Ch. equation

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

$$4 - 5\lambda + \lambda^2 - 4 = 0$$
$$\lambda^2 - 5\lambda = 0$$
$$\lambda(\lambda - 5) = 0 \qquad \lambda = 0, 5$$

$\lambda^2 - 5\lambda = 0 \rightarrow$ Ch. polynomial Eq^n

$\lambda = 0, 5 \rightarrow$ eigen values

$\lambda_1 = 0 \quad \lambda_2 = 5$

for each eigen value. we have to solve HLS

$(A - \lambda I) X = 0$

i) Eigen vector for $\lambda_1 = 0$

$(A - (0)I) X = 0$

$AX = 0$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\downarrow$$

$$\sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \qquad \begin{matrix} b.v \rightarrow x \\ f.v \rightarrow y \end{matrix}$$

$x = -2y$

$X_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is eigen vector for $\lambda_1 = 0$

ii) Eigen vector for $\lambda_2 = 5$

$(A - 5I) X = 0$

$(A - 5I) X = 0$

$$\begin{bmatrix} -4 & 2 \\ \cancel{2} & \cancel{4}\,-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\downarrow$$

$\cancel{R_2 \rightarrow 4R_2 - 3R_1}$ $R_2 \rightarrow R_2 + R_1/2$

$$\cancel{\begin{bmatrix} -4 & 2 \\ 0 & 10 \end{bmatrix}} \quad \begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} +1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} \qquad \begin{matrix} b.v \rightarrow x \\ f.v \rightarrow y \end{matrix}$$

express as smallest integral form

$X_2 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is eigen vector for $\lambda_2 = 5$

WB
Q19

Sol. $\lambda_1 = 1$, $\lambda_2 = 4$ $\qquad$ $\cancel{X_1} =$ $\qquad M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad X_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$(M - \lambda_1 I) X_1 = 0$

$$\begin{bmatrix} a-1 & b \\ c & d-1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

$$\begin{bmatrix} a-1-b \\ c+1-d \end{bmatrix} = 0$$

$(M - \lambda_2 I) X_2 = 0$

$$\begin{bmatrix} a-4 & b \\ c & d-4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 0$$

$MX = \lambda X$

$MX_1 = \lambda_1 X_1$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} a-b \\ c-d \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$a - b = 1$

$c - d = -1$

$MX_2 = \lambda_2 X_2$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 4\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2a+b \\ 2c+d \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

$2a + b = 8$

$2c + d = 4$

$2 + 2b + b = 8$

$b = 2$

$2d - 2 + d = 4$

$3d = 6$

$d = 2$

$a = b + 1 \quad c = d - 1$

$a = 3 \quad c = 1$

$$M = \begin{bmatrix} \cancel{8/3} & \cancel{8/3} \\ \cancel{3/2} & \cancel{5/2} \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$$

Ex $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$ Eigm Vector for eigm Value 1

a) $\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$ b) $\begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix}$ c) $\begin{bmatrix} \sqrt{2} \\ 0 \\ 1 \end{bmatrix}$ d) $\begin{bmatrix} -\sqrt{2} \\ 0 \\ 1 \end{bmatrix}$

Sol:- $\lambda = 1$

by definition $AX = \lambda X$

for $\lambda = 1$

$AX = X$

Check from options by Subsituting values of X

(a) $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 9 \end{bmatrix}$

~~b)~~ $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix}$

We can represent eigm vector as

I) $\{\alpha(4, 2, 1) \mid \alpha \in R, \alpha \neq 0\}$

(or)

II) $4i + 2j + k$

$(4, 2, 1)$ (or) III) $\alpha\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$

Trace of a Matrix

$Tr(A_{n \times n})$ = Sum of main diagonal elements

Properties

$M = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$ $\lambda_1 = 1$ $X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = 4$ $X_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

1) $Tr(M) = 3+2 = 5 = \lambda_1 + \lambda_2$

2) $|M| = 4 = \lambda_1 \cdot \lambda_2$

3) X_1 and X_2 are L.I (Eigen vectors for different eigen values)

$R = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$ $|R| = 1+2 = 3 \neq 0 \Rightarrow$ L.I

ie rank of a matrix tells no of linearly independent rows (OR) columns (OR) vectors.

We need not to form a matrix & then check its determinant.

Ex $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ $\lambda_1 = 0$ $X_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Symmetric $\lambda_2 = 5$ $X_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$Tr(A) = 5 = \lambda_1 + \lambda_2$

$|A| = 0 = \lambda_1 \cdot \lambda_2$

ie if $|A| = 0$ one of the eigen ~~vectors~~ values must be 0

X_1 & X_2 are L.I

$X_1 \cdot X_2 = -2(1) + 1(2) = 0$

X_1 & X_2 are orthogonals (E.vectors for distinct eigen values of symmetric matrix)

WB Q20

Sol: $A = \begin{bmatrix} a & 1 & 0 \\ 1 & a & 1 \\ 0 & 1 & a \end{bmatrix}$

Eigen values = of A = ?. $|A| = a^3 - 2a$

$Tr(A) = 3a$

	Sum = 3a	Product = $a^3 - 2a$
× a)	3a	a^3
× b)	3a	0
× c)	3a	$-4a^3$
d)	3a	$a^3 - 2a$

WB Q22

Sol: $A = \begin{bmatrix} 2 & 1 \\ 1 & k \end{bmatrix}$ Condition for +ve eigen values

~~Tr(A) = 2+k~~ Let a, b be eigen values

$\therefore a > 0 \quad b > 0$

$a + b = Tr(A) = k + 2 > 0$

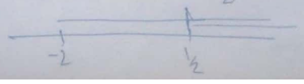
$k > -2$

$a \cdot b = |A| > 0$

$2k - 1 > 0$

$k > 1/2$

$\rightarrow k > 1/2$



Ex:

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 0 & 6 & 5 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \text{upper } \Delta \text{ matrix}$$

E.V of A ________

Sol: ~~TrTA1 = t1 =~~ given A is Upper Δ matrix

∴ eigen value = 4, 6, 1

Ex:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 5 & 6 & 0 \\ 0 & 4 & 3 \end{bmatrix} \longrightarrow \text{lower } \Delta \text{ matrix}$$

eigen value A = 2, 6, 3

Ex:

$$A \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix} \longrightarrow \text{diagonal matrix}$$

eigen value A = 3, 5, 4

Ex:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad E.V \longrightarrow 2, 0$$

E.V = eigen values

Ex:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad E.V \longrightarrow 3, 0, 0$$

Ex:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \qquad E.V \longrightarrow 4, 0, 0, 0$$

Ex:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

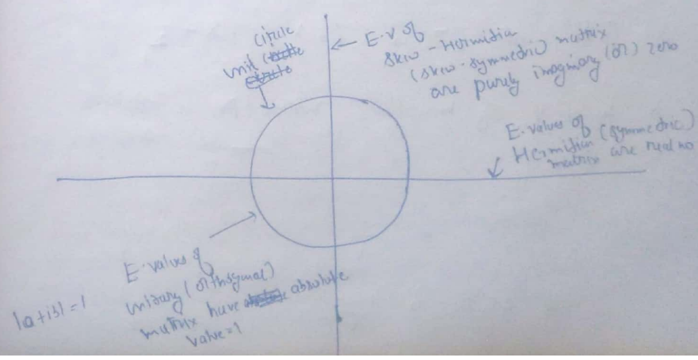
E.V ⟶ ? $(i, -i)$

Sol: $|A - \lambda I| = 0$

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 1 = 0$$

$$\lambda^2 = -1$$

$$\lambda = \pm i$$



Ex. $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ symmetric

E. values $\rightarrow$ 0, 5

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ skew symmetric

E. value $\rightarrow$ $i, -i$

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ orthogonal E.V $\rightarrow \pm i$

$|i| = 1$ $|-i| = 1$

Results (Important)

HO Result

6) λ is E.V of A

E.V of

1) kA is $\underline{k\lambda}$
2) $A + kI$ is $\underline{\lambda + k}$
3) A^m is $\underline{\lambda^m}$
4) $c_0A^2 + c_1A + c_2I$ is $c_0\lambda^2 + c_1\lambda + c_2$

7) λ is E.V of invertible matrix A ie $|A| \neq 0$

then E.V of

1) A^{-1} is $\frac{1}{\lambda}$
2) adj A is $\frac{|A|}{\lambda}$

Ex. $-1, 1, 2$ are E.V of a $A_{3\times3}$

(1) $Tr(A) = -1+1+2 = 2$

(2) $|A| = -1 \times 1 \times 2 = -2$

(3) E.V of $3A \rightarrow -3, 3, 6$ ie $-1(3), 1(3), 2(3)$

(4) E.V of $A+2I \longrightarrow 1, 3, 4$ ie $(-1+2), (1+2), (2+2)$

$Tr(A+2I) = 8$

$|(A+2I)| = 12$

5) EV of $3A^2 - 2A + 4I \rightarrow 3(-1)^2 - 2(-1) + 4 \leftarrow -1$
are $(9, 5, 12)$

$= 3 + 2 + 4$
$= 9$

$3(1)^2 - 2(1) + 4 \leftarrow 1$
$= 5$

$3(2)^2 - 2(2) + 4 \leftarrow 2$
$12 - 4 + 4$
$= 12$

6) Ev of $A^{-1} = \frac{-1}{1}, \frac{1}{1}, \frac{1}{2}$

7) adj$|A| = \frac{-2}{-1}, \frac{-2}{1}, \frac{-2}{2} = 2, -2, -1$ $\because |A| = -2$

(Imp)

8) E.V of $(A+I)^{-1} \longrightarrow$ ~~-1, 1, 2~~ $-1+1, 1+1, 2+1$

0, 2, 3

$|A+I| = 0$

$\therefore |A+I|^{-1}$ does not exist

2) $-1, 1, 0$ are E.V of A

$|A^{100}+I| =$ ______

Sol. $|A^{100}+I| = ?$

E.V → $A^{100} = (-1)^{100}, (1)^{100}, (0)^{100}$

$= 1, 1, 0$

$(A^{100}+I) = 1+1, 1+1, 0+1$

$= 2, 2, 1$

$|A^{100}+I| = 2 \times 2 \times 1 = 4$

3) $A = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$

$\downarrow$

$\begin{bmatrix} 0 & -1 & -1 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$

$1 \begin{bmatrix} -1 & -1 & -3 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

wrong answer → $1((1)(-1)(1+3)) = +4$

Sol:

$A = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$

$A = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}}_{B} + I$

$A = B + I$

E.V of B → 4, 0, 0, 0

E.V of $\underbrace{B+I}_{A}$ → 4+1, 0+1, 0+1, 0+1

5, 1, 1, 1

E.V of A → 5, 1, 1, 1

$|A| = 5 \times 1 \times 1 \times 1$

$= 5$

Q 4)

Sol:

$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \end{bmatrix} \begin{matrix} \rightarrow 15 \\ \rightarrow 15 \\ \rightarrow 15 \\ \rightarrow 15 \\ \rightarrow 15 \end{matrix}$

A has only 1 real eigen value

$\lambda = 15$

Sum of elements in each row is same, then that sum is definitely one of the eigen values

(or) sum of elements in each column is same.

Ex $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix} \begin{matrix} \rightarrow 6 \\ \rightarrow 6 \\ \rightarrow 6 \end{matrix}$ $\quad |A-\lambda I| = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 3 & 2 & 1-\lambda \end{vmatrix}$

$R_1 \rightarrow R_1 + R_2 + R_3$

$\lambda = 6$ is eigen value

$(6-\lambda) \begin{vmatrix} 1 & 1 & 1 \\ 3 & 1-\lambda & 2 \\ 3 & 2 & 1-\lambda \end{vmatrix}$

$\lambda = 6$

★ <u>Algebraic multiplicity of λ</u> $[AM(\lambda)]$

No of times λ is repeated

Geometric multiplicity of λ $(GM(\lambda))$

No of L.I eigen vectors for λ

$GM(\lambda) = n - r = $ (Dimension of $(A-\lambda I)$ ie (soln space))

$r = \rho(A-\lambda I)$

Result

Let $\lambda_1, \lambda_2, \lambda_3, \ldots$ ~~be~~ λ_k be distinct eigen values of $A_{n\times n}$ $(k \le n)$

Then no of L.I eigen vectors of A

$= GM(\lambda_1) + GM(\lambda_2) + \ldots\ldots + GM(\lambda_k)$

(2) $\boxed{GM(\lambda) \le AM(\lambda)}$

Ex: $A = \begin{bmatrix} 4 & 3 & 6 \\ 0 & 5 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ $\quad$ E.V $\rightarrow 4, 5, 3$ (are distinct)

No of L.I eigen vectors of A = 3

Ex: $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

No of L.I eigen vectors of A = ____

Sol: E.V $\rightarrow 2, 2, 3$

$AM(2) = 2 \qquad AM(3) = 1$

$GM(2) = \underline{?} \qquad GM(3) = 1$

$GM(2)$. $\rho(A-\lambda I) = \rho(A-2I) =$

$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$r = 2$

$n = 3$

$GM(2) = n - r$
$= 3 - 2$
$= 1$

No of L.I eigen vectors of A

$= GM(2) + GM(3)$

$= 1+1$

$= 2$

Ex: $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ find no of L.I eigen vectors of A?

Sol: e.v $\rightarrow 1, 1, 2$

$AM(1) = 2$ $AM(2) = 1$

$GM(1) = \underline{?}$ $GM(2) = 1$

$GM(1) \rightarrow \rho(A - \lambda I) = \rho(A - 1I)$ $GM(1) = n - r = 3 - 1 = 2$

$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $r = 1$, $n = 3$

no of L.I eigen vectors of A

$GM(1) + GM(2)$

$= 2 + 1$

$= 3$

Ex: $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Sol:- e.v $\rightarrow 0, 0, 0$

$AM(0) = 3$

$GM(0) = 2$

$GM(0) = \rho(A - \lambda I) = \rho(A - 0I) = \rho(A) = r = 1$

$n = 3$

$GM(0) = 3 - 1 = 2$

No of L.I eigen vectors of $A = GM(0) = 2$

Diagonalization

Ex.

$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$ $\lambda_1 = 1$, $\lambda_2 = 4$ $X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $X_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$

$P^{-1} = \frac{1}{1+2} \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$

$$P^{-1}A\cdot P = \frac{1}{3}\begin{bmatrix}1 & -2\\ 1 & 1\end{bmatrix}\begin{bmatrix}3 & 2\\ -1 & 2\end{bmatrix}\begin{bmatrix}1 & 2\\ -1 & 1\end{bmatrix} \Rightarrow \frac{1}{3}\begin{bmatrix}1 & -2\\ 4 & 4\end{bmatrix}\begin{bmatrix}1 & 2\\ -1 & 1\end{bmatrix}$$

$$\frac{1}{3}\begin{bmatrix}3 & 0\\ 0 & 12\end{bmatrix}$$

$$\begin{bmatrix}1 & 0\\ 0 & 4\end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix}1 & 0\\ 0 & 4\end{bmatrix}$$ → Diagonal Matrix with (main diagonal elements are eigen values.

$\boxed{P^{-1}AP}$ → Diagonalization

Result (Imp)

$A_{n\times n}$ is diagonalizable iff A has n L.I eigen vectors (i.e No of L.I eigen vectors of A = order of A)

~~$P^{-1}AP = D$~~ $\boxed{P^{-1}AP = D}$

1) The n L.I eigen vectors of A are column of P

2) main diagonal of D are eigen values of A

Cayley Hamilton Theorem

Every square matrix satisfies its characteristic equation

$$A = \begin{bmatrix}1 & 0 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}$$

Ch. eqn $|A - \lambda I| = 0$

$$\begin{bmatrix}1-\lambda & 0 & 0\\ 1 & -\lambda & 1\\ 0 & 1 & -\lambda\end{bmatrix} = 0$$

$$(1-\lambda)[(\lambda)^2 - 1] = 0$$

$$\lambda^2 - 1 - \lambda^3 + \lambda = 0$$

$\boxed{\lambda^3 - \lambda^2 - \lambda + 1 = 0}$ Ch. polynomial eqn

Acc. to C-H Theorem

$$A^3 - A^2 - A + I = 0$$

Powers of A

$$\boxed{A^3 = A^2 + A - I}$$

Multiply by A

$A^4 = A^3 + A^2 - A$

$= A^2 + A - I + A^2 - A$

$$\boxed{A^4 = 2A^2 - I}$$

$A^8 = A^4 \cdot A^4$

$= (2A^2 - I)(2A^2 - I)$

$= 4A^4 - 2A^2 - 2A^2 + I$

$= 4A^4 - 4A^2 + I$

$= 4(2A^2 - I) - 4A^2 + I$

$= 8A^2 - 4I - 4A^2 + I$

$$\boxed{A^8 = \cancel{8}4A^2 - 3I}$$

WB Q 38

$A^{50} = ?$

$A^{50} = 25A^2 - 24I$

Only for even powers

$A^{50} \rightarrow$ idea

$A^{16} = A^8 \cdot A^8$

$A^{32} = A^{16} \cdot A^{16}$

$A^{48} = A^{32} \cdot A^{16}$

$A^{50} = A^{48} \cdot A^2$

Find A^{-1} (~~Const~~ Const $\neq 0$)

$A^3 - A^2 - A + I = 0$

Multiply by A^{-1}

$A^2 - A - I + A^{-1} = 0$

$$\boxed{A^{-1} = I + A - A^2}$$

Probability

R.E: Experiment in which outcome is not certain

Sample space → Collection of all possible outcomes of R.E.

Event : Any subset of sample space (or) Outcome of our interest

Random Experiment	Sample space	Events(s)
Single dice rolled	$S = \{1,2,3,4,5,6\}$ $\lvert S\rvert = 6$ No of distinct events $= 2^6 = 64$	E_1 = Getting even no = $\{2,4,6\}$ E_2 = " odd no = $\{1,3,5\}$ E_3 = " prime no = $\{2,3,5\}$ E_4 = " no < 7 = $\{1,2,3,4,5,6\}$ E_5 = " Rajnikant = $\{\}$

Scanned by CamScanner

Result

No of distinct events on S = No of distinct subsets of S
$= 2^n$

RE	Sample space
Single card drawn from pack of 52 cards	$\lvert S \rvert = 52$ Distinct event $= 2^{52}$

Def: Probability of an event A is given by

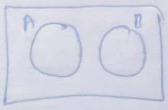
$$P(A) = \frac{\text{favourable no of outcomes}}{\text{total no of outcomes}} = \frac{|A|}{|S|}$$

Axioms (Accepted Rules)

1) for any event A
$0 \le P(A) \le 1$

2) $P(S) = 1$ → this is a certain events

3) If A & B are mutually exclusive $(A \cap B = \phi)$
$P(A \cup B) = P(A) + P(B)$



Results

4) $P(\phi) = 0$

5) $P(A^c) = 1 - P(A)$
↑ prob of not happening of A

Two dice rolled

S = sum

S	2	3	4	5	6	7	8	9	10	11	12
fav	1	2	3	4	5	6	5	4	3	2	1

1) $P(S=5) = \frac{4}{36} = \frac{1}{9}$

2) $P(S = 4 \text{ or } 5) = \frac{3}{36} + \frac{4}{36} = \frac{7}{36}$

WB
Q1
Sol $P(\text{sum exceeds } 8) = \frac{10}{36}$
ie
S = (9 or 10 or 11 or 12)

4) $P(\text{sum is prime no}) = \frac{1+2+4+6+2}{36} = \frac{15}{36}$
(2 or 3 or 5, or 7 or 11)

5) $P(S \text{ is divisible by } 4) = \frac{3+5+1}{36} = \frac{9}{36} = \frac{1}{4}$

6) $P(S \text{ divides } 24) = \frac{1+2+3+5+5+1}{36} = \frac{17}{36}$
S = 2, 3, 4, 6, 8, 12

7) $P(\text{product is } 12) = \frac{4}{36}$

~~(6,2)~~ (2,6),
(3,4)
(6,2)
(4,3)

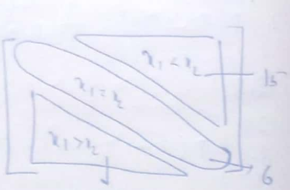
8) $P(\text{product is } 14) = \frac{0}{36}$

Ex Two dice are rolled

1) $P(\text{Equal Outcomes}) = \frac{6}{36} = \frac{1}{6}$

2) $P(\text{Unequal outcomes}) = 1 - \frac{1}{6} = \frac{5}{6}$

3) x_1: outcome on first dice
x_2: outcome on second dice



$P(x_1 > x_2)$

~~$x_1[(2,1)$~~ x_1: 6 $\longrightarrow$ 5
5 $\longrightarrow$ ~~8~~ 4
4 $\longrightarrow$ 3
2 $\longrightarrow$ 1
3 $\longrightarrow$ 2

$= \frac{15}{36}$

Sum = 8 $\longrightarrow$ $x_1 < x_2$
Sum = 8 $\longrightarrow$ (4,4)
Sum = 8 $\longrightarrow$ $x_1 > x_2$
$\downarrow$
5 - Cases
$\uparrow$ it is odd

3) $P[x_1 \le x_2] = \frac{21}{36}$

4) $P[x_1 > x_2 \text{ and } S = 8] = \frac{2}{36}$

Ex: Three dice are rolled

2 3 4 5 6

S	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
fav	1	3	6	10	15	21	25	27	27	25	21	15	10	6	3	1

1) $P\{S = 15\} = \frac{10}{6^3}$

Ex: 4 dice rolled

$P[S = 22] = \frac{10}{6^4}$

1	2	3	4	5	6	
			1		3	$\frac{4!}{3!} = 4$
				2	2	$\frac{4!}{2!2!} = 6$

Ex: $|A| = 5$

$P[\text{A relation on A selected is reflexive}] = \frac{2^{25-5}}{2^{25}} = \frac{2^{20}}{2^{25}} = \frac{1}{2^5} = \frac{1}{32}$

$P(\text{A relation on A selected is symmetric relation}] = \frac{2^{15}}{2^{25}} = \frac{1}{2^{10}} = \frac{1}{1024}$

$P[\text{Relation on A selected is symmetric \& reflexive}) = \frac{2^{10}}{2^{25}} = \frac{1}{2^{15}} = \frac{1}{32}$

Ex Consider simple labelled graphs on 5 vertices

$P(\text{~~getting~~ graph having 6 edges}) = \frac{{}^{10}C_6}{2^{10}} =$

max edges $= {}^nC_2 = {}^5C_2 = 10$

$P(\text{graph is having even no of edges}) = \dfrac{{}^{10}C_0 + {}^{10}C_2 + {}^{10}C_4 + {}^{10}C_6 + {}^{10}C_8 + {}^{10}C_{10}}{2^{10}}$

$= \dfrac{2^{10-1}}{2^{10}} = \dfrac{1}{2}$

Ex 5 objects are arranged. What is the prob. that none of the obj occupy its natural place!

Sol. Total = $5! = 120$

$D_5 = 5!\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right]$

$= 120\left[\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}\right] = 60 - 20 + 5 - 1$
$= 44$

Prob $= \frac{44}{120} = \frac{11}{30}$

WB
Q6
Q2

| 4R 3G | 3B 4G |

$P[1R \ \& \ 1B] = \frac{4}{7} \cdot \frac{3}{7} = \frac{12}{49}$

WB
Q5
Sol: box1 = {3, 6, 9, 12, 15}
box2 = {6, 11, 16, 21, 26}

P(product is even) = ?

| 3, 6, 9, 12, 15 | 6, 11, 16, 21, 26 |

total = $\underset{5}{\square}\ \underset{5}{\square} = 25$

fav: odd product
odd × odd
$\underset{3}{\square}\ \underset{2}{\square}$
$= 6$

$P(\text{product is odd}) = \frac{6}{25}$

$P(\text{product is even}) = 1 - \frac{6}{25} = \frac{19}{25}$

WB
Q2

Sol. total = N = 100

3 child $= \frac{5N}{10}$ 2 child $= \frac{3N}{10}$ 1 child $= \frac{2}{10}N$

$P(2\text{ child}) = \dfrac{\frac{3N}{10}}{\frac{23N}{10}} = \frac{3}{23}$

total children $= 3 \times \frac{5N}{10} + 2 \times \frac{3N}{10} + 1 \times \frac{2N}{10}$

$= \frac{15N}{10} + \frac{6}{10}N + \frac{2N}{10}$

$= \frac{23N}{10}$

50%	30%	20%
1 — 3c	1 — 2c	1 — 1c
$\frac{50}{100} \times 100 = ?$	$\frac{30}{100} \times 100 = ?$	20 — ?
150c	60	20c

N = 100

P[child belongs to family 2 C] = $\frac{60}{230} = \frac{6}{23}$

P(child belongs to family ~~with~~ with 3 C) = $\frac{150}{230} = \frac{15}{23}$

Ex: (1) 10 people are arranged

1) P(certain pair always together) = $\frac{9! \times 2!}{10!} = \frac{2}{10} = \frac{1}{5}$

2) P(certain pair never together) = $\boxed{\frac{10! - 9!\,2!}{10!}}$ $\cancel{\frac{9!\,8}{10!} = 1 - \frac{1}{5} = \frac{4}{5}}$

$1 - \frac{1}{5} = 4/5$

Note: Suppose there are n people arranged

$$P[\boxed{AB}] = \frac{2}{n}$$

$$P[\boxed{A\,B}] = \frac{n-2}{n}$$

Ex 10 people are arranged in circle

1) P(certain pair always together)

2) P(certain pair never together)

Sol 1) $P[AB] = \frac{8! \times 2!}{9!} = \frac{2}{9}$

2) ~~P[AB]~~ $P(A\,B) = 1 - 2/9 = \frac{7}{9}$

Note: n people in circle

1) $P(AB) = \frac{2}{n-1}$

2) $P(A\,B) = \frac{n-3}{n-1}$

Try Home

5 dogs & 4 cats are arranged

1) P(4 cats are together) = $\frac{6! \times 4!}{9!} = \frac{1}{7}$

2) P[no 2 cats are together) = $5/42 = \frac{5! \times {}^6C_4 \times 4!}{9!}$

3) P[dog & cats arranged alternatively] = $\frac{5!\,4!}{9!}$

21/7/18

Ex 5 members selected from 6 monkeys & 7 donkeys. What is prob.

M	D
0	5
1	4
2	3
3	2
4	1
5	0

P[committee has exactly 3 D) = $\frac{{}^7C_3 \times {}^6C_2}{{}^{13}C_5}$

P(Committee has at least 2 D) = $\frac{{}^{13}C_5 - ({}^6C_5 + {}^6C_4 \times {}^7C_1)}{{}^{13}C_5}$

$= 1 - P(\text{at most } 1)$

$= 1 - \frac{({}^6C_5 + {}^6C_4 \times {}^7C_4)}{{}^{13}C_5}$

(5)

WB
Q9
Sol.

4R
6B

$P(1R, 2B)$

I $R\,B_1 B_2$ → $\frac{4}{10} \cdot \frac{6}{9} \cdot \frac{5}{8}$ +
II $B_1 R B_2$ → $\frac{6}{10} \cdot \frac{4}{9} \cdot \frac{5}{8}$ +
III $B_1 B_2 R$ → $\frac{6}{10} \cdot \frac{5}{9} \cdot \frac{4}{8}$ +

$I + II + III = \frac{1}{2}$

Case I: 3 balls selected

$P(1R \& 2B) = \frac{{}^4C_1 \times {}^6C_2}{{}^{10}C_3} = \frac{4 \times 15 \times 6}{10 \times 9 \times 8} = \frac{1}{2}$

(OR)

Case II: 3 balls are selected one after other without replacement

$I + II + III = \frac{1}{2}$

Note: both the cases will be same if order of ball is not ~~mentioned~~, mentioned.

Ex.

4R
6B

3 balls selected one after the other without replacement. What is the prob that first red & next 2 black balls are selected

Sol: P(1^{st} red & next 2 black]

R · B · B

$\frac{4}{10} \cdot \frac{6}{9} \cdot \frac{5}{8} = \frac{1}{6}$

Gate 2018

Q 4R, 4G, & 4B are put in a box
3 balls are pulled out of the box at random. One after the another without replacement.

P(all 3 balls are R)

$= \frac{{}^4C_3}{{}^{12}C_3} = \frac{4 \times 6}{12 \times 11 \times 10} = \frac{1}{55}$

WB
Q10
Sol. 6 — +ve
8 — −ve

Total = ${}^{14}C_4$

$$\left[\begin{array}{l} 1 - P(\text{product } -ve) \\ = 1 - P\left(\frac{{}^6C_1 \times {}^8C_1}{{}^{14}C_2}\right) \\ = 1 - \frac{6 \times 8}{14 \times 13} \times 2 \quad 1 - \frac{48}{91} = \frac{43}{91} \end{array}\right] \times$$

P	N	
4	0	
3	1	— ×
2	2	
1	3	— ×
0	4	

favourable

$= \cancel{1 - P(\text{product is } -ve)}$ ⟹ $1 - P(\text{product is } -ve)$

$= 1 - \left(\frac{{}^6C_3 \times {}^8C_1 + {}^6C_1 \times {}^8C_3}{{}^{14}C_4}\right)$

$= 1 - \left(\frac{20 \times 8 + 6 \times 56}{1001}\right)$

$= \frac{505}{1001}$

Q11 We consider 8 standard 8×8 ...

Sol: total = 2 selected from 64

$= {}^{64}C_2$

In 8×8

8 times $\begin{cases} 7 \leftarrow 1^{st} \text{ row} \\ 7 \leftarrow 2^{nd} \text{ row} \\ \vdots \\ 7 \leftarrow 8^{th} \text{ row} \end{cases}$

8 times $\begin{cases} 7 \leftarrow 1^{st} \text{ column} \\ 7 \leftarrow 2^{nd} \text{ column} \\ \vdots \\ 7 \leftarrow 8^{th} \text{ column} \end{cases}$

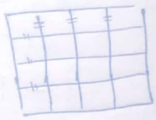
In 3×3

2 ← in 1st row

pair of squares which have sides in common

In 4×4

3 ← in 1st row



$P(\text{have common side}) = \dfrac{7 \cdot 8 + 7 \cdot 8}{{}^{64}C_2}$

Ex. The letters of the word KATTAPPA are arranged

i) $P(\text{3A's are together}) = \dfrac{6!/2!2!}{8!/3!2!2!} = \dfrac{3}{28}$ KTTPP $\boxed{AAA}$

ii) $P(\text{3A's \& also 2 T's are together}) = \dfrac{5!/2!}{8!/3!2!2!} = \dfrac{1}{28}$ KPP $\boxed{TT}$ $\boxed{AAA}$

~~Sol~~

$\text{Total} = C(8;\ \underset{T}{2}, \underset{P}{2}, \underset{A}{3}, \underset{K}{1}) = \dfrac{8!}{2!2!3!}$

WB

Q8

Sol: PROBABILITY

$P(\boxed{BB}\ \boxed{II}) = \dfrac{\frac{9!}{1!1!}}{2!2!} = \dfrac{2}{55}$

$\dfrac{\frac{10!}{2!}}{\frac{11!}{2!2!}} = \dfrac{10! \times 2!}{11!} = \dfrac{\cancel{10} \times 2}{11 \times \cancel{10}} = \dfrac{2}{11}$

PROA~~B~~IL~~I~~TY $\boxed{BI}$ $\boxed{II}$

WB

Q56

Sol: ~~3B 3Blue 3G 3R~~ black, be, G, R

$x_1 \quad x_2 \quad x_3 \quad x_4$

$x_1 + x_2 + x_3 + x_4 = 3 \qquad r = 3 \quad n = 4$

$^{n-1+r}C_r \Rightarrow \cancel{^{3-1+4}C_4} = {}^{5}C_4 =$

$\text{total} = {}^{4-1+3}C_3 = {}^{6}C_3 = 20$

for $\begin{cases} bl, bl, bl \\ be, be, be \\ G, G, G \\ R, R, R \end{cases}$ $= \quad P(\text{3 pens same color}) = \dfrac{4}{20} = \dfrac{1}{5}$

Ex

Sol: 3 ~~similar~~ identical dices are rolled

$\square \ \square \ \square$

R R R

P(same output on 3 dice) =

1 2 3 4 5 6

$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 3$

$n = 6 \quad r = 3 \qquad \text{total} = {}^{n-1+r}C_r = {}^{8}C_3 = 56$

Sam → (1,1,1) (2,2,2) (3,3,3) (4,4,4) (5,5,5) (6,6,6)

$P(\text{same output}) = \frac{6}{56} = \frac{3}{28}$

Addition Theorem

1) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

2) $P(A^c \cap B^c) = 1 - P(A \cup B)$

3) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cup B) - P(B \cup C)$
$- P(C \cup A) + P(A \cap B \cap C)$

4) $P(A^c \cap B^c \cap C^c) = 1 - P(A \cup B \cup C)$

Ex Two dices are rolled

$P(S = 8 \text{ or } 9) = \frac{5}{36} + \frac{4}{36} = \frac{9}{36} = \frac{1}{4}$

WB
(2) $P(\text{Sum is neither } 8 \text{ nor } 9) = 1 - 1/4 = 3/4$

$P(S \cdot \text{divisible by } 6) = \frac{5}{36} + \frac{1}{36} = \frac{1}{6}$
(6 & 12)

WB Q 12
$P[S \text{ is divisible by } 4 \text{ or } 6] = \cancel{\frac{3+4+5}{36}} = \frac{9}{36} + \frac{6}{36} - \frac{1}{36} = \frac{14}{36}$

4 → 4, 8, 12
6 → 6,
4 & 6 = 12, 12

Ex A no is selected from 1 to 1000. What is prob that no is divisible by 4 or 6

Sol $P(\text{no divisible by } 4 \text{ or } 6) = P(\text{div } 4) + P(\text{div } 6) - P(\text{div } 12)$

$P(\text{div } 4) = \left(\frac{250}{1000}\right)$ $P(\text{div } 6) = \frac{166}{1000}$ $P(\text{div } 12) = \cancel{\frac{250}{3000}} = \frac{83}{1000}$

$\frac{333}{1000}$

$\text{Prob} = \frac{(250) + 166}{1000} - \frac{83}{1000} = \cancel{\frac{1850}{3000}} \cancel{\frac{250}{3000}} \cancel{\frac{1000}{3000}} = \cancel{1/3}$

Ex A no ~~divisible by~~ selected from 1 to 1000

$P[\text{no div by } 3, 4 \text{ or } 6]$

$P(\text{div } 3) = \frac{\left\lfloor \frac{1000}{3} \right\rfloor}{1000} = \frac{333}{1000}$ $P(\text{div } 4) = \frac{250}{1000}$ $P(\text{div } 6) = \frac{166}{1000}$

$P(\text{div } 3,4,6) = \frac{\left\lfloor \frac{1000}{12} \right\rfloor}{1000} = \frac{83}{1000}$ $P(\text{div } 12) = \frac{83}{1000}$ $P(\text{div } 12) = \frac{83}{1000}$
$P(\text{div } 6) = \frac{166}{1000}$

$P = \frac{333}{1000} + \frac{250}{1000} + \frac{166}{1000} - \frac{83}{1000} - \frac{83}{1000} - \frac{166}{1000} + \frac{83}{1000} = \frac{1}{2}$

2)

P(~~test~~ P(not div by 3,4 or 6) = $1 - \frac{1}{2} = \frac{1}{2}$

Cards

Total = 52

Suits = 4; spade, club, diamond, heart

spade ♠	club ♣	diamond ♦	heart ♥
A	A	A	A
1	1	1	1
2	2	2	2
⋮	⋮	⋮	⋮
9	10	10	10
10	J	J	J
J	Q	Q	Q
Q	K	K	K
K			
13	13	13	13

Q A card is selected from pack of 52 cards.

Total = 52

$P[\text{Ace}] = \frac{4}{52}$ $\quad P(\text{King}) = \frac{4}{52}$

$P(\text{Ace or King}) = \frac{4}{52} + \frac{4}{52} = \frac{8}{52}$

$P(\text{Heart}) = \frac{13}{52}$

$P(\text{King or Heart}) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52}$

2) Two cards are selected from pack of 52 cards.

Total = $^{52}C_2$

$P(\text{both aces}) = \frac{^4C_2}{^{52}C_2}$ $\quad P(\text{both Hearts}) = \frac{^{13}C_2}{^{52}C_2}$

$P(\text{both kings}) = \frac{^4C_2}{^{52}C_2}$ $\quad P(\text{both black cards}) = \frac{^{26}C_2}{^{52}C_2}$

$P(1\text{ Ace } \& \text{ } 1\text{ King}) = \frac{^4C_1 \times {^4C_1}}{^{52}C_2}$

ME events

$P(\text{both (Aces) or both kings}) = \frac{^4C_2}{^{52}C_2} + \frac{^4C_2}{^{52}C_2}$

$P(\text{both kings or both blacks}) = \frac{^4C_2}{^{52}C_2} + \frac{^{26}C_2}{^{52}C_2} - \frac{^8C_2}{^{52}C_2}$ 2

$P(\text{both kings or Hearts}) = \frac{^4C_2}{^{52}C_2} + \frac{^{13}C_2}{^{52}C_2} - 0$ → P(king & heart) there is only 1 king in Heart

WB
Q47
So: 125 - bolts & 200 - nuts

Defective = $\frac{1}{5} \times 125 = 25$ $\quad$ Defective = $\frac{3}{4} \times 200 = 150$

$$P(D \text{ or nut}) = \frac{175}{325} + \frac{200}{325} - \frac{150}{325}$$

$$= \frac{\cancel{225}}{\cancel{325}} \; \frac{\cancel{45}}{\cancel{65}} \; \frac{9}{13}$$

$$= 9/13$$

$$= 0.69$$

175
150
225

Ex: A single die is rolled

$P(\text{multiple of } 3) = \frac{2}{6} = \frac{1}{3}$

Note: $S = \{1, 2, 3, 4, 5, 6\}$

multiple of $3 = \{3, 6\}$

$P(\text{multiple of } 3) = P[\text{multiple of } 3 \mid S_{\text{happened}}] = \frac{2}{6}$

Ex: Single die is rolled (Conditional Probability)

(1) $P[\text{multiple of } 3 \mid \text{no} > 4] = \frac{1}{2}$

$S = \{5, 6\}$

$\text{fav} = \{6\}$

(2) $P[\text{multiple of } 3 \mid \text{no} > 1] = \frac{2}{5}$

(3) $P[\text{no} > 4 \mid \text{multiple of } 3] = \frac{1}{2}$

4) $P(\text{no} > 1 \mid \text{multiple of } 3) = \frac{2}{2} = 1$

WB Q14

Sol: $P(S=7) = \frac{3}{15} = \frac{1}{5}$

(4,3)
(5,2)
(6,1)

$P(S=7, \text{ when } x_1 > x_2]$

$P(S=7 \mid x_1 > x_2]$

$(S=7) = 6$

$(x_1 > x_2)$: (4, 3), (5, 2), (6, 1)

$(x_1 < x_2)$: (3,4), (2,5), (1,6)

Q15

Sol: Total $= 2^3 = 8$

H, H, T, T, H

$P(\text{Exactly } 2H) = \frac{3}{15}$

ii) $P(S=8 \mid x_1 > x_2) = \frac{2}{15}$

(6,2)
(5,3)

iii) $P(S=6 \mid x_1 \le x_2) = \frac{3}{21}$

(3,3)
~~(5,1)~~
(1,5)
(4,2)

4) It is known that $x_1 \geq x_2$, What is prob of getting $S=9$

$P(S=9 \mid (x_1 \geq x_2)) = \frac{2}{21}$

(5,4) (6,3)

5) $P[x_1 > x_2$, it is known at $S=5] = \frac{2}{4}$

$P[x_1 > x_2 \mid S=5] = \frac{2}{4}$

6) $P(x_1 > x_2 \mid S=6) = \frac{2}{5}$

(4,2)
(5,1)

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WB
Q15

Sol: $P(\text{Exactly } 2H \mid 1^{st} \text{ toss } H) = \frac{2}{4}$

H — $\overline{H}\ \overline{T}$, TH > favourable, TT, HH

$S = \{HHT, HTH, HHH, HTT\}$

Note: $P(A/B) = P(A \text{ given } B)$

$= P[\text{Happening of A when B has already happened}]$

Def: Conditional Probability

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \qquad P(B) \neq 0$$

$$P(B/A) = \frac{P(A \cap B)}{P(A)} \qquad P(A) \neq 0$$

Ex: Single dice is rolled

A = multiple of 3 = $\{3, 6\}$ $\quad P(A) = 1/3$

B = no > 4 = $\{5, 6\}$ $\quad P(B) = 1/3$

$A \cap B = \{6\}$ $\quad P(A \cap B) = 1/6$

$$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{1}{2}$$

$$\frac{1/6}{1/3} = \frac{1}{2}$$

Multiplication Theorem

$$P(A \cap B) = P(A) \cdot P(B/A)$$

(or)

$$= P(B) \cdot P(A/B)$$

WB
Q16
Sol: Paper I $\qquad$ paper 2

$P(\overline{I}) = 0.3$ $\quad P(f\,P_1) = 0.3$

$P(\overline{II}) = 0.2$ $\quad P(f\,P_2) = 0.2$

$P\left(\frac{\overline{I}}{\overline{II}}\right) = 0.6$ $\quad P\left(\frac{f\,P_1}{f\,P_2}\right) = 0.6$

$$P(f\,P_1 \cap f\,P_2) = P(f\,P_2) \cdot P\left(\frac{f\,P_1}{f\,P_2}\right)$$
$$= 0.2 \times 0.6$$
$$= 0.12$$

Independent Events

Def: 2 events A & B are said to be independent if happening or non-happening of A does not effect happening or non-happening of B.

Ex: A card is drawn from pack of 52 cards (I)

Replaced

Card is drawn (II)

Concept:- Here $k\,II$ is independent of $k\,I$.

1) $P[k\,II \mid k\,I] = \frac{4}{52}$ → $P(k\,II)$

2) $P(k\,II \mid \text{not } k\,I) = \frac{4}{52}$ → $P(k\,II)$

Card is drawn

replaced
Card is drawn II

$P(K_{II} | K_I) = \frac{4}{52} = P(K_{II})$

$P(K_{II} | \text{not } K_I) = \frac{4}{52} = P(K_{II})$

K_{II} is independent of K_I

not replaced
Card is drawn

1) $P[K_{II}/K_I] = \frac{3}{51}$

2) $P(K_{II} | \text{not } K_I) = \frac{4}{51}$

Dependent

Results

(1) If A & B are independent then

1) $P(A/B) = P(A)$

2) $P(B/A) = P(B)$

3) $P(A \cap B) = P(A) \cdot P(B)$

4) $P(A \cap B^c) = P(A) \cdot P(B^c)$ [A and B^c are also independent]

5) $P(A^c \cap B) = P(A^c) \cdot P(B)$

6) $P(A^c \cap B^c) = P(A^c) \cdot P(B^c)$.

(2) In ~~general~~ general disjoint events are ~~independent~~ dependent.

Ex ~~2~~ Coin is tossed

A = getting H = {H}

B = getting T = {T}

$A \cap B = \phi$

A & B are dependent.

Ex: Probability of solving a problem by 3 students X, Y & Z are 1/3, 1/4, & 1/5 respectively.

Assuming they solve the problem independently. What is the prob. that problem is solved?

Sol: ~~$P(\text{problem solved}) = P(X) \cdot P(Y) \cdot P(Z)$~~

~~$= \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{5} = \frac{1}{60}$~~

P(problem solved) → at least 1 solved

$P(X \cup Y \cup Z) = P(X) + P(Y) + P(Z) - P(X \cap Y) - P(Y \cap Z) - P(Z \cap X) + P(X \cap Y \cap Z)$

$= \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{12} - \frac{1}{20} - \frac{1}{15} + \frac{1}{60}$

$= \frac{7}{12} + \frac{1}{5} = 3/5$

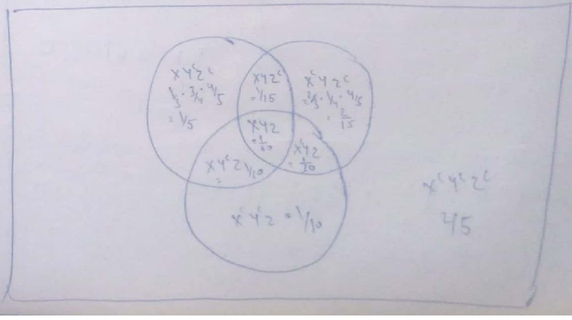
(or)

Better →

$P(X \cup Y \cup Z) = 1 - P(X^c \cap Y^c \cap Z^c) = 1 - P(X^c) \cdot P(Y^c) \cdot P(Z^c)$

$= 1 - 2/3 \cdot 3/4 \cdot 4/5$

$= 3/5$



1) P(only x solved) = 1/15

2) P(Exactly 1 solved) = $\cancel{\frac{1}{15} + \frac{1}{20} + \frac{1}{30}}$ $\cancel{\frac{3}{20}}$ $\frac{1}{5} + \frac{1}{15} + \frac{1}{10} = \frac{22}{60}$

3) P(only x & y solved) = $\frac{1}{15}$

4) P(exactly 2 solved) = $\frac{3}{20}$ $\left(\frac{1}{15} + \frac{1}{60} + \frac{1}{30} \right)$

5) P (at least one) = 3/5

6) P (At least 2 solved) = $\frac{3}{20} + \frac{1}{60} = \frac{1}{6}$

7) P(At most 1 solved) = $\frac{1}{60} + \frac{2}{5} = \frac{25}{60} = \frac{5}{12}$

8) P(A tmost 2 solved) = $\frac{5}{12} + \frac{3}{20} = \frac{25+9}{60} = \frac{34}{60} = \frac{17}{30}$

NOTE:

	Fair (or) Unbiased	Biased (or) Loaded
	all outcomes are equally likely	All outcomes are not equally likely
Ex. Coin	P(H) = ½ P(T) = ½	Ex. P(H) = ⅓ P(T) = ⅔ Ex. P(H) = 4/5 P(T) = 1/5 P(H) = 1 & P(T) = 0

WB
Q20
Sol. H H T T T T T T T T total = 2^{10}

½ ½ ½ ½ ½ ½ ½ ½ ½ ½

$= \left(\frac{1}{2}\right)^{10}$

WB
Q21
Sol. $\boxed{W - 2 \quad , \quad Nuts \to 3 \qquad bolts \to 4}$ ← box

$= \left(\frac{2}{9} \cdot \frac{1}{8} \cdot \frac{3}{7} \cdot \frac{2}{6} \cdot \frac{1}{5} \cdot \frac{4}{4} \right)$

WB
Q22 odd toss
Sol. 1st toss = $\frac{1}{2}$ P(no of toss required is odd) = ?

~~3rd toss = $\frac{1}{2} \cdot \frac{1}{2}$~~

1 or 3 or 5 ......
H TTH TTTTH

3rd toss = $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$

5th toss = $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$ $= \frac{1}{2} + \frac{1}{2} \cdot \left(\frac{1}{2} \cdot \frac{1}{2}\right) + \frac{1}{2} \cdot \left(\frac{1}{2} \cdot \frac{1}{2}\right)\left(\frac{1}{2} \cdot \frac{1}{2}\right) + \dots$

Ans $= \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^7 + \dots$

$= \frac{1}{2}\left[1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \dots \right]$

$= \frac{1}{2}\left(\frac{1}{1 - \frac{1}{4}} \right) = \frac{1}{2}\left(\frac{4}{3} \right) = \frac{2}{3}$

WB
Q23
Sol. A = $\frac{1}{6}$ 1st throw 6 = $\frac{1}{6}$

3rd throw 6 = $\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{1}{6}\right)$

5th throw 6 = $\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{1}{6}\right)$

A → A gets 6 B → B gets 6

P(A) = $\frac{1}{6}$ P(B) = $\frac{1}{6}$

P(A^c) = $\frac{5}{6}$ P(B') = $\frac{5}{6}$

① (or) ③ (or) ⑤

$P(A)$ $\quad$ $P(A')P(B')P(A)$ $\quad$ $P(A')P(B')P(A')P(B')P(A)$

$P(A)$ wins $= \frac{1}{6} + \left(\frac{5}{6}\right)^2\frac{1}{6} + \left(\frac{5}{6}\right)^4\frac{1}{6} + \cdots$

$= \frac{1}{6}\left[1 + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^4 + \cdots\right]$

$\frac{1}{6}\left[\frac{1}{1-\frac{25}{36}}\right]$

$= \frac{1}{6}\left[\frac{36}{11}\right]$

$= \frac{6}{11}$

CSWB

Q 71 If a fair coin is ~~tossed~~ tossed until the same result turns up in the succession (both H or both T). Then find the prob when the no of tosses are even.

Sol: $P(H) = 1/2$ $\quad$ $P(T) = 1/2$

~~for → ② (or) ④~~

~~$P(T)P(H)$ $\quad$ $P(T)P(T)P(H)$~~ $P(H)$

P(coin is tossed even no of times, what is prob of same results in succession)

for situation ② (or) ④

HH or TT $\quad$ HTHH (or) THTT

$\left(\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}\right)$ $\quad$ $\left(\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\right)$

$= \frac{1}{2}$ $\quad$ $= \frac{1}{8}$

$\therefore \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \frac{1}{128} + \frac{1}{512} + \cdots$

$= \cancel{\frac{1}{1-\frac{1}{4}}}\ \cancel{\frac{4}{3}}\ \cancel{\frac{1}{2}}\ \frac{1}{2}\left[1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots\right] = \frac{1}{2}\left(\frac{1}{1-\frac{1}{4}}\right)$

$= 2/3$

Gate 2012

Q Two people P and Q decide to independently roll 2 identical dice, each with 6 faces, numbered 1 to 6. The person with lower no, wins. In case of tie, they rolled the dice repeatedly until there is no tie. Define a trail as a throw of the dice by P and Q. Assume that all 6 no's on each dice are equi-probable and all trail are independent. The prob (rounded to 3 decimal) that one of them wins on the third trail is ______

Sol: Tie in trail $= P(\text{equal}) = \frac{6}{36} = \frac{1}{6}$

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P or Q wins ~~(5/6)~~

$P[x_1 < x_2 \text{ or } x_1 > x_2] = \frac{30}{36} = \frac{5}{6}$

For situation ① ② ③

Tie Tie P or Q wins (ie no tie on 3rd trial)

$= \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{5}{6}$

$= \frac{5}{216}$

WB Q24

Sol:) 2 dice rolled

$P(S=5) = \frac{4}{36} = \frac{1}{9}$

$P(S \neq 5 \cap S \neq 7) = 1 - \frac{10}{36} = \frac{26}{36}$

both dependent

$P(S=7) = \frac{6}{36} = \frac{1}{6}$

ME

$P(S=5 \text{ or } 7) = \frac{4}{36} + \frac{6}{36} = \frac{10}{36}$

and

~~$P(S \text{ not } 5 \text{ or } 7) = \frac{26}{36}$~~

~~P(5 come by 7)~~ P(stop with $S=5$)

~~① ②~~

~~$S=5$ $\left(\frac{26}{36}\right)\left(\frac{10}{}\right.$~~

① $S=5$ (or) ② $(S \neq 5 \cap S \neq 7)(S=5)$

or ③ $(S \neq 5 \cap S \neq 7)(S \neq 5 \cap S \neq 7)(S=5)$

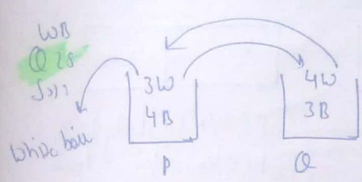
$P(\text{stop with } 5) = \frac{4}{36} + \left(\frac{26}{36}\right)\cdot\left(\frac{4}{36}\right) + \left(\frac{26}{36}\right)\left(\frac{26}{36}\right)\left(\frac{4}{36}\right) + \ldots\ldots$

$= \frac{4}{36}\left[1 + \left(\frac{26}{36}\right) + \left(\frac{26}{36}\right)^2 + \ldots\ldots\right]$

$= \frac{4}{36} \times \left(\frac{1}{1 - \frac{26}{36}}\right) = \frac{4}{36} \times \frac{36}{10} = \frac{2}{5}$

WB Q25

Sol:)



P(White ball) = ?

① ② ③

$\downarrow$ $\downarrow$ W

2 2

① ② ③ (or) ① ② ③ (or) ① ② ③ (or) ① ② ③

W W W WB W B W W B B W

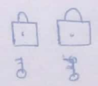
$\frac{3}{7} \cdot \frac{5}{8} \cdot \frac{3}{7} + \frac{3}{7} \cdot \frac{3}{8} \cdot \frac{2}{7} + \frac{4}{7} \cdot \frac{4}{8} \cdot \frac{4}{7} + \frac{4}{7} \cdot \frac{4}{8} \cdot \frac{3}{7}$

$= \frac{45}{7 \cdot 8 \cdot 7} + \frac{18}{7 \cdot 8 \cdot 7} + \frac{64}{7 \cdot 8 \cdot 7} + \frac{48}{7 \cdot 8 \cdot 7}$

$= \frac{25}{56}$

Q26

Sol:-



P(lock can still be opened by drawing one key at random)

$\boxed{2C \quad 4W}$

① ②

lost C

For situation ① ② (or) ① ②

C C W C

lost lost

$\frac{2}{6} \cdot \frac{1}{5} + \frac{4}{6} \cdot \frac{2}{5}$

$= \frac{2}{30} + \frac{8}{30} = \frac{1}{3}$

Total Probability Theorem

WB/49 A student takes MCQ exam where each question has 5 answers

K: knows
G: guess

K	G
70%	30%

Priori Probabilities $\begin{cases} P(K) = 0.7 \\ P(G) = 0.3 \end{cases}$

$\left.\begin{array}{l} P(C/K) = 1 \\ P(C/G) = 1/5 \end{array}\right\}$ likely hoods

$\cancel{P(C) = P(K) \cdot P\left(\frac{C}{K}\right) + P(}$

$P(C) = 0.7 \times 1 + 0.3 \times 1/5 = 0.76$

(or)

$P(C) = P(K \cap C) + P(G \cap C)$

$$\boxed{P(C) = P(K)\, P\left(\frac{C}{K}\right) + P(G) \cdot P\left(\frac{C}{G}\right)}$$

Total Probability Theorem

Bayes Theorem [Posterior Probabilities]

Ex: Given that student gets correct answer. What is the probability that i) student knows the answer
ii) student guess the answer

Sol: i) $P(K|C) = \dfrac{P(K) \cdot P\left(\frac{C}{K}\right)}{P(K) \cdot P\left(\frac{C}{K}\right) + P(G) \cdot P\left(\frac{C}{G}\right)}$

$= \dfrac{0.7 \times 1}{0.7 \times 1 + 0.3 \times 0.2}$

$= \dfrac{0.7}{0.76} = \dfrac{70}{76} = \dfrac{35}{38} = 0.92$

$$\boxed{P(K|C) = \frac{P(K \cap C)}{P(C)} = \cancel{P(K) \cdot P\left(\frac{K}{C}\right)}\ \frac{P(K) \cdot P\left(\frac{C}{K}\right)}{P(C)}}$$

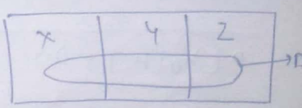
Bayes Theorem

2) $P[G|C] = \dfrac{P(G) \cdot P\left(\frac{C}{G}\right)}{P(K) \cdot P\left(\frac{C}{K}\right) + P(G) \cdot P\left(\frac{C}{K}\right)} = \dfrac{0.3 \times 0.2}{0.7 \times 1 + 0.3 \times 2}$

$= \dfrac{0.06}{0.76} = 0.08$

Ex:

X	Y	Z



$P(X) = 0.6$
$P(Y) = 0.3$
$P(Z) = 0.1$

$P[D/X] = 0.01$
$P[D/Y] = 0.02$
$P[D/Z] = 0.03$

1) $P(D)$ 2) i) $P[X/D]$ ii) $P[Y|D]$ iii) $P[Z/D]$

$P(D) = P(X) \cdot P\left(\frac{D}{X}\right) + P(Y) \cdot P\left(\frac{D}{Y}\right) + P(Z) \cdot P\left(\frac{D}{Z}\right)$

$= 0.6 \times 0.01 + 0.3 \times 0.02 + 0.1 \times 0.03$

$= 0.006 + 0.006 + 0.003$

$= 0.015$

$P[X|D] = \cancel{\dfrac{P[X]P[D]}{P(X)}} = \dfrac{0.6 \times 0.01}{0.015} = \dfrac{0.006}{0.015} = \dfrac{6}{15} = \dfrac{2}{5}$

$P[Y|D] = \dfrac{0.3 \times 0.02}{0.015} = \dfrac{0.006}{0.015} = 0.4$

$P[Z|D] = \dfrac{0.1 \times 0.03}{0.015} = \dfrac{0.003}{0.015} = \dfrac{3}{15} = \dfrac{1}{5} = 0.2$

Gate 2018

The following table gives conditional prob for Delhi temp given Guwahati's temp

	H_D	M_D	L_D
H_G	0.4	0.48	0.12
M_G	0.10	0.65	0.25
L_G	0.01	0.50	0.49

$P[H_D | H_G] = 0.4$ $\quad P[M_D | H_G] = 0.48$ $\quad P(L_D | H_G) = 0.12$

It is also known that $P(H_G) = 0.2$ $\quad P(M_G) = 0.5$

$P(L_G) = 0.3$

P[Guwahati has high tem given High temp in delhi]

$$P[H_G | H_D] = \frac{P(H_G)\, P\left(\frac{H_D}{H_G}\right)}{P(H_D)}$$

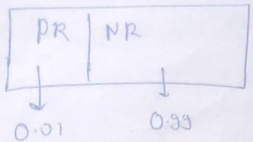
$P(H_G) = 0.2$ $\quad P[H_D | H_G] = 0.4$

$P(M_G) = 0.5$ $\quad P[H_D | M_G] = 0.10$

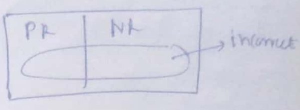
$P(L_G) = 0.3$ $\quad P[H_D | L_G] = 0.01$

$$P[H_G | H_D] = \frac{0.2 \times 0.4}{0.2 \times 0.4 + 0.5 \times 0.1 + 0.5 \times 0.01} = 0.6$$

WB
Q27
Sol:



0.01 $\qquad$ 0.99



$P[P] = 0.01$

$P[N] = 0.99$

$P[I | P] = 0.12$

$P[I | N] = 0.15$

$$P(I) = P(P) \times P\left(\frac{I}{P}\right) + P(N) \times P\left(\frac{I}{N}\right)$$

$= 0.01 \times 0.12 + (0.99) \times 0.15$

$= \cancel{0.0012} + 0.$

$= 0.1497$

Statistics

Statistics : Characterizing data (or) Distribution

1) <u>Measures of Central tendency</u> : Central Value

Ex: Mean , Median , Mode

2) <u>Measures of Dispersion</u> : spread (Range)

Ex: Variance , standard deviation

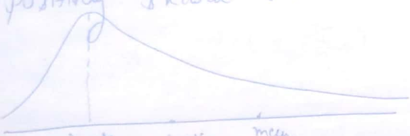
3) <u>Measure of skewness</u> : deviation from symmetry (or) lack of symmetry

Symmetric Distribution

Symmetric distributed across central value

Positively Skewed distribution

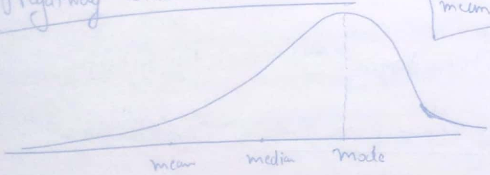
: distribution is lost in positive



$\boxed{\text{mode} < \text{median} < \text{mean}}$

Negatively Skewed Distribution

$\boxed{\text{mean} < \text{median} < \text{mode}}$



Ungrouped data

$x_1, x_2, x_3, \ldots, x_n$

mean : $\boxed{\mu = \frac{\sum x_i}{n}}$

Variance : Average of squared deviations from mean

$\sigma^2 = \frac{\sum (x_i - \mu)^2}{n}$ $\boxed{\sigma^2 = \frac{\sum x_i^2}{n} - \mu^2}$

Standard Deviation

σ : Positive square root of variance ie $\sqrt{\delta^2}$ ie $\sqrt{\sigma^2}$

Ex: 3, 1, 2, 4, 2, 6

mean $= \mu = \frac{3+1+2+4+2+6}{6} = \frac{18}{6} = 3$

Variance $= \sigma^2 = \frac{\sum x_i^2}{n} - \mu^2 =$

$= \frac{3^2+1^2+2^2+4^2+2^2+6^2}{6} - 9$

$= \frac{9+1+4+16+4+36}{6} - 9 = \frac{16}{6}$

$= 8/3 = 2.67$

$\sigma = \sqrt{2.67}$

$= 1.63$

HW

find mean & variance of first n natural numbers

mean $= \frac{n(n+1)}{2}$

variance $= \frac{(n+1)(2n+1)}{2} - \frac{(n+1)^2}{4}$

Random Variables

Random variable associate real no to outcomes of random experiment. (or)

Random variable is a set of possible values from a random experiment.

R.V

Discrete RV
→ takes discrete values

Continuous R.V
→ take all values in an interval

Ex: ~~Two~~ · Coin is tossed twice

$S = \{HH, HT, TH, TT\}$ (2, 1, 1, 0)

Define R.V

X : no of Heads

$X : \{0, 1, 2\}$

Ex. Telephone call duration

Define R.V

X : telephone call duration in minutes

$X : [0, \infty)$

Probability Mass function (Pmf)

The pmf $P(x)$ of a discrete r.v X is defined such that

1) $P(x) \geq 0$ (non-negative)

2) $\sum P(x) = 1$

3) $P(x_i) = P[X = x_i]$ ✓ Prob that X taking value x_i

Ex:

X	0	1	2
P(x)	1/4	2/4	1/4

Ex

X	0	1	2	3
P(x)	k	2k	3k	4k

$\sum P(x) = 1$

$k + 2k + 3k + 4k = 1$

$k = \frac{1}{10} = 0.1$

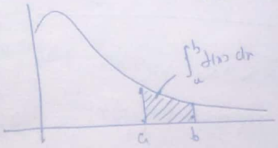
Probability Density function (Pdf)

The pdf $f(x)$ of a continuous random variable is defined such that

1) $f(x) \geq 0$

2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

3) $P(a < x < b) = \int_a^b f(x)\,dx = P(a \leq x \leq b)$



2) $P(1 < x < 3)$

$= P(X = 2) = 3k = 0.1$

Ex

X	0	1	2
P(x)	1/4	2/3	1/3

Wrong

$\sum P(x) \neq 1$

WB

30

Sol:

X	1	2	3	4	5	6
P(x)	k	2k	3k	4k	5k	6k

$k + 2k + 3k + 4k + 5k + 6k = 1$

$21k = 1$

$k = 1/21$

Pmf: $\boxed{P(x) = kx}$

$P(\text{odd}) = \cancel{\frac{k + 3k + 5k}{6k}}$ $= X = 1$ or 3 or 5

$k + 3k + 5k = 9k$

$\frac{1}{21} + \frac{3}{21} + \frac{5}{21} = \frac{9}{21} = \frac{3}{7}$

WB

Q31

Sol: $f(x) = e^{-x}, \quad 0 < x < \infty$

pdf

$P(X > 1) = \int_1^{\infty} f(x)\,dx = \int_1^{\infty} e^{-x}\,dx$

$= -\left[e^{-x}\right]_1^{\infty} = -\left[0 - e^{-1}\right] = e^{-1} = \frac{1}{e}$

WB

Q32

Sol

pdf $= f(x) = \lambda(x-1)(2-x) \qquad 1 \leq x \leq 2$

$\cancel{\int_1^2 f(x)\,dx} = \int_{-\infty}^{\infty} f(x)\,dx = 1$

$\int_1^2 f(x)\,dx = 1$

$\int_1^2 \lambda(-x^2 + 3x - 2)\,dx = 1$

$$\lambda \int_1^2 (-x^2+3x-2)\, dx = 1 \Rightarrow \lambda \left[-\frac{x^3}{3} + \frac{3x^2}{2} - 2x \right]_1^2 = 1$$

$$\lambda \left[\left(-\frac{8}{3} + 6 - 4\right) - \left(-\frac{1}{3} + \frac{3}{2} - 2\right) \right] = 1$$

$$\lambda \left[-\frac{7}{3} + \frac{3}{2} \right] = 1$$

$$\lambda = 6$$

WB
Q 34
Sol:- $f(x) = \frac{1}{5} e^{-x/5}, \quad x \geq 0$

$$P(X > 5) = \int_5^\infty f(x)\, dx = \frac{1}{5} \int_5^\infty e^{-x/5}\, dx$$

$$= \frac{1}{5} \times 5 \left[-e^{-x/5} \right]_5^\infty = +e^{-5/5} = \frac{1}{e}$$

Q 35
Sol:- pdf: $f(x) = \alpha e^{-\alpha t} \quad 0 \leq t \leq \infty$

$$P(100 \leq t \leq 200) = \int_{100}^{200} f(x)\, dx$$

$$\alpha \int_{100}^{200} e^{-\alpha t}\, dx = \frac{\alpha}{\alpha} \left[-e^{-\alpha t} \right]_{100}^{200} = -\left[-e^{-100\alpha} + e^{-200\alpha} \right]$$

$$= e^{-100\alpha} - e^{-200\alpha}$$

Expectation

It is weighted average of all possible value of X, where each value a weighted according to probability of that event occuring

1) Expected value of x (What you would expect the outcome of an experiment to be on an average)

$$E[x] = \begin{cases} \sum x P(x) & X \text{ is discrete R.V where } P(x) \text{ pmf} \\ \int_{-\infty}^{\infty} x f(x)\, dx & x \text{ is continuous R.V where } f(x) \text{ is p.d.f} \end{cases}$$

2) Expected value of x^2

$$E[x^2] = \begin{cases} \sum x^2 P(x) & x \text{ is discrete R.V} \\ \int_{-\infty}^{\infty} x^2 f(x)\, dx & x \text{ is continuous R.V} \end{cases}$$

Note: $E[x^n] \rightarrow n^{th}$ moment

Result

1) Mean of R.V X

$$\boxed{\text{mean} = \mu = E[X]}$$

2) Variance of X

$Var(X) = E[(X-\mu)^2]$ (or) $\boxed{Var(X) = E[X^2] - \mu^2}$

3) $E[C] = C \qquad C = \text{Constant}$

$Var[C] = 0$

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1) Find mean & Variance

$$f(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Sol: mean $\mu = E[x] = \int_0^1 x(2x)\,dx = 2\left[\frac{x^3}{3}\right]_0^1 = 2\left(\frac{1}{3}\right) = \frac{2}{3}$

$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$

Variance $= E[x^2] - \mu^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$

$= \cancel{\int_0^1 x \cdot 2x^2\,dx - \left(\frac{2}{3}\right)^2 \Rightarrow 2\left[\frac{x^3}{3}\right]_0^1 - \frac{4}{9}}$

$\cancel{\Rightarrow \frac{2}{3} - 1 \Rightarrow}$

$= \int_0^1 x^2 \cdot 2x\,dx - \left(\frac{2}{3}\right)^2 \Rightarrow 2\left[\frac{x^4}{4}\right]_0^1 - \frac{4}{9}$

$= \frac{2}{4} - \frac{4}{9} \Rightarrow \frac{1}{18}$

WB
Q50

Sol: $P(X, q) = \begin{cases} q & X = 0 \\ 1-q & X = 1 \\ 0 & \text{otherwise} \end{cases}$

X	0	1
P(X)	0.4	0.6

mean $= \mu = E[x] = \sum x P(x) = 0 \times 0.4 + 1 \times 0.6 = 0.6$

Variance $Var(x) = E(x^2) - \mu^2$

$= \sum x^2 P(x) = [(0)^2 \times 0.4 + (1)^2 \times (0.6)]$

$= 0.6 - (0.6)^2$

$= 0.6 - 0.36$

$= 0.24$

WB
Q52

Sol:

X:	1	2	3	4	5	6
P(X):	1/6	1/6	1/6	1/6	1/6	1/6

mean $= \mu = E[x] = \sum x P(x) = 1\times\frac{1}{6} + 2\times\frac{1}{6} + 3\times\frac{1}{6} + 4\times\frac{1}{6} + 5\times\frac{1}{6} + 6\times\frac{1}{6}$

$= \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6}$

$= \frac{4}{6} = \frac{7}{2} = 3.5$

Discrete Probability Distribution

Bernoulli's Trails

Independent trials of repetative nature in which probability (success (or) failure) of a particular event A does not change. from one trail to next trail are called Bernoullis trails..

1) Binomial Distribution — we conduct experiment n times. And only two outcomes are possible (ie success or failure) prob of success or failure does not change in any trial.

A discrete r.v

$X \sim B(n, p)$

[x follows binomial distribution with parameter n and p] if its pmf $P(x)$ is defined as

$$P(x) = {}^{n}C_{x}\, p^{x} q^{n-x} \qquad x = 0, 1, 2, 3, \dots n$$

Where n : no of Bernoulli's trails

A : event

$p = P(A)$

$q = P(A^c)$

A = success

A^c - failure

R.V X : no of times event A happens.

Note $X \sim B(n, P)$

mean $= nP$

variance $= npq$

Ex: A coin is tossed 6 times.

1) What is Prob of getting exactly 4 heads

ii) P [getting at least 2 heads]

$n=6$

A: getting head

$P = P(A) = \frac{1}{2}$ $\quad P(A^c) = \frac{1}{2}$

X: No of heads $\quad X \sim B(n=6, P=\frac{1}{2})$

$$P(X) = {}^{6}C_{x}\, p^{x} q^{6-x}$$

1) $\cancel{B(n=6, P=\frac{1}{2})} = P(X=4) = P(4) = {}^{6}C_{4} \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^2$

$= \frac{{}^{6}C_{4}}{2^6}$

2) $P[X \geq 2] = 1 - P[X \leq 1]$

$= 1 - \left[{}^{6}C_{1} \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^5 + {}^{6}C_{0} \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^6\right]$

$= 1 - \left[\left(\frac{1}{2}\right)^6 + 6\left(\frac{1}{2}\right)^6\right]$

$= 1 - 7\left(\frac{1}{2}\right)^6 = \frac{57}{64}$

WB Q 36

Sol: $n=4$ $\quad P[\text{no of times } H > \text{no of } T] = ?$

H	T
4	0
3	1
2	2
1	3
0	4

A = getting Head = $P(A) = \frac{1}{2} = P$ $\quad q = \frac{1}{2}$

X: no of heads

$P[X \geq 3] = {}^{4}C_{4} \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^0 + {}^{4}C_{3} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^1$

$= \left(\frac{1}{2}\right)^4 + 4\left(\frac{1}{2}\right)^4 = \left(\frac{1}{2}\right)^4 (5) = \frac{5}{16}$

WB Q 37

Sol.

P(4th Head appears at 10th toss) = ?

A = getting head $\quad p = \frac{1}{2}$ $\quad q = \frac{1}{2}$

X: getting head

1 2 3 . . . . . . 9 10

$\boxed{\text{3 heads in 9 tosses.}}$ H

$n=9$ $\quad p = \frac{1}{2}$ $\quad q = \frac{1}{2}$

$P(X=3) \cdot \frac{1}{2}$

$= {}^{9}C_{3} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^6 \cdot \frac{1}{2}$

$= 84 \times \left(\frac{1}{2}\right)^{10}$

$= \frac{84}{1024}$

Q39

Sol. $n = 11$ $p = 0.4$ $q = 0.6$ A = taking forward ~~step~~ step

P(after 11 steps he is 1 step away from starting position)

after 11 steps he is 1 step forward

after 11 steps he is 1 step backward

X: No of forward steps. $X \sim B(n = 11, p = 0.4)$

take 6F & 5B

take 5F 6B

$P(X = 6F)$

$P(X = 5)$

$= {}^{11}C_6 (0.4)^6 (0.6)^5 + {}^{11}C_5 \times (0.4)^5 (0.6)^6$

$= {}^{11}C_5 (0.4)^5 (0.6)^5 (0.4 + 0.6)$

$= {}^{11}C_5 (0.4)^5 (0.6)^5 (1)$

2) Poisson Distribution

gives discrete probability distribution of number of events occuring in a given time period, given the average number of times the event occurs over the time period

A discrete Random variable X

$$X \sim P(\lambda)$$

[X follows Poisson Distribution with parameter λ]

if its pmf $P(x)$ is given by

$$\boxed{P(x) = \frac{e^{-\lambda} \lambda^x}{x!}} \quad x = 0, 1, 2, \ldots$$

Note: When ~~X follows~~ $X \sim P(\lambda)$

mean $= \lambda$

Variance $= \lambda$

Q Find 2nd moment of Poisson distribution

Sol $E[x^2] = \cancel{\int_{-\infty}^{\infty} x^2 P(x)\, dx}$

$$\text{Variance} = E[x^2] - \mu^2$$

$$\lambda = E[x^2] - \lambda^2$$

$$E[x^2] = \lambda^2 + \lambda$$

3) Limiting case of binomial distribution is Poisson distribution

i.e $n \to \infty$ i.e n is very large & p is very small

$$\boxed{X \sim B(n, p) \approx X \sim P(\lambda) \quad \text{Where } \lambda = np}$$

WB Q41

Sol. X: no of accidents occuring in a month

$X \sim P(\lambda = 5.2)$

$$P[X < 2] = P(0) + P(1)$$

$$= \frac{e^{-5.2} \cdot (5.2)^0}{0!} + \frac{e^{-5.2} \cdot (5.2)^1}{1!}$$

$$= e^{-5.2}(1 + 5.2)$$

$$= 6.2\, e^{-5.2}$$

$$= 0.034$$

Imp Note

1 month — $\lambda = 5.2$

2 month $\to \lambda = 2(5.2)$

⋮

1 year $\to \lambda = 12(5.2)$

$\frac{1}{2}$ month $\to \lambda = \frac{5.2}{2}$

WB
Q43
Sol:

$n = 100 \quad p = 0.01$ (prob of defective condenser)
$q = 0.99$

n is very large
p is very small

$X \sim B(n, p) \approx X \sim P(\lambda)$

$\lambda = np = 100 \times 0.01 = 1$

$$P(x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$P(X \geq 3) = 1 - P(X \leq 2)$$
$$= 1 - [P(0) + P(1) + P(2)]$$
$$= 1 - e\left[\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!}\right]$$
$$= 1 - \frac{5}{2}e^{-1}$$

22/7/18

Continuous Probability Distributions

1) Normal Distribution (Normal or Gaussian or Gauss)

A continuous R.V

$$X \sim N(\mu, \sigma^2)$$

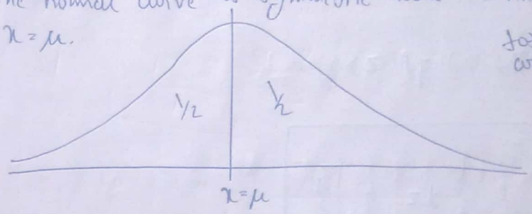
[X follows N.D with mean μ and variance σ^2]

if its pdf $f(x)$ is defined as

$$\boxed{f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}} \quad -\infty < x < \infty$$

Properties

1) The normal curve is symmetric about the line $x = \mu$.



Grade 2017

2) Mean = Median = Mode

3) $P(X \leq \mu) = P(x \geq \mu) = 1/2$

4) Standard normal variate

$$Z = \frac{X - \mu}{\sigma}$$

5) $Z \sim N(\mu = 0, \sigma^2 = 1)$

and its pdf $f(z)$ is given by

$$\boxed{f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}} \quad -\infty < z < \infty$$

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6) The standard normal curve is symmetric about the line $z = 0$

7) $P(Z \le 0) = P(Z \ge 0) = \frac{1}{2}$

8) $\boxed{\int_0^\infty e^{-z^2/2} dz = \sqrt{\frac{\pi}{2}}}$

$$\int_{-\infty}^{\infty} f(z)\, dz = 1$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz = 1$$

$$\int_{-\infty}^{\infty} e^{-z^2/2} dz = \sqrt{2\pi}$$

even fun^n $\left(\because \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx \quad f(-x) = f(x) \right)$

$$2\int_0^\infty e^{-z^2/2} dz = \sqrt{2\pi}$$

$$\int_0^\infty e^{-z^2/2} dz = \sqrt{\frac{\pi}{2}}$$

2) $\boxed{\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}}$

$$\int_0^\infty e^{-z^2/2} dz = \sqrt{\frac{\pi}{2}}$$

$$\frac{z}{\sqrt{2}} = x$$

$$dz = \sqrt{2}\, dx$$

$$\sqrt{2} \int_0^\infty e^{-x^2} dx = \sqrt{\frac{\pi}{2}}$$

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

3) $X_1 \sim N_1(\mu_1, \sigma_1^2)$

$X_2 \sim N_2(\mu_2, \sigma_2^2)$

$Y = c_1 X_1 + c_2 X_2$

$Y \sim N(\mu = c_1\mu_1 + c_2\mu_2,\ \sigma^2 = c_1^2\sigma_1^2 + c_2^2\sigma_2^2)$

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Ex: $X_1 \sim N(\mu_1 = 1, \sigma_1^2 = 1)$

$X_2 \sim N(\mu_2 = 1, \sigma_2^2 = 4)$

$Y = X_1 - X_2$

Sol: $Y \sim N(\mu = (1)(1) + (-1)(1),$

$\sigma^2 = (1)^2(1)^2 + (-1)^2(4)^2)$

$Y \sim N(\mu = 0, \sigma^2 = 5)$

Ex: $X_1 \sim N_1(\mu_1 = 1, \sigma^2 = 1)$

$X_2 \sim N_2(\mu_2 = 1, \sigma_2^2 = 4)$

$Y = 2X_1 + 3X_2$

Sol: $\mu = c_1\mu_1 + c_2\mu_2 \qquad c_1 = 2 \quad c_2 = 3$

$= 2(1) + 3(1)$

$= 5$

$\sigma^2 = c_1\sigma_1^2 + c_2\sigma_2^2$

$= (2)^2(1) + (3)^2(4)$

$= 4 + 36$

$= 40$

$Y \sim N(\mu = 5, \sigma^2 = 40)$

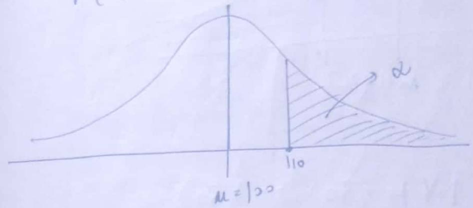
Q44

Sol: $-\infty < X < \infty$

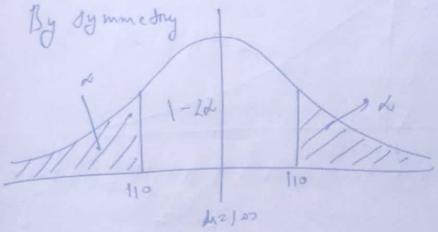
$X \sim N(\mu = 100, \sigma^2)$

$P(X \geq 110) = \alpha$

$P(90 \leq x \leq 110) = ?$



By symmetry



total = 1

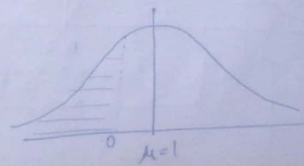
$\therefore P(90 \leq X \leq 110) = 1 - 2\alpha$

Q45

Sol: $X \sim N(\mu = 1, \sigma^2 = 4)$

$P(X < 0) = ?$

less than 0.5 & greater than 0.

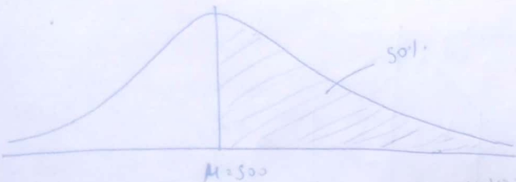


WB
Q46
Sol: $X \sim N(\mu = 1000, \sigma^2 = 200)$

given $\sigma = 200$ m

$P(X > 1200) = \ ' < 50\%$

WB
Q55
Sol: $X \sim N(\mu = 500, \sigma = 50)$

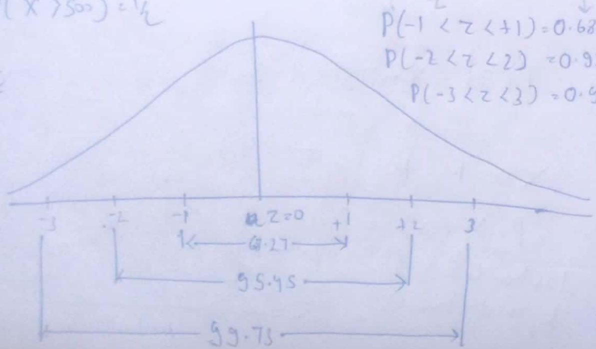


$P(X > 500) = 1/2$

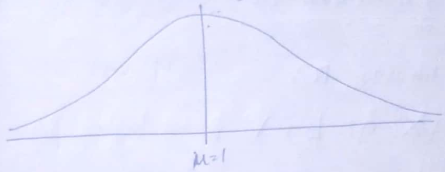
area under the standard normal curve of Random Variable Z

$P(-1 < Z < +1) = 0.6827$
$P(-2 < Z < 2) = 0.9545$
$P(-3 < Z < 3) = 0.9973$

Result
(11)



Ex $X \sim N(\mu = 1, \sigma^2 = 4)$ $P(1 < X < 3) = ?$



$$P\left(\frac{1-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{3-\mu}{\sigma}\right)$$

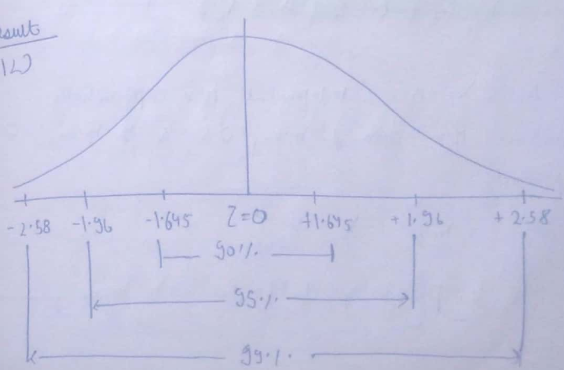
$$P\left(\frac{1-1}{2} < Z < \frac{3-1}{2}\right)$$

$$P(0 < Z < 1)$$

By symmetry

$$\frac{1}{2} P(-1 < Z < 1) = \frac{1}{2} \times 0.6827 = \cancel{0.34135}$$
$$0.34135$$

Result
(12)



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2) Exponential Distribution

A Continuous r.v

$$X \sim E(\lambda)$$

if its pdf $f(x)$ is given by

$$\boxed{f(x) = \lambda e^{-\lambda x}} \qquad x \geq 0$$

1) Mean $= \frac{1}{\lambda}$ Variance $= \frac{1}{\lambda^2}$

2) Variance $= \frac{1}{\lambda^2}$

3) $X_1 \sim E(\lambda_1)$
   $X_2 \sim E(\lambda_2)$

Gratis $\min(X_1, X_2) \sim E(\lambda_1 + \lambda_2)$

Ex: let X_1 & X_2 are 2 independent exponentially distributed r.v with mean$_1$ = 0.5 & mean$_2$ = 0.25 respectively.

$$Y = \min\{X_1, X_2\}$$

is Exponentially distributed with mean ______

Sol: $X_1 \sim E(\lambda_1 = 2)$ $\qquad \lambda_1 = \frac{1}{0.5} = 2$

$X_2 \sim E(\lambda_2 = 4)$ $\qquad \lambda_2 = \frac{1}{0.25} = 4$

mean $= \frac{1}{\lambda} = 0.5$

~~Yeh~~ $Y = \min\{X_1, X_2\}$

$Y \sim E(\lambda = \lambda_1 + \lambda_2)$

$Y \sim E(\lambda = 6)$

mean $= \frac{1}{\lambda} = \cancel{0.66}$ 0.16

3) Uniform Distribution

A Continuous r.v X

$$X \sim U[(a,b)]$$

[X follows Uniform distribution over the interval (a,b)]

if its pdf $f(x)$ is given by

$$\boxed{f(x) = \frac{1}{b-a}} \qquad a < x < b$$

1) Mean $= \frac{b+a}{2}$

2) Variance $= \frac{(b-a)^2}{12}$

WB Q5
Sol:

$X \sim U(0,1)$ Expected length of shorter stick

$f(x) =$

X: length of shorter stick

$$X \sim U(0, \tfrac{1}{2})$$

$$\mu = E[x] = \frac{b+a}{2} = \frac{\frac{1}{2}+0}{2} = \frac{1}{4}$$

LU Decomposition (Linear Algebra)

Let A be a non-singular matrix.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Writing matrix as a product of Lower Δ & upper Δ matrix

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} 1 & u_{12} \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11} u_{12} \\ l_{21} & l_{21} u_{12} + l_{22} \end{bmatrix}$$

$l_{11} = 1$

$u_{12} = \frac{2}{1}$

$l_{21} = 3$

$l_{21} u_{12} + l_{22} = 4$

$3 \times 2 + l_{22} = 4$

$l_{22} = -2$

$$L = \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Note: Set ~~either UΔ or~~ LΔ main diagonal elements as 1 & UΔ matrix with no 0's on the main diagonal.

WB Solutions (Probability)

(3).

H	T	Diff
n	0	n
$n-1$	1	$n-2$
$n-2$	2	$n-4$

$P(\text{diff } n-1) = \frac{0}{2^n} = 0$

(4) $A_1 \to 7$ $A_1 \to 7$, $A_2 \to 7$

(M,M) (T,M) . . . (S,M)
(M,T) (T,T) . . . (S,T)

(M,S) (T,S) . . . (S,S) → 7

favourable $= A_1$ — 7 ways

$A_1 \to 7$, $A_2 \to 7$, $A_3 \to 7$, $A_4 \to 7$, $A_5 \to 7$, $A_6 \to 7$, $A_7 \to 7$ $= 7^7$

$$P(\text{same day}) = \frac{7}{7^7} = \frac{1}{7^6}$$

(17) $P(X) = 0.4$

$P(X \cup Y^c) = 0.7$

$$P(X \cup Y^c) = P(X) + P(Y^c) - P(X \cap Y^c)$$
$$= P(X) + P(Y) - P(X) \cdot P(Y^c)$$

$0.7 = 0.4 + P(Y') \cdot (1 - P(Y))$

$0.3 = P(Y^c) \cdot P(X^c)$

$0.3 = P(X^c \cap Y^c)$

$P(X \cup Y) = 1 - P(X^c \cap Y^c)$

$= 1 - 0.3$

$= 0.7$

(1.7) $P(P) = 1/4$

$P(P/Q) = 1/2$

$P(Q/P) = 1/3$

$P(P^c/Q^c) = ?$

$P(Q/P) = 1/3 = \frac{P(Q \cap P)}{P(P)} = \frac{1}{3}$

$P(Q \cap P) = 1/12$

$P(P/Q) = 1/2 = \frac{P(Q \cap P)}{P(Q)} = \frac{1}{2}$

$P(Q) = 1/6$

$P(P \cup Q) = \frac{1}{4} + \frac{1}{6} - \frac{1}{12}$

$= \frac{1}{3}$

$P(P^c \cap Q^c) = 1 - P(P \cup Q) = 1 - \frac{1}{3} = \frac{2}{3}$

$P(P^c/Q^c) = \frac{P(P^c \cap Q^c)}{P(Q^c)} = \frac{2/3}{5/6} = \frac{4}{5}$

Revision (Imp)

Q 2.75 P-26 (Set Theory)

Sol: $x * x = y * y = x * y * x * y = y * x * y * x = e$

e = identity element

max no. of elements in group?

$x * x = e \quad \therefore x^{-1} = x$ ie x is self inverse

$y * y = e \quad y^{-1} = y$

$(x * y) * (x * y) = (x * y)^{-1} = (x * y)$

$(y * x) * (y * x) = (y * x)^{-1} = (y * x)^{-1}$

$\therefore$ group has 4 distinct elements.

As per definition of group, any element must have only one inverse element (unique)

Q $\lim_{x \to 0} \log_{\sin x} \sin 2x$

Sol: $\lim_{x \to 0} \left(\frac{\log \sin 2x}{\log \sin x}\right) \left(\frac{-\infty}{-\infty}\right) \Rightarrow \lim_{x \to 0} \frac{\left(\frac{2\cos 2x}{\sin 2x}\right)}{\left(\frac{\cos x}{\sin x}\right)}$

$\Rightarrow \lim_{x \to 0} \frac{\left(\frac{2x \cos 2x}{\sin 2x}\right)}{\frac{\cos x}{\left(\frac{\sin x}{x}\right)}} \Rightarrow \lim_{x \to 0} \frac{\cos 2x}{\cos x} = 1$

MT4

Q $\langle D_n, | \rangle$ is ~~a~~ boolean Algebra

$n = 5^{p-n}\, 6^{p+n}\, 7^{n+q+2s}\, 11^{q-2s+n}$

$p, q, n, s \in \mathbb{Z}$

find $(2p + n + 2q + s)$?

Sol: Since D_n is boolean algebra, possible only if n is product of distinct factors.

$n = 5^{p-n} (2^{p+n})(3^{p+n}) \cdot 7^{n+q+2s}\, 11^{q-2s+n}$

$p - n = 1$ $\quad$ $p = 1 \quad n = 0$

$p + n = 1$ $\quad$ $q + 2s = 1$

$n + q + 2s = 1$ $\quad$ $q - 2s = 1$ $\Rightarrow q = 1 \quad s = 0$

$q - 2s + n = 1$

$\therefore 2p + n + 2q + s = 2(1) + 0 + 2(1) + 0 = 4$

Gate-2007

Q How many different non-isomorphic Abelian groups of order 4 are there?

Sol: $O(G) = 4$

prime factorize $4 = 2^2\, 1^1$

# of non-isomorphic abelian groups $= 2 \cdot 1 = 2$

i.e. we can have 2 partitions $\{1,1\}$ $\{2,0\}$

Ex: of order is 600

$O(G) = 600 \Rightarrow 2^3 \times 3^1 \times 5^2$

find # of partitions of all powers

$3 = \{3\}\ \{2,1\}\ \{1,1,1\}$

$2 = \{1,1\}\ \{2,0\}$

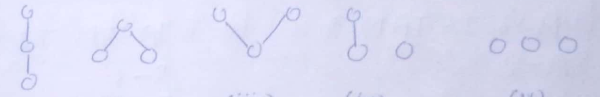
$1 = \{1\}$

# of non-isomorphic abelian group $= 6$ i.e. $3 \cdot 1 \cdot 2 = 6$

Q $A = \{1, 2, 3\}$

find # of partial ordering relations possible?

Sol:- # partial orders = # of Hasse diag possible



(i) $\quad$ (ii) $\quad$ (iii) $\quad$ (iv) $\quad$ (v)

in (i) 1,2,3 can be arranged in $3!$ ways $= 6$

(ii) There are 3 ways to choose top element. And remaining 2 elements can arrange in $2!$ ways $= 3 \cdot 2 = 6$

(iii) 3 ways to choose minimum element. Remaining 2 can arrange in $2!$ ways $= 3 \cdot 2 = 6$

iv) 3 ways to choose, that is not comparable (element). remaining 2 ~~can arra~~ comparable can arrange in $2!$ ways hence total $3 \cdot 2 = 6$

v) ~~elements can~~ represents only one partial order the relation of equality on $\{1,2,3\}$

Total $= 6 + 6 + 6 + 6 + 1 =$ ~~19 ways~~ 25 partial orders

Q $A = \{1, 2, \ldots n\}$

~~f~~ # of equivalence relations on A which are ~~...~~ also surjective.

MT3 (2018)

Q A letter is taken at random from ASSISTANT & STATISTICS. The probability that they are same letters is

Sol: ASSISTANT STATISTICS

ASSISTANT	STATISTICS
A - 2	S - 3
S - 3	T - 3
I - 1	A - 1
T - 2	I - 2
A - 1	C - 1

Prob. they are same :-

$$\left.\begin{aligned}
(A\ A) &\Rightarrow \frac{{}^2C_1}{{}^9C_1} \times \frac{1}{{}^{10}C_1} = \frac{2}{90} \\
&+ \\
(S\ S) &\Rightarrow \frac{{}^3C_1}{{}^9C_1} \times \frac{{}^3C_1}{{}^{10}C_1} = \frac{9}{90} \\
&+ \\
(T\ T) &\Rightarrow \frac{{}^2C_1}{{}^9C_1} \times \frac{{}^3C_1}{{}^{10}C_1} = \frac{6}{90} \\
&+ \\
(I\ I) &\Rightarrow \frac{1}{{}^9C_1} \times \frac{{}^2C_1}{{}^{10}C_1} = \frac{2}{90}
\end{aligned}\right\} \Rightarrow \frac{19}{90}$$

~~To Prob~~ =

MT7 (AD)

Q 59

Sol: # of commutative binary opn on a set with n elements $= n^n \cdot 2^{n(n-1)/2}$

# of non-commutative binary opn on a set with n elements $= n^{n^2} - \left[n^n \cdot 2^{n(n-1)/2}\right]$

$\therefore$ required prob $= \dfrac{n^{n^2} - n^n \cdot 2^{n(n-1)/2}}{n^{n^2}}$

given set of 3 elements $\therefore n = 3$

$$\text{Prob} = \frac{3^9 - 27 \cdot 3^3}{3^9} = 1 - \frac{3^3 \cdot 3^3}{3^9} = 1 - \frac{1}{27} = \frac{26}{27}$$

$$= \cancel{1 - \frac{8}{729}} = \cancel{\frac{721}{729}}$$

$X = \dfrac{26}{27}$ $\qquad 54 \times X = 54 \times \dfrac{26}{27} = 52$

Q. A deck of 5 cards (each carrying distinct no 1 to 5) is shuffled thoroughly. Two cards are then removed one at a time from the deck. What is the probability that the two cards are selected with the number on the first card being one ~~heigh~~ higher than the number on the second card?

Sol:

1	2	3	4	5

$\boxed{1}$ $\boxed{2}$ $\boxed{3}$ $\boxed{4}$ $\boxed{5}$

no of ways to draw 1st card

(5,1) (5,2) (5,3) (5,4)
(4,1) (4,2) (4,3) (4,5)
(3,1) (3,2) (3,4) (3,5)
(2,1) (2,3) (2,4) (2,5)
(1,2) (1,3) (1,4) (1,5)

total cases = 20 = 5.4

favourabe cases
(5,4) (4,3) (3,2) (2,1)

$$Prob = \frac{4}{20} = \frac{1}{5}$$

ii> Prob. no. on first card ~~be~~ higher than no on second card

I	II
5	4,3,2,1
4	3,2,1
3	2,1
2	1

$$\left(\frac{1}{5}\right)\left(\frac{4}{4}\right)+\left(\frac{1}{5}\right)\left(\frac{3}{4}\right)+\left(\frac{1}{5}\right)\left(\frac{2}{4}\right)+\left(\frac{1}{5}\right)\left(\frac{1}{4}\right)$$

$$\frac{4+3+2+1}{20} = \frac{1}{2}$$

Q $A = \begin{bmatrix} 1 & \omega^n & \omega^{2n} \\ \omega^n & \omega^{2n} & 1 \\ \omega^{2n} & 1 & \omega^n \end{bmatrix}$

is singular matrix bcz $1+\omega^n+\omega^{2n}=0$

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