Counting - I

Sum rule (Disjoint)

if an event A can happen in m ways, another event B can happen in n ways. Then A or B can happen in m+n ways.

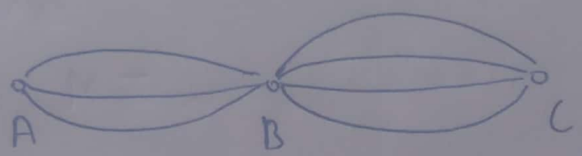
Product rule (Independent)

If an event A can happen in m ways, another event B can happen in n ways, then A followed by B can happen in m × n ways.

university A offers 5 courses

Ex. A – 5
B – 4 How many way we can select a course 5+4+3
C – 3 = 12 ways

Ex.



no of ways to go to C from A via B

1 ——— 4
3 ——— 3×4 = 12

Ex. No of binary strings of length 2 = 4

00
01
10
11

Scanned by CamScanner

$2 \quad 2 \quad = 2^2 = 4$

0 < 0, 1

1 < 0, 1

binary strings of length 3

000
001
010
011
100
101
110
111

Result: No of bs of length ~~of~~ $n = 2^n$

1 2 3 .... n

$2 \quad 2 \quad 2 \quad \ldots \quad 2$

Each position can be filled in 2 ways

$= 2 \cdot 2 \cdot 2 \cdots$ n times

$= 2^n$

Ex. No of bs of length 8 starting with '0' $= 2^7$

0 | 2 2 2 2 2 2 2

$= 2^7$

$= 128$

Ex. IPV4 (32 bit addressing)

32-bit

		netids	host ids
Class A	0 — NID(8), HID(24)	$2^8 - 2$	$2^{24} - 2$
Class B	10 — NID(16), HID(16)	2^{14}	$2^{16} - 2$
Class C	110 — NID(24), HID(8)	2^{21}	$2^8 - 2$

Ex. No of 4-digit integers = ~~10^4~~

$\cancel{0}$: 9 10 10 10

$= 9 \times 10 \times 10 \times 10 = 9000$

[lb up]

up − lb + 1

[1000 − 9999]

[9999 − 1000 + 1]

= 9000

Gate

How many 4 digit even numbers have all 4 digits distinct?

Sol:

$$\left[\begin{array}{l} 9 \quad 10 \quad 10 \quad 2,4,6,8,0 \\ 900 + 900 + 900 + 900 + 900 = 4500 \end{array} \right] \times$$

$\cancel{0}$			
8	8	7	2	$= \cancel{336}$ 448
8	8	7	4	$= \cancel{336}$ 448
8	8	7	6	$= \cancel{336}$ 448
8	8	7	8	$= \cancel{336}$ 448
9	8	7	0	$= 9 \times 56 = 504$

0 X, 2,4,6,8

8 8 7

$4(\overset{448}{\cancel{336}}) + 504 = \cancel{1344 + 504}$

$= \cancel{1848}$ 2296

Ex: 4 digit odd no's have all 4 digit distinct

0 ✗   1, 3, 5, 7, 9

8 8 7

$= (8 \times 8 \times 7) \times 5$

$= 448 \times 5$

$= 2240$

Indirect Counting

even = total − odd

total 4 digit no's having all 4 digit distinct

9 9 8 7 $= 4536$

Ex: No of non-negative integers less than 10^5 = ______

< 100000

10 10 10 10 10

by allowing leading 0's we get 4 digit, 3 digit, 2 digit & also 1 digit no's.

a) no of evens ?, 0, 2, 4, 6, 8

$= 10^4 \times 5$

b) odd ? 1, 3, 5, 7, 9

$= 10^4 \times 5$

Ex: no of non-negative integers $< 10^5$ containing digit 1

Sol: 9 9 9 9 9 $= \cancel{8 \times 10^4}\ 10^5 - 9^5$

Contain digit 1 = total − not containing digit 1

$= 10^5 - 9^5$

Ex: no of palindromes of length 5 using English alphabet?

Sol: 26 26 26 1 1 $= 26 \cdot 26 \cdot 26 = (26)^3$

Ex: No of 5 digit palindromes = ______

0 is not allowed here

~~10~~ 9 10 10 1 1 $= \cancel{(10)^3}\ 9 \times 10 \times 10 = 900$

a) even

2, 4, 6, 8   10   10   1   0, 2, 4, 6, 8

4800

we will not end with 0

b) odd   1, 3, 5, 7, 9

1, 3, 5, 7, 9   10   10   1

$5 \times 100 = 500$

Ex: A flag has to be designed with 6 vertical strips using 4 colors. Red, blue, green and yellow. In how many ways this can be done so that no two adjacent strips have same color?

Sol:

4 3 3 3 3 3

R (B, G, Y) (R, G, Y)

B G

$= 4 \times 3^5$

=

Ex How many positive integers <=100 are

i) divisible by 2 $= \left\lfloor \frac{100}{2} \right\rfloor = 50$ → ceil fun

10 0, 2, 4, 6, 8 $= 10 \times 5 = 50$

ii) divisible by 3 $= \left\lfloor \frac{100}{3} \right\rfloor = 33$

iii) divisible by 2 & 3 $= \left\lfloor \frac{100}{6} \right\rfloor = 16$ LCM(2,3) = 6

iv) divisible by 4 and 6 $= \left\lfloor \frac{100}{12} \right\rfloor = 8$

Ex. In the prime factorization of 300!, what is the exponent of ~~[illegible]~~

a) 11 b) 13 c) 3

$\left\lfloor \frac{300}{11} \right\rfloor = \left\lfloor \frac{27}{11} \right\rfloor \left\lfloor \frac{5}{11} \right\rfloor = 27 + 2 + 0 = 29$

$\downarrow$ 27 $\downarrow$ 2 $\downarrow$ 0

b) $\frac{300}{13} = \frac{23}{13} = \frac{1}{13} = 23 + 1 = 24$

$\downarrow$ 23 $\downarrow$ 1

X

c) $\frac{300}{3} =$

---

a) $\left\lfloor \frac{300}{11} \right\rfloor + \left\lfloor \frac{300}{121} \right\rfloor = 27 + 2 = 29$

b) $\left\lfloor \frac{300}{13} \right\rfloor + \left\lfloor \frac{300}{169} \right\rfloor = 23 + 1 = 24$

c) $\left\lfloor \frac{300}{3} \right\rfloor + \left\lfloor \frac{300}{9} \right\rfloor + \left\lfloor \frac{300}{27} \right\rfloor + \left\lfloor \frac{300}{81} \right\rfloor + \left\lfloor \frac{300}{243} \right\rfloor =$

$= 100 + 33 + 11 + 3 + 1 = 148$

Q No of positive divisors of 300 = ______

$300 = 2^2 \cdot 3^1 \cdot 5^2$

$= (2+1) \cdot (1+1) \cdot (2+1)$

$= 3 \cdot 2 \cdot 3$

$= 18$

b) No of positive odd divisors of 300 = ______

$2^a 3^b 5^c \mid 300$ $0 \le a \le 2$ $0 \le b \le 1$ $0 \le c \le 2$

a	b	c
3	2	3

= 18

for odd divisors put $a = 0$, (b/c we dont want to take even prime no)

a	b	c
1	2	3

= 6

c) No of even positive divisors

$\boxed{\text{even} = \text{total} - \text{odd}}$

$1 \le a \le 2$

2^0 will not be considered

2	2	3

= 12

d) Sum of positive divisors of 300

$= (2^0+2^1+2^2)(3^0+3^1)(5^0+5^1+5^2)$

$= (7)(4)(31)$

$= 28 \times 31$

$= 868$

28
×31
28
84×
868

Qn: No of positive divisors of 2014 = ______

No of divisors of 2100 = ______

Sol: 2 | 2014
19 | 1007
| 53

$2014 = 2^1 \times 19^1 \times 53^1$

$= (1+1)(1+1)(1+1)$

$= 2 \times 2 \times 2$

$= 8$

NOTE:- take $\sqrt{1007} \approx 31$

try by dividing every prime no < 31

2 | 2100
2 | 1050
5 | 525
5 | 105
3 | 21
7 | 7

$= 2^2 \times 3^1 \times 5^2 \times 7^1$

$= (2+1)\cdot(1+1)\cdot(2+1)\cdot(1+1)$

$= 3 \cdot 2 \cdot 3 \cdot 2$

$= 36$

Ex:- P-12 Q 22 (WB)

10 – T_1 15 – T_2 14 – T_3 2 – Girls

G_1	G_2	T_1	T_2	T_3
10	0			
9	1			
8	2			
7	3			
⋮				
0	10		16 ways	15 ways

11

for each of these 11 ways

$= 11 \times 16 \times 15$

Results

1) $1 + 2 + 3 + \ldots\ldots + n = \sum n = \frac{n(n+1)}{2}$

2) $1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \sum n^2 = \frac{n(n+1)(2n+1)}{6}$

3) $1^3 + 2^3 + 3^3 + \ldots\ldots + n^3 = \sum n^3 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{n^2(n+1)^2}{4}$

4) Arithmetic Sequence

$a, a+d, a+2d, \ldots\ldots$

n^{th} term $= t_n = a + (n-1)d$

Sum of n terms $= S_n = \frac{n}{2}(2a + (n-1)d)$

$= \frac{n}{2}[t_1 + t_n]$

5) Geometric Sequence

$a, ar, ar^2, ar^3, \ldots\ldots$

n^{th} term $= t_n = ar^{n-1}$

Sum of n terms $= S_n = \begin{cases} \frac{a(r^n-1)}{r-1} & r > 1 \\ \frac{a(1-r^n)}{1-r} & r < 1 \\ na & r = 1 \end{cases}$

Sum to infinite terms

$S_\infty = a + ar + ar^2 + ar^3 + \ldots\ldots \infty$

$= \frac{a}{1-r} \quad -1 < r < 1$

Subsequence of a string is obtained by deleting 0 (or) more symbols.

Substring of a string is obtained by deleting 0 (or) more symbols from the start (or) end (or) both

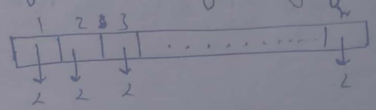
Ex. COMPUTER

~~CO~~MPUTE~~R~~ – MPUTE substring

~~CO~~M~~P~~UTE~~R~~ MUTE subsequence

Result

No of subsequences of a string of length n :



each symbol has 2 choices (Present or absent in subsequence)

$= 2 \cdot 2 \cdot 2 \ldots\ldots n$ times

$= 2^n$

No of substrings of a string of length n = ______

" " " of length $1 = n$

" " " of length $2 = (n-1)$

" " " of length $3 = (n-2)$

⋮

" " " of length $n = 1$

Here we have not taken substring of length 0

$\therefore$ no of substrings $= 1 + (n-1) + (n-2) + \ldots\ldots + 1$

$= \frac{n(n+1)}{2}$

P-12 Q16 (WB)

P-13 Q 34 (WB)

AXIOMATIZABLE

no of subwords $= \frac{n(n+1)}{2}$

$= \frac{13(13+1)}{2}$

$= 91 - 2 - 1$

I repeated once

A repeated twice

NOTE:

→ By default substring of empty string will not be considered (substring of length 0)

By default subsequence of empty string will be included

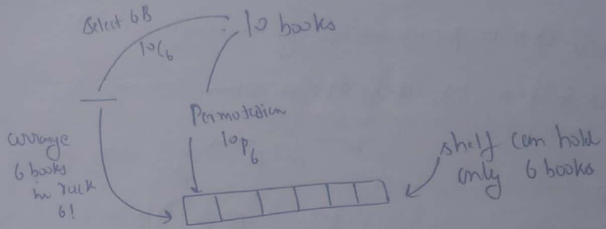
Home Work

P-11, 4, 5, 6,

P-12, 12, 14,

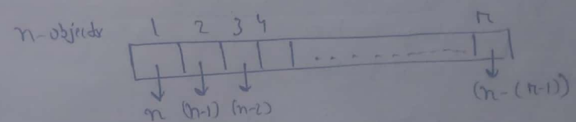
P-13, 27, 28, 35

Permutations and Combinations



Permutations without repetation

^nP_r = r - permutations of n-objects without repetition



$6.5.4 = \frac{6.5.4.3!}{3!} = \frac{6!}{3!}$ ← Concept

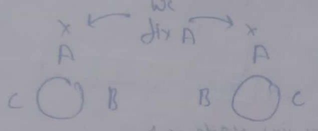
$= \frac{n(n-1).(n-2)(n-3)\ldots n-(r-1)(n-r)!}{(n-r)!}$

$$\boxed{^nP_r = \frac{n!}{(n-r)!}}$$

2) ${}^{n}P_{n}$ = no of (linear) permutations of n-objects $= n!$

3) no of circular permutations of n-objects $= (n-1)!$

Concept :- (we fixed one obj and arrange other remaining obj linearly)



All these are same in circular arrangement
ABC
BCA
CAB

of n objects will cover 1 C.P

$n \text{ L.P} \longrightarrow 1 \text{ C.P}$

$n! \text{ LP} \longrightarrow \frac{n!}{n} \text{ CP} = (n-1)! \text{ CP}$

Q How many ways 10 people can be arranged so that

1) certain pair always together
2) certain pair never together

Sol:- 1) 10 people $\xrightarrow{\text{in line}}$ $10!$

$\boxed{AB}$

1 unit + 8 units = 9 units

9 units $\xrightarrow{\text{in line}}$ $9!$

for each of these arrangement

$AB \xrightarrow{\text{arrange}} 2!$

∴ total arrangements $= 9! \times 2!$

2) pair never together = total − (pair always together)

$= 10! - (9! \times 2!)$

$= 9!(10-2)$

$= 9!(8)$

Q) How many ways 10 people can be arranged in a circle so that

1) certain pair always together
2) certain pair never together

Sol: 10 $\xrightarrow{\text{in circle}}$ $(10-1)! = 9!$

1) $\boxed{AB}$ + 8 unit = 9 unit

9 $\xrightarrow{\text{in circle}}$ $(9-1)! = 8! \times 2!$

↑ AB can arrange themselves in 2 ways.

2) $\cancel{\boxed{AB}}$ $= 9! - 8! \times 2!$

$= 8!(7)$

Q How many ways 5 men & 5 women can be arranged such that

1) 5 M together
2) No two men together
3) men & women arranged alternatively

Sol:- 1) $\boxed{5M}$ + 5 W ⇒ 6 unit ← 5 M can arrange themselves in 5! ways

6 $\xrightarrow{\text{in line}}$ $6! \times 5!$

2) _ M _ M _ M _ M _ M _

$= 5! \times 6!$

×

5) MW MW MW MW MW

---

2) $5W \xrightarrow{\text{in line}} 5!$
(no restriction or condition is imposed on women)

take one such arrangement

× W_1 × W_2 × W_3 × W_4 × W_5 ×   ← no of gaps

5 men in 6 gaps $= {}^6P_5$

No 2 men together $= 5! \times {}^6P_5$

Here for n objects there will be n+1 gaps

3) $5M \xrightarrow{\text{in line}} 5!$

take one such arrangement → 5!

_ M_1 _ M_2 _ M_3 _ M_4 _ M_5 _ → 5!

∴ alternate → [5! + 5!]

NOTE:- alternate arrangement possible i) same no ii) one is just 1 more than other party

total alternate arrangements = 5! [5!+5!]

Ex. how many ways 5M & 5W can be arranged in circle so that
(1) 5 M together
(2) No 2M together
(3) M & W arranged alternatively

Sol:- 1) $\boxed{5M} + 5W \rightarrow 6$ units

6 units $\xrightarrow{\text{in circle}} (6-1)! = 5!$

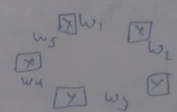
for each of these arrangement

$5M \xrightarrow{\text{arrange}} 5!$

$\boxed{5M} = 5! \times 5!$

2) $5W \xrightarrow{\text{in circle}} (5-1)! = 4!$
(no restriction)

one such arrangement

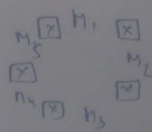


5 M in 5 gaps ⟶ 5!

Here for n objects, no of gaps is also n

No 2M together $= 4! \times 5!$

3) $5M \xrightarrow{\text{in circle}} 4!$



$5W \xrightarrow{\text{women}} 5!$

∴ total alternate $= 4! \times 5!$

P-11 Q 9 & Q 10

Sol:-9 $7G \longrightarrow 6!$

1 unit + 5 units $\longrightarrow 5! \times 2!$

separated = total - together

$= 6! - (5! \times 2!)$

Sol:-10 8 B & 6 G

$8B \xrightarrow{\text{in circle}} 7!$

take one such arrangement

$B_1 \longrightarrow 8$ gaps

$B_8 \quad B_2$

arrange 6 girls in 8 gaps $= {}^8P_6$

No two girls together $= 7! \times {}^8P_6$

Combinations with repetition

${}^nC_r$ = r - combination of n objects without repetition

${}^nP_r = {}^nC_r \times r!$

↑ select r objects

↖ arrange r-objects in line

$$\boxed{{}^nC_r = \frac{n!}{r!(n-r)!}}$$

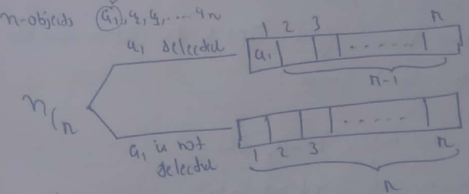
4) ${}^nC_0 = {}^nC_n = 1$

5) ${}^nC_r = {}^nC_{n-r}$ (ways of selecting r objects = ways of ~~rejecting~~ selecting n-r objects for rejection)

* 6) Recurrence relation

$${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$$

n-objects $a_1, a_2, a_3, \dots a_n$



${}^{n-1}C_{r-1}$

${}^{n-1}C_r$

Every item obj has 2 possibility can be selected or can not be selected

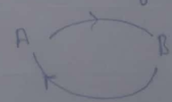
$\cancel{a_1}, \underbrace{a_2, a_3, \dots a_n}_{n-1}$

Q 10 friends meet in a party and shake hands with one another. no of handshakes $= {}^{10}C_2 = 45$

Sol:- no of handshakes = no of 2 combinations of 10 people ie ${}^{10}C_2$

Q 10 friends meet in a party and kick each other. no of kicks = ?

Sol:- no of kicks = no of 2 permutations of 10 people $= {}^{10}P_2$



Q no of ways 5 members can be selected from 10 people.

Sol:- $^{10}C_5$

Q no of ways 3 men & 2 women can be selected from 6 men & 7 women

Sol: $^6C_3 \times {}^7C_2$

Q 5 Members committee selected from 7M & 8W. 1) ~~number~~ (How many) no of such committee can be formed ?.

2) no of such committee contain exactly 3W.

3) no of such committee contain atleast 2 W.

Sol: 1) $^{15}C_5$

2)

7 M	8 W	
5	0	$^7C_5 \times {}^8C_0 +$
4	1	$^7C_4 \times {}^8C_1 +$
3	2	$^7C_3 \times {}^8C_2 +$
2	3	$^7C_2 \times {}^8C_3 +$
1	4	$^7C_1 \times {}^8C_4 +$
0	5	$^7C_0 \times {}^8C_5 +$
		$^{15}C_5$

1) Ans:- $^{15}C_5$ ← 5 members, from 7M & 8W selected

2) $^7C_2 \times {}^8C_3$ <- exactly 3 W

3) $^7C_3 \times {}^8C_2 + {}^7C_2 \times {}^8C_3 + {}^7C_1 \times {}^8C_4 + {}^7C_0 \times {}^8C_5$

$\Downarrow$

Atleast 2 = Total − (atmost 1)

$= {}^{15}C_5 - \left[{}^7C_5 \times {}^8C_0 + {}^7C_4 \times {}^8C_1\right]$

$\boxed{\text{at least } k = \text{total} - (\text{atmost } (k-1))}$

P-11, Q2, Q7 (WB)

Sol:-2

14 players

11 selection

5B	9NB
5	6
~~6~~4	7
3	8
2	9

at least 4B = $^5C_4 \times {}^9C_7 + {}^5C_5 \times {}^9C_6 = 264$

Sol:7 7 selection

	6 G_1	6 G_2
I	6	1
	5	2
	4	3
	3	4
	2	5
II	1	6

total − [I + II]

$^{12}C_7 - \left({}^6C_6 \times {}^6C_1 + {}^6C_1 \times {}^6C_6\right)$

$= 780$

Q1 No of lines ~~segments~~ that can be drawn using n distinct points in a plane

Sol:- $^{n}C_2$

Q no of lines ~~segments~~ can be drawn using 'n' distinct points in a plane in which "m" are collinear

Sol:- $^{n}C_2 - {}^{m}C_2 + 1$ $\qquad$ $^{5}C_2 - {}^{3}C_2$

Q no of triangles that can be formed using n distinct points in a plane.

Sol: $^{n}C_3$

Q no of triangles that can be formed using n distinct points in a plane in which "m" are collinear

Sol:- $^{n}C_3 - {}^{m}C_3$

Handout

Permutation & Combination (~~3~~ 2|7|18)

Q1
Sol:-

at least 1W = total − (no W)

$= {}^{19}C_8 - ({}^{9}C_0 \times {}^{10}C_8)$

Q2
Sol:- $a_1 a_2 \dots a_8$ (ascending order) $a_i < a_{i+1}$

$A \rightarrow a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7\ a_8$

$B \rightarrow 5$ element

$C \rightarrow 3$ element

$\underset{^{8}C_5}{\overset{B}{\downarrow}}\ \underset{^{3}C_3}{\overset{C}{\downarrow}} = {}^{8}C_5 = 56$

Q3
Sol:- 21 - Consonants
5 - vowels

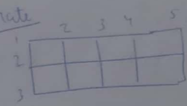
_ _ _ _ _ _ _ _ $\qquad$ 3 - V, 5 - C $\qquad$ 8 letter word

a) $\underset{5C}{\underset{\downarrow}{^{21}C_5}} \times \underset{3V}{\underset{\downarrow}{^{5}C_3}} \times \underset{\text{8 letters arranged}}{\underset{\downarrow}{8!}}$

b) ~~$^{5}C_3 \times$~~ $\underset{1}{\underset{\downarrow}{a}} \cdot \underset{^{4}C_2}{\underset{\downarrow}{2V}} \cdot \underset{^{21}C_5}{\underset{\downarrow}{5C}} \cdot 8!$

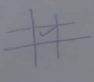
c) $\boxed{a}\underbrace{\boxed{\qquad\qquad}}_{7}$ $\qquad \underset{1}{\underset{\downarrow}{a}}$ $\quad {}^{4}C_2 \cdot {}^{21}C_5 \cdot 7!$

Grate



No of rectangles can be observed?

$^{3}C_2 \times {}^{5}C_2 = 3 \times 10 = 30$



Result: No of rectangle in m×n grid

$=$ ~~$^{m}C_2 \times {}^{n}C_2$~~ $\boxed{^{m+1}C_2 \times {}^{n+1}C_2}$

2C of HL
followed by 2C of VL

no of squares = ?
in 2×4 grid

no of 1 square = 2×4

no of 2 square = 1×3

Ex. no of squares in 4×5 grid ?

$4\times5 + 3\times4 + 2\times3 + 1\times2$

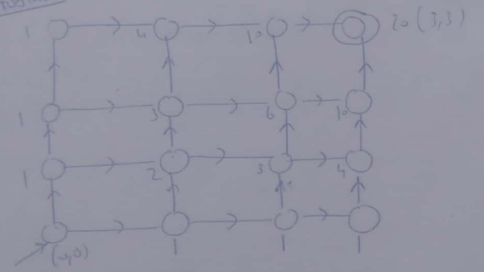
↑ 1 square ↑ 2 square ↑ 3 square ↑ 4 square

Result no of squares in m×n grid

$$\boxed{\sum_{r=1}^{\min(m,n)} (m-r+1)(n-r+1)}$$

<u>Grate</u>

<u>Increasing Path Problem</u>



Total moves = 6

H. moves = (3-0)

V moves = (3-0)

No of paths $= {}^{6}C_{3}$ ← select either H moves or V moves

$\left(\frac{(H+V)!}{H!\,V!}\right)$

no of paths from (x_1, y_1) to (x_2, y_2) in Cartesian plane such that each step we can ~~set~~ move one unit up (or) one unit right only

$${}^{(x_2-x_1)+(y_2-y_1)}C_{(x_2-x_1)}$$

Ex. No of increasing paths from

i) (0,0) to (8,8) → ${}^{8+8}C_8 = {}^{16}C_8$

ii) (0,0) to (10,10) → ${}^{20}C_{10}$

iii) (5,4) to (10,10) → ${}^{11}C_5$

$${}^{20}C_{10} = \binom{20}{10} = (20,10)$$

<u>Handout</u>

5) Paths not using (4,4) to (5,4) = total paths − paths using (4,4) to (5,4)

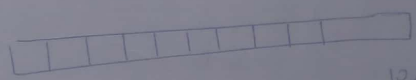
↓

$\left(\text{Paths from } (0,0) \text{ to } (4,4)\right)\left(\text{" } (4,4) \text{ to } (5,4)\right)\left(\text{" } (5,4) \text{ to } (10,10)\right)$

$$= \binom{20}{10} - \binom{8}{4}\binom{11}{5}$$

Q6

Sol: $2^4 \left[\ \underset{1}{\_}\ \underset{1}{\_}\ \underset{1}{\_}\ \underset{1}{\_}\ \underset{1}{\_}\ \underset{1}{\_}\ \underset{2}{\_}\ \underset{2}{\_}\ \underset{2}{\_}\ \underset{2}{\_} = 2^4 = 16 \times \frac{10!}{6!}\right] \times$



$= {}^{10}C_6 \times 1 \times 1$

${}^{10}C_6$ · 1 · 1

↑ select 6 position for 1's

↑ 6 1's in 6 position

↑ 4 0's in 4 position

We can arrange six 1's in 1 ways and we can arrange four 0's in 1 way

Permutation with repetition

r-permutations of n-object with repetition

$= n^r$

Handout

1) a) 4 (2^2)
   b) 2^{10}

2) a) 6^2
   b) 6^{10}

3) 2^n

4) ~~23~~ 3^{25}

Set Theory

Set → It is an unordered collection of well defined objects

$A = \{1, 2, 3\}$ $\quad 1 \in A$

$= \{3, 2, 1\}$ $\quad 5 \notin A$

Subset - $A \subseteq B$ means if $x \in A$ then $x \in B$

NOTE: if $A \subseteq B$ then $B \supseteq A$ ← superset

Equal

$A = B$ means $A \subseteq B$ and $B \subseteq A$

Proper subset

$A \subset B$ means $A \subseteq B$ but $A \neq B$

Empty set (ϕ) : A set with no elements

Ex:- $\phi = \{\ \}$

Universal set : Set with all elements under consideration

Result

1) $\phi \subseteq A$
2) $A \subseteq U$
3) $A \subseteq A$

Cardinality

$|A|$ = no of elements in set A.

Ex $A = \{1, 2, 3\}$

$|A| = 3$

Ex:- I) $\phi \in A$ — F

II) $\phi \subseteq A \rightarrow T$

1) Write all subsets of A

$A = \{1, 2\}$

subset = ϕ, $\{1\}$, $\{2\}$, $\{1, 2\}$

there is one to one correspondence

$|A| = 2$

subset		bs of length 2
ϕ	———	00
$\{1\}$	———	01
$\{1\}$	———	10
$\{1,2\}$	———	11

(ii) $A = \{1,2,3\}$

$|A| = 3$

subset	bs of length 3
ϕ	000
$\{3\}$	001
$\{2\}$	010
$\{2,3\}$	011
$\{1\}$	100
$\{1,3\}$	101
$\{1,2\}$	110
$\{1,2,3\}$	111

Power Set : $P(A)$ Set of all subsets of A

$A = \{1,2\}$; $P(A) = \{\phi, \{1\}, \{2\}, \{1,2\}\}$

$A = \{1,2,3\}$; $P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$

Results

1) $(x+y)^n = {}^nC_0 y^n + {}^nC_1 x y^{n-1} + {}^nC_2 x^2 y^{n-2} + \dots + {}^nC_n x^n$

$= \sum_{r=0}^{n} {}^nC_r x^r y^{n-r}$

2) ${}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \dots + {}^nC_n = 2^n$ put $x=1$ $y=1$

3) ${}^nC_0 + {}^nC_2 + {}^nC_4 + {}^nC_6 + \dots = 2^{n-1}$

4) ${}^nC_1 + {}^nC_3 + {}^nC_5 + {}^nC_7 + \dots = 2^{n-1}$

Results (Imp)

1) $|A| = n$

No of subsets of A = No of bs of length $n = 2^n$

2) $|A| = n$; $|P(A)| = 2^n$

3) $|A| = n$

no of subsets of A containing

(1) 0 element = 1   ie $\{\}$ ${}^nC_0$ ← select 0 element

(2) 1 element = ${}^nC_1$

(3) 2 elements = ${}^nC_2$

(4) r elements = ${}^nC_r$

(5) n elements = ${}^nC_n$

4) $|A| = n$

No of subsets A =

${}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \dots + {}^nC_n = 2^n$

Selecting no. of subset with 0 element

(5) $|A| = n$
no of subsets of A having odd cardinality $= 2^{n-1}$
$^nC_1 + {^nC_3} + {^nC_5} + {^nC_7} + \dots\dots$

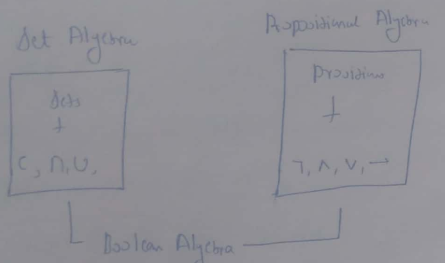
(6) $|A| = n$
no of subsets of A having even cardinality $= 2^{n-1}$
$^nC_0 + {^nC_2} + {^nC_4} + {^nC_6} + \dots\dots$

(7) $|A| = n$ $\quad A = \{1, 2, 3, 4, 6, 7, \dots\dots n\}$

i) No of subsets of A with 3 elements $= {^nC_3}$

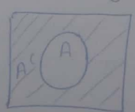
ii) No of subsets of A with 3 elements containing "1" $= {^{n-1}C_2}$ $\quad = \{\underline{1}\ \underbrace{\_\ \_}_{\text{select 2 from remaining } n-1}\}$

iii) No of subsets of A with 3 elements containing "i" $= {^{n-1}C_2}$ $\quad \{\underline{i}\ \underbrace{\_\ \_}_{2}\}$

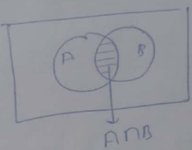


Set Operations

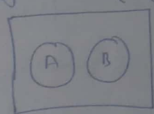
1) $A^c = \{x \mid x \notin A \text{ and } x \in U\}$



2) $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

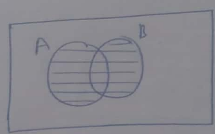


If $A \cap B = \phi$
then A & B are disjoint sets (Mutually exclusive)



NOTE:
$x \in A_1 \cap A_2 \cap \dots\dots \cap A_n$ means
$x \in A_i$ for all i

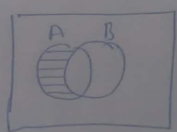
3) $A \cup B = \{x \mid x \in A \text{ or } x \in B \text{ or } x \in A \cap B\}$



$A \cup B$

NOTE: $x \in A_1 \cup A_2 \cup A_3 \cup A_4 \dots\dots\dots A_n$ means
$x \in A_i$ for ~~atti~~ at least one i

4) $A - B = \{x \mid x \in A \text{ but } x \notin B\}$



Result : $\boxed{A - B = A \cap B^c}$

Set Identities

1) Idempotent

$A \cup A = A$

$A \cap A = A$

2) Identity

$A \cup \phi = A$

$A \cap U = A$

3) Domination

$A \cup U = U$

$A \cap \phi = \phi$

4) Complementation

$A \cup A^c = U$

$A \cap A^c = \phi$

5) Double Complementation

$(A^c)^c = A$

6) Commutative

$A \cup B = B \cup A$

$A \cap B = B \cap A$

7) Associative

$A \cup (B \cup C) = (A \cup B) \cup C$

$A \cap (B \cap C) = (A \cap B) \cap C$

8) Absorption

$A \cup (A \cap B) = A$

$A \cap (A \cup B) = A$

9) Demorgan's law

$(A \cup B)^c = A^c \cap B^c$

$(A \cap B)^c = A^c \cup B^c$

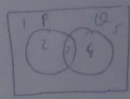
10) Distributive

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Ex:

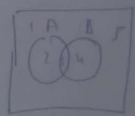
i) $(P \cap Q) \cup P^c \cup Q^c$



$3 \cup \{1,4,5\} \cup \{1,2,5\}$

$\{1,2,3,4,5\} = U$

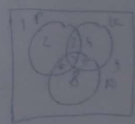
ii) $A \cup (B \cap B^c)$



$\{2,3\} \cup \{\phi\}$

$\{2,3\} = A$

iii) $(P \cup Q) \cap (P \cup R)$



$\{2,3,\overset{4}{6},7,5\} \cap \{2,3,6,7,8,5\} = \cancel{\{6,7\}}$
$\{2,3,6,7,5\}$

$= \cancel{P \cap R}\ P \cup (Q \cap R)$

$\cancel{\{2,3,4,5,6,7\} \cap \{2,3,6,7,8\}}$

$= \cancel{\{2,3,6,7\}}$

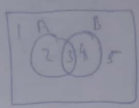
$= \cancel{P}$

$3,4 \cap \{1,3,\ldots$

4) $A^c \cup (B \cap A^c)$

$\{1,4,5\} \cup \{4\}$ $\quad A^c \cup (A^c \cap B)$

$= \cancel{B-A}\ A^c$



5) $(P^c \cup Q \cup R) \cap (P \cup Q \cup R) \cap Q^c \cap R^c$

$= [(P^c \cap P) \cup Q \cup R] \cap Q^c \cap R^c$

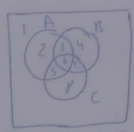
$= [\phi \cup Q \cup R] \cap (Q \cup R)^c$

$= (Q \cup R) \cap (Q \cup R)^c$

$= \phi$

Result

(1) $\boxed{A - (B \cup C) = (A-B) \cap (A-C)}$



$\{2,3,6,5\}$
$\{6,7\}$
$\{1,3,5\}$

a) $(A-B) \cup (A-C)$

b) $(A-B) \cup (A-C)$

c) $(A-B) \cap (A-C)$

d) $(A-B)$

e) $(A-C)$

$A-(B \cup C)$

$A \cap (B \cup C)^c$

$A \cap (B^c \cap C^c)$

$A \cap A \cap (B^c \cap C^c)$

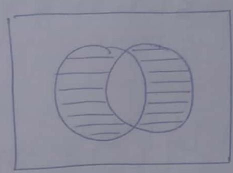
$\rightarrow A \cap B^c \cap A \cap C^c$

$= (A-B) \cap (A-C)$

2) $\boxed{A - (B \cap C) = (A-B) \cup (A-C)}$

Symmetric Difference $\quad A \Delta B$ (or) $A \oplus B$

$A \Delta B = \{x \,/\, x \in A$ or $x \in B$ but $x \notin A \cap B\}$



$A \Delta B$

$(A \Delta B) \Delta C = A \Delta (B \Delta C)$

If $A \Delta C = B \Delta C$
then $A = B$
(cancellation law)

Results:

$A \Delta B = (A-B) \cup (B-A)$

$A \Delta B = (A \cup B) - (A \cap B)$

$A \Delta A = \emptyset$

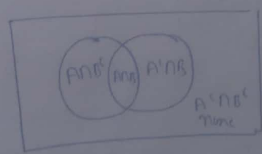
$A \Delta A^c = U$

$A \Delta U = A^c$

$A \Delta \phi = A$

Inclusion - Exclusion Principle

1) $|A \cup B| = |A| + |B| - |A \cap B|$



2) $|A^c \cap B^c| = |U| - |A \cup B|$

3) $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$

4) $|A^c \cap B^c \cap C^c| = |U| - |A \cup B \cup C|$

Q How many +ve integers ≤ 100 are divisible 4 or 6

Sol: ~~no(4) = $\lfloor \frac{100}{4} \rfloor$~~

Divisible 4 $|A| = \lfloor \frac{100}{4} \rfloor = 25$

Div by 6 $|B| = \lfloor \frac{100}{6} \rfloor = 16$

Div by 4 & 6 $|A \cap B| = \lfloor \frac{100}{12} \rfloor = 8$

Div by 4 or 6 $|A \cup B| = |A| + |B| - |A \cap B|$
$= 25 + 16 - 8$
$= 33$

Q No of +ve integers ≤ 100 are not divisible by 4 or 6

Sol: Not div = total - divisible $\qquad |U| = 100$

$(A \cup B)^c = |U| - |A \cup B|$

$A^c \cap B^c = |U| - |A \cup B|$
$= 100 - 33$
$= 67$

Handout
Q8
Sol: $X = \{ n \mid 1 \leq n \leq 123 ;\ n$ not div by either 2, 3 or 5 $\}$

$|U| = 123$

not div = total - div

$|(A \cup B \cup C)^c| = |U| - |A \cup B \cup C|$

$A^c \cup B^c \cup C^c = |U| - |A \cup B \cup C|$

$|A \cup B \cup C| = \lfloor \frac{123}{2} \rfloor + \lfloor \frac{123}{3} \rfloor + \lfloor \frac{123}{5} \rfloor - \lfloor \frac{123}{6} \rfloor - \lfloor \frac{123}{15} \rfloor - \lfloor \frac{123}{10} \rfloor + \lfloor \frac{123}{30} \rfloor$

$= 61 + 41 + 24 - 20 - 8 - 12 + 4$

$= 90$

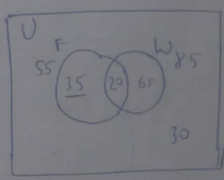
$A^c \cup B^c \cup C^c = 123 - 90 = 33$

HO
Q12
Sol: $|U| = 150$

$|F| = 55$

$|W| = 85$

$|F^c \cap W^c| = 30$



$|F \cup W| = |U| - |F^c \cap W^c|$
$= 150 - 30$
$= 120$

$|F \cap W| = |F| + |W| - |F \cup W|$
$= 55 + 85 - 120$
$= 20$

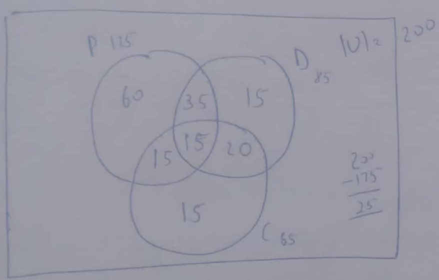
Q 13

Sol:- $|U| = 200$ $|C| = 65$ $|P \cap C| = 30$

$|P| = 125$ $|P \cap DS| = 50$ $|P \cap DS \cap C| = 15$

$|DS| = 85$ $|DS \cap C| = 35$

$|P^c \cap DS^c \cap C^c| = |U| - |P \cup DS \cup C|$



1) Only DS = 15

" P = 60

" C = 15

Exactly one = 60 + 15 + 15 = 90

Only P & DS = 35

Exactly two = 35 + 15 + 20 = 70

At least one = $|P \cup D \cup C| = 175$

At least two = 15 + 20 + 35 + 15
= 85

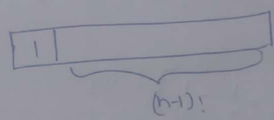
9) At most one = 60 + 15 + 15 + 25
= 115

10) At most two = total − All ~~three~~ three
= 200 − 15 = 185

11) none = 25

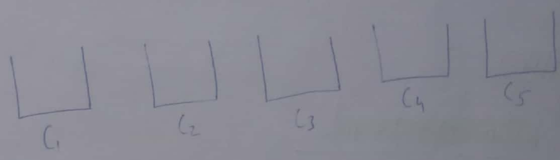
Q 14

Sol:-



Q 15

Sol:-



Derangements

The arrangements of "n" objects ~~must~~ in such a way that none of the objects occupy its natural position.

No of Derangements of n objects

$$D_n = n! \left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} \cdots \frac{(-1)^n}{n!} \right]$$

$D_1 = 0$

$D_2 = 2! \left[\frac{1}{2!}\right] = 1$

$D_3 = 3! \left[\frac{1}{2!} - \frac{1}{3!}\right]$

$= 3! \left(\frac{3-1}{6}\right) = 2$

$D_4 = 4! \left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right] = 4! \left[\frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right]$

Sol:- 15 > $D_5 = 5! \left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right] = 44$

3/7/18

Euler ϕ - function

$\phi(n) =$ no of positive integers $< n$ which are relatively prime to n

GCD $(a, b) = 1$

$\therefore$ a & b are relatively prime

$\phi(8) = 1, 3, 5, 7$

Formula

$$\phi(n) = n\left(1-\frac{1}{P_1}\right)\left(1-\frac{1}{P_2}\right)\left(1-\frac{1}{P_2}\right)\left(1-\frac{1}{P_3}\right) \cdots \left(1-\frac{1}{P_K}\right)$$

Where $P_1, P_2, P_3 \ldots P_K$ are distinct prime factors of n

Ex:- $\phi(24) =$

$24 = 2^3 \cdot 3$

$\phi(24) = 24\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right) = 24\left(\frac{1}{2}\right)\left(\frac{2}{3}\right) = 8$

Ex:- $\phi(7) = 6$

Result

* If P is prime $\phi(P) = P - 1$

Ex $\phi(91) =$

$91 = 13 \times 7$

$= 91\left(1-\frac{1}{13}\right)\left(1-\frac{1}{7}\right)$

$= \overset{7}{\cancel{91}}\left(\frac{12}{\cancel{13}}\right)\left(\frac{6}{\cancel{7}}\right)$

$= 72$

Fermat's Little Theorem

if P is prime no and a is integer not divisible by p

GCD$(a, p) = 1$ ie $(a$ & $P)$ are relatively prime

then $a^{P-1} \bmod p = 1$

Ex:- 7 is prime & 2 is not divisible by 7

Fermats $\rightarrow$ $2^{7-1} \bmod 7 = 1$

$64 \bmod 7 = 1$

Scanned by CamScanner

Ex: $2^{340} \bmod 7 = 0$

Format

$2^{7-1} \bmod 7 = 1$ ie $2^{6} \bmod 7 = 1$ — (i)

$(2^{6})^{56} \cdot 2^{4} \bmod 7 = 1$

Using (i)

$(1)^{56} \cdot 2^{4} \bmod 7$

$2^{4} \bmod 7 = 2$

Homework

Counting

4) i) $2^{340} \bmod 11 = 1$

Format $2^{11-1} \bmod 11 = 1$

$2^{10} \bmod 11 = 1$

$(2^{10})^{34} \bmod 11$

$(1)^{34} \bmod 11 = 1$

(ii) $3^{302} \bmod 5 = 4$

Format

$3^{5-1} \bmod 5 = 1$

$3^{4} \bmod 5 = 1$

$(3^{4})^{75} \cdot 3^{2} \bmod 5$

$(1)^{75} \cdot 3^{2} \bmod 5 \Rightarrow 9 \bmod 5 = 4$

Sol: 5) $13^{99} \pmod{17} = ?$

Format

$13^{16} \bmod 17 = 1$

$(13^{16})^{6} \cdot 13^{3} \bmod 17 \Rightarrow (1)^{6} \cdot 13^{3} \bmod 17$

$13^{3} \bmod 17 = 4$

Logic

→ Logic provide rules to verify the validity of 2 valued arguments.

Proposition (statement)

→ Declarative sentence which is either completely true or or completely false but not both. ← truth value

Ex. 1) P: bhopal is a city [T]

2) q: 2 + 2 = 5 [F]

r: It is raining

s: This statement is false [T/F F/T]

not a proposition

NOTE: self-declarative sentences are not proposition

Truth value : T or F

Propositional Variable

A variable which takes truth value ie T or F

no of truth combinations = no of lines of the Truth table

p ($\uparrow 2$)	q ($\uparrow 2$)
T	T
T	F
F	T
F	F

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

Result

If there are n propositional variable then no of truth combinations $= 2^n$

$\boxed{p_1 \mid p_2 \mid p_3 \mid \dots \mid p_n}$ → $2 \cdot 2 \cdot 2 \cdots 2 = 2^n$

Propositional Algebra

* Logical operators (Connectives)

five fundamental Connectives

$\neg, \wedge, \vee, \rightarrow, \longleftrightarrow$

1) Negation ($\neg$) or ($\sim$)

Not P

P	$\neg P$
T	F
F	T

Representation

p: Einstein is genius

$\neg p$: Einstein is not genius

2) Conjunction (AND, $\wedge$)

"$p \wedge q$ is true only when both p and q are true"

p	q	$p \wedge q$
T	T	T
T	F	
F	T	
F	F	

Representation

Ex: Jack and Jill went up the hill

p: Jack went up the hill

q: Jill went up the hill

$p \wedge q$

Ex: I like tea but not coffee

p: I like tea

q: I like coffee

$(p \wedge \sim q)$ or $(p \wedge \neg p)$

Truth Table

Truth table is a table containing truth values for all the possible truth combinations.

Ex: No of lines in which following proposition is false.

$p \wedge q \wedge r \wedge s$

a) 1 b) 7 c) 9 d) 15

3) Disjunction (OR, $\vee$)

"$p \vee q$" is false only when both p & q are false"

p	q	$p \vee q$
T	T	
T	F	
F	T	
F	F	F

Representation (OR- inclusive OR)

Ex: I like DM or algo

p: I like DM

q: I like algo

$p \vee q$

4) Implication (Conditional, $\rightarrow$)

"A true statement can not imply false"

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Ex:- If you get gate rank then I will give you a chocolate.

NOTE: $p \longrightarrow q$

Antecedent / Hypothesis / Premise — Consequent / Conclusion / Conclusion

Representation

$p \rightarrow q$

1) if p, q
i) p implies q
ii) if p then q
iii) p only if q
1) q when p
iv) q if p
v) q follows from p
vi) q unless $\neg p$ = q if not $\neg p$
q if p
vii) p is sufficient condition for q.
ix) q whenever p
viii) q is necessary condition for p

NOTE: Differentiability $\Rightarrow$ Continuity

Ex: 1) x ~~if~~ only if y
$x \longrightarrow y$

2) a unless $\neg b$
$b \longrightarrow a$

3) a unless b
$\sim b \longrightarrow a$

4) y if z
$z \rightarrow y$

5) L follows M
$M \rightarrow L$

Q1
Sol. i) $\underbrace{\text{if it rain}}_{p}$ then $\underbrace{\text{I dance}}_{q}$ $p \rightarrow q$

Implication : $p \longrightarrow q$
Converse : $q \longrightarrow p$
Inverse : $\neg p \longrightarrow \neg q$
Contrapositive : $\neg q \longrightarrow \neg p$

Q2 (i) $\underbrace{\text{I stay}}_{p}$ only if $\underbrace{\text{you go}}_{q}$.
~~if you g~~

Implication : $p \rightarrow q$: if I stay then you go
Converse : $q \rightarrow p$ if you go then I stay
Inverse : $\neg p \rightarrow \neg p$ if I don't stay then you don't go
Contrapositive : $\neg q \rightarrow \neg p$ if you dont go then I don't stay

Workbook

Q 42 P-7 a)

I stay only if you go
(p: I stay, q: you go)

$p \rightarrow q$ If I stay then you go

Converse $q \rightarrow p$ If you go then I stay

I stay if you go

Ex: No of times in which following proposition is True

i) $[(p \wedge q \wedge r \wedge s) \rightarrow t]$ a) 1 b) 15 c) 17 d) 31

ii) $(p \vee q \vee r \vee s) \rightarrow t$ a) 1 b) 15 ~~c) 17~~ d) 31

Sol:- i)

p	q	r	s	t	$(p \wedge q \wedge r \wedge s) \rightarrow t$
T	T	T	T	F	F
					31 T

$p \rightarrow q$

$T \rightarrow F = F$

Sol:- ii)

p	q	r	s	t	$(p \vee q \vee r \vee s) \rightarrow t$
				F	F
				F	
F	F	F	F	F	$\rightarrow$ T
				T ⋮ T	16 times T

$\overset{T}{p} \rightarrow \overset{F}{Q}$

F

$\therefore$ 17 times

5) Biimplication (Biconditional, $\longleftrightarrow$)

" $p \longleftrightarrow q$ is true only when both p and q have same ~~the~~ truth value"

p	q	$p \longleftrightarrow q$
T	T	T
T	F	
F	T	
F	F	T

Representation

$p \longleftrightarrow q$

p if and only if q

p iff q

Well Formed Formulae (WFF)

1) P is WFF
2) if P is WFF then $\neg P$ is WFF
3) If P and q are WFF then ~~p and q~~
   then 1) $(p \wedge q)$
   2) $(p \vee q)$
   $(p \rightarrow q)$
   $(p \longleftrightarrow q)$
   are also WFF's
4) Only those formulae obtained using rule 1,2,3 are WFF's

Ex: $((p \wedge q) \vee r)$

1) p is WFF
2) q is WFF
3) $(p \wedge q)$ is WFF
4) r is WFF
5) $((p \wedge q) \vee r)$ is WFF

NOTE: we use precedence of operators to avoid unnecessary parenthesis

Operator Precedence

1) $\neg$ (high)
2) $\wedge$
3) $\vee$
4) $\rightarrow$
5) $\leftrightarrow$ (low)

Ex: ~~$p \wedge q \vee r \Leftrightarrow ((p \wedge q) \vee r)$~~

$((p \wedge q) \vee r) \equiv p \wedge q \vee r$

Ex: $(p \wedge (q \vee r)) \equiv p \wedge (q \vee r)$

Parentheses required

Tautology (T, valid)

A proposition which is always true is tautology.

Ex: $p \vee \neg p \equiv T$

Contradiction (absurd, F)

A proposition which is always false.

Ex: $p \wedge \neg p \equiv F$

Satisfiable

A proposition which is true for at least one truth combination is satisfiable.

Contigency (Satisfiable but not Valid)

A proposition which is neither tautology nor contradiction is called contigency.

NOTE:

Tautology $\longrightarrow$ Satisfiable
Contigency $\longrightarrow$ Satisfiable
} - Converse not true

Ex:

p	q	$\neg p$	$\neg q$	$\neg(p \vee q)$ (X)	$\neg p \wedge \neg q$ (Y)	$X \leftrightarrow Y$ (Tautology)
T	T	F	F	F	F	T
T	F	F	T	F	F	T
F	T	T	F	F	F	T
F	F	T	T	T	T	T

Def: $P \equiv Q$ (P Equivalent Q) iff P and Q have same truth table

Result: $P \equiv Q$ iff $P \leftrightarrow Q$ is T (Tautology)

Dual

The dual of a compound proposition involving only the connectives, negation, $\wedge$, and $\vee$ is obtained by replacing every
i) $\wedge$ with $\vee$
ii) $\vee$ with $\wedge$
iii) T with F
iv) F with T

Ex: $P: p \wedge (q \vee r)$

$P^d: p \vee (q \wedge r)$

Ex: $P: \neg p \wedge (\neg q \vee r)$

$P^d: \neg p \vee (\neg q \wedge r)$

Ex: $P(p, q) : p \vee q$

$\neg P(p, q) : \neg (p \vee q)$

$P^d(\neg p, \neg q) : \neg p \vee \neg q$

$P^d(\neg p, \neg q) : \neg p \wedge \neg q$

equivalent

Results

1) $(P^d)^d = P$

*2) $P \equiv Q$ iff $P^d \equiv Q^d$

3) $\neg P(p_1, p_2, p_3 \ldots p_n) \equiv P^d(\neg p_1, \neg p_2, \ldots \neg p_n)$

(Demorgan's law in terms of duals)

4/7/17

Equivalences - I

1) Idempotent

$p \vee p \equiv p$

$p \wedge p \equiv p$

2) Identity

$p \vee F = p$

$p \wedge T = p$

3) Domination

$p \vee T \equiv T$

$p \wedge F \equiv F$

4) Negation

$p \vee \neg p \equiv T$

$p \wedge \neg p \equiv F$

5) Double Negation

$\neg(\neg p) \equiv p$

9) Demorgans

$\neg(p \vee q) = \neg p \wedge \neg q$

$\neg(p \wedge q) = \neg p \vee \neg q$

6) Commutative

$p \vee q \equiv q \vee p$

$p \wedge q \equiv q \wedge p$

7) Associative

$p \vee (q \vee r) \equiv (p \vee q) \vee r$

$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

8) Absorption

$p \vee (p \wedge q) \equiv p$

$p \wedge (p \vee q) \equiv p$

10) Distributive

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

Ex: $(p \vee q \vee r) \vee \neg(p \vee q \vee r) \rightarrow T$

2) $p \vee (p \wedge r) \equiv \cancel{(p \vee p)}$ ~~[illegible]~~

3) $p \vee (q \wedge \neg q) \equiv p$

4) $(p \vee q) \wedge (p \vee r) \equiv p \vee (q \wedge r)$

5) $(p \vee q) \vee (\neg p \vee r) \equiv \underbrace{(p \vee \neg p)}_{T} \vee (q \vee r) \equiv T$

6) $(\neg p \vee q \vee r) \vee (p \vee q \vee r) \vee \neg q \vee \neg r \equiv T$

Equivalences – II (involving implications)

These are the equivalences involving implications

1) Law of implication

$p \to q \equiv \sim p \vee q$

p	q	$p \to q$	$\sim p \vee q$
T	F	F	F

Results

1) $p \to q \equiv \sim p \vee q$
2) $\sim p \to q \equiv \cancel{p \vee \sim q}$ $p \vee q$
3) $p \to \sim q \equiv \sim p \vee \sim q$
4) $\sim p \to \sim q \equiv p \vee \sim q$
5) $p \vee q \equiv \sim p \to q$
6) $\sim p \vee q \equiv p \to q$
7) $p \vee \sim q \equiv \sim p \to \sim q$
8) $\sim p \vee \sim q \equiv p \to \sim q$

Ex $p \wedge q \equiv$

a) $\sim (p \to q)$ b) $\sim (\sim p \to q)$ c) $\sim (p \to \sim q)$ d) $\sim (\sim p \to \sim q)$

$\sim (\sim p \vee \sim q)$

$p \wedge q$

Results

1) $p \to q \equiv \sim p \vee q$
2) $\sim p \to q \equiv p \vee q$
3) $p \to \sim q \equiv \sim p \vee \sim q$
4) $\sim p \to \sim q \equiv p \vee \sim q$

5) $p \wedge q \equiv \sim (p \to \sim q)$
6) $p \wedge \sim q \equiv \sim (p \to q)$
7) $\sim p \wedge q \equiv \sim (\sim p \to \sim q)$
8) $\sim p \wedge \sim q \equiv \sim (\sim p \to q)$

5) $p \wedge q \equiv \sim (p \to \sim q)$
6) $p \wedge \sim q \equiv \sim (p \to q)$
7) $\sim p \wedge q \equiv \sim (\sim p \to \sim q)$
8) $\sim p \wedge \sim q \equiv \sim (\sim p \to q)$

2) Law of Contrapositive

$p \to q \equiv \sim q \to \sim p$

p	q	$p \to q$	$\sim q \to \sim p$
T	F	F	F

Results

1) $p \to q \equiv \sim q \to \sim p$ (implication $\equiv$ contrapositive)
2) $\sim p \to q \equiv \sim q \to p$
3) $p \to \sim q \equiv q \to \sim p$
4) $\sim p \to \sim q \equiv q \to p$ (Inverse $\equiv$ converse)

3) Exportation Law

$p \to (q \to r) \equiv (p \wedge q) \to r$

LHS: $p \to (q \to r)$

$\to p \to (\sim q \vee r)$ implication

$\to \sim p \vee (\sim q \vee r)$ implication

$\to (\sim p \vee \sim q) \vee r$ associativity

$\to \sim (p \wedge q) \vee r$ demorgan

$\to (p \wedge q) \to r$ implication

4) $(p \to q) \vee (p \to r) \equiv$

$(p \to q) \wedge (p \to r) \equiv$

i) $(p \to q) \vee (p \to r) \equiv \underline{p \to (q \vee r)}$

$(p \to q) \vee (\sim p \vee r)$

$(\underline{\sim p} \vee q) \vee (\underline{\sim p} \vee r)$

$\sim p \vee (q \vee r)$

$\sim p \vee (q \vee r)$

$p \longrightarrow (q \vee r)$

ii) $(p \to q) \wedge (p \to r) \equiv \underline{p \to (q \wedge r)}$

$\Rightarrow (\sim p \vee q) \wedge (\sim p \vee r)$

$$\left[\begin{array}{l} \cancel{(\sim p \vee (\sim p \vee q))} \\ (\sim p \vee q) \wedge \sim p \;\vee\; (\sim p \vee q) \wedge r \\ \sim p \vee (\sim p \vee q) \;\vee\; (\sim' \end{array}\right] \times$$

$\Rightarrow \sim p \vee (q \wedge r)$

$\Rightarrow p \longrightarrow (q \wedge r)$

5)

(1) $(p \longrightarrow r) \vee (q \to r) \equiv \underline{(p \wedge q) \longrightarrow r}$

(2) $(p \longrightarrow r) \wedge (q \to r) \equiv \underline{(p \vee q) \longrightarrow r}$

1) $(\sim p \vee r) \vee (\sim q \vee r)$

$(\sim p \vee \sim q) \vee r$

$\sim (p \wedge q) \vee r$

$(p \wedge q) \longrightarrow r$

2) $(p \to r) \wedge (q \to r)$

$(\sim p \vee r) \wedge (\sim q \vee r)$

$(\sim p \wedge \sim q) \vee r$

$\sim (p \vee q) \vee r$

$(p \vee q) \longrightarrow r$

Logic Handout

Q 20

Sol P1: $((A \wedge B) \to C)) \equiv ((A \to C) \wedge (B \to C))$

$$\left[\begin{array}{ll} \text{LHS:- } \sim (A \wedge B) \vee C & \text{RHS: } (\sim A \vee C) \wedge (\sim B \vee C) \\ \quad \sim A \vee \sim B \vee C & \end{array}\right] \times$$

verify from 5th Property

Equivalances - III (Involving Biimplication)

1) $p \longleftrightarrow q \equiv (p \to q) \wedge (q \to p)$

2) $p \longleftrightarrow q \equiv (p \wedge q) \vee (\sim p \wedge \sim q)$

3) $p \longleftrightarrow q \equiv \underbrace{q \longleftrightarrow p}_{\text{Converse}} \equiv \underbrace{\sim p \longleftrightarrow \sim q}_{\text{inverse}} \equiv \underbrace{\sim q \longleftrightarrow \sim p}_{\text{Contrapositive}}$

Logical Implications

An implication which is Tautology (T)

Ex: $p \longrightarrow (p \vee q)$

p	q	$p \vee q$	$p \to (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

Remember

1)

P	Q	P → Q
T	F	F

2)

P	Q	P → Q
F		T

3)

P	Q	P → Q
	T	T

Ex. $(p \to q) \to r$ ← this is not tautology

p	q	r	$(p \to q) \to r$
T	T	T	T
T	F	T	T
F	T	T	T
F	F		
T	F	F	T

Method (to check implication tautology or not } logical implication or not)

I

Try to make p T, Q → Fix F

If (possible) T → F = F
not T
else
T

II

Fix T
p → Q
try to make it F

If (possible) T → F = F
Not T
else: T

Ex. $p \to (p \vee q)$

Method-I

Try to make T, Fix F

p	q	$p \to (p \vee q)$
		[Not possible] F F
F	F	∴ Tautology

1) $p \to (p \vee q)$

Fix T, Try F

p	q	$p \to (p \vee q)$
		T (Not possible)
T		∴ Tautology

2) $[(p \to q) \wedge p] \to q$

Make T, make F, Fix F

p	q	$[(p \to q) \wedge p] \to q$
		F T ✗
F	F	∴ Tautology

3) $[(p \to q) \wedge (q \to r)] \to (r \to p)$

make T, Fix F

p	q	r	$[(p \to q) \wedge (q \to r)] \to (r \to p)$
			T✓ T✓ T F
F		T	∴ T → F = F not T

p	q	r	$[(p \to q) \wedge (q \to r)] \to (r \to p)$
			T T T
T		T	T

When p = T, irrespective of r & q, above logical implication will become True.
Hence Satisfiable but not Valid

Q9 Handout (Logic)

1) $[(p \to q) \wedge (r \to s) \wedge (p \vee r)] \to (q \vee s)$

Fix F

p	q	r	s	$[(p \to q) \wedge (r \to s) \wedge (p \vee r)] \to (q \vee s)$
	F		F	~~Not~~ Tautology

2) $[(p \to q) \wedge p] \to p$

Fix F

p	q	$[(p \to q) \wedge p] \to p$
F.		∴ ~~not~~ tautology

3) $[(p \to q) \wedge (q \to r) \wedge p] \to r$

Fix F

p	q	r	$[(p \to q) \wedge (q \to r) \wedge p] \to r$
		F	∴ tautology

Q11
Sol

Fix F

p	q	r	$[p \to (q \vee r)] \to [(p \wedge q) \to r]$
T	T	F	∴ not tautology

p	q	r	$[p \to (q \vee r)] \to [(p \wedge q) \to r]$
F	F	T	T

∴ Satisfiable but not valid

Q13
Sol: P need not to check, present in all option

Q: $[(\sim p \wedge q) \wedge (q \to (p \to r))] \to \sim r$

Fix

p	q	r	$[(\sim p \wedge q) \wedge (q \to (p \to r))] \to \sim r$
F	T	T	$T \to F = F$ not T

General form of an argument (Inference)

Conjunction of premises $\longrightarrow$ Conclusion

$[P_1 \wedge P_2 \wedge P_3 \ldots\ldots \wedge P_n] \longrightarrow q$

An inference which is tautology (T) is called Valid inference, otherwise it is called Invalid inference

Any valid inference is called rule of inference

✯ Some important rules of inference

Name / Form	Tautology form / proof
1) Conjunction p q ∴ p ∧ q	$(p \wedge q) \longrightarrow (p \wedge q)$

Tautology form

2) Addition

$$\frac{p}{\therefore p \vee q}$$

$p \longrightarrow (p \vee q)$

$p \vdash p \vee q$

3) Simplification

$$\frac{p \wedge q}{\therefore p, \therefore q}$$

$(p \wedge q) \longrightarrow p$

4) Modus Ponens (Rule of detachments)

$$\frac{\begin{array}{c} p \rightarrow q \\ p \end{array}}{\therefore q}$$

$[(p \rightarrow q) \wedge p] \longrightarrow q$

5) Hypothetical Syllogism (Transitive rule)

$$\frac{\begin{array}{c} p \rightarrow q \\ q \rightarrow r \end{array}}{\therefore p \rightarrow r}$$

$[(p \rightarrow q) \wedge (q \rightarrow r)] \longrightarrow (p \rightarrow r)]$

6) Modus Tollens (Denying Consequent)

$$\frac{\begin{array}{c} p \rightarrow q \\ \sim q \end{array}}{\therefore \sim p}$$

$[(p \rightarrow q) \wedge \sim q] \longrightarrow \sim p$

Fallacy (Means invalid)

$$\frac{\begin{array}{c} p \rightarrow \sim q \\ \sim p \end{array}}{\therefore q}$$

In invalid inference which resembles a valid inference is called fallacy

1) Fallacy of affirming Consequent

$$\frac{\begin{array}{c} p \rightarrow q \\ q \end{array}}{\therefore p}$$

INVALID

2) Fallacy of denying anticident

$$\frac{\begin{array}{c} p \rightarrow q \\ \sim p \end{array}}{\therefore \sim q}$$

INVALID

3) Non-Sequitar

$$\frac{p}{\therefore q}$$

Rules Continue

7) Disjunctive Syllogism

$$\frac{\begin{array}{c} p \vee q \\ \neg p \end{array}}{\therefore q}$$

$[(p \vee q) \wedge \sim p] \longrightarrow q$

8) Conditional Proof

$p \rightarrow (q \rightarrow r) \equiv$ if p than r if ~~q~~

$p \wedge q$

$\therefore r$

$[p \rightarrow (q \rightarrow r)] \wedge (p \wedge q)] \longrightarrow r$

$[(p \rightarrow (q \rightarrow r)) \wedge (p \wedge q)] \longrightarrow r$

Proof: $\boxed{p \rightarrow (q \rightarrow r) \wedge (p \wedge q) \quad \text{given} \quad \downarrow p \quad q \rightarrow r}$ ×

$\rightarrow p \rightarrow (q \rightarrow r)$.

$\rightarrow (p \wedge q) \rightarrow r$

$(p \wedge q$

$\rightarrow r$

$(p \wedge q)$

p

q

$p \rightarrow (q \rightarrow r)$

$q \rightarrow r$

r

9) Disjunctive Elimination

$p \rightarrow r$

$q \rightarrow r$

$p \vee q$

$\therefore r$

$[(p \rightarrow r) \wedge (q \rightarrow r) \wedge (p \vee q)] \rightarrow r$

Proof: $p \vee q$

$[(p \rightarrow r) \wedge (q \rightarrow r) \wedge (p \vee q)] \longrightarrow r$

we take contradiction of what we have to proof so we introduce $\sim r$.

Proof:- $(p \rightarrow r)$ given

$(q \rightarrow r)$ given

$\therefore (p \vee q) \longrightarrow r$

$(p \vee q)$

$\therefore r$

$p \rightarrow r$

$\sim r$

$\sim p$

$q \rightarrow r$

$\sim q$

$\sim p \wedge \sim q$

$\sim (q \vee q)$

$(p \vee q)$

$\sim (p \vee q) \wedge (p \vee q)$

F (Contradiction)

10) Resolution

$p \vee q$

$\sim p \vee r$

$\therefore q \vee r$

$[(p \vee q) \wedge (\sim p \vee r)] \longrightarrow (q \vee r)$

Prove:- $[(p \vee q) \wedge (\sim p \vee r)] \longrightarrow (q \vee r)$

$p \vee q$ (given)

$\sim p \vee r$ (given)

$\sim p \rightarrow q \equiv \sim q \rightarrow p$

$p \longrightarrow r \equiv p \rightarrow r$

$\longrightarrow \sim q \rightarrow r = q \vee r$

11) Constructive Dilemma

$p \rightarrow q$

$r \rightarrow s$

$p \vee r$

$\therefore q \vee s$

$[(p \rightarrow q) \wedge (r \rightarrow s) \wedge (p \vee r)] \rightarrow (q \vee s)$

$[\cdot (p \to q) \wedge (r \to s) \wedge (p \vee r)] \longrightarrow (q \vee s)$

$p \to q$

$\sim p \to r$ $\quad \sim p \to s \quad \sim q \to s \cdot \equiv (q \vee s)$

$r \to s$

$\sim q \to \sim p$

12) <u>Destructive Dilemma</u>

$$[(p \to q) \wedge (r \to s) \wedge (\sim q \vee \sim s)] \to \sim p \vee \sim r$$

$p \to q$

$r \to s$

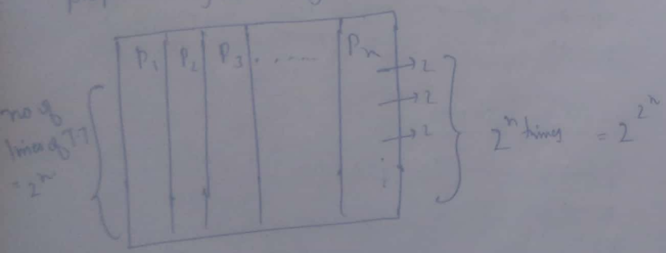
$\sim q \vee \sim s$

$\therefore \sim p \vee \sim r$

<u>5/7/18</u>

<u>Result</u>

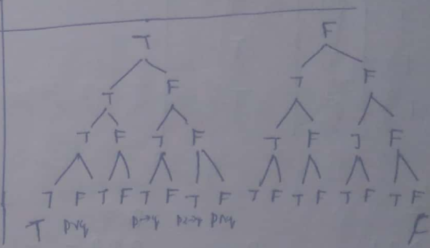
If there are n propositional variables, then no of propositional functions (formula) = ________



<u>Result</u> No of boolean fun^ns involving n boolean variable $= 2^{2^n}$

Ex no of propositional fun^n involving 2 propositional variable $= \underline{2^4}$ (ie 2^{2^2}) $= 16$

p	q
T	T
T	F
F	T
F	F



<u>Additional Connectives</u>

1) Exclusive OR (XOR, $\bar{\vee}$ or $\oplus$)

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

<u>Result</u>

$p \bar{\vee} q \equiv \sim (p \leftrightarrow q)$

Scanned by CamScanner

2) NAND ($\uparrow$ or $|$ sheffer's stroke)

NOT AND

$p \uparrow q \equiv \sim(p \wedge q)$

p	q	$p \uparrow q$
T	T	F
T	F	T
F	T	T
F	F	T

Results

1) $P \uparrow P \equiv \sim P$

Proof:- $P \uparrow P \equiv \sim(P \wedge P)$
$\sim P$

2) $(P \uparrow P) \uparrow (q \uparrow q) = P \vee q$

Proof: $(P \uparrow P) \uparrow (q \uparrow q) \equiv \sim P \uparrow \sim q$
$= \sim(\sim P \wedge \sim q)$
$= P \vee q$

3) $(P \uparrow q) \uparrow (P \uparrow q)$

Proof: $(P \uparrow q) \uparrow (p \uparrow q)$
$\sim(p \wedge q) \uparrow \sim(p \wedge q)$
$\sim(\sim(p \wedge q) \wedge \sim(p \wedge q))$
$((p \wedge q) \vee (p \wedge q))$
$(p \wedge q)$

or

$(p \uparrow q) \uparrow (p \uparrow q)$
$\sim(p \uparrow q)$
$= \sim(\sim(p \wedge q))$
$= p \wedge q$

3) NOR ($\downarrow$ Perice Arrow)

NOT OR

p	q	$p \downarrow q$
T	T	F
T	F	F
F	T	F
F	F	T

Results

1) $P \downarrow P \equiv \sim P$
2) $(P \downarrow P) \downarrow (q \downarrow q) \equiv P \wedge q$
3) $(P \downarrow q) \downarrow (p \downarrow q) \equiv P \vee q$

☆ Functionally Complete

A set of connectives is functionally complete if every compound proposition can be expressed as a new compound proposition involving only the connectives of given set.

Ex:- 1) $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$ F.C
2) $\{\neg, \wedge, \vee\}$ F.C
3) $\{\neg, \wedge\} \rightarrow$ minimal FC — {no proper subset of this is F.C hence it is minimal FC}

(Imp)

$\boxed{\text{minimal means : no proper subset having the property} \\ \text{maximal means : no proper superset having the property}}$

4) $\{\neg, \vee\} \rightarrow$ minimal FC
5) $\{\uparrow\} \rightarrow$ minimal FC
6) $\{\downarrow\} \rightarrow$ minimal FC

They are smallest minimal FC ☆

(Universal gates)

7) $\{\wedge, \vee\} \rightarrow$ not FC

7) $\{\wedge, \vee\} \rightarrow$ not F.C

HO
Q4
Sol: The result of the toss is head

$p \longleftrightarrow q$

c) Type 2 fact $\quad F \longleftrightarrow F \equiv T$ ✗

a) Type 1 head $\quad T \longleftrightarrow T \equiv T$ ✓

Type 2 head $\quad T \longleftrightarrow F \equiv F$ ✓

HO
Q4

p	q	r	$(r \rightarrow p) \rightarrow q$
T	T	F	T
F	T	F	T
F	F	F	F

Concept (Q4)

Note: The one who tells truth, the correct interpretation of his statement should be true

i.e. If statement is spoken by Type-1 (i.e. who always tells truth) then it should evaluate to TRUE

The one who tells ~~to~~ lie, the correct interpretation of his ~~st~~ stmnt should be False

i.e. If stmnt is spoken by Type-2 (i.e. who always lie) then it should evaluate to False.

Predicates (open Proposition)

An open proposition is a proposition except for the fact that it contains variables whose values are to be taken from some universe of discourse

Ex: $P(x): x+2=4 \quad x \in U$

$L(x,y): x+y \leq 6 \quad x,y \in U$

$F(x,y,t)$: x can fool y at time $t \quad x,y,t \in U$

A predicate involving n variables is called n-place predicate

Converting Predicates into Proposition

1) Substitution
2) Quantification
   i) Universal quantifier ($\forall$) (For all)
   ii) ~~Existential~~ Existential quantifier ($\exists$) (There exists or for some)

Ex: $U = \{1, 2, 3, 4, 5\}$

$P(x): x+4 \leq 6$

For all x, $P(x)$ is true [F]

$\forall x, P(x)$

For some x, $P(x)$ is true [T]

$\exists x, P(x)$

Ex $U = \{1, 2\}$

$P(x) : x + 4 \leq 6$

$\forall x\, P(x)$ [T]

NOTE

meaning will remain the same, but truth value depends on ~~propo~~ problem instance

Form	Meaning
1) $\forall x\, P(x)$	all True
2) $\exists x\, P(x)$	Some True (at least 1 true)
3) $\sim \forall x\, P(x)$	Not all True
4) $\sim \exists x\, P(x)$	none True
5) $\forall x \sim P(x)$	all false
6) $\exists x \sim P(x)$	Some false (at least 1 false)
7) $\sim \forall x \sim P(x)$	not all false
8) $\sim \exists x \sim P(x)$	none false

~~$\forall x$~~ Equivalence – IV (Quantifier Predicates)

1) $\forall x\, P(x) \equiv \sim \exists x \sim P(x)$
2) $\exists x\, P(x) \equiv \sim \forall x \sim P(x)$
3) $\sim \forall x\, P(x) \equiv \exists x \sim P(x)$
4) $\sim \exists x\, P(x) \equiv \forall x \sim P(x)$

Negating quantified predicates

$\sim \forall x\, P(x) = \exists x \sim P(x)$

$\sim \exists x\, P(x) = \forall x \sim P(x)$

$\sim \forall x\, [P(x) \longrightarrow Q(x)] = \exists x \sim [P(x) \rightarrow Q(x)]$

$= \exists x\, [P(x) \wedge \sim Q(x)]$

$\sim \exists x\, [P(x) \wedge Q(x)] = \forall x\, [P(x) \longrightarrow \sim Q(x)]$

Handout

Q3

57

$\sim \exists x\, (\forall y\, (\alpha) \wedge \forall z\, (\beta))$

$\forall x \sim (\forall y\, (\alpha) \wedge \forall z\, (\beta))$

$\forall x\, (\forall y\, (\alpha) \longrightarrow \sim \forall z\, (\beta))$

$\forall x\, (\forall y\, (\alpha) \longrightarrow \exists z\, (\sim \beta))$ — (c)

Take contrapositive

$\forall x\, (\sim \exists z\, (\sim \beta) \longrightarrow \sim \forall y\, (\alpha))$

$\forall x\, (\forall z\, (\beta) \longrightarrow \forall y\, (\sim \alpha))$ — (b)

Expressing statements into Symbolic form (Imp)

Ex: Every student in this class studied DM.

Case 1 U = set of students in this class

$D(x)$ = x studied DM

$\forall x \, D(x)$

Case 2 U = set of all people

$S(x)$: x is ~~st~~ student in this class

~~for~~ for all x, if x is student in the class then x studied DM

$\forall x \, (S(x) \longrightarrow D(x))$

Aristotle forms

1) All P's are Q's.

$\forall x \, (P(x) \longrightarrow Q(x)]$

2) Some P's are Q's

~~$\exists x (P(x)$~~ $\exists x \, [P(x) \wedge Q(x)]$

3) Not all P's are Q's

$\sim \forall x \, [P(x) \longrightarrow Q(x)]$

$\exists x \, [P(x) \wedge \sim Q(x)]$

Some P's are not Q's

4) No P's are Q's.

$\sim \exists x \, [P(x) \wedge Q(x)]$

$\forall x \, [P(x) \longrightarrow \sim Q(x)]$

P's are not Q's

Q9 Hand Out

(i) Gold and silver ~~org~~ ornaments are precious

$G(x)$: x is gold

$S(x)$: x is silver

$P(x)$: x is precious

(Gold are precious) and (Silver ornaments are precious)

$\forall x \, [[G(x) \rightarrow P(x)] \wedge [S(x) \rightarrow P(x)]]$ $(p \rightarrow r) \wedge (q \rightarrow r)$

$(p \vee q) \rightarrow r$

$\forall x \, [(G(x) \vee S(x)) \longrightarrow P(x)]$

(ii) All purple mushroom are poisonous

$G(x)$: x is purple

$S(x)$: x is mushroom

$P(x)$: x is poisonous

$\forall x \, [(G(x) \wedge S(x)) \longrightarrow P(x)]$

Ex: gate and cat exams need preparation

~~$\forall x$~~ $((g \rightarrow p) \wedge (c \rightarrow p)$

$((g \vee c) \longrightarrow p)$

Ex: Connected and acyclic graph is tree

$(c \wedge a) \longrightarrow t$

Quantifiers and Connectives

$U = \{1, 2\}$

$\forall x\, P(x) \equiv P(1) \wedge P(2)$

$\exists x\, P(x) \equiv P(1) \vee P(2)$

1) $\forall x\, [P(x) \wedge Q(x)] \equiv \forall x\, P(x) \wedge \forall x\, Q(x)$

2) $\exists x\, [P(x) \vee Q(x)] \equiv \exists x\, P(x) \vee \exists x\, Q(x)$

Ex: $U = \{1, 2\}$

$P(1) = T$ $\quad Q(1) = F$

$P(2) = F$ $\quad Q(2) = T$

$\forall x\, [P(x) \vee Q(x)]$

$= [P(1) \vee Q(1)] \wedge [P(2) \vee Q(2)]$

T F F T

$= T \wedge T$

$= T$

$\forall x\, P(x) \vee \forall x\, Q(x)$

$[P(1) \wedge P(2)] \vee [Q(1) \wedge Q(2)]$

T F F T

$F \vee F$

$= F$

<u>Result</u>

~~$\forall x\, P(x) \forall x\, Q(x)$~~

3) $\forall x\, P(x) \vee \forall x\, Q(x) \longrightarrow \forall x\, [P(x) \vee Q(x)]$

4) $\exists x\, [P(x) \wedge Q(x)] \longrightarrow \exists x\, P(x) \wedge \exists x\, Q(x)$

5) $\forall x\, [P \wedge Q(x)] \equiv P \wedge \forall x\, Q(x)$

6) $\exists x\, [P \vee Q(x)] \equiv P \vee \exists x\, Q(x)$

$\longrightarrow$ P = Proposition, Q(x) = Predicate

7) $\forall x\, [P \vee Q(x)] \equiv P \vee \forall x\, Q(x)$

8) $\exists x\, [P \wedge Q(x)] \equiv P \wedge \exists x\, Q(x)$

Ex: $U = \{1, 2\}$

$P(1) = T$ $\quad Q(1) = F$

$P(2) = F$ $\quad Q(2) = T$

$\forall x\, [P(x) \longrightarrow Q(x)]$	$\forall x\, P(x) \longrightarrow \forall x\, Q(x)$
$[P(1) \rightarrow Q(1)] \wedge [P(2) \rightarrow Q(2)]$	$[P(1) \wedge P(2)] \longrightarrow [Q(1) \wedge Q(2)]$
T F F T	T F F T
$F \wedge T$	$F \longrightarrow$
F	T

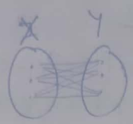
9) $\forall x\, [P(x) \longrightarrow Q(x)] \longrightarrow \forall x\, P(x) \longrightarrow \forall x\, Q(x)$

<u>Results</u>

$\forall x\, [P \rightarrow Q(x)] \equiv P \rightarrow \forall x\, Q(x)$

6/7/2016

x in terms of y



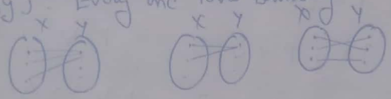
Two place Predicate

L (x, y) : x loves y

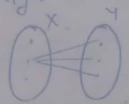
I $\forall x \forall y$ L(x, y) : Every one loves Everybody

II $\forall x \exists y$ L(x, y) : Every one loves Somebody

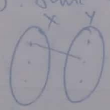
$\forall x \exists y \not\Rightarrow \exists y \forall x$



III $\exists x \forall y$ L(x, y) : Some one loves every body

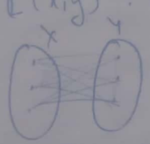


IV $\exists x \exists y$ L(x, y) : Some one loves some body

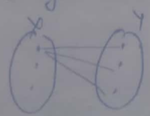


y in terms of x

V $\forall y \forall x$ L(x, y) : Everybody loved by everyone

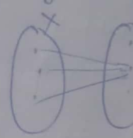


VI $\forall y$ ~~$\forall$~~ $\exists x$ L(x, y) : Everybody loved by someone

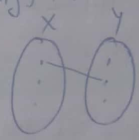


VII $\exists y \forall x$ L(x, y) : Some body loved by every one

$\boxed{\exists y \forall x \Rightarrow \forall x \exists y}$



VIII $\exists y \exists x$ L(x, y) : Some body loved by someone



Ex: $\exists y \forall x\ [x+y=x]$ T $y=0$

$\Downarrow$

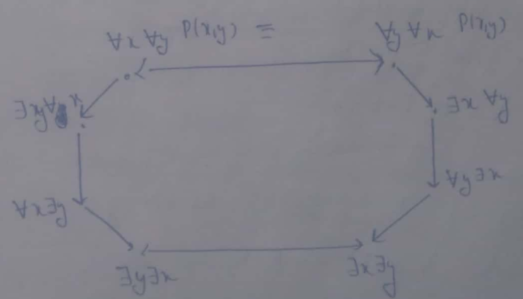
$\forall x \exists y\ [x+y=x]$ T $y=0$

$\forall x \exists y\ [x+y=0]$ T $x=2$ $y=-2$, $x=3$ $y=-3$

$\not\Downarrow$

$\exists y \forall x\ [x+y=0]$ F $\longrightarrow$ (some y) or (can at least by 1 value of y ~~which~~ for all x)

Relation b/w Quantified in 2 places predicates



NOTE : (Important)

1) $\forall x \forall y\, P(x,y)$: x divides y
$\equiv \forall y \forall x\, P(x,y)$: y divisible x

2) $\forall x \forall y\, P(x,y)$: x divides y [T]
$\not\equiv \forall y \forall x\, P(y,x)$: x divisible by y [F]

Handout

Q15
Sol:- Some boys in the class are taller than all girls.
$x \in$ set of all boys
$y \in$ set of all girls
taller (x,y) : x is taller than y
$\exists x \forall y$ taller(x,y)
→ Now use Aristotel form

Q16
Sol: $x \in$ set of all teachers
$y \in$ set of all students
likes(y,x): y likes x
$\forall x \exists y$ likes(y,x)

Q17
Sol: $F(x,y,z)$: x can fool y at time t
~~$\forall x \forall$~~ $\forall \exists y \exists t\, (\sim F(x,y,z))$
Everyone cannot fool someone at sometime

$\sim \exists x \forall y \forall x\, [F(x,y,t)]$
No one fool everyone all the time
a) $\forall x \exists y \exists t\, [F(x,y,t)]$
b) $\forall x \exists y \forall t\, [\sim F(x,y,t)]$
d) $\sim \exists x \exists y \exists t\, [\cdot F(x,y,t)]$

Q 26 (Imp) (Gate 2018)
Sol: Concept: $\boxed{\exists x P(x) \to F \quad \text{for empty universe}}$

$f = \exists s \exists t \exists u \exists v \forall v \forall w \forall x \forall y$
$g(s,t,u,v,w,x,y)$

f has a model with a universe containing 7 element

model ≤ 3 — If all element are same
we can have model with 1 universe

Q16
Sol $p \to (q \to r)$
if p then (r if q)
$(T \to [(H \vee T) \to A]) \wedge (L \to [(H \vee T) \to A])$
$[(T \vee L) \to [(H \vee T) \to A]]$

Workbook

Q54 " Everyone has exactly one best friend "
Everyone has some best friend
$\forall x \exists y\, B(x,y)$
Everyone has exactly one best friend
$\forall x \exists y\, [B(x,y) \wedge \forall z\, [(z \neq y) \to \sim B(x,z)]]$

or

$\forall x \exists y [B(x,y) \wedge \forall z [B(x,z) \longrightarrow (z = y)]]$

(Q.4)

Sol. No female likes a male who does not like an vegetarian

$\neg \exists x \exists y [F(x) \wedge M(y) \wedge likes(x,y)]$

No female likes a male not all veg he likes

$\neg \exists x \exists y [F(x) \wedge M(y) \wedge likes(x,y) \wedge \sim \forall z [V(z) \rightarrow likes(y,z)]$

Scope of a quantifier

The extent up to which a quantifier is applicable

Bound variable

A variable within scope of its quantifier.

Other variables are free variables.

Ex. $\forall x P(\overset{\text{bound}}{x}) \vee Q(\overset{\text{free}}{x})$

scope

$\forall x [P(\overset{\text{bound}}{x}) \vee Q(\overset{\text{bound}}{x})]$

scope

A quantified predicate is proposition if all the variables are bound variables.

Ex. $\forall x [P(\overset{\text{bound}}{x}) \longrightarrow Q(\overset{\text{free}}{y})]$ y should be in the ~~scs~~ scope of y quantifier

P-19 (workbook)

T-11

Sol. d) There is exactly one apple

$\exists x [A(x) \wedge \forall y (Apple(y) \longrightarrow (x = y))]$

or

$\exists x [A(x) \wedge \forall y (x \neq y \longrightarrow \sim Apple(y))]$

b) There are ~~exactly~~ at most two apples

$\exists x \exists y [A(x) \wedge A(y) \wedge \forall z [A(z) \rightarrow (x = z) \vee (y = z)]]$

c) There is at most one apple

$\forall x \forall y [(A(x) \wedge A(y)) \longrightarrow (x = y) \vee (y = x)]$

there can not be 2 apples

or

$\forall x \exists y [B(x,y) \wedge \forall z [B(x,z) \longrightarrow (z=y)]]$

(Q.4)

Sol: No female likes a male who does not like an vegetarian

$\neg \exists x \exists y [F(x) \wedge M(y) \wedge likes(x,y)]$

No female likes a male not all veg he likes

$\neg \exists x \exists y [F(x) \wedge M(y) \wedge likes(x,y) \wedge \sim \forall z [V(z) \rightarrow likes(y,z)]]$

Scope of a Quantifier

The extent up to which a quantifier is applicable

Bound variable

A variable within scope of its quantifier.

Other variables are free variables.

Ex. $\forall x \underbrace{P(\overset{bound}{x})}_{scope} \vee Q(\overset{free}{x})$

$\forall x \underbrace{[P(\overset{bound}{x}) \vee Q(\overset{bound}{x})]}_{scope}$

A quantified predicate is proposition if all the variables are bound variables.

Ex. $\forall x [P(\overset{bound}{x}) \longrightarrow Q(\overset{free}{y})]$ y should be in the ~~...~~ scope of y quantifier

P-19 (workbook)

T-11

Sol d) There is exactly one apple

$\exists x [A(x) \wedge \forall y (Apple(y) \longrightarrow (x=y))]$

or

$\exists x [A(x) \wedge \forall y ((x \neq y \longrightarrow \sim Apple(y))]$

b) There are ~~at most~~ exactly two apples

$\exists x \exists y [A(x) \wedge A(y) \wedge \forall z [A(z) \rightarrow (x=z) \vee (y=z)]]$

c) There is at most one apple

$\forall x \forall y [(A(x) \wedge A(y)) \longrightarrow (x=y) \vee (y=x)]$

there can not be 2 apples

Relations

Cartesian Product

$A \times B = \{ (a,b) \mid a \in A \text{ and } b \in B \}$ → set of all ordered pairs

$A = \{1,2,3\}$ $B = \{3,4,5,6\}$

$A \times B = \{ (1,3)(1,4)(1,5),(1,6)$
$(2,3)(2,4)(2,5)(2,6)$
$(3,3)(3,4)(3,5)(3,6) \}$

Results

1) $|A| = m$ $|B| = n$

$|A \times B| = (m \times n)$

2) ~~|P(A)|~~ $|P(A \times B)| = 2^{m \times n}$

3) "=" $= \{(3,3)\}$

"<" $= \{ (1,3)(1,4)(1,5)(1,6)\ (2,3)(2,4)(2,5)(2,6)\ (3,4)(3,5)(3,6) \}$

"$x+y=5$" $= \{(1,4), (2,3)\}$

"brother of" $= \{ \}$

Def: A relation R from A to B is a subset of $A \times B$

ie $R \subseteq A \times B$

Results $|A| = m$ $|B| = n$

1) no of relations from A to B $= 2^{mn}$

NOTE: 2) A relation R from A to itself is called relation on A

ie $R \subseteq A \times A$

3) No of relations on A $= 2^{m^2}$

Operations

Let R_1 and R_2 be relations from A to B

R^c = set of all ordered pairs which are not in R

1) $R^c = \{ (a,b) \mid (a,b) \notin R \text{ and } (a,b) \in A \times B \}$

2) $R_1 \cap R_2 = \{ (a,b) \mid (a,b) \in R_1 \text{ and } (a,b) \in R_2 \}$

3) $R_1 \cup R_2 = \{(a,b) \mid (a,b) \in R_1 \text{ OR } (a,b) \in R_2 \text{ or } (a,b) \in \text{both} \}$

4) $R_1 - R_2 = \{ (a,b) \mid (a,b) \in R_1 \text{ but } (a,b) \notin R_2 \}$

Inverse

$R \subseteq A \times B$ $\leftarrow$ R is a relation from A to B

then $R^{-1} \subseteq B \times A$ defined as

$R^{-1} = \{ (b,a) \mid (a,b) \in R \}$

NOTE

R	R^{-1}
$=$	$=$
$<$	$>$
$\leq$	$\geq$
divide → $\backslash$	multiple
$\subseteq$	$\supseteq$ ← superset

Composite Relation

~~Let R be relation from A to B~~

Let $R \subseteq A \times B$ and $S \subseteq B \times C$

then composite relation $RS \subseteq A \times C$ defined as

$RS = \{ (a,c) \mid \exists\, b \in B$ such that $(a,b) \in R$ and $(b,c) \in S \}$

Handout Relation

Q1

$\mathcal{U}$: $R_1 \subseteq A \times B$ find $R_1R_2 =$ ______

$R_2 \subseteq B \times C$

$x R_1 y$ iff $x+y$ is divisible 3

$x R_2 y$ iff $x+y$ is even but not divisible by 3

$R_1 = \{(1,2)(1,8),(3,6)(5,4)(7,2)(7,8)\}$

$R_2 = \{(2,2),(4,4),(6,2),(6,4),(8,2)\}$

$a - b - c \Rightarrow a - c$

$R_1 \quad R_2$

$1 - 2 - 2$ $(1,2)$

$1 - 8 - 2$ → no repeat

$3 - 6 - 2$ $(3,2)$

$3 - 6 - 4$ $(3,4)$

$5 - 4 - 4$ $(5,4)$

$7 - 2 - 2$ $(7,2)$

$7 - 8 - 2$ → no repeat

$R_1R_2 = \{(1,2)(3,2)(3,4)(5,4)(7,2)\}$

Relations on A :- (Imp)

Ex1: $A = \{1,2,3\}$

$A \times A = \{(1,1)(1,2)(1,3)$
$(2,1)(2,2)(2,3)$
$(3,1)(3,2)(3,3)\}$ → Universal relation on A

$\phi = \{\}$ → empty relation on A

equal to "=" → $\Delta = \{(1,1),(2,2),(3,3)\}$ ← delta relation

$< = \{(1,2)(1,3)(2,3)\}$

$\leq = \{(1,1)(1,2)(1,3)$ (2] x

$> = \{(3,1)(3,2)(2,1)\}$

$\leq = \{(1,1)(1,2)(1,3)(2,2)(2,3)(3,3)\}$

$\geq = \{(1,1)(2,2)(3,3)(2,1)(3,1)(3,2)\}$

$\backslash = \{(1,1)(1,2)(1,3)(2,2)(3,3)\}$ → b/a ie b divisible by a inverse

multiplication of $= \{(1,1)(2,1)(3,1)(2,2)(3,3)\}$

Def: A relation R on A is

1) Reflexive if $(a,a) \in R \ \forall a \in A$
ie $aRa \ \forall a \in A$

ie all reflexive pairs must be present in R

Ex ① yes → $A \times A, \Delta, \leq, \geq, \backslash$, m.of
No → $\phi, <, >$

$R_{11} = \{(1,1),(2,2),(3,3)\}$ on $A = \{1,2,3\}$
↑ Not reflexive

2) Irreflexive

if $(a,a) \notin R \ \forall a \in A$ ie $a \not R a \ \forall a \in A$

Ex 1 : yes : $\phi, <, >$
No : $A \times A, \Delta, \leq, \geq, \backslash$, m.of

$R_{11} = \{(1,1)(2,2)(2,3)\}$ on $A = \{1,2,3\}$
not irreflexive

ie no reflexive pair must be present in R.

3) Symmetric

$R \subseteq A \times A$ is symmetric if $(a,b) \in R$ then $(b,a) \in R$
~~then~~ where $a, b \in A$

ie if aR_b then bRa

Ex-1 yes - $A \times A, \phi$
no - $<, >, \leq, \geq, \backslash$, m.of

$R_{11} = \{(1,2)(2,1)(2,3)\}$ on $A = \{1,2,3\}$
↖ no symmetric

4) Asymmetric: if $(a,b) \in R$ then $(b,a) \notin R$ where $a, b \in A$ ie if aR_b then $b \not R a$

ie no symmetric pair & no reflexive pairs

Ex: yes :- $\phi, <, >$,
no :- $A \times A, \Delta, \geq, \leq, \backslash$, m.of

~~$R_{11} = \{(1,2)(2,1)(2,3)\}$ on $A =$~~

$R_{11} = \{(1,2)\}$ on $A = \{1,2,3\}$
↑ yes

7/7/2018

5) Antisymmetric: if $(a,b) \in R$ and $(b,a) \in R$ where $a, b \in A$ then $a = b$
ie if aR_b and ~~then~~ bR_a then $a = b$

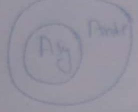
no symmetric pairs, but reflexive pairs are allowed yes – ϕ, Δ, $<$, $>$, $\le$, $\ge$, \, m of

no – $A \times A$

Note

It is asymmetric but allows reflexive elements

ie. every asymmetric is antisymmetric



Ex: $\{(1,2)(2,1)(2,3)\}$ on $A = \{1,2,3\}$

→ Not Sym
Not Asym
Not Antisym

$R_{131} = \{(1,2)(1,1)\}$ → Not Sym → Not asym, Not Antisym

$R_{131} = \{(1,2)\}$ → not sym → Asym → Anti

$R_{111} = \{(1,1)\}$ → Sym → not asym → Anti

6) Transitive : if $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$
where $a,b,c \in A$

ie if aRb and bRc then aRc where $a,b,c \in A$

yes – $A \times A$, ϕ, Δ, $<$, $>$, $\le$, $\ge$, \, m of

no –

Ex: $R_{210} = \{(3,2)(2,3)\}$ on $A = \{1,2,3\}$
← not Tran → (2,2) not exists

* $R_{110} = \{(3,2)(2,3)(3,3)\}$ on $A = \{1,2,3\}$
→ not Trans

$R_{131} = \{(1,2)(1,1)\}$ → Trans

$R_{111} = \{(1,1)\}$ → Trans

Representation

Let $|A| = n$ & R be a relation on A

Matrix relation

$$M_R = [a_{ij}]_{n \times n}$$

$$a_{ij} = \begin{cases} 0 & (i,j) \notin R \\ 1 & (i,j) \in R \end{cases}$$

Digraph

Each element $a \in A$ is represented by vertex.
There is a directed edge from vertex i to vertex j iff $(i,j) \in R$

$i \longrightarrow j$ $(i,j) \in R$

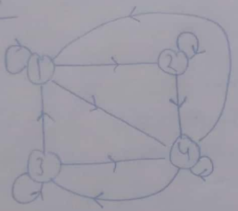
Ex. $A = \{1,2,3,4\}$

$R \subseteq A \times A$

$R = \{ (1,1)(1,2)\cancel{(1,3)}(1,4)$
$\cancel{(2,1)}(2,2)\cancel{(2,3)}(2,4)$
$(3,1)\cancel{(3,2)}(3,3)(3,4)$
$(4,1)\cancel{(4,2)}(4,3)(4,4) \}$

$$M_R = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 & 1 \\ 3 & 1 & 0 & 1 & 1 \\ 4 & 1 & 0 & 1 & 1 \end{array}$$

Digraph



Result

Let R, R_1 & R_2 be relations on A

R, R_1 and R_2 are	R^{-1}	$R_1 \cap R_2$	$R_1 \cup R_2$
Reflexive	✓	✓	✓
Irreflexive	✓	✓	✓
Symmetric	✓	✓	✓
Asymmetric	✓	✓	X
Antisymmetric	✓	✓	X
Transitive	✓	✓	X

X → need not be

Ex:

$R_1 = \{(1,2)\}$ ~~Asym~~	Asym	Anti	Trans
$R_2 = \{(2,1)\}$ ~~Asym~~	Asym	Anti	Trans
$R_1 \cup R_2 = \{(1,2)(2,1)\}$	not Asym	not Ant	not trans

Closures

1) Reflexive closure of R (R_R)

Smallest reflexive relation containing R

$R = \{ (1,1)(2,2)(2,3) \}$ on $A = \{1,2,3\}$

$R_R = \{ (1,1)(2,2)(2,3)(3,3) \}$

Results i) R is reflexive iff $R = R_R$

ii) $R_R = R \cup \Delta$

2) Symmetric Closure of R (R_S)

Smallest symmetric relation containing R

Ex: $R = \{ (1,2)(2,1)(2,3) \}$ on $A = \{1,2,3\}$

$R_S = \{ (1,2)(2,1)(2,3)(3,2) \}$

Result i) R is symmetric iff $R = R_S$

ii) $R_S = R \cup R^{-1}$

NOTE: $R^2 = RR$ ← Composite relation

$R^3 = R^2R$

$\vdots$

$R^{n+1} = R^nR$

3) Transitive closure of R (R^+)

Smallest transitive relation containing R

$R^+ \to$ general
(a,c)
a-b b-c
a-c

Results: i) R is transitive iff $R^+ = R$

$|A| = n$

$R^+ = R \cup R^2 \cup R^3 \cup R^4 \cup \ldots \cup R^n$

Warshall's Algorithm (Complexity $O(n^3)$) (Guide)

Ex. $A = \{1, 2, 3, 4\}$

$R = \{(1,2), (2,1), (3,4), (4,3)\}$

Find Transitive Closure

$$M_R^{(0)} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & \boxed{0} & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 1 & 0 \end{array}$$

✓ examine 1st row & 1st column

Convert uncrossed 0's to 1's

identify zero's in col and cross the corresponding rows

no of iterations $= |A| = n$

$$M_R^{(1)} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & \boxed{0} & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 1 & 0 \end{array}$$

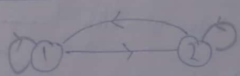
$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 1 & \boxed{0} \end{array}$$

$$M_R^{(3)} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 3 & 0 & 0 & \boxed{0} & 1 \\ 4 & 0 & 0 & 1 & 1 \end{array}$$

$$M_R^{(4)} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 1 \\ 4 & 0 & 0 & 1 & 1 \end{array} = M_{R^+}$$

$R^+ = \{(1,1) (1,2) (2,1) (2,2) (3,3) (3,4) (4,3) (4,4)\}$

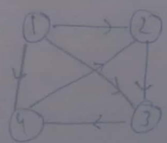
Digraph Method



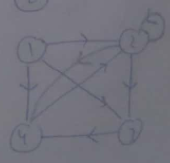
Handout

3) $A = \{1, 2, 3, 4\}$

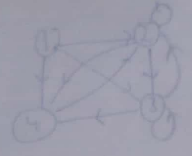
Examine ①



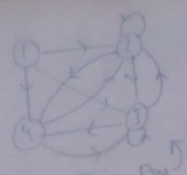
Examine ②



Example ③



Example ④



Final Ans

$R^+ = \{(1,2)(1,4)(1,3)(2,2)(2,3)(2,4)(3,3)(3,4)(3,2)$
$(4,2),(4,3),(4,4)\}$

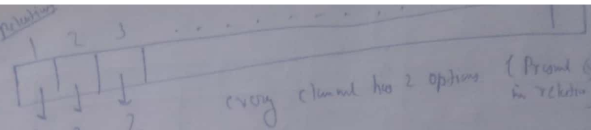
Counting (Imp)

Let $|A| = n$; $|A \times A| = n^2$

1) No of relations on A = 2^{n^2}
2) No of reflexive relations on A = $2^{n^2-n} = 2^{n(n-1)}$
3) No of irreflexive relations on A = $2^{n(n-1)} = 2^{n^2-n}$
4) No of symmetric relations on A = $2^{\frac{n(n+1)}{2}}$
5) No of asymmetric relations on A = $3^{\frac{n(n-1)}{2}}$
6) No of Antisymmetric relations on A = $2^n \cdot 3^{\frac{n(n-1)}{2}}$
7) No of Transitive relations on A = no closed form formula available
8) No of symmetric and reflexive relations on A = $2^{\frac{n(n-1)}{2}}$

9)

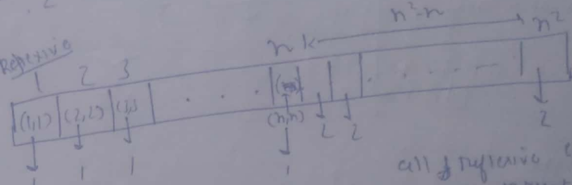
1) Relations



every element has 2 options (Present (or) Absent in relation)

$\therefore 2^{n^2}$

2) Reflexive



all of reflexive element must be present

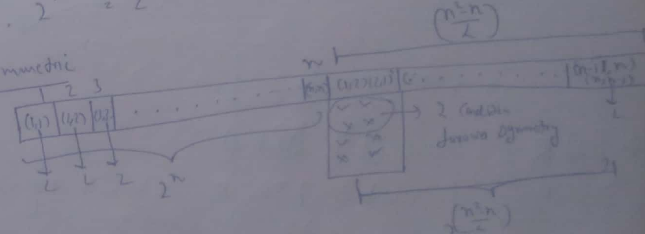
$\therefore 2^{n^2-n} = 2^{n(n-1)}$

3) Irreflexive

Here all reflexive elements must be absent

$\therefore 2^{n^2-n} = 2^{n(n-1)}$

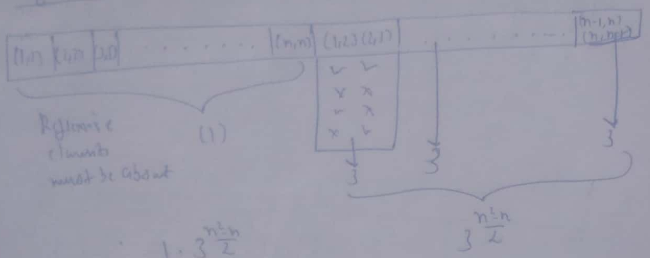
4) Symmetric



$\therefore 2^n \times 2^{\left(\frac{n^2-n}{2}\right)}$

$= 2^{\frac{n(n+1)}{2}}$

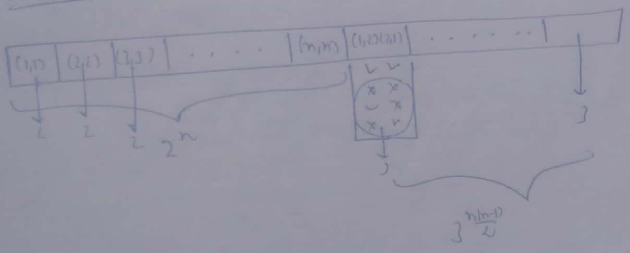
5) Asymmetric



$3^{\frac{n^2-n}{2}}$

$\therefore 1 \cdot 3^{\frac{n^2-n}{2}}$

6) Antisymmetric



$3^{\frac{n(n-1)}{2}}$

$\therefore 2^n \cdot 3^{\frac{n(n-1)}{2}}$

* P.O.R (Partial Order Relation)

<u>Definition</u> A relation R on A which is reflexive

i) Reflexive

ii) Anti Symmetric

iii) Transitive

is called Partially Ordered ~~set (PO)~~ Relation (P.O.R)

Ex:- 1) Relation "$\leq$" on Z is   Z = set of integers

(a) ref only (b) Anti only (c) Trans only ~~(d) P.O.R~~

* 2) "\" on Z $\rightarrow$ ~~P.O.S~~ not P.O.S

Exception ~~B~~ $\because$ 0/0 not reflexive

$-5 | 5$ and $5 | -5$ $-5 \neq 5$

not antisymmetric

$a|b$ & $b|c \Rightarrow a|c$ only Transitive

3) "\" on Z^+ (set of Positive integers) $\longrightarrow$ P.O.R

* 4) "$\subseteq$" on P(A)

$A \subseteq A \rightarrow$ ref

$A \subseteq B$ & $B \subseteq A \Rightarrow A = B \rightarrow$ Anti

$A \subseteq B$ & $B \subseteq C \Rightarrow A \subseteq C \longrightarrow$ Transitive

Hence it is P.O.R

<u>Notation</u> P.O.R is denoted ~~as~~ by $\preccurlyeq$

<u>Definition</u> : A set P together with P.O.R $\preccurlyeq$, $\langle P, \preccurlyeq \rangle$ is called Partially Ordered set (Poset)

Ex: $\langle Z, \leq \rangle$ is Poset

~~(P(A))~~ $\langle P(A), \subseteq \rangle$ is Poset

$\langle Z^+, \backslash \rangle$ is Poset

Scanned by CamScanner

Recall

R	R^{-1}
ref	ref
Anti	Anti
Trans	Trans
POR	P.O.R
$\langle P, R\rangle$ is Poset	$\langle P, R^{-1}\rangle$ is Poset

NOTE: The posets $\langle P, R\rangle$ and $\langle P, R^{-1}\rangle$ are called duals.

Important

	$\langle P, R\rangle$	$\langle P, R^{-1}\rangle$
1)	$\langle Z, \leq\rangle$	$\langle Z, \geq\rangle$
2)	if $A \subseteq Z$ $\langle A, \leq\rangle$	$A \subseteq Z$ $\langle A, \geq\rangle$
3	$\langle Z^+, \backslash\rangle$	$\langle Z^+, \text{m.of}\rangle$
4	if $A \subseteq Z^+$ $\langle A, \backslash\rangle$	$A \subseteq Z^+$ $\langle A, \text{m.of}\rangle$
5	$\langle P(S), \subseteq\rangle$	$\langle P(S), \supseteq\rangle$
6	$\langle D_n, \backslash\rangle$	$\langle D_n, \text{m.of}\rangle$

D_n = set of all +ve divisors of 8

D_n = set of all positive divisors of n

$D_8 = \{1, 2, 4\}$

$D_{24} = \{1, 2, 3, 4, 6, 8, 12\}$

related x, y in order
In
Comparable → both (i.e any order) the direction

Def: Let $\langle P, \preceq\rangle$ be a poset
Two elements $a, b \in P$ are said to be comparable if either $a \preceq b$ (or) $b \preceq a$ (i.e a related to b (or) b related to a)

Ex: $A = \{1, 2, 3, 4\}$

1) $\langle A, \leq\rangle$ is poset	2) $\langle A, /\rangle$ is poset
2, 3 comparable ✓ $2 \leq 3$	2, 4 comparable ✓ $2/4$
4, 1 comparable $4 \nleq 1$ but $1 \leq 4$	2, 3 comparable ✗ $2 \nmid 3$, $3 \nmid 2$
∴ every element of set is comparable hence it becomes "TOSET"	Not every element comparable hence it is still Poset

Definition: A poset $\langle P, \preceq\rangle$ in which every pair of elements are comparable is called Totally Ordered Set i.e Toset (also called Chain)

Ex: $\langle A, \leq \rangle$ $A = \{1,2,3,4\}$

$1 \leq 2 \leq 3 \leq 4$ $1 \backslash 2 \backslash 3 \backslash 4$

~~Hasse Diagram~~

Definition : Let $\langle P, \preceq \rangle$ be a poset

Associated relationship

x is in Associated relationship

$x \prec y$ means $x \preceq y$ but $x \neq y$

We are ignoring reflexive

Covering

y covers x means : $[x \prec y \wedge [x \leq z \leq y \Rightarrow x = z \text{ or } y = z]]$

x and y are related and $x \neq y$

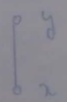
x and y ke beech nothing

Hasse Diagram

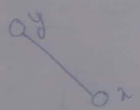
Let $\langle P, \preceq \rangle$ be a poset

→ Every element of A is denoted by "vertex"

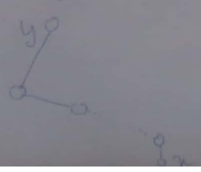
→ if y covers x



(OR)



→ If $x \prec y$ but y does not covers x

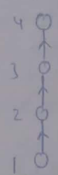


Ex. $A = \{1,2,3,4\}$

$\langle A, \leq \rangle$

NOTE: no element is related to the smallest element

Chain



→ 1 is related to 2 i.e $1 \leq 2$

1, 2

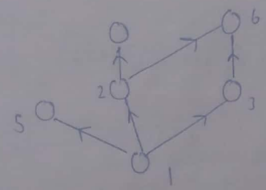
~~1~~, 2, 3

~~1~~, ~~2~~, 3, 4

Start with smallest element

Ex: $A = \{1,2,3,4,5,6\}$

$\langle A, \backslash \rangle$



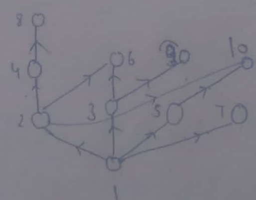
1, 3 → no one covering 1 in this list

Ex: $A = \{1,2,3,4,5,6,7,8,9,10\}$

i) $\langle A, \backslash \rangle$

ii) $\langle D_{24}, \backslash \rangle$ $D_{24} = \{1,2,3,4,6,8,12,24\}$

i)



no of lines = 11

1, 2

1, 3

~~1~~, 2, 4

1, 5

~~1~~, 2, 3, 6

1, 7

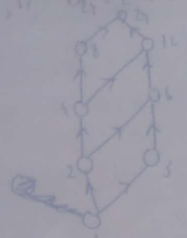
~~1~~, ~~2~~, 4 8

~~1~~, 3, 9

~~1~~, 2, 5 10

Practice

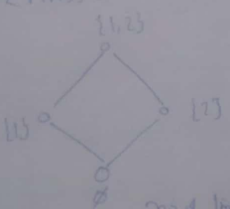
4) $\langle D_{24}, | \rangle$ $D_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$



1, 2
1, 3
~~1~~, 2, 4
~~1~~, 2, 3 6
~~1~~, ~~2~~, 4, 8
~~1~~, ~~2~~, ~~3~~, 4, 6, 12
~~1~~, ~~2~~, ~~3~~, ~~4~~, ~~6~~, ~~8~~, ~~12~~, 24

no of lines = 10

5) $A = \{1, 2\}$
$\langle P(A), \subseteq \rangle$ $P(A) = \{\phi, \{1\}, \{2\}, \{1,2\}\}$

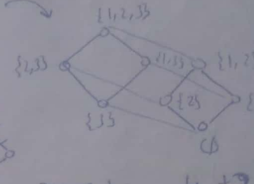


$\phi, \{1\}$
ϕ $\{2\}$
~~ϕ~~, $\{1\}, \{2\}, \{1, 2\}$

no of lines = 4

6) $A = \{1, 2, 3\}$
$\langle P(A), \subseteq \rangle$
$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

Hasse Diag



2^{3-1}

no of lines $= 4 + 2^2 + 4$
$= 12$

$A = \{1, 2, 3, 4\}$
$\langle P(A), \subseteq \rangle$ no of lines $= 12 + 2^3 + 12$ $\longrightarrow 2^{|A|-1}$
ie 2^{4-1}

Special Elements in Hasse Diagram

Let $\langle P, \alpha \rangle$ be a poset

Maximal element

there is no element above it

an element $a \in P$ is maximal if there is no $x \in P$ such that $a \alpha x$

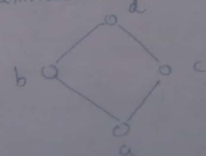
Minimal element

no element below it

an element $a \in P$ is minimal if there is no $x \in P$ such that $x \alpha a$

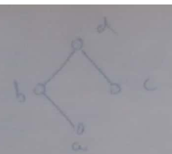
Find maximal & minimal

(i)



max → d
min → a

(2)

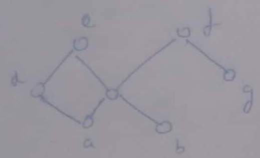


minimal element
↓ element which
do not have tail

maximal - which
do not have heads

Max – d
Min – a, c

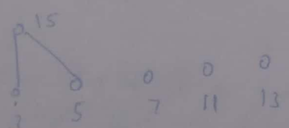
(3)



max – e, f
min → a, b, g

(4) $A = \{3, 5, 7, 11, 13, 15\}$

$\langle A, / \rangle$



max – 15, 7, 11, 13
min – 3, 5, 7, 11, 13

no of lines = 2

Greatest elements

An element $a \in P$ is greatest if $x \preccurlyeq a \; \forall x \in P$

all elements related to this

least element

an element $l \in P$ is least if $l \preccurlyeq x \; \forall x \in P$

it is related
to all elements

NOTE

~~if more than 1 max then we can not have~~

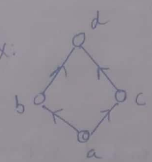
→ if more than 1 maximal then we can not get greatest element

→ if more than 1 minimal then we can not get least element

~~Ex~~ Notations

Greatest → 1
Least → 0

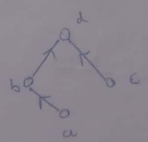
Ex:



max → d
min → a

1 → d
0 → a

Ex:



1 → d
0 → does not exists

max → d
min → a, c

~~Ex~~

NOTE:

1) Every finite poset has maximal & minimal elements.

2) maximal & minimal elements if exists, need not be unique

3) Greatest & least elements if exists are unique

Ex $\langle Z, \leq \rangle$ is poset

max →
min → } does not exists

Note: Infinite posets may or need not have maximal or minimal elements

$\circ a$ ← If set contain only 1 element

max → a | 1 → a
min → a | 0 → a

Def: Let $\langle P, \preceq \rangle$ be a Poset and $A \subseteq P$

An element $U \in P$ is <u>upper bound</u> of A
if $x \preceq U \;\forall\; x \in A$
↳ all elements related to U

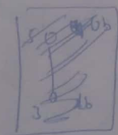
An element $l \in P$ is <u>lower bound</u> of A
if $l \preceq x \;\forall\; x \in A$
↳ l related to all elements of A.

Ex: $P = \{1, 2, 3, 4, 5, 6\}$

$\langle P, \leq \rangle$ is poset

$A = \{3, 5\}$

~~ub of A → 5~~
~~lb of A → 3~~



ub of A → 5, 6 → least upper bound (l.u.b)

lb of A → 1, 2, 3 → g.l.b

<u>Least Upper bound</u> (lub) (Join)

lub $\preceq$ Ub's

(Meet) <u>Greatest lower bound</u> (glb)

lb's $\preceq$ glb

lub $\{3,5\} = 5$
glb $\{3,5\} = 3$
lub $\{2,4\} = 4$
glb $\{2,4\} = 2$

<u>Lattice</u>

Def: A poset $\langle P, \preceq \rangle$ in which every pair of elements (every two element subset) have lub & glb is called <u>Lattice</u>

<u>NOTE</u>: A poset $\langle P, \preceq \rangle$ in which every subset having lub and glb is called <u>Complete lattice</u>

<u>Notation</u>: The lub of 2 elements

lub $\{a,b\}$ · $a \vee b$ · a join b

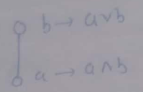
glb $\{a,b\}$ $a \wedge b$ a meet b

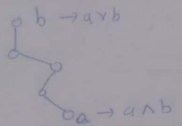
<u>NOTE</u>:

	$\leq$	\	$\subseteq$
lub $\{A,B\}$	max (A,B)	LCM (A,B)	$A \cup B$
glb $\{A,B\}$	min (A,B)	GCD (A,B)	$A \cap B$

Consistency

Let $\langle P, \preceq \rangle$ be a poset and
$a \preceq b$ iff $a \vee b = b$ iff $a \wedge b = a$



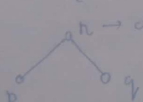


Method (Finding lub & glb of non-comparable elements)

I) i)

$a \wedge b = c$

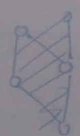
ii)

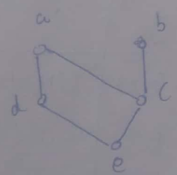


$p \vee q = r$

II

i)





↳ c, e lb's

$a \wedge b = c$

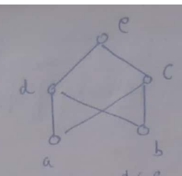
ii

↳ r, t

$p \vee q \to$ ~~r~~

III

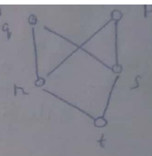
i)



↳ d, c, e ub's

$a \vee b$ → does not exist

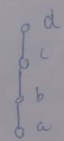
iii



$p \wedge q$ → does not exist
↳ s, r, t

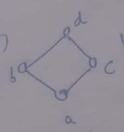
Q Which of the following is / are not lattice

1)



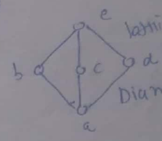
Lattice

2)



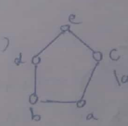
lattice

(3)



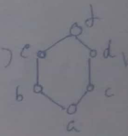
lattice
Diamond

4)



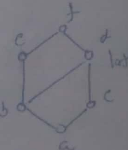
lattice

5)



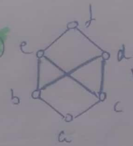
lattice

6)



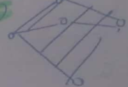
lattice

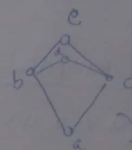
7)



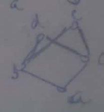
$b \vee c$ does not exist

8)





d & e do ~~are~~ not ~~have~~ have lub & glb



not lattice

NOTE:- Every Toset is lattice

8|7|2018

Result In any lattice $\langle L, \preceq \rangle$ the following property satisfied

1) Idempotent

$a \vee a = a$
$a \wedge a = a$

2) Commutative

$a \vee b = b \vee a$
$a \wedge b = b \wedge a$

3) Associative

$a \vee (b \vee c) = (a \vee b) \vee c$
$a \wedge (b \wedge c) = (a \wedge b) \wedge c$

4) Absorption

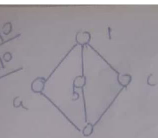
$a \vee (a \wedge b) = a$
$a \wedge (a \vee b) = a$

NOTE: Distributive not satisfied in lattice

Result: In any lattice $\langle L, \preceq \rangle$ following ~~distribution~~ distributive inequalities are satisfied

(1) $a \vee (b \wedge c) \preceq (a \vee b) \wedge (a \vee c)$
$(a \wedge b) \vee (a \wedge c) \preceq a \wedge (b \vee c)$

HQ
Q10



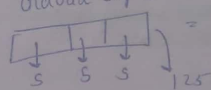
(a,b,c), this triple does not satisfy distributive property

$a \vee (b \wedge c)$ $\quad$ $(a \vee b) \wedge (a \vee c)$

Not all ~~triple~~ ordered triple will satisfy

Ex: $0 \vee (b \wedge c) \cdot (0 \vee b) \wedge (0 \vee c)$

ordered triple



$= \{ (a,b,c) \mid a,b,c \in A \}$

Do not satisfy distributive $= 3! = 6$ $\quad (a,b,c)$

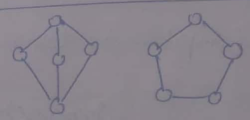
Prob $= \frac{6}{125}$

Prob → $1 - \frac{6}{125} = \frac{119}{125}$

Def: A lattice $\langle L, \preceq \rangle$ in which distributive properties are satisfied is called Distributive lattice

Ex: $\langle P(S), \subseteq \rangle$ is distributive Lattice

Famous non-distributive Lattices



Def: A Lattice $\langle L, \preceq \rangle$ in which greatest & least ~~upper bound~~ elements exist is called bounded Lattice.

ie $0 \preceq a \preceq 1 \quad \forall a \in L$

Scanned by CamScanner

Def: Let $\langle L, \leq \rangle$ be a bounded lattice. Any element $b \in L$ is complement of $a \in L$

if $a \vee b = 1$ and $a \wedge b = 0$ → glb is equal to least element
lub is equal to greatest element

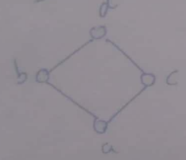
NOTE

1) If "b" is complement of "a" then "a" is complement of "b".

2) $1 \vee 0 = 1$, $1 \wedge 0 = 0$ } greatest & least are complement of each other.

Def: A bounded lattice $\langle L, \leq \rangle$ in which every element having complement is called "**Complemented Lattice**"

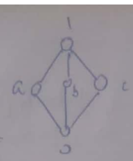
Ex. Find Complements (if exists)



Complemented Lattice

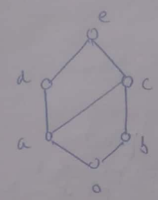
element	Complement
a	d
d	a
b	c
c	b

2)



Complement Lattice

Element	Complement
a	b, c
b	a, c
c	a, b
0	1
1	0



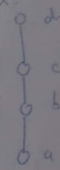
→ not Complemented Lattice

$d \vee c = e$
$d \wedge c = a$

no complement of d also.

no of complements of a = 0
" " " " b = 1

Ex:



not complemented lattice

NOTE: (Imp)
~~Any~~ Every chain with 3 or more elements is non-complemented lattice

Result: In a distributive lattice complements, if exists, are unique. ~~i.e~~ every.

Def: Bounded, distributed, and complemented lattice is Boolean algebra.

Ex: $\langle P(S), \subseteq \rangle$ is boolean Algebra

Result: $\langle D_n, | \rangle$ is BA iff n is product of distinct prime factors.

Ex: $\langle D_{30}, | \rangle$

$30 = 2 \cdot 3 \cdot 5$ ← product of distinct primes

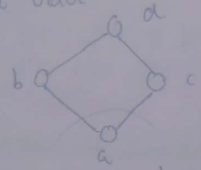
~~Ex~~ Ex: $\langle D_8, \mid \rangle$

$8 = 2^3 \leftarrow$ not product of distinct prime.

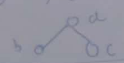
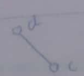
not BA

Topological Sort

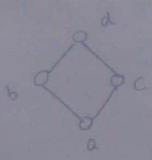
A linear order which contains the given partial order



pick the min and put it in the T.sort

	T. Sort
min $\rightarrow$ a 	a
min $\rightarrow$ b,c 	a α b
	a α b α c
min – c d	a α b α c α d
min – d	

EX

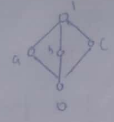


T.sort

$\left\{ \begin{array}{l} a \,\alpha\, b \,\alpha\, c \,\alpha\, d \\ a \,\alpha\, c \,\alpha\, b \,\alpha\, d \end{array} \right\} \alpha$

$\underset{1!}{a} \;\alpha\; \underset{bc}{2!} \;\alpha\; \underset{1!}{d} = 1 \times 2 \times 1 = 2$

Ex



$\underset{1!}{0} \quad \underbrace{a, b, c}_{3!}, \quad \underset{1!}{1} = 6$

Workbook Q19 P-18

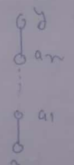
Sol: Q19

$S = \{x, a_1, a_2 \cdots a_n, y\}$ partial order $\leq$ is defined on S

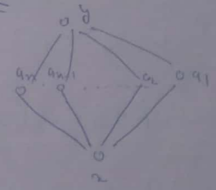
$x \leq a_i \ \forall i$

$a_i \leq y \ \forall i$

find no of total orders on set S which contain the partial order $\leq$



min = 1 orders



max = n! order

(Imp)

Partition: A non-empty collection $\{S_1, S_2, S_3 \ldots S_n\}$ of non-empty subsets of S such that

1) $S_1 \cup S_2 \cup S_3 \cdots \cup S_n = S$ Collectively exhaustive

2) ~~$S_1 \cap S_2 \cap S_3 \cdots \cap S_n = \phi$~~

$S_i \cap S_j = \phi \quad i \neq j$ Mutually exclusive

is called partition of S.

Ex: $S = \{1,2,3,4\}$

$P_1 = \{\{1\},\{2\},\{3\},\{4\}\} \to$ 4-part partition

$P_2 = \{\{1,2\},\{3\},\{4\}\} \to$ 3-part partition

$P_3 = \{\{1,3,4\},\{2\}\} \to$ 2-parts partition

$P_4 = \{\{1,2,3,4\}\} \to$ 1-part partition

$P_5 = \{\{1,2\},\{4\}\} \to$ not a partition

$P_6 = \{\{1,2,3\}\{2,4\}\} \to$ not a partition

✱ Refinement Relation (Grade)

$P_i \preceq P_j$ Let P_i and P_j be two partitions.

$P_i \preceq P_j$ if every part of in P_i is subset of some part in P_j

P_i is refinement of P_j

Ex:- $P_1 \preceq P_2$

P_2 is not refinement of P_3

Result: The refinement relation ($\preceq$) is POR on collection of partitions of a set S.

1) $P_i \preceq P_i$ reflexive

2) $P_i \preceq P_j$ and $P_j \preceq P_i$ only if $P_i = P_j$ → anti sym

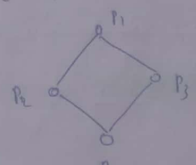
3) $P_i \preceq P_j$ and $P_j \preceq P_k$ then $P_i \preceq P_k$ → transitive

Handout
Q 16

Sol: $P_1 = \{\overline{abcd}\}$ $P_2 = \{\overline{ab}, \overline{cd}\}$ $P_3 = \{\overline{abc}, \bar{d}\}$

$P_4 = \{\bar{a},\bar{b},\bar{c},\bar{d}\}$

Smallest



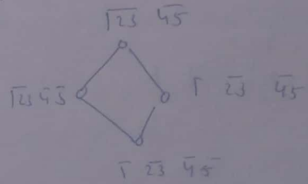
P_4, P_2

P_4, P_3

~~P_4~~, P_2, P_3, P_1

to find meet ie P_4 it must be refinement of P_1 & P_3

to find join ie P_1 ~~it~~ P_2 & P_3 must be refinement of P_1



Equivalence Relation (ER)

A relation R on A which is

i) Reflexive
ii) Symmetric
iii) Transitive

is called an E.R

Def: Let R be an ER on A and $a \in A$

The equivalence ~~class~~ class of a is defined as

$\bar{a}$ (or) $[a] = \{ b \mid (a,b) \in R \}$

Ex:- $A = \{1,2,3,4\}$

$R = \{(1,1)\ (2,2)\ (3,3)\ (4,4)\}$

Ex:- R is ref
R is Sym
R is Tran
} E.R

$[1] = \{1\}$
$[2] = \{2\}$
$[3] = \{3\}$
$[4] = \{4\}$

$P = \{\{1\}, \{2\}, \{3\}, \{4\}\}$ partition of A

NOTE: For every equivalence relation we ~~can for~~ can have partition parts are distinct equivalence classes

Ex:- $A = \{1,2,3,4\}$

$R = \{(1,1)\ (1,3)\ (3,1)\ (3,3)\ (2,2)\ (2,4)\ (4,2)\ (4,4)\}$

E.R

$[1] = \{1,3\} = [3]$
$[2] = \{2,4\} = [4]$

$P = \{\{1,3\}\ \{2,4\}\}$ partition of A

Result: Let R be an E.R on A and $a, b \in A$

(1) $a \in [a]$ ← bcz of reflexive
(2) $a \in [b]$ then $b \in [a]$ ← bcz of sym
(3) $a \in [b]$ then $[a] = [b]$ ← bcz of Transitivity
(4) $[a] = [b]$ (or) $[a] \cap [b] = \phi$

★ **Ex:** Let $P = \{\{1,3\}, \{2\}, \{4\}\}$ be partition of $A = \{1,2,3,4\}$

Find the E.R. on A corresponding to P

NOTE: Take cross product of part with itself

$R = \{1,3\} \times \{1,3\} \cup \{2\} \times \{2\} \cup \{4\} \times \{4\}$

$\{(1,1),(1,3),(3,1),(3,3),(2,2),(4,4)\}$

2) $P = \{\{1,3,4\},\{2\}\}$ is partition of A

$A = \{1,2,3,4\}$. Find E.R. R on A?

Sol: $R = \{1,3,4\} \times \{1,3,4\} \cup \{2\} \times \{2\}$

$= \{(1,1),(1,3),(1,4),(3,1),(3,3),(3,4),(4,1),(4,3),(4,4),(2,2)\}$

(I-P)

Result: given an E.R. R on A we can find a partition P of A and vice versa

2) $|A| = n$

No of E.R on A = No of partitions of A = B_n

(Bell no)

(Since there is 1-1 correspondence b/w No of partitions of A & no of E.R on A)

3) Bell no

$$\boxed{B_n = \sum_{k=0}^{n-1} {}^{n-1}C_k B_k \qquad B_0 = 1 \qquad B_1 = 1}$$

Remember B_2, B_3, B_4

$B_2 = {}^{n}C_0 B_0 + {}^{n}C_1 B_1$

$= 1 \times 1 + 1 \times 1 = 2$

$B_3 = {}^{2}C_0 B_0 + {}^{2}C_1 B_1 + {}^{2}C_2 B_2$

$= 1 \times 1 + 2 \times 1 + 1 \times 2$

$= 1 + 2 + 2$

$= 5$

$B_4 = {}^{3}C_0 B_0 + {}^{3}C_1 B_1 + {}^{3}C_2 B_2 + {}^{3}C_3 B_3$

$= 1 \times 1 + 3 \times 1 + 3 \times 2 + 1 \times 5$

$= 1 + 3 + 6 + 5$

$= 15$

$B_5 = {}^{4}C_0 B_0 + {}^{4}C_1 B_1 + {}^{4}C_2 B_2 + {}^{4}C_3 B_3 + {}^{4}C_4 B_4$

$= 1 \times 1 + 4 \times 1 + 6 \times 2 + 4 \times 5 + 1 \times 15$

$= 1 + 4 + 12 + 20 + 15$

$= 52$

Gate $A = \{1,2,3,4\}$

No of E.R on A = $B_4 = 15$

Gate $A = \{a,b,c\}$

No of partitions of A = $B_3 = 5$

Results $|A| = n$

1) Smallest E.R on A = $\underline{\Delta}$
2) Largest E.R on A = $\underline{A \times A}$
3) No of elements in smallest E.R on A = $\underline{n}$
4) No of element in largest E.R on A = $\underline{n^2}$

5)

R, R_1 & R_2	R^{-1}	$R_1 \cap R_2$	$R_1 \cup R_2$
Ref	Ref	Ref	Ref
Sym	Sym	Sym	Sym
Trans	Trans	Trans	× → need not be
E.R	E.R	E.R	×

WB P-18
Q 24

Sol: $x R y$ iff $x+y$ is even

Ref $x + x = 2x$ even
$x R x$

Sym $x R y$
$x + y$ = even
$\Rightarrow y + x$ = even
$\Rightarrow y R x$

Trans $x R y$ $y R z$
$x + y = 2k$ $y + z = 2m$
$x + 2y + z = 2(k+m)$
$x + z = 2(k + m - y)$
$\therefore x+z$ is even
$x R z$

$\therefore R$ is E.R

To find Equivalence class

$[0] = \{\ldots, -4, -2, 0, 2, 4, 6, \ldots\}$
Set of all even no

$[1] = \{\ldots, -3, -1, 1, 3, 5, \ldots\}$
Set of all odd nos

9/7/18

Algebraic Structures

Let S be a non-empty set and $*$ be an operation

1) Closure (S is closed w.r.t to $*$)
   if $a * b \in S \quad \forall a, b \in S$

2) Associative (S $*$ associative w.r.t $*$)
   if $a * (b * c) = (a * b) * c \quad \forall a, b, c \in S$

3) Existence of Identity (Identity exist in S w.r.t $*$)
   if $\exists e \in S \; \forall a \in S \; [a * e = e * a = a]$

4) Existence of inverse [inverse exist in S w.r.t $*$]
   $\forall a \in S \; \exists b \in S \; [a * b = b * a = e]$

5) Commutative [S is commutative w.r.t $*$]
   $a * b = b * a \quad \forall a, b \in S$

$(S, *)$

1) Closure $\longrightarrow$ $(S, *)$ is a ~~algebraic~~ algebraic structure
   $*$ is binary operators

2) Closure + Associative $\longrightarrow$ $(S, *)$ is semi group

3) Closure + Associative + Identity exists $\longrightarrow$ $(S, *)$ is monoid

4) Closure + Associative + Identity exists + Inverse exist
   $\longrightarrow$ $(S, *)$ is Group

5) Group + Commutative $\longrightarrow$ $(S, *)$ is an ~~algebraic~~ abelian group (or) Commutative ~~gro~~ group

$(S, *)$	Closure	Associative	Identity	Inverse	Commutative	Nature
$(Z, \times)$	✓	✓	$1 \in Z$, $a \times 1 = a$	$\frac{1}{a} \notin Z$		Monoid
$(R, \times)$	✓	✓	$1 \in Z$, $a \times 1 = a$	$\frac{1}{a} \notin R$ inverse does not exist	✓	~~Ab~~ Monoid
$(R - \{0\}, \times)$	✓	✓	✓	$\frac{1}{a} \in R, a \in R$; $a \cdot \frac{1}{a} = 1$	✓	Abelian group
$(M_{n \times n}, \times)$ set of all $n \times n$ matrices w.r.t to multipli'	✓	$A(BC) = (AB)C$	$I_n \in M_{n \times n}$, $A \cdot I_n = A$	$\lvert A \rvert = 0$, A^{-1} does not exists		Monoid
$s(M_{n \times n}, \times)$ set of all non-singular matrices w.r.t to multiplication	✓	✓	✓	$\lvert A \rvert \neq 0$, A^{-1} exists, $AA^{-1} = I_n$	$AB \neq BA$	Group
Set of all bijective fun's w.r.t to composi- ~~tion of func~~ composition of func	✓	$(f \circ g) \circ h = f \circ (g \circ h)$	$I(x)$, $f \circ I = f$	$f \circ f^{-1} = I$	$g \circ f \neq f \circ g$	Group

WB 42 Set of all 2x2 matrices of the form $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ wrt to multiplication	$\begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a_2 & b_2 \\ 0 & 0 \end{bmatrix}$ $= \begin{bmatrix} a_1a_2 & a_1b_2 \\ 0 & 0 \end{bmatrix}$	✓	$\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Identity matrix will not exist			Semi group
WB 16 $(Z, *)$ $n * m = \max(n,m)$	✓	✓	×			Semi group

HO

Q1 $(R - \{1\}, *)$

$a * b = a + b + ab$

Closure ✓

Associative ✓

$(a*b)*c$	$a*(b*c)$
$(a+b+ab)*c$	$a*(b+c+bc)$
$a+b+ab+c+ac+bc+abc$	$a+b+c+bc+ab+ac+abc$

Identity ✓

$a * e = a$

$\cancel{a} + e + ae = \cancel{a}$

$e(1+a) = 0$

$e = 0$

Inverse ✓

$a * b = e$

$a + b + ab = 0$

$b(1+a) = -a$

$b = \left(\frac{-a}{1+a}\right)$

Commutative ✓

$\therefore$ It is abelian group

2) HO

Sol: $(R - \{0\}, \circ)$ $\quad a \circ b = \frac{ab}{5}$

Closure ✓

Associative ✓

$(a \circ b) \circ c$	$a \circ (b \circ c)$
$\left(\frac{ab}{5}\right) \circ c$	$a \circ \left(\frac{bc}{5}\right)$
$\frac{abc}{25}$	$\frac{abc}{25}$

Identity

$a \circ e = a$

$\frac{ae}{5} = a$

$e = 5$

Inverse : $a \circ b = e$

$\frac{ab}{5} = 5$

$b = \frac{25}{a}$

Commutative ✓ Hence it is abelian

Q3>

Sol: $x \oplus y = x^2 + y^2$

Commutative ✓

$x \oplus y = x^2 + y^2$ $\quad$ $y \oplus x = y^2 + x^2$

Associative

$(x \oplus y) \oplus z$ $\quad$ $x \oplus (y \oplus z)$

$(x^2+y^2)^2 + z^2 \neq x^2 + (y^2+z^2)^2$

Modulo m

$a +_m b$ = remainder when m divides $a+b$

$a \times_m b$ = remainder when m divides $a \times b$

~~Def~~ Ex: $2 +_3 5 = 7 \bmod 3 = 1$

$3 +_6 7 = 10 \bmod 6 = 4$

$2 \times_3 5 = 10 \bmod 3 = 1$

$3 \times_7 4 = 12 \bmod 7 = 5$

Ex: $(G = \{0,1,2,3\}, +_4)$

we use concept of

Composition table

Composition Table

Except Associativity we can verify all other properties

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Closure

Elements of table $\in G$

Associative

$a +_m (b +_m c) = (a +_m b) +_m c$

Identity $\quad a \times e = a$

0 is identity

Check column should repeat in the row in same order

Inverse: $a \times b = e$

Every row should contain unique identity

Commutative: $A^T = A$

row → column

column → row

Hence it is abelian group

Ex. ($G = \{0, 1, 2, 3\}$ $\times_4$)

$\times_4$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

e

Closure ✓
Associative ✓
Identity ✓ $e = 1$
Inverse → not exists

Ex. ($G = \{1, 2, 3\}, \times_4$)

$\times_4$	1	2	3
1	1	2	3
2	2	$\boxed{0}$	2
3	3	2	1

not an algebraic structure bcz 0 is present, which is not in G.

Ex. ($G = \{1, 2, 3, 4\}, \times_5$)

$\times_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

Closure ✓
Associ ✓
Identity ✓
Inverse ✓
Commutative ✓

Abelian group

Result

1) The set $G = \{0, 1, 2, \ldots m-1\}$ w.r.t to $+_m$ is an abelian group.

Identity → 0

Inverse of a → $(m-a) \bmod m$

$p_1 p_2 \ldots$ are prime no i.e. prime factorize m

no of proper subgroups

$m = p_1^{n_1} p_2^{n_2} p_3^{n_3} \ldots$

$= [(n_1+1)(n_2+1)(n_3+1)\ldots] - 2$ ← trivial sub groups

Ex: ($G = \{0, 1, 2, \ldots 23\}, +_{24}$)

Abelian group

Identity → 0
inverse 11 → 13
inverse 17 → 7
inverse 0 → 0

subgroups
$24 = 2^3 \times 3^1$
$(3+1)(1+1) = 12$
$12 - 2 = 8$

☆ **Note:** In a group Identity is self inverse (i.e it is inverse of itself)

2) The set $G = \{1, 2, 3, \ldots p-1\}$ w.r.t to $\times_p$ is abelian group when p is prime

Identity is 1

Ex: $G = \{1, -1, i, -i\}$ Fourth roots of unity

(G, ·)

·	1	−1	i	−i
1	1	−1	i	−i
−1	−1	+1	−i	i
i	i	−i	−1	+1
−i	−i	[illegible]	[illegible]	−1

abelian group

Associative
bcz complex multiplication is associative

Subgroup

Def: A non-empty subset S of a group $(G, *)$ is a Subgroup of G if S w.r.t to $*$ opⁿ ie $(S, *)$ is itself a group

Result

(verify closure & inverse)

1) A non-empty subset S of a group $(G, *)$ is a Subgroup of G iff $\underline{a*b^{-1}} \in S \ \forall a,b \in S$

2) A non-empty subset S of a finite group $(G, *)$ is a Subgroup of G iff $a*b \in S \ \forall a,b \in S$

(verify closure)

Ex:- $(G = \{1, -1, i, -i\}, \cdot)$

$H_1 = \{1, -1\}$

closure property verified here

$\cdot$	1	-1
1	1	-1
-1	-1	1

$(H_1, \cdot)$ is a Subgroup

$H_2 = \{1, i\}$

$\cdot$	1	i
1	1	i
i	i	-1

$-1 \to$ not $\in H_2$

Hence not Subgroup

Order of Group

$O(G)$ = no of elements in G

Result

3) Lagrange's Theorem (Necessary Condition)

If H is subgroup of G then $\underbrace{O(H) \mid O(G)}_{O(H) \text{ divides } O(G)}$

Note: The condition is not sufficient

Ex: $O(H_2) \mid O(G)$ but H_2 is not Subgroup

ie If $O(H) \nmid O(G)$ then H cannot be Subgroup of G

4) Let H_1 & H_2 be two Subgroups of G

(1) $H_1 \cap H_2$ is subgroup of G

(2) $H_1 \cup H_2$ need not be a subgroup of G

Ex: $\langle G = \{1,3,5,7\}, \otimes_8 \rangle$
$H_1 = \{1,3\}$ $H_2 = \{3,5\}$
$H_1 \cup H_2$ is not subgroup

HO

Q7

Sol: $O(G) = 15$ L Subgroup G

$O(L) \geq 4$ $L \neq G$

$\therefore O(L)$ should be 5 bcz $O(L) \mid O(G)$

WB

Q4x $O(A) = 4$ $\quad$ $O(B) = 5$

$O(A \cap B) =$ ______

$$\begin{tikzcd}[column sep=small]
 & G \arrow[dl, dash] \arrow[dr, dash] & \\
A \arrow[dr, dash] & & B \arrow[dl, dash] \\
 & A \cap B &
\end{tikzcd}$$

$O(A \cap B) \mid O(A)$ $\quad\Big|\quad$ $O(A \cap B) \mid O(B)$

ie $HCF(4, 5) = 1$

Def: Let G be additive group & e is identity

The smallest positive integer m such that

$$m \cdot a = e$$

is order of a. We write $O(a) = m$

Def: Let G be multiplicative group & e is identity

The smallest positive integer m such that

$$a^m = e$$

is order of a. We write $O(a) = m$

Def: An element $a \in G$ such that $\cancel{O(\ )}$ $O(a) = O(G)$

is called generator of the group

Def: A group having at least one generator is called cyclic group.

Result: Let G be a cyclic group of order n

No of generators of G = $\phi(n)$

Note: (1) $\underbrace{a + a + a + \cdots + a}_{m} = m \cdot a$

(2) $\underbrace{a \cdot a \cdot a \cdots \cdot a}_{m} = a^m$

Ex: $\langle G = \{1, -1, i, -i\}, \times \rangle$

$(G, \cdot)$ abelian group, 1 is identity

$1^1 = 1 \Rightarrow O(1) = 1$

$-1 \times -1 = 1 \Rightarrow (-1)^2 = 1$ $\quad$ $O(-1) = 2$

$i \times i \times i \times i \Rightarrow (i)^4 = 1$ $\quad$ $O(i) = 4$

$-i \times -i \times -i \times -i \Rightarrow (-i)^4 = 1$ $\quad$ $O(-i) = 4$

$O(G) = 4$

$O(i) = O(-i) = O(G)$ $\therefore$

$\therefore$ $i, -i$ are generators of Group

$\therefore$ G is cyclic

$i^1 = i$
$i^2 = -1$
$i^3 = -i$
$i^4 = 1$

all elements of the group generated

Hence i is generator

Ex:- $(G = \{0, 1, 2, 3\}, +_4)$ Abelian group

identity $= 0$

$0^1 = 0$ $\quad$ $O(0) = 1$

$1^4 = 1+1+1+1 = 0$ $\quad$ $O(1) = 4$

$2^2 = 2+2 = 0$ $\quad$ $O(2) = 2$

$3^4 = 3+3+3+3 = 0$ $\quad$ $O(3) = 4$

$O(1) = O(3) = O(G)$

$\therefore$ 1 & 3 are generators

$\therefore$ G is cyclic group

Ex: G is a cyclic group of order 24.

no of generators of G = ______

$\phi(24) = 2^3 \times 3$

$24\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)$

$24 \times \frac{1}{2} \times \frac{2}{3} = 8$