Linear Algebra

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Lecture 23

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Sums and Direct Sums

- **Definition:** Let U and W be subspaces of the vector space V. Then the sum of U and W, $U + W = \{u + w \ : \ u \in U, w \in W\}$. It is easy to see that $U + W$ is a subspace of V. In fact, $U + W$ is the smallest subspace of V containing U and W.
- **Definition:** V is said to be the direct sum of the subspaces U and W if every vector $v \in V$ is uniquely expressible in the form $v = u + w$, where $u \in U, w \in W$. We shall use the notation $V = U \oplus W$ to indicate that V is the direct sum of U and W.
- **Proposition 19:** If U and W are subspaces of the vector space V, then $V = U \oplus W$ if and only if $V = U + W$ and $U \cap W = \{0\}$.
  Proof: Left as an exercise.
- **Remark:** In some books, direct sum is defined as in Proposition 19, and then our definition is derived as a proposition.
- **Remark:** The subspace W in the above is often referred to as a **complement** or **complementary** subspace of U. It should be noted that there is nothing unique about complementary subspaces.

Dimension of the Sum

Proposition 20: If U and W are finite-dimensional subspaces of the vector space V, then
$\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$.

Proof: Put $X = U \cap W$ for convenience. If either U or W is $\{0\}$, then there is nothing to prove. The main idea is to use Prop 15 to construct a basis for $U + W$. Let $B = \{x_1, \ldots, x_k\}$ be a basis for X (note that X is finite dimensional subspace of U and W both by Prop 18). Of course, if X is $\{0\}$, then this step is not needed. Since $X \subseteq U$, we expand B to a basis B_1 for U by adjoining the vectors $u_1, \ldots, u_m$, i.e.
$B_1 = \{x_1, \ldots, x_k, u_1, \ldots, u_m\}$, where $k \geq 0, m > 0$. Similarly, expand B to a basis B_2 of W by adjoining the vectors $w_1, \ldots, w_n$, i.e.
$B_2 = \{x_1, \ldots, x_k, w_1, \ldots, w_n\}$, where $n > 0$. Put
$C = B \cup B_1 \cup B_2 = \{x_1, \ldots, x_k, u_1, \ldots, u_m, w_1, \ldots, w_n\}$. We claim that C is a basis for $U + W$. In order to prove this claim, we need to prove the following:

(i) Span $(C) = U + W$.

(ii) C is L.I.

Dimension of the Sum (Cont'd)

Proof that: Span $(C) = U + W$

Since $B \subseteq X \subseteq U \subseteq U + W$, $B_1 \subseteq U \subseteq U + W$ and $B_2 \subseteq W \subseteq U + W$, we see that $C \subseteq U + W$. Thus, Span $(C) \subseteq U + W$. To prove the other containment, take an arbitrary element $z = u + w \in U + W$, where $u \in U$ and $w \in W$. Now $u \in U$ can be written as:
$u = e_1x_1 + \cdots + e_kx_k + f_1u_1 + \cdots + f_mu_m$ and $w \in W$ can be written as:
$w = g_1x_1 + \cdots + g_kx_k + h_1w_1 + \cdots + h_nw_n$. Therefore,

$$z = (e_1 + g_1)x_1 + \cdots + (e_k + g_k)x_k + f_1u_1 + \cdots + f_mu_m + h_1w_1 + \cdots + h_nw_n,$$

i.e z is a linear combination of elements of C. Thus, $U + W \subseteq$ Span (C), which proves the other containment. Hence $U + W =$ Span (C).

Dimension of the Sum (Cont'd)

Proof that: the set C L. I.

Suppose:

$$a_1x_1 + \cdots + a_kx_k + b_1u_1 + \cdots + b_mu_m + c_1w_1 + \cdots + c_nw_n = 0 \quad (1)$$

$$\text{or } a_1x_1 + \cdots + a_kx_k + b_1u_1 + \cdots + b_mu_m = -c_1w_1 - \cdots - c_nw_n \quad (2)$$

Note that LHS $\in U$ and RHS $\in W$ and thus, this vector belongs to $U \cap W$. Hence we can write:

$$-c_1w_1 - \cdots - c_nw_n = d_1x_1 + \cdots + d_kx_k \quad (3)$$

$$\text{or } \; d_1x_1 + \cdots + d_kx_k + c_1w_1 + \cdots + c_nw_n = 0 \quad (4)$$

Since B_2 is LI, being a basis for W, we must have $d_1 = \cdots = d_k = c_1 = \cdots = c_n = 0$. Therefore (1) becomes

$$a_1x_1 + \cdots + a_kx_k + b_1u_1 + \cdots + b_mu_m = 0. \quad (5)$$

Dimension of the Sum (Cont'd)

Now, since B_1 is a basis for U, we must have
$a_1 = \cdots = a_k = b_1 = \cdots = v_m = 0$. Hence C is LI.
Finally, since $\dim U = k + m, \dim W = k + n$ and $\dim(U \cap W) = k$, we have:

$$\dim U + \dim W = |C| = k + m + n = \dim U + \dim W - \dim(U \cap W).$$

Corollary to Proposition 20: If V is the direct sum of the finite dimensional subspaces U and W, then
$\dim V = \dim(U \oplus W) = \dim U + \dim W$.
Remark: The above result is the analogue for finite-dimensional vector spaces of the familiar result about the cardinality of the union of finite sets

Some Examples

Example 1: Plane $\mathbb{R}^2$

- Consider the subspaces of $\mathbb{R}^2$, say U, W. Note that the only possibility for a proper subspace U is $\dim U = 1$, i.e. it has a basis consisting of one (non-zero vector), say u_1. Geometrically, $u_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ corresponds to the point (x_1, y_1) in the plane or more specifically, the directed line segment joining $(0, 0)$ to (x_1, y_1).
- Taking a particular case, say $u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

  What does U corresponds to:

  Algebraically, $U = \{ku_1 \ : \ k \in \mathbb{R}\}$. Geometrycally, U is a line passing through the origin and $(1, 1)$.
- In fact, any 1-dimensional subspace corresponds to a line through the origin, and conversely.

Some Examples (Cont'd)

- Now, suppose u_2 is a vector which is not a scalar multiple of u_1. Therefore, u_1, u_2 are LI. Thus, $\{u_1, u_2\}$ is necessarily a basis, i.e. any vector u can be uniquely expressed as a linear combination of u_1 and u_2.
- If $U = \text{Span}\{u_1\}$, $W = \text{Span}\{u_2\}$, then $\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$ i.e. $2 = 1 + 1 - \dim(U \cap W)$ and therefore, $\dim(U \cap W) = 0$, i.e. $U \cap W = \{0\}$. One can also verify this geometrically.
- Algebraically, we can see something from a different perspective. Since u_1 and u_2 are linearly independent, the equation $c_1u_1 + c_2u_2 = 0$ holds only for $c_1 = c_2 = 0$. Hence the homogeneous system $AX = 0$, where $A = [u_1, u_2]$, the matrix with u_1 and u_2 as its columns, has only the trivial solution. Hence, by TIMT, A is invertible.

  Also the system $AX = b$ has a solution (actually unique) for every $b \in \mathbb{R}^2$, i.e. every $b \in \text{Span}\{u_1, u_2\}$, i.e. $\{u_1, u_2\}$ is a basis.

Some Examples (Cont'd)

Example 2: Plane $\mathbb{R}^3$

- Let us consider subspaces of $\mathbb{R}^3$, say U, W.
- **Case 1:** $\dim U = 1$
  Then U has a basis consisting of a single vector, say, u_1. As before, $u_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ corresponds to the directed line segment joining $(0, 0, 0)$ and (x, y, z).
- **Case 2:** $\dim U = 2$; then U has a basis consisting of two linearly independent vectors u_1 and u_2, i.e. $U = \text{Span}\{u_1, u_2\}$.
  Then U corresponds to a plane through origin, i.e. corresponds to the geometrical fact that two intersecting lines in $\mathbb{R}^3$ determine a plane. It is obvious that that Span $\{u_1\}$ and Span $\{u_2\}$ intersect at the origin. We can see this using coordinate geometry also.

Some Examples (Cont'd)

- Let us take a plane through the origin, say $x + y + z = 0$. This corresponds to the linear system $AX = 0$ where $A = [1, 1, 1]$. Since A is already an RREF matrix, we solve the corresponding homogenous system:
  $x_1 = -x_2 - x_3$
  $x_2 = x_2$
  $x_3 = x_3$
  to get $X = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, i.e. the plane corresponds to the span of the two linearly independent vectors
  $u_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $u_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

Some Examples (Cont'd)

- Conversely, suppose we take any vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in the span of the linearly independent vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$.

  Thus,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} \tag{6}$$

  Solving for c_1 and c_2 in terms of x and y, i.e. solving the system $\begin{pmatrix} 1 & 1 & : & x \\ 2 & 3 & : & y \end{pmatrix}$ and RREF of this system or matrix is $\begin{pmatrix} 1 & 0 & : & 3x - y \\ 0 & 1 & : & y - 2x \end{pmatrix}$ .i.e $c_1 = 3x - y, c_2 = y - 2x$. Substituting in the third row of system (6), we get $z = 3c_1 + 5c_2 = -x + 2y$, equation of the plane through origin.

Some Examples (Cont'd)

- What about intersection of two planes passing through origin. i.e. two 2-dimensional subspaces U and W.

  We have that

  $$\dim(U+W) = \dim U + \dim W - \dim(U \cap W) = 4 - \dim(U \cap W) \quad (7)$$

  Now, $U + W$ has U and V as subspaces, so $\dim(U + W) = 3$ or 2.

  Thus $\dim(U + W) = 3$ makes (7) into $3 = 4 - \dim(U \cap W) \implies \dim(U \cap W) = 1$, i.e. $U \cap W$ is a line (1-dimensional subspace through the origin).

  Note: If $\dim(U + W) = 2$, then $U + W = U = W \implies U$ and W were actually the same subspace, and (7) becomes $2 = 2 + 2 - 2$, which is obviously true.

Quick Summary

- Concepts: span, linear dependence/independence, basis, dimension
- If V is a finite-dimensional space, U, W subspaces:
  - $|\text{any LI set}| \leq |\text{any spanning set}|$ (Prop 12)
  - Any two bases have the same number of vectors = dimension of the space (Prop 13)
  - Any LI set can be expanded to a basis (Prop 15)
  - Any spanning set can be contracted to a basis (Prop 16)
  - If U is a proper subspace, then $0 < \dim U < \dim V$ (Prop 18)
  - $\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$ (Prop 20)