Linear Algebra and Applications

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Lecture 04

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- You can reduce a matrix to different echelon forms depending on what row operations you use. But there is **only one reduced echelon form** for any matrix.
- **Theorem:** Each matrix is row equivalent to one and only one reduced echelon matrix.

Suppose you have a matrix in echelon form and want to go to reduced echelon form. You already know which entries will be your leading entries. They will be the same in any echelon form and give you your leading $1'$s in reduced echelon form.

- **Definition:** A **pivot position** for a matrix A is a position in A that will contain a leading 1 in the reduced echelon form of A. A **pivot column** of A is one that has a pivot position.
- If A has echelon form

$$\begin{bmatrix} \blacksquare & \bigstar & \bigstar & \bigstar \\ 0 & 0 & \blacksquare & \bigstar \\ o & 0 & 0 & \blacksquare \end{bmatrix}$$

where $\blacksquare$ =any nonzero value and $\bigstar$ =any value (can be 0). Here $\blacksquare$ represents pivot positions and columns $1, 3, 4$ are pivot columns.

The Row Reduction Algorithm (Gauss-Jordan Elimination)

At the start, move all zero rows to the bottom of the matrix using interchange operations, since they will not play any further role.

Goal: To reduce a matrix first to echelon form and then, if desired, to reduced echelon form.

1. Start with the **leftmost** nonzero column. This is a pivot column.
2. Pick a nonzero entry in the column as your pivot and if necessary move this row to the top to put it in the pivot position.
3. Use row replacement to zero out entries below the pivot.
4. Ignore the pivot row and any rows above it. Apply steps $1-3$ to the sub matrix that remains. Repeat untill there are no more nonzero rows to modify. (**Steps** 1 **to** 4 **constitute the forward phase, which produces a matrix in echelon form**– this portion is referred to as Gaussian Reduction).
5. Use scaling operations to make all the pivot elements 1.
6. Starting with the **right-most** pivot, create zeroes in the entire column above it. Repeat this step moving leftward and upward. **(Step 5 and 6 constitute the backward phase, which produces an RREF)**

Conclusion

- **Definition:** If A and B are $m \times n$ matrices, we say that B is row equivalent to A if B can be obtained from A by a finite sequence of row operations.
- **Proposition 1:** Given any $m \times n$ matrix A, there exists an RREF matrix which is row-equivalent to A.
  Proof: The proof is supplied by the above algorithm, i.e. we have given a constructive proof, rather than a pure existence proof.
- **Proposition 2:** Row equivalence is an equivalence relation on the set $\mathbb{R}^{m \times n}$ of $m \times n$ matrices with entries from the field $\mathbb{R}$ of real numbers.
- **Note:** Later on we will occasionally work with the field of complex numbers $\mathbb{C}$. Proposition 2 will continue to hold with $\mathbb{R}$ replaced by $\mathbb{C}$.
- **Remark 1:** Recall that every equivalence relation induces a partition of the underlying set, the parts of the partition being the equivalence classes, i.e. the equivalence classes are pairwise disjoint subsets whose union is the whole set.

- **Remark 2:** In fact, the RREF matrix of any given matrix is unique, i.e. a matrix cannot be row- equivalent to two distinct RREF matrices. Alternatively, two distinct RREF matrices cannot be row-equivalent to each other.
- **Concluding Remark:** So, inside each equivalence class for this equivalence relation, there is a distinctive member, i.e. the one and only RREF matrix in it. This fact can be used to determine whether two matrices are row-equivalent to each other.