Linear Algebra

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Lecture 14

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Sequences

Definition: A sequence of real numbers is a function $a : \mathbb{N} \longrightarrow \mathbb{R}$.

Notation: We usually write a_n for the image of n under a, rather than $a(n)$. The values a_n are often called the elements of the sequence. To make a distinction between a sequence and one of its values it is often useful to denote the entire sequence by $(a_n)_{n=1}^{\infty}$, or just $< a_n >=< a_1, a_2, \ldots, a_n, \cdots >$.

- The nth term a_n is the general term of the sequence $< a_n >$. At times, the general term is given by a formula. For example, $a_n = 1/n$ or $< \frac{1}{n} >$–in this sequence , the nth term is $\frac{1}{n}$.
- The sequence can also be indicated by a pattern, e,g. $s =< 1, 2, 3, 1, 2, 3, 1, 2, 3, \cdots >$. Here the pattern is easily understood, but not so easy to give via a formula.
- As can been seen that terms in a sequence need not be distinct.
- Sequences arise in many mathematical and scientific situations and applications.

Sequences (Conti . . .)

- We will use the notation $\mathbb{R}^\infty$ for the set of all sequences with real terms. Similarly, $\mathbb{C}^\infty$ would be the set of all sequences with complex terms. We will breifly look into two aspects of sequences: algebraic and convergence.
- **Algebra of Sequences:** All sequences under consideration will be in $\mathbb{R}^\infty$. But the approach is similar for $\mathbb{C}^\infty, \mathbb{Q}^\infty$ etc. We will define the following algebraic operations on $\mathbb{R}^\infty$. If $< a_n >, < b_n >$ are sequences in $\mathbb{R}^\infty$ and $c \in \mathbb{R}$, then:
  1. Addition: $< a_n > + < b_n > = < a_n + b_n >$, i.e. the sequence whose general term is $a_n + b_n$.
  2. Scalar multiplication: $c < a_n > = < ca_n >$, i.e. the sequence whose general term is ca_n.
  3. Multiplication: $< a_n >< b_n > = < a_n b_n >$, i.e. the sequence whose general term is $a_n b_n$.

Remark: With respect to operations in (i) and (ii) above, $\mathbb{R}^\infty$ is a vector space over the field $\mathbb{R}$.

Sequences (Conti . . .)

- **Convergence of Sequences:** Convergence is concerned with the behaviour of a sequence as n gets very large. A sequence is said to converge, if there exists a real number L such that the terms of the sequence lie <u>ultimately</u> in any interval about L, however small.
- More formally, the sequence (a_n) is said to converge to a real number L if for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for all $n \geq n_0$.
- L is called the limit of the sequence. It is easy to see that any convergent sequence has precisely one limit.
- The phrases: $< a_n >$ is convergent or L is the limit of $< a_n >$ or $< a_n >$ converges to the limit L means exactly the same thing.
- Notation: $\lim\limits_{n \to \infty} a_n = L$ or $a_n \to L$ as $n \to \infty$
- A sequence which is not convergent is said to be divergent or we can say that the sequence diverges.

Sequences (Conti . . .)

- **Examples:** Example of a convergent sequence:

$$< \frac{1}{n} > \to 0$$

  Examples of Divergent Sequences:
  $< n >$ or $< 1, 2, 3, \cdots >$
  $< 1, 2, 1, 2, 1, 2, \cdots >$
  As the above examples illustrate, divergence can occur because the terms in the sequence are unbounded or because of oscillatory behaviour (or a mixture of both).

- The following basic results about limits are presented without proof. They follow from the basic idea that $a_n \to L$ and ϵ is any positive real number, however, small, the the terms of $< a_n >$ lie ultimately in the small interval $(L - \epsilon, L + \epsilon)$ i.e. there exists +ve integer k such that all the terms $a_k, a_{k+1}, a_{k+2}, \ldots,$ etc must lie in the interval $(L - \epsilon, L + \epsilon)$. Proofs can be found in standard textbook of calculus, advanced calculus, or analysis.

Sequences (Conti . . .)

Proposition: Suppose $< a_n > \to L_1, < b_n > \to L_2$ and $c \in \mathbb{R}$. Then:

1. (i) $< a_n > + < b_n >$ is convergent and $< a_n + b_n > \to L_1 + L_2$.
2. (ii) $c < a_n >$ is convergent and $< ca_n > \to cL_1$.
3. (iii) $< a_n >< b_n >$ is convergent and $< a_n b_n > \to L_1 L_2$.
4. (iv) If $a_n \neq 0$ for all n and $L_1 \neq 0$, then the sequence $< \dfrac{1}{a_n} >$ is well-defined and $< \dfrac{1}{a_n} > \to \dfrac{1}{L_1}$.

We use c (italic) to denote the subset of $\mathbb{R}^\infty$ that consists of all convergent sequences. Using (i) and (ii) above, we can show that c is a (vector) subspace of $\mathbb{R}^\infty$. Using Proposition 8 (to be stated in next few slides!):

1. The zero sequence $< 0 > = < 0, 0, \cdots >$ is convergent, i.e. $< 0 > \in c$.
2. Closed under addition follows from (i).
3. Closed under scalar multiplication follows from (ii).

Sequences (Conti . . .)

- In principle, it is quite difficult to show that a sequence is convergent, since we have to first identify a possible limit L and then verify that L is indeed the limit. Hence, results that show a sequence is convergent without necessarily finding the limit are very useful. One of the most useful such results is:
- **Proposition**: Suppose $< a_n >$ is a monotonically non-decreasing sequence in $\mathbb{R}^\infty$ which is bounded above, i. e. $a_n \leq a_{n+1}$ for all n and $a_n < M$ for some fixed real number M for all n. Then: $< a_n >$ is convergent.
- **Example:** Consider the sequence $< a_n >$ where $a_n = \left(1 + \dfrac{1}{n}\right)^n$. It can be shown that $a_n \leq a_{n+1}$ for all n and $a_n < 3$ for all n. Therefore the sequence is convergent by above proposition.
  Its limit is denoted by e, the real number that plays a major role in mathematics.

Subspaces

Motivation: We may have noticed from the various examples that many of the examples of vector spaces were in fact subsets of each other:

- The space $\mathbb{R}_1[t]$ is a subset of $\mathbb{R}_2[t]$ which is a subset of $\mathbb{R}_3[t]$, etc. Moreover, all of these are subsets of the space $\mathbb{R}[t]$ of all polynomial functions on $\mathbb{R}$: $\mathbb{R}_1[t] \subseteq \mathbb{R}_2[t] \subset \mathbb{R}_3[t] \subseteq \cdots \subseteq \mathbb{R}[t]$.
- The space c of convergent real sequences is a subset of $\mathbb{R}^\infty$ the space of all real sequences.

Definition (Subspace): Let V be a vector space over a field $\mathbb{F}$. A (vector) subspace of V is a non-empty subset W of V which is itself a vector space over $\mathbb{F}$ with the operations of vector addition and scalar multiplication taken from V.