Linear Algebra

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Lecture 20

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Fundamental Results (Conti . . .)

Proposition 13: If V is a finite-dimensional vector space, then any two bases of V have the same number of elements.

Proof: Suppose B_1 and B_2 are two distinct bases of V such that $|B_1| = m$ and $|B_2| = n$. Then by Proposition 12(a), $|B_1| \leq |B_2|$ i.e. $m \leq n$, since B_1 is L. I. and B_2 is spanning set. In a similar way, $B_2 \leq B_1$ i. e. $n \leq m$. Hence we get: $m = n$.

- **Definition:** The dimension of a finite-dimensional space is the number of elements in a basis for V. This is written dim V.
- **Remark:** Proposition 13 ensures that this is a proper definition.
- **Special Case:** The zero subspace $\{0\}$ is defined to have dimension 0. However, it does not have a basis. So our insistence that $\dim\{0\} = 0$ amounts to saying that the **empty** set of vectors is a basis of $\{0\}$. Thus the statement that "the dimension of a vector space is the number of vectors in any basis" holds even for zero space.

Examples

- $\dim \mathbb{R}^n_{\mathbb{R}=n}$
- $\dim \mathbb{R}_{\mathbb{R}} = 1$ and any non-zero real number will act as a basis, e.g., $\{1\}, \{-8\}, \{\sqrt{5}\}, etc.$
- $\dim \mathbb{C}_{\mathbb{C}} = 1$ and any non-zero complex number will act as a basis, e.g., $\{1\}, \{i\}, \{1+i\}, \{5-7i\} etc.$
- $\dim \mathbb{Q}_{\mathbb{Q}} = 1$ and any non-zero rational number will act as a basis, e.g., $\{1\}, \{a \ : \ a \neq 0 \text{ and } a \in \mathbb{Q}\}$.
- $\dim \mathbb{F}_{\mathbb{F}} = 1$ and any non-zero element in $\mathbb{F}$ will act as a basis, e.g., $\{1\}, \{a \ : \ a \neq 0 \text{ and } a \in \mathbb{F}\}$.
- $\dim \mathbb{C}_{\mathbb{R}} = 2$ and the set $\{1, i\}$ is one of its basis.
- $\dim \mathbb{R}^{m\times n}_{\mathbb{R}} = mn$, and one basis consists of all $m \times n$ matrices with exactly one entry equal to 1 and all other entries equal to 0. We call this **standard basis** for $\mathbb{R}^{m\times n}$
- $\dim \mathbb{R}_n[t]_{\mathbb{R}} = n+1$, and the set $\{1, t, t^2, \ldots, t^n\}$ is one of its bases generally known as standard basis for $\mathbb{R}_n[t]$.