Revision of Test (After Mid Sem)

→ $e(A)$ $A \xrightarrow{e} e(A)$ $e^{-1}(e(A)) = A$

elementary opn are reversible

$A \sim B$

↓

$A(RREF) = B(RREF)$

Prop1: given any $m \times n$ matrix A, there exists RREF matrix which is $(RREF \sim A)$

Prop2: row equivalence relation is an equivalence relation on set $R^{m \times n}$ each entry is from set of $\mathbb{R}$.

$AX = 0 \sim RX = 0$

non-zero rows = # variables
only trivial soln

non-zero rows < # variables
we will get free variables
non-trivial soln.

Prop 3: If A is square matrix $n \times n$, then $A \sim I$ iff $AX = 0$ has only trivial soln

Prop4: for NHSLE $AX = B \sim RX = B$
R

$RX = B$ has is consistent iff right most column of R is not pivot column

ie there is no row of the form

$[0\,0\,0 \ldots\ldots b]$.

Invertible matrix $AB = BA = I$

$(A^{-1})^{-1} = A$ $(AB)^{-1} = B^{-1}A^{-1}$

Observation: product of invertible matrices is also invertible

$C = A_1 A_2 A_3 \ldots A_n$

$C^{-1} = A_n^{-1} A_{n-1}^{-1} \ldots\ldots A_2^{-1} A_1^{-1}$

Elementary matrix obtained by applying exactly single elementary row opn on identity matrix.

Prop 5: $A \xrightarrow{e} e(A) \longleftarrow EA$

$I \xrightarrow{e} e(I) = E$

$e(A) = EA$ $\forall$ $m \times n$ matrix A

e f $eI = E$ $(FE) = (FE)I = F(EI)$
$fI = F$ ~~BA~~ = $F(eI)$
$f e I = $

Prop6: Every elementry matrix E is invertible And E^{-1} is an elementry matrix.
of same type

TIMT
give A square matrix $n \times n$

i) A is invertible.
ii) $A \sim I$
iii) $AX = 0$ has only trivial soln
iv) $AX = b$ has soln $\forall\ b \in R^n$

Corollary

Cor 1.1 :- given A invertible matrix
$$A = \underbrace{E_1 E_2 \dots E_n}_{\text{product of elementry matrices}}$$
$$A \xrightarrow{\text{row opn}} I$$

same row opn $I \longrightarrow A^{-1}$

Cor 1.2 :- If A has either left inverse or right inverse then it has an inverse.

Cor 1.3 :- If A can be factored into product of product of square matrices, then
$$A = A_1 A_2 \dots A_n$$
invertible iff each A_i is invertible

Cor 1.4 :- matrix A is invertible iff $AX = b$ has soln $\forall\ b \in R^n$

$B = [v_1 v_2 \dots v_n]$

$$v_1 = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix} \dots\dots v_n = \begin{bmatrix} b_{1n} \\ b_{2n} \\ \vdots \\ b_{nn} \end{bmatrix}$$

$AB = [Av_1\ Av_2 \dots Av_n]$

$a \Rightarrow c \Rightarrow b \Rightarrow a$ $a \Leftrightarrow b$

Cor 1.5 : RREF of every matrix is unique

$Z_p = \{0, 1, 2, \dots p-1\}$
$Z_p^{\times} = \{Z_p - \{0\}\}$
} p is a prime no

$(S, *) \rightarrow$	Algebraic Structure
Closure	
Associativity	Semi group
Identity	Monoid
Inverse	Group
Commutative	Abelian Group

i) $\forall\ a, b \in S$ $a * b \in S$
ii) $\forall\ a, b, c \in S$ $a * (b * c) = (a * b) * c$
iii) $\forall\ a \in S$ $\exists\ e \in S$ s.t $a * e = e * a = a$
iv) $\forall\ a \in S$ $\exists\ b \in S$ s.t $a * b = b * a = e$.
v) $\forall\ a, b \in S$ $a * b = b * a$

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Revise properties Lect-10
of Groups, Composition table - Lect-11

$(\mathbb{Z}_n, +_n) \rightarrow$ abelian group

$\mathbb{Z}_n = \{0, 1, 2, \ldots n-1\}$

$\mathbb{Z}_n^{\times} = \{\cancel{1, 2, 3, \ldots n-1}\}$ all no. relatively prime to n.

$(\mathbb{Z}_n^{\times}, \times_n) \rightarrow$ is a group.

Fields

Non-empty set along with op^n (+) and (×) ~~has~~ satisfying following axioms

1) $\langle F, + \rangle \rightarrow$ abelian group
2) $\langle F^{*}, \times \rangle \rightarrow$ abelian group
3) Distribution law (Mul distribute over addition)
$\forall x, y, z \in F$
$x \cdot (y+z) = x \cdot y + x \cdot z$
$\cancel{(x+y) \cdot z = (x+z) \cdot (y+z)}$

$M^{2\times 2} \rightarrow$ not field, $R^{2\times 2} \rightarrow$ not fields
bcz Commutative property not satisfied

Extended Euclidian

$\gcd(a,b) = d \quad \exists X, Y$ s.t $aX + bY = d$.

$a +_n b = (a+b) \bmod n$
$a \times_n b = (ab) \bmod n$

$a+b = qn + r$
$(a+b) - qn = r$

$\mathbb{Z}_2 = \{0, 1\}$ Smallest field
$\mathbb{Z}_3 = \{0, 1, 2\}$ field

Zero divisor :

A field can not have zero-divisors.

$\cancel{\mathbb{Z}_p^{\times}}$ $\mathbb{Z}_n$ is field iff n is prime.

Vector Space

→ Collection of objects which ~~who~~ are not necessarily vectors but behave just like vectors.
→ they exhibit vector addition & scalar multiplication

Vector addition
i) $\forall u, v \in V \quad u+v \in V$
ii) $\forall u, v, w \in V \quad u+(v+w) = (u+v)+w$
iii) $\forall u \in V \quad \exists \vec{0} \in V$ s.t $u + \vec{0} = \vec{0} + u = \vec{0}$
iv) $\forall u \in V \quad \exists (-u) \in V$ s.t $u + (-u) = (-u) + u = \vec{0}$
v) $\forall u, v \in V \quad u+v = v+u$.

Scalar Multiplication

→ $\forall u \in V,\ \alpha \in \mathbb{F} \therefore \alpha . u \in V$

→ $\alpha, \beta \in \mathbb{F} \quad \forall u \in V \quad$ ~~$\alpha(\beta v) = (\alpha\beta) v$~~

$\alpha(\beta u) = (\alpha\beta) u$

→ $u, v \in V \quad \alpha \in F$

$\alpha(u+v) = \alpha u + \alpha v.$

→ $\alpha, \beta \in \mathbb{F} \quad \forall u \in V$

$(\alpha+\beta) u = \alpha u + \beta u$

→ $1 . u = u \quad \forall u \in V \quad 1 \in \mathbb{F}$

R^n → vector space

$R^{m \times n}$ → m×n matrices.

$C[0,1] = \{ f :$ continuous functions $: f : [0,1] \to R \}$

R^∞ → set of all real seq^n

$R^\infty = \{ \langle a_n \rangle : \langle a_n \rangle$ is seq^n with real no's $\}$

$R_n[t]$

$R_R,\ C_C,\ Q_Q$

$R_R,\ R_Q,\ (R_C \times)$

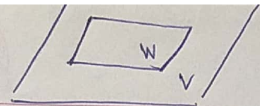
$C_C,\ C_Q,\ C_R$ ✓

$Q_Q,\ (Q_R),\ (Q_C \times)$

×

bcz if we take $\sqrt{2} \in R$ → irrational

W subspace V



$\{a_n\}_{n=1}^{\infty}$ $\quad R_R^\infty$

$\langle a_n \rangle + \langle b_n \rangle = \langle a_n + b_n \rangle$ $\quad C_R^\infty$ ↓

$c \langle a_n \rangle = \langle c a_n \rangle$

$\langle a_n \rangle \langle b_n \rangle = \langle a_n . b_n \rangle$ $\quad$ $\boxed{\text{set of all convergent seq}^n}$

$C^\infty \subseteq R^\infty$

Subspace $\quad$ W is subspace V if

i) $\vec{0} \in W$

ii) $\forall u, v \in W \quad u+v \in W$

iii) $\forall u \in W \quad \alpha \in \mathbb{F} \quad \alpha w \in W$

Imp

Prop 1: A non-empty subset W of V is called vector space iff ~~every element in W can be expressed as~~

$\forall u, v \in W \quad \forall c \in \mathbb{F} \quad cu + v \in W.$

Above 2 tests are equivalent

~~V_R $V_{\mathbb{F}}$ $\{\vec{0}\}$ → zero~~

$\{\vec{0}\} \subseteq V_{\mathbb{F}}$

↓

zero subspace

All Subspaces other than V & $\{\vec{0}\}$ are called proper subspaces.

~~$R^2 = \{(x,y) \mid x+y=0\}$~~

$X = \{(x,y) \in R^2 : x+y=0\}$

subspace R^2

Sym (Square) Subspace $R^{m\times n}$

$\mathbb{R}^2$ not subspace $\mathbb{R}^3$

if U, W are subspaces of V.

then $U \cap W$ is subspace

then $U \cup W$ is not subspace

Ex:- $U = \{(x,0) \mid \forall x_i \in R\}$
$W = \{(0,y) \mid \forall y_i \in R\}$

U subspace $R^2_{\mathbb{R}}$ W subspace $R^2_{\mathbb{R}}$

$U \cap W = \{(0,0)\}$

$U \cup W$ is not subspace

$u \in U$ $u = \{1,0\}$
$w \in W$ $w = \{0,1\}$

$u+w = \{1,0\}+\{0,1\} = \{1,1\} \notin U \cup W$

Hence addition property not satisfied.

U, W $(U \cup W)$ is a subspace iff $U \subseteq W$ or $W \subseteq U$.

Pf: $\Rightarrow$ given $U \cup W$ is subspace V.
then $U \subseteq W$ or $W \subseteq U$

assume $U \not\subseteq W$ and $W \not\subseteq U$
this means that there are elements

$x \in U \setminus W$ && $y \in W \setminus U$.

$\because$ $U \cup W$ is subspace, Hence ~~$x+y \in$~~
hence they must satisfy addition closure

Hence, $x+y \in U \cup W$.

We have either $x+y \in U$ or $x+y \in W$.

I case Let $x+y \in U$.

we can write $y = (x+y) - x$

Now $(x+y) \in U$ && $x \in U$, hence their difference must also belong to U.

hence $y \in U$, this contradicts the fact that $y \in W \setminus U$.

Prop: Span(S) is the smallest subspace that contains S.

II case $x+y \in W$.

We can write $x = (x+y)-y$

$(x+y) \in W$ && $y \in W$ ~~therefore~~ therefore their difference must also $\in W$.
hence $x \in W$.
this contradict the fact that $x \in U \setminus W$.

Hence our assumption is wrong, ~~hence~~ hence either $U \subseteq W$ or $W \subseteq U$.

Pf: ($\Leftarrow$) given ~~to~~ $U \subseteq W$ or $W \subseteq U$.

If $U \subseteq W$ then $U \cup W = W$ which is subspace of V.

If $W \subseteq U$ then $U \cup W = U$ which is subspace of V.

Prop 10: If $S = \{v_1 v_2 \dots v_p\}$ is a set of vectors in a vector space V, then Span(S) is a subspace of V.

~~Remark 1~~:- 91.

Given a set of finite vectors from vector space V

$S = \{v_1 v_2 \dots v_p\}$

Remark 4: A list of vector is L.D, if one of the vectors (at ~~least~~ least) can be represented as linear combination of other

Lect - 18

Remark 1:- If any set of finite vectors contain $\vec{0}$ (zero vector) then it is always linearly dependent

$\vec{0}$ (zero vector is always L.D)

Remark 2:- A single non-zero vector is always L.I (Linearly Independent)

Remark 3: A List of 2 non-zero vectors is L.D if one of ~~them~~ the vectors is scalar multiple of the other.

Remark 5: Any list which contain repeated vectors must be L.D

Important:

Superset of L.D set is always L.D.
Subset of L.I set is always L.I.

Basis & Dimension (Important)

Basis L.I spanning set of vectors.

If $B \subseteq V_F$

i) B is L.I ii) Span(B) = V.

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Ex: R^n — standard basis

Basis = $\{e_1, e_2, \ldots, e_n\}$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \ldots \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Ex: given vectorspace $V = R[t]$. set of all polynomials with real coefficients.

$R[t]$ has infinite dimension (pf BWOC)

BWOC = by way of contradiction

Prop 11: $B = \{v_1, v_2, \ldots v_p\}$ is a basis of vector space V iff each vector $v \in V$ can be uniquely expressed as linear combination of elements of B.

Prop 12: Steinitz Exchange Lemma.

Suppose $v_1, v_2, \ldots, v_n$ are L.I vectors in vector space V, and suppose,

Span $\{w_1, w_2, \ldots w_p\} = V$.

a) $n \leq m$

L.I set is at most as big as spanning set. ie there is no L.I set bigger than spanning set.

b) ~~$\{v_1, v_2, \ldots v_n, w_1, w_2, \ldots w_p\}$~~

$\{v_1, v_2, \ldots v_n, w_{n+1}, w_{n+2}, \ldots w_m\}$ span V, after reordering the w's if necessary

if we take any old bunch of L.I vectors, and any old spanning set of vectors, then I can borrow some spanning set to extend the L.I set into a set that will span the whole vector space.

Prop 13: If V is finite dimensional vector space, then any two bases of V have same no. of elements.

$\dim V$ = # of elements in basis of V.

$\rightarrow \dim(\{0\}) = 0$. dim of zero subspace is 0

Ex: $\dim(R^n) = n$

$\dim(R_R) = 1$ — basis = $\{v\}$ ← non-zero vector

$\dim(C_C) = 1$

$\dim(Q_Q) = 1$

$$\boxed{\dim(\mathbb{F}_{\mathbb{F}}) = 1}$$

$(a+ib)$ $(1,0)\ (0,i)$

$(1, i)$

$\dim(\mathbb{C}_R) = 2$ basis $= \{1, i\}$

$\dim(R_R^{m\times n}) = mn.$

$\dim(R^{2\times 2}) = 4$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \left\{ a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\dim(R_R^{2\times 2}) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$\dim(R_n[t]) = n+1$ $\{1, t, t^2, t^3, \dots t^n\}$

standard basis

Prop 14:- Suppose $S = \{v_1, v_2, \dots v_p\}$ is a linear independent set in a vector space V. Suppose v is a vector which is not in span (S). Then the set obtained by adjoining v to S is L.I

$\Rightarrow S \cup \{v\} \to$ L.I

$c_1v_1 + c_2v_2 + \dots + c_pv_p + cv = 0$

let $c \neq 0$ (BWOC).

Pf $\Rightarrow$

Prop 15:- Any L.I set S in a finite-dimensional vector space can be expanded to a basis.

Prop 16:- Any finite spanning set S in a non-zero vector space can be contracted to a basis

Prop 17: Let V be a finite dimensional vector space with dimension n. then

a). Any subset of V which contain more than n elements is L.D.

b) No subset of V which contain less than n vectors can span V.

Dimension of Subspaces

Prop 18: If W is a proper subspace of a finite-dimensional vector space V, then W is also finite-dimensional and $0 < \dim W < \dim V$

$\rightarrow C[a,b] \rightarrow$ vector space containing all continuous fun^n real-valued fun^n defined in the interval $[a, b]$.

thus C will contain pol. $R[t]$ also. but we know that $\dim(R[t])$ is infinite.

$R[t]$ &

$AX = 0$ Null(A) Null(A):

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Sum and Direct Sum

U subspace V

W subspace V

$U+W = \{ u+w \mid u \in U, w \in W \}$ — it is the smallest subspace that contain U & W.

$V = U \oplus W$

↑

direct sum

if each vector $v \in V$ can be uniquely expressed in the form $v = u + w$ $u \in U$ & $w \in W$.

Sum

EX: $W_1 = \{ (x, 3x) : x \in R \}$

$W_2 = \{ (2x, 0) : x \in R \}$ subspaces

find $w_1 + w_2$

W_1 & W_2 are subspaces of V.

$W_1 + W_2$ will contain all vectors of the form $w_1 + w_2$ $w_1 \in W_1$ & $w_2 \in W_2$

$W_1 + W_2 = \{ x(1,3) + y(2,0) : x \in R,\ y \in R \}$

$W_1 + W_2$ is also subspace of V.

(Null(A))$^{\perp}$ = Null(A)

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Prop 19: If U and W are subspaces of the vector space V, then $V = U \oplus W$ only if $\underbrace{V = U+W}$ and $U \cap W = \{0\}$

i.e. every element of V can be uniquely expressed as sum $v = u + w$ $u \in U$ $w \in W$.

Here we can call W as complement (or) complementary subspace of U.

(Imp) → $U+W$ is the smallest subspace of V, that contain U & W.

Prop 20: If U and W are finite-dimension subspaces of the vector space V, then

$\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)$

Corollary: If V is direct sum of finite dimensional subspace U and W, then

$\dim V = \dim(U \oplus W) = \dim U + \dim W$.

Imp :- Any 1-dimension subspace corresponds to a line through the origin

U subspace R^2

$U = (x, y)$ $U = (1,1)$

U subspace represents a line joining $(0,0)$ to $(1,1)$

(R^2_R) U, W subspaces

$u_1 \in U$ $w_1 \in W$

u_1 & w_1 are independent i.e. u_1 can not be expressed as Linear Combination of w_1
$\{u_1, w_1\} \rightarrow$ form L.I set. Which is necessarily a basis.

$U = Span(u_1)$ $W = Span(w_1)$

$dim(U+W) = dim(U) + dim(W) - dim(U \cap W)$
$2 = 1 + 1 - 0 = 2$

$\therefore dim(U \cap W) = 0$

Ex: R^3_R

Case 1) U subspace R^3.

$dim\, U = 1$
i.e. basis of U contain only one non-zero vector.

$\{u_1\}$ $u_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

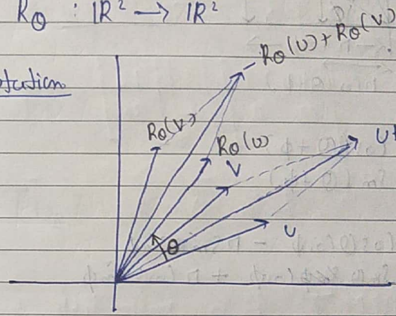
U represents line passing through $(0,0,0)$ to (x,y,z).

Case 2) $dim\, U = 2$
i.e. basis of U contain 2 L.I vectors
$\{u_1, u_2\} \rightarrow$ ~~represents a plane.~~
Here U represents a plane through origin.

Tuesday 22/10/19 (Properties of Linear Transform)

$R_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

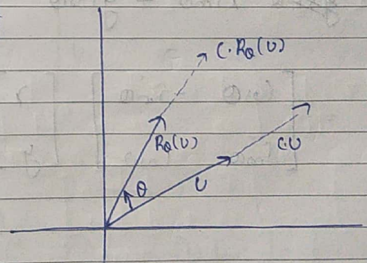
Rotation



$\therefore R_\theta(u) + R_\theta(v) = R_\theta(u+v)$

$R_\theta(u) + R_\theta(v) = R_\theta(u+v)$

Scaling



$c R_\theta(u) = R_\theta(c \cdot u)$

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Revision L.T

Row space $Row(A) = \cancel{Span}$ Span $\{r_1, r_2, \ldots r_m\}$

A $m \times n$ $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$

$Row(A) = Span\{r_1, r_2, \ldots r_m\} \subseteq R^n$
↓
subspace

$Col(A) = Span\{c_1, c_2, \ldots c_n\} \subseteq R^m$

$Nul(A) =$ Collection of all possible soln of $AX=0$
(or)
Solution space of $AX=0$.

$Nul(A) \subseteq R^n$
↓
subspace

→ System of linear Eqn (SLE) $AX=b$ consistent iff $b \in Col(A)$

$x_1c_1 + x_2c_2 + \cdots + x_nc_n = b$

$[c_1\ c_2 \ldots c_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

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→ Elementary row opn do not change null space of a matrix

→ Applying elementary row opn on augment matrix $[A:b]$ does not change soln set of corresponding linear system.

→ Elementary row opn do not change row(A) of a matrix

basis for a vector space ::

find basis for subspace of $\mathbb{R}^5$ spanned by the vectors

$v_1 = (1, -2, 0, 0, 3)$
$v_2 = (2, -5, -3, -2, 6)$
$v_3 = (0, 5, 15, 10, 0)$
$v_4 = (2, 6, 18, 8, 6)$

basis for span $\{v_1\ v_2\ v_3\ v_4\}$

$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$ $\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 1 & 3 & 2 & 0 \\ 1 & 3 & 9 & 4 & 3 \end{bmatrix}$

$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & -1 & -3 & -2 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 5 & 9 & 4 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -6 & -6 & 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

non-zero rows form the basis

$$Row(A) = \left\{ \begin{array}{l} [1, -2, 0, 0, 3] \\ [0, 1, 3, 2, 0] \\ [0, 0, 1, 1, 0] \end{array} \right\}$$

$$\boxed{Col(A) = row(A^T)}$$

dim (Row(A)) = # pivot positions
dim (Col(A)) = # pivot positions
dim (Nul(A)) = # of free variables

Q find subset of vectors which forms basis for space spanned by

$v_1 = (1, -2, 0, 3)$
$v_2 = (2, -5, -3, 6)$
$v_3 = (0, 1, 3, 0)$
$v_4 = (2, -1, 4, -7)$
$v_5 = (5, -8, 1, 2)$

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & -1 & 1 & 3 & 2 \\ 0 & 0 & 0 & -13 & -13 \\ 0 & -3 & 3 & 4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & -1 & 1 & 3 & 2 \\ 0 & -3 & 3 & 4 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & -1 & 1 & 3 & 2 \\ 0 & 0 & 0 & -5 & -5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & -1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

forms basis

$\uparrow$

[ ]

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & +1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

L.D

Rank - Nullity theorem

dim (Row(A)) = dim (Col(A)) = # pivot positions

rank (A) = dim (Row(A)) = dim (Col(A))

Nullity (A) = dim (Nul(A)) = # of free variables.

$\boxed{\text{rank}(A) + \text{nullity}(A) = n}$

$n \to$ # of Columns

rank(A) = # of pivot positions

nullity (A) = n - rank(A)

$\hookrightarrow$ # of parameters in general soln of AX=0

IMT Ver 2.0

given A n×n square matrix

1) A is invertible
2) A ~ I
3) AX=0 has only trivial soln
4) AX=b has soln for $\forall$ $b \in R^n$
5) Nul(A) = {0} or Nullity (A) = 0
6) rank(A) = n = # of columns
7) Col(A) span R^n, Row(A) span R^n
8) det(A) ≠ 0

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→ For $A_{m\times n}$ $\text{rank}(A) \le \min(m,n)$

rank(A) = common dimension of its row and column space

rows of A lie within R^n
columns of A lie within R^m

so $\dim(\text{row}(A)) \le n$
$\dim(\text{col}(A)) \le m$

→ $\text{rank}(A) = \text{rank}(A^T)$

$\text{rank}(A) = \dim(\text{row}(A)) = \dim(\text{col}(A^T)) = \text{rank}(A^T)$
(or)

$\text{rank}(A) = \dim(\text{col}(A)) = \dim(\text{row}(A^T)) = \text{rank}(A^T)$

$$\boxed{\text{rank}(A^T) + \text{nullity}(A^T) = m}$$

$$\boxed{\text{rank}(A) + \text{nullity}(A^T) = m}$$

$\therefore$ nullity $(A) = n - \text{rank}(A)$

nullity $(A^T) = m - \text{rank}(A)$

LT : $V \to W$ s.t $\forall\, u,v \in T$ $\forall$ c scalars

i) $T(u+v) = T(u) + T(v)$ [additivity]

ii) $T(cu) = c\,T(u)$. [homogeneity]

if $W = V$.
then LT is called Linear operator

Prop:-
If T is L.T. , $T : V \to W$, then

$T(0) = 0$
$T(u-v) = T(u) - T(v)$ $\forall\, u,v \in V$.

Pf:- any vector $u \in V$ can be written as
$0 \cdot u = \vec{0}$
$T(0) = T(0\cdot u) = 0 \cdot T(u) = 0$.

$\vec{0} = 0 \cdot u$ (zero vector can be written like)

$T(u-v) = T(u + (-1)v)$
$= T(u) + T((-1)v)$
$T(u) - 1\cdot T(v)$
$= T(u) - T(v)$

Zero Transformation

$0 : V \to W$ $0(v) = 0$. $\forall\, v \in V$.

Identity Operator

$I: V \to V \qquad I(v) = v.$

~~Vec~~ Matrix Transformation from $R^n \to R^m$

$$w_1 = f_1(x_1, x_2 \ldots x_n)$$
$$w_2 = f_2(x_1, x_2 \ldots x_n)$$
$$\vdots = \vdots$$
$$w_m = f_m(x_1 x_2 \ldots x_n)$$

m equations assign a unique point $(w_1 w_2 \ldots w_m)$ in R^m to ~~every point~~ each point $(x_1, x_2 \ldots x_n)$ in R^n

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ \vdots \\ w_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$$

$T_A : R^n \to R^m$

matrix transformation map column vector X in R^n into the column vector w in R^m

$$w = T_A(X)$$

T_A maps 'X' into "w".

$$T(x) = [T]x$$

~~T_A~~: $T: R^n \to R^m$

$$T(x) = [T]X$$

$T \to$ is a matrix transformation with standard matrix $[T]$, and image of x under this transformation is product of the matrix $[T]$ and column vector X.

Kernel

$T: V \to W$ is a linear transformation then set of vector in V that T map ~~to~~ into 0 is called kernel of T.

$$\ker(T) := \{ v \in V \mid T(v) = 0 \}$$

Range of T $\quad R(T)$

Set of all vectors in W that are images ~~of~~ under T of at least one vector in V is called range of T.

~~Ran~~ $R(T) := \{ w \in W \mid T(v) = w \text{ for some } v \in V \}$

$$R(T) \subseteq W$$

subspace of W.

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$T: V \to W$ s.t $\forall\, u, v \in V$ $\forall$ scalars α

$T(u+v) = T(u) + T(v)$ [additivity]

$T(\alpha u) = \alpha T(u)$ [homogeneity]

Properties :- $T: V \to W$ then

1> $T(0) = 0$

2> $T(-v) = -T(v)$

3> T preserves the linear combination of
ie $v_1 v_2 \dots v_k$ are vectors in V

$T(c_1 v_1 + c_2 v_2 + \dots + c_k v_k) = c_1 T(v_1) + \dots + c_k T(v_k)$

Ex: Dilation and Contraction Operators

$T: V \to V$ $\forall\, v \in V$

$T(v) = kv$.

if $0 < k < 1$ then T is contraction
if $k > 1$ then T is dilation

Ex: $T: P_n \to P_{n+1}$ st $\forall\, p \in P_n$

$T(p) = x\,p(x)$.

$P_n \equiv R_n[x]$.

$P_{n+1} \equiv R_{n+1}[x]$.

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Ex: $T: V \to R$ Let v_0 be any ~~vector~~ fixed vector in V.

$\forall\, v \in V$, $T(v) = \langle v, v_0 \rangle$

$= (v^T) \cdot v_0$

$T(u+v) = \langle u+v, v_0 \rangle$

$\langle u, v_0 \rangle + \langle v, v_0 \rangle$

$T(u) + T(v)$

$T(cv) = \langle cv, v_0 \rangle$

$= c \langle v, v_0 \rangle$

$= c\,T(v)$.

Ex: Projection Map: $P_i : R^n \to R^n$

$P_i(x_1, x_2 \dots x_i \dots x_n) = (0, 0, \dots x_i, 0, \dots 0)$

$\to$ $T: V \to W$ is injective $\Leftrightarrow$ $\ker(T) = \{0\}$

Important

Column s: Null space of $A_{m \times n}$ consist of all vectors X in R^n that satisfy $AX = 0$.
ie null space consists of all those vectors in R^n that multiplication by A maps into 0.

Column space of A consists of all vectors b in R^m for which there is at least one vector X in R^n st $AX = b$.
ie column space of A contains of all vectors in

R^m that are images of at least one vector in R^n under ~~no~~ multiplication by A.

if $T: V \to W$ then

1) ker(T) is subspace of V.
2) R(T) is subspace of W.

pf:1) $T(0) = 0$
$\vec{0} \in \ker(T)$.

$v_1, v_2 \in \ker(T)$

$T(v_1 + v_2) = T(v_1) + T(v_2)$
$= 0 + 0$
$= 0$
$v_1 + v_2 \in \ker(T)$

for any scalar c & $v \in \ker(T)$

$T(cv) = cT(v) = c \cdot 0 = 0$
$\therefore cv \in \ker(T)$

Hence ker(T) is subspace of V.

pf 2) $T(0) = 0$
$\therefore 0 \in R(T)$
$w_1, w_2 \in R(T)$
$w_1 = T(v_1)$
$w_2 = T(v_2)$

$w_1 + w_2 = T(v_1) + T(v_2)$
$= T(v_1 + v_2)$
$\because v_1 + v_2 \in V$.
$\therefore w_1 + w_2 \in R(T)$.

for some scalar c, & $w \in R(T)$

$cW = cT(v_1)$
$= T(cv_1)$
$= cv_1 \in V$
$\therefore cW \in R(T)$

Isomorphism

Prop 26 Let V be vector space of finite dim. n & Let $B = \{v_1, v_2, \dots v_n\}$ be basis of V.

a) We can define L.T $T: V \to W$ by simply knowing action of T on basis elements.

b) If $w_1, w_2, \dots w_n$ are vectors W (not necessarily distinct) then $\exists$ L.T $T: V \to W$ s.t. ∃ unique

$T(v_i) = w_i \quad \forall i = 1, 2, \dots n$

ie $T(v_1) = w_1 \quad T(v_2) = w_2 \dots \quad T(v_n) = w_n$

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Isomorphism of Vector Spaces.

A Linear transformation $T: V \to W$ is called isomorphism if it is injective (one-one map) and surjective (onto map).
In this case V is said to be isomorphic to W.

$$V \cong W$$

given $T: V \to W$, then
T is one-one $\iff$ $\ker(T) = \{0\}$

EX: $T: V \to V$
$T(v) = cv$ for some scalar c & $\forall v \in V$.

EX: $T: R_3[t] \to R^4$

$$T(a_0 + a_1 t + a_2 t^2 + a_3 t^3) = (a_0, a_1, a_2, a_3)$$

EX: $T: M_{2\times 2}(R) \to R^4$

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a, b, c, d)$$

Prop: given L.T $T: V \to W$ where V & W are of finite dim. then

a) If $\dim(W) < \dim(V)$ then T can not be one-one map.

b) If $\dim(V) < \dim(W)$ then T can not be onto map.

$T: V \to W$

EX:- $T: R_2[t] \to \mathbb{R}^3$

$$T(a_0 + a_1 x + a_2 x^2) = (a_0, a_1, a_2)$$

Theorem: Every real n dimensional vector space is isomorphic to $\mathbb{R}^n$.

EX: $T: P_{n-1} \to R^n$

EX: $T: M_{2\times 2}(R) \to R^4$

EX: $D: P_3 \to P_2$

$$D(p) = \frac{d(p)}{dx}$$

differential transform

Prop 2.7: given V & W are finite dim vector space.

If T is isomorphism then it maps basis of V to basis of W

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b) If T is L.T, that map basis of V to basis of W, then T is isomorphism

pf :> a) Given isomorphism $T: V \to W$

Let $\{v_1, v_2, \dots v_n\}$ be basis of V.
Let $\{T(v_1), T(v_2), \dots T(v_n)\} = B$

now we need to prove,

Span (B) = W and B is L.I

to prove Span (B) = W, we need to show Span (B) $\subseteq$ W && W $\subset$ Span (B)

Span (B) $\subseteq$ W (Trivial)

to prove W $\subseteq$ Span (B)

We need to show that $\forall w \in W$
$w = c_1 T(v_1) + c_2 T(v_2) + \dots + c_n T(v_n)$

For some $w \in W$
$w = T(v)$ for some $v \in V$.
$v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$

$w = T(v) = T(c_1 v_1 + c_2 v_2 + \dots + c_n v_n)$
$w = c_1 T(v_1) + c_2 T(v_2) + \dots + c_n T(v_n)$

Hence $W \subseteq$ Span (B)

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$\therefore$ Span (B) = W

To prove Span (B) is L.I
Consider expression
$c_1 T(v_1) + c_2 T(v_2) + \dots + c_n T(v_n) = 0$

$T(c_1 v_1 + c_2 v_2 + \dots + c_n v_n) = 0$

$\therefore c_1 v_1 + \dots + c_n v_n \in \ker(T)$
but T is injective ie $\ker(T) = \{0\}$

$\therefore c_1 v_1 + \dots + c_n v_n = 0$
$\Rightarrow c_1 = c_2 = \dots = c_n = 0$

Hence Span (B) is LI.

pf 'b)' given T is LT $T: V \to W$

Let $B = \{v_1 v_2 \dots v_n\}$ be basis of V.
And T maps B to $\{T(v_1), \dots T(v_n)\}$ of W basis

To prove T is one-one && onto.

T is one-one :- We need to ~~need~~ show that $\ker(T) = \{0\}$.

Suppose v is any arbitrary vector $\in \ker(T)$.

$\because v \in \ker(T)$
$T(v) = 0$

Now $v \in V$

$v = c_1 v_1 + c_2 v_2 + \dots + v_n$

$T(v) = T(c_1 v_1 + c_2 v_2 + \dots + c_n v_n) = 0$

$\Rightarrow c_1 T(v_1) + c_2 T(v_2) + \dots + c_n T(v_n) = 0$

$\because T(v_1) \dots T(v_n)$ are L.I.

$\Rightarrow c_1 = c_2 = \dots = c_n = 0$

$\therefore v = 0$

Hence $\ker(T) = \{0\}$

To prove T is onto.

Let $w \in W$

$w = c_1 T(v_1) + c_2 T(v_2) + \dots + c_n T(v_n)$

$w = T(\underbrace{c_1 v_1 + c_2 v_2 + \dots + c_n v_n}_{\text{linear combination of basis element}})$

$\therefore w \in \text{Range}(T)$

Prop 28 given 2 finite dim vector spaces V & W.

$V \cong W \iff \dim V = \dim W$.

isomorphic

☆ Every vector space of dim n over $\mathbb{R}$ is isomorphic to $\mathbb{R}^n$

Pf) $\Rightarrow$ given $V \cong W$ & $\dim V = n$

isomorphism $\Leftarrow$ $\exists\ T: V \to W$.

$\{v_1, v_2, \dots v_n\} \mapsto \{T(v_1) \dots T(v_n)\}$ basis of W.

$\dim(W) = n$

($\Leftarrow$) given $\dim(V) = n$ $\dim(W) = n$

Let $\{v_1, v_2 \dots v_n\}$ basis of V

Let $\{w_1, w_2 \dots w_n\}$ basis of W.

Prop 26: b) $\exists !\ T: V \to W$ s.t

$T(v_i) = w_i \quad \forall\ i = 1, 2, \dots n$

Prop 27 b) T is isomorphism

Rank-Nullity Theorem

$T: V \to W$. $\quad V = \dim(n)$

$\text{rank}(T) = \dim(\text{Range}(T))$

$\ker(T) = \text{Nul}(T) = \{v \in V \mid T(v) = 0\}$

$\ker(T) = \text{Nul}(T) \subset V$.

subspace

$\text{nullity}(T) = \dim(\ker(T))$.

range(T) = [T] matrix of transformation

↑ column space of [T]

T(x) = [T] x

Nul(T) = [T] x = 0 <- solution space of this

$$\boxed{\text{rank}(T) + \text{nullity}(T) = n = \dim(V)}$$

dim vector space V.

Prop:
If ∀ T: V → W, and V is finite dim
then rank(T) ≤ dim V.

Prop: T: V → W
T is injective (one-one) map <=> ker(T) = {0}
(or) Nul(A) = {0}
(or) Nullity(T) = 0

Pf: => given T: V → W is injective
We need to prove ker(T) = {0_V}

∵ T is L.T
$T(0_V) = 0_W$

$T(0_V) = T(0_V + 0_V)$
$= T(0_V) + T(0_V)$
$= 0_W + 0_W$
$= 0_W$

∴ $0_V \in$ Ker(T)

Hence ker(T) = {0_V}

(<=) given ker(T) = {0_V}.
prove T: V → W is one-one

Suppose T is not one-one map
i.e. for some $v_1, v_2 \in V$

$T(v_1) = T(v_2)$
$T(v_1) - T(v_2) = 0$
$T(v_1) + T((-1)v_2) = 0$
$T(v_1 - v_2) = 0$

∵ T is a L.T $T(0_V) = 0$

∴ $v_1 - v_2 = 0$
$v_1 = v_2$

(or)

T takes $(v_1 - v_2)$ to map to 0.
Hence $v_1 - v_2 \in$ ker(T)
but ker(T) contains only 0_V

∴ $v_1 - v_2 = 0_V$
$v_1 = v_2$

Hence ~~proo~~ our assumption is wrong
T is one-one.

$T: V \to W$ $\quad B, C \to$ basis of V & W respectively

B $\quad C$

$\{b_1, b_2, \dots b_n\}$ $\{c_1, c_2, \dots c_m\}$

$[T]_{B \to C} = [\,[T(b_1)]_C \ [T(b_2)]_C \dots [T(b_n)]_C\,]$

$v \in V$ $\quad T(v) \in W$

$\boxed{[T]_{B \to C} \cdot v = T(v)}$ — Verification

Coordinate System

Lect - 30

Ordered basis of a matrix is basis with its elements arranged in same fixed order.

$[V]_B$ = coordinate of vector $v \in V$ = dim(n) wrt to some basis B of vector space V.

new basis

$[V]_B = A^{-1} [V]_S$ $\quad v \in V$.

A = matrix formed by the elements of new basis B as column vectors.

$[V]_S$ = coordinate of v wrt to standard basis

Matrix of Linear Transformation $\quad$ Lect - 31.

$B = \{v_1, v_2, \dots v_n\}$
$C = \{v_1, v_2, \dots v_n\}$

$[T(v)]_C = [T]_{B \to C} [V]_B$

$A = [T]_{B \to C} = [\,[T(v_1)]_C \ [T(v_2)]_C \dots [T(v_n)]_C\,]$

Matrix of Linear Transformation wrt to basis B and C

$T: V \to V$.

$[T(v)]_B = [T]_{B \to B} [V]_B = [T]_B [V]_B$

$[T]_{B \to B}$ can be written as $[T]_B$.

$B = \{b_1, b_2, \dots b_n\}$
$C = \{c_1, c_2, \dots c_n\}$
$v \in V$

$[v]_C = P_{B \to C} [v]_B$

$P_{B \to C} = [\,[b_1]_C \ [b_2]_C \dots [b_n]_C\,]$

$P_{B \to C}$ $\boxed{P_{C \to B} = P_{B \to C}^{-1}}$

$P_{C \to B} = [\,[c_1]_B \ [c_2]_B \dots [c_n]_B\,]$

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Change of Basis

Let $B = \{v_1, v_2, \dots v_n\}$, $C = \{v_1, v_2, \dots v_n\}$ ordered basis of V

be ordered basis of a vector space V. Then for any $x \in V$, there is an invertible $n \times n$ matrix P such that

$[x]_C = P_{B \to C} [x]_B$ $\quad$ invertible matrix

$P_{B \to C}$ = Change of coordinate matrix from B to C.

$[x]_B = P_{C \to B} [x]_C$

$\boxed{P_{B \to C} = P_{C \to B}^{-1}}$ OR $\boxed{P_{C \to B} = P_{B \to C}^{-1}}$

Similarity of Matrices

An $n \times n$ matrix B is said to be similar to an $n \times n$ matrix A if there exists an invertible matrix P such that $B = PAP^{-1}$

Similarity of matrices is an equivalence relation on $\mathbb{F}^{n \times n}$

Satisfy :-
- Reflexive
- Symmetric
- Transitive

★ **Prop 3**: $A \to$ matrix of linear operator $[T]_\alpha$ relative to ordered basis α

$B \to$ matrix of Linear Operator $[T]_\beta$

then A and B are similar matrices

$B = PAP^{-1}$

$P = [P]_{\alpha \to \beta}$ Change of basis matrix

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Eigen Value & Eigen Vectors

eigen vector $X \in R^n$ (column vector in R^n) is a non-zero vector s.t

$$AX = \lambda X \quad \lambda \text{ is some scalar}$$

→ λ is eigen value / characteristic value / latent root

→ X → eigen vector / characteristic vectors.

0 is not considered as eigen vector

→ if matrix $n \times n$ $A \in R^{n \times n}$ has $\lambda = 0$, then $AX = 0X$

⇒ $AX = 0$ will have non-trivial soln

⇒ A is not invertible

$n \times n$ matrix A is invertible ⟺ 0 is not eigen value of A

$\lambda \longrightarrow E_\lambda$ eigen space corresponding to eigen value λ.

→ eigen vector is not unique

set of all eigen vectors corresponding to eigen value λ, along with 0 vector forms a ~~Eigen space~~ V = subspace of $V = R^n$

GM = soln space of $(A - \lambda I)X = 0$

$= n - r$

$E_\lambda = \{ v \in V \mid Av = \lambda v \} \cup \{0\}$

E_λ = ~~eigen subspace~~ eigen space corresponding to λ.

E_λ = null space of $(A - \lambda I)$

$E_\lambda = Nul(A - \lambda I)$

☆ Check λ is eigen value for a given matrix A ?

If λ is eigen value, then $(A - \lambda I) = 0$ must have non-trivial soln

If X_1, X_2 are eigen vectors corresponding to eigen value λ, then $X_1 + X_2$ is also an eigen vector corresponding to same eigen value λ.

~~Then~~ $E_\lambda = \{ \forall X \in R^n \mid AX = \lambda X \} \cup \{0\}$

$= Nul(A - \lambda I)$

$\dim(E_\lambda)$ = G.M of λ

Prop^n If $v_1, v_2, \ldots v_p$ are eigen vectors corresponding to eigen values $\lambda_1, \lambda_2, \ldots \lambda_p$, of the matrix A, then the set $\{v_1, v_2, \ldots v_p\}$ is Linearly Independent

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Basis of Eigen Spaces :- eigen vectors corresponding to eigen values.

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Prop 41 λ is an eigen value of an $n \times n$ matrix A ~~if~~ only if it satisfy characteristic eq^n $\det(A-\lambda I) = 0$

→ $\det(A-\lambda I) = 0$ is polynomial of deg n.
↓
characteristic polynomial

→ A $n \times n$ matrix can have at most n roots ie (at most n eigen values)

→ If A and B are matrices, then they have same eigen values with same multiplicity.

→ If $A \sim B$
then $B^k = PA^kP^{-1}$
A = diagonal matrix $(a_{11}\ a_{22} \dots a_{nn})$.

Diagonalization of Matrix

$A = PDP^{-1}$

P = formed by eigen ~~values~~ vectors of A corresponding to eigen values as column matrix

$D = \text{diag}(\lambda_1\ \lambda_2 \dots \lambda_n)$
eigen values being ~~di~~ diagonal elements

To verify diagonalization ~~Calculate~~, just verify following
$AP = PD$

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→ If D is diag matrix, then
→ diagonal elements are eigen values
→ standard basis vector are its eigen vectors

Matrix A is diagonalizable if $A \sim D$

ie $A = PDP^{-1}$

→ Diagonalization Theorem (Imp)

An $n \times n$ matrix A is diagonalizable $\iff$ A has n L.I eigen vectors
(or)

An $n \times n$ matrix A is diagonalizable $\iff$ A has ~~n~~ enough eigen vectors to form a basis of R^n. Such a basis is called eigen vector basis

→ An $n \times n$ matrix A with distinct eigen values is diagonalizable

→ Algebraic Multiplicity of Eigen Value.

# of times eigen value is repeated.

→ Geometric Multiplicity of Eigen Value.

~~dim~~ $GM = \dim(\text{Nul}(A-\lambda I))$
$= n - r$ ie Nullity

$$\boxed{GM \leq AM}$$

Matrix A is diagonalizable iff GM = AM for each eigen value.

Inner Product

An inner product is a fun^n in a real vector space V, that associates a real no. to with each pair of vectors $u, v \in V$. and satisfy following axioms.

1) $\langle u,v\rangle = \langle v,u\rangle$.
2) $\langle u+v,w\rangle = \langle u,w\rangle + \langle v,w\rangle$
3) $\langle cu,v\rangle = c\langle u,v\rangle$.
4) $\langle v,v\rangle \geq 0$ and $\langle v,v\rangle = 0 \iff v = 0$

Inner product space.

A vector with an inner product is called inner product space.

$\langle , \rangle : V \times V \to R$

$(u,v) \mapsto$ real no.

$\langle u,v\rangle = u^T v$
$= u_1v_1 + u_2v_2 + u_3v_3 + \dots + u_nv_n$
$= |u||v| \cos\theta$

$v = (x_1, x_2, x_3)$

$||v|| = \sqrt{\langle v,v\rangle} = \sqrt{x_1^2 + x_2^2 + x_3^2}$

norm (or) length of a vector

dist b/w 2 vectors
$d(u,v) = ||u-v|| = \sqrt{\langle u-v, u-v\rangle}$

→ Norm and distance b/w vectors depends on Inner product

EX 1) $\langle , \rangle : R \times R \to R$ $\quad \left.\begin{matrix} u = (u_1,u_2) \\ v = (v_1,v_2) \end{matrix}\right\} \in R^2$

$\langle u,v\rangle = 3u_1v_1 + 2u_2v_2$

2) Standard Inner Product on P_n

$p = a_0 + a_1t + a_2t^2 + \dots + a_nt^n$
$q = b_0 + b_1t + b_2t^2 + \dots + b_nt^n$

- $\langle p,q\rangle = a_0b_0 + a_1b_1 + \dots + a_nb_n$

→ $||p|| = \sqrt{a_0^2 + a_1^2 + \dots + a_n^2}$

→ $||cv|| = |c|\,||v||$.

3) Evaluation Inner Product

$p = p(t) = a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n$
$q = q(t) = b_0 + b_1 t + b_2 t^2 + \dots + b_n t^n$

$\langle p, q \rangle = p(t_0) q(t_0) + p(t_1) q(t_1) + \dots + p(t_n) q(t_n)$

$\|p\| = \sqrt{(p(t_0))^2 + (p(t_1))^2 + \dots + (p(t_n))^2}$

where $t_0, t_1, \dots, t_n$ are distinct real no's

4) Inner Product in $C[a,b]$.

$$\langle f, g \rangle = \int_a^b f(x) g(x)\, dx$$

$f = f(x) \quad g = g(x)$.

$$\langle f, g \rangle = \int_a^b f(x) g(x)\, dx$$

$$\langle g, f \rangle = \int_a^b g(x) f(x)\, dx$$

$\therefore \langle f, g \rangle = \langle g, f \rangle$

$\langle f + g, h \rangle = \langle f, h \rangle + \langle g, h \rangle$

$$\langle f + g, h \rangle = \int_a^b (f(x) + g(x))\, h(x)\, dx$$

$$= \int_a^b f(x) h(x) + \int_a^b g(x) h(x)\, dx$$

$= \langle f, h \rangle + \langle g, h \rangle$

$\langle cf, g \rangle = c \langle f, g \rangle$

$$\langle cf, g \rangle = \int_a^b c f(x) \cdot g(x)\, dx$$

$$= c \int_a^b f(x) g(x)\, dx$$

$= c \langle f, g \rangle$

$$\langle f, f \rangle = \int_a^b (f(x))^2\, dx \geq 0 \quad \forall x \in [a,b].$$

$f(x) = \langle f, f \rangle = 0 \iff f(x) = 0$

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Norm of vector in $C[a,b]$

$$\|f\| = \sqrt{\langle f, f\rangle} = \sqrt{\int_a^b f^2(x)\,dx}$$

* Inner Product Space Properties

$u, v, w \in V$ and some scalar c.
↳ inner product space

$\langle 0, v\rangle = \langle v, 0\rangle = 0$
$\langle v+u, w\rangle = \langle u, w\rangle + \langle v, w\rangle$
$\langle v-u, w\rangle = \langle v, w\rangle - \langle u, w\rangle$
$\langle u, cv\rangle = c\langle u, v\rangle$

Normalizing vector $\dfrac{v}{\|v\|}$

Orthogonality

$u \perp v$ iff $\langle u, v\rangle = 0$

A set of vectors $\{u_1, u_2 \dots u_n\}$ is said to be orthogonal set if any two distinct vectors in the set are orthogonal to each other. ie $\langle u_i, u_j\rangle = 0$ $i \neq j$

$c_1u_1 + c_2u_2 + \dots + c_pu_p = 0$
$c_1\langle u_1, u_1\rangle + c_2\langle u_1, u_2\rangle + \dots + c_p\langle u_1, u_p\rangle = 0$
$c_1\langle u_1, u_1\rangle = 0$ $\because \langle u_1, u_1\rangle > 0$ $\therefore c_1 = 0$

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Similarly prove for all $c_i = 0$

Proposition 46

An orthogonal set of non-zero vectors in V is Linearly Independent
↳ (vector space)

If W is a subspace of V, then a vector $v \in V$ is said to be orthogonal to W if v is orthogonal to every vector in W. The set of all vectors orthogonal to W is called Orthogonal complement of W, written $W^\perp$

Prop 47

a) A vector v belongs to $W^\perp$ iff v is orthogonal to every vector in spanning set of for W.

b) $W^\perp$ is subspace of V and $W \cap W^\perp = \{0\}$

a) $(\Rightarrow)$ $\because v \in W^\perp$
$\Rightarrow v \perp w \quad \forall w \in W$.

$(\Leftarrow)$ $v \perp w, \forall w \in W$

$S = \{u_1, u_2 \dots u_p\}$
$\text{Span}(S) = W$.
for $w \in W$
$w = c_1u_1 + c_2u_2 + \dots + c_pu_p$

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$\langle v, w \rangle = \langle v, c_1 u_1 + c_2 u_2 + \dots + c_p u_p \rangle$

$= \cancel{cv}$

$\Rightarrow c_1 \langle v, u_1 \rangle + c_2 \langle v, u_2 \rangle + \dots + c_p \langle v, u_p \rangle$

$\Rightarrow 0$

$\therefore v \in W^\perp$

b) Pf: $v_1, v_2 \in W^\perp$, $w \in W$, $c \in \mathbb{F}$

i) $0 \in W^\perp$ bcz $\langle 0, w \rangle = 0 \ \forall w \in W$.

ii) $\langle v_1 + v_2, w \rangle = \langle v_1, w \rangle + \langle v_2, w \rangle$
$= 0 + 0$
$= 0 \quad \forall w \in W$

iii) $\langle c v_1, w \rangle = c \langle v, w \rangle$
$= c \cdot 0$
$= 0 \quad \forall w \in W$.

$\therefore W^\perp$ is subspace of V.

now we have to prove $W^\perp \cap W = \{0\}$

BWOC, $w \in W^\perp \cap W$.

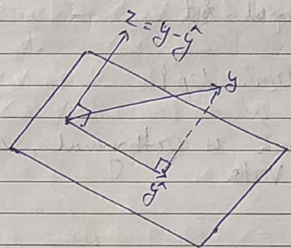
$\langle w, w \rangle = 0 \quad \therefore \cancel{w \neq 0}$ w must be 0.

Orthogonal basis :- basis which is orthogonal set

~~An Orthogonal basis~~

Orthogonal Decomposition Theorem (ODT)

Orthogonal Projection



$\hat{y} = \mathrm{Proj}_u\, y \qquad y = \hat{y} + z$

$z = y - \hat{y}$

$\hat{y}$ = orthogonal projection of y onto u

z = compliment of y orthogonal to u

$$\boxed{\hat{y} = \frac{\langle y, u \rangle}{\langle u, u \rangle} \cdot u}$$

Verification $\langle y, y - \hat{y} \rangle = 0$

$\{y, y - \hat{y}\} \rightarrow$ orthogonal set

Orthogonal matrix A $A^TA = I$
Its columns form orthonormal set $A^{-1} = A^T$

Ex: $M_{n\times n}(R)$

this is also inner ~~vector~~ product space

$A, B \in M_{n\times n}(R)$

$\langle A, B\rangle = \text{trace}(A^TB)$

Orthonormal Set

$S = \{U_1, U_2 \ldots U_p\}$

Set which is orthogonal, and every vector a unit vector

Orthogonal Decomposition Theorem (ODT)

Let W be any finite-dimensional subspace of V. Then each vector y in V can be written uniquely in the form $y = \hat{y} + z$; where $\hat{y}$ is in W and z is in $W^\perp$.

In fact if $\{U_1, U_2 \ldots U_p\}$ is any orthogonal basis of W, then

$$\hat{y} = c_1U_1 + c_2U_2 + \ldots + c_pU_p$$

$$\boxed{c_i = \frac{\langle y, U_i\rangle}{\langle U_i, U_i\rangle}} \quad i = 1, 2, \ldots p$$

and $z = y - \hat{y}$

Theorem 5: Given any finite dim subspace W of V, then we can express

$$V = W + W^\perp \qquad W \cap W^\perp = \{0\}$$

then

$$V = W \oplus W^\perp$$

Every vector can be uniquely expressed as sum of vector in W and a vector in $W^\perp$ i.e. as sum of 2 orthogonal vectors

1) $\hat{y} = \text{Proj}_W y$

if $W = \text{span}\{U\}$

$$\hat{y} = \frac{\langle y, U\rangle}{\langle U, U\rangle} U$$

2) if y belong to W, then $\hat{y} = \text{Proj}_W y = y$

Theorem 6 (Gram-Schmidt Orthogonalization Process)

Given a basis ~~for subspace~~ $\{x_1, x_2 \ldots x_p\}$ for a subspace W of V, we can generate an orthogonal basis ~~$\{v_1, v_2 \ldots v_p\}$~~ $\{v_1, v_2 \ldots v_k\}$ for W such that Span $\{x_1, x_2 \ldots x_{p_k}\}$ = Span $\{v_1, v_2 \ldots v_k\}$ for $k = 1, 2, \ldots p$.

$$v_1 = x_1$$

$$v_2 = x_2 - \frac{\langle x_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$$

$$v_3 = x_3 - \frac{\langle x_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 - \frac{\langle x_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$$

$\vdots$

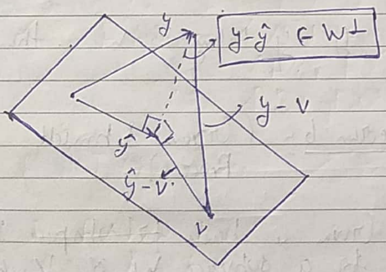
$$v_p = x_p - \frac{\langle x_p, v_{p-1} \rangle}{\langle v_{p-1}, v_{p-1} \rangle} - \frac{\langle x_p, v_{p-2} \rangle}{\langle v_{p-2}, v_{p-2} \rangle} \cdots - \frac{\langle x_p, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 - \frac{\langle x_p, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$$

Pythagorean Theorem

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$

Prop 50: Best Approximation Theorem

W



$y \in V$.

W subspace of V.

$\|y - v\|^2 > 0$
↑ prove

Let W be any finite-dim subspace of V, y any vector in V, and $\hat{y}$ be the orthogonal projection of y onto W. Then $\|y - \hat{y}\| < \|y - v\|$ $\forall\, v \in W$ distinct from $\hat{y}$. (ie $\hat{y}$ is the closest vector (point) in W to y)

Cor 50.1 If v is any vector, and W is finite-dim subspace, then

$$\|\mathrm{proj}_W v\| \le \|v\|$$

Cauchy-Schwarz Inequality

$\forall\, u, v \in V$

$$|\langle u, v \rangle| \le \|u\| \cdot \|v\|$$

Triangular Inequality

$\forall\, u, v \in V$.

$$\|u+v\| \le \|u\| + \|v\|$$

Best A.

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Best Approximation; Least Squares

given a rectangular matrix $A_{m \times n}$

$(\text{row}(A))^{\perp} = \text{Nul}(A)$

$(\text{col}(A))^{\perp} = \text{Nul}(A^T)$.

$\rightarrow$ $A^T A \rightarrow$ square symmetric Matrix

$\text{Nul}(A^T A) = \text{Nul}(A)$

pf :> If $AX = 0 \Rightarrow A^T A X = 0$

ie Vectors X that are present in Nul(A) are also present in ~~col A~~ $\text{Nul}(A^T A)$

Let $A^T A X = 0$

$\langle X, A^T A X \rangle = X^T A^T A X$

$= (AX)^T AX$

$= \langle AX, AX \rangle$.

$\therefore \|AX\|^2 = 0$

or

$AX = 0$

these two Null spaces are identical

Theorem

If columns of A are L.I, then $\text{Nul}(A) = \{0\}$

& A has independent columns $\Longleftrightarrow$ $A^T A$ is square, symmetric and invertible

Least Squares Problem

If $A \in R^{m \times n}$ and $b \in R^m$, &

Suppose the system $AX = b$ is an inconsistent So no exact sol^n possible ; so we look for vector X that ~~comes~~ comes as "close as possible" to being a solution. ~~that~~ in a sense that it minimizes $\|b - AX\|$.

We can think AX as an approximation to b and $\|b - AX\|$ as the error in that approximation - Hence smaller the error, better the approximation

If $A \in R^{m \times n}$, $b \in R^m$, a least-squares solution of $AX = b$ is an $\hat{X}$ in R^n s.t

$$\|b - A\hat{X}\| \leq \|b - AX\|$$

$$\forall X \in R^n$$

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Prop 53 The set of least-squares solutions of $AX = b$ coincide with the non-empty set of sol^n of the normal equations $A^TAX = A^Tb$

Prop 54 The matrix A^TA is invertible $\Leftrightarrow$ columns of A are linearly independent

In this case the equation $AX = b$ has only one least-squares solution $\hat{X}$, and it is given by

$$\boxed{\hat{X} = (A^TA)^{-1} A^Tb}$$

When a least-squares sol^n $\hat{X}$ is used to produce $A\hat{X}$ as an approximation to b, the distance from b to $A\hat{X}$ is called least-squares error of this approximation

SVD (Singular Value Decomposition)

If A is a $m \times n$ matrix, then

i) $Nul(A) = Nul(A^TA)$
ii) $Row(A) = Row(A^TA)$
iii) $Col(A) = Col(A^TA)$
iv) $rank(A) = rank(A^TA)$

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If A is an $m \times n$ matrix, and if $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n$ are eigen values of A^TA

$\sigma_1 = \sqrt{\lambda_1}$, $\sigma_2 = \sqrt{\lambda_2}$ ..... $\sigma_n = \sqrt{\lambda_n}$

are called Singular Values of A

If A is $m \times n$ matrix

a) $Nul(A) = Nul(A^TA)$
b) $Row(A) = Row(A^TA)$
c) $Col(A) = Col(A^TA)$
d) $rank(A) = rank(A^TA)$

e) A^TA is orthogonally diagonalizable
f) The eigen values of A^TA are non-negative

$$A_{m \times n} = \underset{m \times m}{U}\ \underset{m \times n}{\Sigma}\ \underset{n \times n}{V^T}$$

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} \begin{matrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \ddots \\ & & & \sigma_k \end{matrix} & k \times (n-k) \\ (m-k) \times k & (m-k)(n-k) \end{bmatrix}_{m \times n}$$

$V = [v_1\ v_2 \ldots v_n]$ - orthogonal. $V^T = \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix}_{n \times n}$

$U = \left[\ u_1 = \frac{1}{\sigma_1} Av_1 \ldots\ldots u_i = \frac{1}{\sigma_i} Av_i\ \right]_{m \times m}$