Linear Algebra

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Lecture 26

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Linear Transformation

Proposition 28

Two finite-dimensional vector spaces V and W (over the same field $\mathbb{F}$) are isomorphic if and only if $\dim V = \dim W$.

Remark: In particular, it follows that every vector space V of dimension n over $\mathbb{R}$ is isomorphic to $\mathbb{R}^n$. We can certainly exploit our familiarity with $\mathbb{R}^n$ in proving results about finite-dimensional spaces.

Proof of Proposition 28:

$[\Longrightarrow]$

Suppose V and W are isomorphic, i.e., there exists an isomorphism $T : V \longrightarrow W$. If $\dim V = n$ and $\{v_1, \ldots, v_n\}$ is a basis of V. Then by Proposition 27 (a), $\{T(v_1), \ldots, T(v_n)\}$ is a basis of W, and hence, $\dim W = n$.

$[\Longleftarrow]$

Suppose V and W have the same dimension, say n. Let $\{v_1, \ldots, v_n\}$ be a basis of V and let $\{w_1, \ldots, w_n\}$ be a fixed basis of W. By Prop 26(b), there exists a unique linear transformation $T : V \longrightarrow W$ such that $T(v_i) = w_i, i = 1, \ldots, n$. But now, from Prop 27(b), it follows that T is an isomorphism.

Rank of a Linear Transformation

Definition: If $T : V \longrightarrow W$ be a linear transformation, then Range $(T)= \{w \in W : w = T(v)$ for some $v \in V\}$—this is the standard definition of the range of any function.

Remark: Range (T) is a subspace of W (exercise!)

Definition: The **rank** of T is the dimension of the range of T.

Remark: If V is finite-dimensional, then it is easy to see that $\text{rank}(T) \leq \dim V$ (exercise!!)

Recall: We have already defined the kernel or null space of a linear transformation T as Ker (T) =Nul $(T) = \{v \in V : T(v) = 0\}$ and see that it is a subspace of V. If Ker T is finite-dimensional, then its dimension is called the **nullity** of T.

Theorem 3 (Rank-Nullity Theorem for Linear Transformation): Suppose that $T : V \longrightarrow W$ is a linear transformation and V is finite-dimensional. Then:

$$\text{rank}(T) + \text{nullity}(T) = \dim V$$

Proof of Theorem 3 (Rank-Nullity Theorem)

Suppose that $\dim V = n$ and nullity$(T) = k$. Then let $\{v_1, v_2, \ldots, v_k\}$ be a basis for Nul T and expand this to a basis B of V by inserting the additional vectors $v_{k+1}, \ldots, v_n$.

Claim: The vectors $T(v_{k+1}), \ldots, T(v_n)$ form a basis for Range(T).

Firstly, all the vectors $T(v_1), \ldots, T(v_n)$ surely span Range(T), and since $T(v_1) = T(v_2) = \cdots = T(v_k) = 0$, actually $T(v_{k+1}), \ldots, T(v_n)$ span Range(T).

Secondly, suppose that $c_{k+1}T(v_{k+1}) + c_{k+2}T(v_{k+2}) + \cdots + c_nT(v_n) = 0$ Then $T(c_{k+1}v_{k+1} + c_{k+2}v_{k+2} + \cdots + c_nv_n) = 0$. Hence, $c_{k+1}v_{k+1} + c_{k+2}v_{k+2} + \cdots + c_nv_n$ belongs to Nul T. Therefore $c_{k+1}v_{k+1} + c_{k+2}v_{k+2} + \cdots + c_nv_n = b_1v_1 + b_2v_2 + \cdots + b_kv_k$ or $b_1v_1 + b_2v_2 + \cdots + b_kv_k - c_{k+1}v_{k+1} - c_{k+2}v_{k+2} - \cdots - c_nv_n = 0$, which implies that $b_1 = b_2 = \cdots = b_k = c_{k+1} = c_{k+2} = \cdots = c_n = 0$. In particular, $c_{k+1} = c_{k+2} = \cdots = c_n = 0$, which proves the linear independence of the vectors $T(v_{k+1}), \ldots, T(v_n)$. This proves the claim.

Hence Rank $T+$ Nullity $T = (n-k) + k = n = \dim V$.

Observation: We have already noted that if T is a linear transformation on a finite-dimensional space V, then range T is also finite-dimensional of dimension at most $\dim V$. The Rank-Nullity Theorem makes this numerically precise.

Review – Basis and Dimension

- **Definition:** Let V be a vector space. A basis for V is a linearly independent set S of vectors such that $V = \text{Span } S$.
- **Definition:** A space V which has a (finite) basis is said to be finite dimensional.
- **Proposition 13:** If V is a finite-dimensional vector space, then any two bases of V have the same number of elements.
- **Definition:** The dimension of a finite-dimensional space is the number of elements in a basis for V. This is written $\dim V$.
- Three examples of finite dimensional spaces: $\mathbb{R}^n, \mathbb{R}^{m\times n}, \mathbb{R}_n[t]$. Their dimensions are $n, mn, n+1$, respectively.

Very Important Theorem (VIT)–Version 2.0
The Invertible Matrix Theorem (TIMT)

Theorem 1: The following are equivalent for an $m \times m$ square matrix A:

a. A is invertible.
b. A is row equivalent to the identity matrix I_m.
c. The homogeneous system $AX = 0$ has only the trivial solution.
d. The system of equations $AX = b$ has at least one solution for every b in $\mathbb{R}^m$.
e. Nullity $(A) = 0$.
f. Rank $(A) = m$.
g. The columns of A form a basis for $\mathbb{R}^m$.
h. Det $A \neq 0$.