Linear Algebra

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Lecture 26

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Linear Transformation

A Useful Observation about Linear Transformations

Proposition 26

Let V be a finite dimensional vector space of dimension n and let $\{v_1, \ldots, v_n\}$ be a basis of V.

(a) A linear transformation $T : V \longrightarrow W$ is completely determined by its action on a basis of V.

(b) Conversely, given a list of n vectors $w_1, \ldots, w_n$ (not necessarily distinct) in the co-domain space W, there is a unique linear transformation T s. t. $T(v_1) = w_1, T(v_2) = w_2, \ldots, T(v_n) = w_n$.

Proof of 26 (a): If $\{v_1, \ldots, v_n\}$ is a basis of V, then T is completely determined by the n vectors $w_1 = T(v_1), w_2 = T(v_2), \ldots . w_n = T(v_n)$. For if $v \in V$, then $v = c_1 v_1 + \cdots + c_n v_n$ for some scalars c_i, and so

$$T(v) = T(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) = c_1 T(v_1) + c_2 T(v_2) + \cdots + c_n T(v_n)$$
$$:= c_1 w_1 + c_2 w_2 + \cdots + c_n w_n.$$

Note that the vectors $w_1, \ldots, w_n$ need not be distinct.

A Useful Observation about Linear Transformations

Proposition 26 (b): Conversely, given a list of n vectors $w_1, \ldots, w_n$ (not necessarily distinct) in the co-domain space W, there is a unique linear transformation T such that $T(v_1) = w_1$, $T(v_2) = w_2, \ldots, T(v_n) = w_n$.

Proof: Let $B = \{v_1, \ldots, v_n\}$ be a basis of V, and let $w_1, \ldots, w_n$ be a list of n vectors in W, not necessarily distinct.

Existence:
Let v be any vector in V. Then, v can be uniquely expressed as $v = c_1 v_1 + \cdots + c_n v_n$, where $c_i's$ are scalars. Now, we define a map $T \ : \ V \longrightarrow W$ as follows:

$$T(v) = T(c_1 v_1 + \cdots + c_n v_n) = c_1 w_1 + \cdots + c_n w_n. \tag{1}$$

Since $v_i = 0v_1 + \cdots + 0v_{i-1} + 1.v_i + 0v_{i+1} + \cdots + 0v_n$, therefore by the definition (1) of the map, we have

$$T(v_i) = w_i \quad \text{for all the vectors} \quad v_i \in B \tag{2}$$

Proof of Proposition 26 (b) (Cont'd)

It is easy to see that T is a well-defined function. We need to show that T is actually a linear transformation i.e. we need to prove that T satisfies additivity and homogeneity properties.

1. **Additivity:** Suppose $u = d_1 v_1 + \cdots + d_n v_n$ and $v = e_1 v_1 + \cdots + e_n v_n$ are any two vectors of V. Then

$$\begin{aligned} T(u+v) &= T((d_1 v_1 + \cdots + d_n v_n) + (e_1 v_1 + \cdots + e_n v_n)) \\ &= T((d_1 + e_1) v_1 + \cdots + (d_n + e_n) v_n) \\ &= (d_1 + e_1) w_1 + \cdots + (d_n + e_n) w_n, \quad \text{by the def (1)} \\ &= (d_1 w_1 + \cdots + d_n w_n) + (e_1 w_1 + \cdots + e_n w_n) \\ &= T(u) + T(v), \quad \text{again by the def (1).} \end{aligned}$$

2. **Homogeneity:** For any scalar c,

$$\begin{aligned} T(cu) = T(c(d_1 v_1 + \cdots + d_n v_n)) &= T(cd_1 v_1 + \cdots + cd_n v_n) \\ &= cd_1 w_1 + \cdots + cd_n w_n, \quad \text{by the def (1)} \\ &= c(d_1 w_1 + \cdots + d_n w_n) = cT(u), \quad \text{again by the def (1).} \end{aligned}$$

Proof of Proposition 26 (b) (Cont'd)

Uniqueness: Finally, we need to prove uniqueness. Suppose there exists another linear transformation $\bar{T} : V \longrightarrow W$ such that $\bar{T}(v_i) = w_i$ for all $v_i \in B$. Let $v = c_1 v_1 + \cdots + c_n v_n$ be any vector in V. Then

$$\begin{aligned} \bar{T}(v) &= c_1 \bar{T}(v_1) + \cdots + c_n \bar{T}(v_n) \quad \text{by Remark (ii)} \\ &= c_1 w_1 + \cdots + c_n w_n \\ &= T(v) \end{aligned}$$

Since $T(v) = \bar{T}(v)$ for all $v \in V$, it follows that $T = \bar{T}$, proving uniqueness.