Linear Algebra and Applications

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Lecture 02

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Homogeneous and Non-Homogeneous Systems

- If $b = 0$, then the system is said to be **homogeneous**. A homogeneous system always has the trivial solution consisting of all zeroes.
- If $b \neq 0$, the system is said to be **non-homogeneous**. A non-homogeneous system may or may not have any solutions. A system which has at least one solution is said to be **consistent**. Otherwise, it is said to be **inconsistent**.

Matrix Formulation of a system

- Example: The system of equations:

$$
\begin{aligned}
X + Y + Z &= 0 \\
2X + 3Y + Z &= 0
\end{aligned}
$$

can be represented as follows:

$$
\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}}_{\text{Coeff. Matrix}} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

Matrix Formulation of a system

- Example: The system of equations:

$$X + Y = 4$$
$$X + 3Y = 8$$

can be represented as follows:

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}}_{\text{Coeff. Matrix}} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 1 & : & 4 \\ 1 & 3 & : & 8 \end{bmatrix}}_{\text{Augmented Matrix}}$$

Linear System

- Questions to ask about any system of equations
  1. How many solutions?
     - None
     - One
     - More than one? If so finitely many, or infinitely many?
  2. Is there a method to find all solutions?
     If more than one, which is better?
  3. Is there a "good" way to describe the entire solution set?

Linear System

- With regard to Q1 in the previous page, linear systems can exhibit all three types of behaviour:
    - None, for example

$$\begin{aligned} X + Y &= 0 \\ X + Y &= 2 \end{aligned}$$

    - Exactly one, for example

$$\begin{aligned} X + Y &= 4 \\ X + 3Y &= 8 \end{aligned}$$

  The only solution is (2,2).

Linear System

- With regard to Q1 in the previous page, linear systems can exhibit all three types of behaviour:
  - Exactly one, for example

$$\begin{aligned} X + Y &= 4 \\ X + 3Y &= 8 \end{aligned}$$

    The only solution is (2,2).
  - More than one?

$$\begin{aligned} X + Y + Z &= 0 \\ 2X + 3Y + Z &= 0 \end{aligned}$$

    Possible solutions: $(0,0,0), (-2,1,1), (-4,2,2)$ etc. In fact, there are infinitely many solutions as can be see geometrically: the above two equations represent plane through the origin, which intersect in a line, which has infinitely many points on it.

Solving a Linear System

- Small systems of linear equations (with two or three variables) can be solved by a method of "elimination" or a method of "substitution". Our goal now is to evolve a more systematic strategy which can be used in a mechanical way to deal with any system.
- **Observation 1:** In the process of elimination, the variables play no real role. All calculations are done with the coefficients and the RHS scalars. So we should work directly with matrices: the coefficient matrix A and the **augmented matrix** of the system $[A : b]$. So for the time being, we will continue the discussion mostly in terms of matrices (and later come back to the equations and solutions).