Linear Algebra

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Lecture 19

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Basis and Dimension

- **Definition:** Let V be a vector space over $\mathbb{F}$. Then a subset $B \subseteq V$ is a **basis** for V if
  1. B is a linearly independent and,
  2. B spans (or generates) V i. e. $V = \text{Span } B$.
- **Definition:** A space V which has a (finite) basis is said to be **finite dimensional**.
- A space which does not have a finite basis is said to be **infinite dimensional**.
- **Example of Basis:** In $\mathbb{R}^n$, consider the vectors (column vectors written as n-tuples for convenience):
  $e_1 = (1, 0, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \cdots, e_n = (0, 0, \ldots, 0, 1)$
  These vectors are linearly independent and Span (or generate) $\mathbb{R}^n$. Hence, they form a basis for $\mathbb{R}^n$, known as the **standard basis**. Note that these vectors change for different n.
- The plural of basis is bases.

Basis (Conti . . .)

Example of an infinite dimensional space

- The space $\mathbb{R}[t]$ of polynomials with real coefficients.
- **Justification:** Suppose, by way of contradiction, that $\mathbb{R}[t]$ is finite-dimensional. Then it must have a finite basis, say $B = \{p_1(t), p_2(t), \ldots, p_n(t)\}$.

  Put $N = \max\{\deg p_1(t), \deg p_2(t), \ldots, \deg p_n(t)\}$, and let $p(t) = t^{N+1}$. Then we can easily see that $p(t) \notin \text{Span } B$. The contradiction proves the desired result.

Basis (Conti . . .)

Alternative Definition for Basis

- **Recall:** A **basis** for a vector space V is a linearly independent set of vectors which spans the space V. A space V which has a finite basis is said to be **finite dimensional.**
- **Proposition 11:** $B = \{v_1, v_2, \ldots, v_n\}$ is a basis of the vector space V if and only if every vector $v \in V$ is **uniquely** expressible as a linear combination of the elements of B.
- **Remark:** In some books, the above is used as the definition of a basis, and then it is shown that a basis is a linearly independent spanning set.
  Proof: Left as an exercise.

Fundamental Results

Proposition 12 (Steinitz Exchange Lemma): Suppose $v_1, v_2, \ldots, v_n$ are linearly independent vectors in a vector space V, and suppose $V = \text{Span}\{w_1, w_2, \ldots, w_m\}$. Then:

(a) $n \leq m$

(b) $\{v_1, v_2, \ldots, v_n, w_{n+1}, w_{n+2}, \ldots, w_m\}$ spans V, after re-ordering the w's if necessary.

Proof: So we have: $v_1, v_2, \ldots, v_n$ are LI and $w_1, w_2, \ldots, w_m$ span V i.e $V = \text{Span}\{w_1, \ldots, w_m\}$. Since $w_1, w_2, \ldots, w_m$ span V, we must have

$$v_1 = c_1 w_1 + c_2 w_2 + \cdots + c_m w_m, \tag{1}$$

for some sacalars c_i. If $c_i = 0$ for all i, then $v_1 = 0$, which is not possible since any set containing the zero vector is LD. Therefore, $c_i \neq 0$ for at least one i, and re-numbering the w_i's if necessary, we can assume that $c_1 \neq 0$.

Fundamental Results (Conti . . .)

Proof of Proposition 12 (Cont'd)
So we can re-write (1) as:

$$c_1 w_1 = v_1 - c_2 w_2 - c_2 w_2 - \cdots - c_m w_m, \tag{2}$$

and multiplying by c_1^{-1}, we get:

$$w_1 = c_1^{-1} v_1 - c_1^{-1} c_2 w_2 - \cdots - c_1^{-1} c_m w_m, \tag{3}$$

or

$$w_1 = d_1 v_1 + d_2 w_2 + \cdots + d_m w_m, \tag{4}$$

where d_i are scalars. From (4), it follows that:

$$\mathrm{Span}\{v_1, w_2, \ldots, w_m\} = \mathrm{Span}\{w_1, w_2, \ldots, w_m\} = V \tag{5}$$

Fundamental Results (Conti . . .)

Proof of Proposition 12 (Cont'd)

Justification of (5): Suppose $x \in V$, then $x \in \text{Span}\{w_1, \ldots, w_m\}$, i.e.

$$x = b_1 w_1 + b_2 w_2 + \cdots + b_m w_m. \tag{6}$$

Substituting for w_1 in (6) from (4), we get:

$$\begin{aligned} x &= b_1(d_1 v_1 + d_2 w_2 + \cdots + d_m w_m) + b_2 w_2 + \cdots + b_m w_m \\ &= b_1 d_1 v_1 + (b_1 d_2 + b_2) w_2 + \cdots + (b_1 d_m + b_m) w_m \\ &= h_1 v_1 + h_2 w_2 + \cdots + h_m w_m. \end{aligned}$$

Thus $x \in \text{Span}\{v_1, w_2, \ldots, w_m\}$, which implies that $V \subseteq \text{Span}\{v_1, w_2, \ldots, w_m\}$. Hence $V = \text{Span}\{v_1, w_2, \ldots, w_m\}$ as claimed.

Fundamental Results (Conti . . .)

Proof of Proposition 12 (Cont'd)

So, at the next step, we get that $v_2 = \ell_1 v_1 + \ell_2 w_2 + \cdots + \ell_m w_m$ for some scalars ℓ_i. We see that at least one of $\ell_2, \ell_3, \ldots, \ell_m$ is not zero; if all are zero, then $v_2 = \ell_1 v_1$ – contradicting the linear independence of v_i's. By re-numbering w_j's, if necessary, we may assume $\ell_2 \neq 0$. So then: $\ell_2 w_2 = -\ell_1 v_1 + v_2 - \ell_3 w_3 - \cdots - \ell_m w_m$, and arguing as before, we get that:

$$\begin{aligned} \text{Span}\{v_1, v_2, w_3 \ldots, w_m\} &= \text{Span}\{v_1, w_2, \ldots, w_m\} \\ &= \text{Span}\{w_1, w_2, \ldots, w_m\} \\ &= V \end{aligned}$$

Proceeding in this way, we can step-by-step replace w_1 by v_1, w_2 by $v_2, \ldots,$ etc. The process has to stop after n-th step at most (since there are only n of the v vectors).

What is the situation when we come to the stop? There are two possible cases.

Fundamental Results (Conti . . .)

Proof of Proposition 12 (Cont'd)

Case 1: $n \leq m$. In this case we get the following situation:

$$\begin{array}{llllllll} v_1, & v_2, & \ldots, & v_n & & & & \\ \downarrow & \downarrow & \ldots & \downarrow & & & & \\ w_1, & w_2, & \ldots, & w_n, & w_{n+1}, & \ldots, & w_m \end{array}$$

We have replaced n of the w vectors, with re-numbering if necessary, and we get $V = \text{Span}\{v_1, v_2, \ldots, v_n, w_{n+1}, \ldots, w_m\}$. So in case 1, the proposition is proved.

[If $n = m$, then the vectors w_{n+1}, etc are not there in the original spanning set at all.]

Fundamental Results (Conti . . .)

Proof of Proposition 12 (Cont'd)

Case 2: $n > m$. In this case, we are only able to replace $w_1, w_2, \ldots, w_m$ and we are left with the vectors $v_{m+1}, \ldots, v_n$ of the original linearly independent vectors. The situation looks like:

$$\begin{array}{l} v_1, v_2, \ldots, v_m, v_{m+1}, \ldots, v_n \\ \downarrow \quad \downarrow \quad \ldots \downarrow \\ w_1, w_2, \ldots, w_m \end{array}$$

i.e. $\{v_1, v_2, \ldots, v_m\}$ is now a spanning set for V. But, then $v_{m+1} \in \text{Span}\{v_1, \ldots, v_m\}$ or $v_{m+1} = k_1 v_1 + \cdots + k_m v_m$ for some scalars k_i. But this contradicts linear independence of the $v_i' s$. Hence, Case 2 can not happen. Only case 1 can happen, and in this case as we saw before, the Proposition 12 has been proved.

Fundamental Results (Conti . . .)

Proposition 13: If V is a finite-dimensional vector space, then any two bases of V have the same number of elements.

Proof: Suppose B_1 and B_2 are two distinct bases of V such that $|B_1| = m$ and $|B_2| = n$. Then by Proposition 12(a), $|B_1| \leq |B_2|$ i.e. $m \leq n$, since B_1 is L. I. and B_2 is spanning set. In a similar way, $B_2 \leq B_1$ i. e. $n \leq m$. Hence we get: $m = n$.

- **Definition:** The dimension of a finite-dimensional space is the number of elements in a basis for V. This is written dim V.
- **Remark:** Proposition 13 ensures that this is a proper definition.
- **Special Case:** The zero subspace $\{0\}$ is defined to have dimension 0. However, it does not have a basis. So our insistence that $\dim\{0\} = 0$ amounts to saying that the **empty** set of vectors is a basis of $\{0\}$. Thus the statement that "the dimension of a vector space is the number of vectors in any basis" holds even for zero space.