Indian Institute of Technology Jammu

CSD001P5M **Linear Algebra** **Tutorial: 05**

1. In the following is W a subspace of V? Base field is taken as $\mathbb{R}$ in all. Justify your answer.
   (a) $V = R[t]$ = vector space of all polynomials with real coefficients, W = set of all polynomials with integer coefficients.
   (b) $V = \mathbb{R}^2$, $W = \{(x,y) \,:\, x+y \geq 0\}$.
   (c) $V = \mathbb{R}^2$, $W = \{(x,y) \,:\, x^2+y^2 \geq 0\}$.

2. Consider the space V of all 2×2 matrices over $\mathbb{R}$. Which of the following sets of matrices A in V are subspaces of V? Justify (prove) your answers.
   (a) All upper triangular matrices $\left(\text{i.e. matrices of the form } \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\right)$.
   (b) All A such that $AB = BA$ where B is some fixed matrix in V.
   (c) All A such that $BA = 0$ where B is some fixed matrix in V.
   (d) Would the above results hold for all $n\times n$ matrices where n is a general positive integer $(n \geq 2)$?

3. Given the following vectors in $\mathbb{R}^3$: $\mathbf{u} = (1,3,5), \mathbf{v} = (1,4,6), \mathbf{w} = (2,-1,3)$ and $\mathbf{b} = (6,5,17)$.
   (a) Does $\mathbf{b} \in W = \text{span}\{\mathbf{u},\ \mathbf{v},\ \mathbf{w}\}$?
   (b) If the answer to (a) is yes, express $\mathbf{b}$ as a linear combination of **u,v,w**.

4. Let U and W be two subspaces of the vector space V. Show that $U \cap W$ is also a subspace of V.

5. Let U and W be two subspaces of the vector space V. We define $U+W = \{u+w \,:\, u \in U, w \in W\}$. Show that $U+W$ is a subspace of V, and moreover, $U+W$ is the smallest subspace of V which contains both U and W.

6. Prove Remark 6 related to linear dependence/independence: Any list which contains a linearly dependent list is linearly dependent.

7. Prove Remark 7 related to linear dependence/independence : Any subset of a linearly independent set is linearly independent.

8. Determine whether the given matrices in the vector space $\mathbb{R}^{2\times 2}$ are linearly dependent or linearly independent.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

9. In the vector space $V = C[0, 2\pi]$, determine whether the given vectors (i.e. functions) are linearly dependent or linearly independent :

$$f_1(x) = 1, \qquad f_2(x) = \sin(x), \qquad f_3(x) = \sin(2x).$$

(You must justify your answer.)

10. Given the standard basis $B = \{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ and the linearly independent vectors $v_1 = (0, 1, 1)$ and $v_2 = (1, 1, 1)$, apply the method of the Steinitz Exchange Lemma (Proposition 12) to exchange two of the vectors in B and obtain a basis C which includes v_1 and v_2. Show your calculations in detail.

11. Prove Proposition 11: The subset $B = \{v_1, v_2, \cdots, v_n\}$ is a basis of the vector space V if and only if every vector $v \in V$ is uniquely expressible as a linear combination of the elements of B.

12. Let $V = C[0, 1]$, the space of all continuous real-valued functions defined on the closed interval. Is V finite-dimensional? Justify your answer.

13. Let $V = \mathbb{R}^\infty$, $W = \{< a_n > \ : \text{ only finitely many of the terms in } < a_n > \text{ are non-zero}\}$. Show that W is a a subspace of V. Is W finite-dimensional ? Justify your answer.

14. If $(v_1, \cdots, v_m)$ is linearly dependent and $v_1 \neq 0$, there exists an index $j \in \{2, \cdots, m\}$ such that

    (a) $v_j \in \text{Span}(v_1, \cdots, v_{j-1})$.

    (b) If v_j is removed from $(v_1, \cdots, v_m)$, then

$$\text{Span}(v_1, \cdots, v_{j-1}, v_{j+1}, \cdots, v_m) = \text{Span}(v_1, \cdots, v_{j-1}, v_j, v_{j+1}, \cdots, v_m).$$