Linear Algebra

Sartaj Ul Hasan

विद्याधनं सर्वधन प्रधानम्

Department of Mathematics
Indian Institute of Technology Jammu
Jammu, India - 181221

Email: sartaj.hasan@iitjammu.ac.in

Lecture 21

(Sept 13, 2019)

How to find Basis

Proposition 14: Suppose $S = \{v_1, v_2, \ldots, v_n\}$ is a linearly independent set in a vector space V. Suppose v is a vector which is not in Span S. Then the set obtained by adjoining v to S is linearly independent.

Proof: Suppose v is not in Span S, and consider any expression:

$$c_1 v_1 + \cdots + c_n v_n + cv = 0 \tag{1}$$

If $c \neq 0$, we can write: $cv = -c_1 v_1 - \cdots - c_n v_n$
or $c^{-1} cv = -c^{-1} c_1 v_1 - \cdots - c^{-1} c_n v_n$
or $v = -c^{-1} c_1 v_1 - \cdots - c^{-1} c_n v_n$, contradicting the assumption that v is not in Span S. Hence $c = 0$. Then (1) becomes $c_1 v_1 + \cdots + c_n v_n = 0$, and since S is linearly independent, $c_i = 0$ for all i. Thus $S \cup \{v\}$ is linearly independent.

How to find Basis (Cont'd)

As a consequence of Proposition 14 and its proof, we get the following:

Proposition 15: Any linearly independent set S in a finite-dimensional vector space can be expanded to a basis.

Proof: Proof is left as an exercise. The proof proceeds by applying Prop 14 repeatedly; by Prop 12, the process cannot go on indefinitely; it has to stop, and that stage, a basis has been obtained.

In a similar way, we can get a result which works in the "opposite" direction:

Proposition 16: Any finite spanning set S in a non-zero vector space can be contracted to a basis.
Proof: Left as an exercise.

Remark: In view of this proposition, we can say that if a non-zero vector space V has a finite spanning set S, then it must be finite-dimensional.

Remark: We can regard a basis as either a maximal linearly independent set, or as a minimal spanning set.

Summarizing Results about Dimension

Proposition 17: Let V be a finite-dimensional vector space with dimension n. Then:

a. Any subset of V which contains more than n elements is linearly dependent.
b. No subset of V which contains less than n vectors can span V.

Remark: Proposition 17 essentially summarizes Proposition 12 and its consequences in terms of dimension.