Linear Algebra

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Lecture 24

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Three Fundamental Subspaces

Null Space

- **Introduction:** There are three important subspaces related to any given $m \times n$ matrix A.
- **Definition 1:** The null space of an $m \times n$ matrix A, written Nul A, is the set of all solutions to the homogeneous system $AX = 0$.
- **Remark:** Note that Nul A is a subset of $\mathbb{R}^n$. In set notation, we can write Nul $A = \{X \;:\; X \in \mathbb{R}^n \text{ and } AX = 0\}$.
- **Proposition 21:** The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Or equivalently, the set of all solutions of a homogeneous system of m equations in n variables is a subspace of $\mathbb{R}^n$.

  Proof: We show that Nul A contains 0, and is closed under vector addition and scalar multiplication. Consider:

  (i) $A0 = 0$

  (ii) If u, v are in Nul A, then $Au = 0$ and $Av = 0$, hence $A(u + v) = Au + Av = 0 + 0 = 0$

  (iii) Finally, if u is in Nul A and c is any scalar, then $A(cu) = c(Au) = c(0) = 0$

Three Fundamental Subspaces (Cont'd)

- **Remark**: We have to take a homogeneous system of equations to get a subspace. The solution set of a non-homogeneous system is **not** a subspace.
- Nul A is defined implicitly by a condition. To describe Nul A explicitly, we must solve the linear system $AX = 0$.
- The method is to reduce the matrix to an RREF matrix, and express the solution vector of the simplified system as a linear combination where the coefficients are the free variables (same approach as done earlier for solving homogeneous systems). The spanning set produced by this method is a basis for Nul A. **(Why? You should be clear about this.)**
- Either Nul A is the zero subspace or the dimension of Nul A is equal to the number of free variables in the solution.

Three Fundamental Subspaces (Cont'd)

Column Space

- **Definition 2:** The column space of an $m \times n$ matrix A, written Col A, is the set of all linear combinations of the columns of A, i.e. the span of the column vectors obtained from A. If $A = [c_1, c_2, \ldots, c_n]$, then Col $A = \text{Span}\ \{c_1, c_2, \ldots, c_n\}$.
- **Proposition 22:** Col A is a subspace of $\mathbb{R}^m$.

  Proof: Since A is an $m \times n$ matrix, its columns are vectors in $\mathbb{R}^m$. Since Col A is the span of a set of vectors, it is a subspace by a previous proposition.
- **Remark:** Equivalent way to approach Col A: We can also say that Col $A = \{b \in \mathbb{R}^m \ : \ b = AX \text{ for some } X \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$.

Three Fundamental Subspaces (Cont'd)

- **Proposition 23:** The pivot columns of a matrix A form a basis for Col A.
- **Justification (Concise Proof):** Any linear dependence relationship between the columns of A can be expressed in the form $AX = 0$. When A is row reduced to R, the columns change but the equation $RX = 0$ has the same set of solutions. In other words, row reduction does not change the dependence relations between the columns. The pivot columns of A must be linearly independent because the pivot columns of R are linearly independent. Also, non-pivot columns are linear combinations of the preceding (i.e. left) pivot columns.
- Note : We must take the columns of A for the basis (not of its RREF matrix R).

Nul A Versus Col A

Nul A

1. Nul A is a subspace of $\mathbb{R}^n$
2. Nul A is defined implicitly.
3. It takes time to find vectors in Nul A (have to solve an equation).
4. There is no obvious relation between Nul A and entries of A.
5. For v in Nul A, $Av = 0$.
6. Given a specific vector v, we can easily test whether it is in Nul A.
7. Nul $A = \{0\}$ iff $AX = 0$ has only the trivial solution.

Col A

1. Col A is a subspace of $\mathbb{R}^m$
2. Col A is defined explicitly.
3. It is easy to find vectors in Col A because one has to take l. c. of the column vectors.
4. There is a definite relation between the column space i.e. Col A and entries of A.
5. For v in Col A, $AX = v$ has a soln.
6. Given a specific vector v, we cannot easily test whether it is in Col A (have to solve an equation).
7. Col $A = \mathbb{R}^m$ iff $AX = b$ has a solution for every b in $\mathbb{R}^m$.

Three Fundamental Subspaces (Cont'd)

- **Definition 3:** The row space of an $m \times n$ matrix A, written as Row A, is the set of all linear combinations of the rows of A, i.e., the span of the (row) vectors obtained from A. In doing this, we consider each row as an n-tuple, and hence as a vector in $\mathbb{R}^n$.
- If $A = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix}$, then Row $A =$ Span $\{r_1, r_2, \cdots, r_m\}$.
- **Proposition 24:** Row A is a subspace of $\mathbb{R}^n$.
- **Observation:** Elementary row operations replace rows of the original matrix by rows which are the same or linearly dependent on them. Hence, the row space does not get enlarged by row operations. If B has been obtined from A by an elementary row operation, then Row $B \subseteq$ Row A. But since elementary row operations are reversible, we have Row $A \subseteq$ Row B. Therefore Row $B =$ Row A, and we get:
- **Proposition 25:** Row equivalent matrices have the same row space.

Three Fundamental Subspaces (Cont'd)

- **Finding a Basis for the Row Space:** Given a matrix A, reduce it to an RREF matrix R. Then the non-zero rows of R are linearly independent, and they form a basis for the row space of R and also for the row space of A.
- **Alternate Method:** Note that the rows of A correspond to the columns of A^T. Hence, we can find a basis for Row A by using the previous method to find a basis for Col A^T. This method can be used if it is desired to find a basis for Row A consisting of actual rows of A.
- **Final Observation:** There is no direct relationship between Nul A, Col A, and Row A (in case of Nul A and Col A, in general they are not even subspaces of the same space). However, we would have noticed an interesting connection between them, namely:

  $$\textbf{dimension of Col } A = \textbf{dimension of Row } A$$

  We will now formalize this in a theorem.

The Rank-Nullity Theorem

Definition: If A is an $m \times n$ matrix, then the column rank of A is defined to be dim (Col A). Similarly, the row rank of A is defined to be dim (Row A). The nullity of A is defined to be dim (Nul A).

Theorem 2 (Rank-Nullity Theorem for Matrices):

a. The row rank and column rank of a matrix A are equal. This number is called the **rank** of A.
b. The rank of A is equal to the number of pivot positions in the RREF matrix obtained from A.
c. Finally:

$$\textbf{rank } (A) + \textbf{nullity } (A) = n = \textbf{number of columns of } A$$

The Rank-Nullity Theorem (Cont'd)

Observation (Proof): The statements (a) and (b) of the Rank Theorem follow from our discussion of finding the basis for Col A and Row A respectively. In each case, the number of basis vectors corresponded to the number of pivot elements in the RREF matrix R of a given matrix A. For statement (c), observe that pivot columns of R correspond to basis vectors of Col A (leading or basic variables of the homogeneous system), whereas the remaining columns correspond to basis vectors of Nul A (free variables of homogeneous system).

Since the total number of columns = number of variables = n, we get:
$n =$ (number of basis vectors of Col A) + (number of basis vectors of Nul A) = rank (A) + nullity (A), as required.

Important to Remember

Note 1: The column space of a matrix A is all of $\mathbb{R}^m$ if and only if the equation $AX = b$ has a solution for each b in $\mathbb{R}^m$

Note 2: If A is an invertible $m \times m$ matrix, then its columns form a basis for $\mathbb{R}^m$ (from TIMT for invertible matrices)

Corollary to Rank-Nullity Theorem: A square $m \times m$ matrix A is invertible if and only if rank $(A) = m$.

Infinite-Dimensional Vector Spaces

This slide and the next slide are only for some curious students!

- **Remark:** We had earlier seen that the space $\mathbb{R}[t]$ of all polynomials with real coefficients is infinite-dimensional. We had also discussed the case of the space $C[0,1]$ of continuous functions, and by using Proposition 18, we could see that it is also infinite-dimensional. We would like to extend the concepts of linear dependence/independence and bases to infinite-dimensional spaces. We therefore make the following definitions:
- **Definition:** A (possibly infinite) set S of vectors in a vector space V is said to be linearly independent if every finite subset of S is linearly independent.
- **Definition:** If S is a subset of V, then Span $S =$ smallest subspace of V which contains S. This definition covers the case of infinite subsets S and coincides with our earlier definition for finite subsets.

Infinite-Dimensional Vector Spaces (Cont'd)

- **Remark:** Actually, it can be seen that Span S is nothing but the set of all possible finite linear combinations of vectors in S; i.e. Span $S = \{\sum_{\text{finite}} c_i v_i \;:\; v_i \in S, c_i \in F\}$. This also coincides with our earlier definition in the case that S is finite.
- **Definition:** A subset S of a space V is a basis of V if S is linearly independent and Span $S = V$.
- **Example:** The set $B = \{1, t, t^2, t^3, \cdots\} = \{t^n \;:\; n \in \mathbb{N}\}$ is a basis for the space $\mathbb{R}[t]$ of all polynomials with real coefficients.
- **Theorem 3 (Basis Theorem or Fundamental Theorem of Linear Algebra):** Every vector space V has a basis; more precisely, if $v \in V$ is a non-zero vector, then there exists a basis B of V such that $v \in B$.
- **Remark:** The proof of the above requires advanced concepts from set theory, and is usually not given in elementary linear algebra textbooks. Moreover, it is a pure existence proof, it doesn't provide any technique for constructing a basis. For some spaces, it has not been possible to provide a construction for a basis.