Linear Algebra

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Lecture 17

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Linear Dependence

- **Definition 1:** Let $v_1, v_2, \ldots, v_p$ be a finite list of vectors in a vector space V. Then the vectors are said to be (linearly) dependent if there exist scalars $c_1, c_2, \ldots, c_p$, not all zero, such that $c_1 v_1 + c_2 v_2 + \cdots + c_p v_p = 0$.
- **Definition 2:** If a list of vectors is not linearly dependent, it is said to be (linearly) independent. In other words, if the list is linearly independent, and $c_1 v_1 + c_2 v_2 + \cdots + c_p v_p = 0$, then all the c_i's must be 0.
- We usually simply say $v_1, v_2, \ldots, v_p$ are dependent or independent.
- We have given the definition in terms of list of vectors rather than set of vectors, because list is a more general concept in this situation. A finite set can easily be considered as a list and then the above definition can be applied to it.

Linear Dependence (Conti . . .)

Consequences of the Definitions

- Remark 1: Any list which contains the 0 vector has to be linearly dependent. In fact, the single zero vector 0 is always linearly dependent (LD).
  Proof: Let $v_1, \ldots, v_p$ be a list of vectors, and suppose $v_k = 0$ for some $k, k = 1, 2, \ldots, p$. Put $c_k = 1$ and $c_i = 0$ for $i \neq k$. Then:
  $c_1 v_1 + \cdots + c_p v_p = 0v_1 + \cdots + 0v_{k-1} + 1.v_k + 0V_{k+1} + \cdots + 0v_p = 1.0 = 0.$
  Since not all the c_i's are 0, in fact $c_k = 1$, the list of vectors is LD as required.

  Moreover the single zero vector 0 is LD because 1.0=0.
- Remark 2: A single non-zero vector is linearly independent.

  [Hint: For $a \in \mathbb{F}$ and $v \in V$, $av = 0 \implies a = 0$ or $v = 0$.]
- Remark 3: A list of two non-zero vectors is linearly dependent only if one of the vectors is a scalar multiple of the other.

Linear Dependence (Conti . . .)

Consequences of the Definitions

- Remark 4: A list of non-zero vectors is linearly dependent if and only if at least one of the vectors is a linear combination of the others.
  Proof: ($\Longrightarrow$) Given: A list of vectors, say $v_1, \ldots, v_p$, is LD. We have to show that at least one of the vector is expressible as a linear combination of others. Since $v_1, \ldots, v_p$ are LD. Hence,
  $$c_1 v_1 + \cdots + c_p v_p = 0, \tag{1}$$
  where not all $c_i = 0$. Suppose $c_k \neq 0$. Rewrite 1 as:
  $$c_k v_k = -c_1 v_1 - c_2 v_2 - \cdots - c_p v_p, \tag{2}$$
  where the RHS of 2 contains all vectors except v_k. Since $c_k \neq 0$, multiplying by c_k^{-1} on both sides, we get:
  $$1.v_k = -(c_k^{-1} c_1) v_1 - (c_k^{-1} c_2) v_2 - \cdots - (c_k^{-1} c_p) v_p, \tag{3}$$
  i.e. v_k is a linear combination of the rest.

Linear Dependence (Conti . . .)

- Proof of Remark 4 (Cont'd)
  Proof: ($\Longleftarrow$) Given that at least one vector in a list is expressible as a linear combination of the rest. We have to show that the list is linearly dependent. Suppose
  $$v_k = c_1 v_1 + \cdots + c_p v_p, \tag{4}$$
  where the RHS of 4 contains all the vectors other than v_k. Rewrite 4 as: $c_1 v_1 + \cdots - v_k + \cdots + c_p v_p = 0$, we get that the coeff of v_k (c_k) satisfies $c_k = -1 \neq 0$, the list is LD.
- Remark 5: Consequently, any list which contains a repeated vector must be linearly dependent. A list which is linearly independent corresponds to a set.
- Remark 6: Any list which contains a linearly dependent list is linearly dependent or Superset of a linearly dependent set is linearly dependent.
- Remark 7: Any subset of a linearly independent set is linearly independent . [By convention, null-set ϕ is LI]