Linear Algebra

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Self Study Material on the Determinant

The Determinant

- **Remark:** Propositions about determinants will be numbered independently as Prop D1, Prop D2, etc.
- **Definition of the Determinant:** If $A \in \mathbb{F}^{2\times 2}$ where $A = [a_{ij}]$, then $\det A$ is defined to be the scalar $a_{11}a_{22} - a_{12}a_{21}$. Thus det is a function from $\mathbb{F}^{2\times 2}$ to $\mathbb{F}$.
- We extend this definition recursively to $\mathbb{F}^{n\times n}$ as follows:
- **Notation:** If $A \in \mathbb{F}^{n\times n}$, let $A_{i,j}$ denote the $(n-1)\times(n-1)$ matrix obtained from A by omission of the i-th row and j-th column.
- **Column expansion formula:** A formula for the determinant is given by: $\det A = \sum (-1)^{i+j} a_{ij} \det A_{i,j}$, where the summation is taken for $i = 1$ to n.
- **Row expansion formula:** Another formula for the determinant is given by: $\det A = \sum (-1)^{i+j} a_{ij} \det A_{i,j}$, where the summation is taken for $j = 1$ to n.

The Determinant (Conti'd)

- **Proposition D1:** The following holds for the determinant of a square matrix A:
  1. If the matrix A' is obtained from A by interchanging two rows, then $\det A' = -\det A$.
  2. If the matrix A' is obtained from A by multiplying some row by $\lambda \in \mathbb{F}$, then $\det A' = \lambda \det A$.
  3. If the matrix A' is obtained from A by adding a multiple of one row to another row, then $\det A' = \det A$.
- **Remark 1:** The above indicates what happens to the determinant when an elementary row operation – interchange, scaling, or replacement – is applied.
- **Remark 2:** It follows directly from the above that if the rows of A are linearly dependent, then $\det A = 0$.

Procedure for Computing the Determinant

- **Proposition D2:** If an $n \times n$ matrix A is upper triangular, then $\det A = a_{11}a_{22}\dots a_{nn}$
- **Corollary D2.1:** In order to determine the determinant of an $n \times n$ matrix, use elementary row operations of interchange and replacement type only to reduce A to an upper triangular matrix A'. If r is the number of row interchanges carried out, then $\det A = (-1)^r \det A'$.
- **Remark 1:** This follows directly from Proposition 40 and the definition (using the column expansion).
- **Remark 2:** The above method is far less computationally intensive than using either row or column expansion.
- **Note:** there is another formula, and that is equally inefficient.
- **Proposition D3:** An $n \times n$ matrix A is invertible if and only if $\det A \neq 0$.
- **Remark:** The above gives another useful property equivalent to invertibility for square matrices. Consequently, we need to extend our theorem on invertibility of matrices (see next slide).

Very Important Theorem (VIT)–Version 3.0
The Invertible Matrix Theorem (TIMT)

Theorem: The following are equivalent for an $m \times m$ square matrix A:

a. A is invertible.
b. A is row equivalent to the identity matrix I_m.
c. The homogeneous system $AX = 0$ has only the trivial solution.
d. The system of equations $AX = b$ has at least one solution for every b in $\mathbb{R}^m$.
e. Nullity $(A) = 0$.
f. Rank $(A) = m$.
g. The columns of A form a basis for $\mathbb{R}^m$.
h. Det $A \neq 0$.

Further Properties of the Determinant

- **Proposition D4:** For all $A, B \in \mathbb{F}^{n\times n}, \det(AB) = (\det A)(\det B)$
- **Corollary D4.1:** If A is invertible, then $\det A^{-1} = (\det A)^{-1}$.
- **Remark:** While $\det(AB) = (\det A)(\det B)$, in general $\det(A+B) \neq \det A + \det B$. So the determinant is not a linear function or linear transformation.
- **Remark:** A linear transformation from a vector space V to its underlying field $\mathbb{F}$ is known as a linear functional. However, the determinant is not a linear functional.
- **Proposition D5:** For all $A \in \mathbb{F}^{n\times n}$, $\det A^T = \det A$.

Cramer's Rule

- **Remark:** If you have not studied this topic before, it is nicely presented in the book by Lay: Section 3.3
- **Definition:** For any $n \times n$ matrix A and any vector b in $\mathbb{R}^n$, define $A_i(b)$ to be the matrix obtained by replacing the i-th column of A by b.
- **Proposition D6 (Cramer's Rule):** Let A be any invertible $n \times n$ matrix. For any vector b in $\mathbb{R}^n$, the unique solution X of $AX = b$ has entries given by:

$$x_i = (\det A_i(b))/(\det A) \quad \text{for} \quad i = 1, 2, \ldots, n$$

- Cramer's Rule is (usually) not a practical method for solving systems of linear equations since it requires computation of $(n+1)$ determinants.

Application of Cramer's Rule

- **Terminology and Notation:** For any $n \times n$ matrix A, we define the cofactor $C_{ij} = (-1)^{i+j} A_{ij}$.
- **Definition:** The classical adjoint of A (written adj A) is the matrix whose entries are the cofactors of A transposed. In other words, adj A is the matrix:

$$\begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & \dots & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}$$

- **Proposition D7: Inverse Formula:** Let A be any invertible $n \times n$ matrix. Then:

$$A^{-1} = (1/\det A)(\mathrm{adj} A)$$

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Application of Determinants to Areas and Volumes

Proposition D8:

(a) If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$.

(b) If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det A|$.

Proposition D9:

(a) Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then $\{\text{area of T(S)}\} = |\det A| \times \{\text{area of S}\}$.

(b) Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear transformation determined by a 3×3 matrix A. If S is a parallelepiped in $\mathbb{R}^3$, then $\{\text{area of T(S)}\} = |\det A| \times \{\text{area of S}\}$.

Proposition D10: The conclusions of Proposition D9 hold whenever S is a region in $\mathbb{R}^2$ with finite area or a region in $\mathbb{R}^3$ with finite volume. In other words: $\{\text{area or volume of T(S)}\} = |\det A| \times \{\text{area or volume of S}\}$.