Linear Algebra

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Lecture 25

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Linear Transformation

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Definition: Let V and W be vector spaces over the same field $\mathbb{F}$. A map or function $T : V \longrightarrow W$ from a vector space V to a vector space W is said to be a linear transformation (or briefly, linear) if:

a. $T(u+v) = T(u) + T(v)$ for all $u, v \in V$ [Additivity property].
b. $T(cu) = cT(u)$ for all $u \in V$ and for all scalars $c \in \mathbb{F}$ [Homogeneity property].

- **Note 1:** In the special case where $V = W$, the linear transformation T is called a **linear operator** on the vector space V.
- **Note 2:** We write either $T(v)$ or Tv to indicate the image of the vector v under the transformation T.

Linear Transformation (Cont'd)

Example:

1. Two trivial examples of linear transformations:
   - The zero transformation $0 : V \longrightarrow W$ such that $0(u) = 0$ for all u in V.
   - The identity transformation $I : V \longrightarrow V$ such that $I(u) = u$ for all u in V.
2. **Projection Map:** Consider the function $P_i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by $P_i(x_1, x_2, \ldots, x_i, \ldots, x_n) = (0, 0, \ldots, x_i, 0, \ldots, 0)$, where all coordinates other than the i-th coordinate are replaced by 0. Then P_i is a linear transformation.

Note: We can extend this idea by projecting onto any selection of coordinates. Each of the functions so obtained is a linear transformation.

Linear Transformation (Cont'd)

Remarks

1. If T is linear, then $T(0) = 0$ and $T(-v) = -T(v)$, proof left as an exercise.
2. If T is linear, then T "preserves" linear combinations, i.e., $T(c_1 v_1 + c_2 v_2 + \cdots + c_k v_k) = c_1 T(v_1) + c_2 T(v_2) + \cdots + c_k T(v_k)$, proof left as an exercise.
3. **Key subspaces associated with T:**
   - The **kernel** of T, $Ker\ T = \{v \in V \ : \ Tv = 0 \in W\}$ is a subspace of V. Some books call the kernel the null space of T, written Nul T.
   - The **Range** of T, $Range\ T = \{w \in W \ : \ w = Tv \text{ for some } v \in V\}$ is a subspace of W.
4. It is easy to see that T is injective if and only if Ker $T = \{0\}$, proof left as an exercise.