Linear Algebra

Sartaj Ul Hasan

Department of Mathematics
Indian Institute of Technology Jammu
Jammu, India - 181221

Email: sartaj.hasan@iitjammu.ac.in

Lecture 32

(Oct 23, 2019)

Linear Transformation

Change of Basis

Motivation: We would like to know what happens to the matrix of a linear transformation T if the basis gets changed. We will restrict our attention to the case when T is a linear operator from V to V (finite-dimensional). We start with the following preliminary result.

Proposition 29: Let $B = \{u_1, u_2, \ldots, u_n\}$ and $C = \{v_1, v_2, \ldots, v_n\}$ be two ordered bases of a vector space V. Then for any $x \in V$, there is an invertible $n \times n$ matrix P such that $[x]_C = P[x]_B$.

Note: The columns of P are the C-coordinate vectors of the basis B. This matrix P is called the change of coordinates matrix from B to C. If the bases need to be clearly identified, we write it as: $P_{B \to C}$

Important Remark: To change coordinates between the two bases, we need the *coordinate vectors of the "old basis" relative to the "new basis"*.

Proof of Proposition 29

Let $x \in V$; since B is a basis for V, we can write

$$x = b_1 u_1 + \cdots + b_n u_n \tag{1}$$

$$[x]_B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \tag{2}$$

Since C is also a basis for V, we can write:

$$\begin{aligned} u_1 &= A_{11} v_1 + A_{21} v_2 + \cdots + A_{n1} v_n \\ &\vdots \\ u_n &= A_{1n} v_1 + A_{2n} v_2 + \cdots + A_{nn} v_n \end{aligned} \tag{3}$$

From (3), we get that

$$[u_i]_C = \begin{bmatrix} A_{1i} \\ \vdots \\ A_{ni} \end{bmatrix} \quad \text{for} \quad i = 1, 2, \ldots, n. \tag{4}$$

Proof of Proposition 29 (Cont'd)

Substituting from (3) in (1), we get:

$$x = b_1(A_{11}v_1 + \cdots + A_{n1}v_n) + \cdots + b_n(A_{1n}v_1 + \cdots + A_{nn}v_n) \tag{5}$$

Re-arranging and collecting the coefficients of $v_1, \ldots, v_n$, we get:

$$x = (A_{11}b_1 + \cdots + A_{1n}b_n)v_1 + \cdots + (A_{n1}b_1 + \cdots + A_{nn}b_n)v_n \tag{6}$$

Hence,

$$[x]_C = \begin{bmatrix} A_{11}b_1 + \cdots + A_{1n}b_n \\ \vdots \\ A_{n1}b_1 + \cdots + A_{nn}b_n \end{bmatrix} = \left[A_{ij}\right]_{n\times n} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = P[x]_B. \tag{7}$$

As noted already from (4), the columns of the matrix P are nothing but the coordinate vectors of the old basis, i.e., of the u_i's in terms of the new basis, i.e. in terms of v_i's.

Proof of Proposition 29 (Cont'd)

Finally, we note that P must be invertible for the following reason: the coordinate mapping is an isomorphism from V to $\mathbb{F}^n$. Since B is a basis of V, it goes to basis of $\mathbb{F}^n$ under the coordinate mapping with regard to basis C (Prop 27(a)). Since columns of P form a basis of $\mathbb{F}^n$, P is invertible by TIMT (g). This completes the proof of Proposition 29.

Summary – Change of Basis

Important Remark: To change coordinates between two bases, we need the coordinate vectors of the old basis relative to the new basis. These become the columns of the change of coordinates matrix P. In practice, $P = Q^{-1}$, where Q has as its columns the coordinate vectors of the new basis C relative to the old basis B. In most of the applications, the old basis is the standard basis for $\mathbb{R}^n$, so Q can be found directly.

Example for Change of Basis

Consider the old basis $\alpha = \{e_1, e_2\}$ of $\mathbb{R}^2$. The new basis is $\beta = \{u_1, u_2\}$, where $u_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$. Construct the matrix Q which has the vector of β as columns.

$$\therefore Q = \underbrace{\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}}_{\text{new basis in terms of old}} \text{, which is change of coordinate matrix } P_{\beta \to \alpha}.$$

Then, the change of basis matrix $P = P_{\alpha \to \beta} = Q^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$. Let us check with a specific vector, say $u = \begin{bmatrix} 3 \\ 7 \end{bmatrix}_\alpha$. Then,

$$[u]_\beta = P \begin{bmatrix} 3 \\ 7 \end{bmatrix}_\alpha = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix}_\alpha = \begin{bmatrix} -26 \\ 11 \end{bmatrix}_\beta$$

Check that: $-26u_1 + 11u_2 = -26 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 11 \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$.

Example (Cont'd): Verification of Remark

Verification of the remark that columns of P are the coordinate vectors of old basis in terms of the new basis.

- Now $\begin{bmatrix} 3 \\ -1 \end{bmatrix}_\beta = 3u_1 + (-1)u_2 = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = e_1,$

  i.e, $[e_1]_\beta = \begin{bmatrix} 3 \\ -1 \end{bmatrix}_\beta$
- and $\begin{bmatrix} -5 \\ 2 \end{bmatrix}_\beta = (-5)u_1 + 2u_2 = (-5)\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e_2,$

  i.e, $[e_2]_\beta = \begin{bmatrix} -5 \\ 2 \end{bmatrix}_\beta$