Linear Algebra and Applications

Sartaj Ul Hasan

Department of Mathematics
Indian Institute of Technology Jammu
Jammu, India - 181221

Email: sartaj.hasan@iitjammu.ac.in

Lecture 06

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Make-up lecture

Time: 11:00-11:55 am

Invertible Matrices–Quick Revision

Definition

An $m \times m$ (square) matrix A is said to be **invertible** if there exists another square matrix B such that $BA = AB = I_m$ ($m \times m$ identity matrix). B is said to be an **inverse** of A.

- Another terminology: Invertible matrices are also called **nonsingular**. Matrices which are not invertible are said to be **singular**.
- **Observation 1:** The inverse of A if it exists is **unique**, notation A^{-1}.
- **Observation 2:** If A is invertible, then so is A^{-1} and $(A^{-1})^{-1} = A$.
- **Observation 3:** If A and B are invertible, so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.
- **Observation 4 (Generalization of 3):** The product of invertible matrices is invertible, and the inverse is the product of the inverses taken in reverse order. In other words, if $A_1, A_2, \ldots, A_n$ are invertible matrices, then $C = A_1 A_2 \ldots A_n$ is an invertible matrix, and $C^{-1} = A_n^{-1} \ldots A_2^{-1} A_1^{-1}$.

Elementary Matrices

Definition

An $m \times m$ (square) matrix is said to be an **elementary matrix** if it is obtained from the $m \times m$ identity matrix I_m by **a single** elementary row operation.

- **Proposition 5:** If e is an elementary row operation and E is the $m \times m$ elementary matrix $e(I_m)$, then for every $m \times n$ matrix A, $e(A) = EA$.

  Proof: Left as an exercise. (Hint: Try to prove it w.r.t. each row operation!)
- In other words, applying an elementary row operation is the same as left multiplication by the corresponding elementary matrix.

Elementary Matrices (Conti . . .)

Operation	Inverse Operation
I. Interchange row p and q	Interchange row p and q
II. Multiply row p by $k \neq 0$	Multiply row p by $1/k$
III. Add k times row p to row $q \neq p$	Subtract k times row p from row q

- **Example:** Find the inverse of each of the elementary matrices

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- E_1, E_2 and E_3 are of type I, II and III resp. so the table gives:

$$E_1^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1, \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/9 \end{bmatrix}, \quad E_3^{-1} = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Elementary Matrices (Conti . . .)

Proposition 6: Every elementary matrix E is invertible, and E^{-1} is also an elementary matrix (of the same type).

Elementary Matrices (Conti . . .)

Proposition 6: Every elementary matrix E is invertible, and E^{-1} is also an elementary matrix (of the same type).

Proof: Let E be any elementary matrix, and let e be its corresponding elementary row operation. We know that there is another row operation f of the same type that reverses the action of e. Let F be the elementary matrix corresponding to f. Then:

$$FE = (FE)I = F(EI) = f(e(I)) = I$$

Similarly, $EF = I$, so F is E^{-1}. Actually, we have seen that the inverse of an elementary matrix is also an elementary matrix (of the same type).

Very Important Theorem (VIT)–Version 1.0
The Invertible Matrix Theorem (TIMT)

(This is an important theorem and try to memorise it!)

Theorem 1

The following are equivalent for an $m \times m$ square matrix A:

a. A is invertible.
b. A is row equivalent to the identity matrix.
c. The homogenous system $AX = 0$ has only trivial solution.
d. The system of equations $AX = b$ has a solution for every b in $\mathbb{R}^m$.

Note: We will further extend this theorem as we go deep into the theory of vector spaces and related concepts.