# Linear Algebra and Applications

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# Lecture 08

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# Invertible Matrices (Conti . . . )

**Corollary 1.3:** Suppose a square matrix $A$ is factored as a product of square matrices, i.e. $A = A_1 A_2 \ldots A_n$ (all square matrices). Then $A$ is invertible if and only if each $A_i$ is invertible. (Note that the above Corollary 1.3 applies only if the matrices $A_i$ are square.)

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**Proof:** The fact that the product of invertible matrices is invertible was covered previously. So we have only to show that if $A$ is invertible, then each $A_i$ is invertible. We will first show that the last matrix in the product, i.e. $A_n$ is invertible. Consider the homogeneous system $A_n X = 0$. Multiplying on the left by $A_1 A_2 \dots A_{n-1}$, we get: $A_1 A_2 \dots A_{n-1} A_n X = 0$ or $AX = 0$. Since $A$ is invertible, multiplying on the left by $A^{-1}$, we get $(A^{-1}A)X = 0 \implies IX = 0 \implies X = 0$. In short, the homogeneous system $A_n X = 0$ has only the trivial solution. Hence, by TIMT, $A_n$ is invertible. Now putting $A_1 A_2 \dots A_{n-1} A_n = A$ and multiplying on the right by $A_n^{-1}$, we get $A_1 A_2 \dots A_{n-1} = AA_n^{-1} = B$ (say). The matrix $B$ being a product of two invertible matrices is invertible. Thus by what we have shown above, $A_{n-1}$ is invertible. By repeating this step, we get that each $A_i$ is invertible.

# Invertible Matrices (Conti . . . )

**Corollary 1.4:** (Alternative version of last equivalence in TIMT): The matrix $A$ is invertible if and only if the system of equations $AX = b$ has a unique solution for each and every vector $b$ in $\mathbb{R}^m$.

# Invertible Matrices (Conti . . . )

**Corollary 1.4:** (Alternative version of last equivalence in TIMT): The matrix $A$ is invertible if and only if the system of equations $AX = b$ has a unique solution for each and every vector $b$ in $\mathbb{R}^m$.

**Proof:** Suppose that the matrix $A$ is invertible. Then by TIMT, the system of equations $AX = b$ has at least one solution for each $b$. But further, if $u$ is any solution, then:

$$Au = b \implies (A^{-1}A)u = A^{-1}b \implies u = A^{-1}b.$$

In short, the system has the unique solution $A^{-1}b$.

Proof in the reverse direction follows directly from the statement of TIMT.

# Column form of a Matrix

Before we go for a proof of TIMT, let us represent a matrix in the form of columns. Given an $m \times n$ matrix, we can regards it as consisting of $n$ columns, each of which is an $m$-vector, i.e., given

$$B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix}$$

we can write it in the form $B = [v_1, v_2, \dots, v_n]$, where

$$v_1 = \begin{bmatrix} b_{11} \\ \vdots \\ b_{m1} \end{bmatrix}, v_2 = \begin{bmatrix} b_{12} \\ \vdots \\ b_{m2} \end{bmatrix}, \dots, v_m = \begin{bmatrix} b_{1n} \\ \vdots \\ b_{mn} \end{bmatrix}.$$

# Column form of a Matrix (Conti . . . )

Similarly, given an ordered list of $n$ vectors $v_1, v_2, \ldots, v_n$, not necessarily distinct, we can construct a matrix by taking these as columns, i. e. $B = [v_1, v_2, \ldots, v_n]$

**Matrix product in the column form:** If $A$ is a $k \times m$ matrix so that the product $C = AB$ is well-defined, then $C$ can be easily represented in column form as follows:

$$C = AB = A[v_1, v_2, \ldots, v_n] = [Av_1, Av_2, \ldots, Av_n],$$

i.e. $C$ is the matrix whose columns are $Av_1, Av_2, \ldots, Av_n$.