Linear Algebra

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Lecture 30

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Linear Transformation

A Very Important Linear Transformation

- **Left Multiplication by a Matrix:** Let A be a fixed $m \times n$ matrix. Then the function $T_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ defined by $T_A(X) = AX$ is a linear transformation, where X is a column vector in $\mathbb{R}^n$. It is easy to see that AX is a column vector in $\mathbb{R}^m$ for each column vector $X \in \mathbb{R}^n$.
- **Example:** Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1/2 & 3 \end{bmatrix}_{2\times 3}$ matrix over $\mathbb{R}$. Then corresponding to matrix A, one can associate a linear transformation $T_A : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined as follows:

$$T_A\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) := A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ (1/2)x_2 + 3x_3 \end{bmatrix} \in \mathbb{R}^2.$$

- **We now consider the reverse problem:** Suppose V and W are finite-dimensional vector spaces over the field $\mathbb{F}$, and suppose T is a linear transformation $T : V \longrightarrow W$. We will try to associate a matrix with this linear transformation.

Coordinate Systems

- **Observation 1:** Given a basis for a finite-dimensional vector space V, we recall that a vector can be expressed in one and only one way as a linear combination of the basis vectors. Therefore we make the following definitions.
- **Definition:** An ordered basis for a finite-dimensional space V is a finite sequence of vectors which is linearly independent and spans V. In other words, an ordered basis is a basis with the vectors taken in a specified fixed order.
- Given an ordered basis $B = \{u_1, u_2, \ldots, u_n\}$, we can express any vector u **uniquely** in the form $u = x_1 u_1 + x_2 u_2 + \cdots + x_n u_n$. The scalars x_i are called the **coordinates** of u relative to the (ordered) basis B.

Coordinate Mapping

- **Observation 2:** Given a fixed ordered basis B for a finite-dimensional vector space V, we can set up a correspondence between the vectors of V and n-tuples in $\mathbb{F}^n$ as follows:

$$u \to (x_1, x_2, \ldots, x_n),$$

  where the x_i are the coordinates of u relative to B, i.e. the n-tuple we take is precisely the coordinate sequence of the vector u. This correspondence or mapping has the following properties:
- It is a one-to-one correspondence, i.e. each vector has a unique corresponding n-tuple (sequence), and each n-tuple (sequence) has a unique corresponding vector.
- The sum of two vectors corresponds to the sum of the two n-tuples
- The scalar multiple of a vector corresponds to the scalar multiple of the n- tuple.

Coordinate Mapping (Cont'd)

- Rather than the n-tuple $(x_1, x_2, \ldots, x_n)$, for convenience we express this in a column format, i.e. we prefer to take the column vector:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

- This vector is called the **coordinate vector** of u (relative to B) or the B-coordinate vector of u and is written $[u]_B$. In some books it is called the coordinate matrix.
- The mapping $u \to [u]_B$ is called the coordinate mapping determined by B.
- The discussion above indicates that the coordinate mapping is actually an isomorphism from an n-dimensional vector space V over the field $\mathbb{F}$ to $\mathbb{F}^n$.
- **Note:** We get a different isomorphism for each choice of an ordered basis for V.

Example to illustrate coordinate systems

In $\mathbb{R}^2$, given a vector $v = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, its coordinate vector with regard to the standard ordered basis S is nothing but $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_S$. However, suppose we take a different ordered basis, say $B = \{u_1, u_2\}$, where $u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then the vector $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ can be (by inspection) written as $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Therefore $[u]_B = \begin{bmatrix} -1 \\ 2 \end{bmatrix}_B$.

Let's take another vector, say $v = \begin{bmatrix} 15 \\ 24 \end{bmatrix}$. To find $[v]_B$, we need to find scalars x_1 and x_2 such that $x_1 u_1 + x_2 u_2 = v$, i.e., $x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 24 \end{bmatrix}$, i.e.,

Example to illustrate coordinate systems (Cont'd)

$$x_1 + x_2 = 15$$
$$x_2 = 24$$

i.e,

$$AX = v \tag{1}$$

where A is the matrix with u_1 and u_2 as its columns. But now, since the columns of A form a basis (i.e B), A is invertible. Thus, solution of (1) is given by $X = A^{-1}v$. So if we find A^{-1} we can find out the coordinate vector relative to B for any arbitrary vector $y \in \mathbb{R}^2$. Now:
$A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. Therefore, $[v]_B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 15 \\ 24 \end{bmatrix} = \begin{bmatrix} -9 \\ 24 \end{bmatrix}_B$.
Check: $-9u_1 + 24u_2 = -9 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 24 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 24 \end{bmatrix}$.

Outcome of the above example/discussion

To find the coordinate vector of any x w.r.t new ordered basis $B = \{v_1, \ldots, v_k\}$, we set up the matrix $A = [v_1, v_2, \ldots, v_k]$ with the vectors of B as its columns. Since A is invertible by TIMT (or VIT), we can find A^{-1}. Then $[x]_B = A^{-1}[x]_S$; i.e. we assume that x is given in terms of the standard basis S.