Linear Algebra

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Lecture 13

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Introduction to Vector Spaces

Vector Space

Motivation: Think about the properties of usual vectors in 2-dimensional or 3-dimensional space.

Definition: A **vector space** consists of a non-empty set V of objects (called **vectors**) that can be added, that can be multiplied by an element of a field $\mathbb{F}$ (called a **scalar** in this context), and for which certain axioms hold. If $\mathbf{u}$ and $\mathbf{v}$ are two vectors in V, their sum is expressed as $\mathbf{u}+\mathbf{v}$ and the scalar product of $\mathbf{u}$ by an element a in $\mathbb{F}$ is denoted as $a\mathbf{u}$. These operations are called **vector addition** and **scalar multiplication**, respectively, and the following axioms (there are total 10) are assumed to hold.

- **Axioms for vector addition:**

  1. If $\mathbf{u}$ and $\mathbf{v}$ are in V, then $\mathbf{u}+\mathbf{v}$ is in V.
  2. $\mathbf{u}+\mathbf{v}+\mathbf{w}=(\mathbf{u}+\mathbf{v})+\mathbf{w}$ for all $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in V.
  3. An element $\mathbf{0}$ in V exists such that $\mathbf{u}+\mathbf{0}=\mathbf{u}=\mathbf{0}+\mathbf{u}$ for all $\mathbf{u}\in V$.
  4. For each $\mathbf{u}\in V$, an element $-\mathbf{u}$ in V exists such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}=(-\mathbf{u})+\mathbf{u}$.
  5. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ for all $\mathbf{u}$ and $\mathbf{v}$ in V

Vector Space (Conti . . .)

- **Axioms for scalar multiplication:**

  vi. If $\mathbf{u}$ is in V, then $a\mathbf{u}$ is in V for all a in $\mathbb{F}$.
  vii. $a(\mathbf{u}+\mathbf{v}) = a\mathbf{u} + a\mathbf{v}$ for all $\mathbf{u}$ and $\mathbf{v}$ in V and all a in $\mathbb{F}$.
  viii. $(a+b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$ for all $\mathbf{u}$ in V and all a and b in $\mathbb{F}$.
  viv. $a(b\mathbf{u}) = (ab)\mathbf{u}$ for all $\mathbf{u}$ in V and all a and b in $\mathbb{F}$.
  vv. $1\mathbf{u} = \mathbf{u}$ for all $\mathbf{u}$ in V. Here 1 is the multiplicative identity of $\mathbb{F}$.

With these 10 properties, V **is a vector space over** $\mathbb{F}$, some times denoted by $V_{\mathbb{F}}$ or just by V if the field $\mathbb{F}$ is clear from the context.

Most often, we will be dealing with the field $\mathbb{R}$ of real numbers throughout this course.

Vector Space (Conti . . .)

- **Example 1:** Let V consists of a single object, which we denote by $\mathbf{0}$ and define

$$\mathbf{0}+\mathbf{0}=\mathbf{0} \quad \text{and} \quad a\mathbf{0}=\mathbf{0} \quad \text{for all} \quad a \in \mathbb{F}.$$

  Then V is a vector space.

- **Example 2:** The space $V=\mathbb{R}^n$ of n-tuples (for any $n \geq 1$). There are often referred to as (column) vectors. The base field is $\mathbb{F}=\mathbb{R}$.

If $\mathbf{u}=\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$ and $\mathbf{v}=\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}$ are any two vectors in $\mathbb{R}^n$, then

$\mathbf{u}+\mathbf{v}=\begin{bmatrix} X_1+Y_1 \\ X_2+Y_2 \\ \vdots \\ X_n+Y_n \end{bmatrix}$ and $a\mathbf{u}=\begin{bmatrix} aX_1 \\ aX_2 \\ \vdots \\ aX_n \end{bmatrix}$ for any scalar $a \in \mathbb{R}$.

Note: $\mathbb{R}^n$ is usually referred as **Euclidean space**.

Vector Space (Conti . . .)

- **Example 3:** The space $\mathbb{R}^{m \times n}$ of $m \times n$ matrices with real entries. Again, the base field is $\mathbb{R}$. $\mathbb{R}^{m \times n}$ is a vector space over $\mathbb{R}$

  Remark: If we consider an image, it can be regarded as a rectangular array of numbers corresponding to the light intensity at each pixel. Usually, we restrict the values to be positive integers or even just $0 - 1$. However, while doing the computations in image processing, we treat them as real numbers. So these vector spaces play a major role in image processing.

Vector Space (Conti . . .)

- **Example 4:** The space $C[0,1]$ of continuous functions from the closed interval $[0,1]$ on the real line to $\mathbb{R}$, i.e.

$$C[0,1] = \{f \;:\; f \text{ is a continuous function}, f \;:\; [0,1] \longrightarrow \mathbb{R}\}.$$

Remark: This space and related spaces play a major role in signals and systems, since an analogue signal is usually thought of as a continuous function of time. In other words, a signal is nothing but a "vector" in such a vector space.
Note: The above is an example of a "function" space.

Vector Space (Conti . . .)

- **Example 5:** The space $\mathbb{R}^\infty$ of real sequences is a vector space over $\mathbb{R}$, i.e.

$$\mathbb{R}^\infty = \{< a_n > : < a_n > \text{ is a sequence with real number terms}\}.$$

- Of more interest than $\mathbb{R}^\infty$ itself, is c, the subset of convergent sequences. It is also a vector space.
- **Note:** the above are examples of “sequence” spaces. Sequence spaces also play a major role in the study of signals, specifically discrete or digital signals

Vector Space (Conti . . .)

- **Example 6:** The space $\mathbb{R}_n[t]$ of polynomials of degree $\leq n$ with real coefficients.
  Note: The zero polynomial, which technically does not have any degree, is regarded as an element of $\mathbb{R}_n[t]$ for all $n = 0, 1, 2, \ldots$
- **Example 7:** The space $\mathbb{R}[t]$ of all polynomials with real coefficients.
  Note: These two examples are closely related to each other. We can see that $\mathbb{R}_n[t]$ is actually a subset of $\mathbb{R}[t]$ (for all n).
- **Example 8: A set that is NOT a vector space**
  Take $V = \mathbb{R}^2, \mathbb{F} = \mathbb{R}$. Define "vector addition" and "scalar multiplication" as follows:
  If $\mathbf{u} = (u_1, u_2), \mathbf{v} = (v_1, v_2)$, then $\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)$ and $a\mathbf{u} = (au_1, 0)$ for $a \in \mathbb{R}$.
  Since $1(-2, 3) = (-2, 0) \neq (-2, 3)$, V is NOT vector space over $\mathbb{F}$.
  Note that except this property, the remaining 9 properties are satisfied.

Vector Space (Conti . . .)

- **Example 9:** Let $V = \mathbb{R}^+$, the set of all positive real numbers. For $u, v \in \mathbb{R}^+$ and $t \in \mathbb{R}$, define:
  $u + v = uv$ [Vector addition is numerical multiplication]
  $tu = u^t$ [Scalar multiplication is numerical exponentiation]
  Then V is vector space over $\mathbb{R}$.
  Here $0 = 1, -u = \frac{1}{u}$.
- **Example 10:** $\mathbb{R}_\mathbb{R}, \mathbb{Q}_\mathbb{Q}, \mathbb{C}_\mathbb{C}$ are vector spaces. More generally, for any field $\mathbb{F}$, $\mathbb{F}_\mathbb{F}$ is a vector space.
- **Example 11:** $\mathbb{R}_\mathbb{Q}$ and $\mathbb{C}_\mathbb{R}$ are vector spaces.
- **Example 12:** If $\mathbb{F}$ is a field, then for $n \geq 1$, $\mathbb{F}^n$ is a vector space over $\mathbb{F}$. In this example, replace $\mathbb{F}$ by $\mathbb{Z}_p$, $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{C}$ and then verify all the vector space properties.

Vector Space (Conti . . .)

Direct Consequence of the Vector Space Definition Properties

Proposition 7: Let V be a vector space. Then:

(a) The zero vector is unique.

(b) The additive inverse vector of any vector $\mathbf{u}$ is unique; we use the notation $-\mathbf{u}$ for the inverse vector.

(c) $0\mathbf{u} = \mathbf{0}$ for every vector $\mathbf{u}$.

(d) $a\mathbf{0} = \mathbf{0}$ for every scalar a.

(e) $-\mathbf{u} = (-1)\mathbf{u}$ for every vector $\mathbf{u}$.

Proof: Left as an exercise.