Linear Algebra

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Lecture 33

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Linear Transformation

Similarity of Matrices

- **Definition:** An $n \times n$ matrix B is said to be similar to an $n \times n$ matrix A if there exists an invertible matrix P such that $B = PAP^{-1}$.
- **Proposition 30:** Similarity of matrices is an equivalence relation on $\mathbb{F}^{n\times n}$, i.e. the set of $n \times n$ matrices with entries taken from a field $\mathbb{F}$.

  Proof:

  1. Reflexive: If $A \in \mathbb{F}^{n\times n}$, then $A = IAI^{-1}$, so A is smilar to A.
  2. Symmetric: Suppose B is similar to A. Then $\exists$ P s. t. $B = PAP^{-1}$. Put $Q = P^{-1}$. Thus, $QBQ^{-1} = P^{-1}(PAP^{-1})(P^{-1})^{-1} = A$. Therefore A is similar to B.
  3. Transitive: Suppose B is similar to A and C is similar to B. Then $B = PAP^{-1}$ and $C = QBQ^{-1}$, i.e.,
     $C = Q(PAP^{-1})Q^{-1} = (QP)A(P^{-1})Q^{-1}) = (QP)A(PQ)^{-1}$.
- **Remark:** In view of Proposition 30, if B is similar to A, then A is similar to B (*symmetry property of equivalence relations*), so we can simply say that A and B are similar matrices.

Effect of Change of Basis

Proposition 31: Suppose A and B are the matrices of the linear operator T relative to the ordered bases α and β, respectively. Then A and B are similar matrices, in fact $B = PAP^{-1}$, where $P = P_{\alpha \to \beta}$ is the change of basis matrix.

Proof: We use the fact that if P is the change of basis matrix from α to β, then P^{-1} is the change of basis matrix from β to α. Let $A = [T]_\alpha$. Therefore, for any $v \in V$, we have:

$$\begin{aligned}(PAP^{-1})[v]_\beta &= (PA)P^{-1}[v]_\beta = (PA)[v]_\alpha = P(A[v]_\alpha) \\ &= P([T]_\alpha[v]_\alpha) = P[Tv]_\alpha = [Tv]_\beta \\ &= [T]_\beta[v]_\beta\end{aligned}$$

Since the above holds for all vectors $v \in V$, it follows that $PAP^{-1} = [T]_\beta = B$.

Note: In most of the applications, we take α as the standard basis S. However, the result holds in general.

Example on Effect of Change of Basis

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation given by $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ 3x+4y \end{bmatrix}$. Let $\alpha = \{e_1, e_2\}$ be standard basis of $\mathbb{R}^2$ and let $\beta = \{u_1, u_2\}$, where $u_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ another basis of $\mathbb{R}^2$. Let us first determine $A = [T]_\alpha$, matrix of T relative to standard basis. Now

$T(e_1) = T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 1e_1 + 3e_2$ and

$T(e_2) = T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2e_1 + 4e_2$. therefore,

$$[T]_\alpha = A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Applying Prop 31, the matrix relative to the new basis β would be $B = PAP^{-1}$, where $P = P_{\alpha\to\beta}$, the change of basis matrix, i.e,

$$B = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} -38 & -102 \\ 16 & 43 \end{bmatrix}.$$

Example on Effect of Change of Basis (Cont'd)

Let us verify our calculation with an example, say $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_\alpha = \begin{bmatrix} -7 \\ 3 \end{bmatrix}_\beta$,

Since $[v]_\beta = P_{\alpha\to\beta}[v]_\alpha = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -7 \\ 3 \end{bmatrix}_\beta$.

$$\therefore \quad [Tv]_\alpha = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_\alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix}_\alpha = \begin{bmatrix} 5 \\ 11 \end{bmatrix}_\alpha$$

$$\text{and} \quad [Tv]_\beta = \begin{bmatrix} -38 & -102 \\ 16 & 43 \end{bmatrix}_\beta \begin{bmatrix} -7 \\ 3 \end{bmatrix}_\beta = \begin{bmatrix} -40 \\ 17 \end{bmatrix}_\beta$$

Now the vector $w = Tv \in \mathbb{R}^2$ corresponding to coordinate vector $\begin{bmatrix} -40 \\ 17 \end{bmatrix}_\beta$ is given by

$$w = -40u_1 + 17u_2 = -40 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 17 \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix} = 5e_1 + 11e_2 = \begin{bmatrix} 5 \\ 11 \end{bmatrix}_\alpha.$$

$$\text{Thus} \quad w = \begin{bmatrix} -40 \\ 17 \end{bmatrix}_\beta = \begin{bmatrix} 5 \\ 11 \end{bmatrix}_\alpha.$$

Utility of idea of similarity in matrix computation

Suppose $B = PAP^{-1}$. Then

$$B^k = \underbrace{PAP^{-1}.PAP^{-1}.\dots.PAP^{-1}}_{(k \text{ times})} = PA^kP^{-1}.$$

Hence, if it is easy to find the power of A, then it's easy to find power of B.

The easiest Case is when the matrix A is diagonal, i.e., $A = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$. Then $A^k = \operatorname{diag}\{\lambda_1^k, \dots, \lambda_n^k\}$.

Unfortunately, not every matrix is similar to a diagonal matrix. But in many important applications they are.

The idea behind Proposition 31

Think of a matrix A as a system.

The input is a vector X given as coordinate vector with regard to a basis α, and the output is again a vector Y, also given in terms of α.

Diagram:

$$[X]_\alpha \longrightarrow \boxed{\text{A}} \longrightarrow [Y]_\alpha$$

However, now suppose that the input is given as a coordinate vector with regard to basis β, and the output is also desired in this form.

So, we have to proceed as follows (Recall that if the change of basis matrix from α to β is P, the change of basis matrix from β to α is P^{-1}) :

$$[X]_\beta \xrightarrow{P^{-1}} [X]_\alpha \longrightarrow \boxed{\text{A}} \longrightarrow [Y]_\alpha \xrightarrow{P} [Y]_\beta$$

We now express the above system diagram in matrix term. Recall that when a product of matrices is to operate (multiply)on a vector, we proceed from right to left. So

$$[T]_\beta[X]_\beta = (PAP^{-1})[X]_\beta, \text{ i.e., } [T]_\beta = B = PAP^{-1}$$