Linear Algebra

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Lecture 16

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Span of a Set

Span of a Set of Vectors

- **Definition 1**: A **linear combination** of finitely many given vectors is any sum of scalar multiples of the vectors.
- **Definition 2**: Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, , \mathbf{v_p}\}$ be a finite set of vectors in a vector space V. Then the Span of S is the set of all vectors that can be written as linear combinations of the vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_p}$.
- Symbolically, Span $S = \{c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_p\mathbf{v_p} \ : \ c_i \in \mathbb{F}\}$, where $\mathbb{F}$ is the underlying field.

Subspace Spanned by a Set

Proposition 10: If $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, , \mathbf{v_p}\}$ is a set of vectors in a vector space V, then Span $S =$Span $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, , \mathbf{v_p}\}$ is a subspace of V.

Span of a Set(Conti . . .)

Proof of Proposition 10: Let us use the first test for subspaces (Prop 8): The zero vector $\mathbf{0} = 0\mathbf{v_1} + 0\mathbf{v_2} + \cdots + 0\mathbf{v_p}$is a linear combination of the v's. If $\mathbf{w_1} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_p\mathbf{v_p}$ and $w_2 = d_1\mathbf{v_1} + d_2\mathbf{v_2} + \cdots + d_p\mathbf{v_p}$ are two linear combinations, then so is

$$\begin{aligned}\mathbf{w_1} + \mathbf{w_2} &= (c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_p\mathbf{v_p}) + (d_1\mathbf{v_1} + d_2\mathbf{v_2} + \cdots + d_p\mathbf{v_p}) \\ &= (c_1 + d_1)\mathbf{v_1} + (c_2 + d_2)\mathbf{v_2} + \cdots + (c_p + d_p)\mathbf{v_p}\end{aligned}$$

(Note: we have here used some of the axioms without specifically mentioning them)

If c is any scalar, and w_1 is a linear combination as above, then $c\mathbf{w_1} = c(c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_p\mathbf{v_p}) = cc_1\mathbf{v_1} + cc_2\mathbf{v_2} + \cdots + cc_p\mathbf{v_p}$ is again a linear combination.

Span of a Set(Conti . . .)

Corollary10.1: Let V be a vector space.

(a) If U and W are two subspaces of V, then $U \cap W$ (i.e. the intersection of U and W) is also a subspace of V.

(b) If $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_p}\}$ is a set of vectors in a vector space V, then Span S = Span $\{v_1, v_2, \ldots, v_p\}$ is the smallest subspace which contains S, i.e. if W is a subspace such that $S \subseteq W$, then Span $S \subseteq W$.

Remark 1: Proof is left as an exercise (must do!)

Remark 2: In terms of this, Span S is sometimes described as the intersection of all subspaces of V containing S. (Also left as an exercise.)

Remark 3: Show by means of an example that union of two subspaces of a vector space need not be a subspace (Hint: Take two line passing through the origin in $\mathbb{R}^2$).

In fact, if If U and W are two subspaces of V, then $U \cup W$ is subspace of $V \iff$ either $U \subseteq W$ or $W \subseteq U$. (Left as an exercise!)