Linear Algebra

Sartaj Ul Hasan

Department of Mathematics
Indian Institute of Technology Jammu
Jammu, India - 181221

Email: sartaj.hasan@iitjammu.ac.in

Lecture 22

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Dimension of Subspaces

Proposition 18: If W is a proper subspace of a finite-dimensional space V, then W is also finite-dimensional and $0 < \dim W < \dim V$.

(Note: A proper subspace is a subspace different from the zero subspace and the entire space.)

Proof: Since W is a proper subspace, it contains a vector $w_1 \neq 0$. If w_1 spans W, then W is finite-dimensional. If not, there is a vector w_2 in W outside Span w_1, and by adjoining w_2 to w_1, we still have a linearly independent set (by Proposition 14). Continuing in this way, we get a basis of W with at most $\dim V$ elements (upper limit comes because of Proposition 12). Hence, W is finite-dimensional and $\dim W \leq \dim V$. But since W is a proper subspace, there is a vector v outside W. Adjoining v to any basis of W, we still have a linearly independent set. Hence, $\dim W$ must be (strictly) less than $\dim V$, i.e. $\dim W < \dim V$.

Application of Proposition 18

Question: Is $C[a, b]$, i.e. the space of continuous real-valued functions defined on an arbitrary closed interval, finite-dimensional or infinite-dimensional ?

Answer: Infinite-dimensional.

Proof: Suppose BWOC that $C[a, b]$ is finite-dimensional. Now, consider $P[a, b]$, the set of polynomial functions with domain $[a, b]$. Clearly, $P[a, b] \subseteq C[a, b]$ since all polynomial functions are continuous. Furthermore, it is easy to see that $P[a, b]$ is actually a subspace of $C[a, b]$. At this stage, we recall the result that $\mathbb{R}[t]$ is infinite-dimensional. $\mathbb{R}[t]$ and $P[a, b]$ are not exactly the same space, but they are very similar. So the proof that $\mathbb{R}[t]$ is infinite-dimensional can be applied to $P[a, b]$ with minor modifications to show that $P[a, b]$ is infinite dimensional. But by Proposition 18, any subspace of a finite-dimensional vector space must be finite-dimensional. This is a contradiction. Hence, $C[a, b]$ must be infinite-dimensional.

Additional Notes about $P[a, b]$

The importance of $P[a, b]$ follows from the following deep result:

Weierstrass Approximation Theorem: Let $f \in C[a, b]$ and let any $\epsilon > 0$ be given. Then, there exists a polynomial function $p(x) \in P[a, b]$ such that the distance between $p(X)$ and $f(x)$ is less than ϵ, i.e. $|p(x) - f(x)| < \epsilon$ for all $x \in [a, b]$.

Remarks: The above is very useful both in the theory and in practical applications since polynomials are easy to calculate with. Recall Taylor's Theorem:

$$f(a+h) = f(a) + hf^{'}(a) + \frac{h^2}{2!}f^{''}(a) + \cdots + \frac{h^n}{n!}f^n()a + \text{Remainder Term}.$$

This can be used to determine the approximating polynomial provided two conditions hold:

1. $f(x)$ is not just continuous but also has derivatives of all orders, i.e. $f \in C^{\infty}[a, b]$.
2. The Remainder Term converges to 0 as $n \to \infty$ (for small h).

Additional Notes about $P[a, b]$ (Cont'd)

Though we can find functions $f \in C^{\infty}[a, b]$, which do not satisfy (ii) above, many of the functions found in practice, such as exponential, trigonometric, etc do not satisfy (ii). Of course, if $f \notin C^{\infty}[a, b]$ but $f \in C[a, b]$, we can use Weierstrass Approximation Theorem directly.

Sums and Direct Sums

- **Definition:** Let U and W be subspaces of the vector space V. Then the sum of U and W, $U + W = \{u + w \ : \ u \in U, w \in W\}$. It is easy to see that $U + W$ is a subspace of V. In fact, $U + W$ is the smallest subspace of V containing U and W.
- **Definition:** V is said to be the direct sum of the subspaces U and W if every vector $v \in V$ is uniquely expressible in the form $v = u + w$, where $u \in U, w \in W$. We shall use the notation $V = U \oplus W$ to indicate that V is the direct sum of U and W.
- **Proposition 19:** If U and W are subspaces of the vector space V, then $V = U \oplus W$ if and only if $V = U + W$ and $U \cap W = \{0\}$.
  Proof: Left as an exercise.
- **Remark:** In some books, direct sum is defined as in Proposition 19, and then our definition is derived as a proposition.
- **Remark:** The subspace W in the above is often referred to as a **complement** or **complementary** subspace of U. It should be noted that there is nothing unique about complementary subspaces.