Linear Algebra

Sartaj Ul Hasan

Department of Mathematics
Indian Institute of Technology Jammu
Jammu, India - 181221

Email: sartaj.hasan@iitjammu.ac.in

Lecture 34

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Algebra of Linear Transformations

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- Let V and W be vector spaces over a field $\mathbb{F}$.
- **Proposition 32:**
  (a) The set W^V of all functions from V to W is a vector space over $\mathbb{F}$.

  Proof: For any functions f and g from V to W, and any scalar c, we define the functions $(f+g)$ and (cf) by:
  - $(f+g)(u) = f(u) + g(u)$ for all u in V
  - $(cf)(u) = cf(u)$ for all u in V

  It is easy to verify that W^V becomes a vector space over $\mathbb{F}$. But this is so only because W is a vector space over $\mathbb{F}$.

  (b) The set, $L(V, W)$, of all linear transformations from V to W is a subspace of W^V.

  (i) The zero function is a linear transformation, hence belongs to $L(V, W)$.

  (ii) (Closure under addition) Suppose that T and U are two linear transformations. Then:

$$\begin{aligned}(T+U)(u+v) &= T(u+v) + U(u+v) \quad \text{(by definition of addition)}\\ &= T(u) + T(v) + U(u) + U(v) \text{ (Since T and U are L.T.)}\\ &= T(u) + U(u) + T(v) + U(v)\\ &= (T+U)(u) + (T+U)(v)\end{aligned}$$

Proof of Prop 32 (Cont'd)

Similarly,

$$\begin{aligned}(T+U)(cu) &= T(cu)+U(cu) = cT(u)+cU(u) \text{ (since T and U are linear)}\\ &= c(T(u)+U(u)) \quad \text{(because W is a vector space)}\\ &= c(T+U)(u)\end{aligned}$$

(iii) (Closure under scalar multiplication)

$$\begin{aligned}(cT)(u+v) &= cT(u+v)\\ &= c(T(u)+T(v)) \quad \text{(since T is linear)}\\ &= cT(u)+cT(v) = (cT)(u)+(cT)(v)\end{aligned}$$

Similarly,

$$\begin{aligned}(cT)(du) &= cT(du) = cdT(u) \text{ (since T is linear)}\\ &= d(c(T(u)) \quad \text{(because } \mathbb{F} \text{ is a field)}\\ &= d(cT)(u)\end{aligned}$$

Algebra of Linear Transformations (Cont'd)

- **Notation** & **Remark:** This subspace is commonly denoted by $L(V, W)$. It plays a major role in linear algebra, whereas W^V is rarely needed.
- **Remark about the notation used in Prop 32:**

  $W^V = \{f \ : \ V \longrightarrow W\} \longrightarrow$ Why this notation?

  Let $|X| = n$ elements and $|Y| = m$ elements
  Then, the number of functions from X to Y is $= m^n = |Y|^{|X|}$.
  Thus, $|Y^X| = |Y|^{|X|}$ when X and Y are finite sets.

  Note: In our case, W and V are both infinite, considered as sets. But this notation has been adopted for the set of functions from V to W.

Algebra of Linear Transformations (Cont'd)

- **Proposition 33:** Let V, W and Z be vector spaces over a field $\mathbb{F}$. Let T be a linear transformation from V into W, and U be a linear transformation from W into Z. Then the composed function UT from V into Z defined by $(UT)(v) = U(T(v))$ for all v in V is a linear transformation from V into Z.
- **Proof:** Left as an exercise.

Linear Operators

- A special case of primary importance is that of linear transformations of a vector space V into itself, i.e. the space $L(V, V)$. In this case we typically use the terminology linear operator instead of linear transformation, in other words, a linear operator on V is a linear transformation from V into V.
- **Observation:** In the case of the space $L(V, V)$, we can define a "multiplication", i.e. composition of operators. (Note: *We cannot do this in L(V,W) when W is different from V*). As already indicated in Proposition 33, the composition of two linear transformations (provided it is well-defined) is a linear transformation.

Linear Operators (Cont'd)

- Composition of linear operators satisfies the following nice properties:

  (a) $IU = UI = U$ for all linear operators U.
  (b) Associative Law: $(T_1 T_2)T_3 = T_1(T_2 T_3)$
  (c) $U(T_1 + T_2) = UT_1 + UT_2$
  (d) $(T_1 + T_2)U = T_1 U + T_2 U$
  (e) $c(UT_1) = (cU)T_1 = U(cT_1)$
  (f) However, this multiplication is not commutative

- A vector space with a multiplication which satisfies properties (a) through (e) above is commonly referred to as an **algebra**. We will not study algebras in general, but will limit ourselves to $L(V, V)$.

Another Fundamental Isomorphism

- **Proposition 34:** Let V be an n-dimensional vector space over $\mathbb{F}$, and let W be an m-dimensional vector space over $\mathbb{F}$. Then there is an isomorphism between $L(V, W)$ and $\mathbb{F}^{m \times n}$.
- **Idea of proof**: We take a fixed ordered basis $\alpha = \{v_1, v_2, \ldots, v_n\}$ for V, and a fixed ordered basis $\beta = \{w_1, w_2, \ldots, w_m\}$ for W. Let T be any linear transformation in $L(V, W)$. Then we can find the matrix of T with respect to the bases α and β, let us call it $[T]_{\alpha \to \beta}$.

  The mapping $\phi \ : \ L(V, W) \longrightarrow \mathbb{F}^{m \times n}$ which takes a linear transformation T to its matrix $[T]_{\alpha \to \beta}$ is an isomorphism. (*Verification is left as an exercise.*)

  However, note that ϕ is defined in terms of the bases α and β, and is therefore dependent on the choice of α and β.
- **Remark:** The proposition above formalizes our construction of the matrix of a linear transformation, and makes the relationship between matrices and linear transformations very rigorous.

Proof of Proposition 34

Proof of Proposition 34 : We need to verify that the mapping $\phi : L(V, W) \to \mathbb{F}^{m\times n}$ given by:

$$\phi(T) = [T]_{\alpha\to\beta}$$

is an isomorphism. Here V is n-dimensional, W is m-dimensional, and α, β are fixed ordered bases for V and W respectively, $[T]_{\alpha\to\beta}$ is the matrix of T relative to the basis α, β. Recall that the j-th column T_j of $[T]_{\alpha\to\beta}$ is given as follows:
if

$$Tv_j = a_{1j}w_1 + \cdots + a_{mj}w_m \tag{1}$$

then

$$T_j = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix}. \tag{2}$$

Proof of Proposition 34 (Conti $\cdots$) :
In other words, $[T]_{\alpha\to\beta} = [a_{ij}]$.
To show that ϕ is an isomorphism, we need to show: additivity, homogeneity, injectivity, surjectivity.
(i) Additivity: Let $T, U \in L(V, W)$ with $\phi(T) = [T]_{\alpha\to\beta} = [a_{ij}]$ and $\phi(U) = [U]_{\alpha\to\beta} = [b_{ij}]$.
Then:

$$\begin{aligned}(T+U)(v_j) &= Tv_j + Uv_j \qquad \text{(by defn)}\\ &= (a_{1j}w_1 + \cdots + a_{mj}w_m) + (b_{1j}w_1 + \cdots + b_{mj}w_m)\\ &= (a_{1j} + b_{1j})w_1 + \cdots + (a_{mj} + b_{mj})w_m\end{aligned}$$

Therefore, $\phi(T+U) = [a_{ij} + b_{ij}] = [a_{ij}] + [b_{ij}] = \phi(T) + \phi(U)$.

Proof of Proposition 34 (Conti $\cdots$)

Proof of Proposition 34 (Conti $\cdots$) :
(ii) Homogeneity: If $c \in \mathbb{F}$, then using same notation as above,

$$\begin{aligned}(cT)(v_j) &= cTv_j \qquad \text{(by defn)} \\ &= c(a_{1j}w_1 + \cdots + a_{mj}w_m) \\ &= ca_{1j}w_1 + \cdots + ca_{mj}w_m\end{aligned}$$

So $\phi(cT) = [ca_{ij}] = c[a_{ij}] = c\phi(T)$
(iii) Injectivity: It suffices to show that Ker $\phi = \{T_0\}$, where T_0 is the zero transformation.
So suppose $T \in \text{Ker}\phi \implies [T]_{\alpha\to\beta} = [0]$, the zero matrix.
$\implies Tv_j = 0$ for all $v_j \in \alpha$
$\implies Tv = 0$ for all $v \in V$
$\implies T = T_0$.

Proof of Proposition 34 (Conti $\cdots$) :
(iii) Surjectivity: Suppose $A = [a_{ij}] \in \mathbb{F}^{m \times n}$. Then, by Proposition 26 (b), we can define a unique linear transformation $T : V \to W$ by

$$Tv_j = a_{1j} w_1 + \cdots + a_{mj} w_m.$$

Then, $\phi(T) = [T]_{\alpha \to \beta} = A$ by (1).