Indian Institute of Technology Jammu

CSD001P5M **Linear Algebra** **Tutorial: 06**

1. Let $V = C[0,1]$, the space of all continuous real-valued functions defined on the closed interval. Is V finite-dimensional? Justify your answer.

2. Let $V = \mathbb{R}^\infty$, $W = \{< a_n > \ : \text{ only finitely many of the terms in } < a_n > \text{ are non-zero}\}$. Show that W is a a subspace of V. Is W finite-dimensional ? Justify your answer.

3. V is a vector space with $\dim(V) = n$. W_1 and W_2 are subspaces of V such that $\dim(W_1) = \dim(W_2) = n-1$ and $W_1 \cap W_2 = \{0\}$. Find n?

4. Prove Proposition 20: If U and W are subspaces of the vector space V, then $V = U \oplus W$ if and only if $V = U + W$, and $U \cap W = \{0\}$.

5. Given the vector space $\mathbb{R}^3$, let W_1 be the set of vectors of the form $(x, y, 0)$ and let W_2 be the set of vectors of the form $(0, a, b)$.
   (a) Show that W_1 and W_2 are subspaces of $\mathbb{R}^3$.
   (b) Find the dimensions of $W_1, W_2, W_1 + W_2$ and $W_1 \cap W_2$.
   (c) Find two distinct subspaces U_1 and U_2 of $\mathbb{R}^3$ such that $\mathbb{R}^3 = W_1 \oplus U_1 = W_1 \oplus U_2$, i.e. find two distinct complements of V. Justify your answer.

6. Suppose that X, Y and Z are subspaces of V. Then prove that $X + Y + Z$ is a direct sum if and only if $X \cap Y = \{0\}$ and $Z \cap (X+Y) = \{0\}$.

7. Show that dimension of $\mathbb{R}$ over $\mathbb{Q}$ is infinite.

8. For $k = 0, 1, \cdots, n$, let
$$p_k(t) := t^k + t^{k+1} + \cdots + t^n.$$
Then prove that $\{p_0(t), p_1(t), \cdots, p_n(t)\}$ is a basis of $\mathbb{R}_n[t]$.

9. For a fixed $t_0 \in \mathbb{R}$, determine the dimension of the subspace of $\mathbb{R}_n[t]$ defined by
$$\{f \in \mathbb{R}_n[t] : f(t_0) = 0\}.$$

10. Let S be a basis of a vector space $V_{\mathbb{F}}$. Given $v \in V$, show that there exist unique $\{v_1, \cdots, v_m\} \subseteq S$ and unique $\{\alpha_1, \cdots, \alpha_m\} \subseteq \mathbb{F} - \{0\}$ such that $v = \alpha_1 v_1 + \cdots + \alpha_m v_m$.

11. Let V be a vector space of all functions from $\mathbb{R}$ into $\mathbb{R}$. Let W_1 be the subset of even functions, $f(-x) = f(x)$ and W_2 be the subset of odd functions, $f(-x) = -f(x)$.

(a) Prove that W_1 and W_2 are subspaces of V.

(b) Prove that $W_1 + W_2 = V$.

(c) $W_1 \cap W_2 = \{0\}$.

12. Let $S = \{v_1, v_2, v_3, v_4, v_5\}$ where the v_i's are vectors in $\mathbb{R}^3$ given below (they should be taken as column vectors):

$$v_1 = (1,2,3), \quad v_2 = (2,5,7), \quad v_3 = (10,24,34), \quad v_4 = (.1,.5,.6), \quad v_5 = (3,7,11).$$

Let $W = \text{span } S$.

(a) Reduce S to a basis for W. You must explain your method briefly and show your calculations.

(b) Is W all of $\mathbb{R}^3$? Justify your answer (YES or NO) in at most one sentence.

13. Find all the solutions of the following systems of the homogeneous equations over $\mathbb{C}$, the field of complex numbers.

(a)

$$\begin{aligned} 2x_1 + (-1+i)x_2 + x_4 &= 0 \\ (1+\frac{i}{2})x_1 + 8x_2 - ix_3 - x_4 &= 0 \\ 3x_2 - 2ix_3 + 5x_4 &= 0 \\ \frac{2}{3}x_1 - \frac{1}{2}x_2 + x_3 + 7x_4 &= 0 \end{aligned}$$

(b)

$$\begin{aligned} (1-i)x_1 - ix_2 &= 0 \\ 2x_1 + (1-i)x_2 &= 0 \end{aligned}$$

14. Solve the following system of equations over the field $\mathbb{Q}(\sqrt{2})$:

$$\begin{aligned} 2x - (2+3\sqrt{2})y - \sqrt{2}z &= 0 \\ (1+\sqrt{2})x - (3+\sqrt{2})y - 4z &= 0 \\ (2-3\sqrt{2})x - 7\sqrt{2}y + 6z &= 0 \end{aligned}$$

(**Note:** First verify that $\mathbb{Q}(\sqrt{2}) := \{a + b\sqrt{2} \ : \ a, b \in \mathbb{Q}\}$ is a field.)

15. Find all the solutions of the following system of equations over the field $\mathbb{Z}_3$, if any:

$$\begin{aligned} x + 2y &= 0 \\ y + z &= 1 \\ x + y + z &= 2. \end{aligned}$$