Indian Institute of Technology Jammu

CSD001P5M | **Linear Algebra** | **Tutorial: 11**

1. Find the eigenvalues and corresponding eigenvectors for the matrix A given below. Is A diagonalizable ? Justify your answer in at most one sentence.

$$\begin{bmatrix} 3 & -1 & -1 \\ -12 & 0 & 5 \\ 4 & -2 & -1 \end{bmatrix}$$

2. For each matrix find all eigenvalues and a basis of each eigenspace. Which matrix can be diagonalized and why? If yes, indicate the diagonal matrix D and the invertible matrix P such that $A = PDP^{-1}$. [Hint: $\lambda = 4$ is an eigenvalue.]

   (a) $\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$

   (b) $\begin{bmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{bmatrix}$

3. A matrix 7×7 matrix A has three eigenvalues. One eigenspace is 2-dimensional and one of the others is 3-dimensional. Is it possible for A to be not diagonalizable? Justify your answer.

4. Suppose A is an $n \times n$ square matrix and Rank $(A) = k$. Show that A can have at most $(k+1)$ distinct eigenvalues.

5. (a) If A is row-equivalent to the identity matrix, then A must be diagonalizable. Is this statement TRUE or FALSE?

   (b) Justify your answer to (a). Give a proof if TRUE or a concrete counter-example if FALSE. In the second case, you should verify that your counter-example is row-equivalent to identity matrix but not diagonalizable.

6. Let $V = C^{\infty}[\mathbb{R}]$, the vector space of real functions having continuous derivatives of all orders. Let D be the differentiation operator on V. Determine the eigenvalues and corresponding eigenvectors of D.

7. Let $V = \mathbb{F}^{n\times n}$ for a fixed $n \geq 2$, and let $P \in V$ be a fixed but arbitrary invertible matrix. Then the mapping $S_P : V \to V$ given by $S_P(A) = PAP^{-1}$ is known as the similarity transformation induced by P. Show that S_P is an isomorphism. Further, show that S_P is a multiplicative transformation, i.e. $S_P(AB) = S_P(A)S_P(B)$ for all $A, B \in V$.

8. In the 2-dimensional plane, i.e. the vector space $\mathbb{R}^2$, let $\mathbb{R}_\theta$ indicate rotating a vector (regarded as a directed line segment with its tail at the origin) by an angle of θ (radians) in the positive (anti-clock-wise) direction. It is geometrically intuitive that this rotation is a linear transformation. Prove this rigorously by constructing the matrix of the operator $R_\theta : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ relative to the standard basis $S = \{e_1, e_2\}$. If $v_i = R_\theta(e_i)$, find the change of basis matrix $P_{S\to\beta}$ where $\beta = \{v_1, v_2\}$. Finally, find $[v]_\beta$ for any arbitrary $v \in \mathbb{R}^2$.

9. Use the Gram-Schmidt process to find an orthonormal basis given the basis $\{x_1 = (2,1,2), x_2 = (4,1,0), x_3 = (3,1,-1)\}$ for $\mathbb{R}^3$.

10. Let V be the vector space $\mathbb{R}_2[t]$ of polynomials of degree ≤ 2 with real coefficients with the inner product $< p,q >= p(-2)q(-2) + p(0)q(0) + p(2)q(2)$, i.e. the interpolation inner product.

    (a) Find an orthogonal basis for V starting from the standard basis $\{1,t,t^2\}$ using the Gram-Schmidt process.

    (b) Find the coordinates of $p(t) = 1+2t+3t^2$ with respect to the orthogonal basis found in part (a).

11. Find a diagonal matrix D and an orthogonal matrix P such that $A = PDP^T$ for the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}.$$

    (Hint: 3 is an eigenvalue)

12. Let U be an $m\times n$ matrix with orthonormal columns, and suppose x and y are vectors in $\mathbb{R}^n$. Show that:

    (a) $Ux \cdot Uy = x \cdot y$

    (b) $||Ux|| = ||x||$

    (c) $Ux \cdot Uy = 0$ if and only if $x \cdot y = 0$

13. Let W be the subspace of $\mathbb{R}^3$ spanned by the vector $v = (1,2,3)$. Find orthogonal bases for W and $W^\perp$, respectively. Is the union of these two bases a basis for $\mathbb{R}^3$?

14. Let S be a (finite) subset of an inner product vector space V, and define

$$S^{\perp} = \{v \in V : < v, u >= 0 \text{ for every } u \in S\},$$

i.e. $S^{\perp}$ is the set of vectors orthogonal to S. Show that in fact $S^{\perp}$ is a subspace of V. If $W = \text{Span } S$, what is the relationship between $S^{\perp}$ and $W^{\perp}$? Justify your answer.

15. Let $A \in \mathbb{R}^{m\times n}$ i.e. A is an $m \times n$ matrix with real entries. Show that Nul A is the orthogonal complement of Row A.

16. Let S and T be linear operators such that $ST = TS$. Let λ be an eigenvalue of T and let W be its corresponding eigenspace. Show that W is invariant under S, i.e. $S(W) \subseteq W$.

17. Let V be an n-dimensional vector space and let $\beta = \{v_1, \cdots, v_n\}$ be an ordered basis for V. By Proposition 26(b), there exists a unique linear operator $T : V \to V$ given by $Tv_j = v_{j+1}$ for $j = 1, 2, \cdots, n-1; Tv_n = 0$.

    (a) Determine the matrix of T relative to the ordered basis β, i.e. determine $[T]_{\beta}$.

    (b) Show that $T^n = 0$, i.e. the zero operator, but $T^{n-1} \neq 0$.

    (c) Let S be any linear operator on V such that $S^n = 0$, but $S^{n-1} \neq 0$. Show that there exists an ordered basis α for V such that $[S]_{\alpha}$ is exactly the same as the matrix obtained in (a).

18. Prove that if A and B are $n \times n$ matrices over $\mathbb{F}$ such that $A^n = B^n = 0$, but $A^{n-1} \neq 0$ and $B^{n-1} \neq 0$, then A is similar to B.