Linear Algebra and Applications

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Lecture 07

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Very Important Theorem (VIT)–Version 1.0
The Invertible Matrix Theorem (TIMT)

(This is an important theorem and try to memorise it!)

Theorem 1

The following are equivalent for an $m \times m$ square matrix A:

(a) A is invertible.

(b) A is row equivalent to the identity matrix.

(c) The homogenous system $AX = 0$ has only trivial solution.

(d) The system of equations $AX = b$ has a solution for every b in $\mathbb{R}^m$.

Note: We will further extend this theorem as we go deep into the theory of vector spaces and related concepts.

Calculation of the Inverse Matrix

In order to calculate the inverse of a matrix, we use the following result:

- **Corollary 1.1:** An invertible matrix A is a product of elementary matrices. Moreover, any sequence of row operations that reduces A to I also transforms I into A^{-1}.
  (Note that we are implicitly using Theorem 1(b) here.)

Calculation of the Inverse Matrix

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- **Corollary 1.1:** An invertible matrix A is a product of elementary matrices. Moreover, any sequence of row operations that reduces A to I also transforms I into A^{-1}.
  (Note that we are implicitly using Theorem 1(b) here.)
- **Proof of Corollary 1.1:** If A is invertible, then by The Invertible Matrix Theorem (TIMT), A is row equivalent to the identity matrix, i.e. $I = (e_k e_{k-1} \dots e_1)A$ for some sequence of elementary row operations. If E_1 to E_k are the corresponding elementary matrices, then $I = (E_k \dots E_1)A$. Each E_i being invertible, we can write $A = (E_k \dots E_1)^{-1} I = E_1^{-1} \dots E_k^{-1}$ Hence A is a product of elementary matrices. Furthermore,

  $$A^{-1} = (E_1^{-1} \dots E_k^{-1})^{-1} = (E_k \dots E_1) = (E_k \dots E_1)I = (e_k e_{k-1} \dots e_1)I$$

  In other words, the same sequence of row operations that reduces A to I also reduces I to A^{-1}.

Calculation of the Inverse Matrix (Conti . . .)

- **Method:** Form the augmented matrix $[A \quad I]$ (this is sometimes known as the enlarged matrix of A) and carry out elementary row operations till the A part becomes I. The final result has the form $[I \quad A^{-1}]$

- **Example:** Find inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$, if it exists.

$$[A \quad I] = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

Since $A \sim I$, we conclude that A is invertible by TIMT and

$$A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$$

Practice Exercises

- 2.2: Inverse of a Matrix page 126 : 3,6,7,9,13, 29,31.
- 2.3: Characterisations of Invertible Matrices page 132: 3,4,5,6,7,8,11,13. These problems are based on TIMT.

Invertible Matrices (Conti . . .)

- **Corollary 1.2:** If A has a left inverse or a right inverse, then it has an inverse.

Invertible Matrices (Conti ...)

- **Corollary 1.2:** If A has a left inverse or a right inverse, then it has an inverse.
  Proof: (a) Suppose A has a left inverse; then there exists a matrix C such that $CA = I$. Now consider the homogeneous system $AX = 0$. Multiplying on the left by C, we get:

  $$(CA)X = C0 \implies IX = 0 \implies X = 0.$$

  In short, the homogeneous system has only trivial solution. Hence, by TIMT, A is invertible. Furthermore, $I = CA = A^{-1}A$, so multiplying on the right by A^{-1}, we get $C = A^{-1}$.

  (b) Now suppose A has a right inverse; then there exists a matrix D such that $AD = I$. In other words, D has a left inverse, so is invertible by part (a). Hence $(AD)D^{-1} = ID^{-1}$ or $A = D^{-1}$. Thus, A, being inverse of an invertible matrix, is itself invertible, and $A^{-1} = D$.