Linear Algebra and Applications

Sartaj Ul Hasan

Department of Mathematics
Indian Institute of Technology Jammu
Jammu, India - 181221

Email: sartaj.hasan@iitjammu.ac.in

Lecture 09

(Aug 07, 2019)

Time: 10:00 -10:55 am

Make-up lecture

Proof of The Invertible Matrix Theorem (TIMT)

We will proceed as follows:

$$(a) \implies (c) \implies (b) \implies (a),$$

and

$$(a) \iff (d), \quad \text{i.e.,} \quad (a) \implies (d) \implies (a) \quad \text{(using corollaries!)}$$

Proof:

- $(a) \implies (c)$

  Given that A is invertible. Need to show that the system $AX = 0$ has only the trivial solution. Suppose Y is any solution of the homogeneous system. Therefore, $AY = 0$. Multiply on the left by A^{-1}, we get:

$$(A^{-1}A)Y = A^{-1}0 \implies IY = 0 \implies Y = 0.$$

Proof of TIMT (Conti . . .)

- $(c) \implies (b)$

  [**Note:** This is actually Proposition 3 (See Lecture 5, Slides5), but we will now give a proof]

  Suppose $AX = 0$ has only trivial solution, i.e., $X = 0$. If R is the RREF matrix of A, then R has no free variables. Therefore, all variables of R are basic variables. Since no. of variables $=$ no. of columns $=$ no. of rows (since A is square), there must be a basic variable in each row and in each column. Therefore $R = I$, as required.

Proof of TIMT (Conti . . .)

- $(b) \implies (a)$

  Given A is row-equivalent to I, need to show that A is invertible. Since $A \sim I$, there are elementary row operations such that:

$$e_1(e_2(\ldots(e_{n-1}(e_nA))\ldots)) = I$$
$$\therefore \quad (E_1E_2\ldots E_{n-1}E_n)A = I$$

  Put $B = E_1E_2\ldots E_{n-1}E_n$. Then, since each E_i is invertible, so is B.

$$\therefore \quad BA = I$$
$$(B^{-1}B)A = B^{-1}I$$
$$A = B^{-1}$$

  So A being the inverse of an invertible matrix is itself invertible.

Proof of TIMT (Conti . . .)

Proof of $(a) \iff (d)$

- $(a) \implies (d)$

  Suppose A is invertible and $b \in \mathbb{R}^m$, where

$$\mathbb{R}^m = \{(b_1, \ldots, b_m) \; : \; b_i \in \mathbb{R}\} = \left\{ b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}_{m \times 1} \; : \; b_i \in \mathbb{R} \right\}.$$

  Consider the vector $u = A^{-1}b \in \mathbb{R}^m$. Then:

$$Au = A(A^{-1}b) = Ib = b.$$

  Therefore, the system $Ax = b$ has u as a solution.

Proof of TIMT (Conti . . .)

- $(d) \implies (a)$

  Suppose the system $AX = b$ has a solution for every $b \in \mathbb{R}^m$. Let u_i be a solution of the system $AX = e_i$ for $i = 1, 2, \ldots, m$, where e_i denote the column vector having 1 at the i^{th} position and 0 elsewhere. Let B be the matrix whose columns are the u_i, i.e., $B = [u_1, u_2, \ldots, u_m]$. Then:

  $$AB = A[u_1, u_2, \ldots, u_m] = [Au_1, Au_2, \ldots, Au_m] = [e_1, e_2, \ldots, e_m] = I.$$

  Since A has a right inverse, by Corollary 1.2, it is invertible.

Vector Formulation

- A system of linear equations can also be expressed in a vector form: $X_1v_1 + X_2v_2 + \cdots + X_nv_n = b$, where the X_i are scalar unknowns and the v_i are column vectors formed from the coefficients of the original linear system.
- This formulation can be interpreted as: if we can find scalars X_i satisfying the equation, then the given vector b can be expressed in terms of the given vectors v_i . This formulation is not useful for solving the system, but will become very important when we start working with vectors.

Uniqueness of RREF

Corollary 1.5: Uniqueness of RREF matrix of any matrix A.

Proof of Corollary 1.5: We may assume that $A \neq [0]$, i.e., A is non-zero. Let A be any $m \times n$ matrix. We keep m fixed, and prove the result by induction on n.

Base case: $n = 1$

Then the only possible non-zero RREF matrix is $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m \times 1}$, so uniqueness holds.

Induction Step: So suppose $n \geq 2$ and the result holds for all matrices with $n-1$ columns. Let A be an $m \times n$ matrix, and suppose BWOC that A has two distinct RREF matrices, B and C. Let A' be the matrix obtained from A by deleting the n-th column.

Proof of uniqueness of RREF (Conti $\cdots$)

Proof of Corollary 1.5 (Conti $\cdots$) : Because, the row-reduction algorithm proceeds downwards and rightwards column-wise, the sequence which reduces A to an RREF matrix, also reduces A' to an RREF matrix. However, by induction hypothesis, RREF matrix of A' is unique. Hence B and C can differ only in the n-th column. So, since $B \neq C$, there exists some j-th row, $1 \leq j \leq m$ such that the n-th entries of B and C differ, i.e., $b_{jn} \neq c_{jn}$. Now, let v be any vector such that $Bv = 0$, and we may assume $v \neq 0$ (else, the system $Ax = 0$ has only trivial solution $\implies$ both B and C have I_n on top with zero rows below). Now,
$Bv = 0 \implies Cv = 0 \implies (B - C)v = 0$. But the first $(n-1)$ columns of $B - C$ are zero, so by considering the j-th component (coordinate) of v, we get that $(b_{jn} - c_{jn})v_n = 0$. Since $b_{jn} \neq c_{jn}$, we get that $v_n = 0$. Thus, in case $B \neq C$, then any solution $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ of $Bx = 0$ or $Cx = 0$ must have $v_n = 0$.

Proof of uniqueness of RREF (Conti $\cdots$)

Proof of Corollary 1.5 (Conti $\cdots$) :
It follows that x_n can not be a free variable. Recall that if any x_i is a free variable, then we have to insert a dummy equation $x_i = x_i$, and so we get a vector that contain 1 in its i-th coordinate. Therefore, x_n has to be basic variable. In that case, the n-th column of B has to contain a 1 in the first zero row in the RREF matrix of A' (since A' can have at most $(n-1)$ basic variables). But then the n-th column of B consists of a 1 with 0's above and below. The same is true of C. Hence $B = C$, contradicting our assumption that $B \neq C$. Result follows.