Linear Algebra and Applications

Sartaj Ul Hasan

Department of Mathematics
Indian Institute of Technology Jammu
Jammu, India - 181221

Email: sartaj.hasan@iitjammu.ac.in

Lecture 05

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Back to Solving a Linear System

- **Observation 1:**
  - If we obtain a row equivalent matrix to the coefficient matrix (in the case of a homogeneous system), then the solution sets of the two systems are the same.
  - Similarly if we obtain a row equivalent matrix to the augmented matrix (in the non- homogeneous case), then the solution sets of the two systems are the same.
  - This is expressed by saying that the systems are equivalent. In fact, that is why we defined the elementary row operations in the way we did.

Homogeneous Systems

$$AX = 0, \quad \text{where} \quad A = \text{coefficient matrix}$$

Because homogeneous system of linear equations is consistent and always has a trivial solution $(0, \ldots, 0)$, there are only two possibilities for its solutions:

- The system has only the trivial solution.
- The system has infinitely many solutions in addition to the trivial solution.

Homogeneous Systems (Cont . . .)

Suppose that we have row-reduced the coefficient matrix A to an RREF matrix R.

- The leading variables in each non-zero row of R correspond to pivot columns. These are referred to as **basic variables**. Remaining variables, if any, are referred to as **free variables**.
- If we write the matrix equation $RX = 0$ as a linear system, we can obtain the general solution (in terms of parameters) of the system (recall that the system $RX = 0$ is equivalent to the original system $AX = 0$). **The general solution is best expressed in terms of column vector**.

Homogeneous Systems (Cont . . .)

- **Observation 2:** If the number of non-zero rows r of R is less than the number of variables n, then the system has a non-trivial solution as follows:
  - Express basic variables (there will r basic variables) in terms of free variables (there will be $n - r$ free variables).
  - Free variables behave like parameters - i.e. we can choose any values for them, and each such choice gives a solution. So we get infinitely many solutions.
- **Observation 3** (Special case of above): If A is an $m \times n$ matrix with $m < n$, then the homogeneous system $AX = 0$ must have a non-trivial solution (in fact, infinitely many solutions). This is because in this case there have to be free variables.
- **Observation 4:** If the number of non-zero rows of R is equal to the number of variables (i.e. number of columns), then there are no free variables, and the system has a unique solution (only the trivial solution of all zeros).

Homogeneous Systems (Cont . . .)

- **Proposition 3:** If A is a square matrix, then A is row equivalent to the identity matrix if and only if the homogeneous system $AX = 0$ has only the trivial solution.

Homogeneous Systems-Summary

1. System is always consistent.
2. If the system has a unique solution, then it is the trivial solution of all zeroes–in this case the RREF is either the $n \times n$ identity matrix I_n itself or has I_n as its upper portion with only zero rows below.
3. Else, the system contains free variables and has infinitely many solutions (one of which is the trivial solution); this happens when number of non-zero rows in the RREF is less than the number of variables.
4. If number of equations is less than the number of variables, then the system has infinitely many solutions. This is a special case of point 3.

Non-Homogeneous Systems

In this case, we work with the augmented matrix and reduce it to an RREF matrix, say R

- **Proposition 4** (Existence and Nature of Solutions): The system is consistent if and only if the rightmost column of R is not a pivot column, i.e. if there is no row of the form $[0, \ldots, 0, b]$ with b nonzero.
- If the system is consistent, then it has either (i) a unique solution if there are no free variables or (ii) infinitely many solutions when there is at least one free variable.

Vector Interpretation of Solutions

- The general solution of a system of linear equations can be expressed compactly and conveniently in vector form.
- Let $AX = b$ be a non-homogeneous system, and let $AX = 0$ be its associated homogeneous system. Assume that the system $AX = b$ is consistent so that it has at least one solution u. By necessity, $u \neq 0$.

- **Observation 6:** A vector is a solution of the system $AX = b$ if and only if it is of the form $u + v$, where v is a solution of the associated homogeneous system.
- In case the homogeneous system has only the trivial solution, then $v = 0$, and there is a unique solution u. Otherwise we have infinitely many solutions.

Geometric Interpretation of Solutions

- We can have a geometrical interpretation in case we are working with 2-tuples or 3-tuples. In this case, each vector corresponds to a point in either 2-space or 3-space.
- **Observation 7:** Then, the solution of a homogeneous system is either the origin only or all the points on a line or a plane through the origin.
- **Observation 8:** If a non-homogenous system has even a single solution (point), then its entire solution set consists of only that point or the line or plane through that point which is parallel to the solution of the associated homogeneous system.

Example

- Write the general solution of the system:

$$\begin{aligned} X_1 + X_2 + X_3 &= 1 \\ 2X_1 - X_2 + X_3 &= 2 \end{aligned} \tag{1}$$

$$\text{in the form } \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2/3 \\ -1/3 \\ 1 \end{bmatrix} = u + tv \quad \text{(t scalar)}$$

  where u is the solution of the system and v is a solution of the associated homogeneous system.
- Reminder from Coordinate Geometry:
  Equation for the line through $P = (x_0, y_0, z_0)$ prallel to a given vector $\overrightarrow{r} = a\hat{i} + b\hat{j} + c\hat{k}$ (i.e. the line segment from $(0, 0, 0)$ to (a, b, c)) is:

$$x = x_0 + ta, y = y_0 + tb, z = z_0 + tc, t \in \mathbb{R}$$

- The solution we have obtained corresponds to the geometrical equation of the line through $(1, 0, 0)$ which is prallel to the vector determined by $(-2/3, -1/3, 1)$.