Linear Algebra and Applications

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Lecture 03

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Time: 10:00-10:55 am

Make-up lecture

Elementary Row Operations

- Given any $m \times n$ matrix A, we define three **elementary row operations**:
  - Multiplication of one row of A by a **non-zero** scalar c (**scale**).
  - Replacement of one row of A by the sum of the row and a scalar multiple of a **different** row (**replace**).
  - Interchange of two rows of A (**interchange**).
- So by applying an elementary row operation e to A, we get a new matrix $e(A)$.
- **Observation 2:** To each elementary row operation e, there corresponds an elementary row operation e^{-1} of the same type such that $e^{-1}(e(A)) = A$. In other words, the process is reversible.

Row Reduction and Echelon Form

- **Definition:** A row or column of a matrix is a nonzero row or column if at least one entry in the row or column is nonzero. The **leading entry** of a row refers to its first (moving from the left) nonzero entry.
- **Example:**

$$\begin{bmatrix} 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 6 \\ 0 & 0 & 5 & 4 \end{bmatrix}$$

  Rows $1, 3, 4$ are nonzero. Their leading entries are $3, 1$ and 5, respectively.
- In order to solve linear systems we will use row operations to put our augmented matrices into a form that makes it easy to read off the solutions. There will be two forms of interest:
  - Row Echelon Form (REF)
  - Reduced Row Echelon Form (RREF)

REF and RREF

A matrix is in **echelon form** if the following happens:

1. Rows of 0's are at the bottom.
2. If two successive rows are nonzero, then the leading entry of the top row is to the left of the leading entry of the bottom row.
3. All entries in a column beneath a leading entry are 0.

A matrix is in **reduced echelon form** if it is in:

1. in echelon form.
2. all leading entries are 1.
3. Each leading 1 is the only nonzero entry in its column.

Examples:

$$\begin{bmatrix} 5 & 6 & 3 & 0 & 4 \\ 0 & 4 & 1 & 6 & 2 \\ 0 & 0 & 0 & 5 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} (REF) \qquad \begin{bmatrix} 1 & 2 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} (RREF)$$

Some more examples of REF

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Some more examples of REF

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 4 \end{bmatrix} (YES) \qquad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

Some more examples of REF

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 4 \end{bmatrix} (YES) \qquad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 4 \end{bmatrix} (YES) \qquad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 10 & 4 \end{bmatrix}$$

Some more examples of REF

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 4 \end{bmatrix} (YES) \qquad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 4 \end{bmatrix} (YES) \qquad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 10 & 4 \end{bmatrix} (YES)$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 5 \end{bmatrix}$$

Some more examples of REF

$$\begin{bmatrix}1 & 2 & 3 & 4\\0 & 1 & 0 & 5\\0 & 0 & 0 & 4\end{bmatrix} (YES) \qquad \begin{bmatrix}1 & 2 & 3 & 4\\0 & 1 & 2 & 5\\0 & 0 & 1 & 4\end{bmatrix} (YES) \qquad \begin{bmatrix}1 & 2 & 3 & 4\\0 & 1 & 2 & 5\\0 & 0 & 10 & 4\end{bmatrix} (YES)$$

$$\begin{bmatrix}0 & 0 & 0 & 0\\1 & 2 & 3 & 4\\0 & 1 & 2 & 5\end{bmatrix} (NO) \qquad \begin{bmatrix}1 & 2 & 3 & 4\\0 & 1 & 2 & 5\\0 & 1 & 0 & 4\end{bmatrix}$$

Some more examples of REF

$$\begin{bmatrix}1 & 2 & 3 & 4\\0 & 1 & 0 & 5\\0 & 0 & 0 & 4\end{bmatrix} (YES) \qquad \begin{bmatrix}1 & 2 & 3 & 4\\0 & 1 & 2 & 5\\0 & 0 & 1 & 4\end{bmatrix} (YES) \qquad \begin{bmatrix}1 & 2 & 3 & 4\\0 & 1 & 2 & 5\\0 & 0 & 10 & 4\end{bmatrix} (YES)$$

$$\begin{bmatrix}0 & 0 & 0 & 0\\1 & 2 & 3 & 4\\0 & 1 & 2 & 5\end{bmatrix} (NO) \qquad \begin{bmatrix}1 & 2 & 3 & 4\\0 & 1 & 2 & 5\\0 & 1 & 0 & 4\end{bmatrix} (NO)$$