Linear Algebra

Sartaj Ul Hasan

Department of Mathematics
Indian Institute of Technology Jammu
Jammu, India - 181221

Email: sartaj.hasan@iitjammu.ac.in

Lecture 17

(Sept 03, 2019)

Vector Formulation

- A system of linear equations can also be expressed in a vector form: $X_1\mathbf{v_1} + X_2\mathbf{v_2} + \cdots + X_n\mathbf{v_n} = b$, where the X_i are scalar unknowns and the v_i are column vectors formed from the coefficients of the original linear system. The vectors $\mathbf{v_i}$ are the columns of the coefficient matrix A and so we can write $A = [\mathbf{v_1} \quad \mathbf{v_2} \quad \ldots \quad \mathbf{v_n}]$.
- This formulation can be interpreted as: if we can find scalars X_i satisfying the equation, then the given vector b can be expressed in terms of the given vectors $\mathbf{v_i}$. In terms of the concept of the span of a set of vectors, we can say that the non-homogeneous system $Ax = b$ has a solution if and only if the vector $b \in$ Span $\{\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_n}\}$, where the v_i are the columns of A. This formulation is not useful for solving the system, but becomes useful when we are discussing the subspaces associated with a given matrix A.

Span of a Set (Conti . . .)

Example:

- Let $S = \{\mathbf{u}, \mathbf{v}\}$, where $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$.
- Note that $\mathbf{u} \in \text{Span } S$, $\mathbf{v} \in \text{Span } S$ and $S \subseteq \text{Span } S$.
- $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \in \text{Span } S$, $2\mathbf{u} + (-1)\mathbf{v} = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix} \in \text{Span } S$.
- Clearly, while S is finite, Span S is infinite (assuming that the field $\mathbb{F}$ is infinite).
- By the way, $\mathbf{0} = 0\mathbf{u} + 0\mathbf{v} \in \text{Span } S$.

Span of a Set (Conti . . .)

Example (Conti . . .):

- Note that constructing vector in Span S is easy.
- What about the reverse question: given a vector $\mathbf{w}$, does $\mathbf{w} \in$ Span S?
  If $\mathbf{w} \in$ Span S, then $\mathbf{w} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_p\mathbf{v_p}$ for some scalars c_i.
  So we have to solve a linear system!
- As before, let $S = \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\}$
- Put $\mathbf{w} = \begin{bmatrix} 3 \\ 2 \\ 13 \end{bmatrix} \rightarrow$ solve $c_1 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 13 \end{bmatrix}$
- $\begin{bmatrix} 1 & 1 & : & 3 \\ 3 & 1 & : & 2 \\ 2 & 4 & : & 13 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & : & -1/2 \\ 0 & 1 & : & 7/2 \\ 0 & 0 & : & 0 \end{bmatrix}.$

Span of a Set (Conti . . .)

Example (Conti . . .):

- So YES $\rightarrow \mathbf{w} = \begin{bmatrix} 3 \\ 2 \\ 13 \end{bmatrix} = (-1/2) \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + (7/2) \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$.
- On the other hand (OTOH), consider $\mathbf{w_1} = \begin{bmatrix} -3 \\ -2 \\ 7 \end{bmatrix}$.
- Then, $\begin{bmatrix} 1 & 1 & : & -3 \\ 3 & 1 & : & -2 \\ 2 & 4 & : & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & : & -3 \\ 0 & -2 & : & 7 \\ 0 & 0 & : & 20 \end{bmatrix} \rightarrow$ inconsistent!
  So $\mathbf{w_1} \notin$ Span S.