Linear Algebra

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Lecture 28

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Linear Transformation

Isomorphism of Vector Spaces

Definition: A linear transformation $T : V \longrightarrow W$ is said to be an **isomorphism** if it is injective (one-one map) and surjective(onto map). In this case, V is said to be **isomorphic** to W and denoted as $V \cong W$.

Examples:

- The linear transformation $T : \mathbb{R}_3[t] \longrightarrow \mathbb{R}^4$ defined by $T(a_0 + a_1 t + a_2 t^2 + a_3 t^3) = (a_0, a_1, a_2, a_3)$ is an isomorphism and hence, $\mathbb{R}_3[t] \cong \mathbb{R}^4$.
- The linear transformation $T : M_{2\times 2}(\mathbb{R}) \longrightarrow \mathbb{R}^4$ defined by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a, b, c, d)$ is an isomorphism and so, $M_{2\times 2}(\mathbb{R}) \cong \mathbb{R}^4$.

Proposition 27

Let V and W be finite-dimensional vector spaces.

(a) An isomorphism $T : V \longrightarrow W$ takes any arbitrary basis of V to a basis of W.

(b) Conversely, if a linear transformation $T : V \longrightarrow W$ takes some basis of V to a basis of W, then it is an isomorphism.

Proof of Proposition 27

Proof of Part (a): Given that $T : V \longrightarrow W$ is an isomorphism. Let $\{v_1, \ldots, v_n\}$ be any basis of V. Consider now the set $\{T(v_1), \ldots, T(v_n)\} = B$, say. We need to show that B is basis of W, i.e., Span$B = W$ and B is L.I.

- **Span $B = W$:** The containment Span$B \subseteq W$ is obvious. We need to prove the other containment. To this end, let $w \in W$. Then $w = T(v)$ for some $v \in V$ (by surjectivity of T). Now, v can be uniquely expressed as $v = c_1 v_1 + \cdots + c_n v_n$. Thus, we have:

$$\begin{aligned} w = T(v) &= T(c_1 v_1 + \cdots + c_n v_n) \\ &= c_1 T_1(v_1) + \cdots + c_n T_1(v_n) \end{aligned}$$

  This shows that $w \in$ Span B and hence Span$B = W$.

- **B is L.I.:** Suppose that $c_1 T_1(v_1) + \cdots + c_n T_1(v_n) = 0$. Therefore, $T(c_1 v_1 + \cdots + c_n v_n) = 0$, which implies that $c_1 v_1 + \cdots + c_n v_n \in$ Ker T. But T being injective, Ker $T = \{0\}$ and thus, $c_1 v_1 + \cdots + c_n v_n = 0 \implies c_1 = \cdots = c_n = 0$, since v_i's are L.I.

Proof of Proposition 27 (Conti'd)

Proof of Part (b): Suppose T takes some basis of V to a basis of W. Need to show that T is surjective and injective. Let $B = \{v_1, \ldots, v_n\}$ be a basis of V, which is being mapped to a basis $\{T(v_1), \ldots, T(v_n)\}$ of W.

- T **is injective:** It is enough to prove that Ker $T = \{0\}$. Let $v \in$Ker T be an arbitrary element, i.e., $T(v) = 0$. Now, v, being in V, can be uniquely expressed as $v = c_1v_1 + \cdots + c_nv_n$. Thus, $T(v) = 0 \implies T(c_1v_1 + \cdots + c_nv_n) = 0 \implies c_1T_1(v_1) + \cdots + c_nT_1(v_n) = 0 \implies c_1 = \cdots = c_n = 0$, since the vectors $T(v_1), \ldots, T(v_n)$ are LI being the elements of basis of W. Hence $v = 0$, which implies that Ker $T = \{0\}$.
- T **is surjective:** Suppose $w \in W$. Then: $w = c_1T_1(v_1) + \cdots + c_nT_1(v_n) = T(c_1v_1 + \cdots + c_nv_n)$, which implies that $w \in$ Range T.