Indian Institute of Technology Jammu

CSD001P5M | **Linear Algebra** | **Tutorial: 10**

1. (a) Find the matrix relative to the standard basis of the linear operator T on $\mathbb{R}^3$ given by:
$$T(x_1,x_2,x_3) = (x_1+x_3, x_1+2x_2+x_3, -x_1+x_2).$$
   (b) Find the matrix of the same linear operator T relative to the ordered basis $\beta = \{(1,1,1),(1,2,3),(1,3,6)\}$.

   [NB: The change of basis matrix $P_{S\to\beta}$ for this basis was calculated in **Q 2** of tutorial 06.]

2. (a) Let $T : V \to W$ and $U : W \to Z$ be linear transformations, where V, W and Z are finite-dimensional vector spaces over $\mathbb{F}$. Show that rank $(UT) \leq \min\{\text{rank}(T), \text{rank}(U)\}$.
   (b) State an analogous result for matrices A and B, and comment briefly on its proof.
   (c) For (b), give a non-trivial example (i.e. the matrices A, B should be non-zero and non-identity and should be of minimum size 2×2), in which equality is achieved, and a non-trivial example in which strict inequality holds.

3. Prove that there does not exist a linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^2$ such that
$$\text{Ker } T = \{(x_1,x_2,x_3,x_4,x_5) \in \mathbb{R}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}.$$

4. Let $V = \mathbb{R}^{2\times 2}$ = vector space of 2×2 matrices with real entries, and consider the function $U : V \to V$ given by $U(A) = A + A^T$, for all $A \in V$, where A^T indicates the transpose of A.
   (a) Show that U is a linear operator.
   (b) Determine the matrix of U with regard to any suitable ordered basis β of V.
   (c) Determine a basis for Ker U and determine a basis for Range U.
   (d) Determine the dimension of $\text{Sym}_n(\mathbb{R})$, the space of symmetric $n\times n$ matrices with real entries. Briefly explain your answer.

5. Show that a linear transformation $T : V \to W$, where V and W are finite-dimensional with $\dim V = \dim W$, is injective if and only if it is surjective. (NB: This is part of Proposition 39, so you cannot use Prop 39 in your proof.)

6. Give an example of a vector space V, and two linear transformations $T, U : V \to V$, such that T is surjective but not injective, and U is injective but not surjective. (More advanced: this problem should be tried last.)

7. A square matrix A is said to satisfy a polynomial $p(t) \in \mathbb{R}[t]$ if $p(A) = 0$, i.e. if we substitute the matrix A in the polynomial by taking powers of A (in which the constant term is multiplied by identity matrix of appropriate size), then the resultant is the zero-matrix. Show that every $n \times n$ non-zero square matrix with real entries satisfies a non-zero polynomial of degree $\leq n^2$.

8. Let $V = \mathbb{R}^2$, and consider the ordered bases $\alpha = \{u_1, u_2\}$ and $\beta = \{v_1, v_2\}$, where the vectors are as given below. (NB: regard all vectors as column vectors in V)

$$\mathbf{u_1} = (3,1), \qquad \mathbf{u_2} = (\mathbf{11},\mathbf{4}), \qquad \mathbf{v_1} = (\mathbf{3},\mathbf{2}), \qquad \mathbf{v_2} = (\mathbf{7},\mathbf{5})$$

   (a) Find the change of basis matrix $P_{\alpha \to \beta}$.
   (b) Hence find $[v]_\beta$ given that $[v]_\alpha = (10, 20)$.
   (c) Is there some way to check your answer to (b)? Explain your method and use it to check your answer.

9. Given a vector space V over a field $\mathbb{F}$, a linear transformation $f : V \to \mathbb{F}$ is called a functional on V, i.e. the field $\mathbb{F}$ is regarded as a vector space over itself. The vector space $L(V,F) = \{f : f \text{ is a functional on } V\}$ of all functionals is called the dual space of V and is denoted by V^*. We further assume that V is finite-dimensional with $\dim V = n$.

   (a) What is the dimension of V^*? Briefly justify your answer.
   (b) Show that if f is a functional, then its null space is a hyperspace of V.

   [NB: If V is a finite-dimensional space, then a hyperspace of V is a subspace U of V such that $\dim U = \dim V - 1$.]

   (c) Let W be any hyperspace of V. Show that there exists a functional f on V whose null space is exactly W.

10. Let $f_1(x) = x_1 + 2x_2 + 3x_3, f_2(x) = 2x_1 + 3x_2 + 5x_3, f_3(x) = 3x_1 + 2x_2 + 4x_3, x = [x_1 \quad x_2 \quad x_3]^t$ be linear functionals on $\mathbb{R}^3$. Prove that $\{f_1, f_2, f_3\}$ is a basis of $(\mathbb{R}^3)^*$. Find vectors v_1, v_2, v_3 in $\mathbb{R}^3$ such that $f_i(v_j) = \delta_{ij}, i, j = 1, 2, 3$.