Linear Algebra

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Lecture 31

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Linear Transformation

The Matrix of a Linear Transformation

- We proceed as follows. Suppose V and W are finite- dimensional vector spaces over the field $\mathbb{F}$, and $T : V \longrightarrow W$ is a linear transformation . We will associate a matrix with this linear transformation.
- Suppose $\dim V = n$ and $\dim W = m$.
- We take a fixed ordered basis $B = \{v_1, v_2, \ldots, v_n\}$ for V, and a fixed ordered basis $C = \{w_1, w_2, \ldots, w_m\}$ for W. Since $T(v_i)$ belongs to W, we can express it uniquely as a linear combination of the w_j:

$$T(v_i) = A_{1i}w_1 + A_{2i}w_2 + \cdots + A_{mi}w_m, \quad i = 1, \ldots, n$$

  We now form the $m \times n$ matrix A with **these coefficients as columns.**
- The matrix A is called the matrix of T with respect to the bases B and C. We can also use the notation: $[T]_{B \to C}$

The Matrix of a Linear Transformation (Cont'd)

- For any vector v in V, we can find the coordinates of $T(v)$ in W by left multiplying the coordinate vector of v by the matrix $A = [T]_{B\to C}$
- In terms of coordinate vectors, this can be written in the following way:

$$[T(v)]_C = A[v]_B = [T]_{B\to C}[v]_B$$

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- In the special case of a **linear operator**, i.e. a linear transformation from V to itself, the bases B and C are usually taken as the same, and the matrix A is called the B-matrix for T, written $[T]_B$. Then the above equation becomes simply:

$$[T(v)]_B = [T]_B[v]_B.$$

The Matrix of a Linear Transformation – Example

Example 1:

To find the matrix of the linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ given by $T(x, y, z) = (x + y + z, x + 2y + 3z)$, with respect to the standard bases for $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively.
$T(1, 0, 0) = (1, 1) = 1e_1 + 1e_2$
$T(0, 1, 0) = (1, 2) = 1e_1 + 2e_2$
$T(0, 0, 1) = (1, 3) = 1e_1 + 3e_2$
Hence the matrix of T is $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$.
For any vector $v = (x, y, z)$ in $\mathbb{R}^3$ expressed in column form, we have:

$$Av = \begin{bmatrix} x + y + z \\ x + 2y + 3z \end{bmatrix} = T(v).$$

The Matrix of a Linear Transformation – Example

Example 2:

Consider the differentiation transformation $\mathcal{D} : \mathbb{R}_3[t] \longrightarrow \mathbb{R}_2[t]$. We will use the ordered basis $B = \{1, t, t^2, t^3\}$ for $\mathbb{R}_3[t]$ and the ordered basis $C = \{1, t, t^2\}$ for $\mathbb{R}_2[t]$. Note that $\mathcal{D}$ is certainly a linear transformation, since $\mathcal{D}(p(t) + q(t)) = \mathcal{D}(p(t)) + \mathcal{D}(q(t))$ and $\mathcal{D}(cp(t)) = c\mathcal{D}(p(t))$.
Now, we have:
$\mathcal{D}(1) = 0.1 + 0.t + 0.t^2$
$\mathcal{D}(t) = 1.1 + 0.t + 0.t^2$
$\mathcal{D}(t^2) = 0.1 + 2.t + 0.t^2$
$\mathcal{D}(t^3) = 0.1 + 0.t + 3.t^2$
Therefore, the matrix of $\mathcal{D}$ is:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}_{3\times 4}$$

Example 2 (Cont'd)

Suppose we want to find $\mathcal{D}(p(t))$ for any polynomial $p(t) \in \mathbb{R}_3[t]$, e.g. $p(t) = 10 + 5t + 3t^2 - 7t^3$. We see that

$$A[p(t)]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \\ 3 \\ -7 \end{bmatrix}_B = \begin{bmatrix} 5 \\ 6 \\ -21 \end{bmatrix}_C$$

or $\mathcal{D}(p(t)) = 5 + 6t - 21t^2$.
In general, if $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$, we have:

$$A[p(t)]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}_B = \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \end{bmatrix}_C = a_1 + 2a_2 t + 3a_3 t^2.$$