Indian Institute of Technology Jammu

CSD001P5M | **Linear Algebra** | **Tutorial: 07**

1. Do $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}$ form basis for $\mathbb{Z}_2^3$?

2. Do $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}$ form basis for $\mathbb{Z}_3^3$?

3. Given two linearly independent vectors $(1,0,1,0)$ and $(0,-1,1,0)$ of $\mathbb{R}^4$, find a basis for $\mathbb{R}^4$ that includes these two vectors.

4. Let $\{(1,1,1,1),(1,2,1,2)\}$ be linearly independent subset of the vector space $\mathbb{R}^4$. Extend it to a basis for $\mathbb{R}^4$.

5. Extend the set $\{(3,-1,2)\}$ to two different bases for $\mathbb{R}^4$.

6. Given $S_1 = \{(1,2,3),(0,1,2),(3,2,1)\}$ and $S_2 = \{(1,-2,3),(-1,1,-2),(1,-3,4)\}$, determine the dimension and a basis for
   (a) Span $S_1 \cap$ Span S_2.
   (b) Span $(S_1 + S_2)$.

7. (a) Let $U = \{f(t) \in \mathbb{R}_4[t] \mid f(6) = 0\}$. Find a basis of U.
   (b) Extend the basis in part (a) to a basis of $\mathbb{R}_4[t]$.
   (c) Find a subspace W of $\mathbb{R}_4[t]$ such that $\mathbb{R}_4[t] = U \oplus W$.

8. (a) Let $U = \{f(t) \in \mathbb{R}_4[t] \mid \int_{-1}^{1} f(t)dt = 0\}$. Find a basis of U.
   (b) Extend the basis in part (a) to a basis of $\mathbb{R}_4[t]$.
   (c) Find a subspace W of $\mathbb{R}_4[t]$ such that $\mathbb{R}_4[t] = U \oplus W$.

9. (a) Show that if we think of $\mathbb{C}$ as a vector space over $\mathbb{R}$, then the set $\{1+i, 1-i\}$ is linearly independent.
   (b) Show that if we think of $\mathbb{C}$ as a vector space over $\mathbb{C}$, then the set $\{1+i, 1-i\}$ is linearly dependent.

10. Let $V = \mathbb{F}^{2\times 2}$ be the vector space of all 2×2 matrices over the field $\mathbb{F}$. Let W_1 be the set of matrices of the form

$$\begin{bmatrix} x & -x \\ y & z \end{bmatrix}$$

and let W_2 be the set of matrices of the form

$$\begin{bmatrix} a & b \\ -a & c \end{bmatrix}.$$

   (a) Prove that V has dimension 4 by exhibiting a basis for V which has four elements.

   (b) Prove that W_1 and W_2 are subspaces of V.

   (c) Find dimensions of $W_1, W_2, W_1 + W_2$ and $W_1 \cap W_2$.

11. Suppose v_1, v_2, v_3, v_4 spans V. Prove that the list $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$ also spans V.

12. Suppose v_1, v_2, v_3, v_4 is linearly independent in V. Prove that the list $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$ is also linearly independent.

13. Given the matrix A below:

$$A = \begin{bmatrix} 2 & 6 & 3 \\ 4 & 12 & 5 \\ 13 & 39 & 17 \end{bmatrix}$$

   (a) Find a basis for each of the spaces Nul A, Col A and Row A.

   (b) Find a basis for Row A consisting of rows of the given matrix A. This should be different from the one given in part (a).

   (c) Is A invertible? Justify your answer with reference to TIMT.

14. Let $S = \{v_1, v_2, v_3, v_4, v_5\}$ where the v_i's are vectors in $\mathbb{R}^3$ given below (they should be taken as column vectors):

$$v_1 = (1,2,3), \quad v_2 = (2,5,7), \quad v_3 = (10,24,34), \quad v_4 = (.1,.5,.6), \quad v_5 = (3,7,11).$$

   Let $W = \text{span } S$.

   (a) Reduce S to a basis for W. You must explain your method briefly and show your calculations.

   (b) Is W all of $\mathbb{R}^3$? Justify your answer (YES or NO) in at most one sentence.

15. Given the matrix A and B below:

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & -4 & 11 \\ 2 & 4 & -5 & 14 \end{bmatrix}$$

(a) Find a basis for the row space of A and a basis for the row space of B. You must show your calculations.

(b) Let $U = \text{Span}\,\{(1,2,-1,3),(2,4,-1,2),(3,6,3,-7)\}$ and let $W = \text{Span}\,\{1,2,-4,11),(2,4,-5,14)\}$. Is $U = W$? Justify your answer.

16. Given any $m \times n$ matrix A, show that $\text{rank}(A) \leq \min\{m,n\}$. Give a non-trivial example in which equality is achieved, and a non-trivial example in which strict inequality holds.

17. Given any two $m \times n$ matrices A and B, prove that $\text{rank}(A+B) \leq \text{rank}(A)+\text{rank}(B)$. Give a non- trivial example in which equality is achieved, and a non-trivial example in which strict inequality holds.

18. Solve the following system of equations over the field $\mathbb{Z}_5$:

$$\begin{aligned} 2x+4y+z &= 0 \\ 3x+4y+2z &= 0 \\ x+3y+4z &= 0 \end{aligned}$$

19. Solve the following system of equations over the field $\mathbb{Z}_{11}$:

$$\begin{aligned} 2x+y+z &= 0 \\ 7x+2y &= 0 \\ x-2y+2z &= 0 \end{aligned}$$

20. Find all the solutions of the following system of equations over the field $\mathbb{Z}_{13}$, if any:

$$\begin{aligned} 2x+3y+z &= 6 \\ x+y+2z &= 12 \\ 5y+3z &= 3 \end{aligned}$$

21. Let W be the set of all (x_1,x_2,x_3,x_4,x_5) in $\mathbb{R}^5_{\mathbb{R}}$ which satisfy:

$$\begin{aligned} 2x_1 - x_2 + \frac{4}{3}x_3 - x_4 &= 0 \\ x_1 + \frac{2}{3}x_3 - x_5 &= 0 \\ 9x_1 - 3x_2 + 6x_3 - 3x_4 - 3x_5 &= 0 \end{aligned}$$

Find the finite set of vectors which spans W.

22. If A is any 7×7 invertible matrix in $\mathbb{R}^{7\times 7}$, then what is its column space?

23. Construct a matrix A with the required property, or explain why you can't:

(a) Column space of A contains $\begin{bmatrix}1\\1\\0\end{bmatrix}$, $\begin{bmatrix}0\\0\\1\end{bmatrix}$, row space of A contains $\begin{bmatrix}1\\2\end{bmatrix}$, $\begin{bmatrix}2\\5\end{bmatrix}$

(b) Column space of A has basis $\begin{bmatrix}1\\2\\3\end{bmatrix}$, null space of A has basis $\begin{bmatrix}3\\2\\1\end{bmatrix}$.

(c) Row space of A = column space of A, null space of $A \neq$ null space of A^T.

(d) Column space of A contains $\begin{bmatrix}1\\1\\0\end{bmatrix}$, $\begin{bmatrix}1\\0\\1\end{bmatrix}$, but not $\begin{bmatrix}1\\1\\1\end{bmatrix}$.

(e) Dimension of null space of $A = 1+$ dimension of null space of A^T.

(f) Null space of A^T contains $\begin{bmatrix}1\\3\end{bmatrix}$, row space of A contains $\begin{bmatrix}3\\1\end{bmatrix}$.

(g) Null space of $A =$ null space of A^T.

24. Let A be any $m \times n$ matrix. Show that rank $(A) =$ rank (A^T). Also find an equation relating nullity(A) and nullity (A^T).

25. Compute the rank and nullity of the given matrices over the indicated $\mathbb{Z}_p$.

(a) $\begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\\1 & 0 & 1\end{bmatrix}$ over $\mathbb{Z}_2$.

(b) $\begin{bmatrix}1 & 1 & 2\\2 & 1 & 2\\2 & 0 & 0\end{bmatrix}$ over $\mathbb{Z}_3$.

(c) $\begin{bmatrix}1 & 3 & 1 & 4\\2 & 3 & 0 & 1\\1 & 0 & 4 & 0\end{bmatrix}$ over $\mathbb{Z}_5$.

(d) $\begin{bmatrix}2 & 4 & 0 & 0 & 1\\6 & 3 & 5 & 1 & 0\\1 & 0 & 2 & 2 & 5\\1 & 1 & 1 & 1 & 1\end{bmatrix}$ over $\mathbb{Z}_7$.