Linear Algebra

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Lecture 35

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Algebra of Linear Transformations

Another Fundamental Isomorphism (Cont'd)

- **Proposition 35:** If $\dim V = n$, and $\dim W = m$, then $\dim L(V, W) = mn$.
- **Remark:** The above proposition can be proved in two different ways:
  1. **First method:** We use the fundamental isomorphism of Proposition 34. Since $L(V, W)$ is isomorphic to $\mathbb{F}^{m\times n}$, and $\dim(\mathbb{F}^{m\times n}) = mn$, the result follows from Proposition 28.
  2. **Second method:** We take a fixed ordered basis $\alpha = \{v_1, v_2, \ldots, v_n\}$ for V, and a fixed ordered basis $\beta = \{w_1, w_2, \ldots, w_m\}$ for W.

     We define the linear transformation $E_{ij} \; : \; V \longrightarrow W$ by $E_{ij}(v_j) = w_i$, and $E_{ij}(v_k) = 0$ for $k \neq j$. By a lengthy but straightforward calculation, it can be shown that the family

     $$S = \{E_{ij} \; : \; 1 \leq i \leq m, 1 \leq j \leq n\}$$

     forms a basis for $L(V, W)$. Since $|S| = mn$, the result follows.

Another Fundamental Isomorphism (Cont'd)

- In fact, we can say something more about the above isomorphism. Recall that given a finite-dimensional vector space V and a fixed ordered basis β for V, we can determine the matrix of a linear operator T with respect to β, so that $[T(v)]_\beta = [T]_\beta[v]_\beta$ for any vector v in V. Now, we can obtain the following:
- **Proposition 36:** Suppose T and U are linear operators on a finite-dimensional vector space V and β is a fixed ordered basis for V. Then $[UT]_\beta = [U]_\beta[T]_\beta$.

  Proof: Let $\beta = \{v_1, v_2, \ldots, v_n\}$. Suppose that

$$T(v_i) = a_{1i}v_1 + a_{2i}v_2 + \cdots + a_{ni}v_n, i = 1, 2, \ldots, n. \tag{1}$$

  Recall that the coordinate vector $[T(v_i)]_\beta$ is the i-th column of matrix $[T]_\beta$. Similarly,

$$U(v_i) = b_{1i}v_1 + b_{2i}v_2 + \cdots + b_{ni}v_n, i = 1, 2, \ldots, n \tag{2}$$

  The coordinate vector $[U(v_i)]_\beta$ is the i-th column of the matrix $[U]_\beta$.

- **Proof of Propostion 36 (Cont'd):**

$$
\begin{aligned}
\therefore (UT)(v_i) &= U(a_{1i}v_1 + a_{2i}v_2 + \cdots + a_{ni}v_n) \quad \text{from}(1)\\
&= a_{1i}U(v_1) + a_{2i}U(v_2) + \cdots + a_{ni}U(v_n)\\
&= a_{1i}(b_{11}v_1 + b_{21}v_2 + \cdots + b_{n1}v_n)\\
&+ a_{2i}(b_{12}v_1 + b_{22}v_2 + \cdots + b_{n2}v_n) +\\
&\cdots + a_{ni}(b_{1n}v_1 + b_{2n}v_2 + \cdots + b_{nn}v_n) \quad \text{from}(2)\\
&= (a_{1i}b_{11} + a_{2i}b_{12} + \cdots + a_{ni}b_{1n})v_1 +\\
&\cdots + (a_{1i}b_{n1} + a_{2i}b_{n2} + \cdots + a_{ni}b_{nn})v_n
\end{aligned}
$$

  Thus, the coordinate vector $[(UT)(v_i)]_\beta$, i.e., the i-th column of $[UT]_\beta =$ i-th column of $[U]_\beta[T]_\beta$ as we wanted.

- **Proposition 36 (a) (Alternative Statement of Proposition 36):** The mapping $\phi : L(V,V) \longrightarrow \mathbb{F}^{n\times n}$ given by $\phi(T) = [T]_\beta$ is a vector space isomorphism which also preserves products, i.e. $\phi(UT) = \phi(U)\phi(T)$.
- In simple language, the matrix of the product is the product of the matrices. That is why we define the matrix product in the way we do.

Proof of Proposition 36

Proposition 36: Suppose T and U are linear operators on a finite-dimensional vector space V and β is a fixed ordered basis for V. Then $[UT]_\beta = [U]_\beta[T]_\beta$.

Proof of Proposition 36:

The proof that follows is a revised and slightly expanded version of the one given in the previous slides.

By definition of matrix of a linear operator T relative to a fixed ordered basis $\beta = \{v_1, \cdots, v_n\}$ of V, if

$$Tv_j = a_{1j}v_1 + \cdots + a_{nj}v_n \tag{3}$$

then the j-th column of $[T]_\beta = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{bmatrix}$. Note that in writing (3), j has been taken as the second index in the coefficients. Hence $[T]_\beta = [a_{ij}] = A, say$. Similarly if $Uv_j = b_{1j}v_1 + \cdots + b_{nj}v_n$, then $[U]_\beta = [b_{ij}] = B, say$.

Proof of Proposition 36 (Conti $\cdots$)

Proposition 36 (Conti $\cdots$): Let us determine the <u>first</u> column of the matrix $[UT]_\beta$. For this, we consider:

$$\begin{aligned}
(UT)v_1 &= U(Tv_1) = U(a_{11}v_1 + \cdots + a_{n1}v_n) \qquad \text{using (3)} \\
&= a_{11}Uv_1 + \cdots + a_{n1}Uv_n \\
&= a_{11}(b_{11}v_1 + \cdots + b_{n1}v_n) + \cdots + a_{n1}(b_{1n}v_1 + \cdots + b_{nn}v_n) \\
&= (a_{11}b_{11} + \cdots + a_{n1}b_{1n}) + \cdots + (a_{11}b_{n1} + \cdots + a_{n1}b_{nn})
\end{aligned}$$

Hence, the first column of $[UT]_\beta$:

$$\begin{bmatrix} a_{11}b_{11} + a_{21}b_{12} + \cdots + a_{n1}b_{1n} \\ \vdots \\ a_{11}b_{n1} + a_{21}b_{n2} + \cdots + a_{n1}b_{nn} \end{bmatrix} = \begin{bmatrix} b_{11}a_{11} + b_{12}a_{21} + \cdots + b_{1n}a_{n1} \\ \vdots \\ b_{n1}a_{11} + b_{n2}a_{21} + \cdots + b_{nn}a_{n1} \end{bmatrix} \tag{4}$$

Proof of Proposition 36 (Conti $\cdots$)

Proposition 36 (Conti $\cdots$): Now, let us calculate the matrix:

$$[U]_\beta[T]_\beta = BA = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ \vdots & \vdots & \ldots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

The first column of the product $[U]_\beta[T]_\beta$ is:

$$\begin{bmatrix} b_{11}a_{11} + b_{12}a_{21} + \cdots + b_{1n}a_{n1} \\ \vdots \\ b_{n1}a_{11} + b_{n2}a_{21} + \cdots + b_{nn}a_{n1} \end{bmatrix} \tag{5}$$

Comparing (4) and (5), we say that:
Ist column of $[UT]_\beta =$ Ist column of $[U]_\beta[T]_\beta$
Repeating the calculation for column 2 through n, we get the result that $[UT]_\beta = [U]_\beta[T]_\beta$, by comparing columns.

Generalization of Proposition 36

- **Note:** The above result (Proposition 36 – original form) can be extended to the composition of linear transformations $T : V \longrightarrow W$ and $U : W \longrightarrow Z$.
- Then the analogous statement for Proposition 36 in this situation would be:
- **Proposition 36 (b):** Suppose that $\dim V = n, \dim W = m$, and $\dim Z = k$. $UT : V \longrightarrow Z$ would be a linear transformation from a space of dimension n to a space of dimension k, i.e. its matrix would be an $n \times k$ matrix. Let α, β, γ be bases of V, W, Z, respectively. Then:

$$[UT]_{\alpha\to\gamma} = [U]_{\beta\to\gamma}[T]_{\alpha\to\beta}.$$

Invertibility of Linear Transformations

- **Definition:** Any function f from V into W is said to be invertible if there exists a function g from W into V such that gf is the identity function on V and fg is the identity function on W.
- **Observation 1:** In case f is invertible, then the function g is unique, and is called the inverse of f, denoted by f^{-1}.
- **Observation 2:** A function f is invertible if and only if f is injective (old terminology: 1:1 or one-to-one) and surjective (old terminology: onto, i.e. the range of f is all of W), i.e. bijective.
- **Proposition 37:** If T is an invertible linear transformation, its inverse function T^{-1} is also a linear transformation.
  Proof: Consider the function $T^{-1} : W \longrightarrow V$
  (i) Consider $w_1, w_2 \in W$, we have to show that:

$$T^{-1}(w_1 + w_2) = T^{-1}(w_1) + T^{-1}(w_2) \tag{6}$$

- **Proof of part (i) of Proposition 37 (Cont'd):** Let us apply T to LHS of (6):

$$T(T^{-1}(w_1 + w_2)) = (TT^{-1})(w_1 + w_2) = I(w_1 + w_2) = w_1 + w_2 \quad (7)$$

Let us apply T to RHS of (6):

$$\begin{aligned} T(T^{-1}(w_1) + T^{-1}(w_2)) &= T(T^{-1}(w_1)) + T(T^{-1}(w_2)) \\ &= (TT^{-1})(w_1)) + (TT^{-1})(w_2) \qquad (8) \\ &= w_1 + w_2 \end{aligned}$$

Since T is injective, from (7) and (8), we get that:
LHS of (6) = RHS of (6), as required.

(ii) For $c \in \mathbb{F}$ and for $w \in W$, we need to prove that:

$$T^{-1}(cw) = cT^{-1}(w)$$

The proof follows by using the same technique of applying T on both the sides of above equation. The parts (i) and (ii) prove that T^{-1} is linear transformation.

Invertibility of Linear Transformations (Cont'd)

- **Definition:** A linear transformation T from V into W is said to be non-singular if the null space of T is $\{0\}$, i.e., $Tv = 0$ implies $v = 0$.
- **Remark:** This is equivalent to saying that T is injective (we had already noted this when we initially defined the null space or kernel).
- **Proposition 38:** Let T be a linear transformation from V into W. Then T is non-singular if and only if T carries every linearly independent subset of V into a linearly independent subset of W.

  Proof: Left as an exercise.
- **Proposition 39:** Let V and W be finite-dimensional spaces with $\dim V = \dim W$. Let T be a linear transformation from V into W. Then the following are equivalent:
  (a) T is invertible
  (b) T is non-singular
  (c) T is surjective, i.e. the range of T is W
  (d) T carries every basis of V into a basis of W

  Proof: Left as an exercise.

Invertibility of Linear Transformations (Cont'd)

- **Remark:** The essential point in the above Proposition 39 is that for finite-dimensional spaces **with equal dimension**, if the linear transformation is non-singular (i.e. injective) then it must be surjective, and if it is surjective, then it must be injective. However, this holds only for finite-dimensional spaces.
- For infinite-dimensional spaces V, it is possible to find a linear operator $T : V \longrightarrow V$ which is surjective but not injective. Similarly, it is possible to find a linear operator $T : V \longrightarrow V$ which is injective but not surjective. (Left as an exercise.)