Graphical Models

David A. Levin

DAIICT

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- Example: If $X_1, \dots, X_n$ are independent, then the joint density completely factors into marginals:

$$f(x_1, \dots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i)\,. \tag{1}$$

- This is too strong a constraint to require of general applicable probability models.
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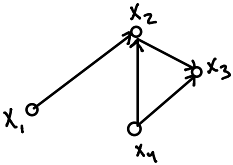
- nodes represent variables
- directed arrows represent dependencies
- Any variable with an arrow pointing to x_i is called a **parent** of x_i; we denote by pa_i all the parents of x_i.



- A joint distribution respects the graphical model encoded by a directed graph if

$$p(x_1, \dots, x_n) = \prod_{i=1}^{n} p(x_i \mid \mathsf{pa}_i)\,.$$

- In the example graph, a joint distribution obeys the graphical model if

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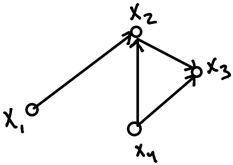

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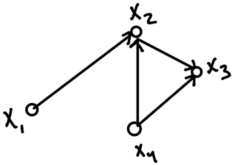

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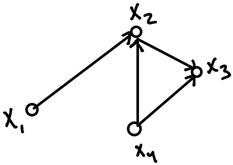

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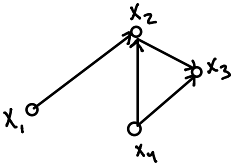

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- The graphical model encodes *conditional independence* statements.
- A set of variables X and Y are conditionally independent given Z if

$$\mathbb{P}(X \in A, Y \in B \mid Z) = \mathbb{P}(X \in A \mid Z)\mathbb{P}(X \in B \mid Z)\,.$$

- In terms of pdfs/pmfs,

$$p(x, y \mid z) = p(x \mid z)\, p(y \mid z)\,.$$

- Or equivalently,

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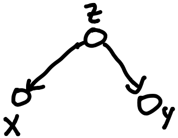

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- Note that

$$p(x, y \mid z) = \frac{p(x, y, z)}{\sum_{x,y} p(x, y, z)} = \frac{p(x \mid z)\, p(y \mid z)\, p(z)}{\sum_x p(x \mid z) \sum_y p(y \mid z)\, p(z)} = p(x \mid z)\, p(y \mid z)$$

- Any time X and Y are separated by a **tail-to-tail** vertex Z, they are conditionally independent.
- Note that X and Y are not independent unconditionally, in general.
- Exercise: Show by example that X and Y are not necessarily independent.

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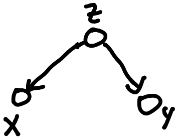

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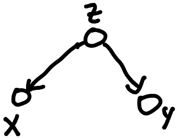

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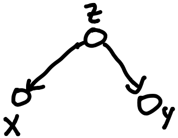

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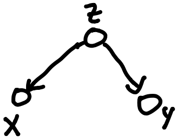

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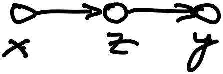

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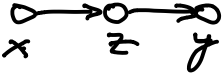

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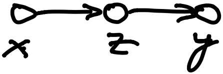

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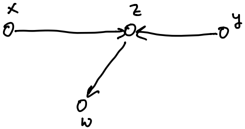

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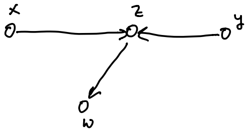

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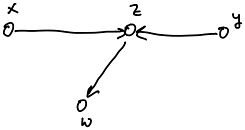

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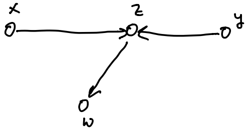

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- Let A, B, C be sets of vertices. The set C **blocks** A to B if every path from a vertex $a \in A$ to a vertex $b \in B$ contains either
  - a vertex $c \in C$ so that c is head-to-tail or tail-to-tail, or
  - a vertex v which is head-to-head, and which neither belongs to C or has a decendent belonging to C.
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Theorem 1 (*d*-separation).

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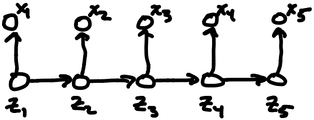

- The above is a **Hidden Markov Model**.
- We can use *d*-separation to establish useful conditional independence relations:
  - $$(X_1, \ldots, X_k) \perp (X_{k+1}, \ldots, X_n) \mid Z_k$$
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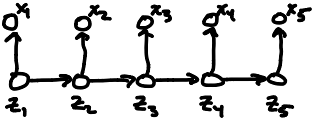

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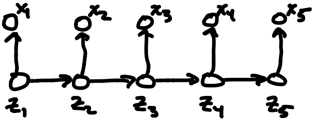

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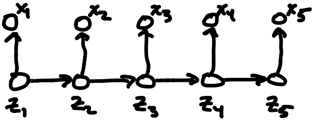

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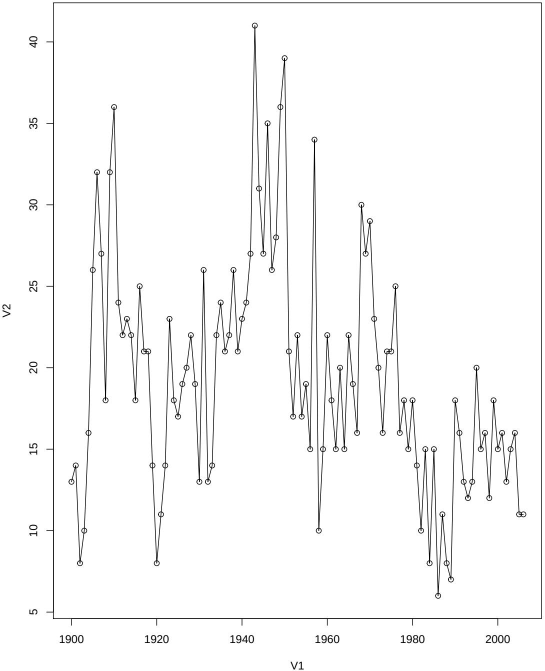

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