# CSL003P1M: Probability and Statistics



## Assignment -V

October 12, 2019

1. Find the moment generating function of the random variable $X$ having the pdf

$$f_X(x) = \begin{cases} \frac{1}{3}, & -1 < x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

2. A number is randomly chosen in the interval $(0, 1)$. Calculate:

   (a) the probability that the first digit to the right of the decimal point is 6;

   (b) the probability that the second digit to the right of the decimal point is 1;

   (c) the probability that the second digit to the right of the decimal point is 8 given that the first digit was 3.

3. Suppose that a certain type of electronic component has an exponential distribution with a mean life of 500 hours. If $X$ denotes the life of this component, then what is the probability that the component will last an additional 600 hours, given that it has operated for 300 hours?

4. (a) Let $X$ be a random variable with mgf $M(t), -h < t < h$. Prove that

$$P(X \geq a) \leq e^{-at}M(t), \quad 0 < t < h, \quad \text{and}$$
$$P(X \leq a) \leq e^{-at}M(t), \quad -h < t < 0.$$

   (b) The mgf of $X$ exists for all real values of $t$ and is given by

$$M(t) = \frac{e^t - e^{-t}}{2t}, \ t \neq 0, \ M(0) = 1.$$

   Use the above result to show that $P(X \geq 1) = 0$ and $P(X \leq -1) = 0$.

   (c) Let $X$ have a Gamma distribution with parameters $\alpha$ and $\beta$. Show that

$$P(X \geq 2\alpha\beta) \leq (2/e)^{\alpha}.$$

5. A student arrives at the bus station at 6:00 AM sharp knowing that the bus will arrive any moment, uniformly distributed between 6:00 AM and 6:20 AM. What is the probability that the student must wait more than 5 minutes? If at 6:10 AM the bus has not arrived yet, what is the probability that the student has to wait at least 5 more minutes?

6. Show that

$$\int_{\mu}^{\infty} \frac{z^{k-1}e^{-z}}{\Gamma(k)}\, dz = \sum_{x=0}^{k-1} \frac{\mu^x e^{-\mu}}{x!}, \ k = 1, 2, 3, \ldots.$$

   This demonstrates the relationship between the cdfs of the Gamma and Poisson distributions.

7. Show that

$$\int_p^1 \frac{n!}{(k-1)!(n-k)!} z^{k-1}(1-z)^{n-k}\ dz = \sum_{x=0}^{k-1} \binom{n}{x} p^x (1-p)^{n-x}.$$

This demonstrates the relationship between the cdfs of the $\beta$ and binomial distributions.

8. Suppose that the marks on an examination are distributed normally with mean 76 and standard deviation 15. Of the best students 15% obtained $A$ as grade and of the worst students 10% lost the course and obtained $P$.

   (a) Find the minimum mark to obtain $A$ as a grade.

   (b) Find the minimum mark to pass the test.

9. (a) If $e^{3t+8t^2}$ is the mgf of the random variable $X$, then find the distribution of $X$ and $P(-1 < X < 9)$.

   (b) If $X \sim N(\mu, \sigma^2)$, then show that $E(|X-\mu|) = \sigma\sqrt{2/\pi}$.

10. Find the uniform distribution of the continuous type on the interval $(b, c)$ that has the same mean ans the same variance as those of a $\chi^2$ distribution with 8 degrees of freedom (the number of independent values or quantities which can be assigned to a statistical distribution). That is, find $b$ and $c$.

11. The time that has elapsed between the calls to an office has an exponential distribution with mean time between calls of 15 minutes.

    (a) What is the probability that no calls have been received in a 30-minute period of time?

    (b) What is the probability of receiving at least one call in the interval of 10 minutes?

    (c) What is the probability of receiving the first call between 5 and 10 minutes after opening the office?

12. (i) Let $X$ be a random variable such that $E(X^m) = (m+1)!2^m, \quad m = 1, 2, 3, \ldots$. Determine the mgf and the distribution of $X$.

    (ii) Let $X$ have a gamma distribution with pdf $f_X(x) = \frac{1}{\beta^2} x e^{-\frac{x}{\beta}}$, $0 < x < \infty$, zero elsewhere. If $X = 2$ is the unique mode of the distribution, find the parameter $\beta$ and the distribution.

13. A gasoline wholesale distributor has bulk storage tanks that hold fixed supplies and are filled every Monday. Of interest to the wholesaler is the proportion of this supply that is sold during the week. Over many weeks of observations, the distributor found that this proportion could be modeled by a beta distribution with $\alpha = 4$ and $\beta = 2$. Find the probability that the wholesaler will sell at least 90% of his stock in a given week.

14. Let $X$ be a normal random variable with parameters 0 and $\sigma^2$. Find the pdf for:

    (a) $Y = |X|$.

    (b) $Y = \sqrt{|X|}$.