CSL003P1M: Probability and Statistics

Assignment -III

August 28, 2019

1. Let X be a random variable with density function given by:

$$f(x) = \begin{cases} 2x^3, & 0 < x \leq 1, \\ 2(2-x)^3, & 1 < x \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

   (i) Calculate $\mu := E(X)$ and $\sigma^2 := Var(X)$.

   (ii) Find $P(\mu - 2\sigma < X < \mu + 2\sigma)$.

2. Let X be a random variable with PMF

$$p_X(x) = \begin{cases} \dfrac{x^2}{a}, & x = -3, -2, -1, 0, 1, 2, 3, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Find a and $E(X)$.

   (b) What is the PMF of the random variable $Z = \big(X - E(X)\big)^2$?

   (c) Using the result from part (b), find the variance of X.

   (d) Find the variance of X using the formula $Var(X) = \sum_x \big(x - E(X)\big)^2 p_X(x)$.

3. A fair coin is tossed four times. Let X denote the number of times a head is followed immediately by a tail. Find the variance of X.

4. If the probability density function of a random variable X is given by

$$f(x) = Ce^{-(x^2+2x+3)}, \quad -\infty < x < \infty,$$

   (i) Find the value of C.

   (ii) Find the expectation and variance of the distribution.

5. **(The Quiz problem)**
   Consider a quiz game where a person is given two questions and must decide which one to answer first. Question 1 will be answered correctly with probability 0.8, and the person will then receive as prize $Rs.100$, while question 2 will be answered correctly with probability 0.5, and the person will then receive as prize $Rs.200$. If the first question attempted is answered incorrectly, the quiz terminates, i.e., the person is not allowed to attempt the second question. If the first question is answered correctly, the person is allowed to attempt the second question.

   Which question should be answered first to maximize the expected value of the total prize money received?

6. Find the mean and variance of the continuous distribution with PDF given by

$$f(x) = \begin{cases} 1 - |1 - x|, & 0 < x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

7. **(Markov's inequality)**
Let X be a real random variable with $X \geq 0$ and such that $E(X)$ exists. Prove that, for all $\alpha > 0$, it is satisfied that:

$$P(X \geq \alpha) \leq \frac{E(X)}{\alpha}.$$

8. Let X be a discrete random variable with values in the non-negative integers such that $E(X)$ and $E(X^2)$ exist.

   (a) Show that:

$$E(X) = \sum_{k=0}^{\infty} P(X > k).$$

   (b) Verify that:

$$\sum_{k=0}^{\infty} kP(X > k) = \frac{1}{2}(E(X^2) - E(X)).$$

9. **(Chebyschev's inequality)**
Let X be a random variable with mean μ and variance σ^2.

   (a) Prove that for all $\epsilon > 0$ it is satisfied that:

$$P(|X - \mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2}.$$

   (b) If $\mu = \sigma^2 = 20$, what can be said about $P(0 \leq X \leq 40)$?

10. Compare the upper bound on the probability $P\Big(|X - E(X)| \geq 2\sqrt{Var(X)}\Big)$ obtained from Chebyschev's inequality with the exact probability if X is uniformly distributed over $(-1, 3)$.