CSL003P1M: Probability and Statistics

Assignment -VII

November 11, 2019

1. Let X be a random variable with expectation μ and variance σ^2. Let $(X_1, X_2, \ldots, X_n)$ be a sample of X. Show that $C\sum_{i=1}^{n-1}(X_{i+1}-X_i)^2$ is an unbiased estimate of σ^2 for an appropriate choice of C. Find that value of C.

2. Suppose that 200 independent observations $X_1, X_2, \ldots, X_{200}$ are obtained from a random variable X. We are told that $\sum_{i=1}^{200} X_i = 300$ and that $\sum_{i=1}^{200} X_i^2 = 3754$. Using these values, obtain an unbiased estimate for $E(X)$ and $V(X)$.

3. A random variable X has pdf $f(x) = (\beta+1)x^\beta,\ ) < x < 1$.
   (a) Obtain the ML estimate of β, based on a sample $X_1, X_2, \ldots, X_n$.
   (b) Evaluate the estimate if the sample values are
   $$0.3, 0.8, 0.27, 0.35, 0.62, \text{ and } 0.55.$$

4. Suppose that T, the time to failure (in hours) of an electronic device, has the following pdf:
   $$f(t) = \begin{cases} \beta e^{-\beta(t-t_0)}, & t > t_0 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$
   (T has an exponential distribution truncated to the left at t_0.) Suppose that n items are tested and the failure times $T_1, T_2, \ldots, T_n$ are recorded.
   (a) Assuming that t_0 is known, obtain the ML estimate of β.
   (b) Assuming that t_0 is unknown but β is known, obtain the ML estimate of t_0.

   Furthermore, suppose N items are tested for T_0 hours ($T_0 > t_0$), and the number of items that fail in that period is recorded, say k. Assuming that t_0 is known, obtain the ML estimate of β.

5. Let X be a continuous random variable with pdf f and cdf F. Let $X_1, X_2, \ldots, X_n$ be a random sample of X and let K and M be the minimum and maximum of the sample, respectively. Then:
   (a) the pdf of M is given by $g(m) = n[F(m)]^{n-1}f(m)$,
   (b) the pdf of K is given by $h(k) = n[1-F(k)]^{n-1}f(k)$.

6. Suppose that X is uniformly distributed over $(0, a)$. Find the ML estimate of a based on a random sample of size $n, X_1, \ldots, X_n$. Is the resulting estimate biassed?

7. A random variable X has distribution $N(\mu, 1)$. Twenty observations are taken on X but instead of recording the actual value we only note whether or not X was negative. Suppose that the event $\{X < 0\}$ occurred precisely 14 times. Using this information, obtain the ML estimate of μ.

8. The capture/recapture method is a way to estimate the size of a population in the wild. The method assumes that each animal in the population is equally likely to be captured by a trap. Suppose 10 animals are captured, tagged and released. A few months later, 20 animals are captured, examined, and released. 4 of these 20 are found to be tagged. Estimate the size of the wild population using the MLE for the probability that a wild animal is tagged.

9. Let $X_1, X_2, \dots, X_n$ be a random sample from a population with density,

$$f(x|\theta) = \begin{cases} \frac{1}{2}\exp(-|x-\theta|), & -\infty < x < \infty, -\infty < \theta < \infty \\ 0, & \text{else.} \end{cases}$$

   Find MLE of θ.

10. Show that if $X_1, X_2, \dots, X_n$ are independent random variables, each having an exponential distribution with parameter α_i, $i = 1, 2, \dots, n$, and if $K = \min(X_1, X_2, \dots, X_n)$, then K has an exponential distribution with parameter $\alpha_1 + \alpha_2 + \cdots + \alpha_n$.

11. A sample of size 5 is obtained from a random variable with distribution $N(12, 4)$.
    (a) What is the probability that the sample mean exceeds 13?
    (b) What is the probability that the minimum of the sample is less than 10?
    (c) What is the probability that the maximum of the sample exceeds 15?

12. The life length (in hours) of an item is exponentially distributed with parameter $\beta = 0.001$. Six items are tested and their times to failure recorded.
    (a) What is the probability that no item fails before 800 hours have elapsed?
    (b) What is the probability that no item lasts more than 3000 hours?

13. Suppose that X has distribution $N(0, 0.09)$. A sample of size 25 is obtained from X, say $X_1, X_2, \dots, X_{25}$. What is the probability that $\sum_{i=1}^{25} X_i^2$ exceeds 1.5?

14. Find the sampling distribution of the mean where the samples are extracted from the population which follows the Poisson distribution.

15. Suppose that for any student the time required for a teacher to grade the final exams of his probability course is a random variable with mean 1 hour and standard deviation 0.4 hours. If there are 100 students in the course, what is the probability that he needs more than 110 hours to grade the exam?