CSL003P1M: Probability and Statistics

Assignment -VI

October 27, 2019

1. Let (X, Y) a random vector with joint pdf

$$f(x,y) = \begin{cases} \frac{3}{2}(x^2+y^2) & \text{if } 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find the joint distribution function.

2. An urn contains 11 balls, 4 of them are red-colored, 5 are black-colored and 2 are blue-colored. Two balls are randomly extracted from the urn without replacement. Let X and Y be the random variables representing the number of red-colored and black-colored balls, respectively, in the sample. Find:

   (a) The joint distribution of X and Y.

   (b) $E(X)$ and $E(Y)$.

3. Let X and Y be independent random variables having the joint distribution given by the following table:

$X \setminus Y$	-1	0	1
0	$\frac{1}{6}$	d	$\frac{1}{6}$
1	a	e	k
2	b	f	h

If $P(X=1) = P(X=2) = \frac{1}{5}$, find the values missing in the table. Calculate $E(XY)$.

4. The random variable X and Y have the joint density function

$$f(x,y) = \begin{cases} 6(1-x-y), & x > 0, y > 0, x+y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Find the marginal distribution of X and Y. Are X and Y independent?

5. The distribution of a two-dimensional random variable (X, Y) is given by

$$f(x,y) = \begin{cases} e^{-x-y}, & x \geq 0, y \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Find

   (i) the joint distribution function,

   (ii) the marginal distribution functions of X and Y,

(iii) $P(X+Y \leq 4)$,

(iv) $P(X \geq 1)$,

(v) $P(X \leq Y)$,

(vi) $P(a < X+Y < b)$ where $0 < a < b$.

Also show that X and Y are independent.

6. (*t-Student distribution*) Let X and Y be independent random variables such that $X \sim^d \mathcal{N}(0,1)$ and $Y \sim \chi^2(k)$. Then find the pdf of $Z = \frac{X}{\sqrt{\frac{Y}{k}}}$.

7. (*F-distribution*) Let X and Y be independent random variables such that $X \sim \chi^2(m)$ and $Y \sim \chi^2(n)$. Find the pdf of $Z = \frac{\frac{X}{m}}{\frac{Y}{n}}$.

8. Suppose 2 balls are drawn without replacement from an urn containing 3 balls numbered $1, 2$, and 3. Let X be the number on the first ball drawn and Y be the larger of the two numbers drawn. Find

   (a) the joint pmf of X and Y,

   (b) the conditional pmf of X given $Y = 3$,

   (c) $Cov(X,Y)$.

9. If X and Y are two independent random variables having the same geometric distribution with the parameter p, find $P(X = Y)$.

10. Let X and Y have the joint density

$$f_{X,Y}(x,y) = \begin{cases} c(1-x-y), & x > 0, y > 0, x+y < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Then find

   (i) the constant c, and the marginal densities of X and Y ,

   (ii) the conditional density of X given $Y = y$, and $E(X|Y = \frac{1}{2})$,

   (iii) the correlation coefficient of X and Y .

11. Let X and Y be two discrete random variables with the joint pmf $f_{X,Y}(x,y) = \frac{x+2y}{24}$, $(x,y) = (0,1), (0,2), (1,1), (1,2), (2,1), (2,2)$, zero elsewhere. Find the conditional mean and conditional variance of X given $Y = 2$. Also determine the correlation of X and Y.

12. Let two random variables X, Y have the joint density

$$f_{X,Y}(x,y) = \begin{cases} e^{-(x+y)}, & x > 0, y > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Then find (a) $P(1 < X+Y < 2)$ (b) $P(X < Y|X > 2Y)$.

13. Let two random variables X, Y have the joint density

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2}xy, & 0 < y < x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find $P(X \leq \frac{3}{2}|Y = 1)$, and the conditional variance of X given $Y = 1$.

(b) Are X, Y independent? Explain.

14. Let X and Y be two discrete random variables such that the pmf of X is $f_X(x) = \frac{x}{3}$, $x = 1, 2$, and the conditional distribution of Y, given $X = x$, is a binomial distribution with parameters x and $\frac{1}{2}$. Find (a) the joint pmf of X and Y (b) $E(Y)$.

15. Three points X_1, X_2 and X_3 are selected at random on a line segment of length L. What is the probability that X_2 lies between X_1 and X_3?

16. Suppose that the joint cdf of two random variables X and Y is

$$F_{X,Y}(x,y) = \begin{cases} (1 - e^{-ax})\, y^2, & 0 \leq x < \infty, 0 \leq y < 1, \\ 1 - e^{-ax}, & 0 \leq x < \infty, 1 \leq y < \infty, \\ 0, & \text{elsewhere,} \end{cases}$$

where $a > 0$ is a constant. Show that X and Y are independent.

17. Let X and Y be random variables such that $Var(X) = 4, Var(Y) = 2$, and $Var(X + 2Y) = 15$. Determine the correlation coefficient of X and Y .

18. Let X and Y have the same variance σ^2 and let ρ be correlation coefficient of X and Y. Show that, for any $k > 0$,

$$P\big[|(X - \mu_x) + (Y - \mu_Y)| \geq k\sigma\big] \leq \frac{2(1+\rho)}{k^2},$$

where μ_X and μ_Y are the mean of X and Y respectively.

19. Prove that

(a) sum of r independent geometric random variables each with parameter p is a negative binomial random variable with the parameters (r, p).

(b) the sum of two independent binomial random variables with the parameters (n_1, p) and (n_2, p) is a binomial random variable with the parameters $(n_1 + n_2, p)$.

(c) sum of two independent Poisson random variables with the parameters λ_1 and λ_2 is a Poisson random variable with the parameter $\lambda_1 + \lambda_2$.

(d) sum of r independent exponential random variables each with the parameter β is a gamma random variable with the parameters (r, β).

(e) sum of two independent gamma random variables with the parameters (α_1, β) and (α_2, β) is a gamma random variable with the parameters $(\alpha_1 + \alpha_2, \beta)$.

(f) sum of two independent χ^2-random variables with n_1 and n_2 degrees of freedom is a χ^2-random variable with $n_1 + n_2$ degrees of freedom.

(g) any linear combination of two independent normal random variables is again a normal random variable.

20. Let X_1, X_2 be two discrete random variables with the joint pmf $f_{X_1,X_2}(x_1, x_2) = \frac{x_1 x_2}{36},\ x_1 = 1, 2, 3,\ x_2 = 1, 2, 3$, zero elsewhere. Find the joint pmf of $Y_1 = X_1 X_2$ and $Y_2 = X_2$.

21. Let X_1, X_2 have the joint pdf $f_{X_1,X_2}(x_1, x_2) = 8x_1 x_2, 0 < x_2 < x_1 < 1$, zero elsewhere. Find the joint pdf of $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$.

22. (a) Let X_1 and X_2 be two independent random variables each following an exponential distribution with the parameter $\beta = 1$. Find the pdf of $X_1 + X_2$.

    (b) Let X and Y be two independent standard normal random variables. Find the pdf of $Z = X^2 + Y^2$.

23. If X and Y are two jointly distributed random variables, show that $E[E(X|Y)] = E(X)$. Verify the result for a random vector (X, Y) which is uniformly distributed over the region $\{(x, y) : 0 < x < y < 1\}$.

24. Let X_1, X_2 and X_3 be three independent random variables each having the pdf $f(x) = 5x^4,\ 0 < x < 1$, zero elsewhere. Let Y be the largest of X_1, X_2 and X_3. Find the pdf of Y.