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Assignment - 4 CS-663

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Q3 Given, matrix A of size $m \times n$

$P = A^T A$ and $Q = AA^T$

(a). To prove : $y^t P y \geq 0$

$y^t P y$

$\Rightarrow y^t (A^T A) y$

$\Rightarrow (y^t A^t)(A y)$

$\Rightarrow (Ay)^t (A y)$

$\Rightarrow \|Ay\|^2 \qquad \{x^t x = X \cdot X = \|X\|^2\}$

$\Rightarrow \geq 0$

Similarly, to prove : $z^t Q z \geq 0$

$z^t \; Q \; z$

$\Rightarrow z^t (A A^t) z$

$\Rightarrow (z^t A)(A^t z)$

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$\Rightarrow (A^t z)^t (A^t z)$

$\Rightarrow \| A^t z \|^2$

$\Rightarrow \geq 0$

Let, v be the eigen vector of P

Now,

$Pv = \lambda v$, λ is eigenvalue

$v^t P v = v^t \lambda v$

L.H.S: $v^t P v \geq 0$, from above

R.H.S: $v^t \lambda v$

$\Rightarrow \lambda \cdot v^t v$

$\Rightarrow \lambda \| v \|^2$

Since, $\| v \|^2 \geq 0$

and L.H.S ≥ 0

$\Rightarrow \lambda \geq 0$

Similarly for Q, eigenvalues are non-negative.

(b). Given,

$$P u = \lambda u$$

To prove,

$$Q A u = \lambda A u$$

Now,

$$P u = \lambda u$$

$$\Rightarrow A^T A u = \lambda u$$

$$\Rightarrow A (A^T A u) = A (\lambda . u)$$

$$\Rightarrow (A A^T) (A u) = \lambda (A u)$$

$$\Rightarrow Q (A u) = \lambda (A u)$$

Given,

$$Q v = \mu v$$

To prove,

$$P A^T v = \mu A^T v$$

Now,

$$Q v = \mu v$$

$$\Rightarrow A A^T v = \mu v$$

$$\Rightarrow A^T (A A^T v) = A^T (\mu v)$$

$$\Rightarrow (A^T A) (A^T v) = \mu (A^T v)$$

$$\Rightarrow P A^T v = \mu A^T v$$

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Size of vectors :

$u \rightarrow n \times 1$

$v \rightarrow m \times 1$

(c). Given, $Q v_i = \lambda v_i$,
where λ is eigen value of Q
and $\lambda \geq 0$ (part-a)

also, $u_i = \dfrac{A^T v_i}{\| A^T v_i \|}$

~~Now~~,

To prove,

$\exists\ \gamma_i$, $A u_i = \gamma_i v_i$ s.t. $\gamma_i \geq 0$

Now,

$A u_i = \gamma_i v_i$

$\Rightarrow A \dfrac{A^T v_i}{\| A^T v_i \|} = \gamma_i v_i$

$\Rightarrow (A A^T) v_i = \gamma_i v_i \| A^T v_i \|$

$\Rightarrow Q v_i = \gamma_i \| A^T v_i \| v_i$

since, $Q v_i = \lambda v_i$

$\therefore \lambda = \gamma_i \| A^T v_i \|$

$$\therefore \quad \gamma_i = \frac{\lambda}{\|A^T v_i\|}$$

since, eigen value λ exist for Q

and

$\lambda \geq 0$

$\therefore \quad \gamma_i$ will exist

and

$\gamma_i \geq 0$

[d]. Given, $U = [v_1 \mid v_2 \mid v_3 \ldots\ldots v_m]$

$V = [u_1 \mid u_2 \mid u_3 \ldots\ldots u_n]$

To prove,

$A = U \Gamma V^T$

where,

$$\Gamma = \begin{bmatrix} \gamma_1 & & & & \\ & \gamma_2 & & & \\ & & \gamma_3 & & \\ & & & \ddots & \\ & & & & \gamma_n \end{bmatrix}$$

Now,

$A u_i = \gamma_i v_i$ (from part-C)

$A^T A u_i = A^T \gamma_i v_i$

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$$\Rightarrow \quad (A^T A)\, u_i = \gamma_i \, A^T v_i$$

Since,

$A^T A = P$ and

$$u_i = \frac{A^T v_i}{\| A^T v_i \|}$$

$$\therefore \quad A^T v_i = u_i \, \| A^T v_i \|$$

$$\Rightarrow \quad P\, u_i = \gamma_i \| A^T v_i \| \, u_i$$

$\Rightarrow$ u_i is eigen vector of P

Now,

$U \Gamma V^T$

$$\Rightarrow \begin{bmatrix} v_{11} & v_{21} & \cdots & v_{m1} \\ v_{12} & v_{22} & & \vdots \\ v_{13} & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ v_{1m} & v_{2m} & \cdots & v_{mm} \end{bmatrix} \begin{bmatrix} \gamma_1 & & & \\ & \gamma_2 & & \\ & & \ddots & \\ & & & \gamma_n \end{bmatrix} V^T$$

$$\Rightarrow \begin{bmatrix} \gamma_1 v_{11} & \gamma_2 v_{21} & \cdots & \gamma_m v_{m1} \\ \gamma_1 v_{12} & \gamma_2 v_{22} & & \vdots \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \gamma_1 v_{1n} & \cdots & \cdots & \gamma_m v_{mn} \end{bmatrix} V^T$$