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Unit - 4 Linear Transformation.

Let U and V be two vector spaces over a field F.
A linear transformation from U into V is a function $T: U \to V$ such that $T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$, $\forall\ \alpha, \beta \in U(F), a, b \in F$

Que Prove that $T: R^2 \to R^2$ defined by $T(x_1, x_2) = (x_1, 0)$ is a linear transformation.

To show, T is linear transformation we prove

$$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta), \quad \forall\ \alpha, \beta \in R^2,\ a, b \in R$$

Let $\alpha, \beta \in R^2$

$$\alpha = (x_1, x_2)$$
$$T(\alpha) = T(x_1, x_2) = (x_1, 0)$$

$$\beta = (y_1, y_2)$$
$$T(\beta) = (y_1, 0)$$

Now,
$$\begin{aligned} T(a\alpha + b\beta) &= T[a(x_1, x_2) + b(y_1, y_2)] \\ &= T[(ax_1, ax_2) + (by_1, by_2)] \\ &= T(ax_1 + by_1, ax_2 + by_2) \\ &= (ax_1 + by_1, 0) \\ &= (ax_1, 0) + (by_1, 0) \\ &= a(x_1, 0) + b(y_1, 0) \\ &= aT(\alpha) + bT(\beta), \quad \forall\ \alpha, \beta \in R^2 \end{aligned}$$

Hence T is linear transformation from R^2 into R^2.

Ques: The function $T: V_3(R) \to V_2(R)$ defined by $T(a, b, c) = (a, b)$ $\forall\ a, b, c \in R$. Show that T is linear transformation from $V_3(R)$ into $V_2(R)$.

We prove $T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$

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Let $\alpha, \beta \in V_3(R)$, $\alpha = (a_1, b_1, c_1) \Rightarrow T(\alpha) = (a_1, b_1)$
$\beta = (a_2, b_2, c_2) \Rightarrow T(\beta) = (a_2, b_2)$.

Now for $a, b \in R$.

$$
\begin{aligned}
L.H.S. &= T(a\alpha + b\beta) \\
&= T[a(a_1, b_1, c_1) + b(a_2, b_2, c_2)] \\
&= T[(aa_1, ab_1, ac_1) + (ba_2, bb_2, bc_2)] \\
&= T(aa_1 + ba_2, ab_1 + bb_2, ac_1 + bc_2) \\
&= (aa_1 + ba_2, ab_1 + bb_2) \quad \text{[By def. of T]} \\
&= (aa_1, ab_1) + (ba_2, bb_2) \\
&= a(a_1, b_1) + b(a_2, b_2) \\
&= aT(\alpha) + bT(\beta), \quad \forall \alpha, \beta \in V_3(R), a, b \in R.
\end{aligned}
$$

Hence T is a linear transformation; $T: V_3(R) \to V_2(R)$.

Que. Show that mapping $T: V_2(R) \to V_3(R)$ defined by
$T(a, b) = (a+b, a-b, 0)$ is L.T. from $V_2(R)$ into $V_3(R)$.

We prove, $T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$.

Let $\alpha, \beta \in V_2(R)$, $\alpha = (a_1, b_1) \Rightarrow T(\alpha) = (a_1+b_1, a_1-b_1, 0)$
$\beta = (a_2, b_2) \Rightarrow T(\beta) = (a_2+b_2, a_2-b_2, 0)$.

Now for $a, b \in R$.

$$
\begin{aligned}
L.H.S. &= T(a\alpha + b\beta) \\
&= T[a(a_1, b_1) + b(a_2, b_2)] \\
&= T[(aa_1, ab_1) + (ba_2, bb_2)] \\
&= T(aa_1 + ba_2, ab_1 + bb_2) \\
&= (aa_1 + ba_2 + ab_1 + bb_2, aa_1 + ba_2 - ab_1 - bb_2, 0) \\
&\cancel{= (aa_1 + ba_2} \\
&= (aa_1 + ab_1, aa_1 - ab_1, 0) + (ba_2 + bb_2, ba_2 - bb_2, 0) \\
&= a(a_1 + b_1, a_1 - b_1, 0) + b(a_2 + b_2, a_2 - b_2, 0) \\
&= aT(\alpha) + bT(\beta)
\end{aligned}
$$

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Q. Show that translation mapping $f : V_2(R) \to V_2(R)$ defined by $f(x, y) = (x+2, y+3)$ is not linear.

$$f(\alpha+\beta) = f(\alpha) + f(\beta), \quad \forall \alpha, \beta \in V_2(R).$$

Let $\alpha, \beta \in V_2(R)$

$$\alpha = (x_1, x_2)$$
$$f(\alpha) = (x_1+2, x_2+3)$$
$$\beta = (y_1, y_2)$$
$$f(\beta) = (y_1+2, y_2+3)$$

Now, $f(\alpha+\beta) = f((x_1, x_2) + (y_1, y_2))$
$$= f[(x_1+y_1, x_2+y_2)]$$
$$= f(x_1+y_1, x_2+y_2)$$
$$= (x_1+y_1+2, x_2+y_2+3)$$

and

$f(\alpha) + f(\beta) = f(x_1, x_2) + f(y_1, y_2)$
$$= (x_1+2, x_2+3) + (y_1+2, y_2+3)$$
$$= [x_1+y_1+4, x_2+y_2+6]$$
$$\neq f(\alpha+\beta)$$

Hence transformation $f : V_2(R) \to V_2(R)$ is not linear.

Q: $T : R^3 \to R^2$ defined as, $T(x, y, z) = [|x|, y-z]$. Show that T is not linear.

$$T(\alpha+\beta) = T(\alpha) + T(\beta)$$

Let $\alpha = (x_1, y_1, z_1)$, $\beta = (x_2, y_2, z_2)$

$T(\alpha) = [|x_1|, y_1 - z_1]$, $T(\beta) = [|x_2|, y_2 - z_2]$

$T(\alpha+\beta) = T[(x_1, y_1, z_1) + (x_2, y_2, z_2)]$
$$= T(x_1+x_2, y_1+y_2, z_1+z_2)$$
$$= [|x_1+x_2|, y_1+y_2-z_1-z_2] \quad \text{—— (i)}$$

and $T(\alpha)+T(\beta) = [|x_1|, y_1-z_1] + [|x_2|, y_2-z_2]$

$= [|x_1|+|x_2|, y_1+y_2-z_1-z_2]$ — (ii)

$\because |x_1|+|x_2| \geq |x_1+x_2|$

Therefore from (i) and (ii), we prove that $T: R^3 \to R^2$ is not linear transformation

Q. Define $T: V_3 \to V_2$ by rule $T(x_1, x_2, x_3) = [x_1-x_2, x_1+x_3]$ Show that T is linear.

To show T is linear,

$$T(a\alpha+b\beta) = aT(\alpha)+bT(\beta)$$

Let $\alpha, \beta \in V_3$, such that,

$$\alpha = (x_1, x_2, x_3)$$
$$\beta = (y_1, y_2, y_3)$$
$$T(\alpha) = [x_1-x_2, x_1+x_3]$$
$$T(\beta) = [y_1-y_2, y_1+y_3]$$

Let $a, b \in R$

Now, $T(a\alpha+b\beta) = T[a(x_1, x_2, x_3)+b(y_1, y_2, y_3)]$

$= T[(ax_1+by_1), (ax_2+by_2), (ax_3+by_3)]$

$= [(ax_1+by_1-ax_2-by_2), (ax_1+by_1+ax_3+by_3)]$

$= [a(x_1-x_2, x_1+x_3) + b(y_1-y_2, y_1+y_3)]$

$= aT(\alpha) + bT(\beta)$

Hence $T: V_3 \to V_2$ is a linear transformation.

Q. Show that $T: V_3(R) \to V_2(R)$ defined as $T(a_1, a_2, a_3) = [3a_1-2a_2+a_3, a_1-3a_2-2a_3]$ is linear transformation from $V_3(R)$ into $V_2(R)$.

To show T is linear,

$$T(a\alpha+b\beta) = aT(\alpha)+bT(\beta)$$

Let $\alpha, \beta \in V_3$ such that,

$\alpha = (a_1, a_2, a_3)$; $T(\alpha) = [3a_1-2a_2+a_3, a_1-3a_2-2a_3]$

$\beta = (b_1, b_2, b_3)$; $T(\beta) = [3b_1-2b_2+b_3, b_1-3b_2-2b_3]$

Let $a, b \in R$,

$$\text{Now, } T(a\alpha+b\beta) = T\left[a(a_1, a_2, a_3) + b(b_1, b_2, b_3)\right]$$
$$= T\left[(aa_1, aa_2, aa_3) + (bb_1, bb_2, bb_3)\right]$$
$$= T\left[(aa_1+bb_1, aa_2+bb_2, aa_3+bb_3)\right]$$

$$= \left[3(aa_1+bb_1) - 2(aa_2+bb_2) + (aa_3+bb_3), (aa_1+bb_1) - 3(aa_2+bb_2) - 2(aa_3+bb_3)\right]$$

$$= \left[3aa_1 + 3bb_1 - 2aa_2 - 2bb_2 + aa_3 + bb_3, aa_1 - bb_1 - 3aa_2 - 3bb_2 - 2aa_3 - 2bb_3\right.$$
$$= \left[a(3a_1 - 2a_2 + a_3) + b(3b_1 - 2b_2 + b_3), a(a_1 - 3a_2 - 2a_3) + b(b_1 - 3b_2 - 2\right.$$
$$= a\left[(3a_1 - 2a_2 + a_3), (a_1 - 3a_2 - 2a_3)\right] + b\left[(3b_1 - 2b_2 + b_3), (b_1 - 3b_2 - 2b\right.$$
$$= aT(\alpha) + bT(\beta)$$

Hence T is linear.

Que: Let $V = C[0\ 1]$ be the vector space of all continuous real valued function from $[0, 1]$ into R i.e. ~~T: [0, 1]~~
Then mapping $\phi: V \to R$ defined as $\phi[f(x)] = \int_0^1 f(x)\,dx$ is a linear transformation from V into R.

To show ϕ is linear we prove, $\forall\, \alpha, \beta \in V$ and $a, b \in R$.

(i) $\phi(\alpha+\beta) = \phi(\alpha) + \phi(\beta)$

(ii) $\phi(a\alpha) = a\phi(\alpha)$

Let $\alpha, \beta \in V$, then, $\alpha = f(x)$, $\beta = g(x)$

$$\phi[f(x)] = \int_0^1 f(x)\,dx$$
$$\phi[g(x)] = \int_0^1 g(x)\,dx$$

$$\phi(\alpha) + \phi(\beta) = \int_0^1 f(x)\,dx + \int_0^1 g(x)\,dx \quad \text{--- (i) [By def.]}$$

$$\phi(\alpha+\beta) = \int_0^1 [f(x) + g(x)]\,dx \quad \text{[By def]}$$

$$= \int_0^1 f(x)\,dx + \int_0^1 g(x)\,dx \quad \text{--- (ii)}$$

From (i) and (ii) $\phi(\alpha+\beta) = \phi(\alpha) + \phi(\beta) \quad \forall\, \alpha, \beta \in V.$

$$\phi(a\alpha) = \int_0^1 a f(x)\,dx$$

$$= a\int_0^1 f(x)\,dx = a\phi(f(x)) = a\phi(\alpha).$$

Hence $\phi(a\alpha) = a\phi(\alpha)$, $\forall\ \alpha \in V$ and $a \in R$.

Therefore $\phi : V \to R$ is a linear transformation.

Q. Show that the mapping $f : V_2(R) \to V_2(R)$ defined by $f(x,y) = [x+2y, 8x-y]$ is linear transformation.

To show $f : V_2(R) \to V_2(R)$ is a linear transformation, we have to prove

$$f(a\alpha + b\beta) = af(\alpha) + bf(\beta)$$

Let $\alpha, \beta \in V_2(R)$, such that,

$$\alpha = (x_1, y_1)$$
$$\beta = (x_2, y_2)$$
$$f(\alpha) = (x_1 + 2y_1, 8x_1 - y_1)$$
$$f(\beta) = (x_2 + 2y_2, 8x_2 - y_2)$$

Let $a, b \in R$,

Now
$$\begin{aligned}
f(a\alpha + b\beta) &= f[a(x_1, y_1) + b(x_2, y_2)] \\
&= f[(ax_1 + bx_2, ay_1 + by_2)] \\
&= [ax_1 + bx_2 + 2(ay_1 + by_2), 8(ax_1 + bx_2) - (ay_1 + by_2)] \\
&= [a(x_1 + 2y_1) + b(x_2 + 2y_2), a(8x_1 - y_1) + b(8x_2 - y_2)] \\
&= a[x_1 + 2y_1, 8x_1 - y_1] + b[x_2 + 2y_2, 8x_2 - y_2] \\
&= a f(\alpha) + b f(\beta)
\end{aligned}$$

Hence, f is a linear transformation.

Rank and Nullity of Linear transformation
Let $T: U(F) \to V(F)$ be a linear transformation

① The dimension of ~~a subspace~~ image [Im] (Range) of T is known as rank of T.

② The dimension of Null space of T is known as Nullity of T

③ Null space $= \{\alpha \in U : T(\alpha) = 0\}$ [Null space or kernal of T]

④ Range of $T = \{T(\alpha) : \alpha \in U\}$

Rank-Nullity theorem:-

$$\dim(U) = \text{Rank}(T) + \text{Nullity}(T).$$

Ques. Show that mapping $T: V_2(R) \to V_3(R)$ defined as $T(a,b) = [a+b, a-b, b]$ is linear transformation, find range, Null space and nullity of T.

To show linear transformation,

$$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta) \quad , \forall \alpha, \beta \in V_2(R) \text{ \& } a, b \in R.$$

Let $\alpha, \beta \in V_2(R)$ such that

$\alpha = (a_1, b_1) \quad ; T(\alpha) = (a_1+b_1, a_1-b_1, b_1)$

$\beta = (a_2, b_2) \quad ; T(\beta) = (a_2+b_2, a_2-b_2, b_2)$

let $a, b \in R$,

Now, $T(a\alpha + b\beta) = T[a(a_1, b_1) + b(a_2, b_2)]$

$= T[(aa_1, ab_1) + (ba_2, bb_2)]$

$= T(aa_1 + ba_2, ab_1 + bb_2)$

$= [aa_1 + ba_2 + ab_1 + bb_2, aa_1 + ba_2 - ab_1 - bb_2, ab_1 + bb_2]$

$= [a(a_1+b_1) + b(a_2+b_2), a(a_1-b_1) + b(a_2-b_2), ab_1 + bb_2]$

$= a(a_1+b_1, a_1-b_1, b_1) + b(a_2+b_2, a_2-b_2, b_2)$

$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta) \quad , \forall \alpha, \beta \in V_2(R) \text{ \& } a, b \in R.$

Hence T is a linear transformation.

let [1,0] and [0,1] is basis of $V_2(R)$ [i.e. Domain] $\dim(V) = 2$

To find image of basis vector,

$$f[1,0] = [1+0, 1-0, 0] = [1,1,0]$$
$$f[0,1] = [0+1, 0-1, 1] = [1,-1,1]$$

$(1,1,0)$ and $(1,-1,1)$ generates $V_3(R)$.

Now we show they are L.I. vectors.

let K_1 and K_2 are two scalars such that $K_1(1,1,0)+K_2(1,-1,1)=(0,0,0)$

$$K_1+K_2=0$$
$$K_1-K_2=0$$
$$K_2=0$$
$$\therefore K_1=0$$

$\Rightarrow K_1 = K_2 = 0$. i.e. Rank of $T = 2$.

$\Rightarrow (1,1,0)$ and $(1,-1,1)$ are linearly independent vectors.

By rank nullity theorem,

$$\dim(V) = \text{Rank}(T) + \text{Nullity}(T)$$
$$\text{Nullity}(T) = 2-2 = 0$$

To find null space,

$$T(a,b) = 0$$
$$[a+b, a-b, b] = [0,0,0]$$

On Equating,

$$a+b=0, \quad a-b=0, \quad b=0$$

$\Rightarrow a=b=0$.

i.e. Null space $= (a,b) = (0,0)$.

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Q. Let F be a field of complex numbers and T be a function from F^3 into F^3 defined by -

$T(x_1, x_2, x_3) = [x_1 - x_2 + 2x_3, 2x_1 + x_2 - x_3, -x_1 - 2x_2]$.

Verify T is linear transformation. Find NULL space of T.

To show L.T. we prove -

(i) $T(\alpha+\beta) = T(\alpha) + T(\beta)$, $\forall\ \alpha, \beta \in F^3$

(ii) $T(a\alpha) = aT(\alpha)$, $\forall\ a \in F$ and $\alpha, \beta \in F^3$.

Let $\alpha = (x_1, x_2, x_3)$

$T(\alpha) = [x_1 - x_2 + 2x_3, 2x_1 + x_2 - x_3, -x_1 - 2x_2]$

$\beta = (y_1, y_2, y_3)$

$T(\beta) = [y_1 - y_2 + 2y_3, 2y_1 + y_2 - y_3, -y_1 - 2y_2]$

$\alpha + \beta = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$

$T(\alpha+\beta) = [x_1 + y_1 - (x_2 + y_2) + 2(x_3 + y_3), 2(x_1 + y_1) + (x_2 + y_2) - (x_3 + y_3), -(x_1 + y_1) - 2(x_2 + y_2)]$

$= [x_1 - x_2 + 2x_3 + y_1 - y_2 + 2y_3, 2x_1 + x_2 - x_3 + 2y_1 + y_2 - y_3, -x_1 - 2x_2 - y_1 - 2y_2]$

$= [x_1 - x_2 + 2x_3, 2x_1 + x_2 - x_3, -x_1 - 2x_2] + [y_1 - y_2 + 2y_3, 2y_1 + y_2 - y_3, -y_1 - 2y_2]$

$= T(\alpha) + T(\beta)$ —— ①

$T(a\alpha) = T[a(x_1, x_2, x_3)]$

$= T(ax_1, ax_2, ax_3)$

$= [ax_1 - ax_2 + 2ax_3, 2ax_1 + ax_2 - ax_3, -ax_1 - 2ax_2]$

$= a[x_1 - x_2 + 2x_3, 2x_1 + x_2 - x_3, -x_1 - 2x_2]$

$= aT(\alpha)$ —— ②

$\therefore$ From ① and ②,

T is linear transformation from F^3 into F^3.

To find NULL space or Kernal :-

$T(\alpha) = 0$ [def. of null space]

$T(x_1, x_2, x_3) = 0$

$(x_1 - x_2 + 2x_3, 2x_1 + x_2 - x_3, -x_1 - 2x_2) = (0, 0, 0)$

$$\left.\begin{aligned} x_1 - x_2 + 2x_3 &= 0 \\ 2x_1 + x_2 - x_3 &= 0 \\ -x_1 - 2x_2 \quad &= 0 \end{aligned}\right\}$$ Homogeneous system of eq^n.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ -1 & -2 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + R_1$

$$\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -5 \\ 0 & -3 & 2 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -5 \\ 0 & 0 & -3 \end{bmatrix}$$

$\rho(A)$ = no. of non-zero rows = 3 = no. of unknowns

$\Rightarrow$ soln is $x_1 = x_2 = x_3 = 0$.

$\alpha = (0, 0, 0)$

Null space of T, $N(T) = (0, 0, 0)$.

Q ~~Show that $T: V_2(R) \to V_3(R)$ is linear transformation~~ when
~~$T(a, b) = [a+b, a-b, b]$. Find range~~

Q Let $T: R^3 \to R^3$ be L.T. defined by $T(x, y, z) = [x+2y-z, y+z, x+y-2z]$
Find basis and dim. of (i) Range of T (ii) Null space of T

Let $B = [(1,0,0), (0,1,0), (0,0,1)]$ is basis of R^3(domain)

$\therefore$ Dimension of domain $R^3 = 3$.

$T(1, 0, 0) = [1, 0, 1]$

$T(0, 1, 0) = [2, 1, 1]$

$T(0, 0, 1) = [-1, 1, -2]$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2$.

Here $\rho(A) = 2$, so 2 vectors are independent ~~to~~

$\Rightarrow$ Only 2 vectors are in basis of range space.

i.e. $B' = \{(1, 0, 1), (2, 1, 1)\} \rightarrow$ Basis of range space.

dim. of range space of T = rank of T

$$\rho(T) = 2.$$

Rank + Nullity = dim. of domain

Nullity $= 3 - 2 = 1$.

$(1, 0, 1)$ and $(2, 1, 1)$ span R^3

Note:

Let W be a subspace of R^3 and $\dim R^3 = 3$, then $\dim W \leq 3$.

i.e. $\dim(W) = 0, 1, 2$ or 3.

- if $\dim W = 0 \Rightarrow W = \{0\}$ is a point.
- if $\dim W = 1 \Rightarrow W$ is a line through origin
- if $\dim W = 2 \Rightarrow W$ is a plane through origin
- if $\dim W = 3 \Rightarrow W$ is entire plane R^3 i.e. $W = R^3$.

Matrix representation of linear transformation

The matrix representation of L.T. is a way of describing a L.T. using a matrix which acts on a vector space.

Let $T: V(F) \rightarrow V(F)$ be L.T. we want to express T as matrix

steps to find matrix representation of L.T. -

1. Choose basis for the domain and co-domain.
2. Find basis vectors for co-domain with the help of T. Let $e_1, e_2, \ldots e_n$ be the basis of domain.

$\therefore T(e_1), T(e_2) \ldots\ldots T(e_n) \rightarrow$ Basis of co-domain.

Find coordinates of $T(e_1), T(e_2) \ldots\ldots T(e_n)$ and form a matrix by arranging these coordinate as column wise.

$$[T]_B = \begin{bmatrix} K_1 & K_4 & \ldots \\ K_2 & K_5 & \ldots \\ K_3 & K_6 & \ldots \end{bmatrix}$$

$T(e_1) = K_1e_1 + K_2e_2 + K_3e_3 + \ldots K_ne_n$

Note,

$T : R^3 \rightarrow R^3$

1. Find basis of R^3 say e_1, e_2, e_3 [Basis of domain space]

2. Find $T(e_1), T(e_2), T(e_3)$

3. $T(e_1) = K_1e_1 + K_2e_2 + K_3e_3$
$T(e_2) = K_4e_1 + K_5e_2 + K_6e_3$
$T(e_3) = K_7e_1 + K_8e_2 + K_9e_3$

Matrix representation of T w.r.t basis B -

$$[T]_B = \begin{bmatrix} K_1 & K_4 & K_7 \\ K_2 & K_5 & K_8 \\ K_3 & K_6 & K_9 \end{bmatrix}$$

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Matrix Representation of Transformation:-

Que. Let T is a linear transformation on R^2 defined by

$$T(x, y) = [4x-2y, 2x+y]$$

Write the matrix of T in the standard ordered basis B for R^2.

$B = [\{1, 0\}, \{0, 1\}]$ is basis for R^2 (Domain)

Now,

$T(1, 0) = [4\times1 - 2\times0, 2\times1+0] = [4, 2]$

$T(0, 1) = [-2, 1]$

$$A = [e_1 \; e_2 \; T(e_1) \; T(e_2)]$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & 4 & -2 \\ 0 & 1 & 2 & 1 \end{array}\right]$$

Matrix representation of T is $[T]_B = \begin{bmatrix} 4 & -2 \\ 2 & 1 \end{bmatrix}$

Ques. Let T be a linear operator on R^3 defined by

$$T[x_1, x_2, x_3] = [3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3]$$

What is the matrix T w.r.t. standard ordered basis B for R^3.

$B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is basis for R^3 (domain).

Now,

$T(1, 0, 0) = [3, -2, -1]$

$T(0, 1, 0) = [0, 1, 2]$

$T(0, 0, 1) = [1, 0, 4]$

$$\therefore A = [e_1 \; e_2 \; e_3 \,\vdots\, T(e_1) \; T(e_2) \; T(e_3)]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 0 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & 4 \end{array}\right]$$

Matrix representation of T is $[T]_B = \begin{bmatrix} 3 & 0 & 1 \\ -2 & 1 & 0 \\ -1 & 2 & 4 \end{bmatrix}$

Q. Find matrix representation of T defined ~~by~~ on R^2 s.t. $T(x,y) = [2y, 3x-y]$. Find matrix representation of T w.r.t. B
$B = \{(1,3), (2,5)\}$

Here, $e_1 = (1,3)$, $e_2 = (2,5)$.
$T(e_1) = [6, 0]$
$T(e_2) = [10, 1]$
Now, $A = [e_1 \; e_2 \; \vdots \; T(e_1) \; T(e_2)]$

$$= \left[\begin{array}{cc|cc} 1 & 2 & 6 & 10 \\ 3 & 5 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 2 & 6 & 10 \\ 0 & -1 & -18 & -29 \end{array}\right] \quad R_2 \to R_2 - 3R_1$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & -30 & -48 \\ 0 & -1 & -18 & -29 \end{array}\right] \quad R_1 \to R_1 + 2R_2$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & -30 & -48 \\ 0 & 1 & 18 & 29 \end{array}\right] \quad R_2 \to R_2/(-1)$$

$$[T]_B = \begin{bmatrix} -30 & -48 \\ 18 & 29 \end{bmatrix}$$

Q. Let T be a linear transformation on R^2 defined by $T(x,y) = [4x-2y, 2x+y]$. Compute matrix of T relative to the basis $\{\alpha_1, \alpha_2\}$ where $\alpha_1 = (1,1)$, $\alpha_2 = (-1,0)$.

Here, $B = [(1,1), (-1,0)]$
$T(\alpha_1) = [2, 3]$
$T(\alpha_2) = [-4, -2]$
Now, $A = [\alpha_1 \; \alpha_2 \; \vdots \; T(\alpha_1) \; T(\alpha_2)]$

$$= \left[\begin{array}{cc|cc} 1 & -1 & 2 & -4 \\ 1 & 0 & 3 & -2 \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} 1 & -1 & 2 & -4 \\ 0 & 1 & 1 & 2 \end{array}\right] \quad R_2 \to R_2 - R_1$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & 3 & -2 \\ 0 & 1 & 1 & 2 \end{array}\right] \quad R_1 \to R_1 + R_2$$

$$[T]_B = \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix}$$

Q. Find matrix of linear transformation on $V_3(R)$ defined by $T(a, b, c) = [2b+c, a-4b, 3a]$ w.r.t. the ordered basis B and B' where

(i) $B = [(1,0,0), (0,1,0), (0,0,1)]$

(ii) $B' = [(1,1,1), (1,1,0), (1,0,0)]$

(i) $T(e_1) = [0, 1, 3]$

$T(e_2) = [2, -4, 0]$

$T(e_3) = [1, 0, 0]$

$A = [e_1 \ e_2 \ e_3 \ \vdots \ T(e_1) \ T(e_2) \ T(e_3)]$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 & -4 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 \end{array}\right]$$

$$[T]_B = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -4 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

(ii) $T(e_1) = [3, -3, 3]$

$T(e_2) = [2, -3, 3]$

$T(e_3) = [0, 1, 3]$

$A = [e_1 \ e_2 \ e_3 \ \vdots \ T(e_1) \ T(e_2) \ T(e_3)]$

$$A = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 3 & 2 & 0 \\ 1 & 1 & 0 & -3 & -3 & 1 \\ 1 & 0 & 0 & 3 & 3 & 3 \end{array}\right]$$

$$A = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 3 & 2 & 0 \\ 0 & 1 & 0 & -6 & -6 & -2 \\ 1 & 0 & 0 & 3 & 3 & 3 \end{array}\right] \quad R_2 \to R_2 - R_3$$

$$A = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 3 & 2 & 0 \\ 0 & 1 & 0 & -6 & -6 & -2 \\ 0 & -1 & -1 & 0 & 1 & 3 \end{array}\right] \quad R_3 \to R_3 - R_1$$

$$A = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 3 & 3 \\ 0 & 1 & 0 & -6 & -6 & -2 \\ 0 & -1 & -1 & 0 & 1 & 3 \end{array}\right] \quad R_1 \to R_1 + R_3$$

$$A = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 3 & 3 \\ 0 & 1 & 0 & -6 & -6 & -2 \\ 0 & 0 & -1 & -6 & -5 & 1 \end{array}\right] \quad R_3 \to R_3 + R_2$$

$$A = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 3 & 3 \\ 0 & 1 & 0 & -6 & -6 & -2 \\ 0 & 0 & 1 & 6 & 5 & -1 \end{array}\right] \quad R_3 \to R_3/(-1)$$

$$[T]_B = \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}$$

Ques Find the matrix representation of T defined on $R^3(R)$ as $T(x, y, z) = (2y + z, x - 4y, 3x)$

(i) $B = [(1,0,0), (0,1,0), (0,0,1)]$

(ii) $B' = [(1,1,1), (1,1,0), (1,0,0)]$

(i) $T(1,0,0) = (0, 1, 3) = 0.(1,0,0) + 1(0,1,0) + 3(0,0,1)$

$T(0,1,0) = (2, -4, 0) = 2(1,0,0) - 4(0,1,0) + 0(0,0,1)$

$T(0,0,1) = (1, 0, 0) = 1(1,0,0) + 0(0,1,0) + 0(0,0,1)$

$$\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$[T]_B = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -4 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

(ii) $T(1,1,1) = (3,-3,3)$
$T(1,1,0) = (2,-3,3)$
$T(1,0,0) = (0,1,3)$
Now,
Let $(a,b,c) = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3$
$(a,b,c) = k_1(1,1,1) + k_2(1,1,0) + k_3(1,0,0)$
$a = k_1 + k_2 + k_3$
$b = k_1 + k_2$
$c = k_1$
$\Rightarrow$ $k_1 = c$
$k_2 = b - c$
$k_3 = a - b$
Put $(a,b,c) = (3,-3,3)$
$\Rightarrow$ $k_1 = 3$, $k_2 = -6$, $k_3 = 6$
Put $(a,b,c) = (2,-3,3)$
$\Rightarrow$ $k_1 = 3$, $k_2 = -6$, $k_3 = 5$
Put $(a,b,c) = (0,1,3)$
$\Rightarrow$ $k_1 = 3$, $k_2 = -2$, $k_3 = -1$.

$$[T]_{B'} = \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}$$

<u>Note</u>

$TX = AX$

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -4 & 0 \\ 3 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2y+z \\ x-4y \\ 3x \end{bmatrix}$$

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Q. Consider the vector space $V(R)$ of all 2×2 matrices over field R of real numbers. let T be the linear transformation on V such that each matrix X onto AX where $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Find the matrix T w.r.t. ordered basis $B = [\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4]$ for V where

$$\alpha_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ \alpha_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ \alpha_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ \alpha_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T(\alpha_1) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = 1\cdot\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T(\alpha_2) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = 0\cdot\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\cdot\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T(\alpha_3) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = 1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T(\alpha_4) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = 0\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[T]_B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Ques If a matrix of a linear transformation T on $V_2(C)$ with respect to the ordered basis $B = [(1,0),(0,1)]$ is $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. What is the matrix of T w.r.t. $B' = [(1,1),(1,-1)]$.

Similarity of matrices.

Let A and B be two square matrices of order n over the same field. F. Then B is said to be similar to A if there exists a non-singular matrix or invertible matrix C with elements in F such that

$$B = C^{-1}AC$$

$$\boxed{CB = AC}$$

* similar matrices have same ~~determinate~~ and have same trace.

Que Examine whether A is similar to B or not where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Let $C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for similarity

$$AC = \cancel{BC}\ CB$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix}$$

On equating.

$$b = a+b$$

$$\Rightarrow \quad a = 0$$

and $\quad d = c+d$

$$\Rightarrow \quad c = 0$$

$$\therefore C = \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}$$

$|C| = 0$ (singular matrix)

Hence A is not similar to B.

Q: $A = \begin{bmatrix} 5 & 5 \\ -2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$.

Let $C = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

For similarity,

$$AC = CB$$

$$\begin{bmatrix} 5 & 5 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 5a+5b & 5c+5d \\ -2a & -2c \end{bmatrix} = \begin{bmatrix} a-3c & 2a+4c \\ b-3d & 2b+4d \end{bmatrix}$$

On equating,

$$5a+5b = a-3c$$

$\Rightarrow$ $4a+5b+3c = 0$ —— (1)

and $5c+5d = 2a+4c$.

$\Rightarrow$ $c+5d-2a = 0$ —— (2)

and $-2a = b-3d$

$\Rightarrow$ $-2a-b+3d = 0$ —— (3)

and $-2c = 2b+4d$

$\Rightarrow$ $-2b-2c-4d = 0$ —— (4)

Let $K = \begin{bmatrix} 4 & 5 & 3 & 0 \\ -2 & 0 & 1 & 5 \\ -2 & -1 & 0 & 3 \\ 0 & -2 & -2 & -4 \end{bmatrix}$

$R_1 \to R_1 + 2R_2$

$K = \begin{bmatrix} 0 & 5 & 5 & 10 \\ -2 & 0 & 1 & 5 \\ -2 & -1 & 0 & 3 \\ 0 & -2 & -2 & -4 \end{bmatrix}$

$$\sim \begin{bmatrix} 0 & 5 & 5 & 10 \\ -2 & 0 & 1 & 5 \\ 0 & -1 & -1 & -2 \\ 0 & -2 & -2 & -4 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$, $R_4 \to R_4 - 2R_3$

$$\sim \begin{bmatrix} -2 & 0 & 1 & 5 \\ 0 & 5 & 5 & 10 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\sim \begin{bmatrix} -2 & 0 & 1 & 5 \\ 0 & -1 & -1 & -2 \\ 0 & 5 & 5 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3 + 5R_2$

$$\sim \begin{bmatrix} -2 & 0 & 1 & 5 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

<u>Soln.</u>

$$-2a + c + 5d = 0$$

$$-b - c - 2d = 0 \Rightarrow b + c + 2d = 0$$

Let $c = K_1$, $d = K_2$

$$-2a + K_1 + 5K_2 = 0$$

$$a = \frac{K_1 + 5K_2}{2}$$

and $b + K_1 + 2K_2 = 0$

$$b = -(K_1 + 2K_2)$$

if $K_1 = 1$, $K_2 = 0$

$a = 1/2$, $c = 1$

$b = -1$, $d = 0$

Now, $C = \begin{bmatrix} 1/2 & 1 \\ -1 & 0 \end{bmatrix}$

$|C| = 1 \neq 0$

Hence A and B are similar.

Similarity of linear transformation

Let A and B be two linear transformations on a vector space V(F). Then B is said to be similar to A if there exists an invertible linear transformation P on V such that

$$B = PAP^{-1}$$

$$\boxed{BP = PA}$$

Equivalent Matrices.

Two matrices A and B of same order n are said to be equivalent if there exists invertible (non-singular) matrices P and Q such that

$$B = PAQ$$

$$B_{m\times n} = \underset{\text{row}}{P_{m\times m}} A_{m\times n} \underset{\text{column operations.}}{Q_{n\times n}}$$

<u>Note</u>: We reduce B into normal form (min order identity matrix) by applying Row and column operation.

A and B have same rank, dimension i.e they represent the same fundamental linear transformation in different bases

A and B do not necessarily have the same determinate, trace or eigen values.

We have $A_{m\times n} = I_{m\times n} A I_{n\times n}$

$\Rightarrow B = Q A P$

$\Rightarrow A = Q^{-1} B P^{-1}$

Q. Whether matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ are equivalent or not?

To find rank of A & B.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$\rho(A) = 1 = \rho(B)$$

Hence A and B are equivalent matrices.

Q. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$R_2 \to R_2 - 4R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{bmatrix}$$

$$\rho(A) = 2 = \rho(B)$$

Hence both matrices are equivalent.

Que. Define $T: R^3 \to R^3$ by $T(x, y, z) = [2x+y, y-z, 2y+4z]$. Verify whether T is linear transformation.

To prove: $T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$

Let $\alpha = (x_1, y_1, z_1)$, $\beta = (x_2, y_2, z_2)$

$a\alpha = (ax_1, ay_1, az_1)$, $b\beta = (bx_2, by_2, bz_2)$

$T(\alpha) = [2x_1+y_1, y_1-z_1, 2y_1+4z_1]$, $T(\beta) = [2x_2+y_2, y_2-z_2, 2y_2+4z_2]$

$$T(a\alpha+b\beta) = T[(ax_1, ay_1, az_1) + (bx_2, by_2, bz_2)]$$
$$= T[(ax_1+bx_2, ay_1+by_2, az_1+bz_2)]$$

$$= [2(ax_1+bx_2) + (ay_1+by_2), (ay_1+by_2) - (az_1+bz_2), 2(ay_1+by_2) + 4(az_1+bz_2)]$$

$$= [(2ax_1+ay_1) + (2bx_2+by_2), (ay_1-az_1) + (by_2-bz_2), (2ay_1+4az_1) + (2by_2+4bz_2)]$$

$$= (2ax_1+ay_1, ay_1-az_1, 2ay_1+4az_1) + (by_2+2bx_2, by_2-bz_2, 2by_2+4bz_2)$$

$$= a(2x_1+y_1, y_1-z_1, 2y_1+4z_1) + b(y_2+2x_2, y_2-z_2, 2y_2+4z_2)$$
$$T(a\alpha+b\beta) = a\,T(\alpha) + b\,T(\beta), \ \forall \alpha, \beta \in R^3 \text{ and } a, b \in R.$$

Hence T is a linear transformation.

3 $T: V_2(R) \to V_2(R)$ define by $T(x, y) = [3x, x-y]$

Let $\alpha_1 = (x_1, y_1)$ and $\beta = (x_2, y_2)$

$T(\alpha) = [3x_1, x_1-y_1]$ $\quad T(\beta) = [3x_2, x_2-y_2]$

let $a, b \in R$,

$$T(a\alpha+b\beta) = T[(ax_1, ay_1) + (bx_2, by_2)]$$
$$= T(ax_1+bx_2, ay_1+by_2)$$
$$= [3(ax_1+bx_2), (ax_1+bx_2-ay_1-by_2)]$$
$$= (3ax_1, ax_1-ay_1) + (3bx_2, bx_2-by_2)$$
$$= a(3x_1, x_1-y_1) + b(3x_2, x_2-y_2)$$
$$T(a\alpha+b\beta) = a\,T(\alpha) + b\,T(\beta), \ \forall a, b \in R \text{ and } \alpha, \beta \in V_2(R).$$

Hence T is a linear transformation.

Q. $T: C \to R$ define by $T(z_1) = (x_1^3 + y_1^3)$, $z = x + iy \in C$.

Let $\alpha = z_1 = (x_1 + iy_1)$ $\quad \beta = z_2 = x_2 + iy_2$.

$T(z_1) = (x_1^3 + y_1^3)$ $\quad T(z_2) = x_2^3 + y_2^3$

let $a, b \in R$.

$$\begin{aligned} T(az_1 + bz_2) &= T[a(x_1 + iy_1) + b(x_2 + iy_2)] \\ &= T[(ax_1 + aiy_1) + (bx_2, biy_2)] \\ &= T[(ax_1 + bx_2, aiy_1 + biy_2)] \\ &= (ax_1 + bx_2)^3 + (ay_1 + by_2)^3 \\ &\neq aT(z_1) + bT(z_2) \end{aligned}$$

Hence T is not a linear transformation.

Q. Find matrix of $T: V_3(R) \to V_2(R)$ defined by $T(x, y, z) = [x+y, y+z]$ w.r.t. standard basis.

$B = [(1,0,0), (0,1,0), (0,0,1)]$ is the standard basis for $V_3(R)$

$$\begin{aligned} T(e_1) &= [1, 0] = K_1 e_1 + K_2 e_2 + K_3 e_3 = 1(1,0,0) + 0(0,1,0) + 0(0,0,1) \\ T(e_2) &= [1, 1] = K_4 e_1 + K_5 e_2 + K_6 e_3 = 1(1,0,0) + 1(0,1,0) + 0(0,0,1) \\ T(e_3) &= [0, 1] = K_7 e_1 + K_8 e_2 + K_9 e_3 = 0(1,0,0) + 1(0,1,0) + 0(0,0,1) \end{aligned}$$

$$\begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} K_4 \\ K_5 \\ K_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} K_7 \\ K_8 \\ K_9 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$[T]_B \begin{bmatrix} K_1 & K_4 & K_7 \\ K_2 & K_5 & K_8 \\ K_3 & K_6 & K_9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[T]_B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{2\times 3}$$

To verify $TX = AX$.

5. Find matrix of $T: V_2(R) \to V_2(R)$ defined by $T(x,y) = (x,-y)$ w.r.t. e_1 and e_2 (standard basis).

Let $B = [\{1,0\},\{0,1\}]$ be the basis for $V_2(R)$ [domain].

$T(e_1) = [1,0]$
$T(e_2) = [0,-1]$

$$A = [e_1 \ e_2 \ \vdots \ T(e_1) \ T(e_2)]$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{array}\right]$$

$$[T]_B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

6. $T: V_3(R) \to V_3(R)$ defined by $T(u_1,u_2,u_3) = [u_1+u_2, u_2+u_3, u_3+u_1]$ with basis ~~of~~ $B' = \{(1,0,1),(0,1,0),(1,0,-1)\}$

$T(e_1) = [1,1,2]$
$T(e_2) = [1,1,0]$
$T(e_3) = [1,-1,0]$

$$A = [e_1 \ e_2 \ e_3 \ \vdots \ T(e_1) \ T(e_2) \ T(e_3)]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 1 & 0 & -1 & 2 & 0 & 0 \end{array}\right]$$

$R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & -2 & 1 & -1 & -1 \end{array}\right]$$

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Find matrix of linear transformation $T: V_2(R) \to V_3(R)$ such that $T(-1,1) = (-1,0,2)$

$T(2,1) = (1,2,1)$

$(-1,1) = -1.e_1 + 1 \cdot e_2$

$(2,1) = 2e_1 + e_2$

Taking transformation,

$T(-1,1) = -T(e_1) + T(e_2)$

$\Rightarrow (-1,0,2) = -T(e_1) + T(e_2)$ ——— ①

Again, $(1,2,1) = 2T(e_1) + T(e_2)$ ——— ②

$(-2,-2,1) = -3T(e_1)$

$T(e_1) = [2/3, 2/3, -1/3]$

Also, multiplying eqn ① by $\times 2$ and adding in ②

$3T(e_2) = (-2,0,4) + (1,2,1)$

$T(e_2) = \left[\frac{-1}{3}, \frac{2}{3}, \frac{5}{3}\right]$

Hence matrix representation,

$[T]_{B'} = [T(e_1) \;\; T(e_2)]$

$$= \begin{bmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ -1/3 & 5/3 \end{bmatrix}$$

For transformation, $TX = AX$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ -1/3 & 5/3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$T(x,y) = \left[\frac{2x-y}{3}, \frac{2(x+y)}{3}, \frac{-x+5y}{3}\right]$$

Q. Let V be the set of ordered pair of real numbers with vector addition defined as $(x_1, y_1) + (x_2, y_2) = \{(x_1 + x_2 + 1), (y_1 + y_2 + 1)\}$

Show that V(R) is a vector space.

Let $\alpha = (x_1, y_1)$ and $\beta = (x_2, y_2) \in V$, $\forall x_1, x_2, y_1, y_2 \in R$.

Closure property :-

$\alpha + \beta = (x_1, y_1) + (x_2, y_2)$

$= \{(x_1 + x_2 + 1), (y_1 + y_2 + 1)\} \in V$

$\therefore \alpha + \beta \in V$, $\forall \alpha, \beta \in V$ and $x_1, x_2, y_1, y_2 \in R$.

Associative property :-

Let $\alpha, \beta, \gamma \in V$ such that.

$\alpha = (x_1, y_1)$, $\beta = (x_2, y_2)$, $\gamma = (x_3, y_3)$

To prove : $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$

$$
\begin{aligned}
L.H.S. &= \alpha + (\beta + \gamma) \\
&= (x_1, y_1) + [(x_2, y_2) + (x_3, y_3)] \\
&= (x_1, y_1) + [(x_2 + x_3 + 1, y_2 + y_3 + 1)] \\
&= (x_1 + 1 + x_2 + x_3 + 1, y_1 + y_2 + y_3 + 1 + 1) \\
&= [(x_1 + x_2 + 1) + x_3 + 1, (y_1 + y_2 + 1) + y_3 + 1] \\
&= [x_1 + x_2 + 1, y_1 + y_2 + 1] + [x_3 + 1, y_3 + 1] \\
&= (\alpha + \beta) + \gamma \\
&= R.H.S.
\end{aligned}
$$