L-1

{16 no - 5th
16 no - 3rd
8 no - 2nd}

Date 21/Nov/2024

UNIT-5

VECTOR CALCULUS

{1. Gradient
2. Divergence
3. ~~Cord~~ Curl
4. Differential directional derivative}

Scalar & Vector functions

Scalar function $f(x,y,z)$ is a function defined at each point in a certain domain D in Space.

NOTE: Its value is real & depends only on at point ★ $P(x,y,z)$ in Space but not on any particular coordinate system being used.

for ex: ★ If to each value of Scalar variable "t" there corresponds a value of vector "$\vec{r}$", then $\vec{r}$ is called vector function of Scalar variable "t".

for ex: Position vector $\vec{r}$ of a particle moving along a curved path is vector function of time "t". where t is scalar.

$$\vec{r}(t) = \vec{f(t)} = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$$

where $f_1(t)$, $f_2(t)$ & $f_3(t)$ are components of $f(t)$ along coordinate axis.

Derivative of Vector function wrt a Scalar

$$\lim_{\delta t \to 0} \frac{\vec{\delta r}}{\delta t} = \lim_{\delta t \to 0} \frac{\vec{f(t+\delta t)} - \vec{f(t)}}{\delta(t)}$$

OR

$$\vec{r(t)} = \vec{f(t)} = \hat{i} f_1(t) + f_2(t)\hat{j} + f_3(t)\hat{k}$$

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}\vec{f(t)} = \hat{i}\frac{d}{dt}f_1(t) + \hat{j}\frac{d}{dt}f_2(t) + \hat{k}\frac{d}{dt}f_3(t)$$

we can also find it's higher order derivatives

NOTE: if $\vec{a}$ & $\vec{b}$ are two vectors of scalar variable t

1. then $\frac{d}{dt}[\vec{a} \pm \vec{b}] = \frac{d}{dt}\vec{a} \pm \frac{d}{dt}\vec{b}$

2. $\frac{d}{dt}[\vec{a} \cdot \vec{b}] = \vec{a} \cdot \frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt} \cdot \vec{b}$

3. $\frac{d}{dt}[\vec{a} \times \vec{b}] = \vec{a} \times \frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt} \times \vec{b}$

4. $\frac{d}{dt}[\phi\vec{a}] = \phi\frac{d\vec{a}}{dt} + \frac{d\phi}{dt}\vec{a}$ [where ϕ is scalar function]

5. $\frac{d}{dt}[\vec{a}\,\vec{b}\,\vec{c}] = \left[\frac{d\vec{a}}{dt}\,\vec{b}\,\vec{c}\right] + \left[\vec{a}\,\frac{d\vec{b}}{dt}\,\vec{c}\right] + \left[\vec{a}\,\vec{b}\,\frac{d\vec{c}}{dt}\right]$

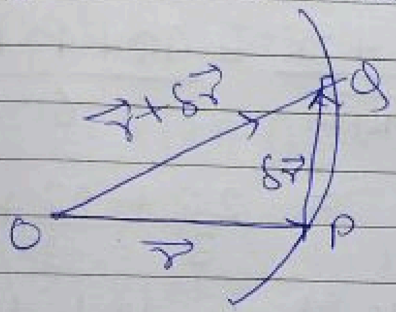
NOTE: ① If $\vec{f(t)}$ has a constant magnitude then $\vec{F} \cdot \frac{d\vec{F}}{dt} = 0$

② If $\vec{f(t)}$ has constant direction, then $\vec{F} \times \frac{d\vec{F}}{dt} = 0$

Geometrical Interpretation of $\frac{d\vec{r}}{dt}$:

$\frac{d\vec{r}}{dt}$ is a vector along the tangent to a curve at point P

→ tangent vector to the curve at P



Velocity and Acceleration

If the scalar variable "t" denote time & $\vec{r}$ is position vector of moving particle P, then $\frac{d\vec{r}}{dt}$ represents velocity vector & its direction is along tangent at P.

$\frac{d^2\vec{r}}{dt^2}$ represents acceleration at point P.

For ex:- find the unit tangent vector at any point on the curve $x = t^2+2$, $y = 4t-5$, $z = 2t^2-6t$ where t is any variable, determine unit tangent vector at the point $t=2$.

Sol: $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{r} = (t^2+2)\hat{i} + (4t-5)\hat{j} + (2t^2-6t)\hat{k}$

$\frac{d\vec{r}}{dt} = (2t)\hat{i} + (4)\hat{j} + (4t-6)\hat{k}$

Date
Page

$$\left(\frac{dr}{dt}\right)_{t=2} = 2\times 2\hat{i} + 4\hat{j} + (4\times 2-6)\hat{k}$$

(tangent vector at $t=2$) $= 4\hat{i} + 4\hat{j} + 2\hat{k}$

Now unit tangent vector, $= \frac{\vec{T}}{|T|}$

$$= \frac{4\hat{i}+4\hat{j}+2\hat{k}}{\sqrt{16+16+4}} = \frac{4\hat{i}+4\hat{j}+2\hat{k}}{6}$$

$$= \frac{2\hat{i}}{3} + \frac{2\hat{j}}{3} + \frac{\hat{k}}{3}$$ Ans

Que→ find the angle b/w the tangents to the curve $\vec{r} = t^2\hat{i} + 2t\hat{j} + t^3\hat{k}$ at the points $t = \pm 1$

Sol:

$$\frac{d\vec{r}}{dt} = 2t\hat{i} + 2\hat{j} + 3t^2\hat{k}$$

at $t=1$

$$\left(\frac{d\vec{r}}{dt}\right)_{t=1} = 2\hat{i} + 2\hat{j} + 3\hat{k}$$

$$\left(\frac{d\vec{r}}{dt}\right)_{t=-1} = (2\times -1)\hat{i} + 2\hat{j} + 3\hat{k}$$

$$= -2\hat{i} + 2\hat{j} + 3\hat{k}$$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{2\times -2 + 2\times 2 + 3\times 3}{\sqrt{4+4+9}\,\sqrt{4+4+9}}$$

8
9 17

$$= \frac{-\cancel{4}+\cancel{4}+9}{17} = \frac{9}{17}$$

$$\theta = \cos^{-1}\left(\frac{9}{17}\right)$$

(jiski direction me chahiye)

Que: A particle moves on a curve $x = 2t^2$ where t is time $y = t^2 - 4t$

find components of velocity & $z = 3t - 5$

acceleration at $t = 1$ in the direction $(\hat{i} - 3\hat{j} + 2\hat{k}) = \vec{b}$

Sol: $\vec{r} = (2t^2)\hat{i} + (t^2 - 4t)\hat{j} + (3t - 5)\hat{k}$

$$\frac{d\vec{r}}{dt} = (4t)\hat{i} + (2t - 4)\hat{j} + 3\hat{k}$$

$$\left(\frac{d\vec{r}}{dt}\right)_{t=1} = 4\hat{i} + (-2)\hat{j} + 3\hat{k}$$

$$\frac{d^2\vec{r}}{dt^2} = (4\hat{i} + 2\hat{j} + 0\hat{k})$$

velocity component in direction

$$\vec{v} = \vec{v}\cdot\hat{b} = \frac{\vec{v}\cdot\vec{b}}{|\vec{b}|} = \frac{(4\hat{i} - 2\hat{j} + 3\hat{k})(\hat{i} - 3\hat{j} + 2\hat{k})}{\sqrt{1+9+4}}$$

$$= \frac{4\times 1 - 2\times -3 + 2\times 3}{\sqrt{14}}$$

$$= \frac{4+6+6}{\sqrt{14}} = \frac{16}{\sqrt{14}}$$

acceleration component in direction

$$\vec{a} = \vec{a}\cdot\hat{b} = \frac{(4\hat{i} + 2\hat{j} + 0\hat{k})(\hat{i} - 3\hat{j} + 2\hat{k})}{\sqrt{1+9+4}}$$

$$= \frac{4\times 1 + 2\times -3 + 0}{\sqrt{14}}$$

$$= \frac{4-6}{\sqrt{14}} = \frac{-2}{\sqrt{14}}$$

L-2

Date 22/Nov/2024

$\vec{\nabla}$ = Vector differential operator (del)

$$\left[\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]$$

$\vec{\nabla}\phi \rightarrow$ gradient $\qquad$ $\vec{\nabla}\times\vec{F} \rightarrow$ ~~curl~~ (Curl)

$\vec{\nabla}\cdot\vec{F} \rightarrow$ divergence

Scalar Point function: Let R be a region ~~of~~ in space at each point of which a scalar ϕ which is a function of $\phi = \phi(x, y, z)$ is given, Then ϕ is called scalar point function & R is called scalar field. **for ex:** Temperature distribution in a medium, atmospheric pressure in space

Vector Point function: Let R be a region in space at each point of which a vector $\vec{v}(\vec{x})$ $\vec{V} = \vec{v}(x, y, z)$ is given then $\vec{V}$ is called vector point function. R is called Vector field.

for ex: velocity of a moving fluid at any instant, Gravitational force.

Gradient of a Scalar field

Let $\phi(x,y,z)$ be a function defining a scalar field then vector $\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$ is called gradient of ϕ.

It is denoted by grad ϕ

i.e. $\text{grad}\,\phi = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]\phi$

(grad $\phi \rightarrow$ Normal vector to the surface)

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

for ex: If $\phi = xyz^2 + xy$ find gradient ϕ.

Sol: $\frac{\partial\phi}{\partial x} = yz^2 + y$

$\frac{\partial\phi}{\partial y} = xz^2 + x$

$\frac{\partial\phi}{\partial z} = 2xyz + 0$

Now, grad $\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$

grad $\phi = \hat{i}(yz^2+y) + \hat{j}(xz^2+x) + \hat{k}(2xyz)$

for ex: find grad ϕ if $\phi = \log(x^2+y^2+z^2)$

Sol: $\frac{\partial\phi}{\partial x} = \frac{1}{x^2+y^2+z^2} \times 2x = \frac{2x}{x^2+y^2+z^2}$

$\frac{\partial\phi}{\partial y} = \frac{2y}{x^2+y^2+z^2}$

$\frac{\partial\phi}{\partial z} = \frac{2z}{x^2+y^2+z^2}$

Now grad $\phi = \hat{i}\left(\frac{2x}{x^2+y^2+z^2}\right) + \hat{j}\left(\frac{2y}{x^2+y^2+z^2}\right) + \hat{k}\left(\frac{2z}{x^2+y^2+z^2}\right)$

$$\vec{\nabla}\phi = \frac{2(x\hat{i} + y\hat{j} + z\hat{k})}{(x^2+y^2+z^2)}$$

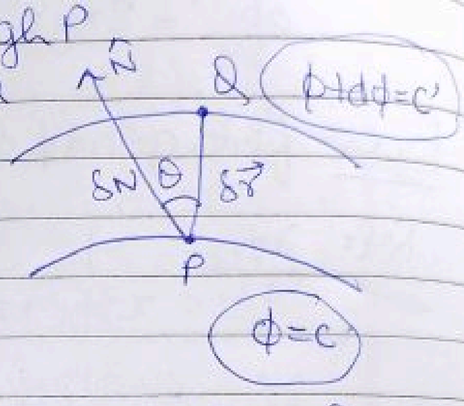
Level Surface: If a surface $\phi(x,y,z) = c$ is drawn through any point P such that at each point on surface the function has the same value as at P, then such a surface is called level surface.

for ex: equipotential surface

NOTE: Through any point passes one and only one living level surface. it means no two level surface can intersect.

Imp Geometrical Interpretation of Gradient

Consider a level surface through P at which the function has value ϕ & another level surface through a neighbourhood point Q where the value is $\phi + d\phi$



Let $\vec{r}$ and $\vec{r} + \delta\vec{r}$ be the position vector of P & Q.
Then $\vec{PQ} = \delta\vec{r}$

$$\text{Now, } \vec{\nabla}\phi \cdot \delta\vec{r} = \left[\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right] \cdot \left[\hat{i}\,\delta x + \hat{j}\,\delta y + \hat{k}\,\delta z\right]$$

$$= \frac{\partial \phi}{\partial x}\cdot \delta x + \frac{\partial \phi}{\partial y}\cdot \delta y + \frac{\partial \phi}{\partial z}\cdot \delta z$$

$$= \delta\phi \quad \text{——— (1)}$$

If $Q \to P$ Then $\delta\phi = 0 \Rightarrow \vec{\nabla}\phi \cdot \delta\vec{r} = 0 \Rightarrow \vec{\nabla}\phi$ is $\perp$r to the surface $\phi(x, y, z) = C$

NOTE: Gradient of a scalar field ϕ is vector normal to the surface ϕ & has a magnitude equal to the rate of change of ϕ along this normal.

Date ______
Page ______

NOTE: ① $\vec{\nabla}\phi$ is Normal vector to the surface

② $|\nabla\phi|$ → maximum rate of change

for ex: find unit Normal vector → gradient (normal to the surface) to the surface

$x^3+y^3\cancel{+z^3}+3xyz=3$ at the point $(1,2,-1)$

Sol: $\phi = x^3+y^3+\cancel{z}+3xyz-3$

$\frac{\partial\phi}{\partial x} = 3x^2+3yz$ $\quad = \left(\frac{\partial\phi}{\partial x}\right)\hat{i} + \left(\frac{\partial\phi}{\partial y}\right)\hat{j} + \left(\frac{\partial\phi}{\partial z}\right)\hat{k}$

$\text{grad}\phi = \vec{\nabla}\phi$

$\frac{\partial\phi}{\partial y} = 3y^2+3xz$ $\quad = (3x^2+3yz)\hat{i} + (3y^2+3xz)\hat{j} + (3xy)\hat{k}$

$\frac{\partial\phi}{\partial z} = 3xy$ $\quad = (3\times1+3\times2\times-1)\hat{i} + (3\times4+3\times1\times-1)\hat{j} + (3\times1\times2)\hat{k}$

$= (3-6)\hat{i} + 9\hat{j} + 6\hat{k}$

$= -3\hat{i}+9\hat{j}+6\hat{k}$

unit Normal vector $= \frac{\nabla\phi}{|\nabla\phi|} = \frac{-3\hat{i}+9\hat{j}+6\hat{k}}{\sqrt{9+81+36}}$

$= \frac{-3\hat{i}+9\hat{j}+6\hat{k}}{\sqrt{126}}$ Ans

for ex: find angle between surfaces $x^2+y^2+z^2=9$ & $z=x^2+y^2-3$ at point $(2,-1,2)$.

Sol: find gradient of both & apply $\cos\theta$

$\cos\theta = \frac{16+4}{6\times\sqrt{20}}$

$= \frac{20}{3\times\sqrt{21}} = \frac{10}{3\sqrt{21}}$

$\text{grad}(1) = (2x)\hat{i} + (2y)\hat{j} + (2z)\hat{k}$

Que: If $\psi(x,y,z) = 4e^{x+5y-13z}$

find grad ϕ at $(1,0,1)$

Sol: $\frac{\partial \psi}{\partial x} = 4e^{x+5y-13z}$

$\frac{\partial \psi}{\partial y} = 4e^{x+5y-13z} \times 5 = 20e^{x+5y-13z}$

$\frac{\partial \psi}{\partial z} = 4e^{x+5y-13z} \times -13 = -52e^{x+5y-13z}$

$$\text{grad}\,\phi = \left(4e^{x+5y-13z}\right)\hat{i} + \left(20e^{x+5y-13z}\right)\hat{j} + \left(-52e^{x+5y-13z}\right)\hat{k}$$

put $(1,0,1)$

$$= \left(4e^{1+0-13}\right)\hat{i} + \left(20e^{1+0-13}\right)\hat{j} - 52e^{1+0-13}\,\hat{k}$$

$$= \left(4e^{-12}\right)\hat{i} + \left(20e^{-12}\right)\hat{j} - \left(52e^{-12}\right)\hat{k}$$

$$= 4e^{-12}\left(\hat{i} + 5\hat{j} - 13\hat{k}\right)$$

Directional Derivative: Directional Derivative is the component of gradient in a particular direction of a vector.

i.e. ~~$\frac{df}{ds}$~~ $\boxed{\frac{df}{ds} = (\text{grad}\,\phi)\cdot\hat{a}}$

is directional derivative of ϕ in the direction of $\vec{a}$ (any vector)

Date ______
Page ______

for ex: find the directional derivative of $f = x^2 - y^2 + 2z^2$ at the point $P(1,2,3)$ in the direction of PQ where $Q(5,0,4)$

Sol: $\overrightarrow{PQ} = (5,0,4) - (1,2,3) = 4\hat{i} - 2\hat{j} + \hat{k}$

$\overrightarrow{PQ} = 4\hat{i} - 2\hat{j} + \hat{k}$

$$\hat{PQ} = \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{16+4+1}} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{21}}$$

jiska dd nikalna hai uska gradient nikal lo.

$\vec{\nabla} f = (2x)\hat{i} + (-2y)\hat{j} + (2 \times 2z)\hat{k}$

$= (2x)\hat{i} + \hat{j}(-2y) + \hat{k}(4z)$

$[\vec{\nabla} f]_{(1,2,3)} = 2\hat{i} - 4\hat{j} + 12\hat{k}$

Directional derivative $= \vec{\nabla} f \cdot \hat{PQ}$

$$= (2\hat{i} - 4\hat{j} + 12\hat{k}) \frac{(4\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{21}}$$

$$= \frac{2 \times 4 - 4 \times -2 + 12 \times 1}{\sqrt{21}}$$

$$= \frac{8+8+12}{\sqrt{21}} = \frac{4\sqrt{21}}{3}$$

$$= \frac{28}{\sqrt{21}} \text{ Ans} \quad \frac{28}{\sqrt{3}\sqrt{7}} \quad \frac{4 \times 7}{\sqrt{3}\sqrt{7}}$$

$$= \frac{4\sqrt{7} \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}}$$

$$= \frac{4\sqrt{21}}{3} \text{ Ans}$$

Date
Page

For ex: find angle between surfaces $x^2+y^2+z^2=9$ & $z=x^2+y^2-3$ at point $(2,-1,2)$.

$\phi=x^2+y^2+z^2-9$

Sol: find gradient of both & apply $\cos\theta$

$$\text{grad}①=\left(\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}\right)$$

$$\vec{u}=\left(\hat{i}(2x)+\hat{j}(2y)+\hat{k}(2z)\right) \text{ --- } ①$$

$\hookrightarrow$ put $x=2, y=-1$ & $z=2$

$$\vec{u}=4\hat{i}-2\hat{j}+4\hat{k} \text{ --- } ①$$

$$\frac{\partial\phi}{\partial x}=\frac{\partial}{\partial x}(x^2+y^2+z^2-9)=2x$$

$$\frac{\partial\phi}{\partial y}=\frac{\partial}{\partial y}(x^2+y^2+z^2-9)=2y$$

$$\frac{\partial\phi}{\partial z}=\frac{\partial}{\partial z}(x^2+y^2+z^2-9)=2z$$

$$\text{grad}②=\left(\hat{i}\frac{\partial r}{\partial x}+\hat{j}\frac{\partial r}{\partial y}+\hat{k}\frac{\partial r}{\partial z}\right) \qquad r=z-x^2-y^2+3$$

$$\frac{\partial r}{\partial x}=\frac{\partial}{\partial x}(z-x^2-y^2+3)=-2x$$

$$\frac{\partial r}{\partial y}=\frac{\partial}{\partial y}(z-x^2-y^2+3)=-2y$$

$$\frac{\partial r}{\partial z}=\frac{\partial}{\partial z}(z-x^2-y^2+3)=1 \longrightarrow \text{put } x=2, y=-1, z=2$$

$$\vec{v}=-4\hat{i}+2\hat{j}+\hat{k} \text{ --- } ②$$

$$\vec{v}=\left(\hat{i}(-2x)+\hat{j}(-2y)+\hat{k}(1)\right) \text{ --- } ②$$

$$=-(4\hat{i}-2\hat{j}-\hat{k})$$

from eqn ① & ②

grad ① : $2x\hat{i}+2y\hat{j}+2z\hat{k}$ $(\vec{u})$

grad ② : $-2x\hat{i}-2y\hat{j}+\hat{k}$ $(\vec{v})$

$$\cos\theta=\frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\,|\vec{v}|}=\frac{2x\times-2x+2y\times-2y+2z\times1}{\sqrt{4x^2+4y^2+4z^2}\sqrt{4x^2+4y^2+1}}$$

put $x=2, y=-1$ & $z=2$

$$\cos\theta = \frac{2\times2\times-2\times2+2\times+1\times+2\times-1+2\times2\times1}{\sqrt{4\times4+4\times1+4\times4}\sqrt{4\times4+4\times1+1}}$$

$$\cos\theta = \left|\frac{-16 \overset{-4}{\cancel{+2}} \overset{+4}{\cancel{+4}}}{\sqrt{16+4+16}\sqrt{16+4+1}}\right|$$

$$\cos\theta = \left|\frac{\cancel{-16+2+4}}{6\times\sqrt{21}}\right| = \left|\frac{\cancel{-14+4}}{6\sqrt{21}}\right| = \frac{\cancel{-10}}{6\sqrt{21}} = \frac{\cancel{5}}{3\sqrt{21}}$$

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \frac{4\times4-2\times-2+4\times-1}{\sqrt{16+4+16}\sqrt{16+4+1}}$$

$$\cos\theta = \frac{16+4-4}{6\times\sqrt{21}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}} \text{ Ans}$$

$$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right) \text{ Ans}$$

L-3

Date: 25/Nov/2024

Que 1 find the directional derivative of $F = e^{2x}\cos(yz)$ at point (0,0,0) in the direction of tangent to the curve $x = a\sin t$, $y = a\cos t$, $z = at$ at $t = \pi/4$.

Sol: find gradient of F

$\vec{r}(t) = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{r}(t) = a\sin t\,\hat{i} + a\cos t\,\hat{j} + at\,\hat{k}$

tangent vector $\vec{T} = \dfrac{d\vec{r}}{dt} = a\cos t\,\hat{i} - a\sin t\,\hat{j} + a\hat{k}$

$$\text{grad}\,\vec{F} = \vec{\nabla}\vec{F} = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$= \hat{i}\frac{\partial}{\partial x}e^{2x}\cos yz + \hat{j}\frac{\partial}{\partial y}e^{2x}\cos yz + \hat{k}\frac{\partial}{\partial z}e^{2x}\cos yz$$

$$= \hat{i}(2e^{2x}\cos yz) - \hat{j}(e^{2x}.z\sin yz) - \hat{k}\frac{\partial}{\partial z}(e^{2x}y\sin yz)$$

$$(T)_{t=\frac{\pi}{4}} = \frac{a}{\sqrt{2}}\hat{i} - \frac{a}{\sqrt{2}}\hat{j} + a\hat{k}$$

$$(\text{grad}\,\vec{F})_{(0,0,0)} = 2\hat{i}$$

Directional derivative of F in the direction of $\vec{T}$

$$= \text{grad}\,F.\hat{\imath} = \frac{(\text{grad}\,\vec{F}).\vec{T}}{|\vec{T}|}$$

$$= \frac{(2\hat{i})\left(\frac{a}{\sqrt{2}}\hat{i} - \frac{a}{\sqrt{2}}\hat{j} + a\hat{k}\right)}{\sqrt{\frac{a^2}{2} + \frac{a^2}{2} + a^2}} = \frac{2a/\sqrt{2}}{\sqrt{\frac{2a^2}{2} + a^2}} = \frac{\sqrt{2}a}{\sqrt{2}a} = 1\ \text{Ans}$$

Date ______
Page ______

Que(2) find DD of $\phi = xy^2 + yz^3$ at point $(2,-1,1)$ in the direction of normal to the surface $x\log z - y^2 + 4$ at point $(2,-1,1)$.

Sol: Normal to the Surface $(x\log z - y^2 + 4) = \text{grad}\,\phi$

$$\phi = x\log z - y^2 + 4$$

$$\text{grad}\,\phi = \nabla\cdot\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$$

$$\frac{\partial\phi}{\partial x} = \log z \qquad \frac{\partial\phi}{\partial z} = \frac{x}{z}$$

$$\frac{\partial\phi}{\partial y} = 0 - 2y$$

$$\text{grad}\,\phi = \hat{i}(\log z) + \hat{j}(-2y) + \hat{k}\left(\frac{x}{z}\right)$$

$$\vec{n} = (\log z)\,\hat{i} - 2y\,\hat{j} + \frac{x}{z}\hat{k}$$

put $x = 2,\ y = -1\ \&\ z = 1$

$$\vec{n} = 0 + 2\hat{j} + 2\hat{k} = 2\hat{j} + 2\hat{k}$$

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{2\hat{j}+2\hat{k}}{\sqrt{4+4}} = \frac{2\hat{j}+2\hat{k}}{2\sqrt{2}} = \frac{\hat{j}+\hat{k}}{\sqrt{2}}$$

$(\phi = xy^2 + yz^3)$

$$\text{grad}\,\phi = \nabla\phi = \frac{\partial}{\partial x}(xy^2+yz^3)\,\hat{i} + \frac{\partial}{\partial y}(xy^2+yz^3)\,\hat{j} + \frac{\partial}{\partial z}(xy^2+yz^3)\,\hat{k}$$

$$\nabla\phi = (y^2)\,\hat{i} + \hat{j}(2xy + z^3) + \hat{k}(0 + 3yz^2)$$

$$(\nabla\phi)_{(2,-1,1)} = \hat{i} + \hat{j}(2\times2\times-1+1) + \hat{k}(3\times-1\times1)$$

$$= \hat{i} + \hat{j}(-4+1) + \hat{k}(-3)$$

$$= \hat{i} - 3\hat{j} - 3\hat{k}$$

$$DD = (\text{grad}\,\phi)\cdot\hat{n} = \frac{(\hat{i}-3\hat{j}-3\hat{k})(\hat{j}+\hat{k})}{\sqrt{2}}$$

$$= \frac{-3-3}{\sqrt{2}} = \frac{-6}{\sqrt{2}} = \frac{-3\times2}{\sqrt{2}} = -3\sqrt{2}\ \text{Ans}$$

jiska DD nikalna hota uska Grad nikalte hai

Ques ③ Find the DD of $\vec{V}^2$ where $\vec{V} = xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}$ at point P(2, 0, 3) in the direction of the outward normal to the Surface $x^2+y^2+z^2=14$ at point (3, 2, 1).

Sol: $\vec{V}^2 \quad \vec{V}\cdot\vec{V} \quad \vec{V}\times\vec{V}$

$$\phi = \vec{V}^2 = \vec{V}\cdot\vec{V} = (xy^2\hat{i}+zy^2\hat{j}+xz^2\hat{k})\cdot(xy^2\hat{i}+zy^2\hat{j}+xz^2\hat{k})$$

$$\phi_1 = x^2y^4 + z^2y^4 + x^2z^4$$

$$\text{grad} = \hat{i}\frac{\partial}{\partial x}(x^2y^4+z^2y^4+x^2z^4) + \hat{j}\frac{\partial}{\partial y}(x^2y^4+z^2y^4+x^2z^4) + \hat{k}\frac{\partial}{\partial z}(x^2y^4+z^2y^4+x^2z^4)$$

$$\phi_1 = \hat{i}(2xy^4+2xz^4) + \hat{j}(4y^3x^2+4y^3z^2) + \hat{k}(2zy^4+x^2\times 4z^3)$$

(2,0,3)

$$\phi_1 = \hat{i}(\cancel{2\times 0}+2\times 2\times 81) + \hat{j}(4\times 0+0) + \hat{k}(2\times 3\times 0+4\times 4\times 27)$$

$$\phi_1 = \hat{i}(324) + \hat{j}(0) + \hat{k}(0+432)$$

$$\phi_1 = 324\hat{i} + 432\hat{k}$$

$$\phi_1 = (81\hat{i} + 108\hat{k})\times 4$$

$$\phi_1 = (9\hat{i} + 12\hat{k})\,9\times 4$$

$$\phi_1 = 36(9\hat{i} + 12\hat{k})$$

$$\phi_1 = 36\times 3(3\hat{i} + 4\hat{k})$$

$$\boxed{\phi_1 = 108(3\hat{i} + 4\hat{k})}$$

$$\phi_2 = \hat{i}\frac{\partial}{\partial x}(x^2+y^2+z^2-14) + \hat{j}\left(\frac{\partial}{\partial y}(x^2+y^2+z^2-14)\right)$$
$$+\hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2-14)$$

$$\phi_2 = \hat{i}(2x) + \hat{j}(2y) + \hat{k}(2z)$$

$(3,2,1)$

$$\phi_2 = \hat{i}(6) + \hat{j}(4) + \hat{k}(2)$$

$$\phi_2 = 6\hat{i} + 4\hat{j} + 2\hat{k}$$

$$= \frac{\vec{\phi_1}\cdot\vec{\phi_2}}{|\vec{\phi_2}|}$$

$= \cancel{108}$

$= \phi_1 \; 2 \; 144$

$$= \frac{324\times 6 + 432\times 2}{\sqrt{36+16+4}} = \frac{1944+864}{\sqrt{56}}$$

$$= \frac{2808\sqrt{56}}{\sqrt{56}\sqrt{56}}$$

$$= \frac{\cancel{1404}\;\cancel{702}\;351}{\cancel{2808}\sqrt{56}}{\cancel{56}}$$

$\cancel{28}$

$\cancel{14}\;7$

$$= \frac{351\sqrt{56}}{7}$$

1944
864
2808

36
20
56

(max change)

Que ⑨ What is the greatest rate of increase of $u = xyz^2$ at point $(1, 0, 3)$.

Sol:
$$\phi = \hat{i}\frac{\partial}{\partial x}(xyz^2) + \hat{j}\frac{\partial}{\partial y}(xyz^2) + \hat{k}\frac{\partial}{\partial z}(xyz^2)$$

$$\phi = \hat{i}(yz^2) + \hat{j}(xz^2) + \hat{k}(2xyz)$$

$$\phi = \hat{i}(0) + \hat{j}(1\times 9) + \hat{k}(0)$$

$$\phi = 9\hat{j}$$

$$\sqrt{81} = 9 \text{ Ans}$$

Que ⑩ find the angle b/w the surfaces $x^2+y^2+z^2 = 9$ & $z = x^2+y^2-3$ at point $(2,-1,2)$

Sol: find grad of both & apply $\cos\theta$

$$\vec{n_1} = x^2+y^2+z^2-9 \qquad x^2+y^2-3-z=0$$

$$\vec{n_1} = \hat{i}(2x) + \hat{j}(2y) + \hat{k}(2z) \rightarrow 4, -2, 4$$

$$\vec{n_2} = \hat{i}(2x) + \hat{j}(2y) + \hat{k}(-1) \rightarrow 4\hat{i} - 2\hat{j} - \hat{k}$$

$$= \cancel{(2x)(2x) + (2y)(2y) + 2z}$$

$$\cos\theta = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|}$$

$$= \frac{(2x)(2x) + (2y)(2y) - 2z}{\sqrt{16+4+1}}$$

$$\cos\theta = \frac{(2\times 2)(2\times 2) + (2\times -1)(2\times -1) - 2\times 2}{\sqrt{21}\ \sqrt{16+4+16}}$$

$$= \frac{4\times 4 - 2\times -2 - 4}{\sqrt{21}\ \sqrt{36}} = \frac{16+4-4}{\sqrt{21}\times 6}$$

$$= \frac{16+4-4}{\sqrt{21}\sqrt{36}} = \frac{16}{\sqrt{21}\times 6}$$

$$= \frac{8}{3\sqrt{21}} \text{ Ans}$$

L-4

Date 27/Nov/2024

Divergence of a Vector point function: Let $\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ be a vector point function and $\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ be vector differential operator then divergence of $\vec{F}$ is denoted by $\operatorname{div}\vec{F}$ and given by

$$\boxed{\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}}$$

ex: If $\vec{F} = x^2y\hat{i} + xy^2z\hat{j} + z^2x\hat{k}$ find $\operatorname{div}\vec{F}$ at point $(1,1,-1)$

Sol: $\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \left[\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right]$

$$= \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(xy^2z) + \frac{\partial}{\partial z}(z^2x)$$

$$\left(\operatorname{div}\vec{F} = 2xy + 2xyz + 2zx\right)$$

at pt $(1,1,-1)$

put $x=1, y=1, z=-1$

$\operatorname{div}\vec{F} = 2-2-2 = -2$

NOTE: if $\operatorname{div}\vec{F} = 0 \Rightarrow$ Solenoid vector or incompressible

Curl: Let $\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ be a vector function and $\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ be vector differential operator then

$$\left(\operatorname{curl}\vec{F} = \vec{\nabla}\times\vec{F}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

for ex: If $\vec{F} = xy^2\hat{i} + yz^2\hat{j} + zx^2\hat{k}$ find curl $\vec{F}$

Sol: $\text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy^2 & yz^2 & zx^2 \end{vmatrix}$

$$\text{Curl } \vec{F} = \hat{i} \begin{vmatrix} \partial/\partial y & \partial/\partial z \\ yz^2 & zx^2 \end{vmatrix} - \hat{j} \begin{vmatrix} \partial/\partial x & \partial/\partial z \\ xy^2 & zx^2 \end{vmatrix} + \hat{k} \begin{vmatrix} \partial/\partial x & \partial/\partial y \\ xy^2 & yz^2 \end{vmatrix}$$

$$\text{Curl } \vec{F} = \hat{i}[0 - 2yz] - \hat{j}[2zx - 0] + \hat{k}[0 - 2xy]$$

$$\boxed{\text{Curl } \vec{F} = -2yz\hat{i} - \hat{j}\,2zx - \hat{k}\,2xy}$$

NOTE: if curl $\vec{F} = 0 \Rightarrow$ irrotational motion
then there exists scalar velocity potential ϕ
such that $\vec{F} = \text{grad}\,\phi$ it means

$$\text{i.e. } \phi = \int \vec{F} \cdot d\vec{r}$$

$$\left(d\vec{r} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz\right)$$

$\left\{ \begin{array}{l} u \to \text{scalar} \\ \vec{a} = \text{vector} \end{array} \right\}$

Date
Page

Imp Identity: $\operatorname{div}(u\vec{a}) = u(\operatorname{div}\vec{a}) + (\operatorname{grad} u)\cdot\vec{a}$

$$\operatorname{curl}(u\vec{a}) = u(\operatorname{curl}\vec{a}) + [\operatorname{grad} u]\times\vec{a}$$

$$\left(d(f_1 f_2) = f_1\, df_2 + f_2\, df_1\right)$$

Que. Show that $\vec{v} = (x+3y)\hat{i} + (y-3z)\hat{j} + (x-2z)\hat{k}$ is solenoid.

Sol: $\operatorname{div}\vec{F} = 0$

$\operatorname{div}\vec{v} = 0$

$$\operatorname{div}\vec{v} = \vec{\nabla}\cdot\vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\vec{\nabla}\cdot\vec{v} = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]\left[v_1\hat{i} + v_2\hat{j} + v_3\hat{k}\right]$$

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-3z) + \frac{\partial}{\partial z}(x-2z)$$

$$= 1 + 1 - 2 = 0$$

$$\boxed{\operatorname{div}\vec{v} = 0}$$

(hence it is a solenoid)

Que. Show that vector field defined by

$$\vec{F} = 2xyz^3\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\hat{k}$$ is irrotational. find the scalar potential u such that $\vec{F} = \operatorname{grad} u$.

$$u = \int \vec{F}\cdot d\vec{r}$$

Sol: Curl $\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xyz^3 & x^2z^3 & 3x^2yz^2 \end{vmatrix}$

$= \hat{i} \begin{vmatrix} \frac{\partial}{\partial y} & \partial/\partial z \\ x^2z^3 & 3x^2yz^2 \end{vmatrix} - \hat{j} \begin{vmatrix} \partial/\partial x & \partial/\partial z \\ 2xyz^3 & 3x^2yz^2 \end{vmatrix} + \hat{k} \begin{vmatrix} \partial/\partial x & \partial/\partial y \\ 2xyz^3 & x^2z^3 \end{vmatrix}$

$= \hat{i}\left[\frac{\partial}{\partial y}(3x^2yz^2) - \frac{\partial}{\partial z}(x^2z^3)\right] - \hat{j}\left[\frac{\partial}{\partial x}(3x^2yz^2) - \frac{\partial}{\partial z}(2xyz^3)\right] + \hat{k}\left[\frac{\partial}{\partial x}(x^2z^3) - \frac{\partial}{\partial y}(2xyz^3)\right]$

$= \hat{i}\left[\cancel{3x^2z^2 - 3z^2x^2}\right] - \hat{j}\left(\cancel{6xyz^2 - 6xyz^2}\right) + \hat{k}\left(\cancel{2xz^3 - 2xz^3}\right)$

$0 = \text{curl}\vec{F}$

To find scaler velocity potential:

$u = \int \vec{F} \cdot \vec{dr} = \int (2xyz^3)dx + (x^2z^3)dy + (3x^2yz^2)dz$

$= \int d(yz^3x^2)$

$\boxed{u = yz^3x^2 + C}$ Ans

$\left[d[f_1f_2f_3] = f_2f_3df_1 + f_1df_2f_3 + f_1f_2df_3\right]$

Date
Page

Ex: Show that the vector field $\vec{F} = \frac{\vec{r}}{r^3}$ is irrotational as well as solenoidal. find scalar potential.

Sol: $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$$

$$\operatorname{div} \vec{r} = \vec{\nabla} \cdot \vec{r}$$

$$= \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y + \frac{\partial}{\partial z} z = 1+1+1$$

$$\boxed{\operatorname{div} \vec{r} = 3}$$

$$\operatorname{curl} \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix}$$

$$= \hat{i}[0-0] - \hat{j}[0-0] + \hat{k}[0-0]$$

$$\boxed{\operatorname{curl} \vec{r} = 0}$$

$$\vec{F} = \vec{r} \cdot \frac{1}{r^3}$$

using identity,

$$\operatorname{div}[u\vec{a}] = u[\operatorname{div} \vec{a}] + [\operatorname{grad} u] \cdot \vec{a}$$

$$\operatorname{div}\left[\frac{\vec{r}}{r^3}\right] = \frac{1}{r^3}[\operatorname{div} \vec{r}] + \left[\operatorname{grad} \frac{1}{r^3}\right] \cdot \vec{r}$$

$$= \frac{3}{r^3} - \frac{3\vec{r}\,\vec{r}}{r^5}$$

$$= \frac{3}{r^3} - \frac{3r^2}{r^5} = \frac{3}{r^3} - \frac{3}{r^3}$$

$$\operatorname{div}\left[\frac{\vec{r}}{r^3}\right] = 0$$

Date ____
Page ____

$\boxed{\mathrm{div}\left[\frac{\vec{r}}{r^3}\right] = 0}$ (Solenoidal)

Now curl $\vec{F}$ = curl $\left[\frac{\vec{r}}{r^3}\right]$

$$\mathrm{Curl}\,\vec{F} = \mathrm{Curl}\left[\frac{\vec{r}}{r^3}\right] = \frac{1}{r^3}\left[\mathrm{curl}\,\vec{r}\right] + \left[\mathrm{grad}\frac{1}{r^3}\right]\times\vec{r}$$

$$= \frac{0}{r^3} - \left[\frac{-3}{r^5}\vec{r}\times\vec{r}\right]$$

$$= 0 - 0 \quad \left[\vec{r}\times\vec{r} = 0\right]$$

curl $\vec{r} = 0$

$\boxed{\mathrm{Curl}\,\vec{F} = 0}$ (is irrotational)

Now Scalar potential:-

$\phi(x,y,z) = c$ $\qquad$ let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

① $\vec{\nabla}\phi$

Now scalar potential,

$$\phi = \int \vec{F}\cdot\vec{dr} = \int \frac{\vec{r}}{r^3}\cdot\vec{dr} = \int \frac{(x\hat{i}+y\hat{j}+z\hat{k})}{(x^2+y^2+z^2)^{3/2}}\cdot(\hat{i}dx+\hat{j}dy+\hat{k}dz)$$

$$= \int \frac{xdx+ydy+zdz}{(x^2+y^2+z^2)^{3/2}}$$

Substitute $x^2+y^2+z^2 \to t$

$$2xdx+2ydy+2zdz = dt$$

$$xdx+ydy+zdz = \frac{dt}{2}$$

$$= \int \frac{\frac{dt}{2}}{(t)^{3/2}} = \frac{1}{2}\int t^{-3/2}dt$$

$$= \frac{1}{2}\left[\frac{t^{-\frac{3}{2}+1}}{-\frac{3}{2}+1}\right]$$

$$= \cancel{\frac{1}{2}}\left[\frac{t^{-1/2}}{-\cancel{\frac{1}{2}}}\right] = -t^{-1/2}$$

$$= \frac{-1}{t^{1/2}}$$

$r = \sqrt{x^2+y^2+z^2}$

$r = \sqrt{t}$

$$\boxed{\phi = -\frac{1}{r} + c}$$ Ans

L-5

Date 28/11/2024

VECTOR DIFFERENTIATION

Point function

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Scalar Point function

$\phi(x,y,z) = c$

① $\vec{\nabla}\phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$

$\vec{\nabla}\phi \rightarrow \text{grad}\,\phi$

grad ϕ → outward normal vector to the surface

② **Directional Derivative:-**

DD of ϕ in the direction of $\vec{a} = \vec{\nabla}\phi \cdot \hat{a}$

Vector Point function

Let $\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$

① $\vec{\nabla}\cdot\vec{F} = \text{div}\,\vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

② $\vec{\nabla}\times\vec{F} = \text{curl}\,\vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

NOTE: if div $\vec{F} = 0$ → solenoid / incompressible

If curl $\vec{F} = 0$ → Irrotational motion or conservative field

if curl $\vec{F} = 0$. Then there exist scaler potential ϕ such that

$$\phi = \int \vec{F}\cdot\vec{dr}$$

Date
Page

Que ①. A fluid motion is given by

$$\vec{V} = [y\sin z - \sin x]\hat{i} + [x\sin z + 2yz]\hat{j} + (xy\cos z + y^2)\hat{k}$$

is the motion is irrotational? If so find velocity potential.

Sol: To check whether the motion is irrotational or not, we have to check the value of $\text{curl}\vec{F}$ if $\text{curl}\vec{F} = 0$ (then the motion is irrotational)

$$\nabla \times \vec{V} = \text{curl}\vec{V} = \begin{vmatrix} \overset{+}{\hat{i}} & \overset{-}{\hat{j}} & \overset{+}{\hat{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$f_1 = y\sin z - \sin x$

$f_2 = x\sin z + 2yz$

$f_3 = xy\cos z + y^2$

$$= \hat{i}\left[\frac{\partial}{\partial y}(f_3) - \frac{\partial}{\partial z}(f_2)\right] - \hat{j}\left[\frac{\partial}{\partial x}(f_3) - \frac{\partial}{\partial z}(f_1)\right] + \hat{k}\left[\frac{\partial}{\partial x}(f_2) - \frac{\partial}{\partial y}(f_1)\right] \quad \text{--- ①}$$

$$\frac{\partial}{\partial y}(f_3) = \frac{\partial}{\partial y}(xy\cos z + y^2) = x\cos z + 2y$$

$$\frac{\partial}{\partial z}(f_2) = \frac{\partial}{\partial z}(x\sin z + 2yz) = x\cos z + 2y$$

$$\frac{\partial}{\partial x}(f_3) = \frac{\partial}{\partial x}(xy\cos z + y^2) = y\cos z$$

$$\frac{\partial}{\partial z}(f_1) = \frac{\partial}{\partial z}(y\sin z - \sin x) = y\cos z$$

$$\frac{\partial}{\partial x}(f_2) = \frac{\partial}{\partial x}(x\sin z + 2yz) = \sin z$$

$\frac{\partial}{\partial y}(F_1) = \frac{\partial}{\partial y}(y\sin z - \sin x) = \sin z$

Now putting all the values in eqn (1)

$$\text{curl}\,\vec{v} = \hat{i}\left[\cancel{(x\cos z + 2y)} - \cancel{(2y + x\cos z)}\right]$$

$$-\hat{j}\left[\cancel{y\cos z} - \cancel{y\cos z}\right] + \hat{k}\left[\cancel{\sin z} - \cancel{\sin z}\right]$$

$\text{curl}\,\vec{v} = 0$

Hence the motion is irrotational.

To find scaler velocity potential ϕ such that

$\vec{F} = \text{grad}\,\phi$ it means

$$\phi = \int \vec{F}\cdot\vec{dr} \qquad (\vec{F} = \vec{v})$$

$$(\vec{dr} = \hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$\phi = \int (y\sin z - \sin x)dx + (x\sin z + 2yz)dy + (xy\cos z + y^2)dz$$

$$= \int y\sin z\,dx - \int \sin x\,dx + \int x\sin z\,dy + \int 2yz\,dy$$

$$+ \int xy\cos z\,dz + \int y^2 dz$$

$$= \int d(y\sin z\,dx) - \int \sin x\,dx + \int d(zy^2) + \int xy\cos z\,dz$$

$$= \int d(xy\sin z) + \cos x + zy^2 + c$$

$$\boxed{\phi = xy\sin z + \cos x + zy^2 + C} \text{ Ans}$$

★ $d(\text{I}\,\text{II}) = d(\text{I})\,\text{II} + \text{I}(d(\text{II}))$

★ $d(\text{I}\,\text{II}\,\text{III}) = d(\text{I})\,\text{II}\,\text{III} + \text{I}\,\text{III}\,d(\text{II}) + \text{I}\,\text{II}\,d(\text{III})$

Date
Page

Que ② Find constants a, b & c so that $(a=4, b=2, c=-1)$

$$\vec{F}=(x+2y+az)\hat{i}+(bx-3y-z)\hat{j}+(4x+cy+2z)\hat{k}$$

is irrotational. If $\vec{F}=\text{grad}\,\phi$.

Show that $\phi=\frac{x^2}{2}-\frac{3y^2}{2}+z^2+2xy+4xz-yz$

Sol: To show that if $\vec{F}$ is irrotational

Show that $\text{curl}\,\vec{F}=0$

$$\text{curl}\,\vec{F}=\nabla\times\vec{F}=\begin{vmatrix} \overset{+}{\hat{i}} & \overset{-}{\hat{j}} & \overset{+}{\hat{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$f_1=(x+2y+az)$

$f_2=(bx-3y-z)$

$f_3=(4x+cy+2z)$

$$=\hat{i}\left[\frac{\partial}{\partial y}(f_3)-\frac{\partial}{\partial z}(f_2)\right]-\hat{j}\left[\frac{\partial}{\partial x}(f_3)-\frac{\partial}{\partial z}(f_1)\right]+\hat{k}\left[\frac{\partial}{\partial x}(f_2)-\frac{\partial}{\partial y}(f_1)\right] \quad \text{—— (1)}$$

$$\frac{\partial}{\partial y}(f_3)=\frac{\partial}{\partial y}(4x+cy+2z)=c$$

$$\frac{\partial}{\partial z}(f_2)=\frac{\partial}{\partial z}(bx-3y-z)=-1$$

$$\frac{\partial}{\partial x}(f_3)=\frac{\partial}{\partial x}(4x+cy+2z)=4$$

$$\frac{\partial}{\partial z}(f_1)=\frac{\partial}{\partial z}(x+2y+az)=a$$

$$\frac{\partial}{\partial x}(f_2)=\frac{\partial}{\partial x}(bx-3y-z)=b$$

$$\frac{\partial}{\partial y}(f_1)=\frac{\partial}{\partial y}(x+2y+az)=2$$

Now putting all the values in eqn (1)

$$\text{curl}\,\vec{F} = \hat{i}[c+1] - \hat{j}[4-a] + \hat{k}[b-2]$$

$c+1=0$ $\qquad$ $4-a=0$ $\qquad$ $b-2=0$

$\boxed{c=-1}$ $\qquad$ $\boxed{a=4}$ $\qquad$ $\boxed{b=2}$

Que (3) → If $u = x^2+y^2+z^2$ and $\vec{v} = x\hat{i}+y\hat{j}+z\hat{k}$

Show that $\text{div}[u\vec{v}] = 5u$.

Sol: ~~using important identity~~;

$$\text{div}(u\vec{v}) = u(\text{div}\,\vec{v}) + (\text{grad}\,u)\vec{v}$$

$$\text{div}\,\vec{v} = \vec{\nabla}\cdot\vec{v} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x\hat{i}+y\hat{j}+z\hat{k})$$

$$\text{div}\,\vec{v} = \vec{\nabla}\cdot\vec{v} = \frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z)$$

$$\text{div}\,\vec{v} = 1+1+1 = 3$$

$$\text{grad}\,u = \left[\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right]u$$

$$= \hat{i}\frac{\partial u}{\partial x}+\hat{j}\frac{\partial u}{\partial y}+\hat{k}\frac{\partial u}{\partial z}$$

$$= \hat{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)+\hat{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)+\hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)$$

$$= \hat{i}(2x)+\hat{j}(2y)+\hat{k}(2z)$$

$$\text{div}(u\vec{v}) = u3 + (2x\hat{i}+2y\hat{j}+2z\hat{k})(x\hat{i}+y\hat{j}+z\hat{k})$$

$$= 3u + 2x^2+2y^2+2z^2$$

$$= 3u + 2(x^2+y^2+z^2)$$

$$= 3u + 2u$$

$$= 5u \text{ (Hence Proved)}$$

Date
Page

Que (4) Show that vector field

$\vec{F} = \nabla = (x+3y)\hat{i} + (y-3z)\hat{j} + (x-2z)\hat{k}$ is solenoidal.

Sol: To show if the vector field is solenoidal we need to show that $\operatorname{div}\vec{F} = 0 = \vec{\nabla}\cdot\vec{F}$

$$\vec{\nabla}\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)$$

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}(F_1) + \frac{\partial}{\partial y}(F_2) + \frac{\partial}{\partial z}(F_3)$$

$$= \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-3z) + \frac{\partial}{\partial z}(x-2z)$$

$$= 1+0+1+0+0-2$$

$$= 2-2 = 0$$

$\vec{\nabla}\cdot\vec{F} = \operatorname{div}\vec{F} = 0$ Hence it is proved that ∇ is solenoidal field.

Que (5) Find directional derivative of $\vec{\nabla}\cdot[\vec{\nabla}\phi]$ at point $(1,-2,1)$ in the direction of the normal to the surface $xy^2z = 3x + z^2$ where $\phi = 2x^3y^2z^4$.

Sol: $$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\phi = 2x^3y^2z^4$$

$$\vec{\nabla}\phi = \hat{i}\frac{\partial}{\partial x}(2x^3y^2z^4) + \hat{j}\frac{\partial}{\partial y}(2x^3y^2z^4) + \hat{k}\frac{\partial}{\partial z}(2x^3y^2z^4)$$

$$\nabla\phi = \hat{i}(6x^2y^2z^4) + \hat{j}(4x^3yz^4) + \hat{k}(8x^3y^2z^3)$$

$$\nabla\cdot[\nabla\phi] = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)[\nabla\phi]$$

$$= \frac{\partial}{\partial x}(6x^2y^2z^4) + \frac{\partial}{\partial y}(4x^3yz^4) + \frac{\partial}{\partial z}(8x^3y^2z^3)$$

$$f = 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$$

To find DD of f, find grad f

$$\text{grad } f = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right] f$$

$$= \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z} = 348\hat{i} - 144\hat{j} + 400\hat{k} \quad \text{—(1)}$$

$$\frac{\partial f}{\partial x} = 12y^2z^4 + 12x^2z^4 + 72x^2y^2z^2$$

$$\frac{\partial f}{\partial y} = 24xyz^4 + 0 + 48x^3yz^2$$

$$\frac{\partial f}{\partial z} = 48xy^2z^3 + 16x^3z^3 + 48x^3y^2z$$

$$\left(\frac{\partial f}{\partial x}\right)_{(1,-2,1)} = 12(-2)^2(1) + 12(1)(1) + 72(1)(-2)^2(1)$$

$$= 12\times4 + 12 + 72\times4$$

$$= 48 + 12 + 288$$

$$= 288 + 60 = 348$$

$$\left(\frac{\partial f}{\partial y}\right)_{(1,-2,1)} = 24(1)(-2)(1) + 0 + 48(1)(-2)(1)$$

$$= -48 - 96$$

$$= -144$$

$$\left(\frac{\partial f}{\partial z}\right)_{(1,-2,1)} = 48(1)(4)(1) + 16(1)(1) + 48(1)(4)$$

$$= 192 + 16 + 192$$

$$= 400$$

Date
Page

$\phi = xy^2z - 3x - z^2 = 0$

↳ Normal to this surface = gradient

$$\text{grad} = \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$$

$$\frac{\partial\phi}{\partial x} = y^2z - 3$$

$$\frac{\partial\phi}{\partial y} = 2xyz$$

$$\frac{\partial\phi}{\partial z} = xy^2 - 2z$$

$$\text{grad}\,\phi = \hat{i}(y^2z-3) + \hat{j}(2xyz) + \hat{k}(xy^2-2z)$$

at $(1,-2,1)$

$$\text{grad}\,\phi = \hat{i}(4-3) + \hat{j}(-4) + \hat{k}(1\times 4 - 2)$$

$$\text{grad}\,\phi = \hat{i} - 4\hat{j} + 2\hat{k}$$

↳ this is the normal vector $\vec{a}$

$$\vec{a} = \hat{i} - 4\hat{j} + 2\hat{k}$$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\hat{i} - 4\hat{j} + 2\hat{k}}{\sqrt{1+16+4}} = \frac{\hat{i} - 4\hat{j} + 2\hat{k}}{\sqrt{21}}$$

$$DD = (\text{grad} f)\,\hat{a}$$

$$= \frac{(340\hat{i} - 144\hat{j} + 400\hat{k})\cdot(\hat{i} - 4\hat{j} + 2\hat{k})}{\sqrt{21}}$$

$$= \frac{340 + 144\times 4 + 800}{\sqrt{21}} = \frac{1140 + 576}{\sqrt{21}} = \frac{1724}{\sqrt{21}}$$

Q-2

Q-2 (part-2) $\vec{F} = \text{grad}\,\phi$

Show that $\phi = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4xz - yz$

$$\vec{F} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$$

put $a=4,\ b=2,\ c=-1$

$$\vec{F} = (x+2y+4z)\hat{i} + (2x-3y-z)\hat{j} + (4x-y+2z)\hat{k}$$

$\vec{F} = \text{grad}\,\phi$ it means

$$\phi = \int \vec{F}\cdot d\vec{r}$$

$$(d\vec{r} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$\phi = \int \left[(x+2y+4z)\hat{i} + (2x-3y-z)\hat{j} + (4x-y+2z)\hat{k}\right]\cdot\left[\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz\right]$$

$$\phi = \int (x+2y+4z)\,dx + \int (2x-3y-z)\,dy + \int (4x-y+2z)\,dz$$

$$\phi = \int x\,dx + \int 2y\,dx + \int 4z\,dx + \int 2x\,dy - \int 3y\,dy - \int z\,dy + \int 4x\,dz - \int y\,dz + \int 2z\,dz$$

$$\phi = \frac{x^2}{2} + 2\int d(xy) + 4\int d(xz) - \int d(yz) - \frac{3y^2}{2} + \frac{2z^2}{2}$$

$$\phi = \frac{x^2}{2} + 2xy + 4xz - yz - \frac{3y^2}{2} + z^2$$

(Hence proved)