Unit-5 → Inner Product Space

Inner Product → Let V be a real vector space. An inner product on V is a function [,] or < , > : $V \times V \to R$ which assigns each ordered pair (u, v) and V×V to a real number (u, v) in (u,v) ∈ V.V in such a way that following axioms hold.

(i) Linearity Prop. :- $\langle au_1 + bu_2, v \rangle = a(u_1, v) + b(u_2, v)$ $\forall u, v \in V$ & $a, b \in R$

(ii) Symmetry → $\boxed{\langle u, v \rangle = \langle v, u \rangle}$ $\forall u, v \in V$

$\langle , \rangle : V \times V \to F$ (Real or Complex)

(iii) Positivity or Positive definiteness :-

$\langle u, v \rangle = \langle v, u \rangle \; \forall u, v \in V \geq 0$

and $\langle u, v \rangle = u^2$ if $u = 0v$

$\langle u, u \rangle > 0$ if $u > 0$ $\quad \langle u, u \rangle = 0$ if $u = 0$

Ex - A vector space occupied with inner product is called inner product space

Ex - Let V be an inner product space and $u, v \in V$, simplify $\langle 2u - 5v, 4u + 6v \rangle$

→ using linearity property of inner product

$\langle 2u - 5v, 4u + 6v \rangle = 2\langle u, 4u + 6v \rangle - 5\langle v, 4u + 6v \rangle$

$= 2\{\langle u, 4u \rangle + 6\langle u, v \rangle\} - 5\{\langle v, 4u \rangle + \langle v, 6u \rangle\}$

$= 2\{4\langle u, u \rangle + 6\langle u, v \rangle\} - 5\{4\langle v, u \rangle + 6\langle v, v \rangle\}$

$= 8\langle u,u\rangle + 12\langle u,v\rangle - 20\langle v,u\rangle - 30\langle v,v\rangle$

$= 8\langle u,u\rangle - 8\langle u,v\rangle - 30\langle v,u\rangle$

Ex- $u = (2,1,0,2) = (u_1, u_2, u_3, u_4)$

$v = (-2,-1,-4,2) = (v_1, v_2, v_3, v_4)$

$w = (0,1,2,3) = (w_1, w_2, w_3, w_4)$

$\langle u, v\rangle = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$

$= -4 - 1 + 0 + 4 = -1$

$\langle v, w\rangle = v_1w_1 + v_2w_2 + v_3w_3 + v_4w_4$

$= 0 - 1 - 8 + 6 = -3$

$\langle u+v, w\rangle = (u_1+v_1)w_1 + (u_2+v_2)w_2 + (u_3+v_3)w_3 + (u_4+v_4)w_4$

OR

$\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$

$= 7 - 3 = 4$ Ans.

Ex- For Matrix Space

$A = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 1 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

find $\langle A, B\rangle = a_{11}b_{11} + a_{12}b_{12} + a_{13}b_{13} + a_{21}b_{21} + a_{22}b_{22} + a_{23}b_{23}$

$= 3\times1 + 0\times3 + 0\times5 + 2\times2 + 1\times4 + 4\times6$

$= 3 + 4 + 4 + 24 = 35$

Ex - For Polynomial Space :- Let $p(x), q(x) \in P_n(x)$

$\langle p(x), q(x)\rangle = p(x_1)\cdot q(x_1) + p(x_2)\cdot q(x_2) + \dots\dots + p(x_n) + q(x_n)$.

Ex -

$p(x) = x^2$, $q(x) = x+1$ & $x_1 = 0$, $x_2 = -1$, $x_3 = 1$

find $\langle p, q \rangle$

$\Rightarrow \langle p,q \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$

$= 1 \times 0 + 0 \times 9 + 1 \times 2 = 2$

Function Space :- Let 'c' be the set (vector space) of all its functions defined on $[a,b]$

Then $\langle f, g \rangle = \int_a^b f(t) \cdot g(t) \, dt \quad a \le t \le b$

Or

$\langle f, g \rangle = \int_a^b f(t) \cdot \overline{g(t)} \, dt$ [Complex f^n].

Q. Let V be inner product space and $u, v \in V$

Simplify. $\langle 5u_1 + 8u_2, 6v_1 - 7v_2 \rangle$

$\langle 5u_1 + 8u_2, 6v_1 - 7v_2 \rangle = 5\langle u_1, 6v_1 - 7v_2 \rangle + 8\langle u_2, 6v_1 - 7v_2 \rangle$

$= 5\{\langle u_1, 6v_1 \rangle - 7\langle u_1, v_2 \rangle\} + 8\{\langle u_2, 6v_1 \rangle - 7\langle u_2, v_2 \rangle\}$

$= 30\langle u_1, v_1 \rangle - 35\langle u_1, v_2 \rangle + 48\langle u_2, v_1 \rangle - 56\langle u_2, v_2 \rangle$

Q. Prove that R^2 is an inner product space with an inner product defined by $\langle u, v \rangle = a_1 b_1 - a_2 b_1 - a_1 b_2 + 2a_2 b_2$ where

$u = (a_1, a_2), \; v = (b_1, b_2) \in R^2$

Sol^n. To show R^2 is inner product space we show following properties,

① Linearity :- Let $u = (a_1, a_2)$, $v = (b_1, b_2)$, $w = (c_1, c_2)$ $\rightarrow \in R^2$ and $\alpha, \beta \in R$

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$B' = \{\alpha_1, \alpha_2\}$

$\alpha_1 = k_1 e_1 + k_2 e_2$

$T(\alpha_1) = kT(e_1) + k_2 T(e_2)$ } matrix Rep.

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we have

$[\alpha u + \beta v, \omega] = \alpha \langle u, \omega \rangle + \beta \langle v, \omega \rangle$

LHS $\rightarrow$ $\langle \alpha u + \beta v, \omega \rangle = (\alpha a_1 + \beta b_1) c_1 - (\alpha a_2 + \beta b_2) c_1$
$- (\alpha a_1 + \beta b_1) c_2 + 2(\alpha a_2 + \beta b_2) c_2$
$= \alpha(a_1 c_1 - a_2 c_2 - a_1 c_1 + 2 a_2 c_2)$
$+ \beta(b_1 c_1 - b_2 c_2 - b_1 c_1 + 2 b_2 c_2)$
$= \alpha(u, \omega) + \beta(v, \omega)$

proved

Q. ② Symmetry $\rightarrow$ $\langle u, v \rangle = a_1 b_1 - a_2 b_1 - a_1 b_2 + 2 a_2 b_2$
$= b_1 a_1 - b_1 a_2 - b_2 a_1 + 2 b_2 a_2$
$= \langle v, u \rangle$

③ Positivity $\rightarrow$ $\langle u, u \rangle = a_1 a_1 - a_2 a_1 - a_1 a_2 + 2 a_2 a_2$
$(a_1, a_1)\ (a, a_1)$
$= a_1^2 - 2 a_2 a_1 + 2 a_2^2$

$\Rightarrow \langle u, u \rangle = 0$ if $a_1 = a_2 = 0 \Rightarrow \boxed{u = 0}$

Q. Define Inner Product Space. Let

$u = \langle u_1, u_2 \rangle,\ v = \langle v_1, v_2 \rangle \in R^2$

define $\langle u, v \rangle = 4 u_1 v_1 + u_2 v_1 + 4 u_1 v_2 + 4 u_2 v_2$

~~Let~~ R^2 is inner Pro. Sp. show

$\rightarrow$ Let $u = (u_1, u_2)$
$v = (v_1, v_2)$
$\omega = (\omega_1, \omega_2)$

we to show

① $[\alpha u + \beta v, \omega] = \alpha \langle u, \omega \rangle + \beta \langle v, \omega \rangle$

$\langle \alpha u + \beta v, \omega \rangle = \cancel{\alpha(4u_1+v}$
$= 4(\alpha u_1 + \beta v_1)\omega_1 + (\alpha u_2 + \beta v_2)\omega_1 +$
$4(\alpha u_1 + \beta v_1)\omega_2 + 4(\alpha u_2 + \beta v_2)\omega_2$
$= \alpha(4 u_1 \omega_1 + u_2 \omega_1 + 4 u_1 \omega_2 + 4 u_2 \omega_2)$
$+ \beta(4 v_1 \omega_1 + v_2 \omega_1 + 4 v_1 \omega_2 + 4 v_2 \omega_2)$
$= \alpha(u, \omega) + \beta(v, \omega)$

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(ii) ~~Similar~~ Symm.

$\langle u, v \rangle = 4u_1v_1 + 4u_2v_1 + 4u_1v_2 + 4u_2v_2$

$= 4v_1u_1 + 4v_1u_2 + 4v_2u_1 + 4v_2u_2$

$= \langle v, u \rangle$

(iii) Positivity → $\langle u, u \rangle$

$(u_1, u_2)(u_1, u_2) = 4u_1u_1 + 4u_2u_1 + 4u_2u_1 + 4u_2u_2$

$= 4u_1^2 + \boxed{8}u_2u_1 + 4u_2^2$

$\langle u, u \rangle = 0$ if $u_1 = u_2 = 0 \Rightarrow u = 0$

Q. $\langle u, v \rangle = 4u_1v_1 + 6u_2v_2$

Linearity → $u = (u_1, u_2)$

$v = (v_1, v_2)$

$w = (w_1, w_2)$

$\langle \alpha u + \beta v, w \rangle = 4(\alpha u_1 + \beta v_1)w_1 + 6(\alpha u_2 + \beta v_2)w_2$

$= \alpha(4u_1w_1 + 6u_2w_2) + \beta(4v_1w_1 + 6v_2w_2)$

$= \alpha\langle u, w \rangle + \beta(v, w)$

Symm. → $\langle u, v \rangle = 4u_1v_1 + 6u_2v_2$

$= 4v_1u_1 + 6v_2u_2$

$= \langle v, u \rangle$

Positivity → $\langle u, u \rangle = 4u_1u_1 + 6u_2u_2$

$= 4u_1^2 + 6u_2^2$

$\langle u, u \rangle = 0$ if $u_1 = u_2 = 0 \Rightarrow u = 0$

NOTE → $\begin{bmatrix} u_1 & u_2 \end{bmatrix} \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

(ii) ~~Similar~~ Symm.

$\langle u, v\rangle = 4u_1v_1 + 4u_2v_1 + 4u_1v_2 + 4u_2v_2$

$= 4v_1u_1 + 4v_1u_2 + 4v_2u_1 + 4v_2u_2$

$= \langle v, u\rangle$

(iii) Positivity → $\langle u, u\rangle$

$(u_1, u_2)(u_1, u_2) = 4u_1u_1 + 4u_2u_1 + 4u_2u_1 + 4u_2u_2$

$= 4u_1^2 + \boxed{8}u_2u_1 + 4u_2^2$

$\langle u, u\rangle = 0$ if $u_1 = u_2 = 0 \Rightarrow u = 0$

Q. $\langle u, v\rangle = 4u_1v_1 + 6u_2v_2$

Linearity → $u = (u_1, u_2)$

$v = (v_1, v_2)$

$w = (w_1, w_2)$

$\langle \alpha u + \beta v, w\rangle = 4(\alpha u_1 + \beta v_1)w_1 + 6(\alpha u_2 + \beta v_2)w_2$

$= \alpha(4u_1w_1 + 6u_2w_2) + \beta(4v_1w_1 + 6v_2w_2)$

$= \alpha\langle u, w\rangle + \beta\langle v, w\rangle$

Symm. → $\langle u, v\rangle = 4u_1v_1 + 6u_2v_2$

$= 4v_1u_1 + 6v_2u_2$

$= \langle v, u\rangle$

Positivity → $\langle u, u\rangle = 4u_1u_1 + 6u_2u_2$

$= 4u_1^2 + 6u_2^2$

$\langle u, u\rangle = 0$ if $u_1 = u_2 = 0 \Rightarrow u = 0$

NOTE → $\begin{bmatrix} u_1 & u_2 \end{bmatrix} \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

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1. Matrix A is real symm. matrix.
2. $|A| \geq 0$
3. Diagonal elements > 0

Ex - $\langle u, v \rangle = 4u_1v_1 + 4u_2v_1 + 4u_1v_2 + 4u_2v_2$

$$A = \begin{bmatrix} u_1v_1 & u_1v_2 \\ u_2v_1 & u_2v_2 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$ symm ✓ $|A| \geq 0$ ✓ D.E > 0 ✓

$\Rightarrow$ Inner prodct ✓

Q. find the value of k so that the following is an inner product on R^2

$\langle u, v \rangle = x_1y_1 - 3x_1y_2 + kx_2y_2 - 3x_2y_1$

where $u = (x_1, x_2)$

$v = (y_1, y_2)$.

$\because$ inner space, By positivity

$\langle u, u \rangle \geq 0$

$x_1x_1 - 3x_1x_2 + kx_2x_2 - 3x_2x_1 \geq 0$

$x_1^2 - 6x_1x_2 + kx_2^2 \geq 0$ $\left(\frac{x_1}{x_2}\right)^2 - \frac{6x_1}{x_2} + k \geq 0$

$\Rightarrow$ $\boxed{k \geq 9}$ $36 - 4k \leq 0$ $(b^2 - 4ac)$

Q. Let $\langle \cdot \rangle$ be the standard inner product on R^2. If $u = (1, 2)$ & $v = (-1, 1) \in R^2$, then find $w \in R^2$ sattisfying $\langle u, w \rangle = -1$ & $\langle v, w \rangle = 3$

$\Rightarrow$ $\langle \alpha u + \beta v, w \rangle = \alpha \langle u, w \rangle + \beta \langle v, w \rangle$

$= (\alpha u_1 + \beta v_1)w_1 + (\alpha u_2 + \beta v_2)w_2 = -\alpha + 3\beta$

$\alpha(u_1w_1$

$\langle u, w \rangle = -1$ $\langle v, w \rangle = 3$

$u_1w_1 + u_2w_2 = -1$ $v_1w_1 + v_2w_2 = 3$

$w_1 + 2w_2 = -1$ $-w_1 + w_2 = 3$

$-w_1 + w_2 = 3$ $w_1 = w_2 - 3$

$3w_2 = 2$ $\boxed{w_2 = \frac{2}{3}}$ $w_1 = \frac{2}{3} - 3$

$\boxed{w_1 = -\frac{7}{3}}$

Imp. Result

(1) if $V \in [a, b]$ be the vector space of all continuous f^n defined on $[a, b]$ then inner product on $[a, b]$ is

$$\langle \varphi_1, \varphi_2 \rangle = \int_b^a f(t) \cdot g(t)\, dt$$

where $f(t), g(t) \in [a, b]$

Q: find $\langle f.g \rangle$ where $f(t) = t - 1$
$g(t) = t$
within the range $[0, 1]$

$$\langle f.g \rangle = \int_0^1 f(t) \cdot g(t)\, dt$$

$$= \int_0^1 (t-1)t\, dt$$

$$= \int_0^1 (t^2 - t)\, dt$$

$$= \left[\frac{t^3}{3} - \frac{t^2}{2}\right]_0^1$$

$$= \left[\frac{1}{3} - \frac{1}{2}\right] = -\frac{1}{6}$$

Norm :- (OR Length of a vector)
Let V be an inner product space and $u \in V$ then $\sqrt{(u.u)} = \sqrt{\langle u, u \rangle}$ is called norm or length of u denoted by $\|u\|$.

NOTE → (1) if Norm = 1 $(\|u\| = 1)$ then u is called unit vector.
(2) Every non-zero vector u is an inner product space

v can be normalized by multiplying it by reciprocal of its length i.e. $u = \frac{1}{\|u\|}$

③ $\|u\| \geq 0$ & $\|u\| = 0$ iff $u = \vec{0}.v$
④ $\|au\| = |a|\,\|u\|$ $\forall\, a \in \mathbb{R}$.
⑤ $|<u+v>| \leq \|u\| + \|v\|$ (Triangle inequality)
⑥ If $\|u+v\| = \|u\| + \|v\|$

$\Rightarrow$ u & v are L.D vectors.

but Converse need not be True.

Distance b/w u & v :-

if $\vec{v}$ is an inner product space then distance b/w two points or vectors can be expressed as $u = (u_1, u_2, \ldots, u_n)$

$v = (v_1, v_2, \ldots, v_n)$

$d(u,v) = \|u-v\| = \sqrt{<(u-v)>}$

Properties :-

① $d(u,v) > 0$
② $d(u,v) = 0$ if $u = v$
③ $d(u,v) < d(u,w) + d(v,w)$

Ex. Let $u = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$, $v = \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix}$, find their inner product & length of each.

Sol^n ① $<u,v> = u^T \cdot v = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix} = 10 - 3 + 4 = 11$

Inner Product

$\|u\| = \text{length} = \sqrt{<u,v>}$

② $<u,u> = 2\times2 + 3\times3 + 4\times4 = 4+9+16 = 29$

length $= \|u\| = \sqrt{29}$

③ $<v,v> = v_1^2 + v_2^2 + v_3^2 = 5^2 + 1^2 + (-1)^2 = 27$

length $= \|v\| = \sqrt{<v,v>} = \sqrt{27}$

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Ex. Use the inner product $\langle f,g\rangle = \int_0^1 f(x)\cdot g(x)\,dx$

Compute $d(f,g)$, $\|f\|$ & $\|g\|$ where $f(x) = x$, $g(x) = e^x$ is $C'[0,1]$

$\Rightarrow$ $\cancel{\|f\|}$ $\cancel{\langle f,g\rangle = \int_0^1 x e^x\,dx = \left[e^x(x-1)\right]_0^1 = 1}$

$$d(f,g) = \|f-g\| = \sqrt{\langle f-g\rangle}$$

$$\langle f,f\rangle = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

$$\|f\| = \sqrt{\langle f,f\rangle} = \frac{1}{\sqrt{3}}$$

$$\langle g,g\rangle = \int_0^1 e^{2x}\,dx = \left[\frac{e^{2x}}{2}\right]_0^1 = \frac{e^2}{2} - \frac{1}{2}$$

$$\|g\| = \sqrt{\langle g,g\rangle} = \sqrt{\frac{e^2}{2} - \frac{1}{2}} = \sqrt{\frac{e^2-1}{2}}$$

$$\sqrt{\langle f-g\rangle} = \|f-g\|$$

$$\Rightarrow \langle f-g\rangle = \langle f-g, f-g\rangle$$

$$\int_0^1 (x-e^x)^2\,dx$$

$$= \int_0^1 x^2 + e^{2x} - 2xe^x\,dx$$

$$= \left[\frac{x^3}{3} + \frac{e^{2x}}{2} - 2e^x(x-1)\right]_0^1$$

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$$= \frac{1}{3} + \frac{e^2}{2} - 2\times 0 - \frac{1}{2}\cancel{+}{-}2$$

$$= \frac{e^2}{2} + \frac{2-3\cancel{+}{-}12}{6} = \frac{e^2}{2}\cancel{+}{-}\frac{13}{6} = \frac{3e^2\cancel{+}{-}13}{6}$$

$$\|f-g\| = \sqrt{\frac{e^2}{2} - \frac{13}{6}}$$

Norm On R^n :-

Three important Norms are

1. $\|(a_1, a_2, \dots, a_n)\|_\infty = \max[|a_i|]$ → Norm Infinity
2. $\|(a_1, a_2, \dots, a_n)\|_1 = |a_1| + |a_2| + \dots + |a_i|$ → Norm one
3. $\|a_1, a_2, \dots, a_n\|_2 = \sqrt{|a_1|^2 + |a_2|^2 + \dots + |a_i|^2}$
   $= \sqrt{\langle u, u\rangle}$ → Norm two

Ex- Consider the vectors $u = (1, -5, 3)$ and $v = (4, 2, -1)$ in R^3.

(i) $\|u\|_\infty = 5$, $\|u\|_1 = 1+5+3 = 9$, $\|u\|_2 = \sqrt{1+25+9}$
$= \sqrt{35}$

(2) $\|v\|_\infty = 4$, $\|v\|_1 = 4+2+1 = 7$, $\|v\|_2 = \sqrt{16+4+1}$
$= \sqrt{21}$

(3) $d(u, v)_\infty = \|u - v\|$
$d(u, v)_1 =$
$d(u, v)_2 =$

Norm On $C[a, b]$

Consider the vector space $V = C[a, b]$ of real cont^n f^n on the interval $a \le t \le b$

$$\langle f, g\rangle = \int_a^b f(t)\, g(t)\, dt$$

i.e.

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① $\|f\|_\infty = \max(|f(t)|)$

② $\|f\|_1 = \int_a^b |f(t)|\,dt$

③ $\|f\|_2 = \sqrt{\langle f, f\rangle} = \sqrt{\int_a^b [f(t)]^2\,dt}$

Geometrically - ① $\|f\|_1$ = Area b/w f^n $|f|$ and 't' axis

② $d(f,g)$ = area b/w f and g.

Orthogonality

Orthogonal Vectors :-

Let V be an inner product space $u, v \in V$. Then u is said to be Orthogonal to v if $\langle u, v\rangle = 0 \Rightarrow u \perp v$

NOTE - Every vector in V is orthogonal to the null vector.

NOTE - Only Null Vector is Orthogonal to itself.

- Two orthogonal are $\perp^r$ to each other.
- A vector u is said to be $\perp^r$ (orthogonal) to subspace S of inner product space V if it is Orthogonal to every vector in S.
i.e $\langle u, v\rangle = 0 \quad \forall\ v \in S$.

Ex. - Let $u = (1, 2, -3)$
$v = (1, 1, 1)$ $\quad w = (-1, 4, -3)$ in R^3

for Orthogonality

$\langle u, v\rangle = 1\times1 + 2\times1 + (-3)\times1 = 0 \Rightarrow u \perp v$

$\langle u, w\rangle = 1\times-1 + 2\times4 + -3\times-3 = 16 \neq 0$ Not Orthogonal vector

$\langle v, w\rangle = 1\times-1 + 1\times4 - 3\times1 = 0 \Rightarrow v \perp w$

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Q. Let $[-\pi, \pi]$ be the inner product space of all Cont^n f^n defined on $[-\pi, \pi]$ with inner prodct defined by $\langle f, g\rangle = \int_{-\pi}^{\pi} f(t)\,g(t)\,dt$.
P.T $\sin t$ & $\cos t$ are orthogonal f^n in $[-\pi, \pi]$

$$\langle \sin t, \cos t\rangle = \int_{-\pi}^{\pi} \underbrace{\overset{\text{odd}}{\sin t} \times \overset{\text{even}}{\cos t}}_{\text{odd}}\,dt$$

$$= 0$$

$$\text{OR} = \frac{1}{2}\int_{-\pi}^{\pi} \sin 2t\,dt$$

$$= -\frac{1}{4}\left[\cos 2t\right]_{-\pi}^{\pi}$$

$$= -\frac{1}{4}\left[\cos 2\pi - \cos 2\pi\right]$$

$$= 0$$

Q. Consider inner Prdt space R^n with std. innner product if $u = (3, 2, k, -5)$
value of k $v = (1, k, 7, 3)$ are Orthogonal - find value of k for orthogonality of u & v.

→ $\langle u, v\rangle = 0$

$$3 \times 1 + 2k + 7k - 15 = 0$$

$$3 + 9k - 15 = 0$$

$$9k = 12$$

$$k = \frac{12}{9} = \frac{4}{3}$$

$$\boxed{k = \frac{4}{3}}$$

Q. $A = \begin{bmatrix} -2 & a & 3 \\ 1 & 0 & -2 \end{bmatrix}$ A & B are orthogonal find a?

$B = \begin{bmatrix} -4 & 3 & a \\ 5 & 7 & 2 \end{bmatrix}$

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$\langle A, B\rangle = 8 + 3a + 1 + 3a + 5 + 0 - 4 = 0$ for orthog.

$$\boxed{a = \frac{3}{2}}$$

Orthogonal sets :- Let V(F) be an inner product space. A set S of non-zero vectors in V(R) is called an Orthogonal set, if each pair of vectors in S is pair of orthogonal vectors.

Orthogonal Basis :- Let V(F) be an inner product space. An orthogonal set of vectors in V is called an orthogonal Basis of V if it is a basis of V. if it is a basis of V.

NOTE - Let V(F) be an inner product space and $\{v_1, v_2, \dots, v_n\}$ be an Orthogonal Basis for V(F). Then for every u & v

$$u = \frac{\langle u, v_1\rangle}{\langle v_1, v_1\rangle} v_1 + \frac{\langle u, v_2\rangle}{\langle v_2, v_2\rangle} v_2 + \dots + \frac{\langle u, v_n\rangle}{\langle v_n, v_n\rangle} v_n$$

$$= \sum_{i=1}^{n} \frac{\langle u, v_i\rangle}{\langle v_i, v_i\rangle} v_i$$

here $\left\{ \frac{\langle u, v_1\rangle}{\langle v_1, v_1\rangle}, \frac{\langle u, v_2\rangle}{\langle v_2, v_2\rangle}, \dots \right\}$

are known as coordinates of u relative to basis S.

Ex- Let S be the set of vectors $v_1 = (1, 2, 1)$, $v_2 = (2, 1, -4)$ & $v_3 = (3, -2, 1)$ in the inner product space R^3 with standard inner product

(i) Show that S is Orthogonal and is the basis for R

(ii) find coordinates of vector (7, 1, 9) relative to Basis S.

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(i) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -2 \\ 1 & -4 & 1 \end{bmatrix}$

$|A| = -42 \neq 0$

$\Rightarrow$ Linearly independent

$S = \{v_1, v_2, v_3\}$ is L.I set of vectors.

$\Rightarrow$ S form Basis for R^3.

To show Orthogonal Basis -

$\langle v_1, v_2 \rangle = 1\times 2 + 2\times 1 + 1\times(-4) = 0$

$\langle v_2, v_3 \rangle = 3-4+1 = 0$

$\langle v_2 v_3 \rangle = 6-2-4 = 0$

$\Rightarrow$ S is orthogonal Basis for R^3.

Coordinates of $u = (7,1,9)$ w.r.t S be (a,b,c)

$a = \dfrac{\langle u, v_1 \rangle}{\langle v_1, v_1 \rangle} = \dfrac{(7+2+9)}{(1+4+1)} = 3$

$b = \dfrac{\langle u, v_2 \rangle}{\langle v_2, v_2 \rangle} = \dfrac{14+1-36}{4+1+16} = \dfrac{-21}{21} = -1$

$c = \dfrac{\langle u, v \rangle}{\langle v_3, v_3 \rangle} = \dfrac{21-2+9}{9+4+1} = \dfrac{28}{14} = 2$

Coordinates of u w.r.t S = $(3, -1, 2)$

Ex - Consider the inner product space R^4 with st. inner product. Let $S = \{v_1, v_2, v_3, v_4\}$ be a subset of R^4, where $v_1 = (1,1,0,-1)$ $v_2 = (1,2,1,3)$ $v_3 = (1,1,-9,2)$ & $v_4 = (16,-13,1,3)$

Show that i) S is orthogonal & Basis of R^4

(ii) find coordinates of $u = (a,b,c,d)$ in R^4 relative to Basis S.

$\rightarrow$ ~~To show Orthogonal Basis~~ -

$A = \begin{bmatrix} 1 & 1 & 1 & 16 \\ 1 & 2 & 1 & -13 \\ 0 & 1 & -9 & 1 \\ -1 & 3 & 2 & 3 \end{bmatrix}$

$R_4 \to R_4 + R_1$, $R_2 \to R_2 - R_1$,

$$\begin{bmatrix} 1 & 1 & 1 & 16 \\ 0 & 1 & 0 & -29 \\ 0 & 1 & -9 & 1 \\ 0 & 4 & 3 & 19 \end{bmatrix}$$

$R_4 \to R_4 - 4R_2$, $R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 1 & 16 \\ 0 & 1 & 0 & -29 \\ 0 & 0 & 3 & 30 \\ 0 & 0 & -9 & +135 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 16 \\ 0 & 1 & 0 & -29 \\ 0 & 0 & 3 & 30 \\ 0 & 0 & 0 & 145 \end{bmatrix}$$

$\rho(A) = 4$ = no. of vectors

L.I. vectors

S form Basis

$\langle v_1, v_2 \rangle = 1 + 2 + 0 - 3 = 0$

$\langle v_1, v_3 \rangle = 1 + 1 + 0 - 2 = 0$

$\langle v_1, v_4 \rangle = 16 - 13 + 0 - 3 = 0$

$\langle v_2, v_3 \rangle = 1 + 2 - 9 + 6 = 0$

$\langle v_2, v_4 \rangle = 16 - 26 + 1 + 9 = 0$

$\langle v_3, v_4 \rangle = 16 - 13 - 9 + 6 = 0$

To find Coordinates of $u = (a, b, c, d)$ relative to S

$A = \dfrac{\langle u\, v_1 \rangle}{\langle v_1, v_1 \rangle} = \dfrac{a+b-d}{1+1+0+1} = \dfrac{a+b-d}{3}$

$B = \dfrac{\langle u\, v_2 \rangle}{\langle v_2\, v_2 \rangle} = \dfrac{a+2b+c+3d}{1+4+1+9} = \dfrac{a+2b+c+3d}{15}$

$C = \dfrac{\langle u\, v_3 \rangle}{\langle v_3\, v_3 \rangle} = \dfrac{a+b-9c+2d}{1+1+81+4} = \dfrac{a+b-9c+2d}{87}$

$d = \dfrac{\langle u\, v_4 \rangle}{\langle v_4\, v_4 \rangle} = \dfrac{16a-13b+c+3d}{256+169+1+9} = \dfrac{a+b-9c+2d}{435}$

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Q. Let $v_1 = (1,-2,3,4)$ and $v_2 = (3,-5,7,8)$ be two vectors in inner product space R^4. find Basis of the subspace of R^4. that is orthogonal to v_1 & v_2.

⇒ Let u be the $= (x, y, z, t)$, v_1 & v_2 form orthogonal Basis for R^4.

Then $\langle u, v_1 \rangle = 0$ & $\langle u, v_2 \rangle = 0$

$x - 2y + 3z + 4t = 0$ — ①

$3x - 5y + 7z + 8t = 0$ — ②

$$A = \begin{bmatrix} 1 & -2 & 3 & 4 \\ 3 & -5 & 7 & 8 \end{bmatrix} \quad R_2 \to R_2 - 3R_1$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & -2 & 4 \end{bmatrix} \quad \rho(A) = 2, \; n = 4$$

$n - r = 4 - 2 = 2$

$x - 2y + 3z + 4t = 0$ Let $t = k_1$

$y - 2z - 4t = 0$ $z = k_2$

~~Let $k_1 = 0$, $k_2 = 1$~~ $y = 2z + 4t$

~~from ② $y = 2k$~~ $y = 2k_2 + 4k_1$

from ① $x = 2y - 3z - 4t$

$x = 2(2k_2 + 4k_1) - 3k_2 - 4k_1$

$\boxed{x = k_2 + 4k_1}$ ~~$k_1 = 0$~~ ~~$k_2 = 0$~~

$k_1 = 0$, $k_2 = 1$	$k_1 = 1$, $k_2 = 0$
$x = 1$	$x = 4$
$y = 2$	$y = 4$
$z = 1$	$z = 0$
$t = 0$	$t = 1$

$$A = \begin{bmatrix} 1 & 3 & 1 & 4 \\ -2 & -5 & 2 & 4 \\ 3 & 7 & 1 & 0 \\ 4 & 8 & 0 & 1 \end{bmatrix}$$

$$\boxed{\begin{aligned} u_1 &= (1,2,1,0) \\ u_2 &= (4,4,0,1) \end{aligned}}$$

Basis $= \{v_1, v_2, u_1, u_2\}$

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Q. Let S be the subspace of R^4 & Orthogonal to the vectors $v_1 = (1, 1, 2, 2)$ & $v_2 = (0, 1, 2, -1)$ find Orthogonal Basis of R^4

$\rightarrow$ Let $S = \{ u = (x, y, z, t)$ and $(u, v_1) = 0$ & $(u, v_2) = 0 \}$

$$\Rightarrow A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 2 & -1 \end{bmatrix}$$

$\rho(A) = 2$ but no. of vector is 4

Hence arbitrary vector be $t = k_1$, $z = k_2$

Let $u = (x, y, z, t)$ be the orthogonal vector such that :-

$$\langle u, v_1 \rangle = 0 \Rightarrow x + y + 2z + 2t = 0$$
$$\langle u, v_2 \rangle = 0 \Rightarrow \underline{y + 2z - t = 0}$$
$$x + 3t = 0$$

$$y + 2(k_2) - k_1 = 0$$
$$y = k_1 - 2k_2$$
$$x + 3k_1 = 0$$
$$x = -3k_1$$

$\therefore x = -3k_1$, $y = k_1 - 2k_2$, $z = k_2$, $t = k_1$

$k_1 = 1$, $k_2 = 0$

$x = -3$
$y = 1$
$z = 0$
$t = 1$

$k_1 = 0$, $k_2 = 1$

$x = 0$
$y = -2$
$z = 1$
$t = 0$

$$B = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 1 & 1 & 1 & -2 \\ 2 & 2 & 0 & 1 \\ 2 & -1 & 1 & 0 \end{bmatrix}_{4 \times 4}$$

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$$\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 4 & -2 \\ 0 & 2 & 6 & 1 \\ 0 & -1 & 6 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 4 & -2 \\ 0 & 0 & -2 & 5 \\ 0 & 0 & 10 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 4 & -2 \\ 0 & 0 & -2 & 5 \\ 0 & 0 & 0 & 23 \end{bmatrix}$$

Projection :-

Projection of a Vector :-

Let V(F) be an inner Product space, the projection of vector v along a non-zero vector w is the scalar multiple cw of w such that (v-cw) is orthogonal to w.

i.e. $\langle v - cw, w \rangle = 0$

$= \langle v, w \rangle - c\langle w, w \rangle = 0$

$\Rightarrow c = \dfrac{\langle v, w \rangle}{\langle w, w \rangle}$

The projection of a vector v along a non-zero vector w is denoted by proj $\langle v, w \rangle$

$\text{proj}\langle v, w \rangle = c = \dfrac{\langle v, w \rangle}{\langle w, w \rangle} \cdot w$

and scalar

$c = \dfrac{\langle v, w \rangle}{\langle w, w \rangle}$ is called Compound of

v and w or fourier Coefficient of v w.r.t. w.

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* Projection of v in the direction w is $= c.w$

$\text{proj}(v,w) = \dfrac{\langle v,w\rangle}{\langle w,w\rangle}.w$ where $c = \dfrac{\langle v,w\rangle}{\langle w,w\rangle}$

Component of v along w or fourier Coeffici (c)

Ex. find Projection of $v = (1,-2,3,-4)$ along a vector $w = (1,2,1,2)$ in the inner Product Space R^4 with standard inner Product.

→ Proj^n of v along $w = \dfrac{\langle v,w\rangle}{\langle w,w\rangle} w$

$$\text{Proj}(v,w) = \frac{1\times1 + 2\times(-2) + 3\times1 + (-4)\times2}{1+4+1+4}(1,2,1,2)$$

$$= \frac{-8}{10}(1,2,1,2)$$

$$= \frac{-4}{5}, \frac{-8}{5}, \frac{-4}{5}, \frac{-8}{5}$$

Ex. Let $V = P_2(t)$ be the inner product space of all polynomial of degree less than or equa to 2 with inner product defined by $\langle f,g\rangle = \int_0^1 f(t)g(t)\,dt$ If $f(t) = t^2$ and $g(t) = t+3$ are two Polynomials in $P_2(t)$ find Projection of $f(t)$ along $g(t)$.

$$\text{Proj}(f,g) = \frac{\langle f,g\rangle}{\langle g,g\rangle} g \quad \text{—— (1)}$$

$$\langle f,g\rangle = \int_0^1 f(t)g(t)\,dt = \int_0^1 t^2(t+3)\,dt$$

$$= \left[\frac{t^4}{4} + \frac{3t^3}{3}\right]_0^1 = \frac{1}{4} + 1 = \frac{5}{4}$$

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$\langle g, g \rangle = \int_0^1 g(t) \cdot g(t)\,dt = \int_0^1 (t+3)^2\,dt$

$= \left[\frac{(t+3)^3}{3}\right]_0^1 = \frac{64}{3} - 9 = \frac{37}{3}$

$\text{proj}[f, g] = \frac{5/4}{37/3}(t+3) = \frac{5 \times 3}{37 \times 4}(t+3)$

Q. In the inner product space $R^{2\times2}$ of all 2×2 matrices over R with inner product defined by $\langle A, B \rangle = tr[B^T A]$ find Proj of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ along $B. = \begin{bmatrix} 1 & 1 \\ 5 & 5 \end{bmatrix}$

$\langle A, B \rangle = 1 + 1\times2 + 3\times5 + 4\times5 = 38$

$\langle B, B \rangle = 1\times1 + 1\times1 + 5\times5 + 5\times5 = 52$

Proj of A along $B = \frac{\langle A, B \rangle}{\langle B, B \rangle} . B = \frac{\cancel{38}^{19}}{\cancel{52}_{26}} \begin{bmatrix} 1 & 1 \\ 5 & 5 \end{bmatrix}$

Q. If S be the subspace of inner product Space R^4 spanned by $v_1 = (1, 1, 1, 1)$ and $v_2 = (-1, 0, 1, 0)$ find projection of u onto S. $u = (1, 2, 5, 7)$

Sol^n → We know v_1 & v_2 are orthogonal vectors and so they form an orthogonal Basis for S.

$\text{Proj}(u, S) = \frac{\langle u, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 + \frac{\langle u, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2$

$= \frac{1+2+5+7}{1+1+1+1}(1,1,1,1) + \frac{-1+0+5+0}{1+0+1+0}(-1,0,1,0)$

$= \frac{15}{4}(1,1,1,1) + \frac{\cancel{4}\,2}{\cancel{2}}(-1,0,1,0)$

$= \left(\frac{15}{4} - 2, \frac{15}{4} + 0, \frac{15}{4} + 2, \frac{15}{4} + 0\right) = \left(\frac{7}{4}, \frac{15}{4}, \frac{23}{4}, \frac{15}{4}\right)$

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Q. Consider the inner product space $P(t)$ of all Polynomial with inner product $\langle f, g \rangle = \int_0^1 f(t) g(t)\, dt$. Let S be a subspace of $P(t)$ spanned by set $\{1, 2t-1, 6t^2-6t+1\}$ find projection of $f(t) = t^3$ onto S.

→ $Proj(t, S) = \frac{\langle f, f_1 \rangle}{\langle f_1, f_1 \rangle} f_1 + \frac{\langle f, f_2 \rangle}{\langle f_2, f_2 \rangle} f_2 + \frac{\langle f, f_3 \rangle}{\langle f_3, f_3 \rangle} f_3$

Orthonormal Vectors -

Two vectors X & Y are said to be orthonormal if (i) Inner product of X & Y is zero i.e. $\langle X, Y \rangle = 0$ → Orthogonal

(ii) They must be of unit length i.e. $\|X\| = 1$ & $\|Y\| = 1$

NOTE - Every Orthonormal set of vectors is Orthogonal but converse need not be true.

NOTE - To make orthonormal to an orthogonal set of vectors we should normalised these vectors (i.e. divide by their norm)

Ex - Let $X_1 = (1, 2, -1)$, $X_2 = (2, 1, 4)$ & $X_3 = (3, -2, -1)$ in R^3 Then show that they form an Orthogonal set under the standard Euclidean inner product for R^3 but not an orthonormal set

(ii) Convert them into a set of vectors under std. Euclidean inner product for R^3.

(i) For Orthogonal

Orthonormal sb ki rank 1 honi chahiye)
Norm = 1

$\langle x_1, x_2 \rangle = 2+2-4 = 0$
$\langle x_1, x_3 \rangle = 3-4+1 = 0$ $\Rightarrow$ Orthogonal set of vectors
$\langle x_2, x_3 \rangle = 6-2-4 = 0$

(iii) $\|x_1\| = \sqrt{\langle x_1, x_1 \rangle} = \sqrt{1+4+1} = \sqrt{6} \neq 1$

$\Rightarrow$ Not Orthonormal set of vectors.

To Construct Orthonormal set :-

$$\frac{\langle x_1 \rangle}{\|x_1\|} = \frac{(1,2,-1)}{\sqrt{6}} = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right)$$

$$\frac{\langle x_2 \rangle}{\|x_2\|} = \frac{(2,1,4)}{\sqrt{21}} = \left(\frac{2}{\sqrt{21}}, \frac{1}{\sqrt{21}}, \frac{4}{\sqrt{21}}\right)$$

$$\frac{\langle x_3 \rangle}{\|x_3\|} = \frac{(3,-2,-1)}{\sqrt{14}} = \left(\frac{3}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}\right)$$

Q. Consider the inner Product Space $P(t)$ of all polynomials with inner product $\langle f, g \rangle = \int_0^1 f(t)\,g(t)\,dt$. Let S be the subspace of $P(t)$ spanned by set $\{\underset{v_1}{1}, \underset{v_2}{2t-1}, \underset{v_3}{6t^2-6t+1}\}$ find the proj. of $f(t) = t^3$ onto S.

$$\text{proj}(f, S) = \frac{\langle f, v_1 \rangle}{\langle v_1 v_1 \rangle} \cdot v_1 + \frac{\langle f, v_2 \rangle}{\langle v_2 v_2 \rangle} \cdot v_2 + \frac{\langle f, v_3 \rangle}{\langle v_3 v_3 \rangle} \cdot v_3$$

$$\langle f, v_1 \rangle = \int_0^1 f \cdot v_1\,dt = \int_0^1 t^3\,dt = \frac{1}{4}$$

$$\langle v_1, v_1 \rangle = \int_0^1 1\,dt = 1$$

$$\langle f, v_2 \rangle = \int_0^1 t^3(2t-1) = \frac{2}{5} - \frac{1}{4} = \frac{3}{20}$$

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$\langle v_2, v_2 \rangle = \int_0^1 (2t-1)^2\,dt = \left[\frac{(2t-1)^3}{3\times 2}\right]_0^1 = \frac{1}{6} + \frac{1}{6}$

$= \frac{1}{3}$

$\langle f, v_3 \rangle = \int_0^1 t^3(6t^2-6t+1)\,dt = \int_0^1 (6t^5-6t^4+t^3)\,dt$

$= \left[t^6 - \frac{6t^5}{5} + \frac{t^4}{4}\right]_0^1 dt$

$= 1 - \frac{6}{5} + \frac{1}{4} - 0 = \frac{20-24+5}{20} = \frac{1}{20}$

steps.

$\langle v_3, v_3 \rangle = \int_0^1 (6t^2-6t+1)^2\,dt$

$= \frac{1}{5}$

$\text{proj}(f, S) = \frac{1/4}{1}\cdot 1 + \frac{3/20}{1/3}(2t-1) + \frac{1/20}{1/5}(6t^2-6t+1)$

$= \frac{1}{4} + \frac{9}{20}(2t-1) + \frac{1}{4}(6t^2-6t+1)$

$= \frac{3}{2}t^2 - \frac{3t}{2} + \frac{1}{4} + \frac{9}{10}t - \frac{9}{20}$

$= \frac{3}{2}t^2 - \frac{3\cancel{6}t}{\cancel{10}_5} - \frac{4}{20}$

$= \frac{3}{2}t^2 - \frac{3t}{5} - \frac{1}{5}$

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Gram-Schmidt Orthogonalization Process :-

Every finite dimensional inner product space has an Orthogonal Basis.

steps → To find an Orthogonal / Orthonormal Basis from Ordinary Basis.

Let $B = \{v_1, v_2, \dots, v_n\}$ be an Ordinary Basis of V (inner product space)

Suppose $B' = \{w_1, w_2, \dots w_n\}$ be an Orthogonal Basis of V.

Let $\boxed{w_1 = v_1}$

$$w_2 = v_2 - \frac{\langle v_2 w_1 \rangle}{\langle w_1 w_1 \rangle} w_1$$

$$w_3 = v_3 - \frac{\langle v_3 w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3 w_2 \rangle}{\langle w_2 w_2 \rangle} w_2$$

$$w_n = v_n - \frac{\langle v_n w_1 \rangle}{\langle w_1 w_1 \rangle} w_1 - \frac{\langle v_n w_2 \rangle}{\langle w_2 w_2 \rangle} w_2 \dots \frac{\langle v_n w_{n-1} \rangle}{\langle w_{n-1} w_{n-1} \rangle} w_{n-1}$$

For Orthonormal Basis

$$w_1 = \frac{w_1}{\|w_1\|}$$

$$w_2 = \frac{w_2}{\|w_2\|}$$

$$\vdots$$

$$w_i = \frac{w_i}{\|w_i\|}$$

$$B'' = \left\{ \frac{w_1}{\|w_1\|}, \frac{w_2}{\|w_2\|}, \dots, \frac{w_n}{\|w_n\|} \right\}$$

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Q. Apply Gram-Schmidt Orthogonalization Process to the inner product space R^3. find an orthonormal Basis of R^3. find an orthonormal Basis of R^3.

Let

$v_1 = (1, 0, 1)$ $\qquad B' = \{w_1, w_2, w_3\}$

$v_2 = (1, 0, -1)$ $\qquad$ Let $w_1 = v_1 = (1, 0, 1)$

$V_3 = (0, 3, 4)$ $\qquad \boxed{w_1 = (1, 0, 1)}$

Now, $w_2 = V_2 - \dfrac{\langle v_2\, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1$

$w_2 = (1, 0, -1) - \dfrac{(1+0-1)}{(1+0+1)} w_1$

$\boxed{w_2 = (1, 0, -1)}$

$w_3 = V_3 - \dfrac{\langle V_3\, w_1 \rangle}{\langle w_1 w_1 \rangle} w_1 - \dfrac{\langle V_3 w_2 \rangle}{\langle w_2 w_2 \rangle} w_2$

$= (0, 3, 4) - \dfrac{4}{2} 2 (1, 0, 1) + \dfrac{4}{2} 2 (1, 0, -1)$

$= (0, 3, 0)$

hence Orthogonal Basis R^3

$B' = [w_1, w_2\ w_3] = [(1, 0, 1), (1, 0, -1), (0, 3, 0)]$

Orthonormal Basis of R^3

$B'' = \left\{ \dfrac{w_1}{\|w_1\|}, \dfrac{w_2}{\|w_2\|}, \dfrac{w_3}{\|w_3\|} \right\}$

$B'' = \left\{ \dfrac{(1, 0, 1)}{\sqrt{2}}, \dfrac{(1, 0, -1)}{\sqrt{2}}, \dfrac{(0, 3, 0)}{3} \right\}$

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$B'' = \left\{ \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right), (0, 1, 0) \right\}$

Ex- Let $V = P_3(t)$ be the vector space of all polynomials $f(t)$ of degree less than and equal to 3. with inner product defined by $\langle f, g \rangle = \int_{-1}^{1} f(t)\, g(t)\, dt$. Apply Gram-Schmidt Orth Process. find Orthonormal Basis of v.

→ Let $B = \{1, t, t^2, t^3\}$

$v_1 = 1, \quad v_2 = t, \quad v_3 = t^2, \quad v_4 = t^4$

Let $B' = \{w_1, w_2, w_3, w_4\}$

let

$w_1 = v_1 \Rightarrow \boxed{w_1 = 1}$

$$w_2 = v_2 - \frac{\langle v_2 w_1 \rangle}{\langle w_1 w_1 \rangle} w_1 \quad —(1)$$

$\langle v_2 w_1 \rangle = \int_{-1}^{1} v_2 w\, dt = \int_{-1}^{1} t\, dt = \cancel{\frac{2}{2}} = 0$ (odd f^n)

$\langle w_1, w_1 \rangle = \int_{-1}^{1} 1\, dt = 2$

from (1) $\quad w_2 = t \cancel{\frac{1}{2}}$

$$w_3 = v_3 - \frac{\langle v_3 w_1 \rangle}{\langle w_1 w_1 \rangle} w_1 - \frac{\langle v_3 w_2 \rangle}{\langle w_2 w_2 \rangle} w_2 - (2)$$

we have

$\langle v_3, w_1 \rangle = \int_{-1}^{1} t^2 dt = 2\int_{0}^{1} t^2 dt = \frac{2}{3}$ (even

$\langle w_2, w_2 \rangle = \int_{-1}^{1} \left(t \cancel{\frac{1}{2}}\right)^2 dt = \frac{2}{3}$

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$\langle V_3, w_3 \rangle = \int_{-1}^{1} t^2 \cdot t \, dt = \int_{-1}^{1} t^3 dt = 0$

from eq^n 2

$$w = t^2 - \frac{2/3}{(2)} \cdot 1 - 0$$

$$w = t^2 - \frac{1}{3}$$

$$\langle v_3 \; w_4 \rangle = V_4 - \frac{\langle V_4 w_1 \rangle}{\langle w_1 w_1 \rangle} w_1 - \frac{\langle v_4 w_2 \rangle}{\langle w_2 w_2 \rangle} -$$

$$\frac{\langle v_4 \; w_3 \rangle}{\langle w_3 w_3 \rangle}$$

$$= t^3 - \frac{3}{5} t$$

Orthonormal Basis

$$B'' = \left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}} t, \frac{1}{2}\sqrt{\frac{5}{2}} (3t^2 - 1), \frac{1}{2}\sqrt{\frac{5}{7}} (5t^2 - 3t) \right\}$$

$4 + \frac{4}{9} + \frac{1}{36} + \frac{49}{36}$

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$\frac{36 \times 4 + 16 + 1 + 49}{36} = \frac{210}{36}$

Q. Let S be the subspace of R^4 spanned by vectors $v_1 = (1,1,1,1)$, $v_2 = (1,-1,2,2)$, $v_3 = (1,2,-3,-4)$. Apply Gram-Schmidt Orth process to find an orthogonal Basis and then an orthonormal Basis of S. hence find projection of $v = (1,2,-3,4)$ on S.

→ Let $S' = \{w_1, w_2, w_3\}$ be the required orthogonal Basis for R^3.

$w_1 = v_1 \Rightarrow w_1 = (1,1,1,1)$

$w_2 = v_2 - \frac{\langle v_2 w_1 \rangle}{\langle w_1 w_1 \rangle} w_1$

$w_2 = (1,-1,2,2) - \frac{\langle (1,-1,2,2)(1,1,1,1) \rangle}{1+1+1+1} (1,1,1,1)$

$= (1,-1,2,2) - \frac{1-1+2+2}{4} (1,1,1,1)$

$= (0,-2,1,1)$

$w_3 = v_3 - \frac{\langle v_3 w_1 \rangle}{\langle w_1 w_1 \rangle} w_1 - \frac{\langle v_3 w_2 \rangle}{\langle w_2 w_2 \rangle} w_2$

$= (1,2,-3,-4) - \frac{1+2-3-4}{4} (1,1,1,1) - \frac{0-4-7}{6} (0,-2,1,1)$

$= (1,2,-3,-4) + (1,1,1,1) + \frac{11}{6} (0,-2,1,1)$

$= \left(2, \frac{-2}{3}, \frac{-1}{6}, \frac{-7}{6}\right)$

$S' = \{w_1, w_2, w_3, \cancel{w_4}\}$ → orthogonal set

$S'' = \left\{ \frac{w_1}{\|w_1\|}, \frac{w_2}{\|w_2\|}, \frac{w_3}{\|w_3\|}, \frac{\cancel{w_4}}{\|w_4\|} \right\}$ → orthonormal

$= \left\{ \frac{(1,1,1,1)}{2}, \frac{(0,-2,1,1)}{\sqrt{6}}, \frac{(2,-2/3,-1/6,-7/6)}{\sqrt{\frac{210}{36}} = \frac{\sqrt{210}}{6}} \right.$

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Projection

$$(s' \ v) = \frac{\langle v, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 + \frac{\langle v, w_1 \rangle}{\langle w_2 w_2 \rangle} w_2 + \frac{\langle v, w_3 \rangle}{\langle w_3 w_3 \rangle} w_3$$

Q. Let S be the subspace of an inner product space R^4, spanned by vectors $v_1 = (1,1,1,1)$ $v_2 = (1,2,4,5)$ and $v_3 = (1,-3,-4,-2)$ in R^4, find orthogonal Basis & then Orthonormal Basis of S. (2)

→ Let $S' = \{w_1, w_2, w_3\}$

$v_1 = w_1 = (1,1,1,1)$

$$w_2 = v_2 - \frac{\langle v_2 \ w_1 \rangle}{\langle w_1, w_1 \rangle} w_1$$

$$= (1,2,4,5) - \frac{1+2+4+5}{4}(1,1,1,1)$$

$$= (1,2,4,5) - 3(1,1,1,1)$$

$$= (-2,-1,1,2)$$

$$w_3 = v_3 - \frac{\langle v_3 \ w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3 w_2 \rangle}{\langle w_2 w_2 \rangle} w_2$$

$$= (1,-3,-4,-2) - \frac{1-3-4-2}{4}(1,1,1,1) - \frac{-2+3-4-4}{4+1+1+4}(-2,-1,1,2)$$

$$= (1,-3,-4,-2) + 2(1,1,1,1) + \frac{7}{10}(-2,-1,1,2)$$

$$= \left(\frac{8}{5}, \frac{-17}{10}, \frac{-13}{10}, \frac{7}{5}\right)$$

$S' = \{w_1 \ w_2 \ w_3\}$ → orthogonal

$S'' = \left\{ \frac{w_1}{\|w_1\|}, \frac{w_2}{\|w_2\|}, \frac{w_3}{\|w_3\|} \right\}$ → orthonormal

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$$= \left\{ \frac{(1,1,1,1)}{2}, \frac{(-2,-1,1,2)}{\sqrt{10}}, \frac{(16,-17,-13,14)}{\sqrt{910}} \right.$$

(2) Let $V = P_2(t)$ be the inner product space of all Polynomials of deg. less than and equal to 2 with inner product $\langle f, g \rangle = \int_0^1 f(t) g(t)\, dt$. If $f(t) = t^2$, $g(t) = 2t+3$ are two poly. in $P_2(t)$, find proj $f(t)$ along $g(t)$.

$\rightarrow$ Let $B = \{1, t, t^2\}$

$v_1 = 1$, $v_2 = t$, $v_3 = t^2$

Let $S = \{w_1, w_2, w_3\}$

$v_1 = w_1 = 1$

$$w_2 = v_2 - \frac{\langle v_2 w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 = t - \frac{1}{2}$$

$\Rightarrow \langle v_2 w_1 \rangle = \int_0^1 t\, dt = \frac{1}{2}$

$$\text{proj}(f, g) = \frac{\langle f, g \rangle}{\langle g, g \rangle} g.$$

$$\langle f, g \rangle = \int_0^1 t^2 (2t+3) = \left[\frac{\cancel{2} t^4}{\cancel{4}2} + \frac{\cancel{3} t^3}{\cancel{3}} \right]_0^1 = \frac{1}{2} + 1 = \frac{3}{2}$$

$$\langle g, g \rangle = \int_0^1 (2t+3)^2\, dt = \left[\frac{(2t+3)^3}{3 \times 2} \right]_0^1 = \frac{125}{6} - \frac{27}{6}$$

$$= \frac{98}{6} = \frac{49}{3}$$

$$\text{proj}(f, g) = \frac{\cancel{49}}{3} \frac{3 \times 3}{2 \times 49} \cdot (2t+3)$$

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Orthogonal Transformation :-

An orthogonal transformation is the linear transformation that preserves the length of vectors and their inner product

$$\begin{cases} \|u\| = \|T(u)\| \\ \|v\| = \|T(v)\| \\ \text{and} \quad \langle u, v \rangle = \langle T(u), T(v) \rangle \end{cases}$$

NOTE - If Transformation is

$$Y = AX$$

Then if $A A^T = I = A^T . A$ (orthogonal Matrix

i.e. $\boxed{A^T = A^{-1}}$

ex

Q. Let $Q = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $u = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$

Show that inner product preserved after and before transformation OR show that this is Orthogonal.

$$Q_u = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$Q_v = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$$

$$\|u\| = \sqrt{\langle u, v \rangle} = \sqrt{5}$$

$$\|v\| = \sqrt{\langle v, v \rangle} = \sqrt{9+16} = 5$$

$$\|Q_u\| = \sqrt{\langle Q_u \; Q_u \rangle} = \sqrt{4+1} = \sqrt{5}$$

$$\|Q_v\| = \sqrt{16+9} = 5$$

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$\|u\| = \|Qu\|$

$\|v\| = \|Qv\|$

$\langle u, v \rangle = 3 + 8 = 11$

$\langle Qu, Qv \rangle = +8 + 3 = 11$

$\Rightarrow \langle u, v \rangle = \langle Qu, Qv \rangle$

~~$\Rightarrow$ Orthogonal Transformation~~

from (1) & (2)

$\therefore$ this preserves length as well as their inner product

$\Rightarrow$ Orthogonal Transformation.

Ex- Show that following is an orthogonal transformation

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

we have $A = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$

$$A^T = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$A \cdot A^T = \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 & -\sin\theta\cos\theta + \cos\theta\cdot\sin\theta \\ 0 & 1 & 0 \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & 0 & \cos^2\theta + \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \quad \text{Similarly } A^T A = I$$

This is Orthogonal Transformation.

Q. Let $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $\theta = 90°$

$u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $v = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

Check whether Transformation is orthogonal or not.

→ we have $Q = \begin{bmatrix} \cos 90 & -\sin 90 \\ \sin 90 & \cos 90 \end{bmatrix}$

$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$Qu = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$Qv = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

$||u|| = \sqrt{\langle u, u \rangle} = \sqrt{1\times1 + 2\times2} = \sqrt{5}$

$||v|| = \sqrt{\langle v, v \rangle} = \sqrt{9+16} = 5$

$||Qu|| = \sqrt{\langle Qu, Qu \rangle} = \sqrt{4+1} = \sqrt{5}$

$||Qv|| = \sqrt{16+9} = 5$

$||u|| = ||Qu||$ & $||v|| = ||Qv||$

$\Rightarrow$ length preserves — (1)

Also, $\langle u, v \rangle = 1\times3 + 2\times4 = 11$

$\langle Qu, Qv \rangle = -2\times-4 + 1\times3 = 11$

Inner Product Preserves — (2)

from (1) & (2) this is an Orthogonal Transformation

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Q. Let $Q = \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix}$

and $u = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

show that this is orthogonal

Rotation :- A rotation is a special type of orthogonal transformation that
* Preserves the Orientation
* Is represented by an orthogonal matrix with det +1.

Note- All rotation are orthogonal transformation but converse need not be true.

Note- In 2D $R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$: this ~~sol~~ rotates vector counter clockwise by an angle θ.

Note- In 3D, a rotation matrix Q is a special kind of orthogonal ~~wt~~ matrix that preserves length and angle and $\det Q = 1$.

Rotation about x-axis.

$$Q_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

Rotation about Y-axis.

$$Q_Y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

When eq^n of T is given

find matrix of transformation

$$T : V_2(R) \to V_2(R)$$

defined by $T(x, y) = (x, -y)$
w.r.t. standard Basis of $V_2(R)$

Matrix of $T = [T(e_1), T(e_2)]$

standard Basis of $V_2(R)$ is

$e_1 = (1, 0)$ & $e_2 = (0, 1)$

$T(e_1) = T(1, 0) = (1, 0)$

$T(e_2) = T(0, 1) = (0, -1)$

Matrix of T

$$T\,A = [T(e_1)\ T(e_2)] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

When Image of α_1 & α_2 is given

Q find the matrix representation $T : V_2(R) \to V_3$

s.t. $T(-1, 1) = (-1, 0, 2)$

$T(2, 1) = (1, 2, 1)$

Let any vector in V_2 can be express as linear Combination of e_1 & e_2

$$(-1, 1) = -1\underset{e_1}{(1, 0)} + 1\underset{e_2}{(0, 1)}$$

$(-1, 1) = -e_1 + e_2$

Taking transformation.

$T(-1, 1) = -T(e_1) + T(e_2)$

$(-1, 0, 2) = -T(e_1) + T(e_2)$

$(2, 1) = 2(1, 0) + 1(0, 1)$

$= 2e_1 + e_2$

$T(2, 1) = 2T(e_1) + T(e_1) = (1, 2, 1)$

$$-2T(e_1) + 2T(e_2) = (-2, 0, 4)$$
$$2T(e_1) + T(e_2) = (1, 2, 1)$$
$$3T(e_2) = (-1, 2, 5)$$
$$T(e_2) = \left(\frac{-1}{3}, \frac{2}{3}, \frac{5}{3}\right)$$

$$T(e_1) = T(e_2) - (-1, 0, 2)$$
$$= \left(\frac{-1}{3}, \frac{2}{3}, \frac{5}{3}\right) - (-1, 0, 2)$$
$$= \left(\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}\right)$$

Matrix of T w.r.t standard Basis

$$T_A = [T(e_1) \;\; T(e_2)]$$
$$= \begin{bmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ -1/3 & 5/3 \end{bmatrix}$$

Q. Find the matrix of T defined by $T : V_2(R) \to V_2(R)$

s.t. $T(1,2) = (3,0)$

$T(2,1) = (1,2)$

$$\Rightarrow (1,2) = 1(1,0) + 2(0,1)$$
$$= e_1 + 2e_2$$
$$T(1,2) = T(e_1) + 2T(e_2) = (3,0)$$
$$(2,1) = 2e_1 + e_2$$
$$T(2,1) = 2T(e_1) + T(e_2) = (1,2)$$

$$\Rightarrow \quad T(e_1) + 2T(e_2) = (3,0)$$
$$4T(e_1) + 2T(e_2) = (2,4)$$
$$-3T(e_1) = (1,-4)$$
$$T(e_1) = (-1/3, 4/3)$$
$$T(e_2) = (1,2) - (-2/3, 8/3)$$
$$= \left(\frac{5}{3}, \frac{-2}{3}\right)$$