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Unit - 1 Matrix

A matrix is defined as a rectangular arrangement of in rows and columns of scalars subject to certain rules of operations.

Denoted by $A = [a_{ij}]_{m \times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ - & - & - & - \end{bmatrix}$$

Types of matrix

1. Square matrix: No. of rows = No. of columns
2. Row matrix : Having one row and any no. of columns
3. Column matrix : Having one column and any no. of rows
4. Null matrix : Having all elements zero.
5. Diagonal matrix: Non diagonal elements should be zero in a square matrix. i.e. $a_{ij} = 0$ if $i \neq j$
6. Scalar matrix : A diagonal matrix is said to be scalar if $a_{ij} = 0$ when $i \neq j$ and all diagonal elements are equal.

   e.g. $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$
7. Unit matrix: A square matrix is said to be unit matrix if all its non-diagonal elements are zero and diagonal elements are unity.

8. Upper Triangular matrix: A square matrix in which all the elements below principle diagonal are zero is called upper triangular matrix.

e.g. $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 6 \\ 0 & 0 & 9 \end{bmatrix}$

9. Lower Triangular matrix: A square matrix in which all the elements above principle diagonal are zero is called lower triangular matrix.

- Transpose of a matrix interchange row into columns
  Denoted by A^T.

Symmetric matrix · if $A^T = A$

Singular matrix if $|A| = 0$

Non-singular matrix if $|A| \neq 0$.

Orthogonal matrix if $AA^T = I = A^TA$

imp: Complex matrix A matrix is said to be complex matrix if atleast one elements is complex number it means in the form of $a+ib$. where a and b belongs to real numbers.

- Hermitian matrix A square matrix is said to be hermitian matrix if $A = [\bar{A}]^T = A^*$

  e.g. $A = \begin{bmatrix} 5 & 2+i \\ 2-i & 3 \end{bmatrix}$

Ques. Check for hermition $A = \begin{bmatrix} 3 & 5+2i & -3 \\ 5+2i & 7 & 4i \\ -3 & -4i & 5 \end{bmatrix}$

$$[\bar{A}] = \begin{bmatrix} 3 & 5-2i & -3 \\ 5-2i & 7 & -4i \\ -3 & 4i & 5 \end{bmatrix}$$

$$A^* = [\bar{A}]^T = \begin{bmatrix} 3 & 5-2i & -3 \\ 5-2i & 7 & 4i \\ -3 & -4i & 5 \end{bmatrix}$$

$\because A \neq A^*$

Thus, $[A]$ is not a hermition matrix

Note For hermition matrix,
- → Diagonal elements are real
- → $a_{ij} = \bar{a}_{ji} \;\forall\; i$ and j

- **Skew Hermition Matrix** :- A square matrix is said to be skew-hermition matrix if $A = -A^* = -[\bar{A}]^T$

e.g. $A = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$

$$\bar{A} = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix}$$

$$[\bar{A}]^T = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix}$$

$$-A^* = -[\bar{A}]^T = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix} = A.$$

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Note: For skew hermition matrix,

- All diagonal elements are pure imaginary
- $a_{ij} = -\bar{a}_{ji} \ \forall \ i \ \& \ j$.

Unitary matrix: A square matrix is said to be unitary matrix if $AA^* = I = A^*A$.

- $|A| \neq 0$ if A is unitary matrix.

Ques: Show that the matrix $A = \begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix}$ is unitary if $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$.

For unitary matrix, $AA^* = I = A^*A$

$$A^* = [\bar{A}]^T = \begin{bmatrix} \alpha - i\gamma & -\beta - i\delta \\ \beta - i\delta & \alpha + i\gamma \end{bmatrix}^T = \begin{bmatrix} \alpha - i\gamma & \beta - i\delta \\ -\beta - i\delta & \alpha + i\gamma \end{bmatrix}$$

Now,

$$AA^* = \begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix} \begin{bmatrix} \alpha - i\gamma & \beta - i\delta \\ -\beta - i\delta & \alpha + i\gamma \end{bmatrix}$$

$$= \begin{bmatrix} \alpha^2 - i\alpha\gamma + i\alpha\gamma + \gamma^2 + \beta^2 + i\beta\delta - i\beta\delta + \delta^2 & \alpha\beta - i\alpha\delta + i\beta\gamma + \gamma\delta - \alpha\beta + i\alpha\delta - i\beta\gamma - \gamma\delta \\ \alpha\beta - i\beta\gamma + i\alpha\delta + \delta\gamma - \alpha\beta - i\alpha\delta + i\beta\gamma - \gamma\delta & \alpha^2 + \beta^2 + \gamma^2 + \delta^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I$$

Hence A is a unitary matrix.

Ques. Show that $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$ is unitary matrix, ω is complex cube root of unity.

For unitary matrix, $AA^* = I$

$$A^* = [\bar{A}]^T$$

$$= \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}^T$$

$$= \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}$$

$$AA^* = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix} \times \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 3 & 1+\omega^2+\omega & 1+\omega+\omega^2 \\ 1+\omega+\omega^2 & 1+\omega^3+\omega^3 & 1+\omega^2+\omega^4 \\ 1+\omega^2+\omega & 1+\omega^4+\omega^2 & 1+\omega^3+\omega^3 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence, A is a unitary matrix.

Theorem: Every square matrix (complex) can be expressed as sum of Hermitian and skew-hermitian matrix.

$$A = \frac{2A}{2}$$

$$A = \frac{1}{2}\left[A + A + A^* - A^*\right]$$

$$A = \frac{1}{2}\left[\underbrace{(A+A^*)}_{\substack{B \\ \text{Hermitian}}} + \underbrace{(A-A^*)}_{\substack{C \\ \text{skew-Hermitian}}}\right] \quad \text{—— (1)}$$

Let $A+A^* = B$

$$B^* = (A+A^*)^*$$
$$= A^* + (A^*)^*$$
$$= A^* + A$$
$$= B$$

Hence B is a hermitian matrix.

Let $C = A - A^*$

$$C^* = (A-A^*)^*$$
$$= A^* - (A^*)^*$$
$$= A^* - A$$
$$= -C$$
$$C = -C^*$$

$\Rightarrow$ C is a skew hermitian matrix.

From (1), A = sum of hermitian and skew hermitian matrix.

Ques Express a matrix $A = \begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix}$, as a sum of hermition and skew-hermition matrix.

$$A^* = [\bar{A}]^T = \begin{bmatrix} -i & 2+3i & 4-5i \\ 6-i & 0 & 4+5i \\ i & 2+i & 2-i \end{bmatrix}^T = \begin{bmatrix} -i & 6-i & i \\ 2+3i & 0 & 2+i \\ 4-5i & 4+5i & 2-i \end{bmatrix}$$

Let $A = \frac{1}{2}\left[A + A + A^* - A^*\right]$

$$A = \frac{1}{2}\left[(A+A^*) + (A-A^*)\right]$$

$$A = \frac{1}{2}\left[P + Q\right]$$

Now, $P = A + A^*$

$$= \begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix} + \begin{bmatrix} -i & 6-i & i \\ 2+3i & 0 & 2+i \\ 4-5i & 4+5i & 2-i \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 8-4i & 4+6i \\ 8+4i & 0 & 6-4i \\ 4-6i & 6+4i & 4 \end{bmatrix}$$

P is hermition matrix by property.

$Q = A - A^*$

$$= -\begin{bmatrix} -i & 6-i & i \\ 2+3i & 0 & 2+i \\ 4-5i & 4+5i & 2-i \end{bmatrix} + \begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix}$$

$$= \begin{bmatrix} 2i & -4-2i & 4+4i \\ 4-2i & 0 & 2-6i \\ -4+4i & -2-6i & 2i \end{bmatrix}$$

Q is a skew hermition matrix by property.

Now, $P+Q = \begin{bmatrix} 0 & 8-4i & 4+6i \\ 8+4i & 0 & 6-4i \\ 4-6i & 6+4i & 4 \end{bmatrix} + \begin{bmatrix} 2i & -4-2i & 4+4i \\ 4-2i & 0 & 2-6i \\ -4+4i & -2-6i & 2i \end{bmatrix}$

$$= \begin{bmatrix} 2i & 4-6i & 8+10i \\ 12+2i & 0 & 8-10i \\ -2i & 4-2i & 4+2i \end{bmatrix}$$

$$\frac{1}{2}[P+Q] = \frac{1}{2}\begin{bmatrix} 2i & 4-6i & 8+10i \\ 12+2i & 0 & 8-10i \\ -2i & 4-2i & 4+2i \end{bmatrix}$$

$$= \begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix}$$

$$= A.$$

Hence proved.

~~Rank~~ Elementary transformation or E-operations

① Interchanging of two rows or two columns R_{ij} / C_{ij}

$R_i \leftrightarrow R_j$

② Multiplication of (each element of) row or column by a non-zero number k

$R_i(k)$ or $C_i(k)$

③ Addition of k-times the element of a row or a column to the corresponding elements of another row or column, $k \neq 0$

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Ques: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

$R_1 \leftrightarrow R_2 \sim \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ → Elementary matrix.

$R_2 \rightarrow R_2 + 2R_1$

$\sim \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

imp **Rank**

A matrix A is said to be of rank r if

- it has atleast one non-zero minor of order r
- All the minors of (r+1) or higher than r are zero.

and rank of A is denoted by $\rho(A)$.

$$\boxed{\rho(A) = r}$$

E.g. $A = \begin{bmatrix} 2 & 5 & 0 \\ 1 & 6 & 1 \\ 4 & 2 & 3 \end{bmatrix}$

* If A is null matrix, then rank of A is zero otherwise $\rho(A) = 1$.

* If A is a non-singular matrix of order $n \times n$, then rank of A is n. $\rho(A) = n$ $[\because |A| \neq 0]$

* If I_n be the $n \times n$ unit matrix then, $\rho(I_n) = n$.

* If A is $m \times n$ matrix, then $\rho(A) \leq \min[m, n]$

* If all minors of order r are equal to zero then $\rho(A) < r$

Methods of finding rank

- Minor Method :- Start with a highest order minor (or minors) of A.
Let their order be r. If any one of them is non-zero. Then
$$\boxed{\rho(A) = r}$$

Ques. $A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 1 \\ 3 & 6 & 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 4 & 0 \\ 2 & 5 & 1 \\ 3 & 6 & 2 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 5 & 1 \\ 6 & 2 \end{vmatrix} - 4 \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix}$$

$$= (10-6) - 4(4-3)$$

$$= 0$$

Minors of 2x2

$$\begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix}, \begin{vmatrix} 4 & 0 \\ 5 & 1 \end{vmatrix}, \dots\dots$$

$= 5-8$

$= -3$

$\neq 0$

$\rho(A) = 2$

Triangular form

- Echelon Form Method: A matrix is reduced in Echelon form as :-

(i) the 1^{st} non-zero element in a row should be unity (if possible)

(ii) All the non-zero rows, if any, precedes the zero row. then rank of matrix A = no. of non-zero rows

$$\boxed{\rho(A) = \text{no. of non-zero rows}}$$

e.g. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 8 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Note We can use only row operations for row ~~echlo~~ echelon form.

→ Column Echelon Form :-

(i) All zero columns are on the right.

(ii) The 1^{st} (if we go down from the top) non-zero entry in each column is one.

e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 2 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$\rho(A) = 2$ $\rho(A) = 2$ $\rho(A) = 1$

Note: Use only column operations for column echelon form

$$\rho(A) = \text{no. of non-zero columns}$$

Ques. Find rank by echelon form.

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ Non-zero rows}$$

$\rho(A)$ = no. of non-zero rows

$\rho(A) = 3$

Ques. $A = \begin{bmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix}$ Find rank by reducing triangular form.

$R_2 \to R_2 + R_3$

$$\sim \begin{bmatrix} 2 & 3 & 4 & -1 \\ 1 & 7 & 12 & -2 \\ -4 & 5 & 12 & -1 \end{bmatrix}$$

$R_3 \to R_3 + 2R_1$

$$\sim \begin{bmatrix} 2 & 3 & 4 & -1 \\ 1 & 7 & 12 & -2 \\ 0 & 11 & 20 & -3 \end{bmatrix}$$

$R_1 \to R_1 - 2R_2$

$$\sim \begin{bmatrix} 0 & -11 & -20 & 3 \\ 1 & 7 & 12 & -2 \\ 0 & 11 & 20 & -3 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

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$$\sim \begin{bmatrix} 1 & 7 & 12 & -2 \\ 0 & -11 & -20 & 3 \\ 0 & 11 & 20 & -3 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 7 & 12 & -2 \\ 0 & -11 & -20 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ → no. of non-zero rows.

$\rho(A) = 2$ Ans

Ques: Find rank. $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

$R_4 \to R_4 - 2R_3$

$$\sim \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 0 & 1 & -6 & -3 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 1 & -2 & 4 & -1 \\ 0 & 1 & -6 & -3 \end{bmatrix}$$

$R_1 \to R_1 - R_2$

$$\sim \begin{bmatrix} 1 & 4 & 1 & 3 \\ 1 & -1 & -2 & -4 \\ 1 & -2 & 4 & -1 \\ 0 & 1 & -6 & -3 \end{bmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 4 & 1 & 3 \\ 0 & -5 & -3 & -7 \\ 0 & -6 & 3 & -4 \\ 0 & 1 & -6 & -3 \end{bmatrix}$$

$R_3 \rightarrow R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 4 & 1 & 3 \\ 0 & -5 & -3 & -7 \\ 0 & -1 & 6 & 3 \\ 0 & 1 & -6 & -3 \end{bmatrix}$$

$R_4 \rightarrow R_4 + R_3$

$$\sim \begin{bmatrix} 1 & 4 & 1 & 3 \\ 0 & -5 & -3 & -7 \\ 0 & -1 & 6 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 5R_3$

$$\sim \begin{bmatrix} 1 & 4 & 1 & 3 \\ 0 & 0 & -33 & -22 \\ 0 & -1 & 6 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \leftrightarrow$

$$\sim \begin{bmatrix} 1 & 4 & 1 & 3 \\ 0 & -1 & 6 & 3 \\ 0 & 0 & -33 & -22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A)$ = no. of non-zero rows

$\rho(A) = 3$ Ans.

Ques $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$ Find Rank

$R_3 \rightarrow R_3 - 3R_1$, $R_2 \rightarrow R_2 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(A)$ = no. of non-zero rows.

$\rho(A) = 2.$

Q. Find the value of p for which the matrix $A = \begin{bmatrix} 3 & p & p \\ p & 3 & p \\ p & p & 3 \end{bmatrix}$ of rank 1.

∵ Rank of A is 1.
So, all the minors of order 2 or higher will be zero:

$$\begin{vmatrix} 3 & p \\ p & 3 \end{vmatrix} = 0 \Rightarrow 9 - p^2 = 0 \Rightarrow p = \pm 3$$

Again,

$$\begin{vmatrix} p & 3 \\ p & p \end{vmatrix} = 0$$

$$p^2 - 3p = 0$$
$$p(p-3) = 0$$
$$p = 0 \text{ or } p = 3.$$

$\therefore \; p = 3.$

Solutions of Linear Equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Consistent : Solution exists
Inconsistent : No soln. exists.

$AX = B \; (B \neq 0) \Rightarrow$ Non-homogeneous
$AX = B \; (B = 0) \Rightarrow$ Homogeneous.

Steps for finding soln of eq $AX = B \; (B \neq 0)$

(i) Let $[A \,\vdots\, B]$ be an augumented matrix.
(ii) Find Rank of $[A \,\vdots\, B]$

ⓐ if $P[A:B] \neq P[A]$

→ System inconsistent i.e. No soln.

ⓑ if $P[A:B] = P[A] = r$ (say)

→ consistent

- if $n = r$ → unique soln.
- if $r < n$ → infinite soln.

$\boxed{AX = B}$

↓ Find $P[A:B]$

$\boxed{P[A:B] \neq P[A]}$ → Inconsistent

$\boxed{P[A:B] = P[A] = r}$ → consistent

- $\boxed{n = r}$ → unique soln
- $\boxed{n \neq r}$ → Infinite no. of soln or many soln.

Note

To find solutions :-

(i) assign arbitrary value k to $(n-r)$ unknowns.

(ii) find soln by Post Substitution Method.

$\boxed{A'X = B'}$, where A' & B' are

Ques. Using matrix method, show that equations are consistent.

$$3x + 3y + 2z = 1$$
$$x + 2y = 4$$
$$10y + 3z = -2$$
$$2x - 3y - z = 5$$

Hence find soln. for x, y, z.

Let $A = \begin{bmatrix} 3 & 3 & 2 \\ 1 & 2 & 0 \\ 0 & 10 & 3 \\ 2 & -3 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 4 \\ -2 \\ 5 \end{bmatrix}$

$n = 3$ (no. of unknow

Let $[A \vdots B] = \begin{bmatrix} 3 & 3 & 2 & \vdots & 1 \\ 1 & 2 & 0 & \vdots & 4 \\ 0 & 10 & 3 & \vdots & -2 \\ 2 & -3 & -1 & \vdots & 5 \end{bmatrix}_{4\times4}.$

$R_1 \to R_1 - 3R_2$, $R_4 \to R_4 - 2R_2$.

$$\sim \begin{bmatrix} 0 & -3 & 2 & -11 \\ 1 & 2 & 0 & 4 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & -3 \end{bmatrix}$$

$R_1 \longleftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & -3 \end{bmatrix}$$

~~$R_3 \to R_3 + R_2 + R_4$~~ $R_3 \to R_3 + 3R_2$ and $R_4 \to R_4 - 2R_2$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 0 & 4 & -16 \\ 0 & -7 & -1 & -3 \end{bmatrix} \quad \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 1 & 9 & -35 \\ 0 & -1 & -5 & 19 \end{bmatrix}$$

$R_3 \to R_3 + R_4$, $R_2 \to R_2 - 3R_4$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 17 & -68 \\ 0 & 0 & 4 & -16 \\ 0 & -1 & -5 & 19 \end{bmatrix}$$

$R_2 \longleftrightarrow R_4$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & -1 & -5 & 19 \\ 0 & 0 & 4 & -16 \\ 0 & 0 & 17 & -68 \end{bmatrix}$$

$R_4 \to R_4 - 4R_3$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & -1 & -5 & 19 \\ 0 & 0 & 4 & -16 \\ 0 & 0 & 1 & -4 \end{bmatrix}$$

$R_3 \to R_3 - 4R_4$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & -1 & -5 & 19 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 0 & 4 \\ 0 & -1 & -5 & 19 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$P[A \vdots B] = 3 = P[A]$$

Hence consistent.

To find soln.,

$n = 3$, $r = 3$

$\Rightarrow$ unique soln. exists.

$$A'X = B'$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & -5 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 19 \\ -4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x + 2y \\ -y - 5z \\ z \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 19 \\ -4 \\ 0 \end{bmatrix}$$

$z = -4$

$-y - 5z = 19$

$\Rightarrow -y + 20 = 19$

$\Rightarrow y = 1$

$x + 2y = 4$

$x + 2 = 4$

$x = 2$

Ans $x = 2$, $y = 1$, $z = -4$.

Q. Test the consistency if yes, then find soln of,

$$x_1 + 2x_2 + 3x_3 + 4x_4 = 5$$
$$6x_1 + 7x_2 + 8x_3 + 9x_4 = 10$$
$$11x_1 + 12x_2 + 13x_3 + 14x_4 = 15$$
$$16x_1 + 17x_2 + 18x_3 + 19x_4 = 20$$
$$21x_1 + 22x_2 + 23x_3 + 24x_4 = 25.$$

Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \\ 21 & 22 & 23 & 24 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, $B = \begin{bmatrix} 5 \\ 10 \\ 15 \\ 20 \\ 25 \end{bmatrix}$

$$[A \vdots B] = \left[\begin{array}{cccc:c} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 & 25 \end{array}\right]$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 - R_1$, $R_5 \to R_5 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 5 & 5 & 5 & 5 \\ 10 & 10 & 10 & 10 & 10 \\ 15 & 15 & 15 & 15 & 15 \\ 20 & 20 & 20 & 20 & 20 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$, $R_4 \to R_4 - 3R_2$, $R_5 \to R_5 - 4R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 5 & 5 & 5 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 5R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -5 & -10 & -15 & -20 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A)$ = no. of non-zero rows = 2.

$r = 2$.

$n = 4$.

$\rho[A \vdots B] = \rho(A) = 2 \rightarrow$ consistent.

$\because n \neq r \rightarrow$ infinite soln. exists.

To find soln.

$$A'X = B'$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -5 & -10 & -15 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ -20 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 + 3x_3 + 4x_4 = 5 \quad \text{--- (1)}$$

$$-5x_2 - 10x_3 - 15x_4 = -20$$

$$\Rightarrow \quad x_2 + 2x_3 + 3x_4 = 4 \quad \text{--- (2)}$$

Let $x_3 = k_1$ and $x_4 = k_2$

From (2), $x_2 + 2k_1 + 3k_2 = 4$.

$\boxed{x_2 = 4 - 2k_1 - 3k_2}$, where k_1 & k_2 are arbitrary

Put this value in eqn (1)

$$x_1 = 5 - 2x_2 - 3x_3 - 4x_4$$

$$x_1 = 5 - 2(4 - 2k_1 - 3k_2) - 3k_1 - 4k_2$$

$$x_1 = 5 - 8 + 4k_1 + 6k_2 - 3k_1 - 4k_2$$

$$\boxed{x_1 = -3 + k_1 + 2k_2}$$

imp (5) Investigate, for what value of λ and μ do the system of equations

$$x + y + z = 6$$

$$x + 2y + 3z = 10$$

$$x + 2y + \lambda z = \mu$$

(i) has no soln.

(ii) unique soln.

(iii) infinite soln.

Let $[A \vdots B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$ $\quad R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right] \quad R_3 \to R_3 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$$

→ No solution :- $\rho[A \vdots B] \neq \rho[A]$
if $\lambda = 3$ and $\mu \neq 10$

→ Unique soln :-
$\rho[A \vdots B] = \rho[A] = r = n = 3$
when $\lambda - 3 \neq 0 \Rightarrow \lambda \neq 3$, $\mu \in R$ [μ can take any value]

→ Infinite soln :-
$\rho[A \vdots B] = \rho[A] = r < n = 3$
$r = 2$
$\lambda = 3$ and $\mu = 10$

Solution of Homogeneous Linear equations.

$\boxed{AX = O}$ Always consistence

Find $\rho[A] = r$

$|A| \neq 0$ $\boxed{r = n}$
Unique soln
or zero soln
or trivial soln

$\boxed{r < n}$ $|A| = 0$
Infinite no. of solns
or Non-trivial soln
or Non-zero soln

Q. Find value of k for which system of equations has non-trivial sol

$$(3k-8)x + 3y + 3z = 0$$
$$3x + (3k-8)y + 3z = 0$$
$$3x + 3y + (3k-8)z = 0$$

For non-trivial soln, $|A| = 0$

$$\begin{vmatrix} 3k-8 & 3 & 3 \\ 3 & 3k-8 & 3 \\ 3 & 3 & 3k-8 \end{vmatrix} = 0$$

$R_1 \to R_1 + R_2 + R_3$

$$\begin{vmatrix} 3k-2 & 3k-2 & 3k-2 \\ 3 & 3k-8 & 3 \\ 3 & 3 & 3k-8 \end{vmatrix} = 0$$

$$(3k-2)\begin{vmatrix} 1 & 1 & 1 \\ 3 & 3k-8 & 3 \\ 3 & 3 & 3k-8 \end{vmatrix} = 0$$

$R_3 \to R_3 - R_2$

$$(3k-2)\begin{vmatrix} 1 & 1 & 1 \\ 3 & 3k-8 & 3 \\ 0 & 5-3k & 3k-11 \end{vmatrix} = 0$$

$R_2 \to R_2 - 3R_1$

$$(3k-2)\begin{vmatrix} 1 & 1 & 1 \\ 0 & 3k-11 & 0 \\ 0 & 5-3k & 3k-11 \end{vmatrix} = 0$$

Expand by C_1,

$$(3k-2) \times 1 \begin{vmatrix} 3k-11 & 0 \\ 5-3k & 3k-11 \end{vmatrix} = 0$$

$$(3k-2)(3k-11)^2 = 0$$

$k = 2/3$ and $k = 11/$

Ques: Solve the equation using matrix method

$$x_1 + 3x_2 + 2x_3 = 0$$
$$2x_1 - x_2 + 3x_3 = 0$$
$$3x_1 - 5x_2 + 4x_3 = 0$$
$$x_1 + 17x_2 + 4x_3 = 0$$

Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 3 & -5 & 4 \\ 1 & 17 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\boxed{n=3}$, $B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

To find rank of A,

$R_4 \to R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 3 & -5 & 4 \\ 0 & 14 & 2 \end{bmatrix}$$

$R_3 \to R_3 - R_2 - R_1$

$$\sim \begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 0 & -7 & -1 \\ 0 & 14 & 2 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & -7 & -1 \\ 0 & 14 & 2 \end{bmatrix}$$

$R_3 \to R_3 - R_2$, $R_4 \to R_4 + 2R_2$

$$\sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(A)$ = no. of non-zero rows = 2 = r.

$\because n = 3$, $r = 2 \Rightarrow r \neq n \Rightarrow$ infinite many soln. exists.

To find solution, $A'X = B'$

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + 3x_2 + 2x_3 \\ -7x_2 - x_3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 3x_2 + 2x_3 = 0 \quad \text{—— (1)}$$
$$-7x_2 - x_3 = 0 \quad \text{—— (2)}$$

Let $x_2 = k$,

$$\boxed{x_3 = -7k}$$

Putting in (1),

$$x_1 + 3k + 2(-7k) = 0$$
$$x_1 - 11k = 0$$
$$\boxed{x_1 = 11k}$$

Q

$$x + 3y - 2z = 0$$
$$2x - y + 4z = 0$$
$$x - 11y + 14z = 0$$

Let $A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

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$$\sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(A)$ = no. of non-zero rows = 2 = r.

$\because$ $n = 3$, $r = 2 \Rightarrow n \neq r \Rightarrow$ Infinite many soln.

$$A'X = B'$$

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x+3y-2z \\ -7y+8z \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + 3y - 2z = 0$$
$$-7y + 8z = 0$$

Let $z = k$,

$$8k = 7y$$

$$\cancel{k = \frac{7y}{8}} \quad y = \frac{8k}{7}$$

$$x + 3\left(\frac{8k}{7}\right) - 2k = 0$$

$$x + \frac{24}{7}k - 2k = 0$$

$$x = 2k - \frac{24}{7}k = \frac{-10k}{7}.$$

Linear Dependence or Independence

A set of n-vectors $X_1, X_2 \ldots X_n$ is said to be linearly dependent if there exists scalars (numbers) $K_1, K_2, \ldots K_n$ (not all zero) such that $K_1X_1 + K_2X_2 + K_3X_3 + \ldots K_nX_n = 0$ otherwise it is linearly independent. $(K_1 = K_2 = K_3 = \ldots K_n = 0)$

Procedure for checking linear dependency or Independency

Step 1: Construct coefficient matrix A with element of given as columns. $A = [X_1 X_2 X_3]$

Step 2: Find $\rho(A)$ by echelon form.

Step 3: If $\rho(A)$ = no. of given vectors then it is linearly independent
or $\rho(A)$ < given no. of vectors then linearly dependent. $|A| = 0$ dependent

~~Ques. Prove that vectors~~
If vectors are linearly dependent then to find relation
consider ~~$A'X$~~ linear combination $a_1X_1 + a_2X_2 + a_3X_3 + \dots a_nX_n = 0$

$$A'X = 0$$

Solve it and find value of $a_1, a_2, \dots a_n$ and put these values

Ques. Show that vectors $X_1 = (1, 2, 4)$ $X_2 = (2, -1, 3)$ $X_3 = (0, 1, 2)$ and $X_4 = (-3, 7, 2)$ are linearly dependent. Also find relation b/w them.

Let $A = \begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix}_{3\times4}$, No. of vectors = 4.

To find rank of A,

$R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 4R_1$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & -5 & 2 & 14 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$\rho(A) = 3$

No. of vectors = 4.

$\Rightarrow$ rank < no. of vectors

$\Rightarrow$ Linearly dependent.

To find relation:-

Let linear combination $a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 = 0$, where a_1, a_2, a_3 & a_4 are scalar.

Let $A'X = O$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix}_{3\times4} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}_{4\times1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{3\times1}$$

$$a_1 + 2a_2 - 3a_4 = 0 \quad \text{---} (2)$$

$$-5a_2 + a_3 + 13a_4 = 0 \quad \text{---} (3)$$

$$a_3 + a_4 = 0 \quad \text{---} (4)$$

Let $a_4 = k$,

$$\boxed{a_3 = -k}$$

Putting values in (3)

$$-5a_2 + a_3 + 13k = 0$$

$$\Rightarrow \quad -5a_2 - k + 13k = 0$$

$$\Rightarrow \quad -5a_2 = -12k$$

$$\Rightarrow \quad a_2 = \frac{12k}{5}$$

Putting values in (2),

$$a_1 + 2\left(\frac{12k}{5}\right) - 3k = 0$$

$$a_1 = 3k - \frac{24k}{5}$$

$$a_1 = \frac{-9k}{5}$$

Putting values in linear combination,

$$-\frac{9}{5}kX_1 + \frac{12}{5}kX_2 - kX_3 + kX_4 = 0$$

$$\frac{k}{5}\left[-9X_1 + 12X_2 - 5X_3 + 5X_4\right] = 0$$

$$-9X_1 + 12X_2 - 5X_3 + 5X_4 = 0 \quad [k \neq 0]$$

Ques. Show that vectors $\eta_1 = (1,2,3)$, $\eta_2 = [3,-2,1]$, $\eta_3 = [1,-6,-5]$ form linear dependent system. Also find relation b/w them.

Let $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & -2 & -6 \\ 3 & 1 & -5 \end{bmatrix}$

To find rank of A,

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 3 & 1 \\ 0 & -8 & -8 \\ 0 & -8 & -8 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 3 & 1 \\ 0 & -8 & -8 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(A)$ = no. of non-zero rows $= 2 = r$.

No. of vectors $= 3 \neq r$.

$\Rightarrow$ Vectors are linearly dependent.

Let linear combination $a_1\eta_1 + a_2\eta_2 + a_3\eta_3 = 0$, where a_1, a_2, a_3 are scaler. — (1)

$$A'X = O$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & -8 & -8 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$a_1 + 3a_2 + a_3 = 0 \quad \text{— ②}$$
$$-8a_2 - 8a_3 = 0 \quad \text{— ③}$$

let $\boxed{a_3 = k}$

$$-8a_2 - 8k = 0$$
$$\boxed{a_2 = -k}$$

Putting values in ②,

$$a_1 + 3(-k) + k = 0$$
$$a_1 - 2k = 0$$
$$\boxed{a_1 = 2k}$$

Putting values in ①,

$$2k\eta_1 - k\eta_2 + k\eta_3 = 0$$
$$k[2\eta_1 - \eta_2 + \eta_3] = 0$$
$$2\eta_1 - \eta_2 + \eta_3 = 0 \quad [k \neq 0]$$

Ques. Find the value of λ for which vectors $(1, -2, \lambda), (2, -1, 5)$ and $(3, -5, 7)$ are L.D.

For linearly dependent $|A| = 0$.

let $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -1 & -5 \\ \lambda & 5 & 7 \end{bmatrix}$

$$|A| = 0$$

$$\begin{vmatrix} 1 & 2 & 3 \\ -2 & -1 & -5 \\ \lambda & +5 & 7 \end{vmatrix} = 0$$

$C_3 \rightarrow C_3 - 3C_1$ and $C_2 \rightarrow C_2 - 2C_1$

$$\begin{vmatrix} 1 & 0 & 0 \\ -2 & 3 & 1 \\ \lambda & 5-2\lambda & 7-3\lambda \end{vmatrix} = 0$$

$$\begin{vmatrix} 3 & 1 \\ 5-2\lambda & 7-3\lambda \end{vmatrix} = 0$$

$$21 - 9\lambda - (5-2\lambda) = 0$$

$$21 - 9\lambda - 5 + 2\lambda = 0$$

$$16 - 7\lambda = 0$$

$$\lambda = \frac{16}{7}$$

Ques. For what value of K the equations $x+y+z=1$, $2x+y+4z=K$, $4x+y+10z=K^2$ have a soln and solve them completely in each case.

let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 4 & 1 & 10 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ K \\ K^2 \end{bmatrix}$

$$[A \vdots B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & K \\ 4 & 1 & 10 & K^2 \end{array}\right]$$

To find rank:-

$R_3 \to R_3 - 2R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & K \\ 0 & -1 & 2 & K^2-2K \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & K-2 \\ 0 & -1 & 2 & K^2-2K \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & K-2 \\ 0 & 0 & 0 & K^2-3K+2 \end{array}\right]$$

if $K^2 - 3K + 2 \neq 0$

$\rho[A \vdots B] = 3$, $\rho[A] = 2$ $\Rightarrow$ No soln.

if $K^2-3K+2=0$ $\Rightarrow K=1,2.$

$\rho[A:B]=2$ and $\rho[A]=2.$

$\Rightarrow$ Infinite no. of soln. exists.

$r=2,\ n=3$

$A'X=B'$ [For $K=1$]

$$\begin{bmatrix}1&1&1\\0&-1&2\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\-1\\0\end{bmatrix}$$

$x+y+z=1$

$-y+2z=-1$

let $z=k$

$-y+2k=-1$

$\boxed{y=2k+1}$

$x+2k+1+k=1$

$\boxed{x=-3k}$

For $K=2$,

$A'X=B'$

$$\begin{bmatrix}1&1&1\\0&-1&2\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$x+y+z=1$

$-y+2z=0$

Let $z=k$,

$\boxed{y=2k}$

$x+2k+k=1$

$\boxed{x=1-3k}$

Gauss Elimination Method

To solve linear eq^n we use gauss elimination method.

$$a_1x + b_1y + c_1z = d_1 \quad \text{---} (1)$$
$$a_2x + b_2y + c_2z = d_2 \quad \text{---} (2)$$
$$a_3x + b_3y + c_3z = d_3 \quad \text{---} (3)$$

First eliminate x from (2) and (3) with the help of eq^n (1).

$$e_1y + f_1z = m_1 \quad \text{---} (4)$$
$$e_2y + f_2z = m_2 \quad \text{---} (5)$$

Eliminate y from (5) with the help of (4), get soln of

$$\boxed{z = N_1}$$

Put this value of z in eq^n (4) get,

$$\boxed{y = N_2}$$

and put z and y in eq^n $\because A'X = B'$, get

$$\boxed{x = N_3}$$

Q. Solve by gauss elimination method

$$6x + 3y + 2z = 6 \quad \text{---} (1)$$
$$6x + 4y + 3z = 0 \quad \text{---} (2)$$
$$20x + 15y + 12z = 0 \quad \text{---} (3)$$

(1) - (2)

$$-y - z = 6 \quad \text{---} (4)$$

Again from (1), divide by 6.

$$x + \frac{y}{2} + \frac{z}{3} = 1$$

Multiply by 20,

$$20x + 10y + \frac{20z}{3} = 20$$

Sub. (3),

$$20x + 10y + \frac{20}{3}z = 20$$
$$20x + 15y + 12z = 0$$
$$-5y - \frac{16}{3}z = 20$$

$\frac{20}{3} - 12$

$$15y + 16z = -60 \quad \text{—— (5)}$$

Eliminating y from (4) and (5),

$$15y + 15z = -90$$
$$15y + 16z = -60$$
$$-z = -30$$
$$\boxed{z = 30}$$

Put $z = 30$ in eq^n (4),

$$y = -6 - z = -36.$$

Put $y = -36$ and $z = 30$ in eq^n (1)

$$x = \frac{1}{6}[6 - 3y - 2z]$$
$$\boxed{x = 9}$$

Gauss-Seidel Method:-

Let $a_1x + b_1y + c_1z = d_1$ —— (1)
$a_2x + b_2y + c_2z = d_2$ —— (2)
$a_3x + b_3y + c_3z = d_3$ —— (3)

Make system diagonally dominant in eq^n (1), (2), (3).

$$a_1 \geq b_1 + c_1$$
$$b_2 \geq a_2 + c_2$$
$$c_3 \geq a_3 + b_3$$

From (1), $x = \frac{1}{a_1}[d_1 - b_1 y - c_1 z]$ — (4)

From (2), $y = \frac{1}{b_2}[d_2 - a_2 x - c_2 z]$ — (5)

From (3), $z = \frac{1}{c_3}[d_3 - a_3 x - b_3 y]$ — (6)

Ist Approximation.

Put $y = 0$, $z = 0$ in eq (4)

$$x^{(1)} = \frac{d_1}{a_1}$$

From (5), put $z = 0$, $x = x^{(1)}$

$$y^{(1)} = \frac{1}{b_2}[d_2 - a_2 x^{(1)}]$$

From (6),

$$z^{(1)} = \frac{1}{c_3}[d_3 - a_3 x^{(1)} - b_3 y^{(1)}]$$

IInd Approximation.

$$x^{(2)} = \frac{1}{a_1}[d_1 - b_1(y^{(1)}) - c_1 z^{(1)}]$$

$$y^{(2)} = \frac{1}{b_2}[d_2 - a_2 x^{(2)} - c_2 z^{(1)}]$$

$$z^{(2)} = \frac{1}{c^3}[d_3 - a_3 x^{(2)} - b_3 y^{(2)}]$$

3rd approximation and so on.

This process of approximation is continued till we get desired accuracy of x, y and z.

Que: Solve by Gauss-Seidal method.

$$\cancel{12x+3y}\ 2x+\cancel{4}y+4z=12$$
$$8x-3y+2z=23$$
$$4x+11y-z=33$$

Rewrite eq^{n}s.

$$8x-3y+2z=23 \quad (8>3+2) \quad \text{—— (1)}$$
$$4x+11y-z=33 \quad (11>4+1) \quad \text{—— (2)}$$
$$2x+y+4z=12 \quad (4>2+1) \quad \text{—— (3)}$$

From (1),

$$x=\frac{1}{8}[23+3y-2z] \quad \text{—— (4)}$$

$$y=\frac{1}{11}[33-4x+z] \quad \text{—— (5)}$$

$$z=\frac{1}{4}[12-2x-y] \quad \text{—— (6)}$$

Ist Approximation,

Put $y=0, z=0$,

$$x^{(1)}=\frac{23}{8}=2.875$$

From (5), Put $x^{(1)}=2.875$ and $z=0$.

$$y^{(1)}=\frac{1}{11}[33-11.5]=1.954$$

Put $x^{(1)}=2.875$ and $y^{(1)}=1.954$ in eq^n (6),

$$z=\frac{1}{4}[12-2(2.875)-(1.954)]=1.074$$

IInd Approximation,

$$x^{(2)}=\frac{1}{8}[23+\overset{5.862}{3(1.954)}-\overset{2.148}{2(1.074)}]=3.339$$

$$y^{(2)}=\frac{1}{11}[33-\overset{13.356}{4(3.339)}+1.074]=1.883$$

$$z^{(2)} = \frac{1}{4}\left[12 - \overset{6.678}{2(3.339)} - 1.883\right] = 0.859$$

IIIrd Approximation,

$$x^{(3)} = \frac{1}{8}\left[23 + \overset{5.649}{3(1.883)} - \overset{1.718}{2(0.859)}\right]$$

$$= 3.366$$

$$y^{(3)} = \frac{1}{11}\left[33 - \overset{13.464}{4(3.366)} + 0.859\right]$$

$$= 1.854$$

$$z^{(3)} = \frac{1}{4}\left[12 - \overset{6.732}{2(3.366)} - 1.854\right]$$

$$= 0.8535$$

$\because$ 3rd approximation is similar to 2nd approximation upto 2 decimal places.

Hence, $x = 3.36$

$y = 1.85$

$z = 0.85$