Linear Algebra For Engineers

Index :-

Unit-1

Matrix Algebra

matrix :

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} \cdots & a_{n3} \cdots & a_{nn} \end{bmatrix}$$

columns

→ Rows

Real and Complex matrices :

1. Real matrices : where all elements are Real.

Ex : $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$

2. Complex matrices:
Atleast one of the elements of the matrix is complex.

Ex: $A = \begin{bmatrix} 0 & 1 & i \\ -2i & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$

* Conjugate Matrix:

Conjugate → jisme complex ka 'i' ko sign change de like:

$$\left.\begin{aligned} z &= x + iy \\ z' &= x - iy \end{aligned}\right\} \quad i \rightarrow -i$$

Ex: $A = \begin{bmatrix} 2-i & 4 \\ -7 & 3i \end{bmatrix}$

$A' = \begin{bmatrix} 2+i & 4 \\ -7 & -3i \end{bmatrix}$

- Symm. Matrix:

$$A = A'$$

- Skew-Symm. Matrix:

$$A' = -A$$

$'$ → Transpose

$-$ → conjugate (sign change of only i)

- Matrix A^{θ} or A^{*} :-

$$A^{\theta} = (\bar{A})' = \overline{(A')}$$

Transpose conjugate of matrix A or Conjugate transpose of matrix A is said to be A^{θ}.

- Hermition Matrix :-

A Square matrix A is said to be Hermition if $\boxed{A^{\theta} = A}$

- Skew-Hermition Matrix :-

A Square matrix A is said to be Skew Hermition if $\boxed{A^{\theta} = -A}$

- Singular matrix :-

$$\boxed{|A| = 0}$$

- Non-Singular matrix :-

$$\boxed{|A| \neq 0}$$

- Diagonal matrix :-

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- Scalar matrix : $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

- Identity matrix :-

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- Triangular matrix :-

1. Upper Triangular matrix :-

$$A = \begin{bmatrix} 1 & 5 & 6 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

2. Lower Triangular Matrix :-

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 7 & 9 & 0 \\ 8 & 2 & 3 \end{bmatrix}$$

Q. If $A = \begin{bmatrix} 2+3i & 1-2i & 2+4i \\ 3-4i & 4+3i & 2-6i \\ 5 & 5+6i & 3 \end{bmatrix}$

find A^{θ}.

Sol :

$$A^{\theta} = \overline{(A')}$$

$$A' = \begin{bmatrix} 2+3i & 3-4i & 5 \\ 1-2i & 4+3i & 5+6i \\ 2+4i & 2-6i & 3 \end{bmatrix}$$

$$\overline{(A')} = \begin{bmatrix} 2-3i & 3+4i & 5 \\ 1+2i & 4-3i & 5-6i \\ 2-4i & 2+6i & 3 \end{bmatrix}$$

- **Unitary Matrix :-**

A Square matrix A is Said to be Unitary if,

$$A^{\theta} \cdot A = I = A \cdot A^{\theta}$$

where, $A^{\theta} = (\bar{A})' = (\overline{A'})$

Q. Show That The matrix

$$A = \begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix}$$ is Unitary

if $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$

Sol:

$$(\bar{A})' = \begin{bmatrix} \alpha + i\gamma & \beta + i\delta \\ -\beta + i\delta & \alpha - i\gamma \end{bmatrix}$$

$$A^{\theta} = (\overline{A'}) = \begin{bmatrix} \alpha - i\gamma & \beta - i\delta \\ -\beta - i\delta & \alpha + i\gamma \end{bmatrix}$$

For Unitary,

$$A^{\theta} \cdot A = \begin{bmatrix} \alpha - i\gamma & \beta - i\delta \\ -\beta - i\delta & \alpha + i\gamma \end{bmatrix} \begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix}$$

$$= \begin{bmatrix} \alpha^2 + \gamma^2 + \beta^2 + \delta^2 & -\beta^2 + \delta^2 + \alpha^2 + \gamma^2 \\ \alpha^2 + \gamma^2 + \beta^2 + \delta^2 & \beta^2 + \delta^2 + \alpha^2 + \gamma^2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Q2. Show That matrix is unitary

$$A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$A' = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$$

$$A^{\theta} = (\overline{A'}) = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$A^{\theta} \cdot A = \frac{1}{3} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$\Rightarrow \frac{1}{3} \begin{bmatrix} 1 + \cancel{1} - \cancel{1} & 1 + i - \cancel{i} - i \\ 1 + 1 - 1 & 1 + \cancel{i} - 1 - 1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$$

$$\Rightarrow \frac{3}{3} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= I$$ Ans

Elementarty transformation :-

* We use only Raw in This method
* Used for matrix which are not Square matrix.
* Baki toh method sikh lo

Date
Page

Rank of a Matrix :-

Echelm Form Method (Row operator only)

- Singular matrix :- (Square matrix)

$|A| = 0$ → Rank $A < n$

- Non-singular matrix:

$|A| \neq 0$

Rank $A = n$ (order of Sq. matrix A)

Ex: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$

$I_3 \neq 0$

$\therefore \boxed{\text{Rank} = 3}$

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$ $I_2 \neq 0$

$\boxed{\text{Rank} = 2}$

→ For Singular matrix
Echelm form method: → After Row operation method, matrix Reduction

Rank A = No. of non-zero Rows

Ex: Find Rank:

$A = \begin{bmatrix} 1 & 2 & 3 & -4 \\ -2 & 3 & 7 & -1 \\ 1 & 9 & 16 & -13 \end{bmatrix} 3 \times 4$

Sol: Step 1: make (1,1) element 1 Through Row operations only

Step 2: with The help of A(1,1) make A(1,2) & A(1,3) zero

$\therefore A = \begin{bmatrix} 1 & 2 & 3 & -4 \\ -2 & 3 & 7 & -1 \\ 1 & 9 & 16 & -13 \end{bmatrix}_{3\times4}$

$R_2 \to R_2 + 2R_1$

$R_3 \to R_3 - R_1$

$A \approx \begin{bmatrix} 1 & 2 & 3 & -4 \\ 0 & 7 & 13 & -9 \\ 0 & 7 & 13 & -9 \end{bmatrix}$

Step 3: check If Any Row is similar If yes, Then Remove That Row.

$R_3 \to R_3 - R_2$

$\approx \begin{bmatrix} 1 & 2 & 3 & -4 \\ 0 & 7 & 13 & -9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

* If there is no option left For Removing, Then directly write Row to zero The Rank

$\therefore$ $\boxed{\text{Rank} = 2}$ (no. of non zero Rows)

2. $A = \begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$

Sol: M_1: Short trick:

~~$R_1 \to R_2 + R_3$~~ $|A| = 0 \to$ Singular

$\therefore \boxed{\text{Rank} = 2}$

M_2: Echelon form:

$R_1 \to \frac{1}{3} R_1$

$\approx \begin{bmatrix} 1 & -1/3 & 2/3 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$

$R_2 \to R_2 + 6R_1$

$R_3 \to R_3 + 3R_1$

$\approx \begin{bmatrix} 1 & -1/3 & 2/3 \\ 0 & 0 & 8 \\ 0 & 0 & 4 \end{bmatrix}$

OR $R_3 \to R_2 - 2R_3$

$\to \begin{bmatrix} 1 & -1/3 & 2/3 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix}$

OR $R_2 \to R_2 - 2R_3$

$\to \begin{bmatrix} 1 & -1/3 & 2/3 \\ 0 & 0 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

$R_2 \to R_3$

$$\begin{bmatrix} 1 & -1/3 & 2/3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$\boxed{\text{Rank } A = 2 = \text{no. of non zero Rows}}$

Rank : The Rank of a matrix is said to be r if

1. It has at least one non-zero minor of order r
2. Every minor of A of order higher than r is zero.

Solution of System of Linear Equations

Consider 3 Eq^n in 3 unknown

$$a_1x + b_1y + c_1z = d_1 \quad -(1)$$
$$a_2x + b_2y + c_2z = d_2 \quad -(2)$$
$$a_3x + b_3y + c_3z = d_3 \quad -(3)$$

In matrix notation,

$$\boxed{AX = B}$$

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Coefficient matrix | column matrix of Unknown | column matrix of constants

Augmented Matrix :

The matrix obtaind by Putting Coefficient matrix of a constant side by side is called Augmented Matrix

$[A \,\vdots\, B]$

- Non-Homogenous :

  In, $\boxed{AX = B}$,

  If d_1, d_2, d_3 are not all zero

- Homogenous :

  If, $\boxed{AX = 0}$

  i.e $d_1 = d_2 = d_3 = 0$

  Then, This type of ~~System~~ System is Known as Homogenous Eqn of System

Gauss - Elimination Method :-

$\boxed{AX = B}$ Non-Homog. eqn

Find Rank $[A \,\vdots\, B]$ and Rank A

Rank $[A \,\vdots\, B]$ = Rank A → System Consistent Soln Exist

Rank $[A \,\vdots\, B] \neq$ Rank A → System Inconsistent No Soln Exist

Rank $[A \,\vdots\, B]$ = Rank A = n (no. of unknown) → Unique Soln

Rank $[A \,\vdots\, B]$ = Rank A < n → Infinite Soln

Homogenous Eqn

$\boxed{Ax = 0}$

Rank A (Soln Always Exist)

Rank A = n → Unique or Trivial Soln Exist

$x = 0, y = 0, z = 0$

Rank A < n → Infinite no. of non Trivial soln Exist.

→ find the x, y with the help of K

- For Non-Trivial Soln,
  $|A| = 0$

Q1. Test For Consistent find the Sol

$$x + 2y - z = 3$$
$$3x - y + 2z = 1$$
$$2x - 2y + 3z = 2$$
$$x - y + z = -1$$

Sol:

Augmented matrix:

$[A \vdots B]$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - 2R_1$

$R_4 \to R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{bmatrix}$$

* For more simplifying, and we cannot make any Row zero further by Row depletion, we move on to next diagonal element for simplication

$R_2 \to -1/7 \, R_2$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -5/7 & -8/7 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{bmatrix}$$

$R_3 \to R_3 + 6R_2, \quad R_4 \to R_4 + 3R_2$

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -5/7 & -8/7 \\ 0 & 0 & 5/7 & -20/7 \\ 0 & 0 & -1/7 & 4/7 \end{bmatrix}$$

$R_4 \to 5R_4$

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -5/7 & -8/7 \\ 0 & 0 & 5/7 & -20/7 \\ 0 & 0 & -5/7 & 20/7 \end{bmatrix}$$

$R_4 \to R_4 + R_3$

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -5/7 & 8/7 \\ 0 & 0 & 5/7 & 20/7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Rank $[A:B] = 3 =$ No. of non-zero Row

Rank $A = 3 =$ No. of non-zero Rows

$\therefore$ Rank $[A:B] =$ Rank $A = n$

$\therefore$ System is consistent and has Unique Solutions.

By Gauss-Elimination Method:

$AX = B$

- Remove zeroth Row,

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -5/7 \\ 0 & 0 & 5/7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 8/7 \\ 20/7 \end{bmatrix}$$

$x + 2y - z = 3 \quad -(1)$

$y - \frac{5}{7}z = \frac{8}{7} \quad -(2)$

$\frac{5}{\cancel{7}}z = \frac{20}{\cancel{7}} \quad -(3)$

$\boxed{z = 4}$

$\therefore y - \frac{20}{7} = \frac{8}{7} \Rightarrow y = \frac{28}{7}$ $\boxed{y = 4}$

$x + 8 - 4 = 3$

$\boxed{x = -1}$

$x = -1, \; y = 4, \; z = 4$ Ans

Q. Solve :

$x + y + z = -3$

$3x + y - 2z = -2$

$2x + 4y + 7z = 7$

Sol: Augmented matrix $[A : B]$

$$\begin{bmatrix} 1 & 1 & 1 & -3 \\ 3 & 1 & -2 & -2 \\ 2 & 4 & 7 & 7 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 2 & 5 & 13 \end{bmatrix}$$

-6
7

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 0 & 0 & 20 \end{bmatrix}$$

$R_2 \to \frac{-1}{2} R_2$

$$\begin{bmatrix} 1 & 1 & 1 & -3 \\ 0 & 1 & -5/2 & 7/2 \\ 0 & 2 & 5 & 13 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & 1 & 1 & -3 \\ 0 & 1 & 5/2 & 11/2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Rank $[A:B] = 3$

Rank $A = 2$

Rank $[A:B] \neq$ Rank A.

Hence, The System is Inconsistent and no Sol^n Exists.

Q3. $4x - 2y + 6z = 8$

$x + y - 3z = -1$

$5x - 3y + 9z = 21$

Sol: Augmented notation $[A:B]$

$$\begin{bmatrix} 4 & -2 & 6 & 8 \\ 1 & 1 & -3 & -1 \\ 5 & -3 & 9 & 21 \end{bmatrix}$$

$R_1 \to \frac{1}{4} R_1$

$$\begin{bmatrix} 1 & -1/2 & -3/2 & 2 \\ 1 & 1 & -3 & -1 \\ 5 & -3 & 9 & 21 \end{bmatrix}$$

Gauss Elimination Method:

To Solve System of linear Eq^n.

Page

Q. Test the consistency and hence solve the Eqn :-

$x_1 + 2x_2 + x_3 = 2$

$3x_1 + x_2 - 2x_3 = 1$

$4x_1 - 3x_2 - x_3 = 3$

$2x_1 + 4x_2 + 2x_3 = 4$

Sol: Augmented matrix

$[A : B]$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 1 & -2 & 1 \\ 4 & -3 & -1 & 3 \\ 2 & 4 & 2 & 4 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - 4R_1$

$R_4 \to R_4 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -5 & -5 & -5 \\ 0 & -11 & -5 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -5 & -5 & -5 \\ 0 & -11 & -5 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to -1/5 R_2$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & -11 & -5 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3 + 11R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 6 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\underbrace{\quad\quad}_{A}$ $\underbrace{\quad}_{B}$

Rank [A : B] = 3 = no. of non zero Rows

Rank [A] = 3 = no. of non zero Rows

∵ Rank (A : B) = Rank A = 3

⇒ System is consistent and have a ~~sol exist~~ unique soln.

Implies

$AX = B$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix}$$

$6x_3 = 6$ $\quad$ $x_3 = 1$ Ans

$x_2 + x_3 = 1$ $\quad$ $x_2 = 0$ Ans

$x_1 + 2x_2 + x_3 = 2$

$x + 1 = 2 \Rightarrow$ $\boxed{x_1 = 1}$ Ans

Q. Solve:

$2x + 3y + 7z = 5$

$3x + y - 3z = 13$

$2x + 19y - 47z = 32$

Sol: Augment notation :

$$\begin{bmatrix} 2 & -3 & 7 & 5 \\ 3 & 1 & -3 & 13 \\ 2 & 19 & -47 & 32 \end{bmatrix}$$

$R_1 \rightarrow R_1 - R_2$

$$\begin{bmatrix} -1 & -4 & 10 & -8 \\ 3 & 1 & -3 & 13 \\ 2 & 19 & -47 & 32 \end{bmatrix}$$

$R_1 \rightarrow -R_1$

$$\begin{bmatrix} 1 & 4 & -10 & 8 \\ 3 & 1 & -3 & 13 \\ 2 & 19 & -47 & 32 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 3R_1$

$R_3 \rightarrow R_3 - 2R_1$

1/8

$-47-20$

-27

$1-12$

$-3-30$

$13-24$

$$\begin{bmatrix} 1 & 4 & -10 & 8 \\ 0 & -11 & 27 & -11 \\ 0 & 11 & -27 & 16 \end{bmatrix}$$

$R_3 \rightarrow R_3 + R_2$

$$\begin{bmatrix} 1 & 4 & -10 & 8 \\ 0 & -11 & 27 & -11 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$R_2 \rightarrow -1/_{11} R_2$

$$\underbrace{\begin{bmatrix} 1 & 4 & -10 \\ 0 & 1 & 27/11 \\ 0 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 8 \\ 1 \\ 5 \end{bmatrix}}_{B}$$

Rank[A:B] = 3 = no. of non zero Rows
Rank [A] = 2 = no. of non-zero Rows

∴ Rank [A:B] ≠ Rank A = 2

Hence, The System is ~~consitent~~ inconsitent and has no soln. Exist.

Q3. $2x + 4y - z = 9$
$3x - y + 5z = 5$
$8x + 2y + 9z = 19$

Sol: Augmented Notation:
[A:B]

$$\begin{bmatrix} 2 & 4 & -1 & 9 \\ 3 & -1 & 5 & 5 \\ 8 & 2 & 9 & 19 \end{bmatrix}$$

$R_1 \to R_1 - R_2$

$$\sim \begin{bmatrix} -1 & +5 & -6 & 4 \\ 3 & -1 & 5 & 5 \\ 8 & 2 & 9 & 19 \end{bmatrix}$$

$R_1 \to -R_1$

$$\sim \begin{bmatrix} 1 & -5 & 6 & -4 \\ 3 & -1 & 5 & 5 \\ 8 & 2 & 9 & 19 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 - 8R_1$

$-1 + 15$ (14)

$2 + 40$ $9 - 30$ $5 + 18$ $5 + 12$

Date
Page

$$\sim \begin{bmatrix} 1 & 5 & 6 & -4 \\ 0 & 14 & -13 & 17 \\ 0 & 42 & -39 & 51 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$

$$\begin{bmatrix} 1 & 5 & 6 & -4 \\ 0 & 14 & -13 & 17 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank $[A;B] = 2$
Rank $[A] = 2$

Rank $[A;B] =$ Rank $A = 2$

Here, The System is Consistent and has ~~unique soln~~ Infinetly many Solution

For Unique Soln,

$AX = B$

$$\begin{bmatrix} 1 & 2 & 6 \\ 0 & 14 & -13 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 17 \\ 0 \end{bmatrix}$$

$x + 2y + 6z = -4 \quad -(1)$
$14y - 13z = 17 \quad -(2)$

Let $\boxed{z = k}$

$y = \frac{17 + 13k}{14}$, $x = -4 + 6k - 2y$
$x = -4 - 6k - \frac{17 + 13k}{7}$

$x =$

Q. Investigate for what value of λ and μ do the System of Eqⁿ.

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + \lambda z = \mu$$

have 1. no solution
2. unique solution
3. Infinite solution.

Sol: $[A \vdots B]$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}$$

$R_2 \to R_2 - R_1, \; R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{bmatrix}$$

1. No Solution,
Rank $[A \vdots B] \neq$ Rank A

$\boxed{\lambda = 3, \; \mu \neq 10}$ Ans

Rank A = 2
Rank $[A \vdots B] = 3$

non-trival soln $\Rightarrow$ infinite soln

2. Unique soln,
Rank [A:B] = Rank [A]
= no. of non-zero Rows.
= 3

$\therefore$ $\boxed{\lambda \neq 3, \mu \text{ can take any value}}$ Ans

3. Infinite soln,
Rank [A:B] = Rank [A] $< n$

$\therefore$ Rank [A:B] = 2
Rank [A] = 2

$\therefore$ $\boxed{\lambda = 3, \mu = 10}$ Ans

$\boxed{AX = 0}$

Q. Test whether the following system of eqn possess a non-trival soln?

$$x_1 + x_2 + 2x_3 + 3x_4 = 0$$
$$3x_1 + 4x_2 + 7x_3 + 10x_4 = 0$$
$$5x_1 + 7x_2 + 11x_3 + 17x_4 = 0$$
$$6x_1 + 8x_2 + 13x_3 + 16x_4 = 0$$

Sol:

Augmented matrix [A B]

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 3 & 4 & 7 & 10 \\ 5 & 7 & 11 & 17 \\ 6 & 8 & 13 & 16 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 5R_1$
$R_4 \to R_4 - 6R_1$

3 : 6 9

Date
Page

$$\sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 2 & 1 & -2 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$
$R_4 \to R_4 - 2R_2$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 4 \end{bmatrix}$$

Cannot Evaluate / make this zero Row

$\therefore$ Rank $A = 4$ = no. of non-zero Rows

$\Rightarrow$ Unique or Trivial Sol^n Exis

$AX = 0$

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + x_2 + 2x_3 + 3x_4 = 0$ — (1)
$x_2 + x_3 + x_4 = 0$ — (2)
$-x_3 = 0$ — (3)
$-x_3 - 4x_4 = 0$ — (4)

From Eq (3), Eq (4), Eq (2), Eq (1)

$\boxed{x_3 = 0, \quad x_4 = 0, \quad x_2 = 0, \quad x_1 = 0}$ Ans

Q. Solve :

$x_1 + 3x_2 + 2x_3 = 0$

$2x_1 - x_2 + 3x_3 = 0$

$3x_1 - 5x_2 + 4x_3 = 0$

$x_1 + 17x_2 + 4x_3 = 0$

Sol: $AX = O$, Homogenous System

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 3 & -5 & 4 \\ 1 & 17 & 4 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$R_4 \to R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -7 \\ 0 & -14 & -2 \\ 0 & 14 & 2 \end{bmatrix}$$

2 6 4
2 -1 3
-3 -9 -6
3 -5 4
-2

$R_2 \to -R_2/7$,

$R_3 \to R_3 + R_4$

$R_4 \to R_4 + R_3$

1 17 4
1 3 2
0 14 2

$$\sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\boxed{\text{Rank } A = 2 = \text{no. of non-zero Rows}}$

$\therefore \Rightarrow$ Infinite no. of solⁿ and non-Trivial solⁿ Exist.

$$AX = O$$

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & +7 & +1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + 3x_2 + 2x_3 = 0$ — (1)

$7x_2 + x_3 = 0$ — (2)

let $\boxed{x_2 = k}$

From Eq (2)

$\boxed{x_3 = -7k}$

From Eq (3)

$\boxed{x_1 = -11k}$

$$\Rightarrow \boxed{\begin{aligned} x_1 &= 11k \\ x_2 &= k \\ x_3 &= -7k \end{aligned}}$$ Ans

where, k arbitary constant

Q3. Find the values of k for which the System of Eqn:

$(3k-8)x + 3y + 3z = 0$

$3x + (3k-8)y + 3z = 0$

$3x + 3y + (3k-8)z = 0$

has a non-trivial solⁿ.

Sol: For Non-Trival Sol^n,

$\boxed{|A| = 0}$ (Singular matrix)

$$\begin{vmatrix} 3k-8 & 3 & 3 \\ 3 & 3k-8 & 3 \\ 3 & 3 & 3k-8 \end{vmatrix} = 0$$

$C_1 \rightarrow C_1 + C_2 + C_3$

$$\begin{vmatrix} 3k-2 & 3 & 3 \\ 3k-2 & 3k-8 & 3 \\ 3k-2 & 3 & 3k-8 \end{vmatrix}$$

$$(3k-2)\begin{vmatrix} 1 & 3 & 3 \\ 1 & 3k-8 & 3 \\ 1 & 3 & 3k-8 \end{vmatrix} = 0$$

$R_2 \rightarrow R_2 - R_1$

$R_3 \rightarrow R_3 - R_1$

$$(3k-2)\begin{vmatrix} 1 & 3 & 3 \\ 0 & 3k-11 & 0 \\ 0 & 0 & 3k-11 \end{vmatrix} = 0$$

$$(3k-2)(3k-11)^2 = 0$$

$$\boxed{k = \frac{2}{3}, \frac{4}{3}, \frac{11}{3}}$$ Ans

Linear Dependence and Indepe[ndence] of Vectors (matrices)

Def: Vector (matrices) $x_1, x_2 \dots x_n$ are said to be dependent if:

1. All the vectors (Rows or colu[mns]) are of the same order
2. n scalars $\lambda_1, \lambda_2, \lambda \dots \lambda_n$ (not all zero) Exist such th[at]

$$\boxed{\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n =}$$

Otherwise, They are linearly Ind[ependent]

Working Rule :-

1. Construct coefficient matrix A [with] element of given vectors as col[umns]
2. Find the Rank of A.
3. If $\boxed{\text{Rank } A = \text{no. of vectors}}$ ⇒ set of vectors are L.I.

b. If $\boxed{\text{Rank } A < \text{no of vectors}}$ ⇒ Set of vector are L.D

Ex: $x_1 = (1, 2, 3)$
$x_2 = (3, 4, 5)$
$x_3 = (7, 8, 9)$.

Date
Page

(4)

Sol. $A = \begin{bmatrix} 1 & 3 & 7 \\ 2 & 4 & 8 \\ 3 & 5 & 9 \end{bmatrix} = [x_1 \quad x_2 \quad x_3]$.

finding Rank of A,

$R_2 \to 2R_2 - 2R_1$
$R_3 \to R_3 - 3R_2$

$$\sim \begin{bmatrix} 1 & 3 & 7 \\ 0 & -2 & -6 \\ 0 & -4 & -12 \end{bmatrix}$$

2 4 8
2 6 14
3 9 21
3 5 9
22

$\sim R_2 \to 2R_2 - R_3$
$R_3 \to \frac{R_3}{2} - R_2$

- If the Vectors are linearly dependent Then one vector can be Expressed as linear combination of Remaining vectors

$$X_1 = a_1 X_2 + a_2 X_3$$

Ex: $(1,2,3) = a_1(3,4,5) + a_2(3,6,7)$

$3a_1 + 3a_2 = 1$
$4a_1 + 6a_2 = 2$
$5a_1 + 7a_2 = 3$

Q. Examine the following vectors for linear dependance and find the Relatives If Exists:-

Jitni vector utne

$x_1 = (1, 2, 4)$, $x_3\ (0, 1, 2)$

$x_2 = (2, -1, 3)$, $x_4\ (-3, 7, 2)$.

Sol: Matrix Relation is given by :-

$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4 = 0$

$$A = \begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 4R_1$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & -5 & 2 & 14 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Rank A = 3 = No. of non-zero Rows

No. of vectors = 4

Rank A < ~~Rank~~ No. of vector

$\therefore \Rightarrow$ vectors are linearly dependent

For homogenous System,

$$AX = 0$$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_1 + 2\lambda_2 - 3\lambda_4 = 0 \quad -(1)$$

$$-5\lambda_2 + \lambda_3 + 13\lambda_4 = 0 \quad -(2)$$

$$\lambda_3 + \lambda_4 = 0 \quad -(3)$$

Let $\lambda_4 = k$

Eq-(3), $\lambda_3 = -\lambda_4 = -k$

$$\boxed{\lambda_3 = -k}$$

Eq-(2)

$$-5\lambda_2 + 13k - k = 0$$

$$\boxed{\lambda_2 = \frac{12k}{5}}$$

Eq-(1)

$$\lambda_1 + \frac{24k}{5} - 3k = 0$$

$$\boxed{\lambda_1 = -\frac{9k}{5}}$$

$\therefore$ Matrix Relation :-

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4 = 0$$

$$-\frac{9k}{5} x_1 + \frac{12k}{5} x_2 - k x_3 + k x_4 = 0$$

$$-9x_1 + 12x_2 - 5x_3 - 5x_4 = 0$$

$$\boxed{9x_1 - 12x_2 + 5x_3 - 5x_4 = 0}$$

$x_1 = [1, 1, 1, 1]$, $x_2 = [0, 1, 1, 1]$,

$x_3 = [0, 0, 1, 1]$, $x_4 = [0, 0, 0, 1]$.

matrix Relation is given by:

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4 = 0 \quad \text{—(I}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_2$

$R_4 \to R_4 - R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\boxed{\text{Rank } A = 4}$ = no. of non-zero Rows

$\boxed{\text{no. of vectors} = 4}$

$\Rightarrow$ Rank $A = 4$ = no. of vectors

$\Rightarrow$ vectors are linearly dependent

Q3. Show that the vectors
$x_1 = [a_1, b_1]$, $x_2 = [a_2, b_2]$
are linearly dependent iff
$a_1 b_2 - a_2 b_1 = 0$.

Sol: Since, vector vector are L.D
$\therefore |A| = 0$

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 0$$

$$\boxed{a_1 b_2 - a_2 b_1 = 0}$$

Hence, Proved

Q. If the vectors:
$(0, 1, a)$, $(1, a, 1)$ and $(a, 1, 0)$ are linearly dependent then find value of a?

Sol:
$$A = \begin{bmatrix} 0 & 1 & a \\ 1 & a & 1 \\ a & 1 & 0 \end{bmatrix}$$

Since, Vectors are linearly dependent (LD)

$\therefore |A| = 0$

$$\begin{vmatrix} 0 & 1 & a \\ 1 & a & 1 \\ a & 1 & 0 \end{vmatrix} \Rightarrow 0 - 1(-a) + a(1 - a^2) = 0$$

$$a + a - a^2 = 0$$
$$2a - a^2 = 0$$
$$a(2 - a) = 0$$
$$\boxed{a = 0,\ a = }$$ Ans

Solution of System of Linear Equation By Iterative method: (Gauss - Siedal Method)

System of Equations should be diagonally Dominant.

Use Gauss - Siedal iterative method to solve the following System of Simultaneous Eqn:

$9x + 4y + z = -17$
$x - 2y - 6z = 14$
$x + 6y = 4$

Perform four Iteration?

$9x + 4y + z = -17$ —(1)
$x - 2y - 6z = 14$ —(2)
$x + 6y = 4$ —(3)

Rearranging:

$9 \cdot x + 4y + z = -17$ —(1)
$x + 6y = 4$ —(2)
$x - 2y - 6z = 0$ —(3)

Gauss-Siedal method Applied;

$$x = \frac{1}{9}(-17 - 4y - z) \quad -(4)$$

$$y = \frac{1}{6}(4 - x) \quad -(5)$$

$$z = \frac{1}{6}(x - 2y - 14) \quad -(3)$$

First approximation:-

Starting with $y = 0, z = 0$

$$x^{(1)} = \frac{-17}{9} = -1.888$$

Put $x^{(1)} = -1.888$, $z = 0$

$$\boxed{y^{(1)} = 0.9815}$$

$$z^{(1)} = \frac{1}{6}\left(x^{(1)} - 2y^{(1)} - 14\right)$$

$$z^{(1)} = \frac{1}{6}\left[(-1.888) - 2(0.9815) - 14\right]$$

$$\boxed{z^{(1)} = -2.9753}$$

2nd Approximation:-

$$x^{(2)} = \frac{1}{9}\left(-17 - 4y^{(1)} - z^{(1)}\right)$$

$$y^{(2)} = \frac{1}{6}\left(4 - x^{(1)}\right)$$

$$z^{(2)} = \frac{1}{6}\left(x^{(1)} - 2y^{(1)} - 14\right)$$

$$x^{(2)} = -1.994$$

$$y^{(2)} = 0.9813$$

$$z^{(2)} = -2.975$$

2. Solve the following system of Eqn using Gauss-Seidal Iterative method.

$2x + 10y + z = 51$ — (2)
$10x + y + 2z = 44$ — (1)
$x + 2y + 10z = 61$. — (3)

Rearranging Eqn to make it diagonally dominant.

$10x + y + 2z = 44$ — (1)
$2x + 10y + z = 51$ — (2)
$x + 2y + 10z = 61$. — (3)

Eq (1)

$$x = \frac{1}{10}(44 - 2z - y) \quad -(4)$$

Eq (2)

$$y = \frac{1}{10}(51 - 2x - z) \quad -(5)$$

Eq (3)

$$z = \frac{1}{10}(61 - x - 2y) \quad -(6)$$

Ist Approximation:-

$y = 0,\ z = 0,\ x = 0$

$\boxed{x^{(1)} = 4.4}$

$\cancel{y^{(1)} = 5.1}$ $\quad y^{(1)} = \frac{1}{10}(51 - 2x^{(1)} -$

$\cancel{z^{(1)} = 6.1}$ $\quad = \frac{1}{10}(51 - 8.8)$

$\boxed{y^{(1)} = 4.22}$

$$z^{(1)} = \frac{1}{10}\left(61 - x^{(1)} - 2y^{(1)}\right)$$

$$= \frac{1}{10}\left(61 - 4.4 - 8.44\right)$$

$$\boxed{z^{(1)} = 4.816}$$

2^{nd} Approximation:-

$$x^{(1)} = 4.4, \quad y^{(1)} = 4.22$$
$$z^{(1)} = 4.816$$

3^{rd} Approximation,

$$x^{(2)} = \frac{1}{10}\left(44 - 2z^{(1)} - y^{(1)}\right)$$

$$= \frac{1}{10}\left(44 - 9.632 - 4.22\right)$$

$$\boxed{x^{(2)} = 3.0148}$$

$$y^{(2)} = \frac{1}{10}\left(51 - 2x^{(2)} - z^{(1)}\right)$$

$$= \frac{1}{10}\left(51 - 3.0148 - 4.816\right)$$

$$\boxed{y^{(2)} = 4.3156}$$

$$z^{(2)} = \frac{1}{10}\left(61 - x^{(2)} - 2y^{(2)}\right)$$

$$= \frac{1}{10}\left(61 - 3.0148 - 12.9498\right)$$

$z^{(2)} = 4.$[illegible]

4^{th} Approx :-

$$x^{(3)} = \frac{1}{10}(44 - 2z^{(2)} - y^{(2)})$$

$$\boxed{x^{(3)} = 2.999}$$

$$y^{(3)} = \frac{1}{10}(51 - 2(x)^{(3)} - z^{(2)})$$

$$\boxed{y^{(3)} = 4.003}$$

$$z^{(3)} = \frac{1}{10}(61 - x^{(3)} - 2y^{(3)})$$

$$\boxed{z^{(3)} = 4.9992}$$

Ans. $\boxed{x=3, y=4, z=5}$

Q3. Solve by Gauss-Seidal Iterati[on] method:

$$20x + y - 2z = 17 \quad -(1)$$
$$3x + 20y - z = -18 \quad -(2)$$
$$2x - 3y + 20z = 25 \quad -(3)$$

Sol:

$$x = \frac{1}{20}(17 + 2z - y) \quad -(4)$$

$$y = \frac{1}{20}(z - 18 - 3x) \quad -(5)$$

$$z = \frac{1}{20}(25 + 3y - 2x) \quad -(6)$$

I^{st} Approx.

$x = 0, y = 0, z = 0$

$$x^{(1)} = \frac{1}{20}(17 + 0 + 0) = 0.85$$

$$\boxed{x^{(1)} = 0.85}$$

$$y^{(1)} = \frac{1}{20}(0 - 18 - 2.55)$$

$$\boxed{y^{(1)} = -1.0275}$$

$$z^{(1)} = \frac{1}{20}(25 + -3.0825 - 1.7)$$

$$\boxed{z^{(1)} = 1.010875}$$

2^{nd} Approximation:

$$x^{(2)} = \frac{1}{20}(17 + 2z^{(1)} - y^{(1)})$$

$$= \frac{1}{20}(17 + 2.02175 + 1.0275)$$

$$\boxed{x^{(2)} = 1.00024625}$$

$$y^{(2)} = \frac{1}{20}(z^{(1)} - 18 - 3x^{(2)})$$

$$= \frac{1}{20}(1.010875 - 18 - 3.00073875)$$

$$= \frac{1}{20}(-19.9965125)$$

$$\boxed{y^{(2)} = -0.999825}$$

$$z^{(2)} = \frac{1}{20}(25 - 2.999476875 - 2004925)$$

$$\boxed{z^{(2)} = 0.99977}$$

Ans. $\boxed{x = 1, y = -1, z = 1}$

Solve by Gauss-Sedial method:

$8x - 3y + 2z = 20$ — (1)

$6x + 3y + 12z = 35$ — (3)

$4x + 11y - z = 33$ — (2)

Rearranging eqn to make it diagonally dominant.

$8x - 3y + 2z = 20$ — (1)

$4x + 11y - z = 33$ — (2)

$6x + 3y + 12z = 35$ — (3)

$$x = \frac{1}{8}(20 - 2z + 3y) \quad -(4)$$

$$y = \frac{1}{11}(33 + z - \cancel{4y}\,4x) \quad -(5)$$

$$z = \frac{1}{12}(35 - 3y - 6x) \quad -(6)$$

Ist Approximation,

Let, $x = 0, y = 0, z = 0$

$$x^{(1)} = \frac{1}{8}(20 - 0 - 0) = 2.5$$

$$y^{(1)} = \frac{1}{11}\left(33 + 0 - 4(x)^{(1)}\right)$$

$$= \frac{1}{11}(33 - 10) = 2.0909$$

$$\boxed{y^{(1)} = 2.0909}$$

Date
Page

$$z^{(1)} = \frac{1}{12}\left(35 - 3(y)^{(1)} - 6x^{(1)}\right)$$

$$= \frac{1}{12}(35 - 6.2727 - 15)$$

$$= \frac{13.7273}{12}$$

$$\boxed{z^{(1)} = 1.4394}$$

IInd Approximation :-

$$x^{(2)} = \frac{1}{8}\left(20 - 2z^{(1)} + 3y^{(1)}\right)$$

$$= \frac{1}{8}(20 - 2.8788 + 6.2727)$$

$$\boxed{x^{(2)} = 1.3560625}$$

$$y^{(2)} = \frac{1}{11}\left(33 + z^{(1)} - 4x^{(2)}\right)$$

$$= \frac{1}{11}(33 + 1.4394 - 5.42425)$$

$$= \frac{29.01515}{11}$$

$$\boxed{y^{(2)} = 2.63774}$$

$$z^{(2)} = \frac{1}{12}\left(35 - 3y^{(2)} - 6x^{(2)}\right)$$

$$= \frac{1}{12}(35 - 7.91322 - 8.1356025)$$