KIET GROUP OF INSTITUTIONS, DELHI – NCR, GHAZIABAD

B. Tech. (I SEM), 2023-24

Calculus for Engineers (K24AS11)

UNIT1(Differential Calculus I)

Syllabus: Introduction to Limits, continuity and differentiability for function of two variables, Higher order Partial derivatives, Euler's Theorem for homogeneous functions, Total derivative of composite functions.

Course Outcomes:

S.NO.	Course Outcome	BL
1	Apply the concept of partial differentiation in application of homogeneous and composite functions	2,3

CONTENT

UNIT I. Differential Calculus - I

Introduction of limits, Continuity and differentiability for function of two variables

LIMIT

A function $f(x,y)$ is said to have a limit L as the point (x,y) approaches (a,b) and is denoted as

$$\lim_{(x,y)\to(a,b)} f(x,y) = L. \quad \text{or} \quad \lim_{\substack{x\to a \\ y\to b}} f(x,y) = L.$$

Method of Obtaining limit

1. Evaluate $\lim f(x,y)$ along path I: $x \to a$ and $y \to b$
2. Evaluate $\lim f(x,y)$ along path II: $y \to b$ and $x \to a$

If the limit values along path I and II are same, then the limit exist. Otherwise not

3. If $a=0$, $b=0$, evaluate limit along say path $y = mx$ and $y = mx^n$.

Q1 Evaluate $\lim\limits_{(x,y)\to(1,2)} \frac{x^2+2y}{x^2+y^2}$

Sol $\lim\limits_{\substack{x\to 1\\ y\to 2}} \frac{x^2+2y}{x^2+y^2}$.

Case 1
Put $x=1$
$\lim\limits_{y\to 2} \frac{1+2y}{1+y^2} = \frac{1+4}{1+4} = \frac{5}{5} = 1 = f_1$.

Case 2 Put $y=2$
$\lim\limits_{x\to 1} \frac{x^2+2\times 2}{x^2+4} = \frac{1+4}{1+4} = \frac{5}{5} = 1 = f_2$.

~~Case 3~~ $\Rightarrow f_1 = f_2 = 1$

$\Rightarrow$ limit exist

$\therefore \lim\limits_{(x,y)\to(1,2)} \frac{x^2+2y}{x^2+y^2} = 1$

Q2. Evaluate $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{x^2y}{x^4+y^2}$.

Sol Case 1 Put $x=0$
$\lim\limits_{y\to 0} \frac{0\cdot y}{0+y^2} = 0 = f_1$

Case 2 Put $y=0$
$\lim\limits_{x\to 0} \frac{x^2\cdot 0}{x^4+0} = 0 = f_2$

Case 3 Put $y=mx$
$\lim\limits_{x\to 0} \frac{x^2(mx)}{x^4+(mx)^2} = \lim\limits_{x\to 0} \frac{mx^3}{x^4+m^2x^2} = 0 = f_3$

Case 4 Put $y=mx^2$.
$\lim\limits_{x\to 0} \frac{x^2(mx^2)}{x^4+(mx^2)^2} = \lim\limits_{x\to 0} \frac{m\cancel{x^4}}{\cancel{x^4}(1+m^2)} = \frac{m}{1+m^2} = f_4$

$\Rightarrow f_1 = f_2 = f_3 \neq f_4$
$\Rightarrow$ limit does not exist

Q3 Evaluate

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)$$

Sol Put $x=0$

Case 1 $\lim_{y\to 0} (0+y^2) = 0 = f_1$

Case 2. Put $y=0$

$\lim_{x\to 0} (x^2+0) = 0 = f_2$

Case 3 Along $y=mx$

$\lim_{x\to 0} x^2+(mx)^2$

$= \lim_{x\to 0} x^2(1+m^2) = 0 = f_3$

Case 4 Along $y=mx^2$

$\lim_{x\to 0} x^2+(mx^2)^2$

$= \lim_{x\to 0} x^2(1+m^2x^2) = 0 = f_4$

$\Rightarrow f_1 = f_2 = f_3 = f_4.$

$\Rightarrow$ limit exists

$\therefore \lim_{(x,y)\to(0,0)} x^2+y^2 = 0$

Q4. Evaluate $\lim_{\substack{x\to 1\\ y\to 2}} \frac{3x^2y}{x^2+y^2+5}$

Sol Case I Put $x=1$

$\lim_{y\to 2} \frac{3(1^2)y}{(1)^2+y^2+5} = \frac{3\times 2}{1+4+5} = \frac{\cancel{6}\,3}{\cancel{10}\,5} = \frac{3}{5} = f_1$

Case 2 Put $y=2$.

$\lim_{x\to 1} \frac{3x^2(2)}{x^2+(2)^2+5} = \frac{6(1)^2}{4+(1^2)+5} = \frac{\cancel{6}\,3}{\cancel{10}\,5} = \frac{3}{5} = f_2$

$\Rightarrow f_1 = f_2 \Rightarrow$ limit exists & value is $\frac{3}{5}$.

Q5 Evaluate $\lim_{\substack{x \to \infty \\ y \to 3}} \frac{2x-3}{x^3+4y^3}$

Sol. $\lim_{\substack{x \to \infty \\ y \to 3}} \frac{2x-3}{x^3+4y^3}$.

Case 1 Put $x \to \infty$.

$$\lim_{y \to 3} \left[\lim_{x \to \infty} \frac{2x-3}{x^3+4y^3} \right]$$

$$\lim_{y \to 3} \left[\lim_{x \to \infty} \frac{\frac{2}{x^2} - \frac{3}{x^3}}{1+4\left(\frac{y}{x}\right)^3} \right]$$

$$= \lim_{y \to 3} \left(\frac{0-0}{1+4(0)} \right) = 0 = f_1$$

Case 2 Put $y = 3$.

$$\lim_{x \to \infty} \left[\lim_{y \to 3} \frac{2x-3}{x^3+4y^3} \right]$$

$$= \lim_{x \to \infty} \left[\frac{2x-3}{x^3+108} \right]$$

$$= \lim_{x \to \infty} \frac{\frac{2}{x^2} - \frac{3}{x^3}}{1+\frac{108}{x^3}} = \frac{0-0}{1+0} = 0 = f_2.$$

$\Rightarrow f_1 = f_2$

Hence limit exist with value zero.

Continuity of two variables

A function $f(x,y)$ is said to be continuous at a point (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

Test of Continuity at a Point (a,b)

1. $f(a,b)$ should be well defined
2. $\lim f(x,y)$ as $(x,y) \to (a,b)$ should exists (must be unique and same along any path)

Q1 If $f(x,y) = \begin{cases} \dfrac{x^2+2y}{x+y^2} \\ 1 \end{cases}$ when $x=1,\ y=2$.

Sol ① $f(x,y) \Rightarrow f(a,b) = f(1,2) = \dfrac{1^2+2(2)}{1+(2)^2}$

$$= \frac{1+4}{1+4} = \frac{5}{5} = 1$$

$\therefore\ f(a,b) = 1 = f_1$

② $\lim\limits_{\substack{x\to 1 \\ y\to 2}} \dfrac{x^2+2y}{x+y^2}$

Case 1 Put $x=1$ $\lim\limits_{y\to 2} \dfrac{1+2y}{1+y^2} = \dfrac{1+4}{1+4} = \dfrac{5}{5} = 1 = f_1$

Case 2 Put $y=2$ $\lim\limits_{x\to 1} \dfrac{x^2+4}{x+4} = \dfrac{1+4}{1+4} = \dfrac{5}{5} = 1 = f_2$.

$\Rightarrow f_1 = f_2 \quad \therefore$ limit exist

$\therefore \lim\limits_{\substack{x\to 1 \\ y\to 2}} \dfrac{x^2+2y}{x+y^2} = 1 = f(1,2)$

$\Rightarrow f(x,y)$ is Continuous

Q2 If $f(x,y) = \dfrac{x^3-y^3}{x^2+y^2}$ when $x \neq 0, y \neq 0$

and $f(x,y) = 0$ when $x=0, y=0$, find out whether the function $f(x,y)$ is Continuous at Origin

Sol. Case I First calculate the limit of the function

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3-y^3}{x^2+y^2} = \lim_{y\to 0}\left(\lim_{x\to 0}\frac{x^3-y^3}{x^2+y^2}\right)$$
$$= \lim_{y\to 0}\left(\frac{-y^3}{y^2}\right) = \lim_{y\to 0}(-y) = 0 = f_1$$

Case 2

$$\lim_{\substack{y\to 0\\ x\to 0}} \frac{x^3-y^3}{x^2+y^2} = \lim_{x\to 0}\left(\lim_{y\to 0}\frac{x^3-y^3}{x^2+y^2}\right) = \lim_{x\to 0}\left(\frac{x^3}{x^2}\right)$$
$$= \lim_{x\to 0} x = 0 = f_2$$

Case 3

$$\lim_{\substack{y=mx\\ x\to 0}} \frac{x^3-y^3}{x^2+y^2} = \lim_{x\to 0}\left[\lim_{y=mx}\frac{x^3-y^3}{x^2+y^2}\right]$$
$$= \lim_{x\to 0}\left[\frac{x^3-m^3x^3}{x^2+m^2x^2}\right] = \lim_{x\to 0}\frac{x^3(1-m^3)}{x^2(1+m^2)}$$
$$= \lim_{x\to 0} \frac{x(1-m^3)}{(1+m^2)} = 0 = f_3$$

Case 4

$$\lim_{\substack{y=mx^2\\ x\to 0}} \frac{x^3-y^3}{x^2+y^2} = \lim_{x\to 0}\left[\lim_{y=mx^2}\frac{x^3-y^3}{x^2+y^2}\right]$$
$$= \lim_{x\to 0}\left[\frac{x^3-m^3x^6}{x^2+m^2x^4}\right] = \lim_{x\to 0}\frac{x^3(1-m^3x^3)}{x^2(1+m^2x^2)}$$
$$= \lim_{x\to 0}\frac{x(1-m^3x^3)}{1+m^2x^2} = 0 = f_4$$

$\Rightarrow$ $f_1 = f_2 = f_3 = f_4$

$\Rightarrow$ limit exist and equal to zero

$\Rightarrow$ $\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = 0 = f(0,0)$

$\Rightarrow$ function f is Continuous at the Origin

Q3 Discuss the Continuity of the function

$$f(x,y) = \frac{x}{\sqrt{x^2+y^2}} \quad \text{when } (x,y) \neq (0,0)$$

and $f(x,y) = 2$ when $(x,y) = (0,0)$

Sol: At first find the limit

① $$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x}{\sqrt{x^2+y^2}} = \lim_{y\to 0}\left(\lim_{x\to 0} \frac{x}{\sqrt{x^2+y^2}}\right) = \lim_{y\to 0} 0 = 0 = f_1$$

② $$\lim_{\substack{y\to 0\\ x\to 0}} \frac{x}{\sqrt{x^2+y^2}} = \lim_{x\to 0}\left(\lim_{y\to 0} \frac{x}{\sqrt{x^2+y^2}}\right) = \lim_{x\to 0} 1 = 1 = f_2$$

$\Rightarrow$ $f_1 \neq f_2$

$\Rightarrow$ The function is discontinuous at the Origin

Differentiability in two Variables

~~If~~ The function $f(x,y)$ is differentiable at point (a,b) if

$$\lim_{\substack{h\to 0\\ K\to 0}} \frac{f(a+h, b+K) - f(a,b) - h\, f_x(a,b) - K\, f_y(a,b)}{\sqrt{h^2+K^2}} \text{ exis}$$

* If $f(x,y)$ is differentiable, then the Partial derivatives f_x and f_y both exist and are finite

$$f_x = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h, y) - f(x,y)}{h}$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{K\to 0} \frac{f(x, y+K) - f(x,y)}{K}.$$

Q1 Let $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$

Show that $f(x,y)$ is Continuous but not differentiable at $(0,0)$

Sol To show $\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = f(0,0)$

$$f_1 = \lim_{\substack{x\to 0\\ y\to 0}} = \lim_{y\to 0} \left(\lim_{x\to 0} \frac{xy}{\sqrt{x^2+y^2}} \right) = 0$$

$$f_2 = \lim_{\substack{x\to 0\\ y\to 0}} = \lim_{x\to 0} \left(\lim_{y\to 0} \frac{xy}{\sqrt{x^2+y^2}} \right) = 0$$

$$f_3 = \lim_{\substack{x\to 0 \\ y = mx.}} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{x\to 0} \frac{x(mx)}{\sqrt{x^2+(mx)^2}}, = 0.$$

$$f_4 = \lim_{\substack{x\to 0 \\ y=mx^2}} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{x\to 0} \frac{x(mx^2)}{\sqrt{x^2+(mx^2)^2}}$$

$$= \lim_{x\to 0} \frac{mx^3}{\sqrt{x^2+mx^4}} = 0$$

$\Rightarrow$ $f_1 = f_2 = f_3 = f_4$

$\therefore$ limit exists and equal to zero

$\therefore$ $\lim_{\substack{x\to 0 \\ y\to 0}} f(x, y) = (0,0)$

$\Rightarrow$ $f(x, y)$ is Continuous at $(0,0)$

Test of differentiability

$$f_x(0,0) = \lim_{x\to 0} \frac{f(0+h, 0) - f(0,0)}{h} = 0$$

$$f_y(0,0) = \lim_{K\to 0} \frac{f(0, 0+K) - f(0,0)}{K} = 0$$

$$\lim_{(h,K)\to(0,0)} \frac{f(0+h, 0+K) - f(0,0) - h f_x(0,0) - K f_y(0,0)}{\sqrt{h^2+K^2}}$$

$$\Rightarrow \lim_{(h,K)\to(0,0)} \frac{f(h,K)}{\sqrt{h^2+K^2}}$$

$$= \lim_{(h,K)\to(0,0)} \frac{\frac{hK}{\sqrt{h^2+K^2}}}{\sqrt{h^2+K^2}} = \lim_{(h,K)\to(0,0)} \frac{hK}{h^2+K^2}$$

let $K = mh$

$$\lim_{h\to 0} \frac{mh^2}{h^2+m^2h^2} = \lim_{h\to 0} \frac{m}{1+m^2}$$

$$= \frac{m}{1+m^2}$$

Limit depends on m

$\Rightarrow$ $f(x,y)$ is not Unique

$\Rightarrow$ function is not differentiable but it is Continuous.

Q2 Show that the function

$$f(x,y) = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is not differentiable at Origin although Partial derivatives $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ exist at $(0,0)$.

Sol. $f_x(0,0) = \lim_{h\to 0} \dfrac{f(0+h,0) - f(0,0)}{h}$

$$= \lim_{h\to 0} \frac{\frac{h^3-0}{h^2-0}}{h} = \lim_{h\to 0} \frac{h^3}{h^3} = 1$$

$f_y(0,0) = \lim_{K\to 0} \dfrac{f(0,0+K) - f(0,0)}{K} = -1.$

Using formula

$$\lim_{(h,K)\to(0,0)} \frac{f(0+h,0+K) - f(0,0) - h f_x(0,0) - K f_y(0,0)}{\sqrt{h^2+K^2}}$$

$$\lim_{(h,K)\to(0,0)} \frac{\frac{h^3-K^3}{h^2+K^2} - 0 - h + K.}{\sqrt{h^2+K^2}}$$

At Origin, Along $K = mh$.

$$= \lim_{h\to 0} \frac{\frac{h^3 - m^3h^3}{h^2+m^2h^2} - h + mh}{\sqrt{h^2+m^2h^2}}$$

$$= \lim_{h\to 0} \frac{\frac{h(1-m^3)}{1+m^2} - h + mh}{h\sqrt{1+m^2}} = \lim_{h\to 0} \frac{\frac{1-m^3}{1+m^2} - 1 + m}{\sqrt{1+m^2}}$$

$\Rightarrow$ limit depends on m

$\Rightarrow$ limit is not Unique

$\Rightarrow f(x,y)$ is not differentiable at $(0,0)$

1.2 Partial Derivatives and its Applications: The concept of partial differentiation is applied for function of two or more independent variables. Partial differentiation is the process of finding various orders partial derivatives of given function. Partial derivatives are very useful in determining Maxima and Minima, Jacobian, Series expansion of function of several variables, Error and Approximation, In solution of wave and heat equations etc.

1.2.1 First order partial derivatives:

If$u = f(x, y)$ be any function of two independent variables x, yhen first order partial derivative of u with respect to $'x'$ is obtained by differentiating u with respect to 'x' treating y and function of 'y' as constant and is denoted by $\frac{\partial u}{\partial x}$

Similarly fist order partial derivative of u with respect to 'y' is obtained by differentiating u with respect to 'y' treating 'x' and functions of 'x' as constant and is denoted by$\frac{\partial u}{\partial y}$.

Example-1: If $u = x^2 + 2xy - y^2 + 2x - 3y + 5$ then find $\frac{\partial u}{\partial x} and \frac{\partial u}{\partial y}$.

Solution: Here we have given $u = x^2 + 2xy - y^2 + 2x - 3y + 5$

Therefore, to get value of $\frac{\partial u}{\partial x}$ we shall differentiate u with respect to 'x' treating 'y' as constant

$$\boldsymbol{\frac{\partial u}{\partial x} = 2x + 2y + 2}$$

Also, to find $\frac{\partial u}{\partial y}$ we shall differentiate u with respect to 'y' treating 'x' as constant

$$\boldsymbol{\frac{\partial u}{\partial y} = 2x - 2y - 3}.$$

1.2.2 Second and higher order partial derivatives:

If $u = f(x, y)$ be any function of two independent variables x, y then second order partial derivative of u with respect to $'x'$ is obtained by differentiating u with respect to 'x' twice in succession treating 'y' and function of 'y' as constant each time and is denoted by $\frac{\partial^2 u}{\partial x^2}$.

Similarly second order partial derivative of u with respect to 'y' is obtained by differentiating u with respect to 'y' twice treating 'x' and functions of 'x' as constant each time and is denoted by$\frac{\partial^2 u}{\partial y^2}$.

Here if we differentiate $\frac{\partial u}{\partial y}$ with respect to 'x' treating 'y' as constant then we get second order derivative of u denoted as$\frac{\partial^2 u}{\partial x \partial y}$. Similarly differentiating $\frac{\partial u}{\partial x}$ with respect to 'y' treating 'x' and function of 'x' as constant we get second order derivative of u denoted as $\frac{\partial^2 u}{\partial y \partial x}$. It is interesting to see that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$. In the similar way third and other higher order partial derivatives can be evaluated.

Example-2: Find all first and second order partial derivatives of$u(x, y) = x^y$. Hence show that$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

Solution: Here we have $u(x, y) = x^y \dots\dots\dots\dots(1)$

Differentiating equation (1) partially with respect to ‘x’ and ‘y’ we get

$$\frac{\partial u}{\partial x} = yx^{y-1} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\; (2)$$

$$\frac{\partial u}{\partial y} = x^y logx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(3)$$

Differentiating equation (2) partially with respect to ‘x’ and ‘y’ we get

$$\frac{\partial^2 u}{\partial x^2} = y(y-1)x^{y-2} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\; (4)$$

$$\boldsymbol{\frac{\partial^2 u}{\partial y \partial x} = yx^{y-1}\, logx + x^{y-1}} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\; (5)$$

Now differentiating equation (3) partially with respect to ‘y’ and ‘x’ we get

$$\frac{\partial^2 u}{\partial y^2} = x^y (logx)^2 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\; (6)$$

$$\frac{\partial^2 u}{\partial x \partial y} = yx^{y-1}\, logx + x^y \left(\frac{1}{x}\right)$$

$$\boldsymbol{\frac{\partial^2 u}{\partial x \partial y} = yx^{y-1}\, logx + x^{y-1}} \;\ldots\ldots\ldots\ldots\ldots\ldots.(7)$$

From equations (5) & (7), it is clear that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

Example-3: If $u = \log(tanx + tany + tanz)$, then prove that $sin2x\frac{\partial u}{\partial x} + sin2y\,\frac{\partial u}{\partial y} + sin2z\,\frac{\partial u}{\partial z} = 2$.

Solution: In this problem, we have $u = \log(tanx + tany + tanz) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$

Differentiating equation (1) partially with respect to ‘x’ we get

$$\frac{\partial u}{\partial x} = \frac{1}{tanx + tany + tanz}(sec^2 x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

Similarly differentiating (1) partially with respect to ‘y’ and ‘z’ we get

$$\frac{\partial u}{\partial y} = \frac{1}{tanx + tany + tanz}(sec^2 y) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(3)$$

$$\frac{\partial u}{\partial z} = \frac{1}{tanx + tany + tanz}(sec^2 z) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(4)$$

Multiplying equations (2), (3) and (4) with $sin2x, sin2y\; and\; sin2z$ respectively and then adding, we get

$$sin2x\,\frac{\partial u}{\partial x} + \text{sin2y}\frac{\partial u}{\partial y} + sin2z\,\frac{\partial u}{\partial z} = sin2x\left\{\frac{1}{tanx+tany+tanz}(sec^2 x)\right\} +$$

$$sin2y\left\{\frac{1}{tanx+tany+tanz}(sec^2 y)\right\} + sin2z\left\{\frac{1}{tanx+tany+tanz}(sec^2 z)\right\}$$

$sin2x\,\frac{\partial u}{\partial x}$+sin2y$\frac{\partial u}{\partial y} + sin2z\,\frac{\partial u}{\partial z} = \frac{2tanx}{tanx+tany+tanz} + \frac{2tany}{tanx+tany+tanz} + \frac{2tanz}{tanx+tany+tanz} +$

$$\boldsymbol{sin2x\,\frac{\partial u}{\partial x} + \mathbf{sin2y}\frac{\partial u}{\partial y} + sin2z\,\frac{\partial u}{\partial z} = 2.}$$

Example-4: If $x = rcos\theta, y = rsin\theta$, then find (a) $\left(\frac{\partial x}{\partial r}\right)_\theta$ and $\left(\frac{\partial y}{\partial \theta}\right)_r$, and (b) $\left(\frac{\partial r}{\partial x}\right)_y$ and $\left(\frac{\partial \theta}{\partial y}\right)_x$.

Solution: Here we have

$x = rcos\theta$ ………………………..(1),

and $y = rsin\theta$ ………………………..(2)

Differentiating equation (1) partially with respect to 'r' treating 'θ' as constant, we get

$$\boldsymbol{\left(\frac{\partial x}{\partial r}\right)_\theta = cos\theta}$$

Similarly differentiating (2) partially with respect to 'θ' treating 'r' as constant we get

$$\boldsymbol{\left(\frac{\partial y}{\partial \theta}\right)_r = rcos\theta}$$

Again, from equations (1) and (2), we get

$r^2 = x^2 + y^2$ ……………………………………… (3)

and, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ …………………………………… (4)

Differentiating equation (3) partially with respect to 'x' treating 'y' as constant, we get

$$2r\left(\frac{\partial r}{\partial x}\right)_y = 2x \quad \boldsymbol{or}\ \boldsymbol{\left(\frac{\partial r}{\partial x}\right)_y = \frac{x}{r}}$$

Similarly Differentiating equation (3) partially with respect to 'y' treating 'x' as constant, we get

$$\left(\frac{\partial \theta}{\partial y}\right)_x = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right)$$

$$\boldsymbol{\left(\frac{\partial \theta}{\partial y}\right)_x = \frac{x}{x^2+y^2}}\ .$$

Example-5: If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that

a). $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$, and b) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$.

Solution: Here we have, $u = \log(x^3 + y^3 + z^3 - 3xyz)$ ……………………… (1)

Differentiating equation partially with respect to 'x' we get

$$\frac{\partial u}{\partial x}=\frac{1}{(x^3+y^3+z^3-3xyz)}(3x^2-3yz)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Similarly differentiating equation (1) partially with respect to ‘y’ and ‘z’ we get

$$\frac{\partial u}{\partial y}=\frac{1}{(x^3+y^3+z^3-3xyz)}(3y^2-3zx)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$\frac{\partial u}{\partial z}=\frac{1}{(x^3+y^3+z^3-3xyz)}(3z^2-3xy)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Adding equations (2), (3) & (4) we get

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{1}{(x^3+y^3+z^3-3xyz)}(3x^2+3y^2+3z^2-3xy-3yz-3zx)$$

$$\boldsymbol{\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3}{x+y+z}}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\qquad(5)$$

Now,

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2u=\frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2u=-\frac{3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}$$

$$\boldsymbol{\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2u=-\frac{9}{(x+y+z)^2}}.$$

Example-6:Find $\frac{\partial u}{\partial r}$ *and* $\frac{\partial u}{\partial \theta}$ if $u=e^{r\cos\theta}.\cos(r\sin\theta)$

Solution: We have $u=e^{r\cos\theta}.\cos(r\sin\theta)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$

Differentiating equation (1) partially with respect to ‘r’ we get

$$\frac{\partial u}{\partial r}=\cos\theta.e^{r\cos\theta}.\cos(r\sin\theta)+e^{r\cos\theta}.\{-\sin(r\sin\theta)\}.\sin\theta$$

$$\frac{\partial u}{\partial r}=e^{r\cos\theta}\{\cos\theta.\cos(r\sin\theta)-\sin\theta.\sin(r\sin\theta\}$$

$$\frac{\partial u}{\partial r}=e^{r\cos\theta}.\cos(r\sin\theta+\theta)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Now, differentiating equation (1) partially with respect to ‘θ’ we get

$$\frac{\partial u}{\partial \theta} = -rsin\theta . e^{rcos\theta} . \cos(rsin\theta) + e^{rcos\theta} . \{-\sin(rsin\theta)\} . rcos\theta$$

$$\frac{\partial u}{\partial \theta} = -re^{rcos\theta}\{sin\theta . . \cos(rsin\theta) + \sin(rsin\theta) . cos\theta\}$$

$$\frac{\partial u}{\partial \theta} = -re^{rcos\theta} . \sin(rsin\theta + \theta) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (3)$$

1.3Total Derivative:

If $u = f(x, y)$ be any function in two variables x, y and if $\delta x, \delta y$ are small increment in values of variables x and y respectively. Then δu corresponding change in value of dependent variable u is given by

$$\delta u = f(x + \delta x, y + \delta y) - f(x, y)$$

$$\delta u = f(x + \delta x, y + \delta y) - f(x, y + \delta y) + f(x, y + \delta y) - f(x, y)$$

$$\delta u = \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x}\delta x + \frac{f(x, y + \delta y) - f(x, y)}{\delta y}$$

Taking limit$\delta x \to 0$ and $\delta y \to 0$, we get

$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$, which is called total derivative of $u = f(x, y)$.

Similarly if $u = f(x, y, z)$.then$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$

1.3.1Composite function and its Derivative: If$u = f(x, y)$, where $x = \varphi(t), y = \omega(t)$ then u iscalled a composite function of single variable t,then the value of total derivative of u with respect to t is given by

$$\frac{du}{dt} = \frac{\partial u}{\partial x} . \frac{dx}{dt} + \frac{\partial u}{\partial y} . \frac{dy}{dt}$$

Also if $z = f(x, y)$, where $x = \varphi(u, v), y = \omega(u, v)$ then z is called a composite function of two variables u , v and the value of total derivative of z with respect to u, v are given by

$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} . \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} . \frac{\partial y}{\partial u}$ and $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} . \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} . \frac{\partial y}{\partial v}$respectively.

Example-7: If $u = x^3 + y^3$where $x = acost$, $y = asint$. Find the value of $\frac{du}{dt}$ and hence verify the result.

Solution: Here we have, $u = x^3 + y^3 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots . . (1)$

Where, $x = acost$, $y = asint$ $\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (2)$

Therefore, u is a composite function in single variable t. So we shall have

$$\frac{du}{dt} = \frac{\partial u}{\partial x} . \frac{dx}{dt} + \frac{\partial u}{\partial y} . \frac{dy}{dt} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (3)$$

Now from equations (1) & (2)

$$\frac{\partial u}{\partial x} = 3x^2 \;\;\&\;\; \frac{\partial u}{\partial y} = 3y^2 \;\; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots . (4)$$

$$\frac{dx}{dt} = -asint \quad \& \quad \frac{dy}{dt} = acost \;\; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Using equations (4) & (5) in equation (3), we get

$$\frac{du}{dt} = 3x^2(-asint) + 3y^2(acost)$$

$$\frac{du}{dt} = 3a^3(sin^2t.cost - cos^2t.sint) \ldots\ldots\ldots. \qquad (6)$$

{Putting values of x and y from equation (2)}

Again, from equations (1) and (2), we get

$$u = a^3cos^3t + a^3sin^3t$$

So, differentiating both sides with respect to t we get

$$\frac{du}{dt} = a^3[(3cos^2t).(-sint) + (3sint^2t).cost$$

$$\frac{du}{dt} = 3a^3(sin^2t.cost - cos^2t.sint) \;\; \ldots\ldots\ldots\ldots\ldots \quad (7)$$

From equations (6) & (7) result is verified

Example-8: If $u = x^2 - y^2 + sinyz$, where $y = e^x, z = logx$, then find $\frac{du}{dx}$.

Solution: Here we have, $u = x^2 - y^2 + sinyz \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$

Where $y = e^x, z = logx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots .. (2)$

Here u will be composite function in single variable x. So we shall have

$$\frac{du}{dx} = \frac{\partial u}{\partial x}.\frac{dx}{dx} + \frac{\partial u}{\partial y}.\frac{dy}{dx} + \frac{\partial u}{\partial z}.\frac{dz}{dx} \;\; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Now, differentiating equations (1) and (2), we get

$$\frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = -2y + zcosyz \,\&\, \frac{\partial u}{\partial z} = ycosyz \;\; \ldots\ldots\ldots\ldots\ldots\ldots\ldots \; (4)$$

$$\frac{dy}{dx} = e^x \& \frac{dz}{dx} = \frac{1}{x} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Using equations (4) and (5) in equation (3), we get

$$\frac{du}{dx} = 2x + (-2y + zcosyz)e^x + ycosyz\left(\frac{1}{x}\right)$$

Putting values of y & z from equation (2), we get

$$\frac{du}{dx} = 2x + (-2e^x + logx.cosyz)e^x + e^x\left(\frac{1}{x}\right)cosyz$$

Example-9: If $v = f(2x - 3y, 3y - 4z, 4z - 2x)$ compute the value of $6v_x + 4v_y + 3v_z$.

Solution: Given that $v = f(2x - 3y, 3y - 4z, 4z - 2x)$

Let $v = f(r, s, t)$ ……………………………………………….(1)

Where $r = 2x - 3y, s = 3y - 4z, t = 4z - 2x$ ………………….(2)

So, from equation (2), we get

$$\frac{\partial r}{\partial x} = 2, \frac{\partial r}{\partial y} = -3, \frac{\partial r}{\partial z} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(3)$$

$$\frac{\partial s}{\partial x} = 0, \frac{\partial s}{\partial y} = 3, \frac{\partial s}{\partial z} = -4 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(4)$$

$$\frac{\partial t}{\partial x} = -2, \frac{\partial t}{\partial y} = 0, \frac{\partial t}{\partial z} = 4 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(5)$$

Now, $v_x = \frac{\partial v}{\partial x} = v_x = \frac{\partial v}{\partial r}.\frac{\partial r}{\partial x} + \frac{\partial v}{\partial s}.\frac{\partial s}{\partial x} + \frac{\partial v}{\partial t}.\frac{\partial t}{\partial x}$

$v_x = \frac{\partial f}{\partial r}(2) + \frac{\partial f}{\partial s}(0) + \frac{\partial f}{\partial t}(-2)$ Or $v_x = 2\left(\frac{\partial f}{\partial r} - \frac{\partial f}{\partial t}\right)$ ………………….(6)

$$v_y = \frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}.\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}.\frac{\partial s}{\partial y} + \frac{\partial v}{\partial t}.\frac{\partial t}{\partial y}$$

$v_y = \frac{\partial f}{\partial r}(-3) + \frac{\partial f}{\partial s}(3) + \frac{\partial f}{\partial t}(0)$ Or $v_y = 3\left(-\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s}\right)$ ………………….(7)

$$v_z = \frac{\partial v}{\partial z} = \frac{\partial v}{\partial r}.\frac{\partial r}{\partial z} + \frac{\partial v}{\partial s}.\frac{\partial s}{\partial z} + \frac{\partial v}{\partial t}.\frac{\partial t}{\partial z}$$

$v_z = \frac{\partial f}{\partial r}(0) + \frac{\partial f}{\partial s}(-4) + \frac{\partial f}{\partial t}(4)$ Or $v_z = 4\left(-\frac{\partial f}{\partial s} + \frac{\partial f}{\partial t}\right)$ ………………….(8)

From equations (6). (7) and (8), we get

$6v_x + 4v_y + 3v_z == 12\left\{\left(\frac{\partial f}{\partial r} - \frac{\partial f}{\partial t}\right) + \left(-\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s}\right) + \left(-\frac{\partial f}{\partial s} + \frac{\partial f}{\partial t}\right)\right\}$or $6v_x + 4v_y + 3v_z = 0.$

Example-10: If $u = u\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, then show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$

Solution: Here we have $u = u\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$ ………………..(1)

Let $u = u(r, s)$ ……………………………………………….(2)

$where\ r = \frac{y-x}{xy}$ or $r = \frac{1}{x} - \frac{1}{y}$ & s= $\frac{z-x}{xz}$ or $s = \frac{1}{x} - \frac{1}{z}$… (3)

From equation(3), we get $\frac{\partial r}{\partial x} = -\frac{1}{x^2}, \frac{\partial r}{\partial y} = \frac{1}{y^2}, \frac{\partial r}{\partial z} = 0$ and $\frac{\partial s}{\partial x} = -\frac{1}{x^2}, \frac{\partial s}{\partial y} = 0, \frac{\partial s}{\partial z} = \frac{1}{z^2}$

Also, $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}.\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}.\frac{\partial s}{\partial x}$

$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\left(-\frac{1}{x^2}\right) + \frac{\partial u}{\partial s}\left(-\frac{1}{x^2}\right)$ or $x^2\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial r} - \frac{\partial u}{\partial s}$ ……………………………..(4)

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}.\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}.\frac{\partial s}{\partial y}$$

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\left(\frac{1}{y^2}\right) + \frac{\partial u}{\partial s}(0)$ or $y^2\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}$ ……………………………….. (5)

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}.\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}.\frac{\partial s}{\partial z}$$

$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}\left(\frac{1}{z^2}\right)$ or $z^2\frac{\partial u}{\partial z} = \frac{\partial u}{\partial s}$ ……………………………….. (6)

Adding equations (4), (5) and (6), we get

$x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = \left(-\frac{\partial u}{\partial r} - \frac{\partial u}{\partial s}\right) + \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}$ or $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0.$

Example-11: If $u = f(r), where\ x = rcos\theta, y = rsin\theta$. Prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r)$.

Solution: Here we have, $u = f(r)$ ………………………….(1)

Also, since $x = rcos\theta, y = rsin\theta$

Therefore $r^2 = x^2 + y^2$ ……………………………… (2)

Differentiating equation (2) partially with respect to 'x' and 'y' we get

$\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}$.

Also, from equations (1) and (2)

$\frac{\partial u}{\partial x} = \frac{df}{dr}.\frac{\partial r}{\partial x}$ or $\frac{\partial u}{\partial x} = \frac{df}{dr}.\left(\frac{x}{r}\right)$ ………………………………….(3)

and

$\frac{\partial u}{\partial y} = \frac{df}{dr}.\frac{\partial r}{\partial y}$ or $\frac{\partial u}{\partial y} = \frac{df}{dr}.\left(\frac{y}{r}\right)$ ………………………………….(4)

Differentiating equation (3) partially with respect to 'x' we get

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left\{\frac{df}{dr}.\left(\frac{x}{r}\right)\right\}$ or $\frac{\partial^2 u}{\partial x^2} = \frac{df}{dr}\frac{\partial}{\partial x}\left\{\left(\frac{x}{r}\right)\right\} + \frac{\partial}{\partial x}\left(\frac{df}{dr}\right).\left(\frac{x}{r}\right)$

$\frac{\partial^2 u}{\partial x^2} = \frac{df}{dr}.\frac{r.1 - x.\frac{\partial r}{\partial x}}{r^2} + \frac{d^2 f}{dr^2}\frac{\partial r}{\partial x}\left(\frac{x}{r}\right)$ or $\frac{\partial^2 u}{\partial x^2} = \frac{r.1 - x.\frac{x}{r}}{r^2} + \frac{d^2 f}{dr^2}.\left(\frac{x}{r}\right).\left(\frac{x}{r}\right)$

$$\frac{\partial^2 u}{\partial x^2} = \frac{df}{dr}.\left(\frac{r^2 - x^2}{r^3}\right) + \frac{d^2 f}{dr^2}\left(\frac{x^2}{r^2}\right) \quad \text{……………………………..(5)}$$

Similarly differentiating equation (4) partially with respect to 'y' we get

$$\frac{\partial^2 u}{\partial y^2} = \frac{df}{dr}\cdot\left(\frac{r^2 - y^2}{r^3}\right) + \frac{d^2 f}{dr^2}\left(\frac{y^2}{r^2}\right) \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots..(6)$$

Adding equations (5) and (6) we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{df}{dr}\cdot\left\{\frac{r^2 - (x^2 + y^2)}{r^3}\right\} + \frac{d^2 f}{dr^2}\left(\frac{x^2 + y^2}{r^2}\right)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{df}{dr}\cdot\left(\frac{1}{r}\right) + \frac{d^2 f}{dr^2}$$

Or $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r)$. Where $\frac{df}{dr}$=$f'(r)$&$\frac{d^2 f}{dr^2} = f''(r)$

1.3.2Differentiation of an implicit function:

For any implicitly defined function $f(x, y) = 0$

We have, $\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$

$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$ or$\frac{dy}{dx} = -\frac{f_x}{f_x}$

Also second order derivative is given by $\frac{d^2 y}{dx^2} = \frac{(f_y)^2 f_{xx} - 2f_x f_y\ f_{xy} + f_{yy}(f_x)^2}{(f_y)^3}$

Example-13: If $x^3 + 3x^2 y + 6xy^2 + y^3 = 1, Find\ \frac{dy}{dx}$.

Solution: Let $f(x, y) \equiv x^3 + 3x^2 y + 6xy^2 + y^3 - 1 = 0 \quad \dots\dots\dots\dots\dots\dots\dots..(1)$

Therefore from equation (1), $f_x = \frac{\partial f}{\partial x} = 3x^2 + 6xy + 6y^2$&$f_y = \frac{\partial f}{\partial y} = 3x^2 + 12xy + 3y^2$

$\therefore\ \frac{dy}{dx} = -\frac{f_x}{f_x}$ will implies that $\frac{dy}{dx} = -\frac{3x^2 + 6xy + 6y^2}{3x^2 + 12xy + 3y^2}$

Or $\frac{dy}{dx} = -\frac{x^2 + 2xy + 2y^2}{x^2 + 4xy + y^2}$.

Example-14: Find$\frac{du}{dx}$, if $u = xlogxy, where\ x^3 + y^3 + 3xy = 1$.

Solution: Here we have $u = xlogxy \quad \dots\dots\dots\dots\dots\dots\dots\dots..(1)$

And let $f(x, y) \equiv x^3 + y^3 + 3xy - 1 = 0 \quad \dots\dots\dots\dots\dots\dots (2)$

From equations (1) & (2), we get

$$\frac{\partial u}{\partial x} = logxy + x\left(\frac{1}{xy}\right)y\ or\ \frac{\partial u}{\partial x} = 1 + logxy\ \ \&\frac{\partial u}{\partial y} = x\ \left(\frac{1}{xy}\right)x\ or\ \frac{\partial u}{\partial y} = \frac{x}{y} \quad \dots\dots\dots\dots\dots\dots\dots\dots (3)$$

$$f_x = \frac{\partial f}{\partial x} = 3x^2 + 3y\ \&f_y = \frac{\partial f}{\partial y} = 3y^2 + 3x \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (4)$$

Also from equation (2), $f(x,y)$ is an implicitly defined function

So $\frac{dy}{dx} = -\frac{f_x}{f_x} = -\left(\frac{x^2+y}{y^2+x}\right)$ putting values from equation (4)

Now, value of $\frac{du}{dx}$ can be determine as

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Putting values from equations (3) & (4) in equation (5), we get

$\frac{du}{dx} = 1 + logxy + \frac{x}{y}\left\{-\left(\frac{x^2+y}{y^2+x}\right)\right\}$ Or $\boldsymbol{\frac{du}{dx} = 1 + logxy - \frac{x}{y}\left(\frac{x^2+y}{y^2+x}\right)}$.

Example-15: If $\varphi(x,y,z) = 0$, show that $\left(\frac{\partial y}{\partial z}\right)_x \cdot \left(\frac{\partial z}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial y}\right)_z = -1$.

Solution: Here we have $f(x,y,z) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$

From equation (1),

$$\left(\frac{\partial y}{\partial z}\right)_x = -\frac{\frac{\partial \varphi}{\partial z}}{\frac{\partial \varphi}{\partial y}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\left(\frac{\partial z}{\partial x}\right)_y = -\frac{\frac{\partial \varphi}{\partial x}}{\frac{\partial \varphi}{\partial z}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$\left(\frac{\partial x}{\partial y}\right)_z = -\frac{\frac{\partial \varphi}{\partial y}}{\frac{\partial \varphi}{\partial x}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

From equations (2), (3) & (4), we get

$$\left(\frac{\partial y}{\partial z}\right)_x \cdot \left(\frac{\partial z}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial y}\right)_z = \left(-\frac{\frac{\partial \varphi}{\partial z}}{\frac{\partial \varphi}{\partial y}}\right) \cdot \left(-\frac{\frac{\partial \varphi}{\partial x}}{\frac{\partial \varphi}{\partial z}}\right) \cdot \left(-\frac{\frac{\partial \varphi}{\partial y}}{\frac{\partial \varphi}{\partial x}}\right)$$

Or $\boldsymbol{\left(\frac{\partial y}{\partial z}\right)_x \cdot \left(\frac{\partial z}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial y}\right)_z = -1}$.

Practice Exercise:

Q-1: Find all first order derivatives of function $u = \sin^{-1}\frac{x}{y}$.

Ans: $\boldsymbol{\frac{\partial u}{\partial x} = \frac{y}{\sqrt{y^2-x^2}}}$ & $\boldsymbol{\frac{\partial u}{\partial y} = \frac{-x}{\sqrt{y^2-x^2}}}$.

Q-2: If $z = f(x+ct) + \varphi(x-ct)$, show that $\frac{\partial^2 z}{\partial t^2} = c^2\frac{\partial^2 z}{\partial x^2}$.

Q-3: If $x = e^{r\cos\theta}.\cos(r\sin\theta)\,, y = e^{r\cos\theta}.\sin(r\sin\theta)$, prove that $\frac{\partial x}{\partial r} = \frac{1}{r}\frac{\partial y}{\partial \theta}\,,\ \frac{\partial y}{\partial r} = \frac{1}{r}\frac{\partial x}{\partial \theta}$.

Q-4: If $u = u(y - z, z - x, x - y)$ show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$

Q-5: $u = f(r, s, t), r = \frac{x}{y}\,, s = \frac{y}{z}\,, t = \frac{z}{x}$. Show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$

Q-6: If $x + y = 2e^{\theta}\cos\varphi\,, x - y = 2i\, e^{\theta}\sin\varphi$, then show that $\frac{\partial^2 v}{\partial \theta^2} + \frac{\partial^2 v}{\partial \varphi^2} = 4xy\frac{\partial^2 v}{\partial x \partial y}$.

Q-7: If $z = u^2 + v^2, u = r\cos\theta, v = r\sin\theta$. Find values of $\frac{\partial z}{\partial r}$ & $\frac{\partial z}{\partial \theta}$.

Ans: $\boldsymbol{\frac{\partial z}{\partial r} = 2r}$ **&** $\boldsymbol{\frac{\partial z}{\partial \theta} = 0}$ **.**

Q-8: If three thermodynamic variables P, V, T are connected by a relation $f(P, V, T) = 0$. Then show that $\left(\frac{\partial P}{\partial T}\right)_V \left(\frac{\partial T}{\partial V}\right)_P \left(\frac{\partial V}{\partial P}\right)_T = -1.$

Q-9: If $u = f(x^2 + 2yz, y^2 + 2zx)$, then prove that $(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0.$

Q-10: If $f(x, y) = 0, \varphi(y, z) = 0$. Show that $\frac{\partial f}{\partial y} . \frac{\partial \varphi}{\partial z} . \frac{dz}{dx} = \frac{\partial f}{\partial x} . \frac{\partial \varphi}{\partial y}$.

1.4. Euler's Theorem for Homogeneous function:

1.4.1. Homogeneous Function:

. A function $u = f(x, y)$ is said to be a homogeneous function in variables x, y of degree n if it is expressible in any one of the following form $u = x^n\varphi\left(\frac{y}{x}\right)$ or $u = y^n\omega\left(\frac{x}{y}\right)$.

Important points about homogeneous function:

1. Any polynomial function in two variables x and y is said to be homogeneous
If all its terms are of the same degree.

For example the function $f(x, y) = x^3 + 3xy^2 - 5x^2y + y^3$ is a homogeneous function in x, y of degree 3 as degree of each term present is same and is equal to 3.

Also, it can be expressed in form $f(x, y) = x^3\left\{1 - 5\left(\frac{y}{x}\right) + 3\left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right)^3\right\}$ or $f(x, y) = x^3\varphi\left(\frac{y}{x}\right)$

2. In case of non-polynomials, a function $u = f(x, y)$ in two variables x and y is said to be a
Homogeneous function of degree n if for any positive number μ , $f(\mu x, \mu y) = \mu^n.f(x, y)$.

Ex: Function $f(x, y) = \cos\left(\frac{x^2+y^2}{2xy}\right)$ will be an homogeneous function in two variables x, y of degree 0 as we have

$$f(\mu x, \mu y) = \cos\left(\frac{\mu^2x^2 + \mu^2y^2}{2\mu^2xy}\right) \quad or \quad f(\mu x, \mu y) = \mu^0\cos\left(\frac{x^2 + y^2}{2xy}\right)$$

For example the function $f(x, y) = x^2\tan^{-1}\frac{y}{x}$ will be a homogeneous function of degree 2 in variables x, y as it is identical with form $f(x, y) = x^n\varphi\left(\frac{y}{x}\right)$ but the function $\cos^{-1}\frac{x^2}{y}$ is not homogeneous function of x, y.

3. If $u = f(x, y, z)$ is a homogeneous function in three variables x, y, z of degree n then it can be expressed in any of these three forms, $u = x^n\varphi\left(\frac{y}{x}, \frac{z}{x}\right)$ $or\ u = y^n\varphi\left(\frac{x}{y}, \frac{z}{y}\right)$ $or\ u = z^n\varphi\left(\frac{x}{z}, \frac{y}{z}\right)$

1.4.2.Euler's Theorem for Homogeneous Function:

Statement: If $u = f(x,y)$ be any homogeneous function in two variables x and y of degree n then

$$\boldsymbol{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.}$$

Proof: Since $u = f(x,y)$ is homogeneous function in two variables x, y of degree n therefore it can be expressed as

$$u(x,y) = x^n\varphi\left(\frac{y}{x}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \quad (1)$$

Differentiating equation (1) partially with respect to x and y we get

$$\frac{\partial u}{\partial x} = nx^{n-1}\varphi\left(\frac{y}{x}\right) + x^n\varphi'\left(\frac{y}{x}\right).\left(\frac{-y}{x^2}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

$$\frac{\partial u}{\partial y} = x^n\varphi'\left(\frac{y}{x}\right).\left(\frac{1}{x}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \quad (3)$$

Multiplying equation (2) & (3) by x & y respectively and then adding we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n\varphi\left(\frac{y}{x}\right) - yx^{n-1}\varphi'\left(\frac{y}{x}\right) + yx^{n-1}\varphi'\left(\frac{y}{x}\right)$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n\varphi\left(\frac{y}{x}\right) \quad or\ \boldsymbol{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu}$$

Note: Similarly, if $u = f(x,y,z)$ is a homogeneous function in three variables x, y, z of degree n then by Euler's theorem we shall have,

$$\boldsymbol{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu}$$

Example-1: If $u = (\sqrt{x} + \sqrt{y})^5$, then find value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

Solution: Here we have $u = (\sqrt{x} + \sqrt{y})^5 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$

From (1) we have $u = (\sqrt{x})^5\left(1 + \sqrt{\frac{y}{x}}\right)^5$ or $u = x^{\frac{5}{2}}\left(1 + \sqrt{\frac{y}{x}}\right)^5$

This is of form $u = x^n\varphi\left(\frac{y}{x}\right)$, so given function u is a homogeneous function in variables x, y of degree $n = \frac{5}{2}$.Therefore from Euler's theorem for homogeneous function of degree n, we shall have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \quad \boldsymbol{or} \quad \boldsymbol{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{5}{2}u}$$

Example-2: If $u = f\left(\frac{y}{x}\right)$, then find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

Solution: Here $u = f\left(\frac{y}{x}\right)$, so it is expressible as $u = x^0 f\left(\frac{y}{x}\right)$

Therefore, u is a homogeneous function in variables x, y of degree $n = 0$.

Hence by Euler's theorem for homogeneous functions

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\,u \quad \boldsymbol{or} \quad \boldsymbol{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0}$$

Example-3: Verify Euler's theorem for homogeneous function for $u=x^{\frac{1}{3}}\,y^{-\frac{4}{3}}\tan^{-1}\left(\frac{y}{x}\right)$.

Solution: Here we have $u=x^{\frac{1}{3}}y^{-\frac{4}{3}}\tan^{-1}\left(\frac{y}{x}\right)$ … … … … … … … … … … … … (1)

Replacing $x\ by\ \mu x$ & $y\ by\ \mu y$ in equation (1) we get

$$u=\mu^{\left(\frac{1}{3}-\frac{4}{3}\right)}x^{\frac{1}{3}}\,y^{-\frac{4}{3}}\tan^{-1}\left(\frac{y}{x}\right) \quad or \quad u=\mu^{(-1)}\,x^{\frac{1}{3}}\,y^{-\frac{4}{3}}\,\tan^{-1}\left(\frac{y}{x}\right)$$

Clearly u is a homogeneous function in variables x, y of degree -1.

Therefore, from Euler's theorem on homogeneous function, we must have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-u \text{ … } (2)$$

Now differentiating equation (1) partially with respect to x we get

$$\frac{\partial u}{\partial x}=\frac{1}{3}x^{\left(\frac{1}{3}-1\right)}\,y^{-\frac{4}{3}}\,\tan^{-1}\left(\frac{y}{x}\right)+\ x^{\frac{1}{3}}\,y^{-\frac{4}{3}}\frac{1}{\left\{1+\left(\frac{y}{x}\right)^2\right\}}\left(\frac{-y}{x^2}\right) \text{ … … … … … … … … } (3)$$

$$\frac{\partial u}{\partial y}=\frac{-4}{3}\,x^{\frac{1}{3}}\,y^{\left(\frac{-4}{3}-1\right)}\,\tan^{-1}\left(\frac{y}{x}\right)+\ x^{\frac{1}{3}}\,y^{-\frac{4}{3}}\frac{1}{\left\{1+\left(\frac{y}{x}\right)^2\right\}}\left(\frac{1}{x}\right) \text{ … … … … … … …. } (4)$$

Multiplying equations (3) & (4) by x & y respectively and then adding we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{3}x^{\frac{1}{3}}\,y^{-\frac{4}{3}}\,\tan^{-1}\left(\frac{y}{x}\right)+\ x^{\frac{1}{3}}\,y^{-\frac{4}{3}}\frac{1}{\left\{1+\left(\frac{y}{x}\right)^2\right\}}\left(\frac{-y}{x}\right)-\frac{4}{3}x^{\frac{1}{3}}\,y^{-\frac{4}{3}}\,\tan^{-1}\left(\frac{y}{x}\right)+\ x^{\frac{1}{3}}\,y^{-\frac{4}{3}}\frac{1}{\left\{1+\left(\frac{y}{x}\right)^2\right\}}\left(\frac{y}{x}\right)$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=(-1)x^{\frac{1}{3}}\,y^{-\frac{4}{3}}\tan^{-1}\frac{y}{x} \quad or\ \boldsymbol{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-u} \text{ … … … … … … } (5)$$

From equations (2) & (5) Euler theorem is verified.

Example-4: If u be a homogeneous function in two variables x, y of degree n, then show that

i) $x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=(n-1)\frac{\partial u}{\partial x}$

ii) $x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}=(n-1)\frac{\partial u}{\partial y}$

iii) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u$

Solution: Here u is a homogenous function in two variables x, y of degree n

∴By Euler's theorem for homogeneous function, we shall have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\,u \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

Differentiating equation (1) partially with respect to x we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=n\frac{\partial u}{\partial x} \quad or \quad x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=(n-1)\frac{\partial u}{\partial x} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \ (2)$$

Again, differentiating equation (1) partially with respect to y, we get

$$x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=n\frac{\partial u}{\partial y} \quad or \quad x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}=(n-1)\frac{\partial u}{\partial y} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.... \ (3)$$

Multiplying equation (2) and (3) by x & y respectively and then adding, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(n-1)x\frac{\partial u}{\partial x}+(n-1)y\frac{\partial u}{\partial y}$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(n-1)\left\{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (4)$$

Using equation (1) in equation (4), we get

$$\boldsymbol{x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u.}$$

Note: For a homogeneous u of two variables x and y we have, $\boldsymbol{x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u}$

This is known as Euler's theorem for second order derivative.

Example-5: If $u=x^n f\left(\frac{y}{x}\right)+y^{-n}\varphi\left(\frac{x}{y}\right)$, then show that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n^2u.$$

Solution: Given that $u=x^n f\left(\frac{y}{x}\right)+y^{-n}\varphi\left(\frac{x}{y}\right)$

Let $u_1=x^n f\left(\frac{y}{x}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \ (1)$

$u_2=y^{-n}\varphi\left(\frac{x}{y}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(2)$

Such that, $u=u_1+u_2$ .... (3)

From equation (1) it is clear that u_1 is a homogenous function in x, y of degree n, therefore from Euler's theorem

$$x\frac{\partial u_1}{\partial x}+y\frac{\partial u_1}{\partial y}=nu_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Also $x^2\frac{\partial^2 u_1}{\partial x^2}+2xy\frac{\partial^2 u_1}{\partial x\partial y}+y^2\frac{\partial^2 u_1}{\partial y^2}=n(n-1)u_1 \quad \ldots\ldots\ldots.(5)$

Similarly,u_2 is a homogenous function in x, y of degree –n, Therefore

$$x\frac{\partial u_2}{\partial x}+y\frac{\partial u_2}{\partial y}=-nu_2 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (6)$$

Also $x^2\frac{\partial^2 u_2}{\partial x^2}+2xy\frac{\partial^2 u_2}{\partial x\partial y}+y^2\frac{\partial^2 u_2}{\partial y^2}=n(n+1)u_2 \quad \dots\dots\dots.(7)$

Adding equations (4) and (6) we get

$$x\frac{\partial}{\partial x}(u_1+u_2)+y\frac{\partial}{\partial y}(u_1+u_2)=n(u_1-u_2) \quad Or\ x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n(u_1-u_2) \quad \dots\dots \quad (8)$$

Again, adding equations (5) & (7), we get

$$x^2\frac{\partial^2}{\partial x^2}(u_1+u_2)+2xy\frac{\partial^2}{\partial x\partial y}(u_1+u_2)+y^2\frac{\partial^2}{\partial y^2}(u_1+u_2)=n(n-1)u_1+n(n+1)u_2$$

Or $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n^2$u- n (($u_1-u_2$) $\quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots. \quad (9)$

At last adding equations (8) & (9), we get

$$\boldsymbol{x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n^2u.} \qquad \text{\{Using equation (8)}$$

Hence the result.

1.4.3Deduction on Euler's Theorem for homogeneous function:

If $u=u(x,y)$ is not homogeneous but $f(u)$ be the homogenous function in variables x, y of degree n, then

$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\ \frac{f(u)}{f'(u)}$ **.**

Proof: Since $v=f(u)$ is given to be homogeneous function in variables x, y of degree n,

Therefore, by Euler's theorem

$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=nv$ Or $x\frac{\partial}{\partial x}f(u)+y\frac{\partial}{\partial y}f(u)=nf(u)$ {since$v=f(u)$

$$f'(u)\left\{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right\}=nf(u)\ Or \quad \boldsymbol{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\ \frac{f(u)}{f'(u)}}$$

Note: (i) If$f(u)$ be homogeneous function in three variables x, y, z. then above result can be written as

$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=n\ \frac{f(u)}{f'(u)}$.

Example-6: If $f(u)$ be the homogenous function in variables x, y of degree n, then show that $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)\{g'(u)-1\}$.

Solution: Since$f(u)$ is homogenous function in variables x, y of degree n, so we shall have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\ \frac{f(u)}{f'(u)}\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\ (1)$$

Let $g(u)=n\ \frac{f(u)}{f'(u)}$, then from equation (1), we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=g(u)\ \dots\dots\dots\dots\dots\dots\dots.\ (2)$$

Differentiating equation (2) partially with respect to x we get,

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=g'(u)\frac{\partial u}{\partial x}\ \ or\ x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=\{g'(u)-1\}\frac{\partial u}{\partial x}\ \dots\dots\dots\dots\dots\dots\dots\dots\dots..\ (3)$$

Again, differentiating equation (2) partially with respect y we get

$$x\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial u}{\partial y}+y\frac{\partial^2 u}{\partial y^2}=g'(u)\frac{\partial u}{\partial y}\ \ or\ x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}=\{g'(u)-1\}\frac{\partial u}{\partial y}\dots\dots\dots\dots\dots\dots\dots\dots\dots..(4)$$

Multiplying equation (3) & (4) by x & y respectively, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\{g'(u)-1\}\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\ (5)$$

Using equation (2) in equation (5) we get

$\boldsymbol{x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)\{g'(u)-1\}}$. Hence **the result**

Example-7: If$u=log\left(\frac{x^4+y^4}{x+y}\right)$, then find the value of $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$.

Solution: Here $u=log\left(\frac{x^4+y^4}{x+y}\right)$, then u is not homogenous function but

$e^u=\left(\frac{x^4+y^4}{x+y}\right)\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.\ (1),$

Here $f(u)=e^u$ is homogeneous function in x, y of degree $n=3$. So, from deduction to Euler's theorem

$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\ \frac{f(u)}{f'(u)}$, Now since $n=3$ & $f(u)=e^u$

Therefore $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3\ \frac{e^u}{e^u}$ $\quad or\quad$ $\boldsymbol{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3}$.

Example-8: If $u=\sin^{-1}\left(\frac{x^2+y^2+z^2}{ax+by+cz}\right)$, then prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=tanu$.

Solution: We have $u=\sin^{-1}\left(\frac{x^2+y^2+z^2}{ax+by+cz}\right)$, clearly u is not homogeneous function

But $\sin u=\left(\frac{x^2+y^2+z^2}{ax+by+cz}\right)\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.\ (1)$

is a homogeneous function in three variables x, y, z of degree $n=1$. Therefore buy deduction to Euler's theorem we have $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=n\ \frac{f(u)}{f'(u)}\ \dots\dots\dots\dots\dots\dots\dots\dots..(2)$

In this problem, f(u) $= \sin u$, so from equation (2) we shall have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=1.\frac{sinu}{cosu}$$

$$\boldsymbol{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=tanu.}$$

Example-9: If $u=\sin^{-1}\left\{\frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{2}}-y^{\frac{1}{2}}}\right\}^{\frac{1}{2}}$**,** then show that

(i) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{12}tanu$

(ii) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{1}{144}tanu(\,sec^2u+12)$

Solution: We have $u=\sin^{-1}\left\{\frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{2}}-y^{\frac{1}{2}}}\right\}^{\frac{1}{2}}$ … … … … … … … … … … …. (1)

Here u is not homogeneous but $f(u)=sinu$ will be homogeneous function in x, y of degree $n=-\frac{1}{12}$.

Therefore, by deduction to Euler's theorem

$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\,\frac{f(u)}{f'(u)}$ Or $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{12}\frac{sinu}{cosu}$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{12}tanu$$

Again we have $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)\{g'(u)-1\}$, where $g(u)=n\,\frac{f(u)}{f'(u)}=-\frac{1}{12}\frac{sinu}{cosu}=--\frac{1}{12}tanu$

Therefore $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-\frac{1}{12}tanu\left(-\frac{1}{12}sec^2u-1\right)$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{1}{144}tanu(\,sec^2u+12)$$

Example-10: If $u=\tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$**,** then prove that

(i) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\,sin2u$

(ii) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=sin4u-sin2u.$

Solution: Here$u=\tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, is not homogenous but $f(u)=tanu$ will be homogeneous function in x, y of degree $n=2$. Therefore, from deduction to Euler's theorem

$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\,\frac{f(u)}{f'(u)}$ or $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\,\frac{tanu}{sec^2u}$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\frac{sinu}{cosu}cos^2u \quad or \quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2sinu.cosuu$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=sin2u \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

From equation (1), $g(u)=n\,\frac{f(u)}{f'(u)}=sin2u$

Also we have $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)\{g'(u)-1\}$

Therefore $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=sin2u(2cos2u-1)$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=2sin2ucos2u-sin2u$$

$$\boldsymbol{x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=sin4u-sin2u}$$

Practice Exercise:

Q-1: State Euler's theorem for homogeneous function and verify it for $u=\sin^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}$.

Q-2: Verify Euler's theorem for $z=\frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{2}}+y^{\frac{1}{2}}}$.

Q-3: If $u=\sin^{-1}\left(\frac{x+2y+3z}{\sqrt{x^8++y^8+z^8}}\right)$. show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}+3\,tanu=0$.

Q-4: Apply Euler's theorem for homogeneous function for $u=(x^2-2xy+y^2)^{\frac{3}{2}}$ to evaluate values of (i) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$ and (ii) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}$.

Ans: (i) 3u and (ii) 6u.

Q-5: If $u=\cos^{-1}\left(\frac{x^3+y^3+z^3}{ax+by+cz}\right)$, then show that $xu_x+yu_y+zu_z=-2\,cotu$.

Q-6: If $u=\csc^{-1}\sqrt{\frac{x^{\frac{1}{2}}+y^{\frac{1}{2}}}{x^{\frac{1}{3}}+y^{\frac{1}{3}}}}$, then evaluate $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$.

Ans: $\boldsymbol{-\frac{1}{12}tanu}$.

Q-7: If $u=\tan^{-1}\frac{y^2}{x}$, show that $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-sin2u\,sin^2u$.

1. https://www.youtube.com/watch?v=XzaeYnZdK5o&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=1
2. https://www.youtube.com/watch?v=9-tir2V3vYY&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=14
3. https://www.youtube.com/watch?v=aqfSOOiO2kI&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=15
4. https://www.youtube.com/watch?v=GoyeNUaSW08&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=16
5. https://www.youtube.com/watch?v=jiEaKYI0ATY&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=17
6. https://www.youtube.com/watch?v=G0V_yp0jz5c&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=18
7. https://www.youtube.com/watch?v=G0V_yp0jz5c&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=18
8. https://www.youtube.com/watch?v=McT-UsFx1Es&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=9
9. https://www.youtube.com/watch?v=XzaeYnZdK5o&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=1
10. https://www.youtube.com/watch?v=btLWNJdHzSQ&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=12
11. https://www.youtube.com/watch?v=pgfcur31PTY
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15. https://www.youtube.com/watch?v=gLWUrF_cOwQ
16. https://www.youtube.com/watch?v=pAb1autRHGA
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KIET GROUP OF INSTITUTIONS, DELHI – NCR, GHAZIABAD

B. Tech. (I SEM), 2024-25

CALCULAS FOR ENGINEERS (K24AS11)

MODULE 2 (Differential Calculus II)

Syllabus: Taylor and Maclaurin's Theorem for function of two variables, Jacobians, properties of Jacobian (without proof), Hessian Matrix, Maxima and Minima of functions of two variables.

course Outcomes:

S.NO.	Course Outcome	BL
CO 2	Apply knowledge of partial differentiation in extrema, series expansion of function and Jacobians.	2,3

CONTENT

2.1.Taylor's and Maclaurin's Theorem for functions of two variables

2.1.1Taylor's Theorem for function of two variables: If $f(x, y)$ be any function in two variables x and y such that f(x, y) and all its partial derivatives up to desired order are finite and continuous for any point (x, y) subject to $a \leq x \leq a+h,\ \ b \leq y \leq b+k$, then

$$f(a+h, b+k) = f(a, b) + \frac{1}{1!}\left\{h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right\} + \frac{1}{2!}\left\{h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right\}$$
$$+\frac{1}{3!}\left\{h^3\frac{\partial^3 f}{\partial x^3} + +3h^2k\frac{\partial^3 f}{\partial x^2 \partial y} + 3hk^2\frac{\partial^3 f}{\partial x \partial y^2} + +k^3\frac{\partial^3 f}{\partial y^3}\right\} + \dots\dots\dots\dots\dots. (1)$$

Here equation (1) represent Taylor series expansion of $f(a+h, b+k)$ in powers of h and k. Also, in this series values of all order partial derivatives are determined at point (a, b).

Taking $a+h = x\ \ or\ h = x-a\ \ \&\ b+k = y\ \ or\ k = y-b$ in equation (1), we get

$$f(x, y) = f(a, b) + \frac{1}{1!}\left\{(x-a)\frac{\partial f}{\partial x} + (y-b)\frac{\partial f}{\partial y}\right\} + \frac{1}{2!}\left\{(x-a)^2\frac{\partial^2 f}{\partial x^2} + 2(x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y} + (y-b)^2\frac{\partial^2 f}{\partial y^2}\right\} + \frac{1}{3!}\left\{(x-a)^3\frac{\partial^3 f}{\partial x^3} + +3(x-a)^2(y-b)\frac{\partial^3 f}{\partial x^2 \partial y} + 3(x-a)(y-b)^2\frac{\partial^3 f}{\partial x \partial y^2} + +(y-b)^3\frac{\partial^3 f}{\partial y^3}\right\} + \dots. (2)$$

Here equation (2) represents Taylor's series expansion of a function $f(x, y)$ about point (a, b). Also, it is said to represent Taylor's series expansion in powers of $(x - a)$and$(y-b)$.

Note: For simplicity sometime we represent first order derivatives $\frac{\partial f}{\partial x}\ by\ f_x, \frac{\partial f}{\partial y}\ by\ f_y$ and second order derivatives $\frac{\partial^2 f}{\partial x^2}\ by\ f_{x^2}, \frac{\partial^2 f}{\partial x \partial y}\ by\ f_{xy}, \frac{\partial^2 f}{\partial y^2}\ by\ f_{y^2}$ and so on other higher order derivatives terms are represented.

By replacing a by x and b by y in equation (1), we obtain Taylor's series expansion of function $f(x+h, y+k)$ in powers of h and k as

$$f(x+h, y+k) = f(x, y) + \frac{1}{1!}\left\{h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right\} + \frac{1}{2!}\left\{h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right\}$$
$$+\frac{1}{3!}\left\{h^3\frac{\partial^3 f}{\partial x^3} + +3h^2k\frac{\partial^3 f}{\partial x^2 \partial y} + 3hk^2\frac{\partial^3 f}{\partial x \partial y^2} + +k^3\frac{\partial^3 f}{\partial y^3}\right\} + \dots\dots\dots\dots\dots\ (3)$$

Note: In Taylor's series expansion represented by equation (3), values of all order partial derivatives are determined at point (x, y).

2.1.2. Maclaurin's Theorem for function of two variables:

Taking $a = \ b = 0$ and replacing h by x & k by y respectively in Taylor's series expansion given by equation (1), we get

$$f(x, y) = f(0,0) + \frac{1}{1!}\left\{x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right\} + \frac{1}{2!}\left\{x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2}\right\} + \frac{1}{3!}\left\{x^3\frac{\partial^3 f}{\partial x^3} + +3x^2y\frac{\partial^3 f}{\partial x^2 \partial y} + 3xy^2\frac{\partial^3 f}{\partial x \partial y^2} + +y^3\frac{\partial^3 f}{\partial y^3}\right\} + \dots$$

This is known as Maclaurin's theorem.

The above series is known as Maclaurin's series expansion of given function $f(x,y)$. This will be important to note here that to find Maclaurin's series expansion of given function we shall require to find out values of all partial derivatives at point (0, 0).
In other words, we can say that Maclaurin's series is nothing but Taylor's series expansion about the point (0,0).

Example-1: Expand $x^2y+3y-2$ in powers of $(x-1)\&\ (y+2)$ by using Taylor's series.
Solution: We know that Taylor's series expansion of any function $f(x,y)$ in powers of

$(x-a)\&\ (y-b)$ is given by

$$f(x,y)=f(a,b)+\frac{1}{1!}\left\{(x-a)\frac{\partial f}{\partial x}+(y-b)\frac{\partial f}{\partial y}\right\}+\frac{1}{2!}\left\{(x-a)^2\frac{\partial^2 f}{\partial x^2}+2(x-a)(y-b)\frac{\partial^2 f}{\partial x\partial y}+(y-b)^2\frac{\partial^2 f}{\partial y^2}\right\}+\frac{1}{3!}\left\{(x-a)^3\frac{\partial^3 f}{\partial x^3}++3(x-a)^2(y-b)\frac{\partial^3 f}{\partial x^2\partial y}+3(x-a)(y-b)^2\frac{\partial^3 f}{\partial x\partial y^2}++(y-b)^3\frac{\partial^3 f}{\partial y^3}\right\}$$
+.. (1)

Let $f(x,y)=x^2y+3y-2$ ……………………………… (2)

To obtain required Taylor's series we take a $=1$ and b $=-2$ and find all values at point (1, -2). Now from equation (2),

$f(x,y)=x^2y+3y-2$ or $f(1,-2)=-10$

$\frac{\partial f}{\partial x}=2xy$ or $\frac{\partial f}{\partial x}(1,-2)=-4$

$\frac{\partial f}{\partial y}=x^2+3$ or $\frac{\partial f}{\partial y}(1,-2)=4$

$$\frac{\partial^2 f}{\partial x^2}=2y \text{ or } \frac{\partial^2 f}{\partial x^2}(1,-2)=-4, \frac{\partial^2 f}{\partial x\partial y}=2x \text{ or } \frac{\partial^2 f}{\partial x\partial y}=2, \frac{\partial^2 f}{\partial y^2}=0 \text{ or } \frac{\partial^2 f}{\partial y^2}(1,-2)=0$$

$$\frac{\partial^3 f}{\partial x^3}(1,-2)=0, \frac{\partial^3 f}{\partial x^2\partial y}=2 \text{ or } \frac{\partial^3 f}{\partial x^2\partial y}(1,-2)=2, \frac{\partial^3 f}{\partial x\partial y^2}(1,-2)=0, \frac{\partial^2 f}{\partial y^3}(1,-2)=0$$

Putting all these values in equation (1), we get

$$x^2y+3y-2=\ -10+\frac{1}{1!}\{(x-1)(-4)+(y+2)(4)\}+\frac{1}{2!}\{(x-1)^2(-4)+2(x-1)(y+2)(2)+(y+2)^2(0)\}+\frac{1}{3!}\{(x-1)^3(0)+3(x-1)^2(y+2)(2)+3(x-1)(y+2)^2(0)+(y+2)^3(0)\}$$

$$\boldsymbol{x^2y+3y-2=-10-4(x-1)+4(y+2)-2(x-1)^2+2(x-1)(y+2)+(x-1)^2(y+2).}$$

Example-2: Expand x^y in powers of $(x-1)\&\ (y-1)$ up to and inclusive of 3rd degree term.

Solution: By Taylor's series we have

$$f(x,y)=f(a,b)+\frac{1}{1!}\left\{(x-a)\frac{\partial f}{\partial x}+(y-b)\frac{\partial f}{\partial y}\right\}+\frac{1}{2!}\left\{(x-a)^2\frac{\partial^2 f}{\partial x^2}+2(x-a)(y-b)\frac{\partial^2 f}{\partial x\partial y}+(y-b)^2\frac{\partial^2 f}{\partial y^2}\right\}+\frac{1}{3!}\left\{(x-a)^3\frac{\partial^3 f}{\partial x^3}++3(x-a)^2(y-b)\frac{\partial^3 f}{\partial x^2\partial y}+3(x-a)(y-b)^2\frac{\partial^3 f}{\partial x\partial y^2}++(y-b)^3\frac{\partial^3 f}{\partial y^3}\right\}+ .. (1)$$

Here let $f(x,y)=x^y$ ……………………………….. (2), To get required series expansion, we take a=b=1and find all values at (1, 1)

Now from equation (2), $f(x,y) = x^y \quad or\ f(1,1) = 1$. Also differentiating equation partially, we get

$$\frac{\partial f}{\partial x} = yx^{y-1} \ \ or\ \ \frac{\partial f}{\partial x}(1,1) = 1\ ,\frac{\partial f}{\partial y} = x^y logx \ \ or\ \ \frac{\partial f}{\partial y}(1,1) = 0$$

$$\frac{\partial^2 f}{\partial x^2} = y(y-1)x^{y-2}\ or\ \frac{\partial^2 f}{\partial x^2}(1,1) = 0, \frac{\partial^2 f}{\partial x \partial y} = yx^{y-1}logx + x^{y-1}\ or\ \frac{\partial^2 f}{\partial x \partial y}(1,1) = 1,$$

$$\frac{\partial^2 f}{\partial y^2} = x^y (logx)^2\ \ or \frac{\partial^2 f}{\partial y^2}(1,1) = 0 \& \frac{\partial^3 f}{\partial x^3} = y(y-1)(y-2)x^{y-2}\ or\ \frac{\partial^3 f}{\partial x^3} = 0$$

$$\frac{\partial^3 f}{\partial x^2 \partial y} = y(y-1)x^{y-2}logx + yx^{y-2} + (y-1)x^{y-2}\ or\ \frac{\partial^3 f}{\partial x^2 \partial y}(1,-1) = 1,$$

$$\frac{\partial^3 f}{\partial x \partial y^2} = yx^{y-1}(logx)^2 + 2logx.x^{y-1}\ or\ \frac{\partial^3 f}{\partial x \partial y^2}(1,1) = 0$$

$$\frac{\partial^3 f}{\partial y^3} = \ x^y (logx)^3\ \ or\ \frac{\partial^3 f}{\partial y^3}(1,1) = 0$$

Using all these values in equation (1) we get

$$\boldsymbol{x^y = 1 + (x-1) + (x-1)(y-1) + \frac{1}{2}(x-1)^2(y-1) + \cdots \dots \dots \dots \dots \dots \dots}$$

Example-3: Obtain Taylor's series expansion off$(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$ about $(1,1)$ upto and including the second-degree terms. Hence, computef(1.1,0.9).

By Taylor's theorem know that

$$f(x,y) = f(a,b) + \frac{1}{1!}\left\{(x-a)\frac{\partial f}{\partial x}(a,b) + (y-b)\frac{\partial f}{\partial y}(a,b)\right\} + \frac{1}{2!}\left\{(x-a)^2\frac{\partial^2 f}{\partial x^2}(a,b) + 2(x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y}(a,b) + (y-b)^2\frac{\partial^2 f}{\partial y^2}(a,b)\right\} + \ \dots\dots\dots\dots\dots\dots \quad (1)$$

Therefore, in this problem we select $a = 1, b = 1$. Also, we have

$$\mathrm{f(x,y)} = \tan^{-1}\left(\frac{y}{x}\right)\ \ or\ f(1,1) = \tan^{-1}(1) = \frac{\pi}{4}$$

Differentiating $\mathrm{f(x,y)}$ partially with respect to x and y we get

$$\frac{\partial f}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right)\ \ or\ \ \frac{\partial f}{\partial x} = \frac{-y}{x^2+y^2}\ \ \ or\ \frac{\partial f}{\partial x}(1,1) = \frac{-1}{2}$$

$$\frac{\partial f}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right)\ \ or\ \ \frac{\partial f}{\partial y} = \frac{x}{x^2+y^2}\ \ \ or\ \frac{\partial f}{\partial y}(1,1) = \frac{1}{2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{y(2x)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}\ \ or\ \frac{\partial^2 f}{\partial x^2}(1,1) = \frac{1}{2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{(x^2+y^2).1 - x.2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \quad or \; \frac{\partial^2 f}{\partial x \partial y}(1,1) = 0$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{-x(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2} \quad or \; \frac{\partial^2 f}{\partial y^2}(1,1) = \frac{-1}{2}$$

Putting all these values in equation (1) we get

$$\mathrm{f(x,y)} = \tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4} + \frac{1}{1!}\left\{(x-1)\left(\frac{-1}{2}\right) + (y-1)\left(\frac{1}{2}\right)\right\} + \frac{1}{2!}\left\{(x-1)^2\left(\frac{1}{2}\right) + 2(x-1)(y-1)\frac{\partial^2 f}{\partial x \partial y}(0) + (y-1)^2\left(\frac{-1}{2}\right)\right\} + \; \ldots\ldots\ldots\ldots\ldots$$

$$\mathbf{f(x,y) = \tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4} - \frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{1}{4}(x-1)^2 - \frac{1}{4}(y-1)^2 + \; \ldots\ldots\ldots\ldots\ldots \; (2)}$$

Putting $x = 1.1$ & $y = 0.9$ in equation (2), we get

$$\mathrm{f}(1.1, 0.9) = \frac{\pi}{4} - \frac{1}{2}(1.1-1) + \frac{1}{2}(0.9-1) + \frac{1}{4}(1.1-1)^2 - \frac{1}{4}(0.9-1)^2$$

$$\mathbf{f(1.1, 0.9) = 0.6857.}$$

Example-4: Expand $\frac{(x+h)(y+k)}{(x+h)+(y+k)}$ in powers of h and k up to and inclusive second-degree terms.

Solution: By Taylor's theorem we have

$$f(x+h, y+k) = f(x,y) + \frac{1}{1!}\left\{h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right\} + \frac{1}{2!}\left\{h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right\} + \; \ldots\ldots\ldots\ldots\ldots \; (1)$$

Here let us take $f(x+h, y+k) = \frac{(x+h)(y+k)}{(x+h)+(y+k)}$ such that $f(x,y) = \frac{xy}{x+y}$ ……… (2)

Differentiating equation (2) partially with respect to x and y we get

$$\frac{\partial f}{\partial x} = \frac{(x+y)y - xy.1}{(x+y)^2} \quad or \quad \frac{\partial f}{\partial x} = \frac{y^2}{(x+y)^2}$$

$$\frac{\partial f}{\partial y} = \frac{(x+y)x - xy.1}{(x+y)^2} \quad or \quad \frac{\partial f}{\partial y} = \frac{x^2}{(x+y)^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{y^2\{-2\}}{(x+y)^3} = -\frac{2y^2}{(x+y)^3}, \; \frac{\partial^2 f}{\partial y^2} = \frac{x^2\{-2\}}{(x+y)^3} = -\frac{2x^2}{(x+y)^3}$$

and $\frac{\partial^2 f}{\partial x \partial y} = \frac{(x+y)^2(2x) - x^2\{2(x+y)\}}{(x+y)^4}$ or $\frac{\partial^2 f}{\partial x \partial y} = \frac{2xy}{(x+y)^3}$

Putting all the values in equation (1), we get

$$\frac{(x+h)(y+k)}{(x+h)+(y+k)} = \frac{xy}{x+y} + \frac{1}{1!}\left\{h\frac{y^2}{(x+y)^2} + k\frac{x^2}{(x+y)^2}\right\}$$

$$+\frac{1}{2!}\left\{h^2.\frac{-2y^2}{(x+y)^3} + 2hk\frac{2xy}{(x+y)^3} + k^2\frac{-2x^2}{(x+y)^3}\right\} + \; \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{(x+h)(y+k)}{(x+h)+(y+k)} = \frac{xy}{x+y} + \frac{y^2}{(x+y)^2}h + \frac{x^2}{(x+y)^2}k - \frac{y^2}{(x+y)^3}h^2 + \frac{2xy}{(x+y)^3}hk - \frac{x^2}{(x+y)^3}k^2 + \cdots$$

Example5: Expand $e^{ax}\sin by$ in powers of x and y as far as terms of third degree.

Solution: By Maclaurin's theorem we have

$$f(x,y) = f(0,0) + \frac{1}{1!}\left\{x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right\} + \frac{1}{2!}\left\{x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2}\right\} + \frac{1}{3!}\left\{x^3\frac{\partial^3 f}{\partial x^3} + +3x^2y\frac{\partial^3 f}{\partial x^2 \partial y} + 3xy^2\frac{\partial^3 f}{\partial x \partial y^2} + +y^3\frac{\partial^3 f}{\partial y^3}\right\} + \ldots\ldots\ldots\ldots \quad (1).$$

In this equation (1), value of all order partial derivatives is evaluated at$(0,0)$.

So, let $f(x,y) = e^{ax}\sin by \qquad or\ f(0,0) = 0$

On differentiating $f(x,y)$ partially with respect to x and y we get

$$\frac{\partial f}{\partial x} = ae^{ax}\sin by \quad or\ f(0,0) = 0$$

$$\frac{\partial f}{\partial y} = be^{ax}\cos by \quad or\ \frac{\partial f}{\partial y}(0,0) = b$$

$$\frac{\partial^2 f}{\partial x^2} = a^2e^{ax}\sin by \quad or\ \frac{\partial^2 f}{\partial x^2}(0,0) = 0$$

$$\frac{\partial^2 f}{\partial y^2} = -b^2e^{ax}\sin by \quad or\ \frac{\partial^2 f}{\partial y^2}(0,0) = 0$$

$$\frac{\partial^2 f}{\partial x \partial y} = abe^{ax}\cos by \quad or\ \frac{\partial^2 f}{\partial x \partial y} = ab$$

$$\frac{\partial^3 f}{\partial x^3} = a^3e^{ax}\sin by \quad or\ \frac{\partial^3 f}{\partial x^3}(0,0) = 0$$

$$\frac{\partial^3 f}{\partial y^3} = -b^3e^{ax}\cos by \quad or\ \frac{\partial^3 f}{\partial y^3}(0,0) = -b^3$$

$$\frac{\partial^3 f}{\partial x^2 \partial y} = a^2be^{ax}\cos by \quad or\ \frac{\partial^3 f}{\partial x^2 \partial y}(0,0) = a^2b$$

$$\frac{\partial^3 f}{\partial x \partial y^2} = -ab^2e^{ax}\sin by \quad or\ \frac{\partial^3 f}{\partial x \partial y^2}(0,0) = 0$$

Putting all these values in equation (1), we get

$$e^{ax}\sin by = 0 + \frac{1}{1!}\{0 + by\} + \frac{1}{2!}\{0 + 2abxy + 0)\} + \frac{1}{3!}\left\{0 + +3a^2b.x^2y\frac{\partial^3 f}{\partial x^2 \partial y} - -b^3y^3\right\} + \cdots$$

$$e^{ax}\sin by = by + ab\,xy + \frac{a^2b}{2}.x^2y - \frac{b^3}{6}y^3 + \cdots$$

Practice Exercise:

Q-1: Expand $e^x cosy$ near the point $\left(1, \frac{\pi}{4}\right)$ by Taylor's theorem.

Ans: $\boldsymbol{e^x cosy = \frac{e}{\sqrt{2}}\left[1 + (x-1) - \left(y - \frac{\pi}{4}\right) + \frac{(x-1)^2}{2} - (x-1)\left(y - \frac{\pi}{4}\right) - \left(y - \frac{\pi}{4}\right)^2 + \ldots\right]}$

Q-2: Expand y^x about $(1, 1)$ up to second degree terms and hence evaluate$(1.02)^{1.03}$.

Ans: $y^x = 1 + (y-1) + (x-1)(y-1) + \ldots\ldots$.&$(1.02)^{1.03} = 1.0206$

Q-3: Expand $cosx\ cosy$ in powers of x & y up to 4th order terms.

Ans: $\boldsymbol{cosx\ cosy = 1 - \frac{x^2}{2} - \frac{y^2}{2} + \frac{x^4}{24} + \frac{x^2y^2}{4} + \frac{y^4}{24} + \ldots\ldots\ldots}$

Q-4: Obtain linearized form $T(x, y)$ of the function $f(x, y) = x^2 - xy + \frac{y^2}{2} + 3$ at the point $(3, 2)$using Taylor's series expansion.

Ans: $\boldsymbol{T(x, y) = 8 + 4(x-3) - (y-2)}$.

Q-5: Expand $e^{-(x^2+y^2)} \cos xy$ about the point $(0, 0)$ up to second order derivative terms.

Ans: $\boldsymbol{e^{-(x^2+y^2)} \cos xy = 1 - x^2 - y^2 +}$**.....................**

2.2.Jacobian

Jacobian is a functional determinant which is very useful to transform of variables from Cartesian to polar, cylindrical, and spherical co-ordinate in multiple integral.

(1) If $u(x,y)$ and $v(x,y)$ are two functions then the jacobian of u and v is denoted by $J(u,v)$ or $\frac{\partial(u,v)}{\partial(x,y)}$ and its value is $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

(2) If $u_1, u_2, u_3, \dots, u_n$ are function $x_1, x_2, x_3, \dots, x_n$ then $\frac{\partial(u_1,u_2,u_3,\dots,u_n)}{\partial(x_1,x_2,x_3,\dots,x_n)}$ is $\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial u_n}{\partial x_1} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$

2.2.1. Properties of Jacobians

1. Chain Rule Property

If $u(r,s)$ and $v(r,s)$ are two functions i.e. u and v are two function of r *and* s and $r(x,y)$ and $s(x,y)$ are two functions i.e. r *and* s are two function of x *and* y then$u$ and v becomes the functions of x and y

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} * \frac{\partial(r,s)}{\partial(x,y)}$$

2. If $\frac{\partial(u,v)}{\partial(x,y)} = J_1$ and $\frac{\partial(x,y)}{\partial(r,s)} = J_2$ then $J_1 J_2 = 1$

3. If u_i and x_i are related as following

$$u_1 = f(x_1)$$

$$u_2 = f(x_1, x_2)$$

$$\dots\dots\dots\dots\dots\dots$$

$$u_n = f(x_1, x_2, x_3, \dots, x_n)$$

Then, $\frac{\partial(u_1,u_2,u_3,\dots,u_n)}{\partial(x_1,x_2,x_3,\dots,x_n)} = \frac{\partial u_1}{\partial x_1} \cdot \frac{\partial u_2}{\partial x_2} \dots \frac{\partial u_n}{\partial x_n}$

4. If $u(x,y)$ and $v(x,y)$ are two dependent functions then $\frac{\partial(u,v)}{\partial(x,y)} = 0$

5. If $u_1, u_2, u_3$are implicit function x_1, x_2, x_3 i.e.

$$F_1(\text{If } u_1, u_2, u_3, x_1, x_2, x_3) = 0$$
$$F_2(\text{If } u_1, u_2, u_3, x_1, x_2, x_3) = 0$$
$$F_3(\text{If } u_1, u_2, u_3, x_1, x_2, x_3) = 0$$

$$\frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)} = (-1)^3 \left[\frac{\partial(F_1,F_2,F_3)}{\partial(x_1,x_2,x_3)} \Big/ \frac{\partial(F_1,F_2,F_3)}{\partial(u_1,u_2,u_3)} \right]$$

2.1.2. Functional Relationship

If $u_1, u_2, u_3, \dots, u_n$ are functions $x_1, x_2, x_3, \dots, x_n$. Then the necessary condition for the existence of a relation of the form $F(\text{If } u_1, u_2, \dots, u_n) = 0$ is that $\frac{\partial(u_1,u_2,u_3,\dots,u_n)}{\partial(x_1,x_2,x_3,\dots,x_n)} = 0$

Example 1: If $u = x + 2y + z, v = x + 2y + 3z$ and $w = 2x + 3y + 5z$ then find the Jacobian $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

Solution: According to the definition of Jacobian $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{vmatrix} = 2$

Example 2: If $u = xyz, v = xy + yz + zx \; and \; w = x + y + z$ then find the jacobian $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

Solution: According to the definition of jacobian $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} yz & zx & xy \\ y+z & z+x & x+y \\ 1 & 1 & 1 \end{vmatrix}$

To solve the above determinant, we want to reduce the matrix in simplest form so we will use the property (elementary operation) of matrix.

By applying the elementary operation $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_2$, we get

$$\begin{vmatrix} yz & z(x-y) & y(x-z) \\ y+z & x-y & x-z \\ 1 & 0 & 0 \end{vmatrix}$$

which can be reduced to $\begin{vmatrix} z(x-y) & y(x-z) \\ x-y & x-z \end{vmatrix} = (x-y)(y-z)(z-x)$

Example 3. If $u = rcos\theta, v = rsin\theta$ find $\frac{\partial(u,v)}{\partial(r,\theta)}$ and $\frac{\partial(r,\theta)}{\partial(u,v)}$. Also $J_1J_2 = 1$

Solution: $J_1 = \frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix} = \begin{vmatrix} cos\theta & -rsin\theta \\ sin\theta & cos\theta \end{vmatrix} = r$

$r = \sqrt{u^2 + v^2}$ and $\theta = tan^{-1}\frac{v}{u}$

$$J_2 = \frac{\partial(r,\theta)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial r}{\partial u} & \frac{\partial r}{\partial v} \\ \frac{\partial \theta}{\partial u} & \frac{\partial \theta}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{u}{\sqrt{u^2+v^2}} & \frac{v}{\sqrt{u^2+v^2}} \\ \frac{-v}{\sqrt{u^2+v^2}} & \frac{u}{\sqrt{u^2+v^2}} \end{vmatrix} = \frac{1}{\sqrt{u^2+v^2}} = \frac{1}{r}$$

Hence $J_1J_2 = 1$

Example 4: Verify the chain rule for Jacobian if $u = x, v = x\tan y, w = z$

Solution: Let$J_1 = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ tany & xsec^2y & 0 \\ 0 & 0 & 1 \end{vmatrix} = xsec^2y$

Solving x, y, z in terms of u, v and w

$x = u, y = tan^{-1}\frac{v}{u}$and $z = w$

$$J_2 = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ \frac{-v}{u^2+v^2} & \frac{u}{u^2+v^2} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{u}{u^2+v^2} = \frac{1}{xsec^2y}$$

So $J_1J_2 = 1$. Therefore, chain rule is verified.

Example 5: If $x + y + z = u, y + z = uv, z = uvw$, then show that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$

Solution: reduce the given value of x, y and z in terms of u, v and w

$$x = u(1-v)$$

$$y = uv(1-w)$$

$$z = uvw$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}$$

To solve the above determinant, we want to reduce the matrix in simplest form so we will use the property (elementary operation) of matrix.

By applying the elementary operation$R_1 \to R_1 + R_2 + R_3$, we get

$$\begin{vmatrix} 1 & 0 & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix} = u^2v$$

Example 6: If u, v and w are the roots of the cubic equation $(x-a)^3+(y-b)^3+(z-c)^3=0$ then find the value of $\frac{\partial(u,v,w)}{\partial(a,b,c)}$

Solution: The cubic equation$(x-a)^3+(y-b)^3+(z-c)^3=0$

$$3x^3-3x^2(a+b+c)+3x(a^2+b^2+c^2)-(a^3+b^3+c^3)=0$$

Given that $u, v\ and\ w$ are the roots of the cubic equation. So

$$u+v+w=a+b+c$$

$$uv+vw+wu=a^2+b^2+c^2$$

$$uvw=\frac{a^3+b^3+c^3}{3}$$

Since u, v and w are the implicit function of x, y and z. so use the property no. 5 for the implicit function.

$$F_1=u+v+w-a-b-c=0$$

$$F_2=uv+vw+wu-a^2-b^2-c^2=0$$

$$F_3=uvw-\frac{a^3+b^3+c^3}{3}=0$$

$$\frac{\partial(u,v,w)}{\partial(a,b,c)}=(-1)^3\left[\frac{\partial(F_1,F_2,F_3)}{\partial(a,b,c)}\Big/\frac{\partial(F_1,F_2,F_3)}{\partial(u,v,w)}\right]$$

$$\frac{\partial(F_1,F_2,F_3)}{\partial(a,b,c)}=\begin{vmatrix}-1 & -1 & -1\\ -2a & -2b & -2c\\ -a^2 & -b^2 & -c^2\end{vmatrix}$$

By applying the elementary operation $C_2 \to C_2-C_1$ and $C_3 \to C_3-C_2$, we get

$$\begin{vmatrix}-1 & 0 & 0\\ -2a & -2b+2a & -2c+2a\\ -a^2 & -b^2+a^2 & -c^2+a^2\end{vmatrix}$$

$$=-2(a-b)(b-c)(c-a)$$

$$\frac{\partial(F_1,F_2,F_3)}{\partial(u,v,w)}=\begin{vmatrix}1 & 1 & 1\\ v+w & u+w & u+v\\ vw & uw & uv\end{vmatrix}$$

By applying the elementary operation $C_2 \to C_2-C_1$ and $C_3 \to C_3-C_2$, we get

$$\begin{vmatrix}1 & 0 & 0\\ v+w & u-v & u-w\\ vw & w(u-v) & v(u-w)\end{vmatrix}$$

$$= -(u-v)(v-w)(w-u)$$

So

$$\frac{\partial(u,v,w)}{\partial(a,b,c)} = (-1)^3 \frac{\frac{\partial(F_1,F_2,F_3)}{\partial(a,b,c)}}{\frac{\partial(F_1,F_2,F_3)}{\partial(u,v,w)}} = -\frac{2(\mathrm{a}-\mathrm{b})(\mathrm{b}-\mathrm{c})(\mathrm{c}-\mathrm{a})}{(\mathrm{u}-\mathrm{v})(\mathrm{v}-\mathrm{w})(\mathrm{w}-\mathrm{u})}$$

Example 7: Prove that $u = x + 2y + z, v = x - 2y + 3z \; and \; w = 2xy - xz + 4yz - 2z^2$ are not independent and find the relation between them.

Solution: To check the dependency of u, v and w we have to calculate $J(u, v, w)$if $J(u, v, w)$ is zero it means the functions are not independent i.e. they are dependent.

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 1 & -2 & 3 \\ 2y-z & 2x+4z & -x+4y-4z \end{vmatrix}$$

By applying the elementary operation $C_2 \rightarrow C_2 - 2C_1$ and $C_3 \rightarrow C_3 - C_2$, we get

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & -4 & 2 \\ 2y-z & 2x+6z-4y & -x+2y-3z \end{vmatrix} = 0$$

i.e. $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$

so the function u, v and w are not independent i.e. dependent. If they are dependent then there exists a relation between them.

$u + v = 2x + 4z \ldots(1)$

$u - v = 4y - 2z \ldots(2)$

By multiplying both equations, we get

$$u^2 - v^2 = 4(2xy - xz + 4yz - 2z^2)$$

$$u^2 - v^2 = 4w$$

is the relation between u, v and w.

Example 8: Prove that $u = \frac{x+y}{1-xy}$ and $v = tan^{-1}x + tan^{-1}y$ are functionally related. Find the relation between them.

Solution: As we discussed above question to check the dependency of u, v and w we have to calculate $J(u, v)$ if $J(u, v)$ is zero it means the functions are not independent i.e. they are dependent.

$$J(u,v) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix} = 0$$

So u and v are functionally related

$$tan^{-1}x + tan^{-1}y = tan^{-1}\frac{x+y}{1-xy}$$

$$v = tan^{-1}u$$

So u and v are functionally related with the relation $v = tan^{-1}u$.

Practice Exercise:

Q-1: If $u_1 = x_1 + x_2 + x_3 + x_4$, $u_1u_2 = x_2 + x_3 + x_4$, $u_1u_2u_3 = x_3 + x_4$, $u_1u_2u_3u_4 = x_4$ the show that $\frac{\partial(x_1,x_2,x_3,x_4)}{\partial(u_1,u_2,u_3,u_4)} = {u_1}^3{u_2}^2u_3$.

Q-2: If $u^3 + v^3 = x + y$, $u^2 + v^2 = x^3 + y^3$ then show that $\frac{\partial(u,v)}{\partial(x,y)} = \frac{y^2-x^2}{2uv(u-v)}$.

Q-3: If $u_1 = x_1 + x_2 + x_3 + x_4$, ${u_1}^2u_2 = x_2 + x_3$ and ${u_1}^3u_3 = x_3$then find the value of $\frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)}$

Ans: $\boldsymbol{{u_1}^{-5}}$

Q-4: If $x = v^2 + w^2, y = w^2 + u^2 z = u^2 + v^2$ then show that $\frac{\partial(x,y,z)}{\partial(u,v,w)} . \frac{\partial(u,v,w)}{\partial(x,y,z)} = 1$

Q-5: Use the Jacobian to prove that the function $u = x + y - z, v = x - y + z$ and$w = x^2 + y^2 + z^2 - 2yz$ are not independent of one another.

Q-6: Show that $u = x + y + z$, $v = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$ and $w = x^3 + y^3 + z^3 - 3xyz$ are functionally related.

Q-7: If $u = x + y + z$, $v = x^2 + y^2 + z^2$ and $w = x^3 + y^3 + z^3 - 3xyz$, prove that u, v *and* w are not independent and hence find the relation between them. **Ans:** $\boldsymbol{2w = u(3v - u^2)}$

Q-8: Show that the functions: $u = x + y + z$, $v = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$ and $w = x^3 + y^3 + z^3 - 3xyz$ are functionally related. Find the relation between them. **Ans:** $\boldsymbol{w = \frac{u(u^2+3v)}{4}}$

Q-9: If $u = x^2 - y^2$, $v = 2xy$ and If $x = r\cos\theta, y = r\sin\theta$ then $\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^3$.

Q-10: If $u = \sin^{-1}x + \sin^{-1}y$ and $v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$, find $\frac{\partial(u,v)}{\partial(x,y)}$. Prove that u *and* v functionally related, find the relation between them. **Ans:** $\boldsymbol{v = \sin u}$

2.3.Maxima and minima of function of two variables

A function $f(x,y)$ is said to have a maximum value at $x = a, y = b$ if

$f(a,b) > f(a+h, b+k)$, for small and independent values of h and k, positive or negative.

A function $f(x,y)$ is said to have a minimum value at $x = a, y = b$ if

$f(a,b) < f(a+h, b+k)$, for small and independent values of h and k, positive or negative.

Thus, f (x, y) has a maximum or minimum value at a point (a,b) according as

$$F = f(a+h, b+k) - f(a,b) < or > 0$$

The maximum and minimum value of a function is called its extreme value.

2.3.1. Rule to find the extreme values of a function $z = f(x,y)$

1. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$
2. Solve $\frac{\partial z}{\partial x} = 0$ by $\frac{\partial z}{\partial y} = 0$ simultanously
   Let $(a,b), (c,d)$ .... Be the solution of these equations
3. For each solution in step 2 find$r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$ $t = \frac{\partial^2 z}{\partial y^2}$
4. (a) If $rt - s^2 > 0$ and $r < 0$ for a particular (a,b) of step 2, then z has maximum value at(a,b).
   (b) If $rt - s^2 > 0$ and $r > 0$ for a particular (a,b) of step 2, then z has minimum value at (a,b)
   (c) If $rt - s^2 < 0$ for a particular (a,b) of step 2, then z has no extreme value at (a,b)

Example 1. Find the extreme value for the function $x^2 + y^2 + 6x + 12$

Solution. Let $f(x,y) = x^2 + y^2 + 6x + 12$

By taking the partial derivative with respect to x and y respectively

$\frac{\partial f}{\partial x} = 2x + 6, \frac{\partial f}{\partial y} = 2y, r = \frac{\partial^2 f}{\partial^2 x} = 2, s = \frac{\partial^2 f}{\partial x \partial y} = 0$ and $t = \frac{\partial^2 f}{\partial^2 y} = 2$

By equating the first order partial derivative to 0, we get

$2x + 6 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$ and

$$2y = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (2)$$

After solving equation (1) and (2)

We get $x = -3$ and $y = 0$

The stationary point is $(-3,0)$

At $(-3,0)$

$rt - s^2 = 4 > 0 \; and \; r = 2 > 0$

According to working rule, $f(x,y)$ has minimum value at $(-3,0)$ and the minimum value is 3

Example 2: Find the extreme values of function $x^3 + y^3 - 3axy$.

Solution: Here $f(x,y) = x^3 + y^3 - 3axy$

$$f_x = 3x^2 - 3ay, f_y = 3y^2 - 3ax, r = f_{xx} = 6x, s = f_{xy} = -3a, t = f_{yy} = 6y$$

By equating the first order partial derivative to zero

$x^2 - ay = 0 \quad \dots\dots\dots\dots\dots\dots\dots.(1)$ and

$y^2 - ax = 0 \quad \dots\dots\dots\dots\dots\dots\dots.(2)$

From equation (1) $y = \frac{x^2}{a}$

Put the value of y in equation (2) $\frac{x^4}{a^2} - ax = 0 \; or \; x(x^3 - a^3) \; or \; x = 0, a$

When $x = 0, y = 0$; when $x = a, y = a$

There are two stationary points $(0,0)$ and (a,a)

Now, $rt - s^2 = 36xy - 9a^2$

At $(\mathbf{0},\mathbf{0})$ $rt - s^2 = -9a^2 < 0$

It means $f(x,y)$ has no extreme value at $(0,0)$

At $(\boldsymbol{a},\boldsymbol{a})$ $rt - s^2 = 27a^2 > 0$

$f(x,y)$ has extreme value at $(0,0)$

$$r = 6a$$

if $a > 0, r > 0$ so that $f(x,y)$ has a minimum value at (a,a) and minimum value = $-a^3$

if $a > 0, r < 0$ so that $f(x,y)$ has a maximum value at (a,a) and maximum value = a^3

Example 3: Examine for minimum and maximum values $sin\,x + sin\,y + sin\,(x + y)$

Solution: Here $f(x,y) = sin\,x + sin\,y + sin\,(x + y)$

$$f_x = \cos x + \cos(x+y)$$

$$f_y = \cos y + \cos(x+y)$$

$$r = f_{xx} = -\sin x - \sin(x+y)$$

$$s = f_{xy} = \cos x - \sin(x+y)$$

$$t = f_{yy} = -\sin y - \sin(x+y)$$

By equating the first order partial derivative to zero

$\cos x + \cos(x+y) = 0$ ....(1)

$\cos y + \cos(x+y) = 0$...........................(2)

By subtracting equation (2) from (1)

$cos\ x = \ cos\ y\ or\ \ x = y$

put the value in equ (1) $cos\ 2x\ = \ -cosx\ = \ cos(\pi - x)$

$$x = \frac{\pi}{3} = y$$

The stationary point is $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right) r = -\sqrt{3}, s = \frac{\sqrt{3}}{2}, t = -\sqrt{3}$

$$rt - s^2 = \frac{9}{4} > 0\ and\ r < 0$$

So $f(x, y)$ has a maximum value at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ and maximum value is$\frac{3\sqrt{3}}{2}$.

Example 4. In a plane triangle ABC, find the maximum value of $cosA\ cosB\ cosC$.

Solution. In a triangle $A + B + C = \pi$

Convert the given function into two variables A and B

So $\cos A \cos B \cos C = -cosA \cos B\ cos\ (\pi - A - B) = f(A, B)$

Taking the partial derivative of f

$$\frac{\partial f}{\partial A} = -\cos B[-\sin A \cos(A+B) - \cos A\ sin(A+B)] = \cos B \sin(2A+B)]$$

$$\frac{\partial f}{\partial B} = -\cos A[-\sin B \cos(A+B) - \cos B\ sin(A+B)] = \cos A \sin(A+2B)]$$

$$r = 2\cos B\ \cos\ (2A+B), s = \square \cos\ (2A+2B), t = 2\cos A\ \cos(A+2B)$$

By equating the first order partial derivative to zero

$\frac{\partial f}{\partial A} = 0$ and $\quad \frac{\partial f}{\partial B} = 0$

$\cos B \sin(2A + B) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$ and

$$\cos A \sin(A + 2B) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Solving the equation (1) and (2)

if $\cos B = 0$ then $B = \frac{\pi}{2}$ then $\cos A \sin(A + \pi) = 0 \text{ or} - \cos A \sin A = 0$

either $\cos A = 0$ this implies $A = \frac{\pi}{2}$ it means $C = 0$ which is not possible in triangle

or $\sin A = 0$ this implies $A = 0 \text{ or } \pi$ not possible in triangle

so $\cos A \neq 0$ similarly $\cos B \neq 0$

$\sin(2A + B) = 0 \text{ or } 2A + B = \pi$..............(3)

$\sin(A + 2B) = 0 \text{ or } A + 2B = \pi$...(4)

solving equation (3) and (4) $A = B = \frac{\pi}{3}$

so $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ is the stationary point

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

$$r = -1, s = -\frac{1}{2}, t = -1$$

$$rt - s^2 = \frac{3}{4} > 0 \text{ and } r = -1 < 0$$

So $f(A, B)$ is maximum at $A = B = \frac{\pi}{3}$ and maximum value is $\frac{1}{8}$.

Example 5. A rectangular box, open at the top, is to have a given capacity. Find the dimensions of the box requiring last material for its construction.

Solution. Let $x, y, \text{and } z$ be the length, breadth, and height of a rectangular box respectively. Let V be the given capacity and S is surface

Capacity is given so V is constant

$V = xyz$ or $z = \frac{V}{xy}$

$$S = xy + 2xz + 2yz = xy + \frac{2V}{y} + \frac{2V}{x} = f(x, y)$$

By taking the partial derivative

$f_x = y - \frac{2V}{x^2}, f_y = x - \frac{2V}{y^2}, r = f_{xx} = \frac{4V}{x^3}, s = f_{xy} = 1, t = f_{yy} = \frac{4V}{y3}$

By equating the first order partial derivative to zero

$f_x = y - \frac{2V}{x^2} = 0$……………………..(1) and

$f_y = x - \frac{2V}{y^2} = 0$ ……………………..(2)

From equation (1) $y = \frac{2V}{x^2}$

Put the value of y in equation (2) we get $x - 2V\frac{x^4}{4V^2} = 0$

$x\left(1 - \frac{x^3}{2V}\right) = 0$

$x = (2V)^{1/3}$ and $y = \frac{2V}{x^2}$ so $y = (2V)^{1/3}$

Hence $\left((2V)^{1/3}, (2V)^{1/3}\right)$ is the stationary point

At $\left((2V)^{1/3}, (2V)^{1/3}\right)$

$r = 2, s = 1$ and $t = 2$

$rt - s^2 = 3 > 0\ and\ r = 2 > 0$

So $f(x, y)$ has minimum value at $x = y = (2V)^{1/3}$and$z = \frac{V}{xy} = \frac{y}{2}$

Example 6. Find the shortest distance between the lines $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$ and

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$$

Solution: Let $P(x, y, z)$ and $Q(x, y, z)$ be the points on the given line respectively.

$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} = \delta$ and $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1} = \mu$

So $P(\delta + 3, 5 - 2\delta, 7 + \delta), Q(7\mu - 1, -1 - 6\mu,\ \mu - 1)$are two points on the first and second line respectively.

The distance between points P and Q is

$$PQ = \sqrt{(\delta + 3 - 7\mu + 1)^2 + (5 - 2\delta + 6\mu + 1)^2 + (\delta + 7 - \mu + 1)^2}$$

$$= \sqrt{6\delta^2 + 86\mu^2 - 40\delta\mu + 116}$$

If the distance is max or min so it will be the square of the distances.

$$f(\delta,\mu) = PQ^2 = 6\delta^2 + 86\mu^2 - 40\delta\mu + 116$$

Taking the partial derivatives

$$\frac{\partial f}{\partial \delta} = 12\delta - 40\mu$$

$$\frac{\partial f}{\partial \mu} = 172\mu - 40\delta$$

$$r = \frac{\partial^2 f}{\partial \delta^2} = 12$$

$$s = \frac{\partial^2 f}{\partial \delta \partial \mu} = -40$$

$$t = \frac{\partial^2 f}{\partial \mu^2} = 172$$

Equate the first order partial derivative to zero

$$\frac{\partial f}{\partial \delta} = 12\delta - 40\mu = 0 \ldots\ldots..(1)$$

$$\frac{\partial f}{\partial \mu} = 172\mu - 40\delta = 0 \ldots\ldots(2)$$

By solving equation (1) and (2) , we get $\delta = 0, \mu = 0$

The stationary point is $(0,0)$

At $(0,0)$ $\quad rt - s^2 = 12 * 172 - (-40)^2 > 0 \; and \; r = 12 > 0$

So $f(x,y)$ is minimum at $(0,0)$ and the shortest distance is $PQ = \sqrt{116} = 2\sqrt{29}$

Practice Exercise:

Q-1: Examine for extreme values $f(x,y) = x^3 + y^3 - 3xy$**Ans: Min value = -1 at** $(\mathbf{1},\mathbf{1})$

Q-2: Examine for extreme values $f(x,y) = 3x^2 - y^2 + x^3$**Ans: Max value = 4 at** $(\mathbf{-2},\mathbf{0})$

Q-3: Examine for extreme values $f(x,y) = x^3y^2(1-x-y)$**Ans: Max value =** $\mathbf{\frac{1}{432}}$ **at** $\left(\mathbf{\frac{1}{2}},\mathbf{\frac{1}{3}}\right)$

Q-4: Determine the points where the function $f(x,y) = x^2 + y^2 + 6x + 12$ has a maximum or minimum.

Ans: Min value = 3 at $(\mathbf{-2},\mathbf{0})$

Q-5: Find the maximum value of the function $f(x,y) = 1 - x^2 - y^2$.**Ans:1**

Q-6: Prove that if the perimeter of a triangle is constant, its area is maximum when the triangle is equilateral

Q-8: A rectangular box, open at the top, is to have a volume of 32c.c. Find the dimensions of the box requiring least material for its construction. **Ans:4,4,2**

2.4. Hessian matrix

In mathematics, the Hessian matrix, Hessian or (less commonly) Hess matrix is **a square matrix of second-order partial derivatives of a scalar-valued function**.

If f: $R^n \rightarrow R$ is a function, then Hessian matrix of f is denoted by H_f or $H_{F(x,y)}$

Hessian matrix is often used in machine learning and data science algorithms for optimizing a function of interest.

Example1: If $f(x, y) = x^3 + 2y^2 + 3xy^2$ Find H_f

$$H_f = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 6x & 6y \\ 6y & 4+6x \end{bmatrix}.$$

E- Link for more understanding

1. https://www.youtube.com/watch?v=XzaeYnZdK5o&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=1
2. https://www.youtube.com/watch?v=9-tir2V3vYY&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=14
3. https://www.youtube.com/watch?v=aqfSOOiO2kI&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=15
4. https://www.youtube.com/watch?v=GoyeNUaSW08&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=16
5. https://www.youtube.com/watch?v=jiEaKYI0ATY&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=17
6. https://www.youtube.com/watch?v=G0V_yp0jz5c&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=18
7. https://www.youtube.com/watch?v=G0V_yp0jz5c&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=18
8. https://www.youtube.com/watch?v=McT-UsFx1Es&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=9
9. https://www.youtube.com/watch?v=XzaeYnZdK5o&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=1
10. https://www.youtube.com/watch?v=btLWNJdHzSQ&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=12
11. https://www.youtube.com/watch?v=pgfcur31PTY
12. https://www.youtube.com/watch?v=hJ0FMHVZVSc
13. https://www.youtube.com/watch?v=6iTAY9i_v9E
14. https://www.youtube.com/watch?v=NpR91wexqHA
15. https://www.youtube.com/watch?v=gLWUrF_cOwQ
16. https://www.youtube.com/watch?v=pAb1autRHGA
17. https://www.youtube.com/watch?v=HeKB72M2Puw
18. https://www.youtube.com/watch?v=eTp5wq-cSXY
19. https://www.youtube.com/watch?v=6tQTRlbkbc8
20. https://www.youtube.com/watch?v=8ZAucbZscNA

CALCULUS FOR ENGINEERS (K24AS11)

UNIT 3 : COMPLEX VARIABLE - DIFFERENTIATION

Complex Number → It is defined as an ordered pair (x, y) of real numbers and is expressed as.

$$\boxed{z = x + iy} \quad \text{where} \quad i = \sqrt{-1}$$

In polar coordinates it is expressed as.

$\boxed{z = re^{i\theta}}$ where r is modulus of z and θ is argument of z and given as.

$$\boxed{r = \sqrt{x^2 + y^2}}, \quad \boxed{\theta = \tan^{-1}\frac{y}{x}}$$

* The set of points (z) which satisfies the equation $\boxed{|z - a| = r}$ defines a circle C with centre at the point 'a' and radius 'r'.

Function of a compex variable → Let S and S^* be two non empty sets of complex numbers. If correspondind to each value of complex variable z in S, there correspond one more values of complex variable w in S^* then w is called function of complex variable z and is written as

$$w = f(z)$$

where $z = x + iy$ and w is considered as $u + iv$.

Limit of a function $f(z)$ → A fu^n $f(z)$ tends to the limit l as z tends to z_0 along any path, if to each positive arbitrary number ϵ, however small, there corresponds a positive number δ, such that

$$|f(z) - l| < \epsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta$$

and we write $\lim\limits_{z \to z_0} f(z) = l$ where l is finite.

Continuity of function f(z) → A function which is single valued, is said to be continuous at the point $z = z_0$ if following three conditions are satisfied:

(i) $f(z_0)$ exist (ii) $\lim_{z \to z_0} f(z)$ exist (iii) $\lim_{z \to z_0} f(z) = f(z_0)$

Differentiability of function f(z) → Let $f(z)$ be a single valued function defined in a domain D. The function $f(z)$ is said to be differentiable at a point z_0, if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{(z - z_0)} \quad \text{exists.}$$

This limit is called as derivative of $f(z)$ at $z = z_0$)

$$\Rightarrow \boxed{f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}} \text{ or } \boxed{f'(z) = \lim_{\delta z \to 0} \frac{f(z_0 + \delta z) - f(z_0)}{\delta z}}$$

Analyticity of function f(z) → A single valued function $f(z)$ of a complex variable z is said to be analytic at a point z_0 if it is differentiable at the point z_0 and also at each point in some neighbourhood of the point z_0.

→ A function $f(z)$ is said to be analytic in certain domain D if it is analytic at every point of D.

→ If a function is analytic in a domain D except for a finite number of points then these points are said to be regular points or singularities.

Entire function → A fu^n $f(z)$ which is analytic at every point of the finite complex plane is called an entire function.

Cauchy-Riemann equations (C-R Equations) →

for a function of complex variable

$$w = f(z) = u(x,y) + i\,v(x,y)$$

C-R equations are given as $\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$ and $\boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$

Necessary and Sufficient condition for f(z) to be analytic →

Necessary Condition → "The necessary condition for a function f(z) to be analytic is that C-R equations must be satisfied."

But this is only necessary condition not sufficient.

Sufficient Condition → With the truthfullness of C-R equations

"The sufficient condition for the funⁿ f(z) to be analytic is that four partial derivatives U_x, U_y, V_x, V_y must exist and must be continuous at all points of region."

Proof of C-R Equation → Let $w = f(z) = u + iv$ be anaytic in R then $\frac{dw}{dz} = f'(z)$ exist at every point of R

we know

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = \lim_{\delta z \to 0} \frac{(u+\delta u) + i(v+\delta v) - (u+iv)}{\delta z}$$

$$= \lim_{\delta z \to 0} \left(\frac{\delta u}{\delta z} + i \frac{\delta v}{\delta z} \right) \quad \text{------ (1)}$$

Since $w = f(z)$ is analytic in R, the limit must exist independent of path in which $\delta z \to 0$

(i) First let $\delta z \to 0$ along a line parallel to x-axis so that $\delta y = 0$ and $\delta z = \delta x$ $\quad (\because \delta z = \delta x + i\delta y)$

From (i),

$$f'(z) = \lim_{\delta z \to 0} \left(\frac{\delta u}{\delta x} + i \frac{\delta v}{\delta x} \right) = \boxed{\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}} \quad \text{------ (2)}$$

(ii) Again let $\delta z \to 0$ along a line parallel to y axis so that $\delta x = 0$ and $\delta z = i\delta y$ $\quad (\because \delta z = \delta x + i\delta y)$

From (i)

$$f'(z) = \lim_{\delta z \to 0} \left(\frac{\delta u}{i\delta y} + i \frac{\delta v}{i\delta y} \right) = \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$= \boxed{\frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}} \quad \text{------ (3)}$$

from (2) and (3), we have

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{or} \quad \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

<u>C-R Equations in Polar Coordinates</u> →
Let $x = r\cos\theta$, $y = r\sin\theta$ such that
$z = re^{i\theta}$ then C-R eq^n in Polar coordinates are given as $\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}}$ and $\boxed{\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}}$

<u>Harmonic Function</u> → A fu^n of x, y which posseses continuous partial derivatives of the first and second orders and satisfies Laplace equation is called a Harmonic function.

<u>THEOREMS</u> → (1) If $f(z) = u+iv$ is an analytic function then u and v are both harmonic function.

(2) An analytic function with constant modulus is constant.

(3) <u>Orthogonal System</u> → Every analytic function $f(z) = u+iv$ defines two families of curves $u(x,y) = c_1$ and $v(x,y) = c_2$, which form an orthogonal system.
i.e. the curves from two families intersects at right angle

* If $f(z) = u+iv$ is analytic fu^n then u and v are said to be harmonic conjugate of each other.

<u>Determination of conjugate function</u> → If $f(z) = u+iv$ is an analytic function where both $u(x,y)$ and $v(x,y)$ are conjugate function. Being given one of these say $u(x,y)$ we can find the other $v(x,y)$ as following

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \quad \text{by C-R equation}$$

And now dv can be integrated to get "v".

Similarly we cand find $u(x,y)$ if $v(x,y)$ is given and consequently $w = f(z) = u + iv$ may be determined.

MILNE'S THOMSON METHOD → (To construct $f(z)$ when one conjugate is given.)

In this method we determine $f(z)$ as a whole directly without determining the other conjugate function.

Case-I → When real part $u(x,y)$ is given → We follow the steps.

(i) Find $\frac{\partial u}{\partial x}$ and write it as equal to $\phi_1(x,y)$

(ii) Find $\frac{\partial u}{\partial y}$ and write it as equal to $\phi_2(x,y)$

(iii) Find $\phi_1(z,0)$ and $\phi_2(z,0)$ by replacing x by z and y by 0 in $\phi_1(x,y)$ and $\phi_2(x,y)$.

(iv) $f(z)$ is calculated as

$$\boxed{f(z) = \int \{\phi_1(z,0) - i\,\phi_2(z,0)\}\,dz + c}$$

Case-II → When imaginary part $v(x,y)$ is given →

(i) Find $\frac{\partial v}{\partial y}$ and write it as equal to $\psi_1(x,y)$

(ii) Find $\frac{\partial v}{\partial x}$ and write it as equal to $\psi_2(x,y)$

(iii) Find $\psi_1(z,0)$ and $\psi_2(z,0)$ by replacing x by z and y by 0 in $\psi_1(x,y)$ and $\psi_2(x,y)$

(iv) $f(z)$ is calculated as

$$\boxed{f(z) = \int \{\psi_1(z,0) + i\,\psi_2(z,0)\}\,dz + c}$$

Case-III → When $(u-v)$ is given →

(i) $f(z) = u + iv \Rightarrow if(z) = iu - v$

(ii) adding, $(1+i)f(z) = (u-v) + i(u+v)$

or $F(z) = U + iV$

where $F(z) = (1+i)f(z)$, $U = u - v$, $V = u + v$

(iii) As $(u-v)$ or U is given we apply the same process as for real part and find $F(z)$.

(iv) We find $\boxed{f(z) = \frac{1}{1+i}F(z)}$

Case IV → When $u+v$ is given →

(i) $f(z) = u + iv \Rightarrow if(z) = iu - v$

(ii) adding, $(1+i)f(z) = (u-v) + i(u+v)$

or $F(z) = U + iV$

where $F(z) = (1+i)f(z)$, $U = u - v$, $V = u + v$

(iii) As $(u+v)$ or V is given we apply the same process as for imaginary part and find $F(z)$.

(iv) We find $f(z)$ as $\boxed{f(z) = \frac{1}{(1+i)} F(z)}$

Q:- Prove that the function $\sinh z$ is analytic and find its derivative.

Sol:- $f(z) = u + iv = \sinh z = \sinh(x+iy) = \sinh x \cos y + i \cosh x \sin y$

Here $u = \sinh x \cos y$, $v = \cosh x \sin y$

$$\frac{\partial u}{\partial x} = \cosh x \cos y, \quad \frac{\partial u}{\partial y} = -\sinh x \sin y$$

$$\frac{\partial v}{\partial x} = \sinh x \sin y, \quad \frac{\partial v}{\partial y} = \cosh x \cos y$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

C-R. equations are satisfied.

$\because$ $\sinh x$, $\cosh x$, $\sin y$ and $\cos y$ are continuous functions

$\therefore$ $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$ are also continuous functions.

Hence $f(z) = \sinh z$ is analytic.

Now $f'(z)$ is given as

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \cosh x \cos y + i \sinh x \sin y$$

$$= \cosh(x+iy)$$

$$\boxed{f'(z) = \cosh z}$$

Q: Verify if $f(z) = \dfrac{xy^2(x+iy)}{x^2+y^4}$, $z \neq 0$; $f(0)=0$ is analytic or not.

Sol^n ⇒ $f(z) = u+iv = \dfrac{xy^2(x+iy)}{x^2+y^4}$ ⇒ $u = \dfrac{x^2y^2}{x^2+y^4}$, $v = \dfrac{xy^3}{x^2+y^4}$

At origin,

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

Since, $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

Hence C-R equations are satisfied at origin.

Now
$$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{\substack{x\to 0\\ y\to 0}} \left[\frac{xy^2(x+iy)}{x^2+y^4} - 0\right]\frac{1}{x+iy}$$

$$f'(0) = \lim_{\substack{x\to 0\\ y\to 0}} \frac{xy^2}{x^2+y^4}$$

Let $z\to 0$ along the real axis $y=0$, then $f'(0)=0$

again let $z\to 0$ along the curve $x=y^2$ then

$$f'(0) = \lim_{x\to 0} \frac{x^2}{x^2+x^2} = \frac{1}{2}$$

Here we observe that $f'(0)$ does not exist as the limit is not unique along the two paths.

Hence $f(z)$ is not analytic at origin although C-R equations are satisfied there.

Q: Show that $u = \frac{1}{2}\log(x^2+y^2)$ is Harmonic and find its Harmonic conjugate.

Sol^n ⇒ $u = \frac{1}{2}\log(x^2+y^2)$

Now
$$\frac{\partial u}{\partial x} = \frac{1}{2}\frac{1}{(x^2+y^2)}\cdot 2x = \frac{x}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{2}\frac{1}{x^2+y^2}2y = \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)\cdot 1 - y\cdot 2y}{(x^2+y^2)^2} = \frac{x^2 - y^2}{(x^2+y^2)^2}$$

Hence we get $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\because$ u satisfied Laplace eq^n $\therefore$ u is Harmonic fu^n.

To find v,

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

$$= \left(\frac{-y}{x^2+y^2}\right)dx + \left(\frac{x}{x^2+y^2}\right)dy$$

$$= \frac{x\,dy - y\,dx}{(x^2+y^2)} = d\left(\tan^{-1}\frac{y}{x}\right)$$

Integrating $\boxed{v = \tan^{-1}\frac{y}{x} + c}$

Q: If $u = e^x(x\cos y - y\sin y)$ is harmonic function, find an analytic function $f(z) = u + iv$ such that $f(1) = e$. (Use Milne's Thomson method)

Sol^n ⇒ Here $u = e^x(x\cos y - y\sin y)$.

$$\frac{\partial u}{\partial x} = e^x(x\cos y - y\sin y) + e^x(\cos y) = \phi_1(x,y)$$

$$\frac{\partial u}{\partial y} = e^x[-x\sin y - y\cos y - \sin y] = \phi_2(x,y)$$

Now $\phi_1(z,0) = z e^z + e^z = (z+1)e^z$

$\phi_2(z,0) = 0$

By Milne's Thomson method $f(z) = \int\{\phi_1(z,0) - i\phi_2(z,0)\}dz + c$

$$f(z) = \int (z+1)e^z dz + c = (z-1)e^z + e^z + c = \boxed{ze^z + c}$$

Now $f(1) = e + c$ ⇒ $e = e + c$ ⇒ $c = 0$.

⇒ $\boxed{f(z) = ze^z}$

Transformation or Mapping → If we take two complex planes w plane and z plane then then $w=f(z)$ defines a mapping or transformation of z plane into the w plane.

for example for the transformation $w=z+(1-i)$. we will determine D' of w plane corresponding to the rectangle region D in z plane bounded by $x=0, y=0, x=1, y=2$

Since, $w=z+(1-i)$, we have

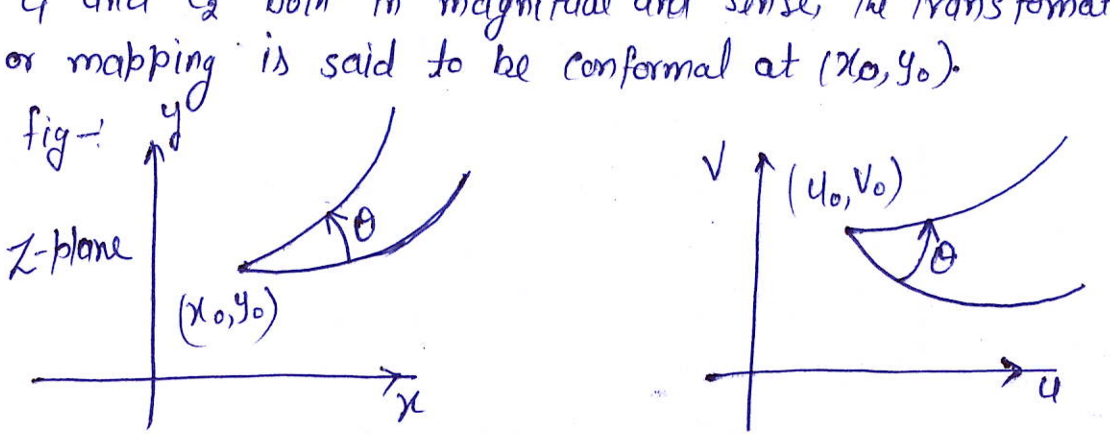
$$u+iv=(x+iy)+(1-i)=(x+1)+i(y-1)$$

$$u=x+1 \quad \text{and} \quad v=y-1$$

Using these relations we obtain that the lines $x=0, y=0$, $x=1$ and $y=2$ in z plane are mapped onto the lines $u=1, v=-1, u=2, v=1$ in w-plane.

Conformal Transformation → Let $w=f(z)$ be a transformation which maps the point (x_0,y_0) of z plane into point (u_0,v_0) of w plane while curves C_1 and C_2 [intersecting at (x_0,y_0)] are mapped into curves C_1' and C_2' [intersecting at (u_0,v_0)]. Then if the transformation is such that the angle at (x_0,y_0) between C_1 and C_2 is equal to the angle at (u_0,v_0) between C_1' and C_2' both in magnitude and sense, the transformation or mapping is said to be conformal at (x_0,y_0).

fig →



* A harmonic function remains harmonic under a conformal transformation.

<u>Coefficient of Magnification</u> → Coefficient of magnification for the conformal transformation $w=f(z)$ at $z=\alpha+i\beta$ is given by $\boxed{|f'(\alpha+i\beta)|}$.

<u>Angle of Rotation</u> → Angle of rotation for conformal transformation $w=f(z)$ at $z=\alpha+i\beta$ is given by $\boxed{\text{amp}\cdot[f'(\alpha+i\beta)]}$

<u>Theorem</u> → If $f(z)$ is analytic and $f'(z)\neq 0$ in a region R of the z plane, then the mapping $w=f(z)$ is conformal at all point of R.

<u>Q:</u> for the conformal mapping $w=z^2$, show that
(a) the coefficient of magnification at $z=2+i$ is $2\sqrt{5}$
(b) the angle of rotation at $z=2+i$ is $\tan^{-1} 0.5$

<u>Solⁿ:</u> Let $w=f(z)=z^2$

$$\Rightarrow f'(z)=2z \Rightarrow f'(2+i)=2(2+i)=4+2i$$

(a) coefficient of magnification at $(z=2+i)$ is $=|f'(2+i)|$

$$=|4+2i|=\sqrt{16+4}=\sqrt{20}=2\sqrt{5}$$

(b) angle of rotation at $(z=2+i)$ is $=\text{amp}[f'(2+i)]=\text{amp}[4+2i]$

$$=\tan^{-1}\frac{2}{4}=\tan^{-1}\frac{1}{2}$$

<u>Some General Transformation</u> →

1→ <u>Translation</u> → $w=z+c$ (c is complex constant)
By this transformation, figures in z plane are displaced or translated in the direction of vector c.

2→ <u>Rotation</u> → $w=e^{i\theta}z$ (θ is real constant)
By this transformation, figures in z plane are rotated through an angle θ. If $\theta>0$ the rotation is counterclockwise, while if $\theta<0$ then rotation is clockwise.

3→ <u>Stretching</u> → $w=az$ (a is real constant)
By this transformation, figures in z plane are stretched (or contracted) in the direction z if $a>1$ (or $0<a<1$).

4 → **Inversion** → $w = \frac{1}{z}$

By this transformation, figures in z-plane are mapped upon the reciprocal figure in w-plane.
i.e. interior of circle $|z|=1$ into the exterior of the circle $|w|=1$ and the exterior of $|z|=1$ is mapped into the interior of $|w|=1$.

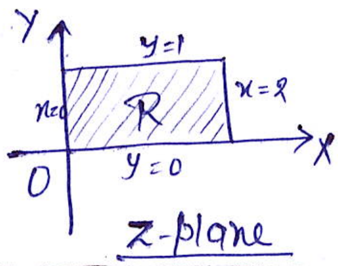
Question → Let a rectangular domain R be bounded by $x=0, y=0$, $x=2, y=1$. Determine the region R' of w-plane into which R is mapped under transformation $w = z+(1-2i)$. [TRANSLATION]

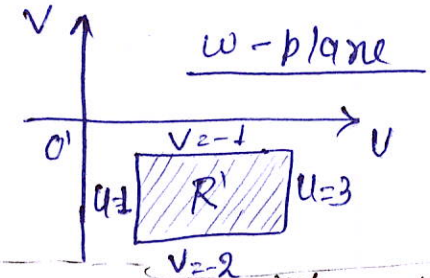
Sol^n ⇒ $w = z+(1-2i) \Rightarrow u+iv = (x+iy)+(1-2i)$

$\Rightarrow u+iv = (x+1)+i(y-2)$

$\Rightarrow \boxed{u=x+1}$ and $\boxed{v=y-2}$

by the map $u=x+1$, lines $x=0$ and $x=2$ are mapped on the lines $u=1$ and $u=3$ respectively.
by the map $v=y-2$ lines $y=0$ and $y=1$ are mapped on the lines $v=-2$ and $v=-1$ respectively.
Hence we can show R and R' as.





Question → Consider the transformation $w = ze^{i\pi/4}$ and determine the region R' in w plane corresponding to the triangular region R bounded by the lines $x=0$, $y=0$ and $x+y=1$ in z plane. [ROTATION]

Sol^n ⇒ $w = ze^{i\pi/4} \Rightarrow u+iv = (x+iy)(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$

$$u+iv = (x+iy)\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\left[(x-y)+(x+y)\right]$$

$\Rightarrow u = \frac{x-y}{\sqrt{2}}, \quad v = \frac{x+y}{\sqrt{2}}$

for $x=0$, $\quad u = -\frac{1}{\sqrt{2}}y, \quad v = \frac{1}{\sqrt{2}}y \quad$ or $\quad v=-u$

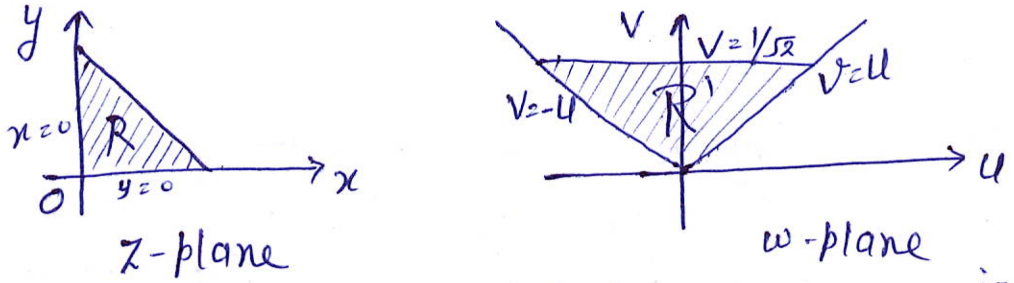
for $y=0$, $\quad u = \frac{1}{\sqrt{2}}x, \quad v = \frac{1}{\sqrt{2}}x \quad$ or $\quad v=u$

for $x+y=1$ we have $v=\frac{1}{\sqrt{2}}$

Hence the region R' in w plane will be bounded by

$v=u$, $v=-u$, $v=\frac{1}{\sqrt{2}}$

Hence R and R' may be shown as.



Hence we observe that the mapping $w=z\,e^{i\pi/4}$ performs a rotation of R through the angle $\pi/4$.

Question → Consider the transfomation $w=2z$ and determine the region R' of w plane into which the triangular region R bounded by the lines $x=0, y=0, x+y=1$. in the z-plane is mapped. [STRETCHING]

Sol^n ⇒ $w=2z \Rightarrow u+iv = 2x+i2y.$

$\Rightarrow u=2x$ and $v=2y.$

Mapping of lines will be

for $\boxed{x=0 \Rightarrow u=0}$

for $\boxed{y=0 \quad , \quad v=0}$

for $x+y=1$, $2x+2y=u+v \Rightarrow 2(x+y)=u+v$

$\Rightarrow$ for $\boxed{x+y=1 \quad , \quad u+v=2}$

Hence region R' may be shown as (which is bounded by $u=0$, $v=0$, $u+v=2$.

This transformation $w=2z$ performs a magnification of R into R'.

Question → Find the image of infinite strip $\frac{1}{4} \leq y \leq \frac{1}{2}$ under the transformation $w = \frac{1}{z}$. Also show the region graphically. [INVERSION]

Sol^n ⇒ $w = \frac{1}{z} \Rightarrow z = \frac{1}{w} \Rightarrow x + iy = \frac{1}{u+iv}$

$$\Rightarrow x + iy = \frac{u - iv}{u^2+v^2} \Rightarrow x = \frac{u}{u^2+v^2}, \quad y = -\frac{v}{u^2+v^2}$$

Now $\boxed{y < \frac{1}{2}} \Rightarrow -\frac{v}{u^2+v^2} < \frac{1}{2} \Rightarrow -2v < u^2+v^2$

$$\Rightarrow u^2+v^2+2v > 0 \Rightarrow \boxed{u^2+(v+1)^2 > 1}$$

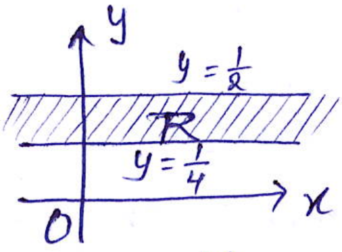
and $\boxed{y > \frac{1}{4}} \Rightarrow -\frac{v}{u^2+v^2} > \frac{1}{4} \Rightarrow -4v > u^2+v^2$

$$\Rightarrow u^2+v^2+4v < 0 \Rightarrow \boxed{u^2+(v+2)^2 < 4}$$

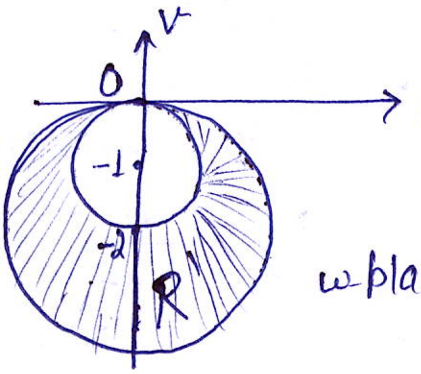
Hence $\frac{1}{4} < y < \frac{1}{2} \Rightarrow u^2+(v+1)^2 > 1$ and $u^2+(v+2)^2 < 4$

this shows that region R' in w plane will be mapped outside the circle $u^2+(v+1)^2 = 1$ and inside $u^2+(v+2)^2 = 4$

R and R' may be shown as below.



z - plane



w plane

Linear Transformation →

The transformation $\boxed{w = az + b}$ where a and b are complex constant is called a linear transformation.

BILINEAR TRANSFORMATION → A transformation of the form

$$\boxed{w = \frac{az+b}{cz+d}}$$, where a, b, c, d are complex constants and $ad - bc \neq 0$, is called a bilinear transformation.

This transformation may be considered as a combination of transformations of Translation, Rotation, Stretching and Inversion.

Cross Ratio – If four points z_1, z_2, z_3, z_4 are taken in order, then the ratio $\dfrac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$ is called cross ratio.

Invarient or fixed points → The points which coincide with their transformations are called invariant points of the transformation. such points for a transformation $w = f(z)$ may be obtained as the sol^n of the equation $z = f(z)$

* for example for $w = z^2$, invariant points may be obtained as. $z = z^2 \Rightarrow z = 0, 1$ are the fixed or invariant points.

Theorem → A bilinear transformation preserves cross ration of four points. i.e. $\boxed{\dfrac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)} = \dfrac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}}$

Question → Find the bilinear transformation which maps the points $z = 1, -i, -1$ to the points $w = i, 0, -i$ respectively. Show that transformation maps the region outside the circle $|z| = 1$ into the half plane $R(w) \geq 0$.

Sol^n ⇒ Required transformation is given as.

$$\frac{(w-i)(0+i)}{(i-0)(-i-w)} = \frac{(z-1)(-i+1)}{(1+i)(-1-z)} \quad \left\{\begin{array}{l}\text{Here points are taken as}\\ (w, i, 0, -i) \;\&\; (z, 1, -i, -1)\end{array}\right\}$$

$$\Rightarrow \frac{i-w}{i+w} = \frac{(z-1)(i-1)}{(z+1)(i+1)} \Rightarrow \frac{i-w}{i+w} = \frac{(i-1)z + (1-i)}{(i+1)z + (1+i)}$$

$$\frac{2i}{-2w} = \frac{2iz+2}{-2z-2i} \quad \text{(applying componendo and dividendo)}$$

on solving $\boxed{w = \dfrac{iz-1}{iz+1}}$ This is the required transformation.

Now this transformation may also be given as.

$$z = i\left(\frac{w+1}{w-1}\right)$$

$|z| \geq 1$ is transformed into $\left|\dfrac{w+1}{w-1}\right| |i| \geq 1$

$$\Rightarrow |w+1|^2 \geq |w-1|^2 \Rightarrow |u+iv+1|^2 \geq |u+iv-1|^2$$

$$\Rightarrow |(u+1)+iv|^2 \geq |(u-1)+iv|^2 \Rightarrow (u+1)^2 + v^2 \geq (u-1)^2 + v^2$$

$$\Rightarrow (u+1)^2 \geq (u-1)^2 \Rightarrow u \geq 0 \Rightarrow \boxed{R(w) \geq 0}$$

Thus exterior of circle $|z| = 1$ is mapped into half plane $R(w) \geq 0$

Question → Find a bilinear transformation which maps the points, $i, -i, 1$ of z plane into $0, 1, \infty$ of w plane respectively.

Sol^n ⇒ Transformation is given as

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

or
$$\frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{(w_1-w_2)\left(1-\frac{w}{w_3}\right)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\Rightarrow \frac{(w-0)\left(\frac{1}{\infty}-1\right)}{(1-0)\left(1-\frac{w}{\infty}\right)} = \frac{(z-i)(-i-1)}{(i+i)(1-z)}$$

$$\Rightarrow w = \frac{(z-i)(i+1)}{(2i)(z-1)} = \boxed{\frac{(i-1)z+(i+1)}{(-2z+2)}}$$

Question → Find the invariant points of the transformation
$$w = -\left(\frac{2z+4i}{iz+1}\right).$$

Sol^n ⇒ Invariant points are given as $z = f(z)$

$$\Rightarrow z = -\left(\frac{2z+4i}{iz+1}\right) \Rightarrow iz^2+3z+4i=0$$

$$\Rightarrow z^2-3iz+4=0 \Rightarrow (z-4i)(z+i)=0$$

$$\Rightarrow z=4i,\ z=-i$$

Hence $z=4i$ and $z=-i$ are two invariant points.

UNIT-4
Multiple Integral

Syllabus: Evaluation of double integrals, change of order of integration, Change of variable (double -integral). Application of double integrals to find area of a region.

Employ the concept of multiple integral to find area of bounded region.

TABLE OF CONTENTS

4.2 INTRODUCTION TO MULTIPLE INTEGRAL

Multiple integral is a natural extension of a definite integral to a function of two variables (Double integral) or more variables.

4.2.1 APPLICATIONS

- Double integrals are useful in finding **Area**.
- Double integrals are useful in finding **Volume**.
- Double integrals are useful in finding **Mass**.
- Double integrals are useful in finding **Centroid**.
- In finding **Average value of a function**.
- In finding **Distance, Velocity, Acceleration.**
- Useful in calculating **Kinetic energy** and **Improper Integrals**.
- In finding **Arc Length** of a curve.
- The most important application of Multiple Integrals involves finding **areas bounded by a curve** and **coordinate axes** and **area between two curves.**
- It includes finding solutions to various complicated problems of **work** and **energy.**
- Multiple integrals are used in many applications in physics. The **gravitational potential** associated with a mass distribution given by a mass **measure on three-dimensional Euclidean space R3** is calculated by **multiple integration.**
- In **electromagnetism, Maxwell's equations** can be written using multiple integrals to calculate the total **magnetic** and **electric fields**.
- We can determine **the probability of an event** if we know the **probability density function** using **double integration.**

4.3 MULTIPLE INTEGRATION

4.3.1 Double integral

A double integral is its counterpart in two dimensions. Let a single valued and bounded function f(x, y) of two independent variables x, y defined in a closed region R.

Then double integral of f(x, y) over the region R is denoted by,

$$\boxed{\iint_R f(x,y)\,dA}$$

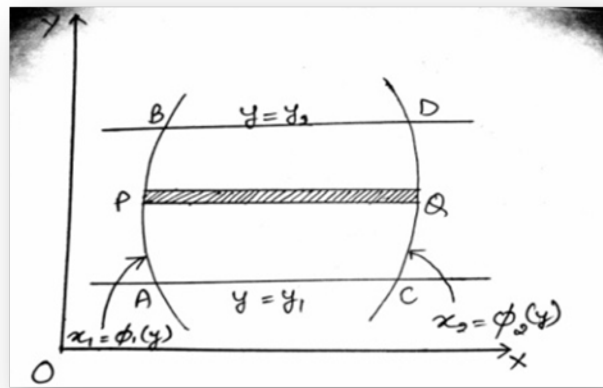
Also we can express as $\iint_R f(x,y)dx\,dy$ or $\iint_R f(x,y)dy\,dx$

4.3.1.2 Evaluation of Double Integral in Cartesian Coordinates

The method of evaluating the double integrals depends upon the nature of the curves bounding the region R. Let the region be bounded by the curves $x = x_1, x = x_2$ and $y = y_1, y = y_2$.

(i) ***When x_1, x_2 are functions of y and y_1, y_2 are constants***: If we have functional limits of x in terms of dependent variable y $[x_1 = \emptyset_1(y), x_2 = \emptyset_2(y)]$ and constant limits of variable y then we will first integrate with respect to variable x in case of double integral, as follows:

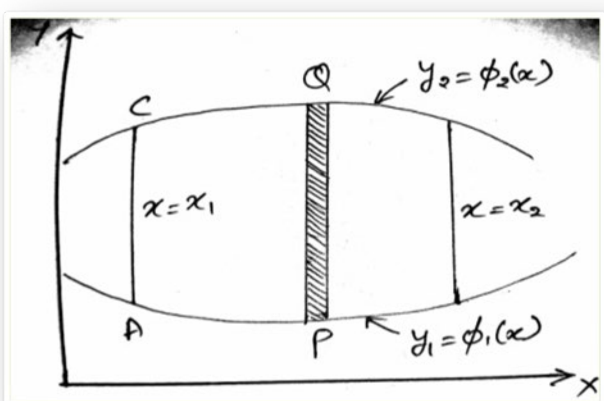
$$\boxed{\iint_R f(x,y)\,dx\,dy = \int_{y_1}^{y_2}\{\int_{x_1=\emptyset_1(y)}^{x_2=\emptyset_2(y)} f(x,y)dx\}\,dy}$$



(Here we have drawn the strip parallel to x axis, because variable limits $[x_1 = \emptyset_1(y), x_2 = \emptyset_2(y)]$ are provided.)

(ii) ***When y_1, y_2 are functions of x and x_1, x_2 are constants:*** If we have functional limits of y in terms of dependent variable x $[\,y = \emptyset_1(x), y_2 = \emptyset_2(x)]$ and constant limits of variable x then we will first integrate with respect to variable y in case of double integral, as follows:

$$\boxed{\iint_R f(x,y)\,dx\,dy = \int_{x_1}^{x}\{\int_{y_1=\emptyset_1(x)}^{y_2=\emptyset_2(x)} f(x,y)dy\}\,dx}$$

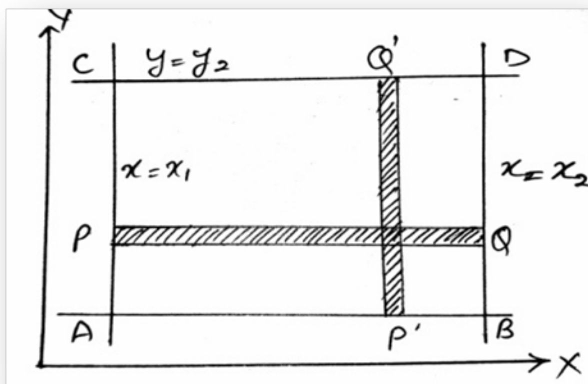


(Here we have drawn the strip parallel to y axis, because variable limits [$y = \emptyset_1(x), y_2 = \emptyset_2(x)$] are provided.)

(iii) ***When x_1, x_2, y_1, y_2 are constants:*** If we have both the variables x and y with constant limits x_1, x_2, y_1, y_2 then we can first integrate with respect to any variable x or y in case of double integral, as follows:

$$\boxed{\iint_R f(x,y)dx\,dy = \int_{x_1}^{x2}\{\int_{y_1}^{y_2} f(x,y)dy\}\,dx}$$

$$\boxed{\iint_R f(x,y)dx\,dy = \int_{y_1}^{y_2}\{\int_{x_1}^{x_2} f(x,y)dx\}\,dy}$$



(Here we can draw the strip parallel to any of the axes, because both x and y are having constant limits.)

From case no. (i) and (ii) discussed above, we observe that integration is to be performed w.r.t. the variable limits first and then w.r.t. the variable with constant limits.

4.3.1.3 Solved examples

Example1: Evaluate $\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)(1-y^2)}}$.

Solution: $\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)(1-y^2)}} = \int_0^1 \left[\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-y^2)}}\right] dy$ (Here we have constant limits for both x and y variables, so we may integrate w.r.t. any variable the first)

$$= \int_0^1 \frac{1}{\sqrt{(1-y^2)}} [\sin^{-1} x]_0^1 \, dy$$

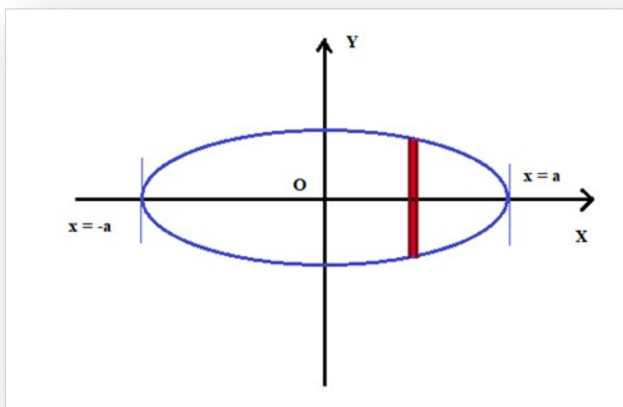
$$= \int_0^1 \frac{1}{\sqrt{(1-y^2)}} \frac{\pi}{2} dy$$

$$= \frac{\pi}{2} [\sin^{-1} y]_0^1 = \frac{\pi}{2} . \frac{\pi}{2} = \frac{\pi^2}{4}$$

Example 2: Evaluate $\iint (x+y)^2 \, dxdy$ over the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution: (Here we have area bounded by the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, depending on variables x and y so we have to construct a strip parallel to any one axis to observe variable limits of one variable.)

For the ellipse we may write $\frac{y}{b} = \pm\sqrt{1 - \frac{x^2}{a^2}}$ or $y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$



∴ The region of integration R can be expressed as

$-a \leq x \leq a, \ -\frac{b}{a}\sqrt{a^2 - x^2} \leq y \leq \frac{b}{a}\sqrt{a^2 - x^2}$,

where we have chosen variable limits of y and constant limits of x.

So first we will integrate w.r.t. y,

$$\therefore \iint (x+y)^2 \, dxdy = \iint_R (x^2 + y^2 + 2xy) dxdy$$

$$= \int_{-a}^{a} \int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2 + y^2 + 2xy) dxdy$$

$$= \int_{-a}^{a} \int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2)dxdy + \int_{-a}^{a} \int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (2xy)dxdy$$

$$= \int_{-a}^{a} \int_{0}^{\frac{b}{a}\sqrt{a^2-x^2}} 2(x^2+y^2)dxdy + 0$$ [using the property of even and odd functions.]

$$= \int_{-a}^{a} \left[2\left(x^2y+\frac{y^3}{3}\right)\right]_0^{\left(\frac{b}{a}\right)\sqrt{a^2-x^2}} dx$$

$$= 2\int_{-a}^{a} \left[x^2\frac{b}{a}\sqrt{a^2-x^2}+\frac{1}{3}\frac{b^3}{a^3}(a^2-x^2)^{3/2}\right]dx$$

$$= 4\int_{0}^{a} \left[\frac{b}{a}x^2\sqrt{a^2-x^2}+\frac{b^3}{3a^3}(a^2-x^2)^{3/2}\right]dx$$ Putting $x = a\sin\theta$

$\Rightarrow dx = a\cos\theta d\theta$

$$= 4\int_{0}^{\pi/2} \left[\frac{b}{a}a^2\sin^2\theta . a\cos\theta + \frac{b^3}{3a^3}.a^3\cos^3\theta\right] \times a\cos\theta d\theta$$

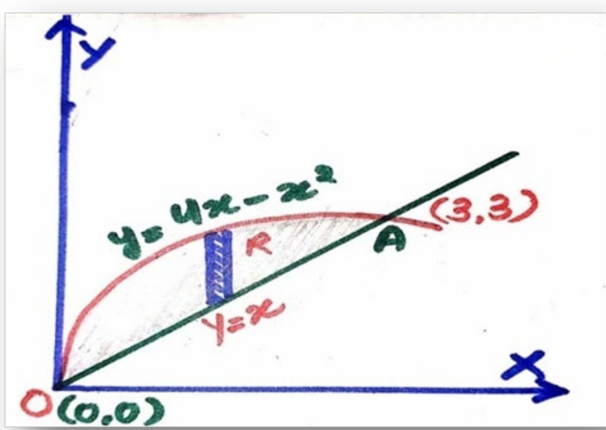
$$= 4\int_{0}^{\pi/2} \left[a^3b\sin^2\theta.\cos^2\theta + \frac{ab^3}{3}.\cos^4\theta\right]d\theta$$

$$= 4\ [a^3b.\frac{1}{4}.\frac{1}{2}.\frac{\pi}{2} + \frac{ab^3}{3}\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} = \frac{\pi}{4}(a^3b+ab^3) = \frac{\pi}{4}ab(a^2+b^2)$$

Example3: Evaluate $\iint y\ dxdy$ over the part of the plane bounded by the lines y= x and the parabola y = 4x - x^2.

Solution: The line y =x and the parabola y = 4x - x^2 intersect each other at two distinct points O (0,0) and A (3,3). Now in the intersected area we will construct a strip suitably parallel to x-axis or y-axis to have variable limit of one variable in terms of other variable.

$$\iint_R y\ dxdy = \int_0^3 \int_x^{4x-x^2} y\ dydx$$



$= \int_0^3 \left(\frac{y^2}{2}\right)_x^{4x-x^2} dx$

$= \frac{1}{2}\int_0^3 [(4x - x^2)^2 - x^2]\, dx$

$= \frac{1}{2}\int_0^3 [15x^2 + x^4 - 8x^3]\, dx$

$= \frac{1}{2}\left(5x^3 + \frac{x^5}{5} - 2x^4\right)_0^3 \quad = \frac{54}{5}$

Example 4: Evaluate $\iint y\, dxdy$ over the region R bounded by the parabolas $y^2 = 4x\; x^2 = 4y$.

Solution: Solving $y^2 = 4x \; and \; x^2 = 4y$, we have

$$\left(\frac{x^2}{4}\right)^2 = 4x \;\; or \;\; x(x^3 - 64) = 0$$

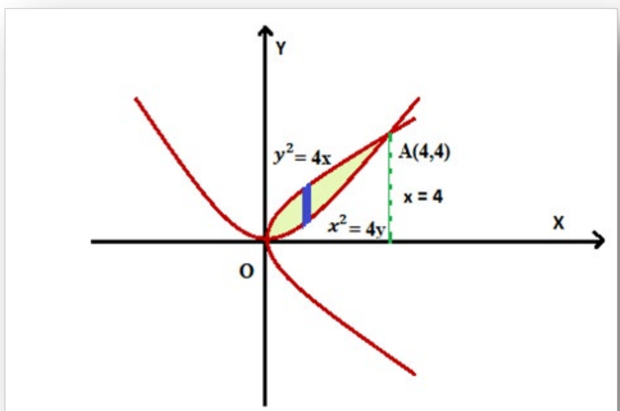
∴ x = 0, 4

When x = 4, y = 4

∴ Co-ordinates of A (intersection point of parabolas) are (4, 4)

The region R can be expressed as

$0 \leq x \leq 4, \; \frac{x^2}{4} \leq y \leq 2\sqrt{x}$



∴ $\iint_R y\, dx\, dy$ y dx $= \int_0^4 \int_{x^2/4}^{2\sqrt{x}} y\, dydx$

$= \int_0^4 \frac{1}{2}[y^2]_{x^2/4}^{2\sqrt{x}}\, dx = \frac{1}{2}\int_0^4 \left(4x - \frac{x^4}{16}\right) dx$

$= \frac{1}{2}\left[2x^2 - \frac{x^5}{80}\right]_0^4 = \frac{1}{2}\left[32 - \frac{1024}{80}\right] = \frac{48}{5}$

Example 5: Evaluate $\iint_S \sqrt{xy - y^2\, dxdy}$, where S is a triangle with vertices (0, 0), (10, 1) and (1, 1).

Solution: Let OAB be the triangle formed by given vertices (0, 0), (10, 1) and (1, 1) as shown in the figure through shaded area.

The equation of the line joining O (0, 0) and A (1, 1) can be find as follows,

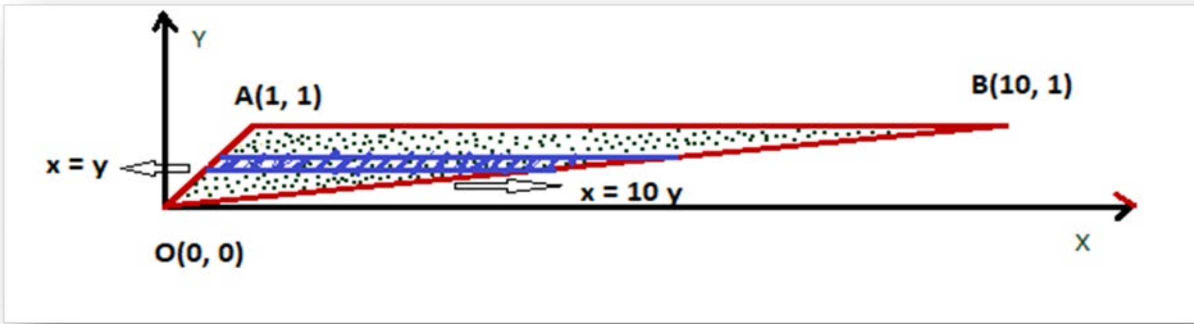
$y - 0 = \frac{1-0}{1-0}(x-0) \quad \Rightarrow y = x$

The equation of the line joining O (0, 0) and B (10, 1) can be calculated as follows,

$y - 0 = \frac{1-0}{10-0}(x-0) \quad \Rightarrow x = 10\,y$

Here we have taken strip intentionally parallel to x-axis, so that the strip bounded by x = y and x = 10 y may cover the complete shaded area from y = 0 to y = 1.

Hence the region of integration can be expressed as $y \le x \le 10y, 0 \le y \le 1$



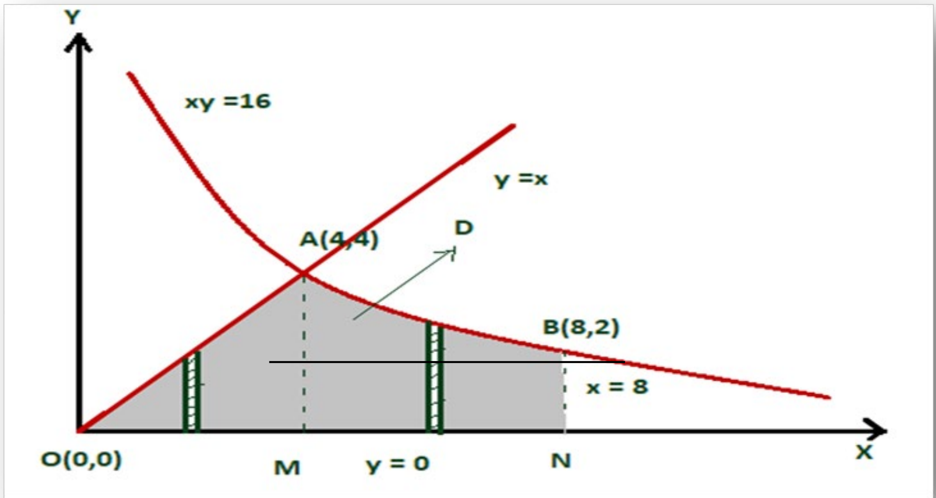
$$\iint_S \sqrt{xy - y^2}\,\text{dxdy} = \int_0^1 \int_y^{10y} \sqrt{xy - y^2}\,\text{dxdy}$$

$$= \int_0^1 \left[\frac{2}{3}\frac{(xy-y^2)^{3/2}}{y}\right]_y^{10y} \text{dy} = \int_0^1 \frac{2}{3y}(9y^2)^{3/2}\,dy$$

$$= 18\int_0^1 y^2 dy = 18\left(\frac{y^3}{3}\right)_0^1 = 6$$

Example 6: Let D be the region in the first quadrant bounded by the curves xy = 16, x = y, y = 0 and x = 8.Sketch the region of integration of the following integral $\iint_D x^2\,dxdy$ and evaluate it by expressing it as an appropriate repeated integral.

Solution: In this question we have to integrate the given function within the region bounded by the straight line x = y, hyperbola xy = 16, y = 0 and x = 8, so first we will draw the figure for clarity finding all intersection points of curves provided in question.



Here we can see the equations x = y and xy = 16, on solving give intersection point at A (4, 4)

Similarly on solving xy =16 and x = 8, we get the intersection point at B (8, 2)

Drawing the curves we get the intersection area as shown in figure.

Now we are to decide with respect to which variable we should first integrate, we construct strips in such a manner that the complete area may be covered.

Here we cannot cover the whole shaded area using single strip (Neither parallel to x- axis nor parallel to y- axis).

Because area is changing from dotted lines, if we plot strip parallel to y-axis and also area is changing from lines drawn, if we plot the strip parallel to x-axis. So in both cases we need to draw two strips.

Here we are splitting the area OABNO in two parts by AM as shown in figure and plotted strips parallel to y- axis from x = 0 to x = 4 and from x= 4 to x = 8

Then, $\iint_D x^2\,dxdy = \int_{x=0}^{x=4}\int_{y=0}^{y=x} x^2\,dydx + \int_{x=4}^{x=8}\int_{y=0}^{y=16/x} x^2\,dydx$

$$= \int_{x=0}^{x=4} x^2\,dx \int_{y=0}^{y=x} dy + \int_{x=4}^{x=8} x^2\,dx \int_{y=0}^{y=16/x} dy$$

$$= \int_{x=0}^{x=4} x^2\,(y)_0^x\,dx + \int_{x=4}^{x=8} x^2\,(y)_0^{16/x}\,dx$$

$$= \int_0^4 x^3 dx + \int_4^8 16x\,dx = \left(\frac{x^4}{4}\right)_0^4 + (8x^2)_4^8 = 64 + 8(64 - 16) = 64 + 384 = 448$$

4.3.1.4 Practice problems

1. Evaluate $\int_1^2 \int_0^x \frac{dydx}{x^2+y^2}$ Ans : $\pi/_2 \log 2$

2. Evaluate $\int_1^{\log 8} \int_0^{\log y} e^{x+y}\, dx\, dy$. Ans. 8(log8-2)+e

3. Evaluate:

(i) $\int_0^1 dx \int_0^x e^{y/x}\; dy$ Ans. ½

(ii) $\int_0^1 \int_{y^2}^{y} (1+xy^2) dxdy$ Ans. 41/210

(iii) $\int_1^a \int_1^b \frac{1}{xy} dxdy$ Ans. log a log b

(iv) $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2) dydx$ Ans.3/ 35

4. Evaluate $\iint_R \left(1-\frac{x^2}{a^2}+\frac{y^2}{b^2}\right) dxdy$ over the first quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=0$. Ans. $\frac{\pi ab}{4}$

5. Evaluate $\iint xy(x+y)dxdy$ over the area between $y=x^2\; and\; y=x$. Ans. 3/56

6. Evaluate $\iint_A\; xy\; dxdy$, where A is the domain bounded by x-axis, ordinate x = 2a and the curve $x^2=4ay$.

Ans. $\frac{a^4}{3}$

4.3.1.5 Evaluation of Double Integrals in Polar Coordinates

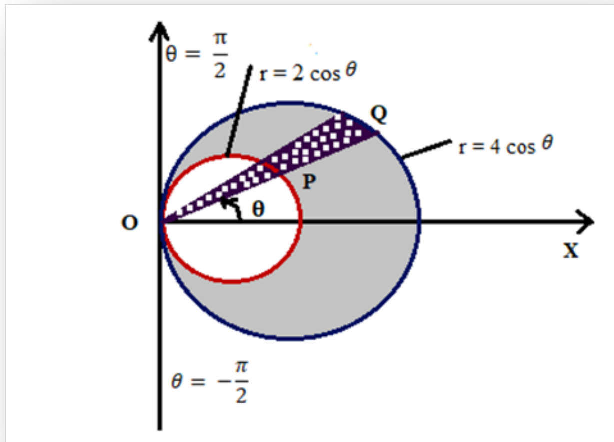
In polar coordinates we know that, x = r cos θ and y = r sin θ

Sometimes integration can be easier by converting Cartesian form to polar form. In such cases we may evaluate integral by polar coordinates using variable r and θ in the same manner as we done earlier. Here we draw radial strip to decide the limit in order to cover the whole area.

4.3.1.6 Solved Examples

Example 1: Evaluate $\iint r^3\, dr d\theta$, over the area bounded between the circles r = 2 cos θ and r = 4 cos θ.

Solution: The region of integration R is shown shaded. Here r varies from r = 2 cos θ to r = 4 cos θ while θ varies from $-\frac{\pi}{2}\; to \frac{\pi}{2}$.



$$\iint_R r^3 \,\mathrm{dr}\,\mathrm{d}\,\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{2\cos\theta}^{4\cos\theta} r^3\, dr d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{r^4}{4}\right]_{2\cos\theta}^{4\cos\theta} d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{4}(256\cos^4\theta - 16\cos^4\theta)\; d\theta$$

$$= 60 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta\, d\theta = 120 \int_0^{\frac{\pi}{2}} \cos^4\theta d\theta = 120 \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} = \frac{45}{2}\pi$$

Reference formula: $\int_0^{\pi/2} \sin^n\theta\, d\theta = \int_0^{\pi/2} \cos^n\theta\, d\theta = \begin{cases} \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}\ldots\ldots\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} & if\ n\ is\ even \\ \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}\ldots\ldots\frac{4}{5}.\frac{2}{3} & if\ n\ is\ odd \end{cases}$

Example 2: Evaluate $\int_0^{\frac{\pi}{2}} \left[\int_0^{a\cos\theta} r\sqrt{a^2 - r^2}\, dr\right] d\theta$.

Solution: $\mathrm{I} = \int_0^{\pi/2} \left[\int_0^{a\cos\theta} -\frac{1}{2}(a^2 - r^2)^{1/2}(-2r) dr\right] d\theta$

$$= \int_0^{\pi/2} \left[-\frac{1}{2}.\frac{(a^2-r^2)^{3/2}}{3/2}\right]_0^{a\cos\theta} d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2}(a^3\sin^3\theta - a^3) d\theta = \frac{a^3}{3}\left[\frac{2}{3} - \frac{\pi}{2}\right] = \frac{a^3}{3}(3\pi - 4).$$

4.3.1.7 Practice problems

1. Evaluate $\int_0^{\pi/2} \int_0^{a\cos\theta} r\sin\theta \quad dr d\theta$ Ans: $\frac{a^2}{6}$

2. Evaluate $\int_0^{\pi} \int_0^{a(1-cos\theta)} r^2 \sin\theta \, dr \, d\theta$ Ans: $\frac{4}{3}a^3$

3. Evaluate $\iint r^3 \, dr \, d\theta$, over the area bounded between the circles r = 2 sinθ and r = 4 sin θ. Ans: $\frac{45\pi}{2}$

4.3.2 CHANGE OF ORDER OF INTEGRATION

In double integral, if the limits of integration are constant, then the order of integration does not matter, provided the limits of integration are changed accordingly. Thus,

$$\int_c^d \int_a^b f(x,y) dx dy = \int_a^b \int_c^d f(x,y) dx dy$$

But if the limits of integration are variable, then in order to change the order of limits of integration we have to construct the rough figure of given region of integration and re construct the strip parallel to that axis with respect to which we want to first integrate.

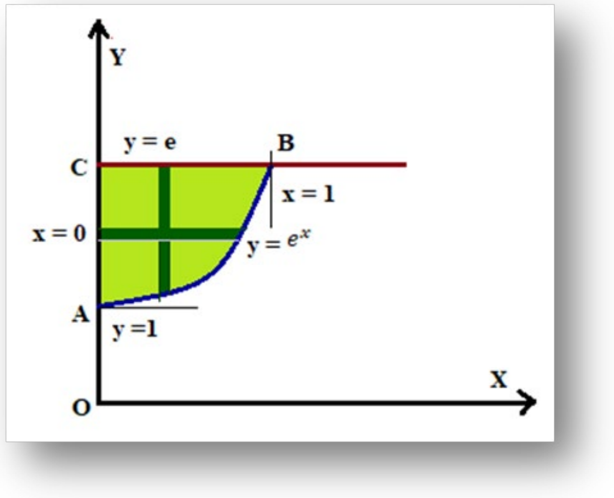
Mostly we need this process to make our integration simpler if possible.

4.3.2.1 Solved examples

Example 1: Evaluate the following integral by changing the order of integration

$$\int_0^1 \int_{e^x}^{e} \frac{dx dy}{\log y} .$$

Solution: The given limits show that the region of integration is bounded by the curves y = e^x, y = e, x = 0, x = 1. Plotting these curves we have the shaded region of integration as shown in figure.



In given problem we had variable limits of y in terms of x, so we had to integrate w.r.t. y the first. But we are instructed to solve this problem changing the order of integration.

Now to integrate first w.r.t. x we have to find variable limits of x in terms of y. So we have to construct a strip parallel to x- axis in order to find variable limits of x.

From the strip we can see lower limit lies on x = 0 and y = $e^x \Rightarrow x = \log y$ in between the constant limits of y from y = 1 to y = e.

Hence $\int_0^1 \int_{e^x}^{e} \frac{dxdy}{\log y} = \int_1^e \int_0^{\log y} \frac{dxdy}{\log y}$

$$= \int_1^e \left(\frac{x}{\log y}\right)_0^{\log y} dy$$

$$= \int_1^e 1.\, dy = (y)_1^e = (e-1).$$

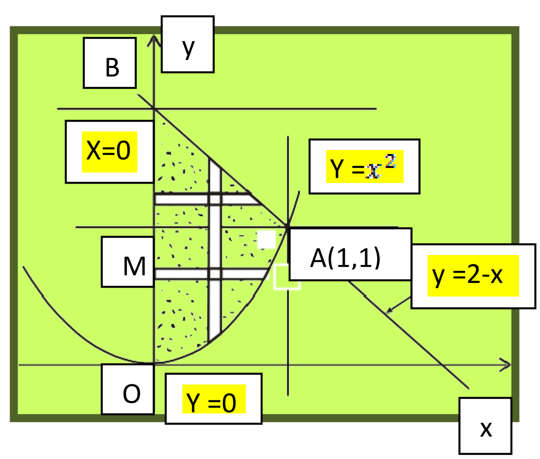
Example 2: Change the order of integration in I = $\int_0^1 \int_{x^2}^{2-x} xy\, dydx$ and hence evaluate the same

Solution: From the variable limits of integration, it is clear that we have to integrate first w.r. to y which varies from y $=x^2$ to y = 2-x and then with respect to x which varies from x = 0 to x = 1. The region of integration is divided into vertical strips.

For changing the order of integration, we divide the region of integration into horizontal strips.

Solving y = x^2 and y = 2-x, the co-ordinates of A are (1,1).

Draw AM ⊥ OY. The region of integration is divided into two parts, OAM and MAB.



For the region OAM, x varies from 0 to 2-y and y varies from 1 to 2.

$$\therefore \int_0^1 \int_{x^2}^{2-x} xy\, dydx = \int_0^1 \int_0^{\sqrt{y}} xy\, dydx + \int_1^2 \int_0^{2-y} xy\, dydx$$

$$= \int_0^1 y\left(\frac{x^2}{2}\right)_0^{\sqrt{y}} dy + \int_1^2 \left(\frac{x^2}{2}\right)_0^{2-y} dy = \frac{1}{2}\int_0^1 y^2 dy + \frac{1}{2}\int_1^2 y(2-y)^2 dy$$

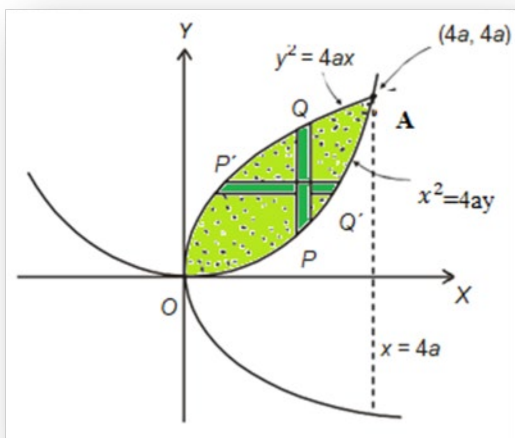
$$= \frac{1}{2}\left(\frac{y^3}{3}\right)_0^1 + \frac{1}{2}\int_1^2 (4y - 4y^2 + y^3)dy = \frac{1}{6} + \frac{1}{2}\left[2y^2 - \frac{4y^3}{3} + \frac{y^4}{4}\right]_1^2$$

$$= \frac{1}{6} + \frac{1}{2}\left[\left(8 - \frac{32}{3} + 4\right) - \left(2 - \frac{4}{3} + \frac{1}{4}\right)\right] = \frac{3}{8}$$

Example 3: Change the order of integration in the following integral and evaluate:

$$\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dydx\, .$$

Solution: From the limit of integral, it is clear that we have to first integrate with respect to y, having variable limits of y in terms of x ($y = x^2/4a$ to $y = 2\sqrt{ax}$). So integration is performed along the strip PQ shown in figure.



In above figure we drawn both the parabolas intersecting at A(4a,4a).

In order to change the order of integration we draw a strip P'Q', which extend from P' on parabola $x = y^2/4a$ to Q' on parabola $x = 2\sqrt{ay}$ and varying from y = 0 to y = 4a.

$$\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dydx = \int_0^{4a} \int_{y^2/4a}^{2\sqrt{ay}} dxdy$$

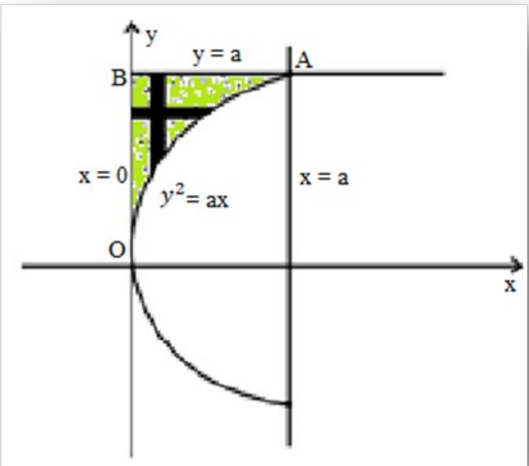
$$= \int_0^{4a} (x)_{y^2/4a}^{2\sqrt{ay}} dy = \int_0^{4a} (2\sqrt{ay} - {y^2}/{4a}) dy$$

$$= \left[2\sqrt{a}.\frac{y^{3/2}}{3/2} - \frac{y^3}{12a}\right]_0^{4a} = \frac{4}{3}\sqrt{a}(4a)^{3/2} - \frac{64a^3}{12a} = \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}$$

Example 4: Change the order of integration and hence evaluate $\int_0^a \int_{\sqrt{ax}}^a \frac{y^2 dy dx}{\sqrt{y^4 - a^2x^2}}$.

Solution: The given limits shows that the area of integration lies between$y^2 = ax$, y = a, x = 0, x = a.

We can consider it as lying between y =0, y = a, x = 0, x = $\frac{y^2}{a}$



by changing the order of integration. Hence the given integral,

$$\int_{x=0}^{a} \int_{y=\sqrt{ax}}^{a} \frac{y^2 dy dx}{\sqrt{y^4 - a^2x^2}} = \int_{y=0}^{a} \int_{x=0}^{\frac{y^2}{a}} \frac{y^2 dy dx}{\sqrt{y^4 - a^2x^2}}$$

$$= \frac{1}{a} \int_{y=0}^{a} \int_{x=0}^{\frac{y^2}{a}} \frac{y^2 dy dx}{\sqrt{\left(\frac{y^2}{a}\right)^2 - x^2}}$$

$$= \frac{1}{a} \int_0^a y^2 \left[sin^{-1}\left(\frac{ax}{y^2}\right)\right]_0^{y^2/a} dy$$

$$= \frac{1}{a} \int_0^a y^2 [sin^{-1}(1) - sin^{-1}(0)] dy$$

$$= \frac{\pi}{2a} \int_0^a y^2 dy = \frac{\pi}{2a}\left(\frac{y^3}{3}\right)_0^a = \frac{\pi}{6a}(a^3) = \frac{\pi a^2}{6}$$

Example5: Evaluate the following integral by changing the order of integration:

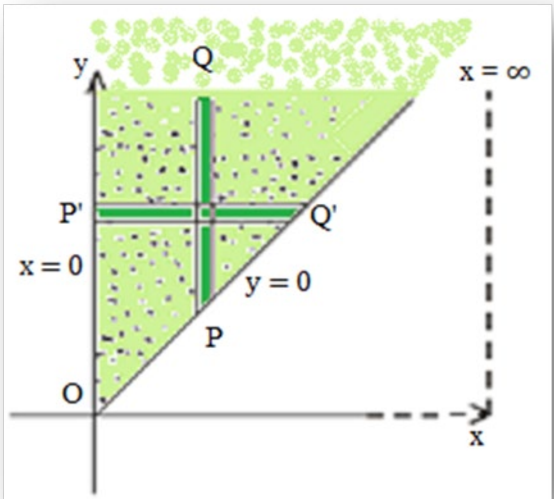
$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dy dx$$

Solution: The given limits shows that the area of integration lies between y = x, y = ∞, x = 0 and x = ∞.

We can consider it as lying between x = 0, x = y, y = 0 and y = ∞ by changing the order of integration.

Hence the given integral,

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dy dx = \int_0^\infty \int_0^y \frac{e^{-y}}{y} dx dy$$



$$=\int_0^\infty \frac{e^{-y}}{y}(x)_0^y\,dy$$

$$=\int_0^\infty e^{-y}\,dy$$

$$=(-e^{-y})_0^\infty$$

$$= 1-0 = 1$$

4.3.2.2 Practice problems

1. Evaluate the integrals by changing the order of integration:

(i)	$\int_0^1\int_x^{\sqrt{2-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2-y^2}}$	Ans: $1-\frac{1}{\sqrt{2}}$
(ii)	$\int_0^a\int_{x^2/a}^{2a-x} xy\,dy\,dx$	Ans: $\frac{3a^4}{8}$
(iii)	$\int_0^1\int_{y^2}^{2-y} xy\,dy\,dx$	Ans: 3/8
(iv)	$\int_0^1\int_x^1 \sin y^2\,dy\,dx$	Ans: ½(1- cos 1)
(v)	$\int_0^\infty\int_0^x x\,e^{\frac{-x^2}{y}}\,dy\,dx$	Ans: $\frac{1}{2}$
(vi)	$\int_0^2\int_{\frac{x^2}{4}}^{3-x} xy\,dy\,dx$	Ans: $\frac{8}{3}$
(vii)	$\int_0^{2a}\int_{\frac{x^2}{4a}}^{3a-x}(x^2+y^2)\,dy\,dx$	Ans: $\frac{314}{35}a^4$
(viii)	$\int_0^1\int_0^{\sqrt{1-x^2}} \frac{e^y\,dy\,dx}{(e^y+1)\sqrt{x^2-y^2}}$	Ans: $\frac{\pi}{2}\log\frac{(e+1)}{2}$
(ix)	$\int_0^3\int_0^{6/x} x^2\,dy\,dx$	Ans: 27
(x)	$\int_0^1\int_{x^2}^{2-x} f(x,y)\,dy\,dx$	Ans: $\int_0^1\int_0^{2\sqrt{y}} f(x,y)dxdy+\int_1^2\int_0^{2-y} f(x,y)\,dxdy$

4.3.3 CHANGE OF VARIABLES

In simple integration, we use substitution to make our integration simpler than before. Similarly, in double or triple integration we use suitable change of variables to make the evaluation of integration simple.

In general, there are following four types of transformation:

i. To change Cartesian co-ordinates (x, y) to some given co-ordinates (u, v).

ii. To change Cartesian co-ordinates (x, y) to polar co-ordinates (r, θ).

iii. To change Cartesian co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ).

iv. To change Cartesian co-ordinates (x, y, z) to cylindrical co-ordinates (r, ϕ, z).

4.3.3.1 To change Cartesian co-ordinates (x, y) to some given co-ordinates (u, v):

Let there be two variables (x, y) in the double integral $\iint_R f(x, y)dxdy$ .We are to change these variables to some variables (u, v) under the transformation $x = \phi(u, v), y = \varphi(u, v)$. Under this transformation the given integral takes the form $\iint_{R'} f\{\phi(u, v), \varphi(u, v)\}|J|dudv$ where $J = \dfrac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$ is called the Jacobian of given transformation from (x, y) to (u, v). Also R' is the region in the uv-plane corresponding to the region R in *xy-plane*.

4.3.3.2 ***To change Cartesian co-ordinates (x, y) to polar co-ordinates (r, θ) :***

We know that $x = r\cos\theta, y = r\sin\theta$ and $x^2 + y^2 = r^2$,so

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left(\cos^2\theta + \sin^2\theta\right) = r.$$

$$\Rightarrow \iint_R f(x, y)dxdy = \iint_{R'} f(r\cos\theta, r\sin\theta)r\,drd\theta.$$

4.3.3.3 Solved examples

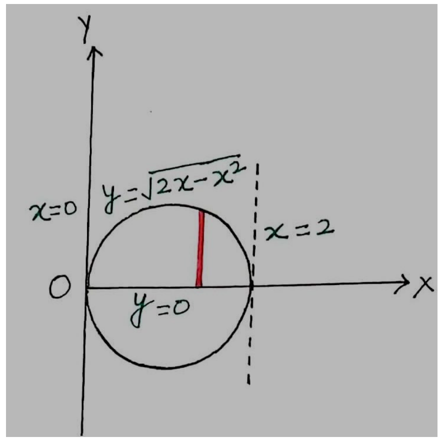
Example 1: Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} \dfrac{xdydx}{\sqrt{x^2 + y^2}}$ by changing into polar co-ordinates.

Solution: In the given integral,

x varies from 0 to 2

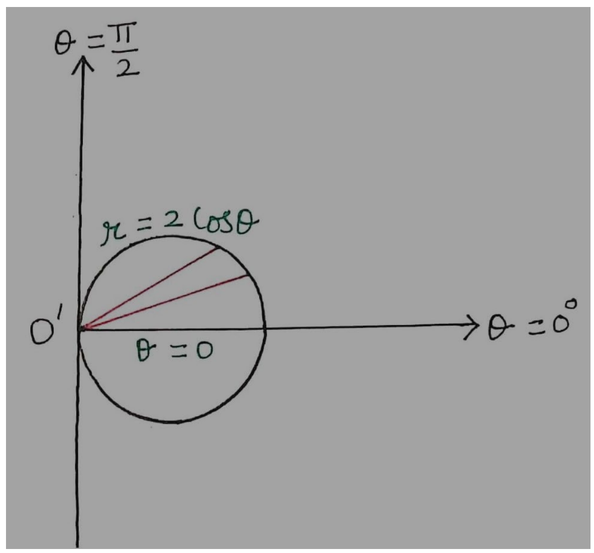
y varies from 0 to $\sqrt{2x - x^2}$

Now, $y = \sqrt{2x - x^2} \Rightarrow y^2 = 2x - x^2$ *or* $x^2 + y^2 = 2x$.



In polar co-ordinates, we have $x = r\cos\theta;\ y = r\sin\theta$,

Therefore, in polar coordinates $x^2 + y^2 = 2x$ becomes $r^2 = 2r\cos\theta$ *or* $r = 2\cos\theta$.



For this region of integration r varies from 0 to $2\cos\theta$ and θ varies from 0 to $\dfrac{\pi}{2}$.

So for polar coordinates, put $x = r\cos\theta;\ y = r\sin\theta$ and $dxdy = rdrd\theta$ in the given integral,

$$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{2\cos\theta} \frac{r\cos\theta}{r} rdrd\theta = \int_{\theta=0}^{\pi/2} \int_{r=0}^{2\cos\theta} r\cos\theta drd\theta$$

$$= \int_{\theta=0}^{\pi/2} \cos\theta \left(\frac{r^2}{2}\right)_0^{2\cos\theta} d\theta = \int_{\theta=0}^{\pi/2} 2\cos^3\theta d\theta = 2.\frac{2}{3} = \frac{4}{3}$$

Example 2: Evaluate $\iiint z(x^2 + y^2)dxdydz$ over the volume of the cylinder $x^2 + y^2 = 1$ intercepted by the planes $z = 2$ and $z = 3$.

Solution: Here,

$$I = \iiint z\left(x^2 + y^2\right)dxdydz$$

Now using cylindrical polar coordinates,

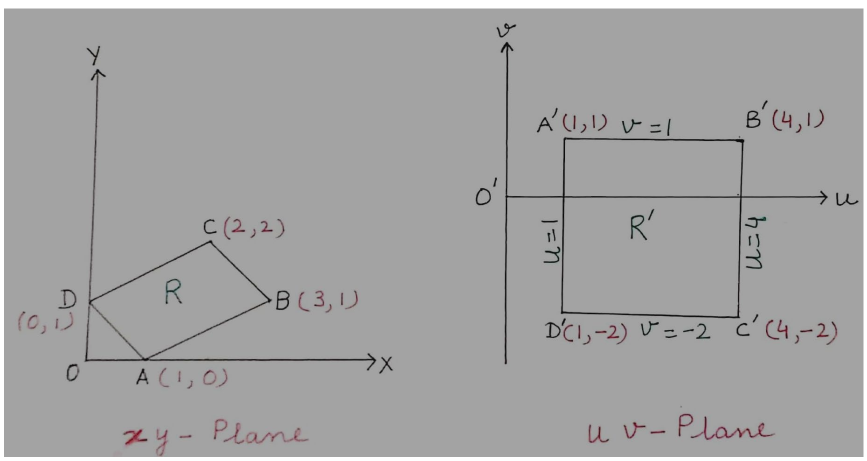
$$I = \int_{z=2}^{3}\int_{\phi=0}^{2\pi}\int_{r=0}^{1} z.r^2.rdrd\phi dz$$

$$I = \int_{z=2}^{3}\int_{\phi=0}^{2\pi} z\left(\frac{r^4}{4}\right)_0^1 d\phi dz$$

$$I = \frac{1}{4}\int_{z=2}^{3} z(\phi)_0^{2\pi} dz = \frac{1}{4}.2\pi\left(\frac{z^2}{2}\right)_2^3 = \frac{\pi}{4}(9-4) = \frac{5\pi}{4}$$

Example 3: Evaluate $\iint\limits_R (x+y)^2 dx\,dy$, where R is the parallelogram in the xy-plane with vertices (1, 0), (3, 1), (2, 2), (0, 1) using the transformation $u = x + y$ and $v = x - 2y$.

Solution: The region R in xy-plane i.e., parallelogram ABCD with vertices $A\,(1,0), B\,(3,1), C\,(2,2), D\,(0,1)$ becomes region R' in uv-plane i.e., rectangle $A'B'C'D'$ with vertices $A'\,(1,1), B'\,(4,1), C'\,(4,-2), D'\,(1,-2)$.



Solving the given equations for x and y, we get $x = \frac{1}{3}(2u+v), y = \frac{1}{3}(u-v)$.

Here

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{3}.$$

$$\therefore \iint_R (x+y)^2 dx\,dy = \iint_{R'} u^2 |J| du\,dv = \int_{-2}^{1}\int_{1}^{4} u^2 \frac{1}{3} du\,dv = \int_{-2}^{1} 7dv = 21.$$

Example 4: Using the transformation $x - y = u, x + y = v$, show that $\iint_R \sin\left(\frac{x-y}{x+y}\right) dx\,dy = 0$, where R is the region bounded by the co-ordinate axes and $x + y = 1$ in first quadrant.

Solution: Here, region R is a triangle OAB in *xy-plane* having sides $x = 0, y = 0$ and $x + y = 1$.

Also

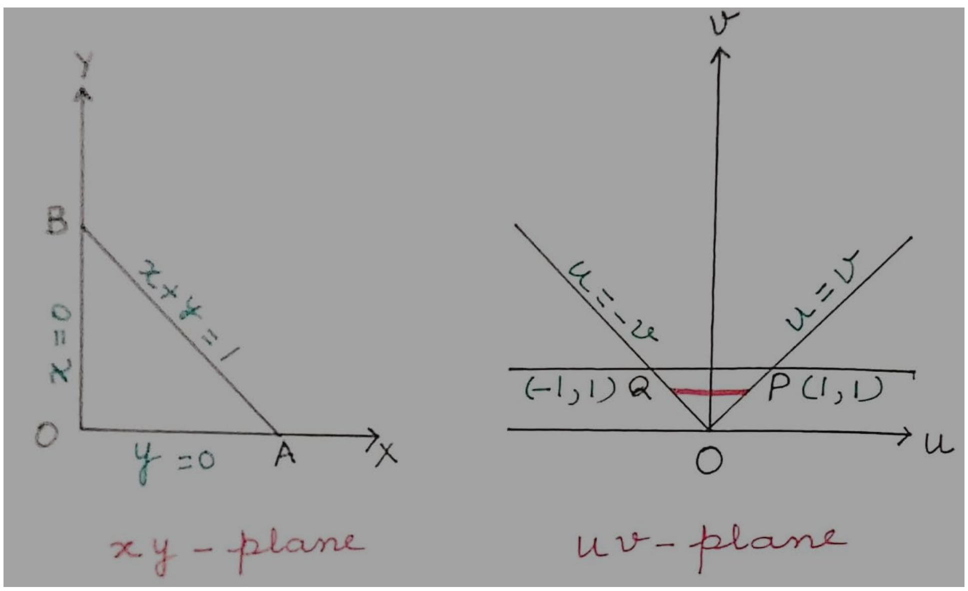
$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2.$$

Using given transformation, we get

If $x = 0, y = 0$ then $u = -v, u = v$.

If $x + y = 1$ then $v = 1$.

Thus corresponding region $R^{'}$ in *uv*-plane is a triangle $O^{'}A^{'}B^{'}$ bounded by $u = -v, u = v, v = 1$.



Therefore, in *uv*-plane the given integral becomes

$$I=\int_0^1\int_{-v}^{v}\sin\left(\frac{u}{v}\right)\frac{1}{2}\,du\,dv$$

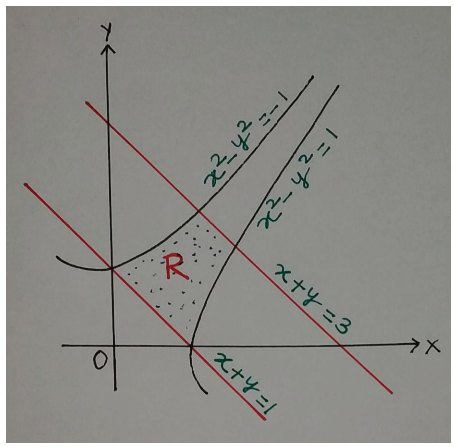
$$I=\frac{1}{2}\int_0^1\left[\frac{-\cos\left(\frac{u}{v}\right)}{\left(\frac{1}{v}\right)}\right]_{-v}^{v}dv$$

$$I=\frac{1}{2}\int_0^1 v[-\cos 1+\cos(-1)]dv$$

$I=0.$

Example 5: Using the transformation $u=x-y$ and $v=x+y$, evaluate the integral $\iint_R (x-y)e^{x^2-y^2}\,dx\,dy$ where R is the region bounded by the lines $x+y=1$ and $x+y=3$ and the curves $x^2-y^2=-1$ and $x^2-y^2=1$.

Solution: Region R in $xy-plane$ is shown below:



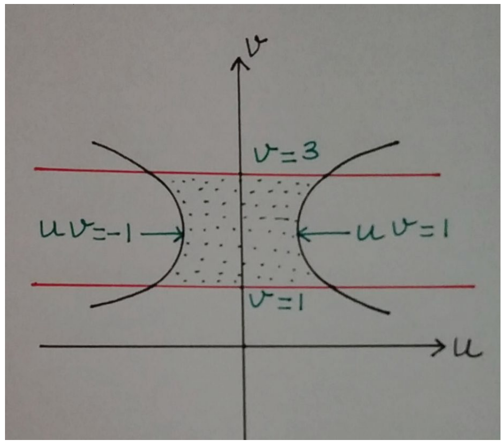
From given transformation,

$u=x-y;v=x+y$

$\Rightarrow x=\frac{u+v}{2};y=\frac{v-u}{2}$

Using this, $x+y=1;x+y=3$ become $v=1;v=3$. Also the curves $x^2-y^2=-1;x^2-y^2=1$ become $uv=1;uv=-1$.

Thus, under the given transformation region R in $xy-plane$ becomes region R' in $uv-plane$.



Also $J(x, y) = \dfrac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{1}{2} & -\dfrac{1}{2} \\ \dfrac{1}{2} & \dfrac{1}{2} \end{vmatrix} = \dfrac{1}{2}$

Therefore, under the given transformation

$$\iint_R (x-y)e^{x^2-y^2}\,dx\,dy = \frac{1}{2}\int_{v=1}^{3}\int_{u=-1/v}^{1/v} ue^{uv}\,du\,dv$$

$$\iint_R (x-y)e^{x^2-y^2}\,dx\,dy = \frac{2}{3e}.$$

4.3.3.6 Practice problems

1. Evaluate the following by changing into polar coordinates:

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} y^2\sqrt{x^2+y^2}\,dx\,dy$$ Ans: $\dfrac{\pi a^5}{20}$

2. Change into polar coordinates and evaluate $\int_0^{\infty}\int_0^{\infty} e^{-\left(x^2+y^2\right)}\,dy\,dx.$ Ans: $\dfrac{\pi}{4}$

3. Let D be the region in the first quadrant bounded by $x = 0, y = 0$ and $x + y = 1$. Change the variables x, y to u, v where $x + y = u$, $y = uv$ and evaluate

$$\iint_D xy(1-x-y)^{1/2}\,dx\,dy.$$ Ans: 16/945

4. Prove that the area in the positive quadrant bounded by the curves $y^2 = 4ax, y^2 = 4bx, xy = c^2$ and $xy = d^2$ is $\dfrac{1}{3}\left(d^2 - c^2\right)\log\left(\dfrac{b}{a}\right); d > c, b > a.$

5. Determine the value of the integral $\iiint_D e^{\sqrt{x^2+y^2+z^2}}\,dV$ where D is the region bounded by the planes

$y = 0, z = 0, y = x$ and the sphere $x^2 + y^2 + z^2 = 9.$ Ans: $\dfrac{\pi\left(5e^3 - 2\right)}{4}$

Hint: $0 \le r \le 3; 0 \le \theta \le \pi/4; 0 \le \phi \le \pi/2$

4.4 APPLICATIONS

4.4.1 AREA

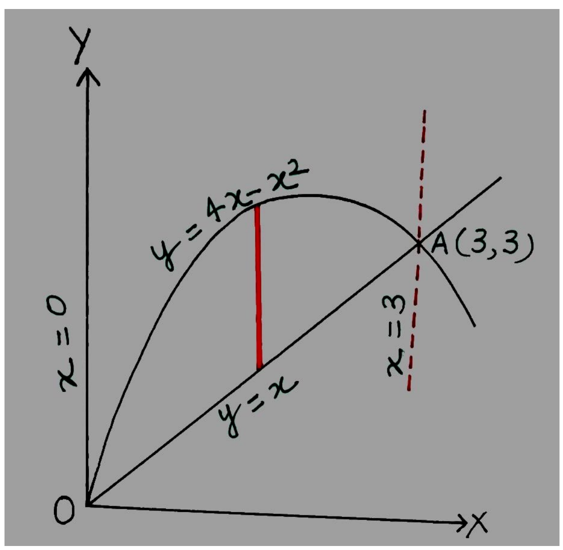
4.4.1.1 Cartesian Co-ordinates: The area A of the region bounded by two curves $y = f_1(x), y = f_2(x)$ and the lines $x = a, x = b$ is given by $A = \int_a^b \int_{f_1(x)}^{f_2(x)} dy\,dx$.

4.4.1.2 Polar Co-ordinates: The area A of the region bounded by two curves $r = f_1(\theta), r = f_2(\theta)$ and the lines $\theta = \alpha, \theta = \beta$ is given by $A = \int_\alpha^\beta \int_{f_1(\theta)}^{f_2(\theta)} r dr\,d\theta$.

4.4.1.3 Solved Examples

Example 1: Find the area lying between the parabola $y = 4x - x^2$ and the line $y = x$.

Solution: Solving the equations of given curves, we get $x = 0,3.$



Selecting the vertical strip, the required area lies between $x = 0, x = 3$ and $y = x, y = 4x - x^2$.

Therefore, required area

$$A = \int_0^3 \int_x^{4x-x^2} dy\,dx$$

$$A = \int_0^3 [y]_x^{4x-x^2} dx$$

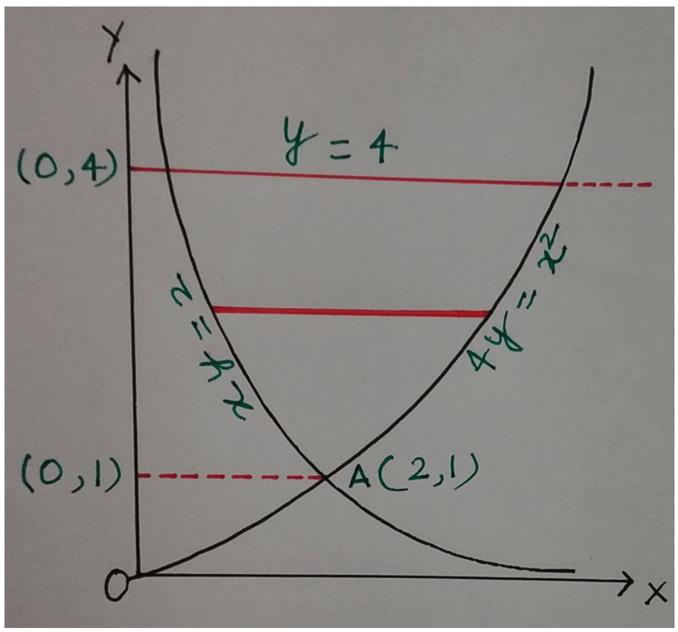
$$A = \int_0^3 \left(3x - x^2\right) dx$$

$$A = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3 = \frac{27}{2} - 9 = \frac{9}{2}.$$

Example 2: Determine the area of region bounded by the curves $xy = 2, 4y = x^2, y = 4$.

Solution: Selecting horizontal strip, the required area lies between

$xy = 2, 4y = x^2, y = 4$ alongwith point of intersection of curves $xy = 2, 4y = x^2$, *i.e.*, $x = 2, y = 1$.



Therefore, required area

$$A = \int_{y=1}^{4} \int_{x=2/y}^{2\sqrt{y}} dx\,dy$$

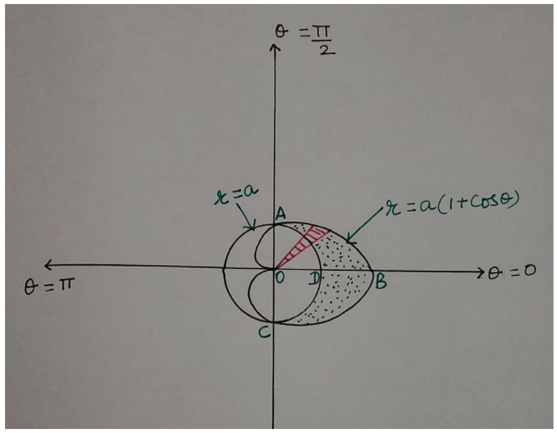
$$A = \int_1^4 \left(2\sqrt{y} - 2/y\right) dy$$

$$A = 2\left(\frac{2}{3}y^{3/2} - \log y\right)_1^4$$

$$A = 2\left[\left(\frac{16}{3} - 2\log 2\right) - \frac{2}{3}\right] = \frac{28}{3} - 4\log 2$$

Example 3: Find, by double integration, the area lying inside the cardioid $r = a(1 + \cos\theta)$ and outside the circle $r = a$.

Solution: Since the bounded region is symmetric about initial line, we calculate the area lying above the initial line only.



Required area

$$A = 2\int_{\theta=0}^{\pi/2}\int_{r(circle)}^{r(cardioid)} r\,d\theta\,dr$$

$$A = 2\int_{\theta=0}^{\pi/2}\int_{a}^{a(1+\cos\theta)} r\,d\theta\,dr$$

$$A = 2\int_{\theta=0}^{\pi/2}\left[\frac{r^2}{2}\right]_a^{a(1+\cos\theta)} d\theta$$

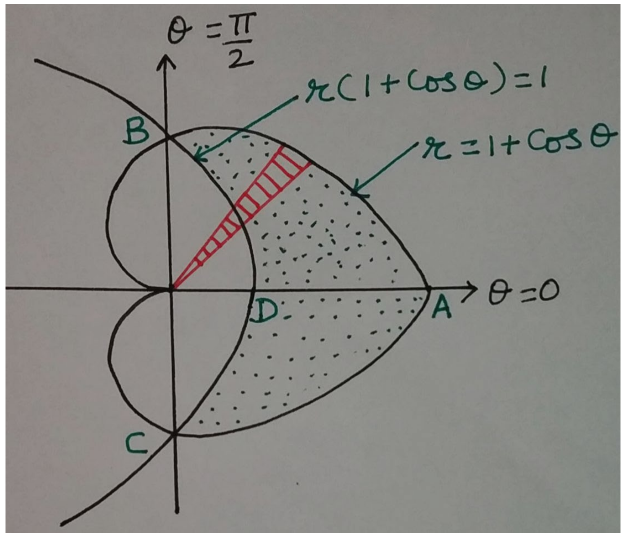
$$A = a^2\int_{\theta=0}^{\pi/2}\left[(1+\cos\theta)^2 - 1\right]d\theta$$

$$A = a^2\int_{\theta=0}^{\pi/2}\left[\cos^2\theta + 2\cos\theta\right]d\theta$$

$$A = a^2\left(\frac{1}{2}.\frac{\pi}{2}+2\right) = \frac{a^2}{4}(\pi+8)\left(\text{using Walli's formula}\right)$$

Example 4: Find by the double integration, the area lying inside a cardioid $r = (1+\cos\theta)$ and outside the parabola $r(1+\cos\theta) = 1$.

Solution: Since bounded region is symmetric about initial line,



Therefore, required area=2area above the initial line

$$A = 2\int_{\theta=0}^{\pi/2}\int_{r=parabola}^{r=cardioid} r\,dr\,d\theta$$

$$A = 2\int_{0}^{\pi/2}\left(\frac{r^2}{2}\right)_{\frac{1}{1+\cos\theta}}^{1+\cos\theta} d\theta$$

$$A = \int_{0}^{\pi/2}\left[(1+\cos\theta)^2 - \frac{1}{(1+\cos\theta)^2}\right]d\theta$$

$$A = \int_{0}^{\pi/2}(1+\cos\theta)^2\,d\theta - \int_{0}^{\pi/2}\frac{1}{(1+\cos\theta)^2}\,d\theta$$

$$A = \int_{0}^{\pi/2}\left(1+\cos^2\theta + 2\cos\theta\right)d\theta - \frac{1}{4}\int_{0}^{\pi/2}\sec^4\frac{\theta}{2}\,d\theta$$

Let $I_1 = \int_0^{\pi/2} \left(1 + \cos^2\theta + 2\cos\theta\right) d\theta$

and $I_2 = \frac{1}{4}\int_0^{\pi/2} \sec^4 \frac{\theta}{2} d\theta$, then

$$I_1 = \int_0^{\pi/2} \left(1 + \cos^2\theta + 2\cos\theta\right) d\theta$$

$$I_1 = \frac{\pi}{2} + \frac{1}{2}.\frac{\pi}{2} + 2 = \frac{3\pi}{4} + 2$$

$$I_2 = \frac{1}{4}\int_0^{\pi/2} \sec^4 \frac{\theta}{2} d\theta$$

$$I_2 = \frac{1}{4}\int_0^{\pi/4} \sec^4 \phi .2d\phi \qquad let \frac{\theta}{2} = \phi$$

$$\Rightarrow d\theta = 2d\phi$$

$$I_2 = \frac{1}{2}\int_0^{\pi/4} \left(1 + \tan^2\phi\right)\sec^2\phi \, d\phi \qquad let\, t = \tan\phi$$

$$\Rightarrow dt = \sec^2\phi d\phi$$

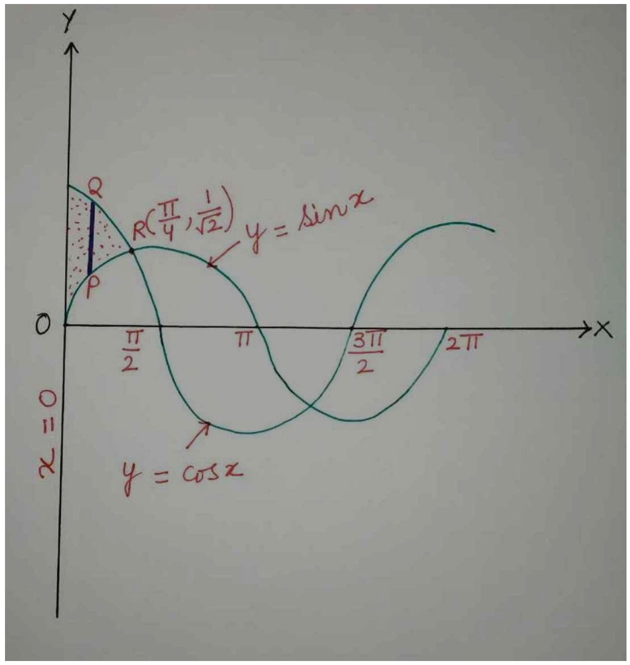
$$I_2 = \frac{1}{2}\int_0^{1} \left(1 + t^2\right) dt$$

$$I_2 = \frac{1}{2}\left(t + \frac{t^3}{3}\right)_0^1 = \frac{2}{3}$$

Hence required area= $\frac{3\pi}{4} + 2 - \frac{2}{3} = \frac{3\pi}{4} + \frac{4}{3}$.

Example 5: Find the area bounded by the lines $y = \sin x, y = \cos x, x = 0$.

Solution: Clearly, dotted portion is the required area.



Therefore,

$$A = \int_0^{\pi/4} \int_{\sin x}^{\cos x} dy\, dx$$

$$A = \int_0^{\pi/4} (\cos x - \sin x) dx$$

$$A = (\sin x + \cos x)_0^{\pi/4}$$

$$A = \left(\frac{1}{\sqrt{2}} - 0\right) + \left(\frac{1}{\sqrt{2}} + 1\right) = \left(\sqrt{2} - 1\right)$$

4.4.1.4 Practice problems

1. Find the area between the curves $y = x^2$ and $y = x^3$. Ans:1/12

2. Using double integral find the area bounded by the curves $y = 1 - x^2$ and $y = x^2 - 3$. Ans: $\frac{16\sqrt{2}}{3}$

Hint: $-\sqrt{2} \le x \le \sqrt{2}; x^2 - 3 \le y \le 1 - x^2$

3. Find, by double integration, the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Ans: πab

4. Evaluate the area enclosed between the parabola $y = x^2$ and the straight line $y = x$. Ans: $\frac{1}{6}$

5. Find, by double integration, the area of the region enclosed by the curves $x^2 + y^2 = a^2; x + y = a$ in the first quadrant. Ans: $\frac{a^2(\pi - 2)}{4}$

6. Show, by double integration, that the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$.

7. Find the area bounded by the circles $r = 2\sin\theta$ and $r = 4\sin\theta$. Ans: 3π

8. Find the area of one loop of the lemniscate $r^2 = a^2\cos 2\theta$. Ans: $\frac{a^2}{2}$

9. Find the area of the loop of the curve $x^3 + y^3 = 3xy$. Ans: $\frac{3a^2}{2}$

4.5 RELATED LINKS

1. https://www.youtube.com/watch?time_continue=1&v=4rc3w1sGoNU&feature=emb_logo
2. https://www.youtube.com/watch?time_continue=1&v=wtY5fx6VMGQ&feature=emb_logo
3. https://www.youtube.com/watch?v=6ntZ1KQL04A&feature=emb_logo
4. https://www.khanacademy.org/math/multivariable-calculus/integrating-multivariable-functions/double-integrals-topic/v/double-integral-1
5. https://www.khanacademy.org/math/multivariable-calculus/integrating-multivariable-functions/double-integrals-topic/v/double-integrals-2
6. https://www.khanacademy.org/math/multivariable-calculus/integrating-multivariable-functions/double-integrals-topic/v/double-integrals-3
7. https://www.khanacademy.org/math/multivariable-calculus/integrating-multivariable-functions/double-integrals-topic/v/double-integrals-4
8. https://www.khanacademy.org/math/multivariable-calculus/integrating-multivariable-functions/double-integrals-topic/v/double-integrals-5
9. https://www.khanacademy.org/math/multivariable-calculus/integrating-multivariable-functions/double-integrals-topic/v/double-integrals-6
10. https://www.khanacademy.org/math/multivariable-calculus/integrating-multivariable-functions/double-integrals-a/a/double-integrals
11. https://www.khanacademy.org/math/multivariable-calculus/integrating-multivariable-functions/double-integrals-a/a/double-integrals-over-non-rectangular-regions
12. https://www.khanacademy.org/math/multivariable-calculus/integrating-multivariable-functions/double-integrals-a/a/double-integrals-beyond-volume
13. https://www.khanacademy.org/math/multivariable-calculus/integrating-multivariable-functions/double-integrals-a/v/polar-coordinates-1
14. https://www.khanacademy.org/math/multivariable-calculus/integrating-multivariable-functions/double-integrals-a/a/double-integrals-in-polar-coordinates

KIET GROUP OF INSTITUTIONS, DELHI NCR, GHAZIABAD

B. Tech. (I Sem.), 2024-25

CALCULUS FOR ENGINEERS (K24AS11)

UNIT-5 (Vector Differentiation)

SYLLABUS: Scalar point function, Vector point function, Gradient of a scalar field, Directional derivatives, Application of divergence, Curl to solenoidal and irrotational vectors respectively.

Course Outcome:

Apply the concept of vector differentials to study the properties of point functions.

Application in Engineering:

Vector differentiation is a powerful mathematical tool that provides insights into dynamic systems, enabling predictions and optimizations across various disciplines. Vector differentiation has a wide range of applications in various fields of science, engineering, and mathematics. Here, some applications are given in real-life scenarios:

1. **Physics**:

   Motion Analysis: In physics, vector differentiation is used to analyse the motion of objects. The position vector of an object can be differentiated with respect to time to obtain the velocity vector and differentiating the velocity vector gives the acceleration vector. This is fundamental in kinematics.
2. **Electromagnetism**: The behaviour of electric and magnetic fields can be described using vector calculus. For example, Maxwell's equations, which govern electromagnetism, involve vector differentiation.
3. **Engineering**:

   Fluid Dynamics: In fluid mechanics, the velocity field of a fluid can be described using vector functions. Differentiating these functions helps in understanding the flow characteristics, such as calculating the rate of change of velocity (acceleration) or analysing flow patterns.
4. **Structural Analysis**: Engineers use vector differentiation to analyse forces and moments acting on structures. This is crucial for determining stress, strain, and stability.

5. **Computer Graphics**:

   Animation and Motion: In computer graphics, vector differentiation is used to calculate the motion of objects. For instance, differentiating position vectors over time helps in determining velocity and acceleration for realistic animations.

   Surface Rendering: Techniques like normal mapping rely on vector differentiation to calculate surface normal, which are essential for rendering light reflections and shading.

6. **Robotics**:

   Path Planning: In robotics, vector differentiation is used to compute trajectories and optimize the movement of robots. By analysing the velocity and acceleration vectors, robots can navigate efficiently in their environment.

7. **Economics and Social Sciences**:

   Modelling Change: In economics, vector differentiation can be used to model changes in multi-dimensional systems, such as how various economic indicators change over time in response to different factors.

8. **Machine Learning**:

   Gradient Descent: In optimization problems, particularly in training machine learning models, vector differentiation is used to calculate gradients. This helps in minimizing loss functions by updating model parameters in the direction of steepest descent.

9. **Geophysics**:

   Seismic Analysis: Vector differentiation is used in geophysics to analyse seismic waves. Understanding how these waves propagate through different materials involves differentiating vector fields representing displacement.

CONTENTS

Vector Differentiation

Introduction: Vector calculus or vector analysis is concerned with differentiation and integration of vector fields. It is used extensively in physics and engineering, especially in the description of electromagnetic fields, gravitational fields and fluid flow.

1. **Point Function:** A variable quantity whose value at any point in a region of space depends upon the position of the point, is called a point function.

1.1 Scalar Point Function: If to each point *P (x, y, z)* of a region *R* in space there corresponds a unique scalar *f(P)*, then *f* is called a scalar point function.

Examples.

(i) Temperature distribution in a heated body,

(ii) Density of a body & **(iii)** Potential due to gravity.

1.2 Vector Point Function: If to each point *P (x, y, z)* of a region *R* in space there corresponds a unique vector *f(P)*, then *f* is called a vector point function.

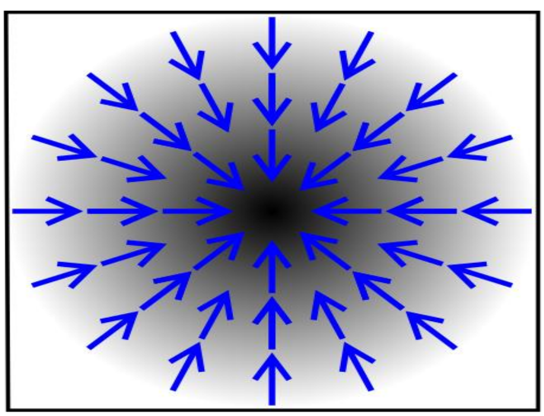
Examples.

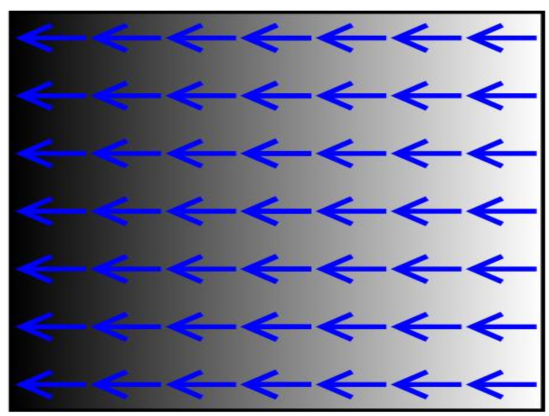
(i) Forest wind **(ii)** The velocity of a moving fluid **(iii)** Gravitational force.

2. Gradient of a scalar point function

The gradient is closely related to the derivative, but it is not itself a derivative.

The gradient can be interpreted as the "direction and rate of fastest increase"





Vector Differential Operator Del (∇): It is defined as:

$$\nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Gradient of a scalar function:

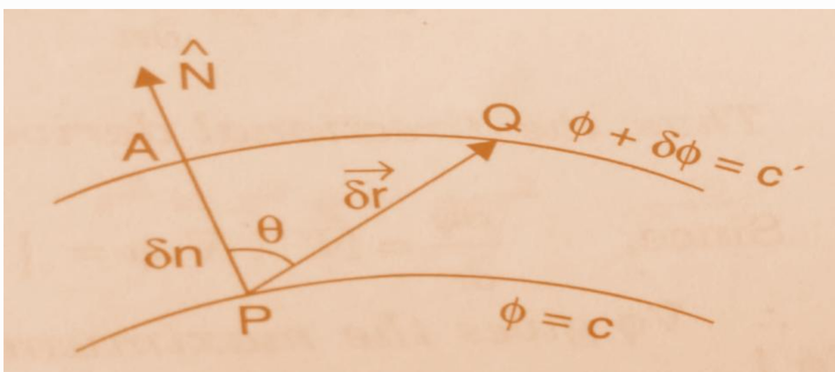
Let $\emptyset(x, y, z)$ be a scalar function, then the vector $\hat{\imath}\frac{\partial \emptyset}{\partial x} + \hat{\jmath}\frac{\partial \emptyset}{\partial y} + \hat{k}\frac{\partial \emptyset}{\partial z}$ is called the gradient of a scalar function $\emptyset$. Thus, $grad\ \emptyset = \nabla\emptyset$.

2.1 Geometrical Interpretation of Gradient: If a surface $\emptyset(x, y, z) = c$, passes through a point P. The value of function at each point of the surface is the same as at P. Then such a surface is called a *level surface* through P.

Example. If $\emptyset(x, y, z)$ represents potential at point P. Then equipotential surface $\emptyset(x, y, z) = c$, is a level surface.

Note: Two level surfaces can't intersect.

Let the level surface pass through P at which the value of function is $\emptyset$.



Consider another level surface passing through Q, Where the value of function $\emptyset + d\emptyset$.

Let $\vec{r}$ and $\vec{r} + \delta\vec{r}$ be the position vector of P and Q then $\overrightarrow{PQ} = \delta\vec{r}$

$$\nabla\phi . d\vec{r} = \left(\hat{\imath}\frac{\partial \phi}{\partial x} + \hat{\jmath}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right).\left(\hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz\right)$$

$$= \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = d\phi \text{-------- (1)}$$

If Q lies on the level surface of P, then $d\phi = 0$

From equation (1), we get

$\nabla\phi . d\vec{r} = 0$, then $\nabla\phi \perp to\ d\vec{r}\ (tangent)$

Hence $\nabla\phi$ is the **Normal** to the surface $\phi(x, y, z) = c$

2.2 Properties of gradient

(1) If ϕ is a constant scalar point function, then $\nabla\phi = \vec{0}$

(2) If ϕ_1 and ϕ_2 are two scalar point functions, then

(a) $\nabla(\phi_1 \pm \phi_2) = \nabla\phi_1 \pm \nabla\phi_2$

(b) $\nabla(c_1\phi_1 \pm c_2\phi_2) = c_1\nabla\phi_1 \pm c_2\nabla\phi_2$, where c_1& c_2 are constants.

(c) $\nabla(\phi_1\phi_2) = \phi_1\nabla\phi_2 + \phi_2\nabla\phi_1$

(d) $\nabla\left(\frac{\phi_1}{\phi_2}\right) = \frac{\phi_2\nabla\phi_1 - \phi_1\nabla\phi_2}{\phi_2{}^2}$, $\phi_2 \neq 0$

Example 1. Find *grad ø,* when ø is given by $\emptyset = 3x^2y - y^3z^2$ at the point $(1, -2, -1)$.

Solution. Here $\phi = 3x^2y - y^3z^2$

$$\therefore\ grad\phi = \nabla\phi = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(3x^2y - y^3z^2)$$

$$= \hat{\imath}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \hat{\jmath}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \hat{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$= \hat{\imath}(6xy) + \hat{\jmath}(3x^2 - 3y^2z^2) + \hat{k}(-2y^3z)$$

$$grad\phi]_{(1,-2,-1)} = -12\hat{\imath} - 9\hat{\jmath} - 16\hat{k}$$

Example 2. What is the greatest rate of increase of $u = xyz^2$ at the point $(1,0,3)$?

Solution. Here, we have

$$grad\, u = \nabla u = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(xyz^2) = yz^2\hat{\imath} + xz^2\hat{\jmath} + 2xyz\hat{k}$$

Now, $grad\, u]_{(1,0,3)} = 9\hat{\jmath}$

The greatest rate of increase of u at the point $(1,0,3) = |\nabla u| = |9\hat{\jmath}| = 9$

Example 3. Find a unit vector normal to the surface $yx^2 + 2xz = 4$ at the point $(2, -2, 3)$.

Solution. Let $\emptyset = yx^2 + 2xz - 4$

$$grad\,\emptyset = \nabla\emptyset = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(yx^2 + 2xz - 4)$$

$$= (2xy + 2z)\hat{\imath} + x^2\hat{\jmath} + 2x\hat{k}$$

Now, $grad\,\emptyset]_{(2,-2,3)} = -2\hat{\imath} + 4\hat{\jmath} + 4\hat{k}$ and $|grad\,\emptyset| = \sqrt{4 + 16 + 16} = 6$

The unit vector normal to the surface $\emptyset = \frac{grad\,\emptyset}{|grad\,\emptyset|}$

$$= \frac{-2\hat{\imath} + 4\hat{\jmath} + 4\hat{k}}{6} = \frac{-\hat{\imath} + 2\hat{\jmath} + 2\hat{k}}{3}.$$

Example 4. If $\nabla\phi = (y^2 - 2xyz^3)\hat{\imath} + (3 + 2xy - x^2z^3)\hat{\jmath} + (6z^3 - 3x^2yz^2)\hat{k}$, find ϕ.

Solution. Let $\vec{F} = \nabla\phi \Rightarrow \vec{F}.d\vec{r} = \nabla\phi.d\vec{r}$ [taking dot product with $d\vec{r}$]

$$\vec{F}.d\vec{r} = \left(\hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right).\left(dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}\right) = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$\Rightarrow \vec{F}.d\vec{r} = d\phi \qquad \left[as\ \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\phi\right]$$

Or $d\phi = \vec{F}.d\vec{r} = \nabla\phi.d\vec{r}$

$$d\phi = \left[(y^2 - 2xyz^3)\hat{\imath} + (3 + 2xy - x^2z^3)\hat{\jmath} + (6z^3 - 3x^2yz^2)\hat{k}\right].\left(dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}\right)$$

$$= (y^2 - 2xyz^3)dx + (3 + 2xy - x^2z^3)dy + (6z^3 - 3x^2yz^2)dz \quad \text{-----------------} (1)$$

$$= 3dy + 6z^3dz + y^2dx + 2xydy - 2xyz^3dx - x^2z^3dy - 3x^2yz^2dz$$

$$d\phi = 3dy + 6z^3dz + d(xy^2) - d(x^2yz^3)$$

Integrate, we get $\phi = 3y + \frac{3}{2}z^4 + xy^2 - x^2yz^3 + C$

Example 5. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$.

Solution. Let $\phi_1 \equiv x^2 + y^2 + z^2 - 9 = 0$ and $\phi_2 \equiv x^2 + y^2 - 3 - z = 0$

then $\nabla\phi_1 = 2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k}$ and $\nabla\phi_2 = 2x\hat{\imath} + 2y\hat{\jmath} - \hat{k}$

Let $\quad \overrightarrow{n_1} = \nabla\phi_1]_{(2,-1,2)} = 4\hat{\imath} - 2\hat{\jmath} + 4\hat{k}$

and $\quad \overrightarrow{n_2} = \nabla\phi_2]_{(2,-1,2)} = 4\hat{\imath} - 2\hat{\jmath} - \hat{k}$

Let θ is the angle between the vectors $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$ which are normal to the given surfaces ϕ_1 and ϕ_2 respectively. Then

$$\cos\theta = \frac{\overrightarrow{n_1}.\overrightarrow{n_2}}{|\overrightarrow{n_1}||\overrightarrow{n_2}|} = \frac{4.4 + (-2).(-2) + 4.(-1)}{\sqrt{16+4+16}\sqrt{16+4+1}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\therefore \boldsymbol{\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)}$$

Example 6. If $u = x + y + z,\ v = x^2 + y^2 + z^2,\ w = yz + zx + xy$, prove that $grad\ u,\ grad\ v\ and\ grad\ w\ are\ coplanar\ vectors.$

Solution. Here, we have

$$grad\ u = \nabla u = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x + y + z) = \hat{\imath} + \hat{\jmath} + \hat{k}$$

$$grad\ v = \nabla v = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2) = 2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k}$$

$$grad\ w = \nabla w = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(yz + zx + xy)$$

$$= (z + y)\hat{\imath} + (z + x)\hat{\jmath} + (y + x)\hat{k}$$

Now, $grad\, u\,.(grad\, v\, \times grad\, w) = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ z+y & z+x & y+x \end{vmatrix} = 0$

Hence, $grad\, u,\ grad\, v\ and\ grad\, w\ are\ coplanar\ vectors$ **.**

Exercise

1. Show that $\nabla(\vec{a}.\vec{r}) = \vec{a}$ where $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ and $\vec{a}$ is a constant vector.

2. Find a unit vector normal to the surface $x^3 + y^3 + 3xyz = 3$ at the point $(1,2,-1)$.

3. Find a unit vector normal to the surface $xy^3z^2 = 4$ at the point $(-1,-1,2)$.

4. Find a unit vector normal to the surface $z^2 = 4(x^2 + y^2)$ at the point $(1,0,2)$.

5. Calculate the angle between the normal to the surface $xy = z^2$ at the points $(4,1,2)\ and\ (3,3,-3)$.

Answers

2. $-\frac{1}{\sqrt{14}}(\hat{\imath} - 3\hat{\jmath} - 2\hat{k})$ **3.** $-\frac{1}{\sqrt{11}}(\hat{\imath} + 3\hat{\jmath} - \hat{k})$ **4.** $\frac{1}{\sqrt{5}}(2\hat{\imath} - \hat{k})$ **5.** $\cos^{-1}\left(-\frac{1}{\sqrt{22}}\right)$

3. Directional derivative

Directional derivative of a scalar field f at a point $P(x,y,z)$ in the direction of unit vector $\hat{a}$ is given by $\frac{df}{ds} = (grad\, f).\hat{a}$

Example 1: Find the directional derivative of $\emptyset = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$ at the point $P(3,1,2)$ in the direction of the vector $yz\hat{\imath} + zx\hat{\jmath} + xy\hat{k}$.

Solution. Here, $\emptyset = (x^2 + y^2 + z^2)^{-\frac{1}{2}} \Rightarrow grad\, \emptyset = -\frac{x\hat{\imath}+y\hat{\jmath}+z\hat{k}}{(x^2+y^2+z^2)^{\frac{3}{2}}}$

Let $\hat{a}$ be the unit in the given direction, then $\hat{a} = \frac{yz\hat{\imath}+zx\hat{\jmath}+xy\hat{k}}{\sqrt{x^2+y^2+z^2}}$

$grad\, \emptyset]_{(3,1,2)} = -\frac{3\hat{\imath}+\hat{\jmath}+2\hat{k}}{14\sqrt{14}}$ and $\hat{a}]_{(3,1,2)} = \frac{2\hat{\imath}+6\hat{\jmath}+3\hat{k}}{7}$

Directional derivative $= \frac{d\emptyset}{ds} = (grad\, \emptyset).\hat{a} = -\frac{6+6+6}{98\sqrt{14}} = -\frac{\mathbf{9}}{\mathbf{49\sqrt{14}}}$

Example 2: Find the directional derivative of $\emptyset = 5x^2y - 5y^2z + \frac{5}{2}z^2x$ at the point $P(1,1,1)$ in the direction of the line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$.

Solution. Here, $\emptyset = 5x^2y - 5y^2z + \frac{5}{2}z^2x$

$\Rightarrow grad\, \emptyset = \left(10xy + \frac{5}{2}z^2\right)\hat{\imath} + (5x^2 - 10yz)\hat{\jmath} + (-5y^2 + 5xz)\hat{k}$

$\Rightarrow grad\, \emptyset]_{(1,1,1)} = \frac{25}{2}\hat{\imath} - 5\hat{\jmath}$

Let $\vec{a}$ be a vector in the direction of the line $\frac{x-1}{2}=\frac{y-3}{-2}=\frac{z}{1}$, then

$\vec{a}=2\hat{\imath}-2\hat{\jmath}+\hat{k} \Rightarrow \hat{a}=\frac{2\hat{\imath}-2\hat{\jmath}+\hat{k}}{3}$ and $\hat{a}]_{(1,1,1)}=\frac{2\hat{\imath}-2\hat{\jmath}+\hat{k}}{3}$

Directional derivative $=\frac{d\emptyset}{ds}=(grad\ \emptyset).\hat{a}=\frac{25}{3}+\frac{10}{3}=\frac{35}{6}$

Example 3: Find the directional derivative of $\emptyset=x^2-y^2+2z^2$ at the point $P(1,2,3)$ in the direction of the line PQ where Q is the point $(5,0,4)$.
In what direction will it be maximum? Find also the magnitude of this maximum.

Solution. Here, $\emptyset=x^2-y^2+2z^2 \Rightarrow \nabla\emptyset=2x\,\hat{\imath}-2y\hat{\jmath}+4z\hat{k}$

$\nabla\emptyset]_{(1,2,3)}=2\,\hat{\imath}-4\hat{\jmath}+12\hat{k}$ and $\overrightarrow{PQ}=4\hat{\imath}-2\hat{\jmath}+\hat{k}$

Let $\hat{a}$ be a unit vector in the direction of $\overrightarrow{PQ}$, then

$\hat{a}=\frac{4\hat{\imath}-2\hat{\jmath}+\hat{k}}{\sqrt{21}}$ and $\hat{a}]_{(1,2,3)}=\frac{4\hat{\imath}-2\hat{\jmath}+\hat{k}}{\sqrt{21}}$

Directional derivative $=\frac{d\emptyset}{ds}=(\nabla\emptyset).\hat{a}=\frac{8+8+12}{\sqrt{21}}=\frac{28}{\sqrt{21}}$

Directional derivative will be maximum in the direction of normal to the given surface i.e., in the direction of $\nabla\emptyset$.

The maximum value of this directional derivative $=|\nabla\emptyset|=2\sqrt{41}$

Example 4: Find the directional derivative of $\emptyset=xy^2+yz^3$ at the point $P(2,-1,1)$ in the direction of the normal to the surface $x\log z-y^2+4=0$ at $(2,-1,1)$.

Solution. Here, $\emptyset=xy^2+yz^3 \Rightarrow \nabla\emptyset=y^2\hat{\imath}+(2xy+z^3)\hat{\jmath}+3yz^2\hat{k}$

Let $\emptyset_1\equiv x\log z-y^2+4=0 \Rightarrow \nabla\emptyset_1=\log z\,\hat{\imath}-2y\hat{\jmath}+\frac{x}{z}\hat{k}$

$\nabla\emptyset]_{(2,-1,1)}=\hat{\imath}-3\hat{\jmath}-3\hat{k}$ and $\nabla\emptyset_1]_{(2,-1,1)}=2\hat{\jmath}+2\hat{k}$

Let $\hat{a}$ be a unit vector in the direction of the normal to the surface $\emptyset_1$ at the point $(2,-1,1)$, then

$\hat{a}=\frac{\nabla\emptyset_1]_{(2,-1,1)}}{|\nabla\emptyset_1]_{(2,-1,1)}|}=\frac{2\hat{\jmath}+2\hat{k}}{2\sqrt{2}}$

Required Directional derivative $=(\nabla\emptyset).\hat{a}$

$$=(\hat{\imath}-3\hat{\jmath}-3\hat{k}).\left(\frac{2\hat{\jmath}+2\hat{k}}{2\sqrt{2}}\right)$$

$$=\frac{-6-6}{2\sqrt{2}}=-\frac{6}{\sqrt{2}}=-\mathbf{3}\sqrt{\mathbf{2}}$$

Example 5: Find the directional derivative of $\vec{V}^2$ where $\vec{V}=xy^2\hat{\imath}+zy^2\hat{\jmath}+xz^2\hat{k}$ at the point $P(2,0,3)$ in the direction of the outward normal to the sphere $x^2+y^2+z^2=14$ at the point $(3,2,1)$.

Solution. Here, $\vec{V}=xy^2\hat{\imath}+zy^2\hat{\jmath}+xz^2\hat{k} \Rightarrow \vec{V}^2=x^2y^4+z^2y^4+x^2z^4$

$\nabla\vec{V}^2=2x(y^4+z^4)\hat{\imath}+4y^3(x^2+z^2)\hat{\jmath}+(2zy^4+4z^3x^2)\hat{k}$

Let $\emptyset_1\equiv x^2+y^2+z^2-14 \Rightarrow \nabla\emptyset_1=2x\hat{\imath}+2y\hat{\jmath}+2z\hat{k}$

$$\nabla\vec{V}^2\big]_{(2,0,3)} = 324\hat{\imath} + 432\hat{k} \quad \text{and} \quad \nabla\phi_1\big]_{(3,2,1)} = 6\hat{\imath} + 4\hat{\jmath} + 2\hat{k}$$

Let $\hat{a}$ be a unit vector in the direction of the normal to the surface ϕ_1 at the point $(3,2,1)$, then

$$\hat{a} = \frac{\nabla\phi_1]_{(3,2,1)}}{|\nabla\phi_1]_{(3,2,1)}|} = \frac{6\hat{\imath}+4\hat{\jmath}+2\hat{k}}{2\sqrt{14}} = \frac{3\hat{\imath}+2\hat{\jmath}+\hat{k}}{\sqrt{14}}$$

Required Directional derivative $= \left(\nabla\vec{V}^2\right).\hat{a}$

$$= \left(324\hat{\imath} + 432\hat{k}\right).\left(\frac{3\hat{\imath}+2\hat{\jmath}+\hat{k}}{\sqrt{14}}\right) = \boldsymbol{\frac{1404}{\sqrt{14}}}$$

Example 6: Find the directional derivative of $\nabla.(\nabla\phi)$ at a point $(1,-2,1)$ in the direction of the normal to the surface $xy^2z = 3x + z^2$, where $\phi = 2x^3y^2z^4$.

Solution. Here, $\phi = 2x^3y^2z^4 \Rightarrow \nabla\phi = 6x^2y^2z^4\hat{\imath} + 4x^3yz^4\hat{\jmath} + 8x^3y^2z^3\hat{k}$

$$\nabla.(\nabla\phi) = 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$$

Now, $\nabla\{\nabla.(\nabla\phi)\} = (12y^2z^4 + 12x^2z^4 + 72x^2y^2z^2)\hat{\imath} + (24xyz^4 + 48x^3yz^2)\hat{\jmath}$

$$+(48xy^2z^3 + 16x^3z^3 + 48x^3y^2z)\hat{k}$$

$\nabla\{\nabla.(\nabla\phi)\}]_{(1,-2,1)} = 348\hat{\imath} - 144\hat{\jmath} + 400\hat{k}$

Let $\phi_1 \equiv xy^2z - 3x - z^2 = 0 \Rightarrow \nabla\phi_1 = (y^2z - 3)\hat{\imath} + 2xyz\hat{\jmath} + (xy^2 - 2z)\hat{k}$

$\nabla\phi_1]_{(1,-2,1)} = \hat{\imath} - 4\hat{\jmath} + 2\hat{k} \;\Rightarrow \hat{a} = \frac{\nabla\phi_1]_{(1,-2,1)}}{|\nabla\phi_1]_{(1,-2,1)}|} = \boldsymbol{\frac{\hat{\imath}-4\hat{\jmath}+2\hat{k}}{\sqrt{21}}}$

Where, $\hat{a}$ is a unit vector in the direction of the normal to the surface ϕ_1 at the point $(1,-2,1)$.

Required Directional derivative $= \left(348\hat{\imath} - 144\hat{\jmath} + 400\hat{k}\right).\left(\frac{\hat{\imath}-4\hat{\jmath}+2\hat{k}}{\sqrt{21}}\right) = \frac{1724}{\sqrt{21}}v$

Exercise

1. If the directional derivative of $\phi = ax^2y + by^2z + cz^2x$ at the point $(1,1,1)$ has maximum magnitude 15 in the direction parallel to the line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$ find the values of $a, b\ and\ c$.
2. For the function $\phi = \frac{y}{x^2+y^2}$, find the magnitude of the directional derivative making an angle $30°$ with the positive $x - axis$ at the point $(0,1)$.
3. Find the values of constants a, b, c so that the maximum value of the directional derivative of $\phi = axy^2 + byz + cz^2x^3$ at $(1,2,-1)$ has a magnitude 64 in the direction parallel to $z - axis$.
4. Find the directional derivative of $f(x,y,z) = 2x^2 + 3y^2 + z^2$ at the point $P(2,1,3)$ in the direction of the vector $\vec{a} = \hat{\imath} - 2\hat{k}$.
5. Find the directional derivative of $\Psi(x,y,z) = 4e^{x+5y-13z}$ at the point $(1,2,3)$ in the direction towards the point $(-3,5,7)$.

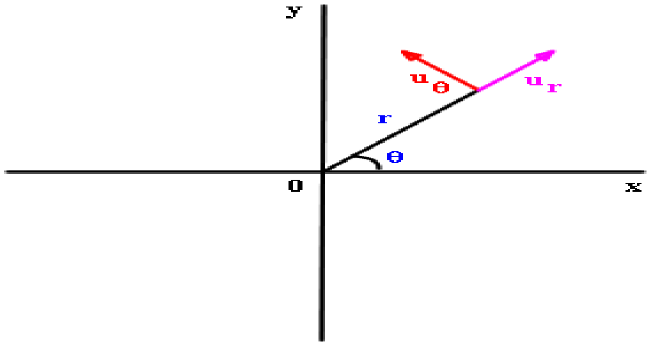
Answers

1. $a = \pm\frac{20}{9}, b = \mp\frac{55}{9}, c = \pm\frac{50}{9}$ **2.** $-\frac{1}{2}$ **3.** $a = 6, b = 24, c = -8$ **4.** $-\frac{4}{\sqrt{5}}$ **5.** $-4\sqrt{41}e^{-28}$

Gradient in Polar Form

Sometimes the surface is given in the form of $\emptyset = f(r,\theta)$. We can find $grad\ \emptyset$ directly without changing into cartesian form.

Let $u_r\ and\ u_\theta$ be the unit vectors along and perpendicular to $\vec{r}$.



Directional derivative along $(u_r) = \nabla\emptyset . u_r$

Thus $\frac{\partial\emptyset}{\partial s} = \frac{\partial\emptyset}{\partial r} = \nabla\emptyset . u_r$ ----------------- (1) [$ds = dr$ in the direction of $\vec{r}$]

Directional derivative along $(u_\theta) = \nabla\emptyset . u_\theta$

Thus $\frac{\partial\emptyset}{\partial s} = \frac{\partial\emptyset}{rd\theta} = \nabla\emptyset . u_\theta$ ----------------- (2) [$ds = rd\theta$ in the direction of $\vec{r}$]

Now, $\nabla\emptyset = (\nabla\emptyset . u_r)u_r + (\nabla\emptyset . u_\theta)u_\theta$

Or $\nabla\emptyset = \left(\frac{\partial\emptyset}{\partial r}\right)u_r + \left(\frac{\partial\emptyset}{rd\theta}\right)u_\theta$ [from equation (1) and (2)]

Example 1. Find (i) $\nabla\left(\frac{e^{\mu r}}{r}\right)$ (ii) $\nabla|\vec{r}|^2$ (iii) $\nabla \log r^n$ (iv) $grad f'(r) \times \vec{r}$

Solution. (i) $\nabla\left(\frac{e^{\mu r}}{r}\right) = \frac{\partial}{\partial r}\left(\frac{e^{\mu r}}{r}\right)\hat{r} = \left[\frac{r\mu e^{\mu r} - e^{\mu r}}{r^2}\right]\frac{\vec{r}}{r} = \frac{e^{\mu r}(\mu r - 1)\vec{r}}{r^3}$

(ii) $\nabla|\vec{r}|^2 = \nabla r^2 = \frac{\partial}{\partial r}(r^2)\hat{r} = 2r\frac{\vec{r}}{r} = 2\vec{r}$

(iii) $\nabla \log r^n = \frac{\partial}{\partial r}(\log r^n)\hat{r} = \left(\frac{1}{r^n} n r^{n-1}\right)\frac{\vec{r}}{r} = \frac{n\vec{r}}{r^2}$

(iv) $grad f'(r) \times \vec{r} = \left[\frac{\partial}{\partial r}\{f'(r)\}\hat{r}\right] \times \vec{r} = f''(r)\frac{\vec{r}}{r} \times \vec{r} = \vec{0}$

Example 2. If $V(x,y) = \frac{1}{2}\log(x^2 + y^2)$ prove that $gradV = \frac{\vec{r} - \vec{k}(\vec{k}.\vec{r})}{\{\vec{r} - \vec{k}(\vec{k}.\vec{r})\}.\{\vec{r} - \vec{k}(\vec{k}.\vec{r})\}}$

Solution. Here, $V = \frac{1}{2}\log(x^2 + y^2) == \frac{1}{2}\log r^2 = \log r$

L.H.S. $= grad\ V = \nabla V = \frac{\partial}{\partial r}(\log r)\hat{r} = \left(\frac{1}{r}\right)\frac{\vec{r}}{r} = \frac{\vec{r}}{r^2}$

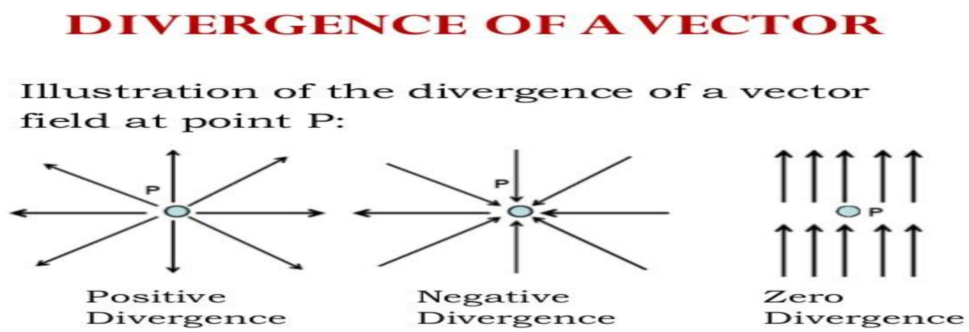
R.H.S. $= \frac{\vec{r} - \vec{k}(\vec{k}.\vec{r})}{\{\vec{r} - \vec{k}(\vec{k}.\vec{r})\}.\{\vec{r} - \vec{k}(\vec{k}.\vec{r})\}}$

$= \frac{\vec{r} - 0}{\{\vec{r} - 0\}.\{\vec{r} - 0\}}$ [as $\vec{k}.\vec{r} = \vec{k}.(x\hat{\imath} + y\hat{\jmath}) = 0$]

$= \frac{\vec{r}}{\vec{r}.\vec{r}} = \frac{\vec{r}}{r^2}$

4. Divergence of a vector point function

Introduction: The divergence of a vector field is the extent to which the vector field flux behaves like a source/sink at a given point.



Definition: The divergence of a differentiable vector point function $\vec{V}$ is denoted by $div\vec{V}$ and defined as:

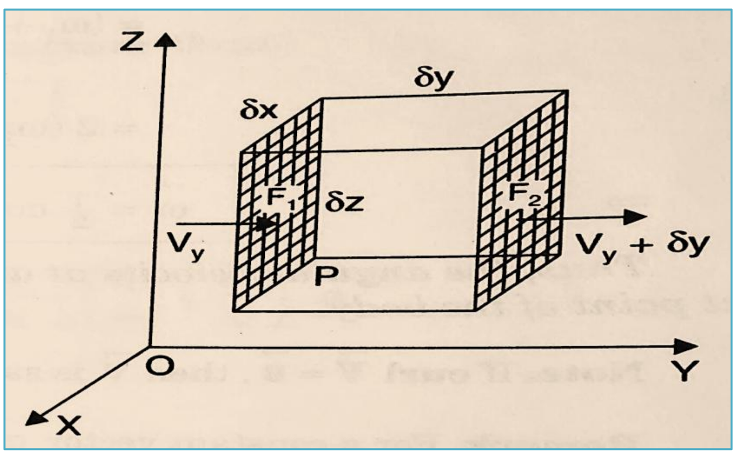
$$div\vec{V} = \nabla.\vec{V} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\vec{V} = \left(\hat{\imath}\frac{\partial\vec{V}}{\partial x} + \hat{\jmath}\frac{\partial\vec{V}}{\partial y} + \hat{k}\frac{\partial\vec{V}}{\partial z}\right)$$

4.1. Physical Interpretation of Divergence

Let ρ *and* $\vec{V}$ are the density and velocity at a point $P(x, y, z)$ at any time t respectively, of a moving fluid in a rectangular parallelepiped.

Let $\vec{V} = V_x\hat{\imath} + V_y\hat{\jmath} + V_z\hat{k} = \rho\vec{v}$, then $\vec{V}$ have the same direction as $\vec{v}$ and its magnitude $|\rho\vec{v}|$ is known as flux.

Let us consider a small parallelepiped with edges $\delta x, \delta y, \delta z$ parallel to the coordinate axes.



The mass of the fluid entering through the face F_1 is $= V_y\delta x\delta z$

The mass of the fluid out through the opposite face F_2 is $= V_{y+\delta y}\delta x\delta z$

The net decrease in the mass of fluid flowing across these faces

$$= V_{y+\delta y}\delta x\delta z - V_y\delta x\delta z$$

$$= \left(V_{y+\delta y} - V_y\right)\delta x\delta z$$

$$= \left[V_y + \frac{\partial}{\partial y}(V_y)\delta y + \cdots \ldots \ldots - V_y\right]\delta x \delta z$$

$$= \frac{\partial}{\partial y}(V_y)\delta y \delta x \delta z = \frac{\partial V_y}{\partial y}\delta x \delta y \delta z$$

Similarly,

The decrease in mass of the fluid to the flow along $x - axis = \frac{\partial V_x}{\partial x}\delta x \delta y \delta z$

The decrease in mass of the fluid to the flow along $z - axis = \frac{\partial V_z}{\partial z}\delta x \delta y \delta z$

Total decrease of the amount of the fluid per unit time $= \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)\delta x \delta y \delta z$

Thus, the rate of loss of the fluid per unit volume $= \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(\hat{\imath}V_x + \hat{\jmath}V_y + \hat{k}V_z\right)$$

$$= \nabla.\vec{V} = div\,\vec{V}$$

Hence, $div\,\vec{V}$ gives the rate of outflow per unit volume at a point of the fluid.

Note: If $div\,\vec{V} = 0$, then **(i)** $\vec{V}$ is called solenoidal vector **(ii)** fluid is called compressible

5. Curl of a vector point function

Introduction: The curl is a vector operator that describes the infinitesimal rotation of a vector field in three-dimensional space.

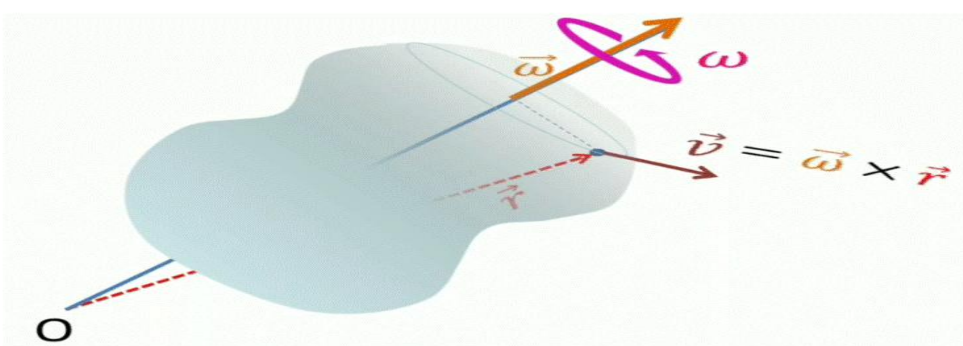
Definition: The curl of a differentiable vector point function $\vec{F}$ is denoted by curl $\vec{F}$ and defined as:

$curl\,\vec{F} = \nabla \times \vec{F} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \vec{F}$

5.1. Physical Interpretation of Curl

Let us consider a rigid body rotating about a fixed axis through O with uniform angular velocity $\vec{\omega} = \omega_1\hat{\imath} + \omega_2\hat{\jmath} + \omega_3\hat{k}$. Then we have,

$\vec{v} = \vec{\omega} \times \vec{r}$, where $\vec{v}$ is the linear velocity at any point and $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ is a position vector for that point.



$$\therefore\ \vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} = (\omega_2 z - \omega_3 y)\hat{\imath} + (\omega_3 x - \omega_1 z)\hat{\jmath} + (\omega_1 y - \omega_2 x)\hat{k}$$

Now, $curl\,\vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$

$$= 2\omega_1\hat{\imath} + 2\omega_2\hat{\jmath} + 2\omega_3\hat{k}$$

$$= 2\vec{\omega} \quad \Rightarrow \quad \vec{\omega} = \tfrac{1}{2} curl\,\vec{v}$$

Thus, the angular velocity at any point is equal to the half of the $curl$ of the linear velocity at that point of the body.

Note: If $\vec{v} = \vec{0}$, then $\vec{v}$ is said to be an irrotational vector.

Example 1. Find the divergence and curl of $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$.

Solution. $div\,\vec{r} = \nabla.\vec{r}$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right)$$

$$= 1 + 1 + 1 = \mathbf{3}$$

$$curl\,\vec{r} = \nabla \times \vec{r} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right)$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= (0-0)\hat{\imath} + (0-0)\hat{\jmath} + (0-0)\hat{k} = \vec{\mathbf{0}}$$

Example 2. Find the divergence and curl of the vector.
$\vec{R} = (x^2 + yz)\hat{\imath} + (y^2 + zx)\hat{\jmath} + (z^2 + xy)\hat{k}$

Solution. $div\,\vec{R} = \nabla.\vec{R}$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left[(x^2 + yz)\hat{\imath} + (y^2 + zx)\hat{\jmath} + (z^2 + xy)\hat{k}\right]$$

$$= 2x + 2y + 2z = \mathbf{2(x + y + z)}$$

$$curl\,\vec{F} = \nabla \times \vec{R}$$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left[(x^2 + yz)\hat{\imath} + (y^2 + zx)\hat{\jmath} + (z^2 + xy)\hat{k}\right]$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + yz & y^2 + zx & z^2 + xy \end{vmatrix}$$

$$= (x - x)\hat{\imath} + (y - y)\hat{\jmath} + (z - z)\hat{k} = \vec{\mathbf{0}}$$

6. Application of divergence and curl

In vector calculus, a solenoidal vector field (also known as a divergence-free or incompressible vector field or a transverse vector field) is one where the divergence is zero at all points. This means that the field has no sources or sinks. Here are some points about the application of divergence to a solenoidal vector field:

1. A vector field is said to be solenoidal if $div\,\vec{V} = 0$. This implies that the net flux of the vector field through any closed surface is zero.
2. The velocity field of an incompressible fluid is solenoidal, meaning the volume of the fluid remains constant over time.

An irrotational vector field is one where the curl is zero everywhere. This means that the field has no rotational component. Here are some points about the application of curl to an irrotational vector field:

1. A vector field is irrotational if $curl\,\vec{F} = \vec{0}$. This implies that the field can be expressed as the gradient of a scalar potential function.
2. In fluid dynamics, if the velocity field of a fluid is irrotational, it means that the fluid elements do not exhibit any rotational motion about their own axes.

Example 1. Prove that $(y^2 - z^2 + 3yz - 2x)\hat{\imath} + (3xz + 2xy)\hat{\jmath} + (3xy - 2xz + 2z)\hat{k}$ is both solenoidal and irrotational.

Solution. Let $\vec{V} = (y^2 - z^2 + 3yz - 2x)\hat{\imath} + (3xz + 2xy)\hat{\jmath} + (3xy - 2xz + 2z)\hat{k}$

$div\,\vec{V} = \nabla.\vec{V} = -2 + 2x - 2x + 2 = 0 \quad \therefore \vec{V}$ is solenoidal

$$curl\,\vec{V} = \nabla \times \vec{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 + 3yz - 2x & 3xz + 2xy & 3xy - 2xz + 2z \end{vmatrix}$$

$$= (3x - 3x)\hat{\imath} + (-2z + 3y - 3y + 2z)\hat{\jmath} + (3z + 2y - 2y - 3z)\hat{k}$$

$$= \vec{\mathbf{0}} \qquad \therefore \vec{V} \text{ is irrotational.}$$

Example 2. Show that the vector field $\vec{A}$, where $\vec{A} = (x^2 - y^2 + x)\hat{\imath} - (2xy + y)\hat{\jmath}$ is irrotational. And find a scalar $\emptyset$ such that $\vec{A} = grad\,\emptyset$.

Solution. Here, $curl\,\vec{A} = \nabla \times \vec{A}$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2 - y^2 + x) & -(2xy + y) & 0 \end{vmatrix} = \vec{\mathbf{0}} \qquad \therefore \vec{A} \text{ is irrotational.}$$

Let $\vec{A} = \nabla\emptyset$, where $\emptyset$ is scalar potential.

$\therefore \vec{A}.d\vec{r} = \nabla\emptyset.d\vec{r}$

$\Rightarrow [(x^2 - y^2 + x)\hat{\imath} - (2xy + y)\hat{\jmath}].(dx\,\hat{\imath} + dy\,\hat{\jmath}) = d\emptyset \quad [as\ \nabla\emptyset.d\vec{r} = d\emptyset]$

$\Rightarrow d\emptyset = (x^2 - y^2 + x)dx - (2xy + y)dy$ -------------------------- (1)

Integrate equation (1) we get, $\emptyset = \frac{x^3}{3} - y^2x + \frac{x^2}{2} - \frac{y^2}{2} + C$

Example 3. Find the constants a, b, c so that

$\vec{F} = (x + 2y + az)\hat{\imath} + (bx - 3y - z)\hat{\jmath} + (4x + cy + 2z)\hat{k}$ is irrotational.

If $\vec{F} = grad\ \emptyset$, show that $\emptyset = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4xz - yz + C$.

Solution. Here, $\vec{F}$ is irrotational $\therefore curl\ \vec{F} = \vec{0}$

$$\Rightarrow \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix} = \vec{\mathbf{0}}$$

$\Rightarrow (c + 1)\hat{\imath} + (a - 4)\hat{\jmath} + (b - 2)\hat{k} = 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k} \quad \Rightarrow a = 4, b = 2, c = -1$

It is given that $\vec{F} = grad\ \emptyset$ or $\vec{F} = \nabla\emptyset$

$\therefore \vec{F}.d\vec{r} = \nabla\emptyset.d\vec{r}$

$\Rightarrow \left[(x + 2y + 4z)\hat{\imath} + (2x - 3y - z)\hat{\jmath} + (4x - y + 2z)\hat{k}\right].\left(dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}\right) = d\emptyset$

$\Rightarrow d\emptyset = (x + 2y + 4z)dx + (2x - 3y - z)dy + (4x - y + 2z)dz$ ----------- (1)

Integrate eq^n (1) we get, $\emptyset = \frac{x^2}{2} + 2xy + 4xz - \frac{3y^2}{2} - yz + z^2 + C$.

Vector Identities

If $\vec{a}, \vec{b}$ are vector functions and $\emptyset$ is a scalar function, then

(1) $div(\emptyset\vec{a}) = \emptyset div\vec{a} + (grad\emptyset).\vec{a}$

Proof. $div(\emptyset\vec{a}) = \sum \hat{\imath}.\frac{\partial}{\partial x}(\emptyset\vec{a}) = \sum \hat{\imath}.\left(\emptyset\frac{\partial\vec{a}}{\partial x} + \frac{\partial\emptyset}{\partial x}\vec{a}\right)$

$$= \emptyset \sum \hat{\imath}.\frac{\partial\vec{a}}{\partial x} + \left(\sum \hat{\imath}\frac{\partial\emptyset}{\partial x}\right).\vec{a} = \emptyset div\vec{a} + (grad\emptyset).\vec{a}$$

(2) $curl(\emptyset\vec{a}) = \emptyset curl\vec{a} + (grad\emptyset) \times \vec{a}$

Proof. $curl(\emptyset\vec{a}) = \sum \hat{\imath} \times \frac{\partial}{\partial x}(\emptyset\vec{a}) = \sum \hat{\imath} \times \left(\emptyset\frac{\partial\vec{a}}{\partial x} + \frac{\partial\emptyset}{\partial x}\vec{a}\right)$

$$= \emptyset \sum \hat{\imath} \times \frac{\partial\vec{a}}{\partial x} + \left(\sum \hat{\imath}\frac{\partial\emptyset}{\partial x}\right) \times \vec{a} = \emptyset curl\vec{a} + (grad\emptyset) \times \vec{a}$$

(3) $div\left(\vec{a} \times \vec{b}\right) = \vec{b}.curl\vec{a} - \vec{a}.curl\vec{b}$

Proof. $div\left(\vec{a} \times \vec{b}\right) = \sum \hat{\imath}.\frac{\partial}{\partial x}\left(\vec{a} \times \vec{b}\right) = \sum \hat{\imath}.\left(\vec{a} \times \frac{\partial\vec{b}}{\partial x} + \frac{\partial\vec{a}}{\partial x} \times \vec{b}\right)$

$$= \sum \hat{\imath}.\left(-\frac{\partial \vec{b}}{\partial x} \times \vec{a}\right) \ + \sum \hat{\imath}.\left(\frac{\partial \vec{a}}{\partial x} \times \vec{b}\right)$$

$$= \sum \hat{\imath}.\left(\frac{\partial \vec{a}}{\partial x} \times \vec{b}\right) - \sum \hat{\imath}.\left(\frac{\partial \vec{b}}{\partial x} \times \vec{a}\right)$$

$$= \sum \left(\hat{\imath} \times \frac{\partial \vec{a}}{\partial x}\right).\vec{b} - \sum \left(\hat{\imath} \times \frac{\partial \vec{b}}{\partial x}\right).\vec{a}$$

$$= (curl\vec{a}).\vec{b} - \left(curl\vec{b}\right).\vec{a} = \vec{b}.(curl\vec{a}) - \vec{a}.\left(curl\vec{b}\right)$$

(4) $curl\left(\vec{a} \times \vec{b}\right) = \vec{a}div\vec{b} - \vec{b}div\vec{a} + \left(\vec{b}.\nabla\right)\vec{a} - (\vec{a}.\nabla)\vec{b}$

Proof. $curl\left(\vec{a} \times \vec{b}\right) = \sum \hat{\imath} \times \frac{\partial}{\partial x}\left(\vec{a} \times \vec{b}\right) = \sum \hat{\imath} \times \left(\vec{a} \times \frac{\partial \vec{b}}{\partial x} + \frac{\partial \vec{a}}{\partial x} \times \vec{b}\right)$

$$= \sum \hat{\imath} \times \left(\vec{a} \times \frac{\partial \vec{b}}{\partial x}\right) \ + \sum \hat{\imath} \times \left(\frac{\partial \vec{a}}{\partial x} \times \vec{b}\right)$$

$$= \sum \left[\left(\hat{\imath}.\frac{\partial \vec{b}}{\partial x}\right)\vec{a} - (\hat{\imath}.\vec{a})\frac{\partial \vec{b}}{\partial x}\right] + \sum \left[\left(\hat{\imath}.\vec{b}\right)\frac{\partial \vec{a}}{\partial x} - \left(\hat{\imath}.\frac{\partial \vec{a}}{\partial x}\right)\vec{b}\right]$$

$$= \sum \left(\hat{\imath}.\frac{\partial \vec{b}}{\partial x}\right)\vec{a} - \sum(\vec{a}.\hat{\imath})\frac{\partial \vec{b}}{\partial x} + \sum\left(\vec{b}.\hat{\imath}\right)\frac{\partial \vec{a}}{\partial x} - \sum \left(\hat{\imath}.\frac{\partial \vec{a}}{\partial x}\right)\vec{b}$$

$$= \left(div\vec{b}\right)\vec{a} - \left(\vec{a}.\sum \hat{\imath}\frac{\partial}{\partial x}\right)\vec{b} + \left(\vec{b}.\sum \hat{\imath}\frac{\partial}{\partial x}\right)\vec{a} - (div\vec{a})\vec{b}$$

$$= \vec{a}div\vec{b} - \vec{b}div\vec{a} + \left(\vec{b}.\nabla\right)\vec{a} - (\vec{a}.\nabla)\vec{b}$$

(5) $div(grad\emptyset) = \nabla^2\emptyset$

Proof. $div(grad\emptyset) = \nabla.(\nabla\emptyset) = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(\hat{\imath}\frac{\partial\emptyset}{\partial x} + \hat{\jmath}\frac{\partial\emptyset}{\partial y} + \hat{k}\frac{\partial\emptyset}{\partial z}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{\partial\emptyset}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\emptyset}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\emptyset}{\partial z}\right) = \frac{\partial^2\emptyset}{\partial x^2} + \frac{\partial^2\emptyset}{\partial y^2} + \frac{\partial^2\emptyset}{\partial z^2}$$

$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\emptyset = \nabla^2\emptyset$; ∇^2 is known as Laplacian operator

(6) $curl(grad\emptyset) = \nabla \times (\nabla\emptyset) = \vec{0}$

Proof. curl$(grad\emptyset) = \nabla \times (\nabla\emptyset) = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(\hat{\imath}\frac{\partial\emptyset}{\partial x} + \hat{\jmath}\frac{\partial\emptyset}{\partial y} + \hat{k}\frac{\partial\emptyset}{\partial z}\right)$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\emptyset}{\partial x} & \frac{\partial\emptyset}{\partial y} & \frac{\partial\emptyset}{\partial z} \end{vmatrix} = \sum \hat{\imath}\left(\frac{\partial^2\emptyset}{\partial y\partial z} - \frac{\partial^2\emptyset}{\partial z\partial y}\right) = \vec{0}$$

(7) $div\left(curl\,\vec{V}\right) = 0$

Proof. Let $\vec{V} = V_1\hat{\imath} + V_2\hat{\jmath} + V_3\hat{k} \Rightarrow curl\,\vec{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$

$$= \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\hat{\imath} + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\hat{k}$$

Now, $div(curl\,\vec{V}) = \frac{\partial}{\partial x}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$

$$= \left(\frac{\partial^2 V_3}{\partial x \partial y} - \frac{\partial^2 V_2}{\partial x \partial z}\right) + \left(\frac{\partial^2 V_1}{\partial y \partial z} - \frac{\partial^2 V_3}{\partial y \partial x}\right) + \left(\frac{\partial^2 V_2}{\partial z \partial x} - \frac{\partial^2 V_1}{\partial z \partial y}\right) = 0$$

Example 1. If $u = x^2 + y^2 + z^2$ and $\vec{V} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, show that $div(u\vec{V}) = 5u$

Solution. $div(u\vec{V}) = u\, div\vec{V} + (grad\, u).\vec{V}$

$$= 3u + \left(\hat{\imath}\frac{\partial u}{\partial x} + \hat{\jmath}\frac{\partial u}{\partial y} + \hat{k}\frac{\partial u}{\partial z}\right).\vec{V}$$

$$= 3u + (2x\hat{\imath} + 2y\hat{\jmath} + 2z\,\hat{k}).(x\hat{\imath} + y\hat{\jmath} + z\hat{k})$$

$$= 3u + (2x^2 + 2y^2 + 2z^2)$$

$$= 3u + 2(x^2 + y^2 + z^2)$$

$$= 3u + 2u = \mathbf{5u}$$

Example 2. If $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, prove that
(i) $curl(r^n\vec{r}) = \vec{0}$ (ii) $\nabla^2(r^n\vec{r}) = n(n+3)r^{n-2}\vec{r}$

Solution. (i) $curl(r^n\vec{r}) = r^n curl\,\vec{r} + (grad\, r^n) \times \vec{r}$

$$= \vec{0} + (nr^{n-1}\hat{r}) \times \vec{r} = nr^{n-1}\frac{\vec{r}}{r} \times \vec{r} = \vec{0}$$

(ii) $\nabla^2(r^n\vec{r}) = \nabla[\nabla.(r^n\vec{r})] = grad[div(r^n\vec{r})]$

$$= grad[r^n div\,\vec{r} + (grad r^n).\vec{r}]$$

$$= grad[3r^n + (nr^{n-1}\hat{r}).\vec{r}] = grad\left[3r^n + nr^{n-1}\frac{\vec{r}}{r}.\vec{r}\right]$$

$$= grad\left[3r^n + nr^{n-1}\frac{r^2}{r}\right] = grad[(n+3)r^n]$$

$$= (n+3)nr^{n-1}\frac{\vec{r}}{r} = n(n+3)r^{n-2}\vec{r}$$

Example 3. Find the most general $f(r)$ such that $f(r)\vec{r}$ is solenoidal.

Solution. The vector $f(r)\vec{r}$ will be solenoidal, if

$$div[f(r)\vec{r}] = 0$$

$$\Rightarrow f(r)div\,\vec{r} + [grad f(r)].\vec{r} = 0$$

$$\Rightarrow 3f(r) + [f'(r)\hat{r}].\vec{r} = 0 \Rightarrow 3f(r) + \left[f'(r)\frac{\vec{r}}{r}\right].\vec{r} = 0$$

$$\Rightarrow 3f(r) + f'(r)\frac{r^2}{r} = 0 \Rightarrow \frac{f'(r)}{f(r)} = -\frac{3}{r} \qquad \text{----------------------- (1)}$$

Integrate eq^n (1), we get

$$\log[f(r)] = -3\log r + \log c$$

$$\Rightarrow f(r) = \frac{c}{r^3}$$

Example 4. Prove that the vector $f(r)\vec{r}$ is irrotational.

Solution. The vector $f(r)\vec{r}$ will be irrotational, if $curl[f(r)\vec{r}] = \vec{0}$

Now, $curl[f(r)\vec{r}] = f(r) curl\,\vec{r} + [grad f(r)] \times \vec{r}$

$$= \vec{0} + [f'(r)\hat{r}] \times \vec{r}$$

$$= \vec{0} + \left[f'(r)\frac{\vec{r}}{r}\right] \times \vec{r}$$

$$= \vec{0} + f'(r)\frac{\vec{r}}{r} \times \vec{r}$$

$$= \vec{0} + \vec{0} = \vec{0}$$

$\therefore$ The vector $f(r)\vec{r}$ is irrotational.

Example 5. Prove that $\nabla^2 f(r) = f''(r) + \frac{2}{r}f'(r)$.

Hence evaluate $\nabla^2(\log r)$ if $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$

Solution. $\nabla^2 f(r) = \nabla.\nabla f(r) = div[\nabla f(r)] = div[grad\, f(r)]$

$$= div[f'(r)\hat{r}] = div\left[f'(r)\frac{\vec{r}}{r}\right] = div\left[\left\{\frac{f'(r)}{r}\right\}\vec{r}\right]$$

$$= \frac{f'(r)}{r} div\vec{r} + \left[grad\left\{\frac{f'(r)}{r}\right\}\right].\vec{r} = 3\frac{f'(r)}{r} + \left[\frac{rf''(r) - f'(r)}{r^2}\hat{r}\right].\vec{r}$$

$$= \frac{3}{r}f'(r) + \left[\frac{rf''(r) - f'(r)}{r^2}\left(\frac{\vec{r}}{r}\right)\right].\vec{r} = \frac{3}{r}f'(r) + \frac{rf''(r) - f'(r)}{r^2}\left(\frac{\vec{r}.\vec{r}}{r}\right)$$

$$\nabla^2 f(r) = \frac{3}{r}f'(r) + \frac{rf''(r) - f'(r)}{r} = f''(r) + \frac{2}{r}f'(r)$$

Put $f(r) = \log r$, we get, $\nabla^2(\log r) = -\frac{1}{r^2} + \frac{2}{r}\frac{1}{r} = \boldsymbol{\frac{1}{r^2}}$**.**

Exercise

1. If $\vec{F}(x,y,z) = xz^3\hat{\imath} - 2x^2yz\hat{\jmath} + 2yz^4\hat{k}$ find $divergence\ and\ curl$ of $\vec{F}(x,y,z)$.

2. Find the $divergence\ and\ curl$ of the vector field $\vec{V} = x^2y^2\hat{\imath} + 2xy\hat{\jmath} + (y^2 - xy)\hat{k}$.

3. A fluid motion is given by $\vec{V} = (y+z)\hat{\imath} + (z+x)\hat{\jmath} + (x+y)\hat{k}$

(i) Is this motion irrotational? If so, find the velocity potential.

(ii) Is the motion possible for an incompressible fluid?

4. If $\vec{V} = \frac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}$, show that $\nabla.\vec{V} = \frac{2}{\sqrt{x^2+y^2+z^2}}$ and $\nabla \times \vec{V} = \vec{0}$

5. Show that $\vec{A} = (6xy + z^3)\hat{\imath} + (3x^2 - z)\hat{\jmath} + (3xz^2 - y)\hat{k}$ is irrotational. Find the velocity potential $\emptyset$ such that $\vec{A} = \nabla\emptyset$.

6. A fluid motion is given by $\vec{v} = (y\sin z - \sin x)\hat{\imath} + (x\sin z + 2yz)\hat{\jmath} + (xy\cos z + y^2)\hat{k}$. Is the motion irrotational? If so, find the velocity potential.

7. If $\vec{E}$ *and* $\vec{H}$ are irrotational, prove that $\vec{E} \times \vec{H}$ is irrotational.

8. If $u\vec{F} = \nabla v$, where u, v are scalar field and $\vec{F}$ is a vector field show that $\vec{F}.curl\,\vec{F} = 0$.

9. If $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ and $r = |\vec{r}|$, show that

(i) $div(\vec{r})\emptyset = 3\emptyset + \vec{r}.grad\emptyset$ **(ii)** $div(\hat{r}) = \frac{2}{r}$

10. Prove that $div(grad\, r^n) = \nabla^2 r^n = n(n+1)r^{n-2}$, where $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$. Hence show that $\nabla^2\left(\frac{1}{r}\right) = 0$. Hence or otherwise evaluate $\nabla \times \left(\frac{\vec{r}}{r^2}\right)$.

Answers

1. $div\,\vec{F} = z^3 - 2x^2z + 8yz^3, curl\,\vec{F} = 2(x^2y + z^4)\hat{\imath} + 3xz^2\hat{\jmath} - 4xyz\hat{k}$
2. $div\,\vec{V} = 2xy^2 + 2x,\ curl\,\vec{F} = (2y - x)\hat{\imath} + y\hat{\jmath} + 2y(1 - x^2)\hat{k}$
3. **(i)** Yes; velocity potential $\emptyset = xy + yz + zx + c$

   (ii) Yes

5. $\emptyset = x^2yz^3 + c$ **6.** Yes; $\emptyset = xy\sin z + \cos x + y^2z + c$

10. $\vec{0}$

7. E-resources:

https://www.youtube.com/watch?v=fZ231k3zsAA&t=57s

https://www.youtube.com/watch?v=qOcFJKQPZfo

https://www.youtube.com/watch?v=3TkKm2mwR0Y

https://www.youtube.com/watch?v=ynzRyIL2atU

https://www.youtube.com/watch?v=Cxc7ihZWq5o

https://www.youtube.com/watch?v=vvzTEbp9lrc

https://www.youtube.com/watch?v=ZtQyuN7DdKE