CALCULUS FOR ENGINEERS (K24AS11)

UNIT 3 : COMPLEX VARIABLE - DIFFERENTIATION

Complex Number → It is defined as an ordered pair (x, y) of real numbers and is expressed as.

$$\boxed{z = x + iy} \quad \text{where} \quad i = \sqrt{-1}$$

In polar coordinates it is expressed as.

$\boxed{z = re^{i\theta}}$ where r is modulus of z and θ is argument of z and given as.

$$\boxed{r = \sqrt{x^2 + y^2}}, \quad \boxed{\theta = \tan^{-1}\frac{y}{x}}$$

* The set of points (z) which satisfies the equation $\boxed{|z - a| = r}$ defines a circle C with centre at the point 'a' and radius 'r'.

Function of a compex variable → Let S and S^* be two non empty sets of complex numbers. If correspondind to each value of complex variable z in S, there correspond one more values of complex variable w in S^* then w is called function of complex variable z and is written as

$$w = f(z)$$

where $z = x + iy$ and w is considered as $u + iv$.

Limit of a function $f(z)$ → A fu^n $f(z)$ tends to the limit l as z tends to z_0 along any path, if to each positive arbitrary number ϵ, however small, there corresponds a positive number δ, such that

$$|f(z) - l| < \epsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta$$

and we write $\lim\limits_{z \to z_0} f(z) = l$ where l is finite.

Continuity of function f(z) → A function which is single valued, is said to be continuous at the point $z=z_0$ if following three conditions are satisfied:

(i) $f(z_0)$ exist (ii) $\lim_{z\to z_0} f(z)$ exist (iii) $\lim_{z\to z_0} f(z) = f(z_0)$

Differentiability of function f(z) → Let $f(z)$ be a single valued function defined in a domain D. The function $f(z)$ is said to be differentiable at a point z_0, if

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{(z-z_0)} \text{ exists.}$$

This limit is called as derivative of $f(z)$ at $z=z_0$)

$$\Rightarrow \boxed{f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}} \text{ or } \boxed{f'(z) = \lim_{\delta z\to 0} \frac{f(z_0+\delta z)-f(z_0)}{\delta z}}$$

Analyticity of function f(z) → A single valued function $f(z)$ of a complex variable z is said to be analytic at a point z_0 if it is differentiable at the point z_0 and also at each point in some neighbourhood of the point z_0.

→ A function $f(z)$ is said to be analytic in certain domain D if it is analytic at every point of D.

→ If a function is analytic in a domain D except for a finite number of points then these points are said to be regular points or singularities.

Entire function → A fu^n $f(z)$ which is analytic at every point of the finite complex plane is called an entire function.

Cauchy-Riemann equations (C-R Equations) →

for a function of complex variable

$$w = f(z) = u(x,y) + i\,v(x,y)$$

C-R equations are given as $\boxed{\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}}$ and $\boxed{\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}}$

Necessary and Sufficient condition for $f(z)$ to be analytic →

Necessary Condition → "The necessary condition for a function $f(z)$ to be analytic is that C-R equations must be satisfied."

But this is only necessary condition not sufficient.

Sufficient Condition → With the truthfullness of C-R equations

"The sufficient condition for the fu^n $f(z)$ to be analytic is that four partial derivatives u_x, u_y, v_x, v_y must exist and must be continuous at all points of region."

Proof of C-R Equation → Let $w = f(z) = u + iv$ be anaytic in R then $\dfrac{dw}{dz} = f'(z)$ exist at every point of R

we know $$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = \lim_{\delta z \to 0} \frac{(u+\delta u) + i(v+\delta v) - (u+iv)}{\delta z}$$

$$= \lim_{\delta z \to 0} \left(\frac{\delta u}{\delta z} + i \frac{\delta v}{\delta z} \right) \quad \text{------ (1)}$$

Since $w = f(z)$ is analytic in R, the limit must exist independent of path in which $\delta z \to 0$

(i) First let $\delta z \to 0$ along a line parallel to x-axis so that $\delta y = 0$ and $\delta z = \delta x$ $\quad (\because \delta z = \delta x + i\delta y)$

From (i), $$f'(z) = \lim_{\delta z \to 0} \left(\frac{\delta u}{\delta x} + i \frac{\delta v}{\delta x} \right) = \boxed{\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}} \quad \text{--- (2)}$$

(ii) Again let $\delta z \to 0$ along a line parallel to y axis so that $\delta x = 0$ and $\delta z = i\delta y$ $\quad (\because \delta z = \delta x + i\delta y)$

From (i) $$f'(z) = \lim_{\delta z \to 0} \left(\frac{\delta u}{i\delta y} + i \frac{\delta v}{i\delta y} \right) = \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$= \boxed{\frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}} \quad \text{------ (3)}$$

from ② and ③, we have

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{or} \quad \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

C-R Equations in Polar Coordinates →

Let $x = r\cos\theta$, $y = r\sin\theta$ such that $z = re^{i\theta}$ then C-R eq^n in Polar coordinates are given as

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}} \quad \text{and} \quad \boxed{\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}}$$

Harmonic Function → A fu^n of x, y which possesses continuous partial derivatives of the first and second orders and satisfies Laplace equation is called a Harmonic function.

THEOREMS → ① If $f(z) = u + iv$ is an analytic function then u and v are both harmonic function.

② An analytic function with constant modulus is constant.

③ **Orthogonal System →** Every analytic function $f(z) = u + iv$ defines two families of curves $u(x,y) = c_1$ and $v(x,y) = c_2$, which form an orthogonal system.

i.e. the curves from two families intersects at right angles

* If $f(z) = u + iv$ is analytic fu^n then u and v are said to be harmonic conjugate of each other.

Determination of conjugate function → If $f(z) = u + iv$ is an analytic function where both $u(x,y)$ and $v(x,y)$ are conjugate function. Being given one of these say $u(x,y)$ we can find the other $v(x,y)$ as following

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \quad \text{by C-R equation}$$

And now dv can be integrated to get "v".

Similarly we cand find $u(x,y)$ if $v(x,y)$ is given and consequently $w = f(z) = u + iv$ may be determined.

MILNE'S THOMSON METHOD → (To construct $f(z)$ when one conjugate is given.)

In this method we determine $f(z)$ as a whole directly without determining the other conjugate function.

Case-I → When real part $u(x,y)$ is given → We follow the steps.

(i) Find $\frac{\partial u}{\partial x}$ and write it as equal to $\phi_1(x,y)$

(ii) Find $\frac{\partial u}{\partial y}$ and write it as equal to $\phi_2(x,y)$

(iii) Find $\phi_1(z,0)$ and $\phi_2(z,0)$ by replacing x by z and y by 0 in $\phi_1(x,y)$ and $\phi_2(x,y)$.

(iv) $f(z)$ is calculated as

$$\boxed{f(z) = \int \{\phi_1(z,0) - i\,\phi_2(z,0)\}\,dz + c}$$

Case-II → When imaginary part $v(x,y)$ is given →

(i) Find $\frac{\partial v}{\partial y}$ and write it as equal to $\psi_1(x,y)$

(ii) Find $\frac{\partial v}{\partial x}$ and write it as equal to $\psi_2(x,y)$

(iii) Find $\psi_1(z,0)$ and $\psi_2(z,0)$ by replacing x by z and y by 0 in $\psi_1(x,y)$ and $\psi_2(x,y)$

(iv) $f(z)$ is calculated as

$$\boxed{f(z) = \int \{\psi_1(z,0) + i\,\psi_2(z,0)\}\,dz + c}$$

Case-III → When $(u-v)$ is given →

(i) $f(z) = u + iv \Rightarrow if(z) = iu - v$

(ii) adding, $(1+i)f(z) = (u-v) + i(u+v)$

or $F(z) = U + iV$

where $F(z) = (1+i)f(z)$, $U = u - v$, $V = u + v$

(iii) As $(u-v)$ or U is given we apply the same process as for real part and find $F(z)$.

(iv) We find $\boxed{f(z) = \frac{1}{1+i}F(z)}$

Case IV → When $u+v$ is given →

(i) $f(z) = u + iv \Rightarrow if(z) = iu - v$

(ii) adding, $(1+i)f(z) = (u-v) + i(u+v)$

or $F(z) = U + iV$

where $F(z) = (1+i)f(z)$, $U = u-v$, $V = u+v$

(iii) As $(u+v)$ or V is given we apply the same process as for imaginary part and find $F(z)$.

(iv) We find $f(z)$ as $\boxed{f(z) = \frac{1}{(1+i)} F(z)}$

Q:- Prove that the function $\sinh z$ is analytic and find its derivative.

Sol:- $f(z) = u + iv = \sinh z = \sinh(x+iy) = \sinh x \cos y + i \cosh x \sin y$

Here $u = \sinh x \cos y$, $v = \cosh x \sin y$

$\frac{\partial u}{\partial x} = \cosh x \cos y$, $\frac{\partial u}{\partial y} = -\sinh x \sin y$

$\frac{\partial v}{\partial x} = \sinh x \sin y$, $\frac{\partial v}{\partial y} = \cosh x \cos y$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

C-R equations are satisfied.

$\because$ $\sinh x$, $\cosh x$, $\sin y$ and $\cos y$ are continuous functions

$\therefore$ $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$ are also continuous functions.

Hence $f(z) = \sinh z$ is analytic.

Now $f'(z)$ is given as

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \cosh x \cos y + i \sinh x \sin y$$

$$= \cosh(x+iy)$$

$$\boxed{f'(z) = \cosh z}$$

Q: Verify if $f(z) = \frac{xy^2(x+iy)}{x^2+y^4}$, $z \neq 0$; $f(0)=0$ is analytic or not.

Sol^n ⇒ $f(z) = u+iv = \frac{xy^2(x+iy)}{x^2+y^4} \Rightarrow u = \frac{x^2y^2}{x^2+y^4}$, $v = \frac{xy^3}{x^2+y^4}$

At origin,

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

Since, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Hence C-R equations are satisfied at origin.

Now $f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{\substack{x\to 0\\ y\to 0}} \left[\frac{xy^2(x+iy)}{x^2+y^4} - 0\right]\frac{1}{x+iy}$

$$f'(0) = \lim_{\substack{x\to 0\\ y\to 0}} \frac{xy^2}{x^2+y^4}$$

Let $z\to 0$ along the real axis $y=0$, then $f'(0)=0$

again let $z\to 0$ along the curve $x=y^2$ then

$$f'(0) = \lim_{x\to 0} \frac{x^2}{x^2+x^2} = \frac{1}{2}$$

Here we observe that $f'(0)$ does not exist as the limit is not unique along the two paths.

Hence $f(z)$ is not analytic at origin although C-R equations are satisfied there.

Q: Show that $u = \frac{1}{2}\log(x^2+y^2)$ is Harmonic and find its Harmonic conjugate.

Sol^n ⇒ $u = \frac{1}{2}\log(x^2+y^2)$ Now $\frac{\partial u}{\partial x} = \frac{1}{2}\frac{1}{(x^2+y^2)}\cdot 2x = \frac{x}{x^2+y^2}$

$$\frac{\partial u}{\partial y} = \frac{1}{2}\frac{1}{x^2+y^2}2y = \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)\cdot 1 - y\cdot 2y}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

Hence we get $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\because$ u satisfied Laplace eq^n $\therefore$ u is Harmonic fu^n.

To find v,

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

$$= \left(\frac{-y}{x^2+y^2}\right)dx + \left(\frac{x}{x^2+y^2}\right)dy$$

$$= \frac{x\,dy - y\,dx}{(x^2+y^2)} = d\left(\tan^{-1}\frac{y}{x}\right)$$

Integrating $\boxed{v = \tan^{-1}\frac{y}{x} + c}$

Q: If $u = e^x(x\cos y - y\sin y)$ is harmonic function, find an analytic function $f(z) = u + iv$ such that $f(1) = e$. (Use Milne's Thomson method)

Sol^n $\Rightarrow$ Here $u = e^x(x\cos y - y\sin y)$

$$\frac{\partial u}{\partial x} = e^x(x\cos y - y\sin y) + e^x(\cos y) = \phi_1(x,y)$$

$$\frac{\partial u}{\partial y} = e^x[-x\sin y - y\cos y - \sin y] = \phi_2(x,y)$$

Now $\phi_1(z,0) = z\,e^z + e^z = (z+1)e^z$

$\phi_2(z,0) = 0$

By Milne's Thomson method $f(z) = \int\{\phi_1(z,0) - i\phi_2(z,0)\}dz + C$

$$f(z) = \int (z+1)e^z dz + C = (z-1)e^z + e^z + c = \boxed{ze^z + c}$$

Now $f(1) = e + c$ $\Rightarrow$ $e = e + c$ $\Rightarrow$ $c = 0$.

$\Rightarrow$ $\boxed{f(z) = z\,e^z}$

Transformation or Mapping → If we take two complex planes w plane and z plane then $w = f(z)$ defines a mapping or transformation of z plane into the w plane.

for example for the transformation $w = z + (1-i)$. we will determine D' of w plane corresponding to the rectangle region D in z plane bounded by $x=0, y=0, x=1, y=2$

Since, $w = z + (1-i)$, we have

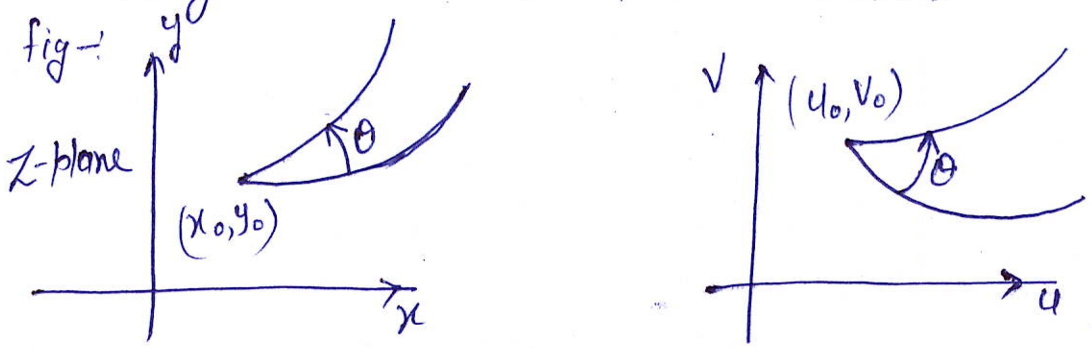
$$u + iv = (x+iy) + (1-i) = (x+1) + i(y-1)$$

$$u = x+1 \quad \text{and} \quad v = y-1$$

Using these relations we obtain that the lines $x=0, y=0$, $x=1$ and $y=2$ in z plane are mapped onto the lines $u=1$, $v=-1$, $u=2$, $v=1$ in w-plane.

Conformal Transformation → Let $w=f(z)$ be a transformation which maps the point (x_0, y_0) of z plane into point (u_0, v_0) of w plane while curves C_1 and C_2 [intersecting at (x_0, y_0)] are mapped into curves C_1' and C_2' [intersecting at (u_0, v_0)]. Then if the transformation is such that the angle at (x_0, y_0) between C_1 and C_2 is equal to the angle at (u_0, v_0) between C_1' and C_2' both in magnitude and sense, the transformation or mapping is said to be conformal at (x_0, y_0).

fig→



* A harmonic function remains harmonic under a conformal transformation.

Coefficient of Magnification → Coefficient of magnification for the conformal transformation $w = f(z)$ at $z = \alpha + i\beta$ is given by $\boxed{|f'(\alpha + i\beta)|}$.

Angle of Rotation → Angle of rotation for conformal transformation $w = f(z)$ at $z = \alpha + i\beta$ is given by $\boxed{amp \cdot [f'(\alpha + i\beta)]}$

Theorem → If $f(z)$ is analytic and $f'(z) \neq 0$ in a region R of the z plane, then the mapping $w = f(z)$ is conformal at all point of R.

Q-! for the conformal mapping $w = z^2$, show that

(a) the coefficient of magnification at $z = 2+i$ is $2\sqrt{5}$

(b) the angle of rotation at $z = 2+i$ is $\tan^{-1} 0.5$

Solⁿ) Let $w = f(z) = z^2$

$$\Rightarrow f'(z) = 2z \Rightarrow f'(2+i) = 2(2+i) = 4+2i$$

(a) coefficient of magnification at $(z = 2+i)$ is $= |f'(2+i)|$

$$= |4+2i| = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$$

(b) angle of rotation at $(z = 2+i)$ is $= amp[f'(2+i)] = amp[4+2i]$

$$= \tan^{-1}\frac{2}{4} = \tan^{-1}\frac{1}{2}$$

Some General Transformation →

1 → Translation → $w = z + c$ (c is complex constant)

By this transformation, figures in z plane are displaced or translated in the direction of vector c.

2 → Rotation → $w = e^{i\theta} z$ (θ is real constant)

By this transformation, figures in z plane are rotated through an angle θ. If $\theta > 0$ the rotation is counterclockwise, while if $\theta < 0$ then rotation is clockwise.

3 → Stretching → $w = az$ (a is real constant)

By this transformation, figures in z plane are stretched (or contracted) in the direction z if $a > 1$ (or $0 < a < 1$).

4 → Inversion → $w = \frac{1}{z}$

By this transformation, figures in z-plane are mapped upon the reciprocal figure in w-plane.
i.e. interior of circle $|z|=1$ into the exterior of the circle $|w|=1$ and the exterior of $|z|=1$ is mapped into the interior of $|w|=1$.

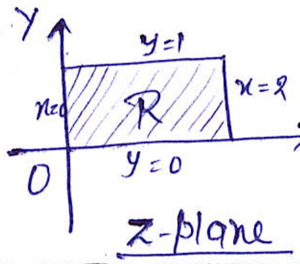
Question → Let a rectangular domain R be bounded by $x=0, y=0$, $x=2, y=1$. Determine the region R' of w-plane into which R is mapped under transformation $w = z+(1-2i)$. [TRANSLATION]

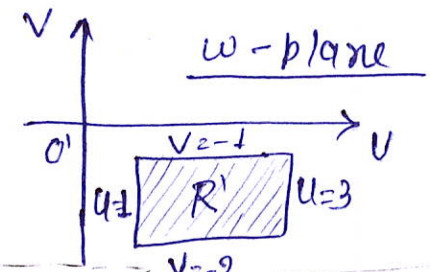
Sol^n ⇒ $w = z+(1-2i) \Rightarrow u+iv = (x+iy)+(1-2i)$

$\Rightarrow u+iv = (x+1)+i(y-2)$

$\Rightarrow \boxed{u = x+1}$ and $\boxed{v = y-2}$

by the map $u = x+1$, lines $x=0$ and $x=2$ are mapped on the lines $u=1$ and $u=3$ respectively.
by the map $v = y-2$ lines $y=0$ and $y=1$ are mapped on the lines $v=-2$ and $v=-1$ respectively.
Hence we can show R and R' as.





Question → Consider the transformation $w = ze^{i\pi/4}$ and determine the region R' in w plane corresponding to the triangular region R bounded by the lines $x=0$, $y=0$ and $x+y=1$ in z plane. [ROTATION]

Sol^n ⇒ $w = ze^{i\pi/4} \Rightarrow u+iv = (x+iy)(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$

$u+iv = (x+iy)(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}}[(x-y)+(x+y)]$

$\Rightarrow u = \frac{x-y}{\sqrt{2}}$, $v = \frac{x+y}{\sqrt{2}}$

for $x=0$, $u = -\frac{1}{\sqrt{2}}y$, $v = \frac{1}{\sqrt{2}}y$ or $v=-u$

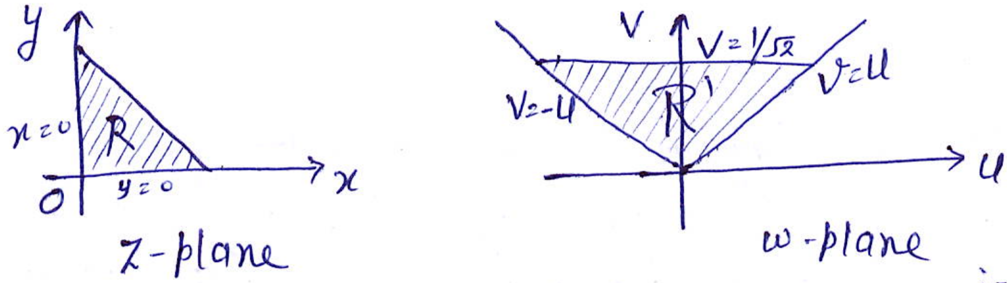
for $y=0$, $u = \frac{1}{\sqrt{2}}x$, $v = \frac{1}{\sqrt{2}}x$ or $v=u$

for $x+y=1$ we have $v=\frac{1}{\sqrt{2}}$

Hence the region R' in w plane will be bounded by

$v=u$, $v=-u$, $v=\frac{1}{\sqrt{2}}$

Hence R and R' may be shown as.



Hence we observe that the mapping $w=ze^{i\pi/4}$ performs a rotation of R through an angle $\pi/4$.

Question → Consider the transfomation $w=2z$ and determine the region R' of w plane into which the triangular region R bounded by the lines $x=0, y=0, x+y=1$. in the z-plane is mapped. [STRETCHING]

Sol^n ⇒ $w=2z \Rightarrow u+iv = 2x+i2y$.

$\Rightarrow u=2x$ and $v=2y$.

Mapping of lines will be

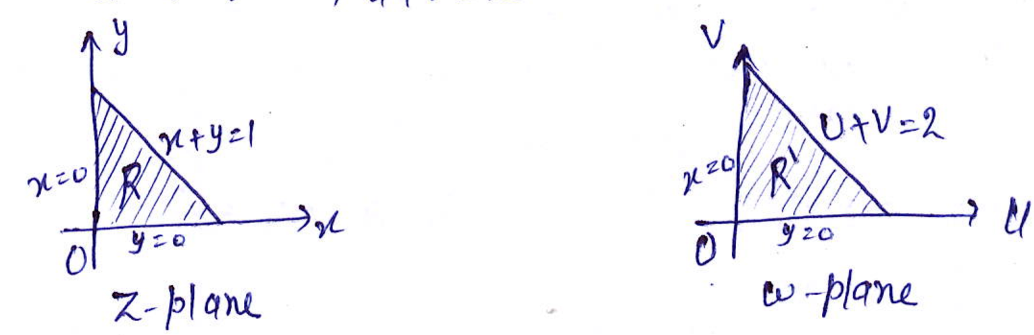
for $\boxed{x=0 \Rightarrow u=0}$

For $\boxed{y=0 \quad , \quad v=0}$

for $x+y=1 \quad , \quad 2x+2y=u+v \Rightarrow 2(x+y)=u+v$

$\Rightarrow$ for $\boxed{x+y=1 \quad , \quad u+v=2}$

Hence region R' may be shown as (which is bounded by $u=0$, $v=0$, $u+v=2$.



This transformation $w=2z$ performs a magnification of R into R'.

Question → Find the image of infinite strip $\frac{1}{4} \leq y \leq \frac{1}{2}$ under the transformation $w = \frac{1}{z}$. Also show the region graphically. [INVERSION]

Sol^n ⇒ $w = \frac{1}{z} \Rightarrow z = \frac{1}{w} \Rightarrow x + iy = \frac{1}{u+iv}$

$\Rightarrow x + iy = \frac{u - iv}{u^2+v^2} \Rightarrow x = \frac{u}{u^2+v^2}, \quad y = -\frac{v}{u^2+v^2}$

Now $\boxed{y < \frac{1}{2}} \Rightarrow -\frac{v}{u^2+v^2} < \frac{1}{2} \Rightarrow -2v < u^2+v^2$

$\Rightarrow u^2+v^2+2v > 0 \Rightarrow \boxed{u^2+(v+1)^2 > 1}$

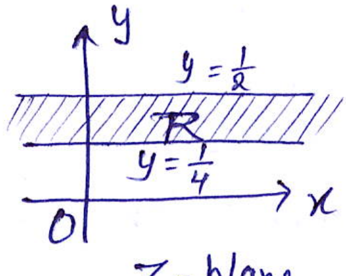
and $\boxed{y > \frac{1}{4}} \Rightarrow -\frac{v}{u^2+v^2} > \frac{1}{4} \Rightarrow -4v > u^2+v^2$

$\Rightarrow u^2+v^2+4v < 0 \Rightarrow \boxed{u^2+(v+2)^2 < 4}$

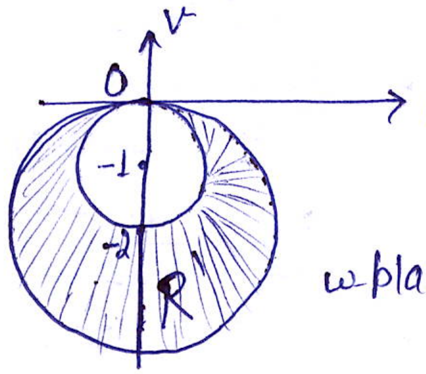
Hence $\frac{1}{4} < y < \frac{1}{2} \Rightarrow u^2+(v+1)^2 > 1$ and $u^2+(v+2)^2 < 4$

this shows that region R' in w plane will be mapped outside the circle $u^2+(v+1)^2 = 1$ and inside $u^2+(v+2)^2 = 4$

R and R' may be shown as below.



z - plane



w plane

Linear Transformation →

The transformation $\boxed{w = az + b}$ where a and b are complex constant is called a linear transformation.

BILINEAR TRANSFORMATION → A transformation of the form

$\boxed{w = \frac{az+b}{cz+d}}$, where a, b, c, d are complex constants and $ad - bc \neq 0$, is called a bilinear transformation.

This transformation may be considered as a combination of transformations of Translation, Rotation, Stretching and Inversion.

Cross Ratio - If four points z_1, z_2, z_3, z_4 are taken in order, then the ratio $\dfrac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$ is called cross ratio.

Invarient or fixed points → The points which coincide with their transformations are called invariant points of the transformation. such points for a transformation $w=f(z)$ may be obtained as the solⁿ of the equation $z=f(z)$

* for example for $w=z^2$, invariant points may be obtained as. $z=z^2 \Rightarrow z=0,1$ are the fixed or invariant points.

Theorem → A bilinear transformation preserves cross ration of four points. i.e. $\boxed{\dfrac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)} = \dfrac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}}$

Question → Find the bilinear transformation which maps the points $z=1,-i,-1$ to the points $w=i,0,-i$ respectively. Show that transformation maps the region outside the circle $|z|=1$ into the half plane $R(w)\geq 0$.

Solⁿ ⇒ Required transformation is given as.

$$\frac{(w-i)(0+i)}{(i-0)(-i-w)} = \frac{(z-1)(-i+1)}{(1+i)(-1-z)} \quad \left\{\begin{array}{l}\text{Here points are taken as}\\ (w,i,0,-i)\ \&\ (z,1,-i,-1)\end{array}\right\}$$

$$\Rightarrow \frac{i-w}{i+w} = \frac{(z-1)(i-1)}{(z+1)(i+1)} \Rightarrow \frac{i-w}{i+w} = \frac{(i-1)z+(1-i)}{(i+1)z+(1+i)}$$

$$\frac{2i}{-2w} = \frac{2iz+2}{-2z-2i} \quad \text{(applying componendo and dividendo)}$$

on solving $\boxed{w = \dfrac{iz-1}{iz+1}}$ this is the required transformation.

Now this transformation may also be given as.

$$z = i\left(\frac{w+1}{w-1}\right)$$

$|z|\geq 1$ is transformed into $\left|\dfrac{w+1}{w-1}\right||i|\geq 1$

$$\Rightarrow |w+1|^2 \geq |w-1|^2 \Rightarrow |u+iv+1|^2 \geq |u+iv-1|^2$$

$$\Rightarrow |(u+1)+iv|^2 \geq |(u-1)+iv|^2 \Rightarrow (u+1)^2+v^2 \geq (u-1)^2+v^2$$

$$\Rightarrow (u+1)^2 \geq (u-1)^2 \Rightarrow u\geq 0 \Rightarrow \boxed{R(w)\geq 0}$$

Thus exterior of circle $|z|=1$ is mapped into half plane $R(w)\geq 0$

Question → Find a bilinear transformation which maps the points, $i, -i, 1$ of z plane into $0, 1, \infty$ of w plane respectively.

Solⁿ ⇒ transformation is given as

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

or
$$\frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{(w_1-w_2)\left(1-\frac{w}{w_3}\right)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\Rightarrow \frac{(w-0)\left(\frac{1}{\infty}-1\right)}{(1-0)\left(1-\frac{w}{\infty}\right)} = \frac{(z-i)(-i-1)}{(i+i)(1-z)}$$

$$\Rightarrow w = \frac{(z-i)(i+1)}{(2i)(z-1)} = \boxed{\frac{(i-1)z+(i+1)}{(-2z+2)}}$$

Question → Find the invariant points of the transformation
$$w = -\left(\frac{2z+4i}{iz+1}\right).$$

Solⁿ ⇒ Invariant points are given as $z = f(z)$

$$\Rightarrow z = -\left(\frac{2z+4i}{iz+1}\right) \Rightarrow iz^2+3z+4i=0$$

$$\Rightarrow z^2-3iz+4=0 \Rightarrow (z-4i)(z+i)=0$$

$$\Rightarrow z=4i, \quad z=-i$$

Hence $z=4i$ and $z=-i$ are two invariant points.