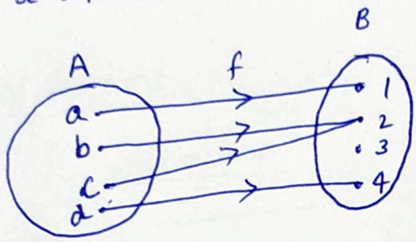
Unit-2 (Functions)

§. <u>Function (Definition)</u>: Let A and B be two non-empty sets. A function f from A to B is a set of ordered pairs $f \subseteq A \times B$ with the property that for each element x in A there is a unique element y in B such that $(x, y) \in f$.

The statement "f" is a function from A to B" is usually represented symbolically by

$$f : A \longrightarrow B$$

A function can be represented pictorially as shown in figure below



It must be noted

(i) that there may be some elements of the set B which are not associated to any element of the set A

(ii) That each element of the set A must be associated to one and only one element of the set B.

If f is a function from A to B, then A is called the domain of f denoted by dom f, its members are the first co-ordinates of the ordered pairs belonging to f and the set B is called the co-domain.

If $(x, y) \in f$, then we write $y = f(x)$, y is called the image of x; and x is a pre-image of y.

y is also called the value of f at x. The set consisting of all the images of the elements of A under the function f is called the range of f. It is denoted by $f(A)$

Thus range of $f = \{f(x) : \text{for all } x \in A\}$

Note that range of f is a subset of B (co-domain) which may or may not be equal to B.

<u>Another definition of function</u>: A function $f: A \to B$ is a rule or formula that assigns to each element x in A a unique element in B, denoted by $f(x)$.

e.g. $f(x) = x^2$ for $x \in R$ represents a function where R is the set of real numbers and $f: R \to R$.

§. <u>Classification of functions:</u>

- Functions
  - (I) Algebraic function
    - (i) Polynomial functions
    - (ii) Rational functions
    - (iii) Irrational functions
  - (II) Transcendental functions
    - (a) Trigonometric functions
    - (b) Inverse trigonometric functions
    - (c) Exponential functions
    - (d) Logarithmic functions

(I) <u>Algebraic function</u>: A function which consists of a finite number of terms involving powers and roots of the independent variable x and the four fundmental operations of addition, subtraction, multiplication and division is called algebraic function.

Three particular cases of algebraic functions are:

(i) <u>Polynomial functions</u>: A function of the form $a_0x^n + a_1x^{n-1} + \dots + a_n$, where n is a positive integer and $a_0, a_1, \dots, a_n$ are real constants and $a_0 \neq 0$ is called a polynomial ~~of x in~~ in x of degree n.

e.g. $f(x) = 5x^3 + 2x^2 + 2x - 5$ is a polynomial of degree 3.

(ii) Rational functions: A function of the form $\frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are polynomials in x, $g(x) \neq 0$ is called a rational function.

e.g. $f(x) = \frac{5x^2+x+1}{x+3}$

(iii) Irrational functions: The functions involving radicals are called irrational functions.

e.g. $f(x) = \sqrt[3]{x}+2$ is an irrational function.

(II) Transcendental function: A function which is not algebraic is called transcendental function.

(a) Trigonometric functions: The six functions $\sin x$, $\cos x$, $\tan x$, $\sec x$, $\operatorname{cosec} x$, $\cot x$ where the angle x is measured in radian are called trigonometric functions.

(b) Inverse trigonometric functions: The six functions $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$, $\sec^{-1}x$, $\operatorname{cosec}^{-1}x$ are called inverse trigonometric functions.

(c) Exponential functions: A function $f(x) = a^x$, $(a>0)$ satisfying the law $a^1 = a$ and $a^x \cdot a^y = a^{x+y}$ is called the exponential function.

(d) Logarithm functions: The inverse of the exponential function is called the logarithm function.

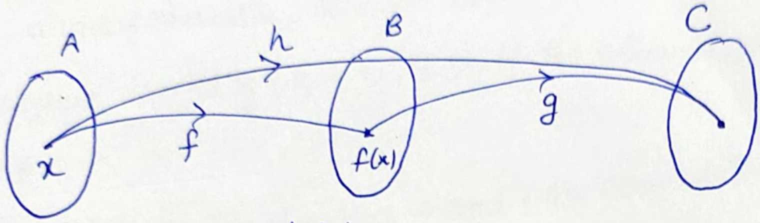
So, if $y = a^x$ $(a>0, x \in R, y>0)$ then $x = \log_a y$ is called Logarithm function.

§. Composition of functions:

Def: Let $f: A \rightarrow B$ and $g: B \rightarrow C$. The composition of f and g, denoted by gof, read as 'g of f' results in a new function from A to C and is given by

$(gof)(x) = g(f(x))$ for all x in A.

Hence, the composition gof first applies f to map A into B and it then employs g to map B to C. In other words, the range space of f becomes the domain space of g.



$(gof)(x) = g(f(x)) = h(x)$

Q1: If $f: R \rightarrow R$ and $g: R \rightarrow R$ are defined by the formulas

$f(x) = x+2$ for all x in R and $g(x) = x^2$ for all x in R.

Then find $(gof)x$ and $(fog)(x)$.

Sol We have $(gof)(x) = g(f(x)) = g(x+2) = (x+2)^2 = x^2+4x+4$

and $(fog)(x) = f(g(x)) = f(x^2) = x^2+2$

Note: $(gof)(x) \neq (fog)(x)$

Thus the composition of functions is not commutative.

§. Associative Law of function composition:

Let $f: A \rightarrow B$, $g: B \rightarrow C$ and $h: C \rightarrow D$. Then $ho(gof) = (hog)of$

Q:1 If $f: R \rightarrow R$ and $g: R \rightarrow R$ defined by $f(x) = x^3 - 4x$, $g(x) = \frac{1}{x^2+1}$, $h(x) = x^4$

Find the following composition functions.

(a) $(fogoh)(x)$ (b) $(hogof)(x)$ (c) $(gog)(x)$ (d) $(goh)(x)$

Unit-2 (Boolean Algebra)

- → Introduction
- → Axioms and Theorems of Boolean algebra
- → Algebraic manipulation of Boolean expressions
- → Simplification of Boolean Functions
- → Karnaugh Maps

Boolean Algebra: (Definition): Let B be a non-empty set with two binary operations $+$ and $\cdot$, a unary operation $'$, and two distinct elements 0 and 1. Then B is called a Boolean algebra, denoted by $(B, +, \cdot, ', 0, 1)$, if the following axioms hold for any $a, b, c \in B$

[B_1] **Commutative laws:** The operations $+$ and $\cdot$ are commutative.

In other words,

$$a+b = b+a \quad \text{and} \quad a \cdot b = b \cdot a \quad \forall a, b \in B$$

[B_2] **Identity laws:** For any $a \in B$,

$$a+0 = a \quad \text{and} \quad a \cdot 1 = a$$

that is both operations $+$ and $\cdot$ have identity elements denoted by 0 and 1 respectively.

[B_3] **Distributive laws:** Each binary operation is distributive over the other

ie. for any $a, b, c \in B$

$$a+(b \cdot c) = (a+b) \cdot (a+c) \quad \text{and} \quad a \cdot (b+c) = (a \cdot b) + (a \cdot c)$$

[B_4] **Complement laws:** For each a in B, there exists an element in B such that

$$a+a' = 1 \quad \text{and} \quad a \cdot a' = 0$$

Note: The elements 0 and 1 are called zero element (identity for $+$) and unit element (identity for $\cdot$) of B respectively while a' is called complement of a in B.

Q1: Show that the power set $P(S)$ of a non empty set S forms a Boolean algebra with the binary operations $\cup$ and $\cap$ and the unary operation $'$ of complementation.

Sol We know that from set theory that

$A \cup B$, $A \cap B$ and $A' \in P(S)$ for $A, B \in P(S)$.

So the closure property holds.

(i) Commutative laws: We know from set theory that

$A \cup B = B \cup A$ and $A \cap B = B \cap A$ $\forall A, B \in P(S)$

Thus commutative laws are satisfied.

(ii) Identity laws: We know that ϕ and S belongs to $P(S)$ such that

$A \cup \phi = A$ and $A \cap S = A$ for any $A \in P(S)$

Thus ϕ and S act as 0 and 1 respectively.

(iii) Distributive laws: From the set theory, we know that

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

for all $A, B, C \in P(S)$

Hence both operations $\cup$ and $\cap$ are distributive over each other.

(iv) Complement laws: For any $A \in P(S)$, $S - A \in P(S)$ such that

$A \cup (S - A) = S$ and $A \cap (S - A) = \phi$

Thus every element A in $P(S)$ has complement $A' = S - A$ in $P(S)$

Hence $(P(S), \cup, \cap, ', \phi, S)$ is a Boolean algebra.

Q2: Let $B = (1, 5, 7, 35)$ be the set of positive integers and operations $+$ and $\bullet$ are defined as follows:

$a + b = lcm(a, b)$, $a \bullet b = gcd(a, b)$ $\forall a, b \in B$.

A unary operation $'$ on B is defined as $a' = \frac{35}{a}$ $\forall a \in B$

Show that $(B, +, \bullet, ')$ is a Boolean algebra.

Sol Given $B=(1,5,7,35)$, $a+b=\text{lcm}(a,b)$, $\gcd(a,b)=a\cdot b$ and $a'=\frac{35}{a}$

The composition tables of binary operations are given by

'+'	1	5	7	35
1	1	5	7	35
5	5	5	35	35
7	7	35	7	35
35	35	35	35	35

•	1	5	7	35
1	1	1	1	1
5	1	5	1	5
7	1	1	7	7
35	1	5	7	35

a	1	5	7	35
$a'=\frac{35}{a}$	35	7	5	1

From the composition table we see that all the entries in the tables are elements of the set B. Therefore both + and • are binary operations on B and ′ is a unary operation on B.

(i) Commutative laws: Since the composition tables for + and • are symmetrical with respect to main diagonals, therefore operations + and • are commutative.

(ii) Identity laws: From the composition tables we see that

$a+1=a$ and $a\cdot 35=a \quad \forall a\in B$

a	1	$a+1$
1	1	1
5	1	5
7	1	7
35	1	35

a	35	$a\cdot 35$
1	35	1
5	35	5
7	35	7
35	35	35

(iii) Distributive laws: With the help of composition tables for + and • it can be verified that $a+(b\cdot c)=(a+b)\cdot(a+c)$ $\quad \forall a,b,c\in B$

and $a\cdot(b+c)=(a\cdot b)+(a\cdot c)$

For arbitrary elements, $a=7$, $b=5$, $c=1$

we have $a+(b\cdot c)=7+(5\cdot 1)=7+1=7$

$(a+b)\cdot(a+c)=(7+5)\cdot(7+1)=35\cdot 7=7$

Also $a\cdot(b+c)=7\cdot(5+1)=7\cdot 5=1$

$(a\cdot b)+(a\cdot c)=(7\cdot 5)+(7\cdot 1)=1+1=1$

(iv) <u>Complement laws:</u> For each $a \in B$, there exists $a' = \frac{35}{a}$ in B such that

$a + a' = 35$ and $a \cdot a' = 1$

Thus complement of every element exists in B.

a	a'	$a+a'$
1	35	35
5	7	35
7	5	35
35	1	35

a	a'	$a \cdot a'$
1	35	1
5	7	1
7	5	1
35	1	1

Hence $(B, +, \cdot, ', 1, 35)$ is a Boolean algebra.

Questions for practice

<u>Q1:</u> If $B = \{1, 3, 5, 15\}$, then show that $(B, +, \cdot, ')$ is a Boolean algebra, where $a + b = lcm(a, b)$, $a \cdot b = gcd(a, b)$ and $a' = \frac{15}{a}$

<u>Q2:</u> Let B be a set of positive integers being divisors of 30 and operations $\vee, \wedge$ on it defined as

$a \vee b = c$, where c is the l.c.m of a and b

$a \wedge b = d$, where d is the g.c.d of a, b

$a' = \frac{30}{a}$

then show that B is a Boolean algebra.

§. Basic theorems: The following holds in a Boolean algebra B.

1. Idempotent laws:

$a+a=a$ and $a\cdot a=a$ $\forall a\in B$

Proof: To prove $a+a=a$

we have $a=a+0$ by Identity law
$=a+a\cdot a'$ by complement law
$=(a+a)\cdot(a+a')$ by Distributive law
$=(a+a)\cdot 1$ by complement law
$=a+a$ by identity law

Hence $\boxed{a+a=a}$

To prove $a\cdot a=a$

we have $a=a\cdot 1$ by identity law
$=a\cdot(a+a')$ by complement law
$=a\cdot a+a\cdot a'$ by Distributive law
$=a\cdot a+0$ by complement law
$a=a\cdot a$ by identity law

Hence $\boxed{a\cdot a=a}$

[2] Boundedness laws:

$a+1=1$ and $a\cdot 0=0$ $\forall a\in B$

Proof: we have $1=a+a'$ by complement law
$=a+a'\cdot 1$ by identity law ($\because a'\cdot 1=a'$)
$=(a+a')\cdot(a+1)$ by Distributive law
$=1\cdot(a+1)$ by complement law
$=a+1$ by identity law

Thus $\boxed{a+1=1}$

To show $a\cdot 0=0$

we have $0=a\cdot a'$ by complement law
$=a\cdot(0+a')$ by identity law
$=a\cdot 0+a\cdot a'$ by Distributive law
$=a\cdot 0+0$ by complement law
$\boxed{0=a\cdot 0}$ by identy law

[3] Absorption laws:

$a+(a \cdot b)=a$ and $a \cdot (a+b)=a$ $\forall\, a, b \in B$

Proof: To prove $a+(a \cdot b)=a$

We have
$$\begin{aligned}
a &= a \cdot 1 && \text{by identity law}\\
&= a \cdot (1+b) && \text{by boundedness law}\\
&= a \cdot 1 + a \cdot b && \text{by Distributive law}\\
&= a + a \cdot b && \text{by idenity law}
\end{aligned}$$

Thus $\boxed{a+a \cdot b=a}$

To prove $a \cdot (a+b)=a$

We have
$$\begin{aligned}
a &= a+0 && \text{by identity law}\\
&= a+(0 \cdot b) && \text{by boundedness law}\\
&= (a+0) \cdot (a+b) && \text{by Distributive law}\\
a &= a \cdot (a+b) && \text{by identity law}
\end{aligned}$$

Thus $\boxed{a \cdot (a+b)=a}$

[4] Associative laws: $(a+b)+c = a+(b+c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ $\forall\, a, b, c \in B$.

For any element a in a Boolean algebra B, $(a')'=a$

[5] Involution law: For any element a in a Boolean algebra B, a' is unique.

Theorem: For each element a in a Boolean algebra B is unique.

In other words, complement of an element a in Boolean algebra B. If possible suppose ~~that~~ x and

Proof: Let a be any element in a Boolean algebra B.

y be two complements of a in B. Then

$a+x=1$, $a \cdot x=0$

and $a+y=1$, $a \cdot y=0$

Now, we have
$$\begin{aligned}
x &= x \cdot 1 && \text{by identity law}\\
&= x \cdot (a+y) && \text{by assumption}\\
&= x \cdot a + x \cdot y && \text{by Distributive law}\\
&= 0 + x \cdot y && \text{by assumption}\\
&= x \cdot y && \text{by identity law}\\
&= x \cdot y + 0 && \text{by Identity law}\\
&= x \cdot y + a \cdot y && \text{by assumption}\\
&= (x+a) \cdot y && \text{by distributive law}\\
&= 1 \cdot y && \text{by assumption}
\end{aligned}$$

$\boxed{x=y}$ Thus complement of a is unique.

Note: In any Boolean algebra, $0'=1$ and $1'=0$

Theorem: (De Morgan's laws): For every pair of elements a and b in a Boolean algebra B, show that

$(a+b)' = a' \cdot b'$ and $(a \cdot b)' = a' + b'$

Proof: (i) In order to prove that $(a+b)' = a' \cdot b'$, we must show

$(a+b) + (a' \cdot b') = 1$ and $(a+b) \cdot (a' \cdot b') = 0$

we have $(a+b) + (a' \cdot b') = [(a+b) + a'] \cdot [(a+b) + b']$ by Distributive law

$= [(b+a)+a'] \cdot [a + (b+b')]$ by associative law

$= [b + (a+a')] \cdot [a + (b+b')]$

$= (b+1) \cdot (a+1)$ by complement law

$= 1 \cdot 1$ by boundedness law

$= 1$

$\therefore \boxed{(a+b) + (a' \cdot b') = 1}$

we have $(a+b) \cdot (a' \cdot b') = a \cdot (a' \cdot b') + b \cdot (a' \cdot b')$ by Distributive law

$= (a \cdot a') \cdot b' + b \cdot (b' \cdot a')$ by Associative law

$= 0 \cdot b' + (b \cdot b') \cdot a'$ by complement & Associative laws

$= 0 + 0 \cdot a'$ by complement law

$= 0 + 0$

$\boxed{(a+b) \cdot (a' \cdot b') = 0}$

Thus $(a+b)' = a' \cdot b'$

(ii) In order to prove that $(a \cdot b)' = a' + b'$, we must show that

$(a \cdot b) + (a' + b') = 1$ and $(a \cdot b) \cdot (a' + b') = 0$

we have $(a \cdot b) + (a' + b') = [a + (a' + b')] \cdot [b + (a' + b')]$ by Distributive law

$= [(a+a') + b'] \cdot [b + (b' + a')]$ by Associative & commutative law

$= (1 + b') \cdot [(b+b') + a']$ by complement & Associative law

$= (1+b') \cdot (1 + a')$ by boundedness law

$= 1 \cdot 1$ by boundedness law

$= 1$

$\therefore \boxed{(a \cdot b) + (a' + b') = 1}$

Now, we have

$$
\begin{aligned}
(a\cdot b)\cdot(a'+b') &= (a\cdot b)\cdot a' + (a\cdot b)\cdot b' && \text{by Distributive law}\\
&= (b\cdot a)\cdot a' + (a\cdot b)\cdot b' && \text{by commutative law}\\
&= b\cdot(a\cdot a') + a\cdot(b\cdot b') && \text{by associative law}\\
&= b\cdot 0 + a\cdot 0 && \text{by complement law}\\
&= 0 + 0 && \text{by boundedness law}\\
&= 0
\end{aligned}
$$

Hence $(a\cdot b)' = a' + b'$

§. Boolean Function: A Boolean function or Boolean polynomial is an expression derived from a finite number of applications of the operations $+$, $\cdot$ and $'$ to the elements of a Boolean algebra.

Examples: Expressions such as ab, $(a+b)'+ab'$ and $a'+b'$ are Boolean functions.

§. Sum of Products and Product of sums form:

SOP: A Boolean expression E is said to be in a sum of products if E is a sum of two or more product of variables (complemented or uncomplemented), none of which is included in another.

Examples: $E_1 = xz' + x'yz' + xy'z$

$E_2 = abc + ac + a'bc'$

§. Product of sums form (POS)

A Boolean expression E is said to be in a product of sums if it consists of several sum terms logically multiplied. The variables may or may not be complemented.

Examples: $E_1 = (x+y)(x'+y)$

$E_2 = (x+y'+z)(x+z)$

$E_3 = (a+b')(c'+d)$

§. Minterm: A minterm of n variables is a product of n literals in which each variable appears exactly once in either true or complemented form, but not both.

Examples: (i) The list of all of the minterms of the two variables x and y are

$xy, x'y, xy', x'y'$

(ii) The list of all of the minterms of three variables x, y and z are

$xyz, xyz', xy'z, x'yz, xy'z', x'y'z, x'yz', x'y'z'$

In a similar way, n variables can be combined to form 2^n minterms.

§. Maxterm: A maxterm of n variables is a sum of n literals in which each variable appears exactly once in either true or complemented form, but not both.

Examples: (i) All maxterm of two variables x and y are

$x+y$, $x'+y$, $x+y'$, $x'+y'$

(ii) all max terms of three variables x, y and z are $x+y'+z'$, $x+y'+z$, $x+y+z'$, $x'+y+z$, $x+y+z$, $x'+y'+z'$, $x'+y'+z$, $x'+y+z'$, $x+y'+z'$

In a similar manner, n variables forming maxterms, with each variable being complemented or uncomplemented, provide 2^n possible combinations.

§. Normal Form

(I) Disjunctive Normal Form (DNF)

(II) Conjunctive Normal Form (CNF)

(I) Disjunctive Normal Form (DNF): When a Boolean function f is written as a sum of minterms, it is referred to as a minterm expansion or the ~~conjunctive~~ disjunctive normal form of the Boolean ~~algebra~~ function. It is also called canonical sum of products or standard sum of Products.

Examples: (i) $f(x,y) = xy + x'y$

(ii) $f(x,y,z) = x'y'z + x'yz + xy'z' + xyz$

(II) Conjunctive Normal Form (CNF): When a Boolean function f is written as a product of maxterms, it is referred to as a maxterm expansion or the conjunctive normal form of the Boolean function. It is also called canonical product of sums and standard product of sums.

Examples: (i) $f(x,y) = (x+y)(x'+y)$

(ii) $f(x,y,z) = (x+y+z)(x+y'+z)(x'+y+z')(x'+y'+z)$

Note(i) Boolean functions expressed as a sum of minterms or product of maxterms are said to be in canonical form.

(ii) A Boolean function when expressed as a sum of all 2^n minterms of n variables is called complete ~~nos~~ minterm expansion or the complete disjunctive normal form.

The complete minterm expansion in two variables x and y is

$$xy + xy' + x'y + x'y'$$

(iii) Similarly, a Boolean function when expressed as a product of all 2^n maxterms of n variables is called the complete maxterm expansion or complete conjunctive normal form.

*****

§. Expression of a Boolean function as a Canonical form: (Algebraic Method)

(I) To obtain the minterm expression by algebraic method, first write the expression as a sum of products and then introduce the missing variable in each term by applying the theorem $a + a' = 1$.

(II) For finding maxterm expansion, factor the function to obtain a product of sums, introduce the missing variable in each sum term by using $aa' = 0$ and then factor again to obtain the maxterms.

Q1: Express the Boolean function $f(x, y, z) = x + y'z$ in a sum of minterms.

Sol We have,

$$f(x, y, z) = x + y'z \qquad [\because y + y' = 1,\ z + z' = 1,\ x + x' = 1]$$

$$= x(y + y')(z + z') + (x + x')y'z$$

$$= x(yz + yz' + y'z + y'z') + xy'z + x'y'z$$

$$= xyz + xyz' + xy'z + xy'z' + xy'z + x'y'z \qquad [a + a = a]$$

$$[\because xy'z + xy'z = xy'z]$$

$$\boxed{f(x, y, z) = xyz + xyz' + xy'z + xy'z' + x'y'z}$$

Q2: Express the Boolean function $f(a,b,c) = ab + a'c$ as a product of maxterms.

Sol. We have

$$f(a,b,c) = ab + a'c$$
$$= (ab + a')(ab + c)$$
$$= (a + a')(b + a')(a + c)(b + c)$$
$$= (b + a')(a + c)(b + c) \quad (\because a + a' = 1)$$
$$= (a' + b + cc')(a + c + bb')(b + c + aa')$$
$$= (a' + b + c)(a' + b + c')(a + b + c)(a + b' + c)(a + b + c)(a' + b + c) \quad [\because aa = a]$$

$$\boxed{f(a,b,c) = (a+b+c)(a+b'+c)(a'+b+c)(a'+b+c')}$$

Simplification of Boolean expression by Algebraic Method

Basically simplification of Boolean expression mean elimination of terms and literals which can be done by combining terms, deliberate introduction of redundant terms like, adding xx', multiplying by $(x+x')$, adding yz to $xy + x'z$ or adding xy to x in the original expression.

Q1: Simplify $f(x,y) = x'y + xy' + xy$ or $f(x,y) = xy' + xy + x'y$

Sol. We have

$$f(x,y) = xy' + xy + x'y$$
$$= x(y' + y) + x'y$$
$$= x + x'y$$
$$= (x + x')(x + y)$$

$$\boxed{f(x,y) = x + y}$$

Q2: Simplify the following expression using Boolean algebra:

(i) $(AB'C' + AB'C + ABC + ABC')(A+B)$

(ii) $P + P'QR' + (Q+R)'$

Sol. Let $f(A,B,C) = (AB'C' + AB'C + ABC + ABC')(A+B)$
$$= [AB'(C' + C) + AB(C' + C)](A+B)$$
$$= (AB' + AB)(A+B)$$
$$= A(B' + B)(A+B)$$
$$= A(A+B) = A + AB = A(1+B) = A \cdot 1 = A$$

(ii) Let $f(P,Q,R) = P + P'QR' + (Q+R)'$

$= P + P'QR' + Q'R'$ (By De Morgan's law)

$= P + (P'Q + Q')R'$

$= P + (P'+Q')\cdot(Q+Q')R'$

$= P + (P'+Q')R'$ $(Q+Q'=1)$

$= P + P'R' + Q'R'$

$= (P+P')\cdot(P+R') + Q'R'$

$= P + R' + Q'R'$

$= P + R'(1+Q')$

$\boxed{f(P,Q,R) = P+R'}$

Q3: Simplify the Boolean expression:

*** (a) $XY + X'Z + YZ$

(b) $C(B+C)(A+B+C)$ Ans: C

(c) $A + B(A+B) + A(A'+B)$ Ans: $A+B$

Sol

(a) Let $f(X,Y,Z) = XY + X'Z + YZ$ **** $(\because X+X'=1)$

$= XY + X'Z + (X+X')YZ$

$= XY + X'Z + XYZ + X'YZ$

$= XY(1+Z) + X'Z(1+Y)$

$= XY + X'Z$

which is the simplified form.

§. Conversion of Disjunctive Normal form (DNF) of a Boolean function to its conjunctive normal form (CNF) and vice-versa.

(I) Conversion of DNF to CNF

The following steps can be followed to convert a Boolean function given in DNF to CNF.

S-1: Denote the given function which is in DNF by f.

S-2: Find f', which is the sum of those terms of the complete DNF in the variables of f, that are not present in the given function.

S-3: Take complement f'' of f'.

S-4. Simplify RHS of f'' by using De Morgan's laws. This will transform the R.H.S. into its CNF.

S-5. Since $f''=f$, the RH.S. is the required CNF of the given function in DNF.

(II) Conversion of CNF to DNF:

S-1. Denote the given function which is in CNF by f.

S-2. Find f', which is the product of those terms of the complete CNF in the variables of f, that are not present in the given function.

S-3. Take complement f'' of f'.

S-4. Simplify the RH.S. of f' by using De Morgan's laws. This will transform the RH.S. into its DNF.

S-5. Since $f''=f$, the RHS is the required DNF of the given function in CNF.

Q1: Convert the Boolean function $f(x,y)=xy'+x'y+x'y'$ to its conjunctive normal form.

Sol The complete DNF in two variables x and y is $= xy + xy' + x'y + x'y'$

Missing term

Given $f(x,y)=xy'+x'y+x'y'$

$\therefore\ f'(x,y)=xy$

$\Rightarrow f''(x,y)=(xy)' = x'+y'$ (By De Morgan's law)

$\Rightarrow \boxed{f(x,y)=x'+y'}$ $(\because f''=f)$

Which is the required CNF of given function.

Method-2

$f(x,y)=xy'+x'y+x'y'$

$=xy'+x'(y+y')$

$=xy'+x'$

$=(x+x')(x'+y')$

$\boxed{f(x,y)=x'+y'}$

****

Q2: In the Boolean algebra $(B, +, \cdot, ')$ express the Boolean function $f(x,y,z) = (xy' + xz)' + z'$ its conjunctive normal form.

Sol Given $f(x,y,z) = (xy' + xz)' + z'$

$$= (xy')' \cdot (xz)' + z' \quad \text{[By De Morgan's laws]}$$
$$= (x'+y) \cdot (x'+z') + z'$$
$$= (x'+y+z') \cdot (x'+z'+z') \quad \text{[by Distributive law]}$$
$$= (x'+y+z') \cdot (x'+z') \quad (\because a+a=a)$$
$$= (x'+y+z')(x'+z'+yy') \quad (yy'=0)$$
$$= (x'+y+z')(x'+z'+y)(x'+z'+y')$$
$$\boxed{f(x,y,z) = (x'+y+z')(x'+y+z')} \quad [\because aa=a]$$

Which is the required CNF of given function.

Q3: In the Boolean algebra $(B, +, \cdot, ')$ express the Boolean function $f(x,y,z) = x(y'z)'$ in its disjunctive normal form.

Sol Given $f(x,y,z) = x(y'z)'$

$$= x(y+z') \quad \text{(By De Morgan's law)}$$
$$= xy + xz'$$
$$= xy(z+z') + xz'(y+y')$$
$$= xyz + \underline{xyz'} + \underline{xyz'} + xy'z'$$
$$\boxed{f(x,y,z) = xyz + xyz' + xy'z'} \quad (\because a+a=a)$$

Which is the required DNF.

Q4 Convert the Boolean function $f(x,y,z) = (x'+y+z')(x'+y+z)(x+y'+z)$ in disjunctive normal form.

Sol $f(x,y,z) = \underline{(x'+y+z')(x'+y+z)}(x+y'+z)$

$$= (x'+y+zz')(x+y'+z) \quad (\because zz'=0)$$
$$= (x'+y)(x+y'+z)$$
$$= x'y' + x'z + xy + yz \quad (\because xx'=0,\ yy'=0)$$
$$= x'y'(z+z') + x'(y+y')z + xy(z+z') + (x+x')yz$$
$$= \underline{x'y'z} + x'y'z' + \underline{\underline{x'yz}} + \underline{x'y'z} + \underline{xyz} + xyz' + \underline{xyz} + \underline{\underline{x'yz}} \quad [\because a+a=a]$$
$$= x'y'z + x'y'z' + x'yz + xyz' + xyz \quad \underline{\text{Ans}}$$

Another Method:

Given $f(x,y,z) = (x'+y+z')(x'+y+z)(x+y'+z)$

The complete CNF in three variables x, y and z

$$= (x+y+z)(x+y+z')(x+y'+z)(x'+y+z)(x+y'+z')(x'+y+z')(x'+y'+z)(x'+y'+z')$$

$$\therefore f'(x,y,z) = (x+y+z)(x+y+z')(x+y'+z')(x'+y'+z)(x'+y'+z')$$

$$\Rightarrow [f'(x,y,z)]' = [(x+y+z)(x+y+z')(x+y'+z')(x'+y'+z)(x'+y'+z')]'$$

$$\boxed{f = x'y'z' + x'y'z + x'yz + xyz' + xyz}$$ Ans

Q:-5 In the Boolean algebra $(B, +, \cdot, ')$ express the Boolean function

$f(x,y,z) = (x+y)(x+z') + y + z'$ in its disjunctive normal form.

[Ans: $f(x,y,z) = xyz + xy'z + xyz' + x'yz + x'yz' + xy'z' + x'yz'$]