KIET GROUP OF INSTITUTIONS, DELHI NCR, GHAZIABAD

B. Tech. (I Sem.), 2024-25

CALCULUS FOR ENGINEERS (K24AS11)

UNIT-5 (Vector Differentiation)

SYLLABUS: Scalar point function, Vector point function, Gradient of a scalar field, Directional derivatives, Application of divergence, Curl to solenoidal and irrotational vectors respectively.

Course Outcome:

Apply the concept of vector differentials to study the properties of point functions.

Application in Engineering:

Vector differentiation is a powerful mathematical tool that provides insights into dynamic systems, enabling predictions and optimizations across various disciplines. Vector differentiation has a wide range of applications in various fields of science, engineering, and mathematics. Here, some applications are given in real-life scenarios:

1. **Physics**:

   Motion Analysis: In physics, vector differentiation is used to analyse the motion of objects. The position vector of an object can be differentiated with respect to time to obtain the velocity vector and differentiating the velocity vector gives the acceleration vector. This is fundamental in kinematics.
2. **Electromagnetism**: The behaviour of electric and magnetic fields can be described using vector calculus. For example, Maxwell's equations, which govern electromagnetism, involve vector differentiation.
3. **Engineering**:

   Fluid Dynamics: In fluid mechanics, the velocity field of a fluid can be described using vector functions. Differentiating these functions helps in understanding the flow characteristics, such as calculating the rate of change of velocity (acceleration) or analysing flow patterns.
4. **Structural Analysis**: Engineers use vector differentiation to analyse forces and moments acting on structures. This is crucial for determining stress, strain, and stability.

5. **Computer Graphics**:

   Animation and Motion: In computer graphics, vector differentiation is used to calculate the motion of objects. For instance, differentiating position vectors over time helps in determining velocity and acceleration for realistic animations.

   Surface Rendering: Techniques like normal mapping rely on vector differentiation to calculate surface normal, which are essential for rendering light reflections and shading.

6. **Robotics**:

   Path Planning: In robotics, vector differentiation is used to compute trajectories and optimize the movement of robots. By analysing the velocity and acceleration vectors, robots can navigate efficiently in their environment.

7. **Economics and Social Sciences**:

   Modelling Change: In economics, vector differentiation can be used to model changes in multi-dimensional systems, such as how various economic indicators change over time in response to different factors.

8. **Machine Learning**:

   Gradient Descent: In optimization problems, particularly in training machine learning models, vector differentiation is used to calculate gradients. This helps in minimizing loss functions by updating model parameters in the direction of steepest descent.

9. **Geophysics**:

   Seismic Analysis: Vector differentiation is used in geophysics to analyse seismic waves. Understanding how these waves propagate through different materials involves differentiating vector fields representing displacement.

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Vector Differentiation

Introduction: Vector calculus or vector analysis is concerned with differentiation and integration of vector fields. It is used extensively in physics and engineering, especially in the description of electromagnetic fields, gravitational fields and fluid flow.

1. **Point Function:** A variable quantity whose value at any point in a region of space depends upon the position of the point, is called a point function.

1.1 Scalar Point Function: If to each point $P\ (x, y, z)$ of a region R in space there corresponds a unique scalar $f(P)$, then f is called a scalar point function.

Examples.

(i) Temperature distribution in a heated body,

(ii) Density of a body & **(iii)** Potential due to gravity.

1.2 Vector Point Function: If to each point $P\ (x, y, z)$ of a region R in space there corresponds a unique vector $f(P)$, then f is called a vector point function.

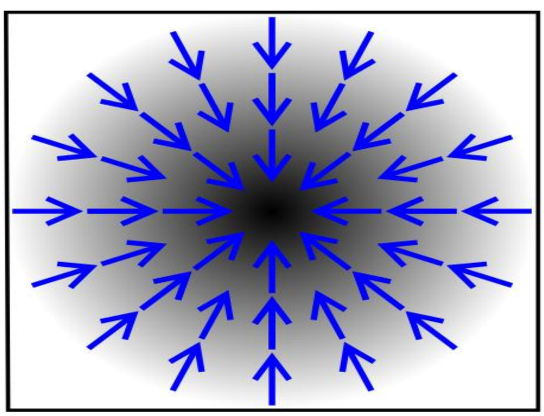
Examples.

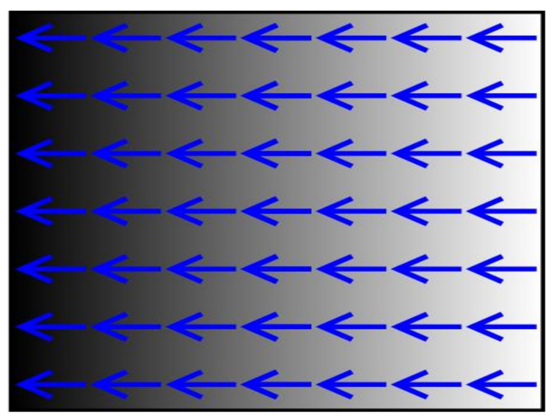
(i) Forest wind **(ii)** The velocity of a moving fluid **(iii)** Gravitational force.

2. Gradient of a scalar point function

The gradient is closely related to the derivative, but it is not itself a derivative.

The gradient can be interpreted as the "direction and rate of fastest increase"





Vector Differential Operator Del (∇): It is defined as:

$$\nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Gradient of a scalar function:

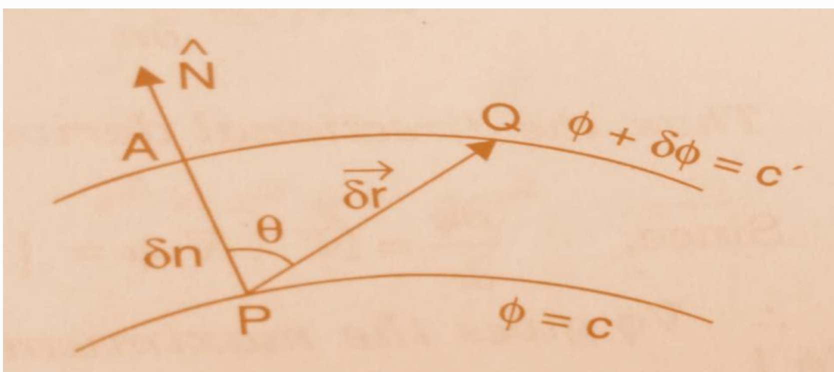
Let $\emptyset(x, y, z)$ be a scalar function, then the vector $\hat{\imath}\frac{\partial\emptyset}{\partial x} + \hat{\jmath}\frac{\partial\emptyset}{\partial y} + \hat{k}\frac{\partial\emptyset}{\partial z}$ is called the gradient of a scalar function $\emptyset$. Thus, $grad\ \emptyset = \nabla\emptyset$.

2.1 Geometrical Interpretation of Gradient: If a surface $\emptyset(x, y, z) = c$, passes through a point P. The value of function at each point of the surface is the same as at P. Then such a surface is called a *level surface* through P.

Example. If $\emptyset(x, y, z)$ represents potential at point P. Then equipotential surface $\emptyset(x, y, z) = c$, is a level surface.

Note: Two level surfaces can't intersect.

Let the level surface pass through P at which the value of function is $\emptyset$.



Consider another level surface passing through Q, Where the value of function $\emptyset + d\emptyset$.

Let $\vec{r}\ and\ \vec{r} + \delta\vec{r}$ be the position vector of P and Q then $\overrightarrow{PQ} = \delta\vec{r}$

$$\nabla\phi.d\vec{r} = \left(\hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right).\left(\hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz\right)$$

$$= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\phi \text{-------- (1)}$$

If Q lies on the level surface of P, then $d\phi = 0$

From equation (1), we get

$\nabla\phi.d\vec{r} = 0$, then $\nabla\phi \perp to\ d\vec{r}\ (tangent)$

Hence $\nabla\phi$ is the **Normal** to the surface $\phi(x, y, z) = c$

2.2 Properties of gradient

(1) If ϕ is a constant scalar point function, then $\nabla\phi = \vec{0}$

(2) If ϕ_1 and ϕ_2 are two scalar point functions, then

(a) $\nabla(\phi_1 \pm \phi_2) = \nabla\phi_1 \pm \nabla\phi_2$

(b) $\nabla(c_1\phi_1 \pm c_2\phi_2) = c_1\nabla\phi_1 \pm c_2\nabla\phi_2$, where c_1& c_2 are constants.

(c) $\nabla(\phi_1\phi_2) = \phi_1\nabla\phi_2 + \phi_2\nabla\phi_1$

(d) $\nabla\left(\frac{\phi_1}{\phi_2}\right) = \frac{\phi_2\nabla\phi_1 - \phi_1\nabla\phi_2}{{\phi_2}^2}$, $\phi_2 \neq 0$

Example 1. Find *grad ø*, when ø is given by $\emptyset = 3x^2y - y^3z^2$ at the point $(1,-2,-1)$.

Solution. Here $\phi = 3x^2y - y^3z^2$

$$\therefore\ grad\phi = \nabla\phi = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(3x^2y - y^3z^2)$$

$$= \hat{\imath}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \hat{\jmath}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \hat{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$= \hat{\imath}(6xy) + \hat{\jmath}(3x^2 - 3y^2z^2) + \hat{k}(-2y^3z)$$

$$grad\phi]_{(1,-2,-1)} = -12\hat{\imath} - 9\hat{\jmath} - 16\hat{k}$$

Example 2. What is the greatest rate of increase of $u = xyz^2$ at the point (1,0,3) ?

Solution. Here, we have

$$grad\ u = \nabla u = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(xyz^2) = yz^2\hat{\imath} + xz^2\hat{\jmath} + 2xyz\hat{k}$$

Now, $grad\ u]_{(1,0,3)} = 9\hat{\jmath}$

The greatest rate of increase of u at the point $(1,0,3) = |\nabla u| = |9\hat{\jmath}| = 9$

Example 3. Find a unit vector normal to the surface $yx^2 + 2xz = 4$ at the point $(2,-2,3)$.

Solution. Let $\emptyset = yx^2 + 2xz - 4$

$$grad\ \emptyset = \nabla\emptyset = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(yx^2 + 2xz - 4)$$

$$= (2xy + 2z)\hat{\imath} + x^2\hat{\jmath} + 2x\hat{k}$$

Now, $grad\ \emptyset]_{(2,-2,3)} = -2\hat{\imath} + 4\hat{\jmath} + 4\hat{k}$ and $|grad\ \emptyset| = \sqrt{4 + 16 + 16} = 6$

The unit vector normal to the surface $\emptyset = \frac{grad\ \emptyset}{|grad\ \emptyset|}$

$$= \frac{-2\hat{\imath} + 4\hat{\jmath} + 4\hat{k}}{6} = \frac{-\hat{\imath} + 2\hat{\jmath} + 2\hat{k}}{3}.$$

Example 4. If $\nabla\emptyset = (y^2 - 2xyz^3)\hat{\imath} + (3 + 2xy - x^2z^3)\hat{\jmath} + (6z^3 - 3x^2yz^2)\hat{k}$, find $\emptyset$.

Solution. Let $\vec{F} = \nabla\emptyset \Rightarrow \vec{F}.d\vec{r} = \nabla\emptyset.d\vec{r}$ [taking dot product with $d\vec{r}$]

$$\vec{F}.d\vec{r} = \left(\hat{\imath}\frac{\partial\emptyset}{\partial x} + \hat{\jmath}\frac{\partial\emptyset}{\partial y} + \hat{k}\frac{\partial\emptyset}{\partial z}\right).\left(dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}\right) = \frac{\partial\emptyset}{\partial x}dx + \frac{\partial\emptyset}{\partial y}dy + \frac{\partial\emptyset}{\partial z}dz$$

$$\Rightarrow \vec{F}.d\vec{r} = d\emptyset \qquad \left[as\ \frac{\partial\emptyset}{\partial x}dx + \frac{\partial\emptyset}{\partial y}dy + \frac{\partial\emptyset}{\partial z}dz = d\emptyset\right]$$

Or $d\emptyset = \vec{F}.d\vec{r} = \nabla\emptyset.d\vec{r}$

$$d\emptyset = \left[(y^2 - 2xyz^3)\hat{\imath} + (3 + 2xy - x^2z^3)\hat{\jmath} + (6z^3 - 3x^2yz^2)\hat{k}\right].\left(dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}\right)$$

$$= (y^2 - 2xyz^3)dx + (3 + 2xy - x^2z^3)dy + (6z^3 - 3x^2yz^2)dz \quad \text{-----------------} \ (1)$$

$$= 3dy + 6z^3dz + y^2dx + 2xydy - 2xyz^3dx - x^2z^3dy - 3x^2yz^2dz$$

$$d\emptyset = 3dy + 6z^3dz + d(xy^2) - d(x^2yz^3)$$

Integrate, we get $\emptyset = 3y + \frac{3}{2}z^4 + xy^2 - x^2yz^3 + C$

Example 5. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$.

Solution. Let $\emptyset_1 \equiv x^2 + y^2 + z^2 - 9 = 0$ and $\emptyset_2 \equiv x^2 + y^2 - 3 - z = 0$

then $\nabla\emptyset_1 = 2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k}$ and $\nabla\emptyset_2 = 2x\hat{\imath} + 2y\hat{\jmath} - \hat{k}$

Let $\quad \overrightarrow{n_1} = \nabla\emptyset_1]_{(2,-1,2)} = 4\hat{\imath} - 2\hat{\jmath} + 4\hat{k}$

and $\quad \overrightarrow{n_2} = \nabla\emptyset_2]_{(2,-1,2)} = 4\hat{\imath} - 2\hat{\jmath} - \hat{k}$

Let θ is the angle between the vectors $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$ which are normal to the given surfaces $\emptyset_1$ and $\emptyset_2$ respectively. Then

$$\cos\theta = \frac{\overrightarrow{n_1}\,.\,\overrightarrow{n_2}}{|\overrightarrow{n_1}||\overrightarrow{n_2}|} = \frac{4.4 + (-2).(-2) + 4.(-1)}{\sqrt{16+4+16}\,\sqrt{16+4+1}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\therefore \boldsymbol{\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)}$$

Example 6. If $u = x + y + z,\ v = x^2 + y^2 + z^2,\ w = yz + zx + xy$, prove that $grad\ u,\ grad\ v\ and\ grad\ w\ are\ coplanar\ vectors.$

Solution. Here, we have

$$grad\ u = \nabla u = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x + y + z) = \hat{\imath} + \hat{\jmath} + \hat{k}$$

$$grad\ v = \nabla v = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2) = 2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k}$$

$$grad\ w = \nabla w = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(yz + zx + xy)$$

$$= (z + y)\hat{\imath} + (z + x)\hat{\jmath} + (y + x)\hat{k}$$

Now, $grad\, u\,.(grad\, v\, \times grad\, w) = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ z+y & z+x & y+x \end{vmatrix} = 0$

Hence, $grad\, u,\ grad\, v\ and\ grad\, w\ are\ coplanar\ vectors$ **.**

Exercise

1. Show that $\nabla(\vec{a}.\vec{r}) = \vec{a}$ where $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ and $\vec{a}$ is a constant vector.

2. Find a unit vector normal to the surface $x^3 + y^3 + 3xyz = 3$ at the point $(1,2,-1)$.

3. Find a unit vector normal to the surface $xy^3z^2 = 4$ at the point $(-1,-1,2)$.

4. Find a unit vector normal to the surface $z^2 = 4(x^2 + y^2)$ at the point $(1,0,2)$.

5. Calculate the angle between the normal to the surface $xy = z^2$ at the points $(4,1,2)\ and\ (3,3,-3)$.

Answers

2. $-\frac{1}{\sqrt{14}}(\hat{\imath} - 3\hat{\jmath} - 2\hat{k})$ **3.** $-\frac{1}{\sqrt{11}}(\hat{\imath} + 3\hat{\jmath} - \hat{k})$ **4.** $\frac{1}{\sqrt{5}}(2\hat{\imath} - \hat{k})$ **5.** $\cos^{-1}\left(-\frac{1}{\sqrt{22}}\right)$

3. Directional derivative

Directional derivative of a scalar field f at a point $P(x, y, z)$ in the direction of unit vector $\hat{a}$ is given by $\frac{df}{ds} = (grad\, f).\hat{a}$

Example 1: Find the directional derivative of $\emptyset = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$ at the point $P(3,1,2)$ in the direction of the vector $yz\hat{\imath} + zx\hat{\jmath} + xy\hat{k}$.

Solution. Here, $\emptyset = (x^2 + y^2 + z^2)^{-\frac{1}{2}} \Rightarrow grad\, \emptyset = -\frac{x\hat{\imath}+y\hat{\jmath}+z\hat{k}}{(x^2+y^2+z^2)^{\frac{3}{2}}}$

Let $\hat{a}$ be the unit in the given direction, then $\hat{a} = \frac{yz\hat{\imath}+zx\hat{\jmath}+xy\hat{k}}{\sqrt{x^2+y^2+z^2}}$

$grad\, \emptyset]_{(3,1,2)} = -\frac{3\hat{\imath}+\hat{\jmath}+2\hat{k}}{14\sqrt{14}}$ and $\hat{a}]_{(3,1,2)} = \frac{2\hat{\imath}+6\hat{\jmath}+3\hat{k}}{7}$

Directional derivative $= \frac{d\emptyset}{ds} = (grad\, \emptyset).\hat{a} = -\frac{6+6+6}{98\sqrt{14}} = -\frac{\mathbf{9}}{\mathbf{49\sqrt{14}}}$

Example 2: Find the directional derivative of $\emptyset = 5x^2y - 5y^2z + \frac{5}{2}z^2x$ at the point $P(1,1,1)$ in the direction of the line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$.

Solution. Here, $\emptyset = 5x^2y - 5y^2z + \frac{5}{2}z^2x$

$\Rightarrow grad\, \emptyset = \left(10xy + \frac{5}{2}z^2\right)\hat{\imath} + (5x^2 - 10yz)\hat{\jmath} + (-5y^2 + 5xz)\hat{k}$

$\Rightarrow grad\, \emptyset]_{(1,1,1)} = \frac{25}{2}\hat{\imath} - 5\hat{\jmath}$

Let $\vec{a}$ be a vector in the direction of the line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$, then

$\vec{a} = 2\hat{\imath} - 2\hat{\jmath} + \hat{k} \Rightarrow \hat{a} = \frac{2\hat{\imath}-2\hat{\jmath}+\hat{k}}{3}$ and $\hat{a}]_{(1,1,1)} = \frac{2\hat{\imath}-2\hat{\jmath}+\hat{k}}{3}$

Directional derivative $= \frac{d\emptyset}{ds} = (grad\ \emptyset).\hat{a} = \frac{25}{3} + \frac{10}{3} = \frac{35}{6}$

Example 3: Find the directional derivative of $\emptyset = x^2 - y^2 + 2z^2$ at the point $P(1,2,3)$ in the direction of the line PQ where Q is the point $(5,0,4)$.
In what direction will it be maximum? Find also the magnitude of this maximum.

Solution. Here, $\emptyset = x^2 - y^2 + 2z^2 \Rightarrow \nabla\emptyset = 2x\,\hat{\imath} - 2y\hat{\jmath} + 4z\hat{k}$

$\nabla\emptyset]_{(1,2,3)} = 2\,\hat{\imath} - 4\hat{\jmath} + 12\hat{k}$ and $\overrightarrow{PQ} = 4\hat{\imath} - 2\hat{\jmath} + \hat{k}$

Let $\hat{a}$ be a unit vector in the direction of $\overrightarrow{PQ}$, then

$\hat{a} = \frac{4\hat{\imath}-2\hat{\jmath}+\hat{k}}{\sqrt{21}}$ and $\hat{a}]_{(1,2,3)} = \frac{4\hat{\imath}-2\hat{\jmath}+\hat{k}}{\sqrt{21}}$

Directional derivative $= \frac{d\emptyset}{ds} = (\nabla\emptyset).\hat{a} = \frac{8+8+12}{\sqrt{21}} = \frac{28}{\sqrt{21}}$

Directional derivative will be maximum in the direction of normal to the given surface i.e., in the direction of $\nabla\emptyset$.

The maximum value of this directional derivative $= |\nabla\emptyset| = 2\sqrt{41}$

Example 4: Find the directional derivative of $\emptyset = xy^2 + yz^3$ at the point $P(2,-1,1)$ in the direction of the normal to the surface $x\log z - y^2 + 4 = 0$ at $(2,-1,1)$.

Solution. Here, $\emptyset = xy^2 + yz^3 \Rightarrow \nabla\emptyset = y^2\hat{\imath} + (2xy + z^3)\hat{\jmath} + 3yz^2\hat{k}$

Let $\emptyset_1 \equiv x\log z - y^2 + 4 = 0 \Rightarrow \nabla\emptyset_1 = \log z\,\hat{\imath} - 2y\hat{\jmath} + \frac{x}{z}\hat{k}$

$\nabla\emptyset]_{(2,-1,1)} = \hat{\imath} - 3\hat{\jmath} - 3\hat{k}$ and $\nabla\emptyset_1]_{(2,-1,1)} = 2\hat{\jmath} + 2\hat{k}$

Let $\hat{a}$ be a unit vector in the direction of the normal to the surface $\emptyset_1$ at the point $(2,-1,1)$, then

$\hat{a} = \frac{\nabla\emptyset_1]_{(2,-1,1)}}{|\nabla\emptyset_1]_{(2,-1,1)}|} = \frac{2\hat{\jmath}+2\hat{k}}{2\sqrt{2}}$

Required Directional derivative $= (\nabla\emptyset).\hat{a}$

$$= (\hat{\imath} - 3\hat{\jmath} - 3\hat{k}).\left(\frac{2\hat{\jmath}+2\hat{k}}{2\sqrt{2}}\right)$$

$$= \frac{-6-6}{2\sqrt{2}} = -\frac{6}{\sqrt{2}} = \mathbf{-3\sqrt{2}}$$

Example 5: Find the directional derivative of $\vec{V}^2$ where $\vec{V} = xy^2\hat{\imath} + zy^2\hat{\jmath} + xz^2\hat{k}$ at the point $P(2,0,3)$ in the direction of the outward normal to the sphere $x^2 + y^2 + z^2 = 14$ at the point $(3,2,1)$.

Solution. Here, $\vec{V} = xy^2\hat{\imath} + zy^2\hat{\jmath} + xz^2\hat{k} \Rightarrow \vec{V}^2 = x^2y^4 + z^2y^4 + x^2z^4$

$\nabla\vec{V}^2 = 2x(y^4 + z^4)\hat{\imath} + 4y^3(x^2 + z^2)\hat{\jmath} + (2zy^4 + 4z^3x^2)\hat{k}$

Let $\emptyset_1 \equiv x^2 + y^2 + z^2 - 14 \Rightarrow \nabla\emptyset_1 = 2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k}$

$$\nabla\vec{V}^2\Big]_{(2,0,3)} = 324\hat{\imath} + 432\hat{k} \quad \text{and} \quad \nabla\emptyset_1]_{(3,2,1)} = 6\hat{\imath} + 4\hat{\jmath} + 2\hat{k}$$

Let $\hat{a}$ be a unit vector in the direction of the normal to the surface $\emptyset_1$ at the point (3,2,1), then

$$\hat{a} = \frac{\nabla\emptyset_1]_{(3,2,1)}}{|\nabla\emptyset_1]_{(3,2,1)}|} = \frac{6\hat{\imath}+4\hat{\jmath}+2\hat{k}}{2\sqrt{14}} = \frac{3\hat{\imath}+2\hat{\jmath}+\hat{k}}{\sqrt{14}}$$

Required Directional derivative $= \left(\nabla\vec{V}^2\right).\hat{a}$

$$= \left(324\hat{\imath} + 432\hat{k}\right).\left(\frac{3\hat{\imath}+2\hat{\jmath}+\hat{k}}{\sqrt{14}}\right) = \frac{\mathbf{1404}}{\sqrt{\mathbf{14}}}$$

Example 6: Find the directional derivative of $\nabla.(\nabla\emptyset)$ at a point $(1,-2,1)$ in the direction of the normal to the surface $xy^2z = 3x + z^2$, where $\emptyset = 2x^3y^2z^4$.

Solution. Here, $\emptyset = 2x^3y^2z^4 \Rightarrow \nabla\emptyset = 6x^2y^2z^4\hat{\imath} + 4x^3yz^4\hat{\jmath} + 8x^3y^2z^3\hat{k}$

$$\nabla.(\nabla\emptyset) = 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$$

Now, $\nabla\{\nabla.(\nabla\emptyset)\} = (12y^2z^4 + 12x^2z^4 + 72x^2y^2z^2)\hat{\imath} + (24xyz^4 + 48x^3yz^2)\hat{\jmath}$

$$+(48xy^2z^3 + 16x^3z^3 + 48x^3y^2z)\hat{k}$$

$\nabla\{\nabla.(\nabla\emptyset)\}]_{(1,-2,1)} = 348\hat{\imath} - 144\hat{\jmath} + 400\hat{k}$

Let $\emptyset_1 \equiv xy^2z - 3x - z^2 = 0 \Rightarrow \nabla\emptyset_1 = (y^2z - 3)\hat{\imath} + 2xyz\hat{\jmath} + (xy^2 - 2z)\hat{k}$

$\nabla\emptyset_1]_{(1,-2,1)} = \hat{\imath} - 4\hat{\jmath} + 2\hat{k} \;\Rightarrow \hat{a} = \frac{\nabla\emptyset_1]_{(1,-2,1)}}{|\nabla\emptyset_1]_{(1,-2,1)}|} = \frac{\hat{\imath}-4\hat{\jmath}+2\hat{k}}{\sqrt{21}}$

Where, $\hat{a}$ is a unit vector in the direction of the normal to the surface $\emptyset_1$ at the point $(1,-2,1)$.

Required Directional derivative $= \left(348\hat{\imath} - 144\hat{\jmath} + 400\hat{k}\right).\left(\frac{\hat{\imath}-4\hat{\jmath}+2\hat{k}}{\sqrt{21}}\right) = \frac{1724}{\sqrt{21}}v$

Exercise

1. If the directional derivative of $\emptyset = ax^2y + by^2z + cz^2x$ at the point (1,1,1) has maximum magnitude 15 in the direction parallel to the line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$ find the values of a, b *and* c.
2. For the function $\emptyset = \frac{y}{x^2+y^2}$, find the magnitude of the directional derivative making an angle 30° with the positive $x - axis$ at the point (0,1).
3. Find the values of constants a, b, c so that the maximum value of the directional derivative of $\emptyset = axy^2 + byz + cz^2x^3$ at $(1,2,-1)$ has a magnitude 64 in the direction parallel to $z - axis$.
4. Find the directional derivative of $f(x,y,z) = 2x^2 + 3y^2 + z^2$ at the point $P(2,1,3)$ in the direction of the vector $\vec{a} = \hat{\imath} - 2\hat{k}$.
5. Find the directional derivative of $\Psi(x,y,z) = 4e^{x+5y-13z}$ at the point (1,2,3) in the direction towards the point $(-3,5,7)$.

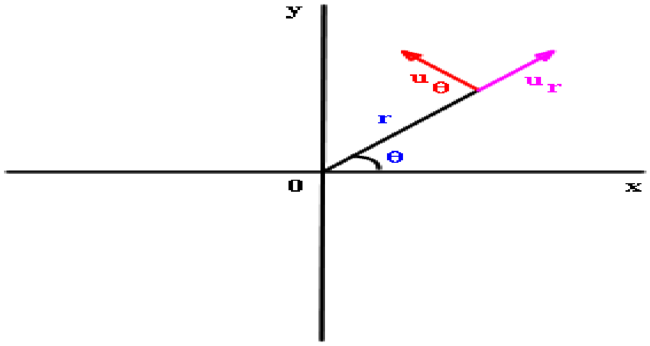
Answers

1. $a = \pm\frac{20}{9}, b = \mp\frac{55}{9}, c = \pm\frac{50}{9}$ **2.** $-\frac{1}{2}$ **3.** $a = 6, b = 24, c = -8$ **4.** $-\frac{4}{\sqrt{5}}$ **5.** $-4\sqrt{41}e^{-28}$

Gradient in Polar Form

Sometimes the surface is given in the form of $\emptyset = f(r, \theta)$. We can find $grad\ \emptyset$ directly without changing into cartesian form.

Let u_r *and* u_θ be the unit vectors along and perpendicular to $\vec{r}$.



Directional derivative along $(u_r) = \nabla\emptyset.u_r$

Thus $\frac{\partial\emptyset}{\partial s} = \frac{\partial\emptyset}{\partial r} = \nabla\emptyset.u_r$ ----------------- (1) [$ds = dr$ in the direction of $\vec{r}$]

Directional derivative along $(u_\theta) = \nabla\emptyset.u_\theta$

Thus $\frac{\partial\emptyset}{\partial s} = \frac{\partial\emptyset}{rd\theta} = \nabla\emptyset.u_\theta$ ----------------- (2) [$ds = rd\theta$ in the direction of $\vec{r}$]

Now, $\nabla\emptyset = (\nabla\emptyset.u_r)u_r + (\nabla\emptyset.u_\theta)u_\theta$

Or $\nabla\emptyset = \left(\frac{\partial\emptyset}{\partial r}\right)u_r + \left(\frac{\partial\emptyset}{rd\theta}\right)u_\theta$ [from equation (1) and (2)]

Example 1. Find (i) $\nabla\left(\frac{e^{\mu r}}{r}\right)$ (ii) $\nabla|\vec{r}|^2$ (iii) $\nabla \log r^n$ (iv) $grad f'(r) \times \vec{r}$

Solution. (i) $\nabla\left(\frac{e^{\mu r}}{r}\right) = \frac{\partial}{\partial r}\left(\frac{e^{\mu r}}{r}\right)\hat{r} = \left[\frac{r\mu e^{\mu r} - e^{\mu r}}{r^2}\right]\frac{\vec{r}}{r} = \frac{e^{\mu r}(\mu r - 1)\vec{r}}{r^3}$

(ii) $\nabla|\vec{r}|^2 = \nabla r^2 = \frac{\partial}{\partial r}(r^2)\hat{r} = 2r\frac{\vec{r}}{r} = 2\vec{r}$

(iii) $\nabla \log r^n = \frac{\partial}{\partial r}(\log r^n)\hat{r} = \left(\frac{1}{r^n}nr^{n-1}\right)\frac{\vec{r}}{r} = \frac{n\vec{r}}{r^2}$

(iv) $grad f'(r) \times \vec{r} = \left[\frac{\partial}{\partial r}\{f'(r)\}\hat{r}\right] \times \vec{r} = f''(r)\frac{\vec{r}}{r} \times \vec{r} = \vec{0}$

Example 2. If $V(x, y) = \frac{1}{2}\log(x^2 + y^2)$ prove that $gradV = \frac{\vec{r} - \vec{k}(\vec{k}.\vec{r})}{\{\vec{r} - \vec{k}(\vec{k}.\vec{r})\}.\{\vec{r} - \vec{k}(\vec{k}.\vec{r})\}}$

Solution. Here, $V = \frac{1}{2}\log(x^2 + y^2) == \frac{1}{2}\log r^2 = \log r$

L.H.S. $= grad\ V = \nabla V = \frac{\partial}{\partial r}(\log r)\hat{r} = \left(\frac{1}{r}\right)\frac{\vec{r}}{r} = \frac{\vec{r}}{r^2}$

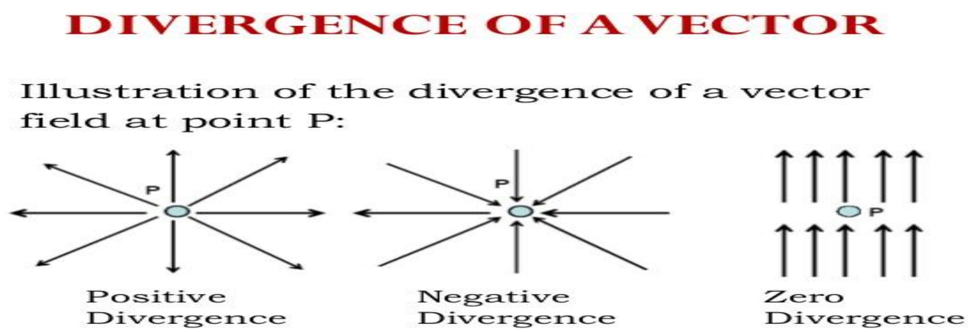
R.H.S. $= \frac{\vec{r} - \vec{k}(\vec{k}.\vec{r})}{\{\vec{r} - \vec{k}(\vec{k}.\vec{r})\}.\{\vec{r} - \vec{k}(\vec{k}.\vec{r})\}}$

$= \frac{\vec{r} - 0}{\{\vec{r} - 0\}.\{\vec{r} - 0\}}$ [as $\vec{k}.\vec{r} = \vec{k}.(x\hat{\imath} + y\hat{\jmath}) = 0$]

$= \frac{\vec{r}}{\vec{r}.\vec{r}} = \frac{\vec{r}}{r^2}$

4. Divergence of a vector point function

Introduction: The divergence of a vector field is the extent to which the vector field flux behaves like a source/sink at a given point.



Definition: The divergence of a differentiable vector point function $\vec{V}$ is denoted by $div\vec{V}$ and defined as:

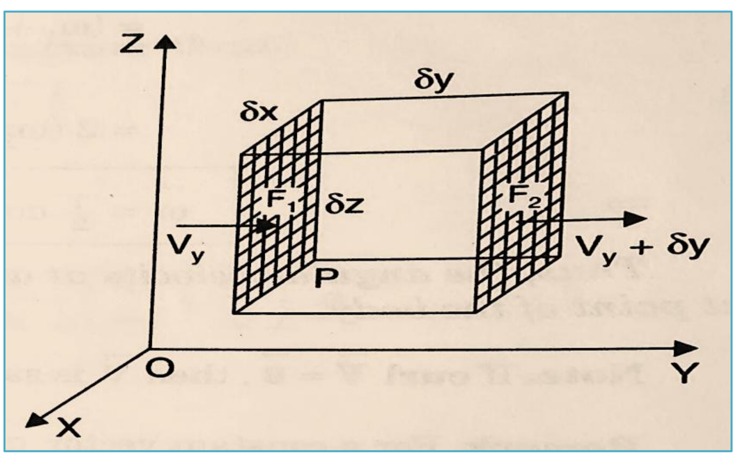
$$div\vec{V} = \nabla.\vec{V} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\vec{V} = \left(\hat{\imath}\frac{\partial\vec{V}}{\partial x} + \hat{\jmath}\frac{\partial\vec{V}}{\partial y} + \hat{k}\frac{\partial\vec{V}}{\partial z}\right)$$

4.1. Physical Interpretation of Divergence

Let ρ and $\vec{V}$ are the density and velocity at a point $P(x, y, z)$ at any time t respectively, of a moving fluid in a rectangular parallelepiped.

Let $\vec{V} = V_x\hat{\imath} + V_y\hat{\jmath} + V_z\hat{k} = \rho\vec{v}$, then $\vec{V}$ have the same direction as $\vec{v}$ and its magnitude $|\rho\vec{v}|$ is known as flux.

Let us consider a small parallelepiped with edges $\delta x, \delta y, \delta z$ parallel to the coordinate axes.



The mass of the fluid entering through the face F_1 is $= V_y\delta x\delta z$

The mass of the fluid out through the opposite face F_2 is $= V_{y+\delta y}\delta x\delta z$

The net decrease in the mass of fluid flowing across these faces

$$= V_{y+\delta y}\delta x\delta z - V_y\delta x\delta z$$

$$= \left(V_{y+\delta y} - V_y\right)\delta x\delta z$$

$$= \left[V_y + \frac{\partial}{\partial y}(V_y)\delta y + \cdots \ldots \ldots - V_y\right]\delta x \delta z$$

$$= \frac{\partial}{\partial y}(V_y)\delta y \delta x \delta z = \frac{\partial V_y}{\partial y}\delta x \delta y \delta z$$

Similarly,

The decrease in mass of the fluid to the flow along $x - axis = \frac{\partial V_x}{\partial x}\delta x \delta y \delta z$

The decrease in mass of the fluid to the flow along z $- \, axis = \frac{\partial V_z}{\partial z}\delta x \delta y \delta z$

Total decrease of the amount of the fluid per unit time $= \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)\delta x \delta y \delta z$

Thus, the rate of loss of the fluid per unit volume $= \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(\hat{\imath}V_x + \hat{\jmath}V_y + \hat{k}V_z\right)$$

$$= \nabla.\vec{V} = div\,\vec{V}$$

Hence, $div\,\vec{V}$ gives the rate of outflow per unit volume at a point of the fluid.

Note: If $div\,\vec{V} = 0$, then **(i)** $\vec{V}$ is called solenoidal vector **(ii)** fluid is called compressible

5. Curl of a vector point function

Introduction: The curl is a vector operator that describes the infinitesimal rotation of a vector field in three-dimensional space.

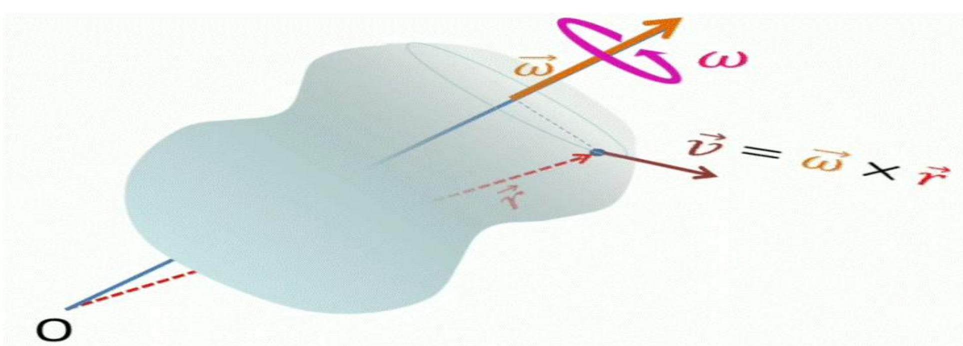
Definition: The curl of a differentiable vector point function $\vec{F}$ is denoted by curl $\vec{F}$ and defined as:

$$curl\,\vec{F} = \nabla \times \vec{F} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \vec{F}$$

5.1. Physical Interpretation of Curl

Let us consider a rigid body rotating about a fixed axis through O with uniform angular velocity $\vec{\omega} = \omega_1\hat{\imath} + \omega_2\hat{\jmath} + \omega_3\hat{k}$. Then we have,

$\vec{v} = \vec{\omega} \times \vec{r}$, where $\vec{v}$ is the linear velocity at any point and $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ is a position vector for that point.



$$\therefore\ \vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} = (\omega_2 z - \omega_3 y)\hat{\imath} + (\omega_3 x - \omega_1 z)\hat{\jmath} + (\omega_1 y - \omega_2 x)\hat{k}$$

Now, $curl\,\vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$

$$= 2\omega_1\hat{\imath} + 2\omega_2\hat{\jmath} + 2\omega_3\hat{k}$$

$$= 2\vec{\omega} \quad \Rightarrow \quad \vec{\omega} = \frac{1}{2} curl\,\vec{v}$$

Thus, the angular velocity at any point is equal to the half of the $curl$ of the linear velocity at that point of the body.

Note: If $\vec{v} = \vec{0}$, then $\vec{v}$ is said to be an irrotational vector.

Example 1. Find the divergence and curl of $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$.

Solution. $div\,\vec{r} = \nabla.\vec{r}$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right)$$

$$= 1 + 1 + 1 = \mathbf{3}$$

$$curl\,\vec{r} = \nabla \times \vec{r} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right)$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= (0-0)\hat{\imath} + (0-0)\hat{\jmath} + (0-0)\hat{k} = \vec{\mathbf{0}}$$

Example 2. Find the divergence and curl of the vector.
$\vec{R} = (x^2 + yz)\hat{\imath} + (y^2 + zx)\hat{\jmath} + (z^2 + xy)\hat{k}$

Solution. $div\,\vec{R} = \nabla.\vec{R}$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left[(x^2 + yz)\hat{\imath} + (y^2 + zx)\hat{\jmath} + (z^2 + xy)\hat{k}\right]$$

$$= 2x + 2y + 2z = \mathbf{2(x + y + z)}$$

$$curl\,\vec{F} = \nabla \times \vec{R}$$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left[(x^2 + yz)\hat{\imath} + (y^2 + zx)\hat{\jmath} + (z^2 + xy)\hat{k}\right]$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + yz & y^2 + zx & z^2 + xy \end{vmatrix}$$

$$= (x - x)\hat{\imath} + (y - y)\hat{\jmath} + (z - z)\hat{k} = \vec{\mathbf{0}}$$

6. Application of divergence and curl

In vector calculus, a solenoidal vector field (also known as a divergence-free or incompressible vector field or a transverse vector field) is one where the divergence is zero at all points. This means that the field has no sources or sinks. Here are some points about the application of divergence to a solenoidal vector field:

1. A vector field is said to be solenoidal if $div\,\vec{V} = 0$. This implies that the net flux of the vector field through any closed surface is zero.
2. The velocity field of an incompressible fluid is solenoidal, meaning the volume of the fluid remains constant over time.

An irrotational vector field is one where the curl is zero everywhere. This means that the field has no rotational component. Here are some points about the application of curl to an irrotational vector field:

1. A vector field is irrotational if $curl\,\vec{F} = \vec{0}$. This implies that the field can be expressed as the gradient of a scalar potential function.
2. In fluid dynamics, if the velocity field of a fluid is irrotational, it means that the fluid elements do not exhibit any rotational motion about their own axes.

Example 1. Prove that $(y^2 - z^2 + 3yz - 2x)\hat{\imath} + (3xz + 2xy)\hat{\jmath} + (3xy - 2xz + 2z)\hat{k}$ is both solenoidal and irrotational.

Solution. Let $\vec{V} = (y^2 - z^2 + 3yz - 2x)\hat{\imath} + (3xz + 2xy)\hat{\jmath} + (3xy - 2xz + 2z)\hat{k}$

$div\,\vec{V} = \nabla.\vec{V} = -2 + 2x - 2x + 2 = 0 \quad \therefore \vec{V}$ is solenoidal

$$curl\,\vec{V} = \nabla \times \vec{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 + 3yz - 2x & 3xz + 2xy & 3xy - 2xz + 2z \end{vmatrix}$$

$$= (3x - 3x)\hat{\imath} + (-2z + 3y - 3y + 2z)\hat{\jmath} + (3z + 2y - 2y - 3z)\hat{k}$$

$$= \vec{\mathbf{0}} \qquad \therefore \vec{V} \text{ is irrotational.}$$

Example 2. Show that the vector field $\vec{A}$, where $\vec{A} = (x^2 - y^2 + x)\hat{\imath} - (2xy + y)\hat{\jmath}$ is irrotational. And find a scalar $\emptyset$ such that $\vec{A} = grad\,\emptyset$.

Solution. Here, $curl\,\vec{A} = \nabla \times \vec{A}$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2 - y^2 + x) & -(2xy + y) & 0 \end{vmatrix} = \vec{\mathbf{0}} \quad \therefore \vec{A} \text{ is irrotational.}$$

Let $\vec{A} = \nabla\emptyset$, where $\emptyset$ is scalar potential.

$\therefore \vec{A}.d\vec{r} = \nabla\emptyset.d\vec{r}$

$\Rightarrow [(x^2 - y^2 + x)\hat{\imath} - (2xy + y)\hat{\jmath}].(dx\,\hat{\imath} + dy\,\hat{\jmath}) = d\emptyset \quad [as\ \nabla\emptyset.d\vec{r} = d\emptyset]$

$\Rightarrow d\emptyset = (x^2 - y^2 + x)dx - (2xy + y)dy$ -------------------------- (1)

Integrate equation (1) we get, $\emptyset = \frac{x^3}{3} - y^2x + \frac{x^2}{2} - \frac{y^2}{2} + C$

Example 3. Find the constants a, b, c so that

$\vec{F} = (x + 2y + az)\hat{\imath} + (bx - 3y - z)\hat{\jmath} + (4x + cy + 2z)\hat{K}$ is irrotational.

If $\vec{F} = grad\ \emptyset$, show that $\emptyset = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4xz - yz + C$.

Solution. Here, $\vec{F}$ is irrotational $\therefore curl\,\vec{F} = \vec{0}$

$$\Rightarrow \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix} = \vec{\mathbf{0}}$$

$\Rightarrow (c + 1)\hat{\imath} + (a - 4)\hat{\jmath} + (b - 2)\hat{k} = 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k} \quad \Rightarrow a = 4, b = 2, c = -1$

It is given that $\vec{F} = grad\ \emptyset$ or $\vec{F} = \nabla\emptyset$

$\therefore \vec{F}.d\vec{r} = \nabla\emptyset.d\vec{r}$

$\Rightarrow \left[(x + 2y + 4z)\hat{\imath} + (2x - 3y - z)\hat{\jmath} + (4x - y + 2z)\hat{K}\right].\left(dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}\right) = d\emptyset$

$\Rightarrow d\emptyset = (x + 2y + 4z)dx + (2x - 3y - z)dy + (4x - y + 2z)dz$ ----------- (1)

Integrate eq^n (1) we get, $\emptyset = \frac{x^2}{2} + 2xy + 4xz - \frac{3y^2}{2} - yz + z^2 + C$.

Vector Identities

If $\vec{a}, \vec{b}$ are vector functions and $\emptyset$ is a scalar function, then

(1) $div(\emptyset\vec{a}) = \emptyset div\vec{a} + (grad\emptyset).\vec{a}$

Proof. $div(\emptyset\vec{a}) = \sum \hat{\imath}.\frac{\partial}{\partial x}(\emptyset\vec{a}) = \sum \hat{\imath}.\left(\emptyset\frac{\partial\vec{a}}{\partial x} + \frac{\partial\emptyset}{\partial x}\vec{a}\right)$

$$= \emptyset\sum \hat{\imath}.\frac{\partial\vec{a}}{\partial x} + \left(\sum \hat{\imath}\frac{\partial\emptyset}{\partial x}\right).\vec{a} = \emptyset div\vec{a} + (grad\emptyset).\vec{a}$$

(2) $curl(\emptyset\vec{a}) = \emptyset curl\vec{a} + (grad\emptyset) \times \vec{a}$

Proof. $curl(\emptyset\vec{a}) = \sum \hat{\imath} \times \frac{\partial}{\partial x}(\emptyset\vec{a}) = \sum \hat{\imath} \times \left(\emptyset\frac{\partial\vec{a}}{\partial x} + \frac{\partial\emptyset}{\partial x}\vec{a}\right)$

$$= \emptyset\sum \hat{\imath} \times \frac{\partial\vec{a}}{\partial x} + \left(\sum \hat{\imath}\frac{\partial\emptyset}{\partial x}\right) \times \vec{a} = \emptyset curl\vec{a} + (grad\emptyset) \times \vec{a}$$

(3) $div\left(\vec{a} \times \vec{b}\right) = \vec{b}.curl\vec{a} - \vec{a}.curl\vec{b}$

Proof. $div\left(\vec{a} \times \vec{b}\right) = \sum \hat{\imath}.\frac{\partial}{\partial x}\left(\vec{a} \times \vec{b}\right) = \sum \hat{\imath}.\left(\vec{a} \times \frac{\partial\vec{b}}{\partial x} + \frac{\partial\vec{a}}{\partial x} \times \vec{b}\right)$

$$= \sum \hat{\imath}.\left(-\frac{\partial \vec{b}}{\partial x}\times \vec{a}\right) + \sum \hat{\imath}.\left(\frac{\partial \vec{a}}{\partial x}\times \vec{b}\right)$$

$$= \sum \hat{\imath}.\left(\frac{\partial \vec{a}}{\partial x}\times \vec{b}\right) - \sum \hat{\imath}.\left(\frac{\partial \vec{b}}{\partial x}\times \vec{a}\right)$$

$$= \sum \left(\hat{\imath}\times\frac{\partial \vec{a}}{\partial x}\right).\vec{b} - \sum \left(\hat{\imath}\times\frac{\partial \vec{b}}{\partial x}\right).\vec{a}$$

$$= (curl\vec{a}).\vec{b} - \left(curl\vec{b}\right).\vec{a} = \vec{b}.(curl\vec{a}) - \vec{a}.\left(curl\vec{b}\right)$$

(4) $curl\left(\vec{a}\times\vec{b}\right) = \vec{a}div\vec{b} - \vec{b}div\vec{a} + \left(\vec{b}.\nabla\right)\vec{a} - (\vec{a}.\nabla)\vec{b}$

Proof. $curl\left(\vec{a}\times\vec{b}\right) = \sum \hat{\imath}\times\frac{\partial}{\partial x}\left(\vec{a}\times\vec{b}\right) = \sum \hat{\imath}\times\left(\vec{a}\times\frac{\partial \vec{b}}{\partial x} + \frac{\partial \vec{a}}{\partial x}\times\vec{b}\right)$

$$= \sum \hat{\imath}\times\left(\vec{a}\times\frac{\partial \vec{b}}{\partial x}\right) + \sum \hat{\imath}\times\left(\frac{\partial \vec{a}}{\partial x}\times\vec{b}\right)$$

$$= \sum \left[\left(\hat{\imath}.\frac{\partial \vec{b}}{\partial x}\right)\vec{a} - (\hat{\imath}.\vec{a})\frac{\partial \vec{b}}{\partial x}\right] + \sum \left[\left(\hat{\imath}.\vec{b}\right)\frac{\partial \vec{a}}{\partial x} - \left(\hat{\imath}.\frac{\partial \vec{a}}{\partial x}\right)\vec{b}\right]$$

$$= \sum \left(\hat{\imath}.\frac{\partial \vec{b}}{\partial x}\right)\vec{a} - \sum(\vec{a}.\hat{\imath})\frac{\partial \vec{b}}{\partial x} + \sum\left(\vec{b}.\hat{\imath}\right)\frac{\partial \vec{a}}{\partial x} - \sum\left(\hat{\imath}.\frac{\partial \vec{a}}{\partial x}\right)\vec{b}$$

$$= \left(div\vec{b}\right)\vec{a} - \left(\vec{a}.\sum \hat{\imath}\frac{\partial}{\partial x}\right)\vec{b} + \left(\vec{b}.\sum \hat{\imath}\frac{\partial}{\partial x}\right)\vec{a} - (div\vec{a})\vec{b}$$

$$= \vec{a}div\vec{b} - \vec{b}div\vec{a} + \left(\vec{b}.\nabla\right)\vec{a} - (\vec{a}.\nabla)\vec{b}$$

(5) $div(grad\emptyset) = \nabla^2\emptyset$

Proof. $div(grad\emptyset) = \nabla.(\nabla\emptyset) = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(\hat{\imath}\frac{\partial\emptyset}{\partial x} + \hat{\jmath}\frac{\partial\emptyset}{\partial y} + \hat{k}\frac{\partial\emptyset}{\partial z}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{\partial\emptyset}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\emptyset}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\emptyset}{\partial z}\right) = \frac{\partial^2\emptyset}{\partial x^2} + \frac{\partial^2\emptyset}{\partial y^2} + \frac{\partial^2\emptyset}{\partial z^2}$$

$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\emptyset = \nabla^2\emptyset$; ∇^2 is known as Laplacian operator

(6) $curl(grad\emptyset) = \nabla\times(\nabla\emptyset) = \vec{0}$

Proof. curl$(grad\emptyset) = \nabla\times(\nabla\emptyset) = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\left(\hat{\imath}\frac{\partial\emptyset}{\partial x} + \hat{\jmath}\frac{\partial\emptyset}{\partial y} + \hat{k}\frac{\partial\emptyset}{\partial z}\right)$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\emptyset}{\partial x} & \frac{\partial\emptyset}{\partial y} & \frac{\partial\emptyset}{\partial z} \end{vmatrix} = \sum \hat{\imath}\left(\frac{\partial^2\emptyset}{\partial y\partial z} - \frac{\partial^2\emptyset}{\partial z\partial y}\right) = \vec{0}$$

(7) $div\left(curl\,\vec{V}\right) = 0$

Proof. Let $\vec{V} = V_1\hat{\imath} + V_2\hat{\jmath} + V_3\hat{k} \Rightarrow curl\,\vec{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$

$$= \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\hat{\imath} + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\hat{k}$$

Now, $div\left(curl\,\vec{V}\right) = \frac{\partial}{\partial x}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$

$$= \left(\frac{\partial^2 V_3}{\partial x \partial y} - \frac{\partial^2 V_2}{\partial x \partial z}\right) + \left(\frac{\partial^2 V_1}{\partial y \partial z} - \frac{\partial^2 V_3}{\partial y \partial x}\right) + \left(\frac{\partial^2 V_2}{\partial z \partial x} - \frac{\partial^2 V_1}{\partial z \partial y}\right) = 0$$

Example 1. If $u = x^2 + y^2 + z^2$ and $\vec{V} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, show that $div\left(u\vec{V}\right) = 5u$

Solution. $div\left(u\vec{V}\right) = u\, div\vec{V} + (grad\, u).\vec{V}$

$$= 3u + \left(\hat{\imath}\frac{\partial u}{\partial x} + \hat{\jmath}\frac{\partial u}{\partial y} + \hat{k}\frac{\partial u}{\partial z}\right).\vec{V}$$

$$= 3u + \left(2x\hat{\imath} + 2y\hat{\jmath} + 2z\,\hat{k}\right).\left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right)$$

$$= 3u + (2x^2 + 2y^2 + 2z^2)$$

$$= 3u + 2(x^2 + y^2 + z^2)$$

$$= 3u + 2u = \boldsymbol{5u}$$

Example 2. If $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, prove that
(i) $curl(r^n\vec{r}) = \vec{0}$ (ii) $\nabla^2(r^n\vec{r}) = n(n+3)r^{n-2}\vec{r}$

Solution. (i) $curl(r^n\vec{r}) = r^n curl\,\vec{r} + (grad\, r^n) \times \vec{r}$

$$= \vec{0} + (nr^{n-1}\hat{r}) \times \vec{r} = nr^{n-1}\frac{\vec{r}}{r} \times \vec{r} = \vec{0}$$

(ii) $\nabla^2(r^n\vec{r}) = \nabla[\nabla.(r^n\vec{r})] = grad[div(r^n\vec{r})]$

$$= grad[r^n div\,\vec{r} + (grad r^n).\vec{r}]$$

$$= grad[3r^n + (nr^{n-1}\hat{r}).\vec{r}] = grad\left[3r^n + nr^{n-1}\frac{\vec{r}}{r}.\vec{r}\right]$$

$$= grad\left[3r^n + nr^{n-1}\frac{r^2}{r}\right] = grad[(n+3)r^n]$$

$$= (n+3)nr^{n-1}\frac{\vec{r}}{r} = n(n+3)r^{n-2}\vec{r}$$

Example 3. Find the most general $f(r)$ such that $f(r)\vec{r}$ is solenoidal.

Solution. The vector $f(r)\vec{r}$ will be solenoidal, if

$$div[f(r)\vec{r}] = 0$$

$$\Rightarrow f(r) div\,\vec{r} + [grad f(r)].\vec{r} = 0$$

$$\Rightarrow 3f(r) + [f'(r)\hat{r}].\vec{r} = 0 \Rightarrow 3f(r) + \left[f'(r)\frac{\vec{r}}{r}\right].\vec{r} = 0$$

$$\Rightarrow 3f(r) + f'(r)\frac{r^2}{r} = 0 \Rightarrow \frac{f'(r)}{f(r)} = -\frac{3}{r} \qquad \text{----------------------- (1)}$$

Integrate eq^n (1), we get

$$\log[f(r)] = -3\log r + \log c$$

$$\Rightarrow f(r) = \frac{c}{r^3}$$

Example 4. Prove that the vector $f(r)\vec{r}$ is irrotational.

Solution. The vector $f(r)\vec{r}$ will be irrotational, if $curl[f(r)\vec{r}] = \vec{0}$

$$\text{Now, } curl[f(r)\vec{r}] = f(r)curl\,\vec{r} + [grad f(r)] \times \vec{r}$$

$$= \vec{0} + [f'(r)\hat{r}] \times \vec{r}$$

$$= \vec{0} + \left[f'(r)\frac{\vec{r}}{r}\right] \times \vec{r}$$

$$= \vec{0} + f'(r)\frac{\vec{r}}{r} \times \vec{r}$$

$$= \vec{0} + \vec{0} = \vec{0}$$

∴ The vector $f(r)\vec{r}$ is irrotational.

Example 5. Prove that $\nabla^2 f(r) = f''(r) + \frac{2}{r}f'(r)$.

Hence evaluate $\nabla^2(\log r)$ if $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$

Solution. $\nabla^2 f(r) = \nabla.\nabla f(r) = div[\nabla f(r)] = div[grad\, f(r)]$

$$= div[f'(r)\hat{r}] = div\left[f'(r)\frac{\vec{r}}{r}\right] = div\left[\left\{\frac{f'(r)}{r}\right\}\vec{r}\right]$$

$$= \frac{f'(r)}{r}div\vec{r} + \left[grad\left\{\frac{f'(r)}{r}\right\}\right].\vec{r} = 3\frac{f'(r)}{r} + \left[\frac{rf''(r) - f'(r)}{r^2}\hat{r}\right].\vec{r}$$

$$= \frac{3}{r}f'(r) + \left[\frac{rf''(r) - f'(r)}{r^2}\left(\frac{\vec{r}}{r}\right)\right].\vec{r} = \frac{3}{r}f'(r) + \frac{rf''(r) - f'(r)}{r^2}\left(\frac{\vec{r}.\vec{r}}{r}\right)$$

$$\nabla^2 f(r) = \frac{3}{r}f'(r) + \frac{rf''(r) - f'(r)}{r} = f''(r) + \frac{2}{r}f'(r)$$

Put $f(r) = \log r$, we get, $\nabla^2(\log r) = -\frac{1}{r^2} + \frac{2}{r}\frac{1}{r} = \mathbf{\frac{1}{r^2}}$.

Exercise

1. If $\vec{F}(x, y, z) = xz^3\hat{\imath} - 2x^2yz\hat{\jmath} + 2yz^4\hat{k}$ find $divergence\ and\ curl$ of $\vec{F}(x, y, z)$.

2. Find the $divergence\ and\ curl$ of the vector field $\vec{V} = x^2y^2\hat{\imath} + 2xy\hat{\jmath} + (y^2 - xy)\hat{k}$.

3. A fluid motion is given by $\vec{V} = (y + z)\hat{\imath} + (z + x)\hat{\jmath} + (x + y)\hat{k}$

(i) Is this motion irrotational? If so, find the velocity potential.

(ii) Is the motion possible for an incompressible fluid?

4. If $\vec{V} = \frac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}$, show that $\nabla.\vec{V} = \frac{2}{\sqrt{x^2+y^2+z^2}}$ and $\nabla \times \vec{V} = \vec{0}$

5. Show that $\vec{A} = (6xy + z^3)\hat{\imath} + (3x^2 - z)\hat{\jmath} + (3xz^2 - y)\hat{k}$ is irrotational. Find the velocity potential $\emptyset$ such that $\vec{A} = \nabla\emptyset$.

6. A fluid motion is given by $\vec{v} = (y \sin z - \sin x)\hat{\imath} + (x \sin z + 2yz)\hat{\jmath} + (xy \cos z + y^2)\hat{k}$. Is the motion irrotational? If so, find the velocity potential.

7. If $\vec{E}$ *and* $\vec{H}$ are irrotational, prove that $\vec{E} \times \vec{H}$ is irrotational.

8. If $u\vec{F} = \nabla v$, where u, v are scalar field and $\vec{F}$ is a vector field show that $\vec{F}.curl\,\vec{F} = 0$.

9. If $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ and $r = |\vec{r}|$, show that

(i) $div(\vec{r})\emptyset = 3\emptyset + \vec{r}.grad\emptyset$ **(ii)** $div(\hat{r}) = \frac{2}{r}$

10. Prove that $div(grad\, r^n) = \nabla^2 r^n = n(n+1)r^{n-2}$, where $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$. Hence show that $\nabla^2\left(\frac{1}{r}\right) = 0$. Hence or otherwise evaluate $\nabla \times \left(\frac{\vec{r}}{r^2}\right)$.

Answers

1. $div\,\vec{F} = z^3 - 2x^2z + 8yz^3, curl\,\vec{F} = 2(x^2y + z^4)\hat{\imath} + 3xz^2\hat{\jmath} - 4xyz\hat{k}$
2. $div\,\vec{V} = 2xy^2 + 2x,\; curl\,\vec{F} = (2y - x)\hat{\imath} + y\hat{\jmath} + 2y(1 - x^2)\hat{k}$
3. **(i)** Yes; velocity potential $\emptyset = xy + yz + zx + c$

   (ii) Yes

5. $\emptyset = x^2yz^3 + c$ **6.** Yes; $\emptyset = xy\sin z + \cos x + y^2z + c$

10. $\vec{0}$

7. E-resources:

https://www.youtube.com/watch?v=fZ231k3zsAA&t=57s

https://www.youtube.com/watch?v=qOcFJKQPZfo

https://www.youtube.com/watch?v=3TkKm2mwR0Y

https://www.youtube.com/watch?v=ynzRyIL2atU

https://www.youtube.com/watch?v=Cxc7ihZWq5o

https://www.youtube.com/watch?v=vvzTEbp9lrc

https://www.youtube.com/watch?v=ZtQyuN7DdKE