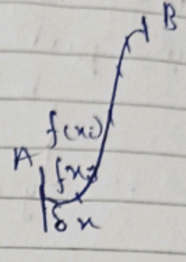
UNIT-3 → Multiple Integrals

Double Integral

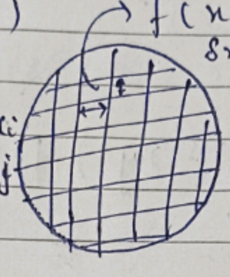
As we know for line Integral

$$I = \sum_{i=1}^{n} f(x_i) = \int_a^b f(x_i)\,dx$$



$$I = \sum_{i=1}^{n} \sum_{j=1}^{n} f(x_i, y_j) \rightarrow f(x_i, y_j)\ \delta x_i, \delta y_j$$

$$= \int_a^b \int_c^d f(x_i, y_j)\,dx_i\,dy_j$$



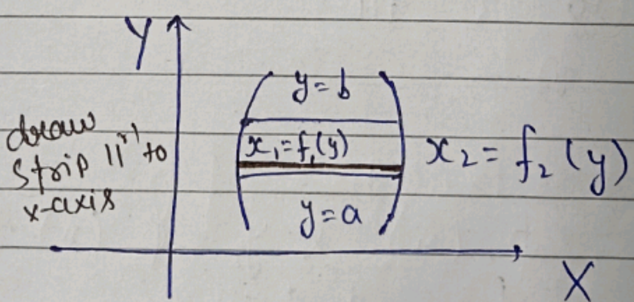
$$\Rightarrow \quad I = \int_a^b \int_c^d f(x, y)\,\partial x\,\partial y$$

Evaluation of double Integral —

The methods of evaluating the double integral depends upon the nature of curves bounding the region R.

Let region R be bounded by $x = x_1, x = x_2$ & $y = y_1$ to $y = y_2$

Case 1 → when $x_1 = f_1(y)$ & $x_2 = f_2(y)$ and $y_1 = a$ and $y_2 = b$



variable limit inside

Constant lim. outside.

$$I = \iint_R f(x, y)\,dx\,dy$$

$$= \int_{y=a}^{b} \int_{f_1(y)=x}^{f_2(y)} f(x, y)\,dx\,dy$$

The area covered by strip while rotating it is the region of Integration.

First Int w.r.t x (keeping y const.) then put limit, Then Int w.r.t y.

$$= \int_{y=a}^{b}\left[\int_{x_1=f_1(y)}^{f_2(y)} f(x,y)\,dx\right] dy.$$

Ex. Evaluate - $\displaystyle\int_{y=0}^{1}\int_{x=y^2}^{y}(1+xy^2)\,dx\,dy$

$$= \int_0^1\left[\int_{y^2}^{y}(1+xy^2)\,dx\right]dy \rightarrow \text{Int. w.r.t. } x \text{ keeping } y \text{ const.}$$

$$= \int_0^1\left[x + \frac{x^2}{2}y^2\right]_{y^2}^{y} dy = \int_0^1\left[y + \frac{y^4}{2} - y^2 - \frac{y^6}{2}\right]dy$$

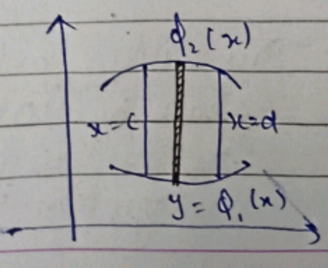
$$= \int_0^1\left[y + \frac{y^4}{2} - y^2 - \frac{y^6}{2}\right]dy$$

$$= \left[\frac{y^2}{2} + \frac{y^5}{10} - \frac{y^3}{3} - \frac{y^7}{14}\right]_0^1$$

$$= \frac{1}{2} + \frac{1}{10} - \frac{1}{3} - \frac{1}{14} - 0$$

$$= \frac{41}{210} \text{ Ans.}$$

Case 2: If $y_1 = \phi_1(x)$ & $y_2 = \phi_2(x)$ & $x_1 = c$ & $x_2 = d$



$$\boxed{I = \int_{x=c}^{d}\int_{y=\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\,dx}$$

draw strip $\parallel^{rl}$ to Y-axis from $\phi_1(x)$ to $\phi_2(x)$

$$Ex - \int_0^1 \int_{x^2}^{x} (x^2+3y+2)\,dy\,dx$$

$$= \int_{x=0}^{1} \left[\int_{x^2}^{x} x^2+3y+2 \, dy \right] dx$$

Int. wrt. y keeping x const.

$$= \int_0^1 \left[x^2 y + \frac{3y^2}{2} + 2y \right]_{x^2}^{x} dx$$

$$= \int_0^1 -\left(x^4 + \frac{3x^4}{2} + 2x^2\right) + x^3 + \frac{3x^2}{2} + 2x \quad dx$$

$$= \int_0^1 -\frac{5x^4}{2} + \frac{x^3}{1} - \frac{x^2}{2} + 2x \quad dx$$

$$= \left[-\frac{\cancel{5}x^5}{2\times\cancel{5}} + \frac{x^4}{4} - \frac{x^3}{3\times 2} + \frac{2x^2}{2} \right]_0^1$$

$$= \frac{-1}{2} + \frac{1}{4} - \frac{1}{6} + 1 = \frac{-12+6-4+24}{24}$$

$$= \frac{14}{24} \qquad \frac{7}{12}$$

Case. 3. when $x_1=a$, $x_2=b$ & $y_1=c$, $y_2=d$ are Const.

$$I = \int_{y=c}^{d} \underbrace{\int_{x=a}^{b} f(x,y)\,dx}_{\text{Int. wrt. } x \text{ (y Const.)}}\,dy$$

$$= \int_{x=a}^{b} \underbrace{\int_{y=c}^{d} f(x,y)\,dy}_{\text{Int. wrt. } y \text{ (x Const.)}}\,dx$$

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$$\Rightarrow \int_{x=a}^{b}\int_{y=c}^{d} f(x,y)\,dy\,dx = \int_{y=c}^{d}\int_{x=a}^{b} f(x,y)\,dx\,dy$$

NOTE → ① In Case (i) & Case (ii) integration is to be performed w.r.t. that variable having limit first and then w.r.t. variable with Constant limits.

② If $f(x,y)$ is discontⁿ. within or on the boundary of the region of Int. then

$$\int_{y=a}^{b}\int_{x=c}^{d} f(x;y)\,dx\,dy \neq \int_{x=c}^{d}\int_{y=a}^{b} f(x,y)\,dy\,dx$$

Ex. Evaluate $\displaystyle\int_{x=0}^{1}\int_{y=x}^{\sqrt{x}} (x^2+y^2)\,dy\,dx$

= Int. wrt-y (x const.)

$$\int_0^1\left[\int_x^{\sqrt{x}}(x^2+y^2)\,dy\right]dx$$

$$= \int_0^1\left[x^2y+\frac{y^3}{3}\right]_x^{\sqrt{x}}dx$$

$$= \int_0^1\left(x^2\sqrt{x}+\frac{x\sqrt{x}}{3} - x^3 - \frac{x^3}{3}\right)dx = \int_0^1\left(x^{5/2}+\frac{x^{3/2}}{3} - \frac{4x^3}{3}\right)dx$$

$$= \left[\frac{2x^{7/2}}{7}+\frac{2x^{5/2}}{3\times5}-\frac{4x^4}{3\times4}\right]_0^1$$

$$= \frac{2}{7}+\frac{2}{15}-\frac{1}{3}$$

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Q. Evaluate $\int_{x=0}^{1}\int_{y=0}^{y=x^2} e^{y/x}\, dy\, dx$

Int wrt. y

$$\int_0^1 \left[\int_0^{x^2} e^{y/x}\, dy\right]_0^{x^2} dx = \int_0^1 \left[\underline{xe^{y/x}}\right]_0^{x^2} dx$$

$$= \int_0^1 \left(x e^{x^2/x} - x\right) dx = \int_0^1 (xe^x - x)\, dx$$

$$= \cancel{\left[e^x \cdot \frac{x^2}{2} - \int e^x \cdot \frac{x^2}{2}\right.}$$

$$= \int_0^1 \left[\left(xe^x - \int e^x\, dx\right) - \frac{x^2}{2}\right]$$

$$= \left[xe^x - e^x - \frac{x^2}{2}\right]_0^1$$

$$= \cancel{e} - \cancel{e} - \frac{1}{2} + 1 = \frac{1}{2} \quad \text{Ans.}$$

Q. $\int_{y=0}^{1}\int_{x=y}^{y^2+1} x^2 y\, dx\, dy$

Int. wrt. x (y const.)

$$\int_0^1 \left[\int_y^{y^2+1} x^2 y\, dx\right] dy = \int_0^1 \left[\frac{x^3}{3} y\right]_y^{y^2+1} dy$$

$$= \int_0^1 \left[\frac{(y^2+1)^3}{3} y - \frac{y^4}{3}\right] dy$$

$$= \int_0^1 \left[\frac{y^7}{3} + \frac{y}{3} + \frac{\cancel{3}y^5}{\cancel{3}} + \frac{\cancel{3}y^3}{\cancel{3}} - \frac{y^4}{3}\right] dy$$

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$$= \left[\frac{y^8}{3\times 8} + \frac{y^2}{3\times 2} + \frac{y^6}{6} + \frac{y^4}{4} - \frac{y^5}{3\times 5}\right]_0^1$$

$$= \frac{1}{24} + \frac{1}{6} + \frac{1}{6} + \frac{1}{4} - \frac{1}{15}$$

$$= \frac{5 + 20 + 20 + 30 - 8}{120}$$

$$= \frac{67}{120}$$

Q. Evaluate $\displaystyle\int_{x=0}^{1}\int_{y=0}^{y=\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$

Int. w.r.t. y

$$\int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{dy}{1+x^2+y^2}\right] dx$$

$$= \int_0^1 \left[\int \frac{dy}{(\sqrt{1+x^2})^2 + y^2}\right] dx = \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1} \frac{y}{\sqrt{1+x^2}}\right]_0^{\sqrt{1+x^2}} dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \tan^{-1} \frac{\sqrt{1+x^2}}{\sqrt{1+x^2}} - \frac{1}{\sqrt{1+x^2}} \tan^{-1} 0$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \frac{\pi}{4}\, dx$$

$$\cancel{= \frac{\pi}{2}\left[\tan^{-1} x\right]_0^1 = \frac{\pi}{2}\left[\frac{\pi}{2}\right] = \frac{\pi}{4}}$$

$$= \frac{\pi}{4}\left[\log\left(x + \sqrt{1+x^2}\right)\right]_0^1$$

$$= \frac{\pi}{4}\left[\log\left[1+\sqrt{2}\right]\right]$$

Q. 1. Show that $\int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3}\, dy \neq \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3}\, dx$

$\Rightarrow \quad \int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3}\, dy$

Int. w.r.t. y (x const.)

$$= \int_0^1 dx \left[\int_0^1 \frac{x-y}{(x+y)^3}\, dy\right] = \int_0^1 dx \left[\int_0^1 \frac{x-y+x-x}{(x+y)^3}\, dy\right]$$

$$= \int_0^1 dx \left[\int_0^1 \frac{2x}{(x+y)^3}\, dy - \int_0^1 \frac{\cancel{(x+y)}}{(x+y)^{\cancel{3}2}}\, dy\right]$$

$$= \int_0^1 dx \left[\frac{\cancel{2}x(x+y)^{-2}}{-\cancel{2}} + (x+y)^{-1}\right]_0^1$$

$$= \int_0^1 dx \left[-\frac{\cancel{1}}{\cancel{2}} + \frac{\cancel{1}}{\cancel{2}}\, x\left(\frac{1}{1+x}\right)^2 + \frac{1}{(1+x)} - \frac{x}{x^2} - \frac{1}{x}\right)$$

$$= \int_0^1 dx \left[\frac{\cancel{(1+x)}}{(1+x)^2} - \frac{1}{(1+x)^2} + \frac{1}{(1+x)} - \frac{2}{x}\right]$$

$$= \left[2\log(1+x) + \frac{1}{\cancel{3}(1+x)^{\cancel{3}}} - \cancel{\frac{1}{2(1+x)^2}} - 2\log x\right]_0^1$$

$$= \cancel{2\log 2} + \frac{1}{\cancel{3}\times 8} - \cancel{\frac{1}{2\times 4}} - \cancel{2\log 1} - \frac{1}{3} + \cancel{\frac{1}{2}}\Big]$$

$$= \cancel{2\log 2 - \frac{1}{12}} \qquad 2\log 2 + \frac{1}{2} - 1$$

$$= 2\log 2 - \frac{1}{2}$$

Q.2. Show that $\int_1^2 \int_3^4 (xy + e^y)\, dy\, dx = \int_3^4 \int_1^2 (xy+e^y)\, dx\, dy$

$$= \int_1^2 \int_3^4 (xy + e^y)\, dy\, dx$$

Int. wrt y (x Const.)

$$= \int_1^2 \left[\int_3^4 (xy + e^y)\, dy \right] dx$$

$$= \int_1^2 \left[\frac{xy^2}{2} + e^y \right]_3^4 dx = \int_1^2 \left[8x + e^4 - \frac{9x}{2} - e^3 \right] dx$$

$$= \left[\frac{7}{2} \frac{x^2}{\times 2} + xe^4 - xe^3 \right]_1^2$$

$$= 7 + 2e^4 - 2e^3 - \frac{7}{4} - e^4 + e^3$$

$$= \frac{21}{4} + e^4 - e^3$$

$$\int_3^4 \int_1^2 (xy + e^y)\, dx\, dy$$

Int. wrt. x (y Const.

$$= \int_3^4 \left[\int_1^2 (xy + e^y)\, dx \right] dy$$

$$= \int_3^4 \left[\frac{x^2}{2} y + xe^y \right]_1^2 dy$$

$$= \int_3^4 \left[\frac{4}{2} y + 2e^y - \frac{y}{2} - e^y \right] dy$$

$$= \int_3^4 \left[\frac{3y}{2} + e^y \right] dy = \left[\frac{3y^2}{4} + e^y \right]_3^4$$

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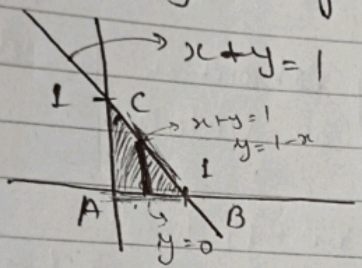
Double Integration.

(1) Evaluate → $\iint e^{2x+3y}\,dx\,dy$ over the triangle bounded by $x=0$, $y=0$ & $x+y=1$

⇒ (1) Identify Region of Integ.



ABCA = R of Int.

(2) Now,
Limit. $\int_0 \int_0 e^{2x+3y}$

(case 3. only possible when fig is sq. or rect. Otherwise case 1 or case 2)

$x \to$ Constant lim. — U.L = $\boxed{x=1}$, L.L = $\boxed{x=0}$

$y \to$ variable lim. — U.L $\boxed{y=1-x}$, L.L $\boxed{y=0}$

draw a strip $\parallel^{r}$ to variable lim.
if no. of curve pt. ↑ No. of strips ↑.

⇒ Now, $$\int_{x=0}^{x=1}\int_{y=0}^{y=1-x} e^{2x+3y}\,dx\,dy$$

$$= \int_0^1 \left[\int_0^{1-x} e^{2x}\cdot e^{3y}\,dy\right]dx$$

$$= \int_0^1 \left[\frac{e^{2x}\cdot e^{3y}}{3}\right]_0^{1-x} dx$$

$$= \frac{1}{3}\int_0^1 e^{2x}\left[\frac{e^3}{e^{3x}} - 1\right]dx$$

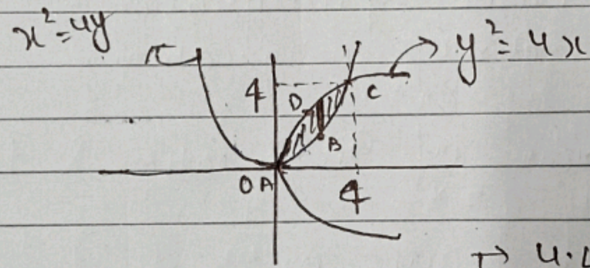
$$= \frac{1}{3}\int_0^1 \left[e^3\cdot e^{-x} - e^{2x}\right]dx$$

$$= \frac{1}{3}\left[-e^3\cdot e^{-x} - \frac{e^{2x}}{2}\right]_0^1 = \frac{1}{3}\left[-e^3\cdot e^{-1} - \frac{e^2}{2} + e^3 + \frac{1}{2}\right]$$

$$= \frac{1}{3}\left[-e^2 - \frac{e^2}{2} + e^3 + \frac{1}{2}\right] = \frac{1}{3}\left[-\frac{3e^2}{2} + e^3 + \frac{1}{2}\right]$$

$$= -\frac{e^2}{2} + \frac{e^3}{3} + \frac{1}{6}$$ Ans.

(2) Evaluate - $\iint_R y\,dx\,dy$ where R is the region bounded by the Parabollas $y^2=4x$ & $x^2=4y$



$y = \frac{4}{x^2}$

$y^2 = \frac{4}{16}$

region of Int. = ABCDA

lim $x \to$ constant $\to$ U.L = 4, L.L = 0

y lim $\to$ variable $\to$ U.L = $\cancel{x^2/4}$, L.L = 0

ADC : $2\sqrt{x}$

ABC $x^2/4$

$$\int_0^4\int y\,dx\,dy = \int_0^4\left[\int_{x^2/4}^{2\sqrt{x}} y\,dy\right]dx$$

$$= \int_0^4\left[\frac{y^2}{2}\right]_{x^2/4}^{2\sqrt{x}} dx = \frac{1}{2}\int_0^4\left[4x - \frac{x^4}{16}\right]dx$$

$$= \frac{1}{2}\left[\frac{4x^2}{2} - \frac{x^5}{16\times 5}\right]_0^4 = 16 - \frac{4\times 4^3}{5\times 16\times 2}$$

$$= \cancel{\frac{80-64}{5}} = \cancel{\frac{16}{5}}\quad 16 - \frac{32}{5}$$

$$= \frac{80-32}{5} = \frac{48}{5}$$ Ans.

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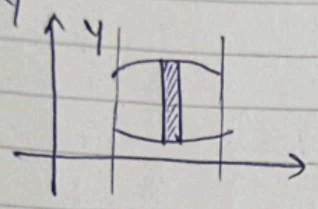
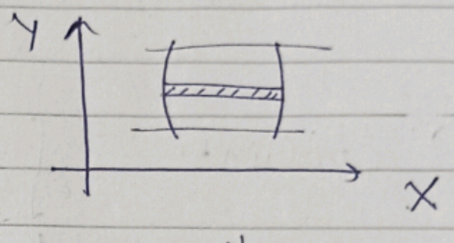
Change of Order of Integration -

$$\int_{x=a}^{b}\int_{y=\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\,dx = \int_{y=c}^{d}\int_{x=f_1(y)}^{f_2(y)} f(x,y)\,dx\,dy$$



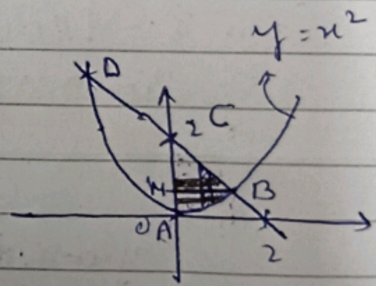


Strip $\parallel^{r}$ to Y-axis   Strip $\parallel^{r}$ to X-axis.

Note → change the order of integration = change the dir. of integration.

Ex - Change the order of Integration & hence Evaluate

$$I = \int_{0}^{1}\int_{x^2}^{2-x} xy\,dy\,dx$$



→ we have $x = 0$ to $x = 1$

$y = x^2$ to $y = 2 - x$

parabola   st. line

Strip $\parallel^{r}$ to y-axis is given

Intersection point

$x^2 = 2 - x$

$x^2 + x - 2 = 0$

$(x+2)(x-1) = 0 \Rightarrow x = 1, -2$

when $x = 1$ $y = 1$

when $x = -2$ $y = 4$

$\Rightarrow$ Int. point $= (1,1)$ & $(-2,4)$

ABCA = region of Int.

Change the order of strip. -
we draw 2 new strip $\parallel^{\text{el}}$ to X-axis
1^{st} strip → MABM from $x=0$ (L.L) to $x=\sqrt{y}$ (U.L)

2^{nd} strip → MBCM from $x=0$ to $x=2-y$
Now ABCA = MABM + MBCM
I = I_1 I_2

$$\Rightarrow \int_{x=0}^{1}\int_{x^2}^{2-x} xy\,dy\,dx = \int_0^1\int_0^{\sqrt{y}} xy\,dy\,dx + \int_1^2\int_0^{2-y} xy\,dx\,dy$$

$$= \int_0^1 y\left[\frac{x^2}{2}\right]_0^{\sqrt{y}} dy + \int_1^2 \left[\frac{x^2}{2}\right]_0^{2-y} y\,dy$$

$$= \int_0^1 y\cdot\frac{y}{2}\,dy + \int_1^2 \frac{(2-y)^2}{2}\,y\,dy$$

$$= \left[\frac{y^3}{6}\right]_0^1 + \frac{1}{2}\left[\cancel{2}4\frac{y^2}{2} + \frac{y^4}{4} - \frac{4y^3}{3}\right]_1^2$$

$$= \frac{1}{6} + \frac{1}{2}\left[8 + \frac{\cancel{16}^{4}}{\cancel{4}} - 4\cdot\frac{8}{3} - 2 - \frac{1}{4} + \frac{4}{3}\right]$$

$$= \frac{1}{6} + \cancel{3}5 - \frac{16}{3} + \frac{2}{3} - \frac{1}{8} = \frac{1}{6} + 5 - \frac{14}{3} - \frac{1}{8}$$

$$= -\frac{\cancel{27}9}{\cancel{6}2} - \frac{1}{8}$$

Q. Change the order of Int. & hence evaluate

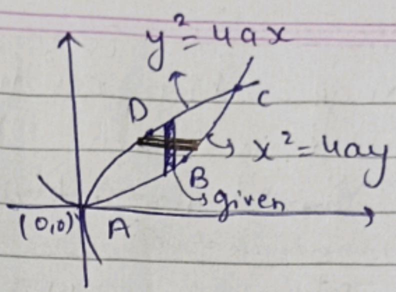
$$\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx$$

⇒ we have $x=0$ & $x=4a$

$y = \frac{x^2}{4a} \Rightarrow x^2 = 4ay$ parabola & $y = 2\sqrt{ax} \Rightarrow y^2 = 4ax$ parabola

strip $\parallel^{\text{el}}$ to y-axis.

Int. pt.



$y^2 = 4ax$ & $x^2 = 4ay$

$\Rightarrow \quad \frac{x^2}{4a} = 2\sqrt{ax}$

$\frac{x^4}{16a^2} = 4ax$

$x^4 - 64a^3x = 0$

$x(x^3 - 64a^3) = 0$

$\Rightarrow x = 0$ & $x = 4a$

when $x = 0 \Rightarrow y = 0$

when $x = 4a \Rightarrow y = 4a$

→ Change the Order of Int.

we draw 1 strip $\|^{el}$ to X-axis from $x = 2\sqrt{ay}$ to $x = \frac{y^2}{4a}$

$$\Rightarrow \int_0^4 \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx = \int_0^{4a} \int_{x = \cancel{2\sqrt{ay}}\ y^2/4a}^{\cancel{y^2/4a}\ 2\sqrt{ay}} dx\,dy$$

$$= \int_0^{4a} [x]_{2\sqrt{ay}}^{y^2/4a}\, dy$$

$$= \int_0^{4a} \left[\frac{-y^2}{4a} + 2\sqrt{ay}\right] dy$$

$$= \left[\frac{-y^3}{12a} + 4\sqrt{a}\,\frac{y^{3/2}}{3}\right]_0^{4a}$$

$$= -\frac{\cancel{4}\cdot 16a^{\cancel{3}2}}{3\cancel{12}\cancel{a}} + \frac{4}{3}\sqrt{a}\cdot a\sqrt{a}\cdot 8$$

$$= \frac{16a^2}{3} \quad \text{Ans.}$$

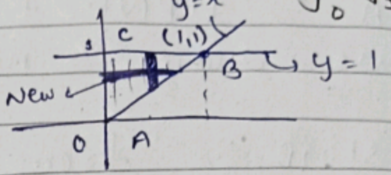
Q. Chang order of Int. & hence solve $\int_0^1 \int_x^1 \sin y^2 \, dy\, dx$

→ given $x=0$ & $x=1$

$y = x$ & $y = 1$

Strip $||^{r}$ to y-axis

Region of Int. ABCA



Change the Order of Int.

we draw new strip $||^{r}$ to X-axis

from $x = 0$ to $x = y$

L.L   U.L

Now, $\int_0^1 \int_x^1 \sin y^2 \, dy\, dx = \int_0^1 \int_0^y \sin y^2 \, dx\, dy$

$= \int_0^1 [x]_0^y \sin y^2 \, dy$

$= \frac{1}{2}\int_0^1 2y \sin y^2 \, dy$

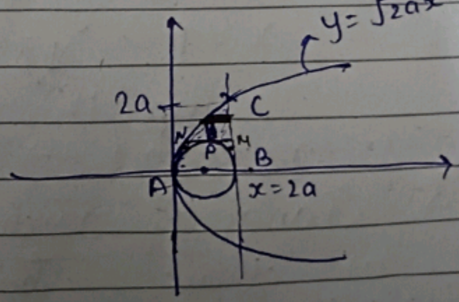
$= -\frac{1}{2}[\cos y^2]_0^1$

$= -\frac{1}{2}[\cos 1 - 1]$

$= \frac{1 - \cos 1}{2}$

Ex - change the order of int. ~~& hence solve~~

$\int_0^{2a} \int_{\sqrt{2ax - x^2}}^{\sqrt{2ax}} f(x,y)\, dy\, dx$



→ we have $x = 0$ & $x = 2a$

$y = \sqrt{2ax - x^2}$ & $y = \sqrt{2ax}$

strip $||^{r}$ to Y-axis

Int. pt.

$2ax - x^2 = 2ax$

$x^2 = 0$

when $x = 0 \to y = 0$

$y^2 + x^2 - 2ax + a^2 - a^2 = 0$

$y^2 + (x-a)^2 = a^2$

$\Rightarrow \boxed{r = a}$ centre $= (a, 0)$

To change ~~the~~ the Order of Int.

$\int ABCA =$ APNA + BMPB + NMCN

We draw 3 new strips $\parallel^{el}$ to X-axis

I^{st} strip $\rightarrow$ from $x = \frac{y^2}{2a}$ to $x = a - \sqrt{a^2 - y^2}$

II^{nd} strip $\rightarrow$ from $x = a + \sqrt{a^2 - y^2}$ to $x = 2a$

III^{rd} strip $\rightarrow$ from $x = \frac{y^2}{2a}$ to $x = 2a$

$$I = I_1 + I_2 + I_3$$

$$\int_0^{2a} \int_{\sqrt{2ax - x^2}}^{\sqrt{2ax}} f(x,y)\, dy\, dx = \iint_{y^2/2a}^{a - \sqrt{a^2 - y^2}} dx\, dy + \int_{y=0}^{a} \int_{a + \sqrt{a^2 - y^2}}^{2a} dx\, dy + \int_{y=a}^{2a} \int_{y^2/2a}^{2a} dx\, dy$$

Q. Change the Order of Integr. $\int_0^1 \int_{x^2}^{2-x} f(x,y)\, dy\, dx$

$\Rightarrow$ we have $x = 0$ & $x = 1$

$y = x^2$ & $y = 2 - x$

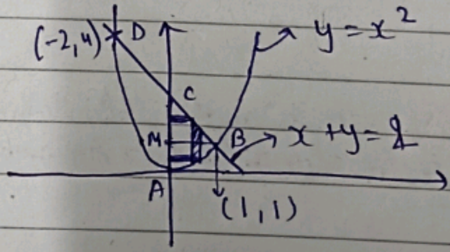
pt. of Inter.

$x^2 = 2 - x$

$x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

$x = -2$ & 1



when $\Rightarrow x = -2 \quad y = 4$

$x = 1 \quad y = 1$

strip $\parallel^{el}$ to Y-axis

$\therefore x = 0$ to $1 \Rightarrow$ Reg. of Int. ABCA

$\rightarrow$ Change of Order.

We draw 2 strip $\parallel^{el}$ to X-axis

Strip 1 from $x = 0$ to $x = \sqrt{y}$

strip 2 from $x = 0$ to $x = 2 - y$

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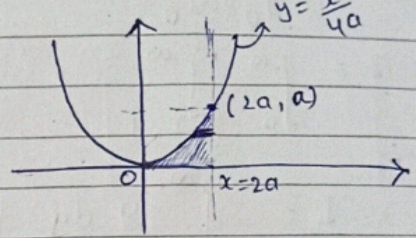
$$\Rightarrow I = \int_0^1\int_0^{\sqrt{y}} dx\,dy + \int_1^2\int_0^{2-y} dx\,dy$$ Ans.

(2) $$\int_0^{2a} dx \int_0^{x^2/4a} (x+y)^3\, dy.$$



we have, $x=0$ to $x=2a$
& $y=0$ & $y=\frac{x^2}{4a}$

strip $\|^{r}$ to Y-axis

pt. of int.

$\frac{x^2}{4a}=0$

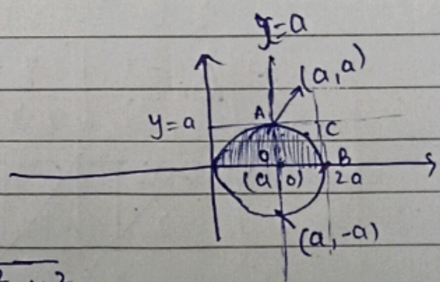
$x=0$

$\Rightarrow$ when $x=0 \Rightarrow y=0$

Change of order
strip $\|^{r}$ to X-axis
from $x=2\sqrt{ay}$ to $x=2a$

$$I=\int_0^a\int_{2\sqrt{ay}}^{2a}(x+y)^3\,dx\,dy$$

(3) $$\int_0^a\int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} xy\,dx\,dy.$$



we have $y=0$ & $a=y$
& $x=a+\sqrt{a^2-y^2}$ & $x=a-\sqrt{a^2-y^2}$

pt. of int.

$a+\sqrt{a^2-y^2}=a-\sqrt{a^2-y^2}$

$\Rightarrow 2\sqrt{a^2-y^2}=0$

$\Rightarrow \boxed{y^2=a^2} \Rightarrow y=\pm a$

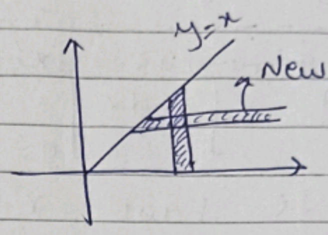
Region of Int. = OACBO

when $y=a \to x=a$
$y=-a \to x=a$

Change of order of Int.
draw 2 strip $\|^{r}$ to Y-axis
Strip1 $\to$ from $y=-\sqrt{2ax-x^2}$ to $y=a$
Strip 2 $\to$ from $y=a$ to $y=\sqrt{2ax-x^2}$

$x=a-\sqrt{a^2-y^2}$
$(x-a)^2=a^2-y^2$
$y^2=a^2-(x-a)^2$
$(x-a)^2+y^2=a^2$
$C\to(a,0) \to y=\pm\sqrt{2ax-x^2}$

(4) $$\int_0^\infty \int_{y=0}^{x} x e^{-x^2/y}\, dy\, dx$$



$$I = \int_0^\infty \int_{x=y}^{\infty} x e^{-x^2/y}\, dx\, dy$$

$$\frac{x^2}{y} = t \qquad x^2 = yt \qquad 2x\,dx = y\,dt$$

$$I = \int_0^\infty \frac{y}{2}\, dy \int_{x=y}^{\infty} e^{-t}\, dt$$

$$= \int_0^\infty \frac{y}{2}\, dy \left[\frac{e^{-x^2/y}}{-1}\right]_{x=y}^{\infty}$$

$$= \frac{1}{2}\int_0^\infty y e^{-y}\, dy$$

$$= \frac{1}{2}\left[-y e^{-y} - e^{-y}\right]_0^\infty = -\frac{1}{2}\lim_{y\to\infty}\left[-y e^{-y}\right] - \frac{1}{2}\left[e^{-\infty} - e^{0}\right]$$

$$= \lim -\frac{1}{2}\left[\frac{y}{e^y}\right]_\infty + \frac{1}{2} = 0 - \frac{1}{2}\left[\frac{1}{e^y}\right]_\infty + \frac{1}{2}$$

$$= \cancel{-\frac{1}{2}\left[\frac{1}{e^y}\right]_\infty + \frac{1}{2}} = 0 + \frac{1}{2} = \frac{1}{2}$$

Change of Variable - if $x = \phi_1(u,v)$ & $y = \phi_2(u,v)$ then

To Convert the $\iint_R f(x,y)\,dx\,dy$ into $\iint f(\phi_1(u,v), \phi_2(u,v)) \cdot (|J|\,du\,dv)$

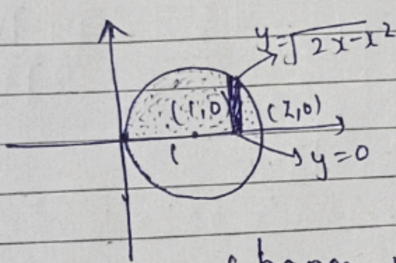
Ex- Evaluate double integral $\int_0^2 \int_{y=0}^{y=\sqrt{2x-x^2}} (x^2+y^2)\,dy\,dx$

$\Rightarrow$ we have $x=0$ to $x=2$

& $y=0$ to $y=\sqrt{2x-x^2}$

$\Rightarrow x^2+y^2-2x=0$ [circle]

if circle is boundary $\rightarrow$ use polar $\rightarrow$ becomes very easy ques.



Change into Polar Coordinate

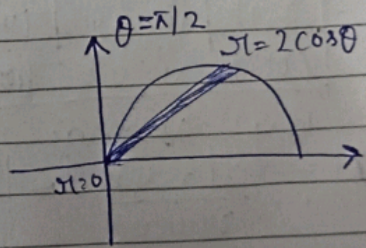
Put $x = r\cos\theta$, $y = r\sin\theta$

① $dx\,dy = |J|\,dr\,d\theta = r\,dr\,d\theta$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\sin\theta \end{vmatrix} = r$$

$\boxed{|J| = r}$

② $\theta = 0$ to $\theta = \pi/2$



$x^2+y^2-2x=0$

$r^2[\cos^2\theta+\sin^2\theta] - 2r\cos\theta = 0$

$r^2 - 2r\cos\theta = 0$

$\Rightarrow r=0$ & $r = 2\cos\theta$

③ $I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{2\cos\theta} [(r\cos\theta)^2 + (r\sin\theta)^2]\,|J|\,dr\,d\theta$

$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{2\cos\theta} r^2 \cdot r\,dr\,d\theta = \int_{\theta=0}^{\pi/2} d\theta \cdot \left[\frac{r^4}{4}\right]_0^{2\cos\theta}$

$= \int_0^{\pi/2} 4\cos^4\theta\,d\theta = \left[\cancel{4\ldots}\right]_0^{\pi/2} =$

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$$I = 4 \times \frac{\Gamma\frac{4+1}{2} \cdot \Gamma\frac{0+1}{2}}{2\,\Gamma\frac{4+0+2}{2}}$$

$$= 2 \cdot \frac{\Gamma 5/2 \cdot \Gamma 1/2}{\Gamma 3}$$

$$= \frac{\cancel{2} \times 3/2 \times \frac{1}{2}\sqrt{\pi} \times \sqrt{\pi}}{\cancel{\lfloor 2}}$$

$$\boxed{I = \frac{3\pi}{4}}$$

gamma 5

$\Gamma 5 = \lfloor 4 = 4 \times 3 \times 2 \times 1$

$\Gamma\frac{5}{2} = \frac{3}{2} \times \frac{1}{2}\sqrt{\pi}$

$$\int_0^{\pi/2} (\sin\theta)^m (\cos\theta)^n \, d\theta = \frac{\Gamma\frac{m+1}{2} \cdot \Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}}$$

ex- $\int_0^{\pi/2} \sin^7\theta \, d\theta$ $m=7$, $n=0$

$$= \frac{\Gamma\frac{7+1}{2} \cdot \Gamma\frac{0+1}{2}}{2\,\Gamma\frac{7+0+2}{2}} = \frac{\Gamma 4 \cdot \Gamma 1/2}{2\,\Gamma 9/2}$$

$$= \frac{\lfloor 3 \cdot \sqrt{\pi}}{2 \times \frac{7}{2} \times \frac{5}{2} \pi \frac{3}{3} \times \frac{1}{2}\sqrt{\pi}}$$

Imp.

Q. ~~Using~~ Evaluate $\iint_R (x+y)^2 \, dx\, dy$ where R is ||gm in the X-Y plane with vertices (1,0) (3,1) (2,2) & (0,1) using transformation $u = x+y$, $v = x-2y$.

$\Rightarrow$

(x,y)	$u = x+y$, $v = x-2y$	(u,v)
$(1,0)$	$u=1$, $v = 1-0=1$	$(1,1)$
$(3,1)$	$u=4$, $v = 3-2=1$	$(4,1)$
$(2,2)$	$u=4$, $v = 2-4=-2$	$(4,-2)$
$(0,1)$	$u=1$, $v = 0-2=-2$	$(1,-2)$

$1 \le u \le 4$

$-2 \le v \le 1$

By change of variable

$$|J| = \frac{\partial(x,y)}{\partial(u,v)} = \left| \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} \right| = \frac{1}{\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}} = \frac{1}{\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}}$$

$$= \left| -\frac{1}{3} \right| = \frac{1}{3} \text{ Ans}$$

$$\boxed{|J| = \frac{1}{3}}$$

$$\iint_R (x+y)^2 \, dx\, dy = \iint u^2 |J| \, du\, dv$$

$$= \int_{v=-2}^{1} \int_{u=1}^{4} \frac{u^2}{3} \, du\, dv = \frac{1}{3}\int_{-2}^{1} \left[\frac{u^3}{3}\right]_1^4 dv$$

$$= \frac{1}{9}\left[63\, v\right]_{-2}^{1} = 7[1+2] = 21 \text{ Ans}$$

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Q. Evaluate $\iint_R xy\,dx\,dy$ where R is the region in I^{st} quad. bounded by hyperbola $x^2-y^2=a^2$ & $x^2-y^2=b^2$ & the Circle $x^2+y^2=c^2$ & $x^2+y^2=d^2$ where $0<a<b<c<d$.

$\Rightarrow$ Let $x^2-y^2=u$ & $x^2+y^2=v$

$\Rightarrow$ $u=a^2$ (L.L) & $u=b^2$ (u.L)

$v=c^2$ (L.L) & $v=d^2$ (u.L)

$$|J| = \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}} = \frac{1}{\begin{vmatrix} 2x & -2y \\ 2x & 2y \end{vmatrix}}$$

$$|J| = \frac{1}{8xy}$$

By change of var.

Now,

$$I = \iint xy \cdot |J| \cdot du\,dv = \int_{v=c^2}^{d^2}\int_{u=a^2}^{b^2} \cancel{xy} \times \frac{1}{8\cancel{xy}}\,du\,dv$$

$$= \frac{1}{8}[u]_{a^2}^{b^2}[v]_{c^2}^{d^2}$$

$$= \frac{[b^2-a^2][d^2-c^2]}{8}$$

Q. using $x+y=u$, $x-y=v$ evaluate $\iint_R (x^2+y^2)\,dx\,dy$ integration being taken over the area enclosed by $x+y=2$, $x+y=0$, $x-y=2$, $x-y=0$

$\Rightarrow$ $x=\frac{u+v}{2}$, $y=\frac{u-v}{2}$

① $J = \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2}$

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(2) $I = \iint_R (x^2+y^2)\,dx\,dy$

$$= \frac{1}{4}\int_{v=0}^{2}\int_{u=0}^{2}\left[(u+v)^2+(u-v)^2\right]|J|\,du\,dv$$

$$= \frac{1}{8}\times 2\int_{v=0}^{2}\int_{u=0}^{2}(u^2+v^2)\,du\,dv$$

$$= \frac{1}{4}\int_0^2\left[\frac{u^3}{3}+v^2u\right]_0^2 dv$$

$$= \frac{1}{4}\int_0^2 \frac{8}{3}+2v^2\,dv$$

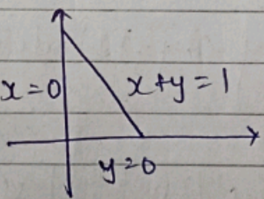
$$= \frac{1}{4}\left[\frac{16}{3}+\frac{16}{3}\right] = \frac{8}{3}$$

(3) Evaluate $\int_0^1\int_0^{1-x} e^{\frac{y}{x+y}}\,dy\,dx$ using transformation $x+y=u$, $y=uv$

→ we have

$x=0$ to $x=1$

$y=0$ to $y=1-x$



we have → $\boxed{x = u-uv}$

& $y = uv$

put $x=0 \Rightarrow u-uv=0$

$\Rightarrow u=0$ & $v=1$

$0\le u\le 1$

$0\le v\le 1$

$y=0 \Rightarrow uv=0$

$u=0$, $v=0$

$x+y=1 \Rightarrow \boxed{u=1}$

$$|J| = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u-vu+uv$$

$$= u$$

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By change of var.

$$I = \int_{v=0}^{1}\int_{u=0}^{1} e^{\frac{uv}{4}} \cdot |J|\, du\, dv$$

$$= \int_0^1\int_0^1 e^{\frac{4v}{4}} \cdot u\, du\, dv$$

$$= \int_0^1 e^{v} \cdot \left[\frac{u^2}{2}\right]_0^1 dv = \left[\frac{e^v}{2}\right]_0^1$$

$$= \frac{e-1}{2} \text{ Ans.}$$

Ex – Evaluate $\iint (x+y)^2\, dx\, dy$ using $u = x+y$, $v = y$

$\Rightarrow$ $x = u - v$, $y = v$

$$|J| = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 2$$

$|J| = 2$

By change of vari.

$$I = \iint u^2 |J|\, du\, dv = \iint 2u^2\, du\, dv$$

$$= \int \left[\frac{2u^3}{3}\right] dv$$

Ex. Evaluate $\iint (x^2+y^2)\, dx\, dy$ over the area in +ve quadrant by $x^2-y^2=1$, $x^2-y^2=2$, $xy=4$, $xy=2$

$\Rightarrow$ $u = x^2-y^2$ $\quad v = xy$

$$\Rightarrow |J| = \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix}} = \frac{1}{2x^2+2y^2}$$

$$\Rightarrow I = \int_{v=2}^{4}\int_{1}^{2} \cancel{(x^2+y^2)} \times \frac{1}{2\cancel{(x^2+y^2)}}\, du\, dv = \int_2^4 \left[\frac{1}{2} \cdot u\right]_1^2 dv$$

$$= 1 \text{ Ans}$$

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Ex. Evaluate $\int_0^1 \int_0^x \sqrt{x^2+y^2}\, dx\, dy$ using $x = u$, $y = uv$

$\Rightarrow |J| = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 0 \\ v & u \end{vmatrix} = u$

$y=0$
$x=0 \to u=0$
$x=1 \to u=1$
$y=0 \to v=0$
$y=x \to uv=u$, $v=1$

$I = \int_{u=0}^{1} \int_{v=0}^{1} u\sqrt{1+v^2} \cdot u\, du\, dv$

$= 2\int_0^1 u\left[\dfrac{(1+v^2)^{3/2}}{3}\right]_0^u du$

$= \dfrac{2}{3}\int \left[u \left((1+u^2)^{3/2} - 1 \right) \right] du$

$= \dfrac{2}{3}\int_0^1 u(1+u^2)^{3/2} du - \dfrac{2}{3}\int_0^1 u\, du$

$= \dfrac{2}{3} \int_0^1 \int_0^1 u^2 \left[\dfrac{v^2}{2}\sqrt{1+v^2} + \dfrac{1}{2}\log|v+\sqrt{1+v^2}| \right]$

$= \int_0^1 u^2 \left[\dfrac{\sqrt{2}}{2} + \dfrac{1}{2}\log|1+\sqrt{2}| \right] du$

$= \left[\dfrac{u^3}{3}\left[\dfrac{1}{\sqrt{2}} + \dfrac{1}{2}\log|1+\sqrt{2}| \right] \right]_0^1$

$= \dfrac{1}{3}\left[\dfrac{1}{\sqrt{2}} + \dfrac{1}{2}\log|1+\sqrt{2}| \right]$ Ans.

Ex. using transformation $x+y=u$, $y=v$ evaluate $\int_0^\pi \int_0^\pi |\cos(x+y)|\, dx\, dy$

$\Rightarrow$ $x = u - v$, $y = v$

$x=0$, $u=v$
$x=\pi$, $u-v=\pi$, $u=\pi+v$
$y=0$, $v=0$
$y=\pi$, $v=\pi$

$|J| = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$

$I = \int_0^\pi dv \int_0^{\pi+v} |\cos u|\, du$

$$= \int_0^\pi dv \left[\int_{v}^{\frac{\pi}{2}} \cos u \, du - \int_{\pi/2}^{3\pi/2} \cos u \, du + \int_{3\pi/2}^{\pi+v} -\cos u \, du \right]$$

$$= \int_0^\pi dv \left[[\sin u]_{v}^{\pi/2} - [\sin u]_{\pi/2}^{3\pi/2} - [\sin u]_{3\pi/2}^{\pi+v} \right]$$

$$= \int_0^\pi dv \left[1 - 0 + 0 + 1 - \cancel{\sin v} + 1 + \cancel{\sin v} - 1 \right]$$

$$= \cancel{8\int_0^\pi \sin v \, dv = 8[\cos v]_0^\pi = 8[-1-1]}$$

$$2\pi$$

$$= \cancel{\pi} . \qquad = 2\pi \text{ Ans.}$$

Q. Prove that the area in the +ve quadrant bdd by curves $y^2 = 4ax$, $y^2 = 4bx$, $xy = c^2$ & $xy = d^2$ is $\frac{1}{3}(d^2 - c^2)\log\frac{b}{a}$, $d > c$, $b > a$.

→ Let $xy = u \Rightarrow \underset{(L.L)}{u = c^2}$ & $u = d^2$ (U.L)

& $\frac{y^2}{4x} = \underset{(L.L)}{a}$ & $\frac{y^2}{4x} = b$ (U.L)

Let $\frac{y^2}{4x} = v \Rightarrow v = a$ (L.L) & $v = b$ (U.L)

Now, Area $= \iint dx\, dy$

$= \iint |J| \, du\, dv$ [By change of var.]

$$|J| = \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\begin{vmatrix} y & x \\ -\frac{y^2}{4x^2} & \frac{y}{2x} \end{vmatrix}} = \frac{1}{\frac{y^2}{2x} + \frac{y^2}{4x}} = \frac{4x}{3y^2}$$

$$|J| = \frac{4x}{3y^2} = \frac{1}{3v}$$

$$\text{Area} = \int_{c^2}^{d^2}\int_a^b \frac{1}{3v} \, du\, dv = \frac{1}{3}\int_{c^2}^{d^2} [\log v]_a^b$$

$$= \frac{1}{3}(d^2 - c^2)\log\frac{b}{a} \quad \text{P.}$$

Area :-

① $Z = f(x, y)$

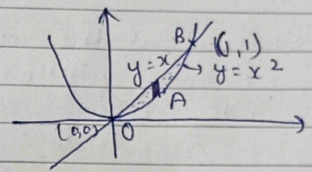
Area $= \iint dx\,dy$ or $\iint dy\,dx$

② If $r = f(\theta)$

Area $= \iint r\,dr\,d\theta$

Ex - find Area enclosed b/w $y = x^2$ & $y = x$. region of integration OABO

Area $= \iint dy\,dx$



draw strip ‖^{el} to Y-axis from $y = x^2$ (L.L) to $y = x$ (U.L)

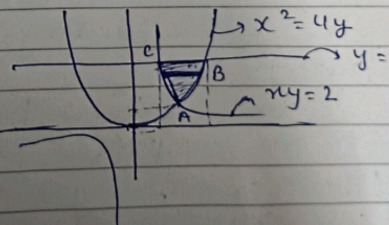
$$\text{Area} = \int_{x=0}^{1}\int_{y=x^2}^{x} dy\,dx$$

$$= \int_0^1 dx\,[x^2 - x] = \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_0^1 = \frac{1}{3} - \frac{1}{2} = \frac{-1}{6}\,m^2$$

Ex. Determine Area bdd by curves $xy = 2$, $4y = x^2$ & $y = 4$.

→ Region of Int. = ABCA

Draw strip ‖^{el} to X-axis from $x = \frac{2}{y}$ to $x = 2\sqrt{y}$



$$\text{Area} = \iint dx\,dy$$

$$= \int_{y=1}^{4}\int_{2/y}^{2\sqrt{y}} dx\,dy = \int_1^4 2\sqrt{y} - \frac{2}{y}\,dy$$

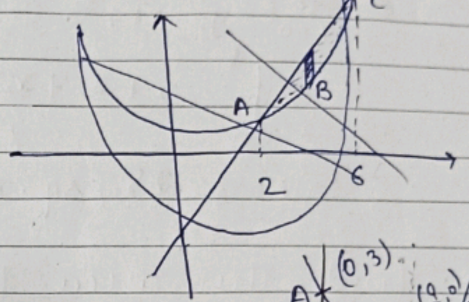
$$= 2 \times \frac{2y^{3/2}}{3} - 2\log y$$

$= \frac{4}{3}[8-1] - 4\log 2$

$= 4\left[\frac{7}{3} - \log 2\right]$ Ans.

Q. Find Area b/w curves $y = x^2-6x+3$ & $y = 2x-9$

→ Region of Int. → ABCA



$\cancel{(2x-9)^2 = x^2-6x+3}$

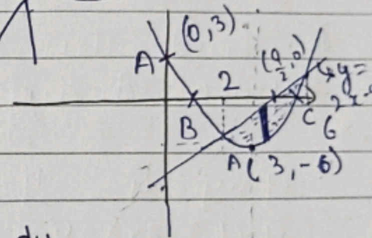
$\cancel{4x^2+81-36x = x^2-6x+3}$

$\cancel{3x^2-30x+78=0}$

$\cancel{x^2-10x+26=0}$

$x^2-8x+12=0$

$x = 6$ & $x = 2$ ✓



when $x = 2$ $y = -5$

$x = 6$ $y = 3$

Area $= \iint dx\,dy = \iint dx\,dy$

$y = 2x-9$

x	0	9/2
y	-9	0

$y = x^2-6x+3$

$y = (x^2-6x+9)+3-9$

$y+6 = (x-3)^2$

$Y = X^2$

For vertex $Y=0 \Rightarrow y=-6$

$X=0 \Rightarrow x=3$

Vertex $= (3,-6)$

if $x=0 \Rightarrow y = 0-0+3 \Rightarrow y=3$ A.pt $(0,3)$

if $y=0 \Rightarrow 0 = x^2-6x+3 \Rightarrow x = \frac{6\pm3\sqrt{6}}{2} = 3\pm\sqrt{6}$

$B = (3-\sqrt{6}, 0)$ & $C = (3+\sqrt{6}, 0)$

Now, Area $= \int_2^6 \int_{x^2-6x+3}^{2x-9} dx\,dy = \int_2^6 2x-9-x^2+6x-3$

Strip $\parallel^{r}$ to y

$= \int_2^6 -x^2+8x-12$

$= -\frac{36\times6}{3} + \frac{8\times36}{2} - 12\times6 + \frac{8}{3} - 8 + 24$

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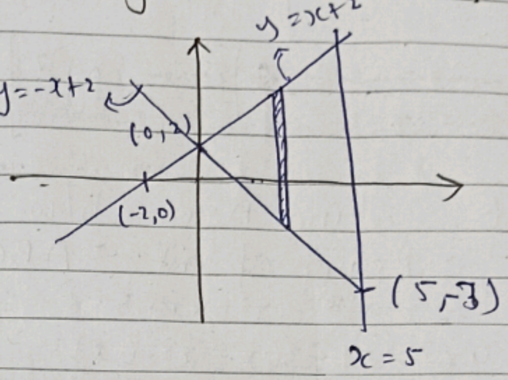
(2) Find Area bdd by lines $y = x+2$ & $x = 5$ & $y = -x+2$



$$\text{Area} = \iint dy\,dx$$

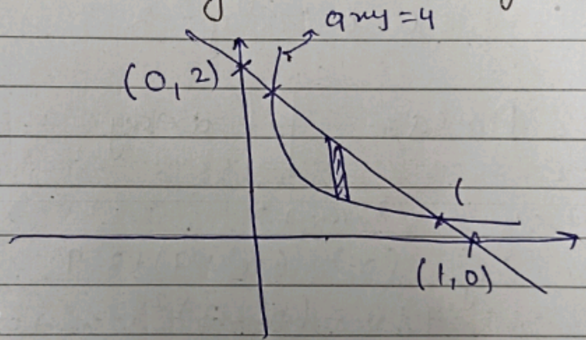
$$= \int_{x=0}^{5} dx \int_{y=-x+2}^{x+2} dy$$

$$= \int_0^5 x+2+x-2$$

$$= 25 \text{ sq. unit. Ans.}$$

Q. find the area by double integration enclosed by the curves $9xy = 4$ & $2x+y=2$

→



$9xy = 4 \qquad y = 2-2x$

$9x(2-2x) = 4$

$18x - 18x^2 = 4$

$\Rightarrow 9x^2 - 9x + 2 = 0$

$9x^2 - 6x - 3x + 2 = 0$

$3x(3x-2) - 1(3x-2) = 0$

$\Rightarrow x = \frac{1}{3}, \frac{2}{3}$

draw strip ॥'' to Y-axis. from $y = \frac{4}{9x}$ to $y = 2-2x$

$$\text{req. Area} = \int_{1/3}^{2/3} \int_{4/9x}^{2-2x} dy\,dx = \int_{1/3}^{2/3} 2-2x - \frac{4}{9x}\,dx$$

$$= \left[2x - x^2 - \frac{4}{9}\log x\right]_{1/3}^{2/3}$$

$$= \frac{4}{3} - \frac{4}{9} - \frac{4}{9}\log\frac{2}{3} - \frac{2}{3} + \frac{1}{9} + \frac{4}{9}\log\frac{1}{3}$$

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$$= \frac{2}{3} - \frac{1}{3} + \frac{4}{9}\log\frac{\cancel{3}}{2}\times\frac{1}{\cancel{3}} = \frac{1}{3} + \frac{4}{9}\log\frac{1}{2} \quad \text{Ans.}$$

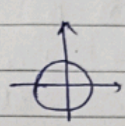
$$= \frac{1}{3} - \frac{4}{9}\log 2 \quad \text{Ans.}$$

Area in polar Coord.

$$A = \iint r\, dr\, d\theta$$

Circle

(1) $x^2 + y^2 = a^2$



To change into polar

put $x = r\cos\theta,\ y = r\sin\theta$

$r^2[\cos^2\theta + \sin^2\theta] = a^2$

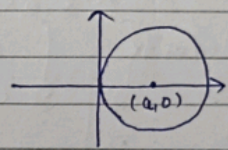
$\boxed{r = a}$

Centre → $(0,0)$

radius → a

(2) $x^2 + y^2 - 2ax = 0$

$(x-a)^2 + y^2 = a^2$



$x = r\cos\theta,\ y = r\sin\theta$

$= r^2[\cos^2\theta + \sin^2\theta] - 2ar\cos\theta = 0$

$\Rightarrow \boxed{r = 2a\cos\theta}$

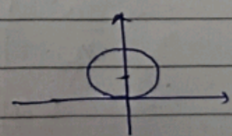
Centre → $(a,0)$

radius → $2a\cos\theta$

(3) $x^2 + y^2 - 2ay = 0$

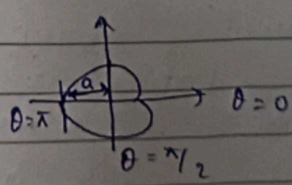
$\Rightarrow$ centre → $(0,a)$

radius → $2a\sin\theta$

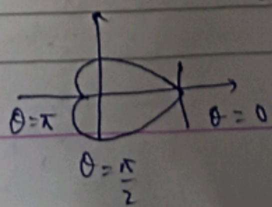


(4) $r = a[1-\cos\theta]$

(cardioid)



$r = a[1+\cos\theta]$



Loop curves – $r = a\sin n\theta$, $r = a\cos n\theta$ $\Big\}$ no. of loops

$\downarrow$

$n \to n$ is odd

$2n \to n$ is even

Ex – find the area of the curve $r = a\sin 3\theta$

$\Rightarrow$

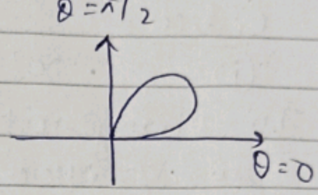
$\to$ no. of loops $= 3$

$\because$ at Origin $\vec{r} = 0$

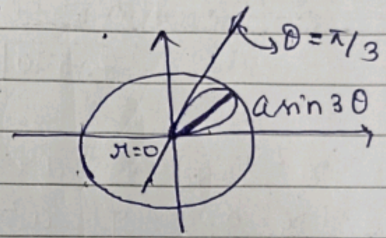
$\Rightarrow r = 0 \Rightarrow a\sin 3\theta = 0$

$3\theta = 0, \pi, \ldots$

$\theta = 0, \frac{\pi}{3}, \ldots$



$\Rightarrow$ All three loops will be in circle



Required Area $= 3\times\int_0^{\pi/3}\int_{r=0}^{a\sin 3\theta} r\,dr\,d\theta$

$$= 3\int_0^{\pi/3}\left[\frac{r^2}{2}\right]_0^{a\sin 3\theta} d\theta = \int_0^{\pi/3} \frac{a^2\sin^2 3\theta}{2}\,d\theta$$

$$= 3\int_0^{\pi/3} \frac{a^2}{2}\left[\frac{1-\cos 6\theta}{2}\right] d\theta$$

$$= \frac{3a^2}{4}\left[\frac{\pi}{3} - \cancel{1} + \cancel{1}\right]$$

$$= \frac{\cancel{3}a^2}{4}\frac{\pi}{\cancel{3}} = \frac{\pi a^2}{4}$$

Q. Find Area of Curve $r = a\cos 3\theta$

$\Rightarrow$ $\to$ No. of Loops $= 3$

Required Area $= 3 \times$ Area of 1 Loop

If $r = 0 \Rightarrow a\cos 3\theta = 0$

$3\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

$\theta = \frac{\pi}{6}, \frac{\pi}{2}$

1st Loop from

$\because$ sym. along x-axis

$\boxed{-\frac{\pi}{6} < \theta < \frac{\pi}{6}}$

$$\text{Required Area} = 3 \times \iint r\,dr\,d\theta$$

$$= 3 \int_{\theta=\frac{\pi}{6}}^{\pi/6} \int_{r=0}^{a\cos 3\theta} r\,dr\,d\theta$$

$$= 3 \int_{-\pi/6}^{\pi/6} \left[\frac{r^2}{2}\right]_0^{a\cos 3\theta}$$

$$= \frac{3}{2} \int_{-\pi/6}^{\pi/6} a^2 \cos^2 3\theta$$

$$= \frac{3a^2}{2} \int_{-\pi/6}^{\pi/6} \frac{1+\cos 6\theta}{2}\,d\theta$$

$$= \frac{3a^2}{4}\left[\frac{\pi}{6} + \frac{\pi}{6} + \cancel{\int_{-\pi/6}^{\pi/6} \cos 6\theta}^{\,0}\right]$$

$$= \frac{\cancel{3}a^2}{4} \times \frac{\pi}{\cancel{3}}$$

$$= \frac{\pi a^2}{4}$$

Q. Find area common to the circle $r=a$ & $r=2a\cos\theta$

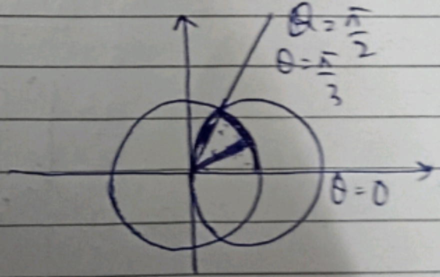
$\Rightarrow$ For intersection pt of

$r=a$ & $r=2a\cos\theta$

$a = 2a\cos\theta$

$\Rightarrow \cos\theta = \frac{1}{2}$

$\theta = \pi/3$



we draw two strips from

I) $r=0$ to $r=a$

II) $r=0$ to $r=2a\cos\theta$

$$\text{Area} = \iint r\,dr\,d\theta = 2\left[\int_{\theta=0}^{\frac{\pi}{3}} \int_{r=0}^{a} r\,dr\,d\theta + \int_{\frac{\pi}{3}}^{\pi/2} \int_{r=0}^{2a\cos\theta} r\,dr\,d\theta\right]$$

$$= 2\left[\int_0^{\frac{\pi}{3}} \frac{a^2}{2}\,d\theta + \int_{\pi/2}^{\pi/2} 2a^2\cos^2\theta\,d\theta\right]$$

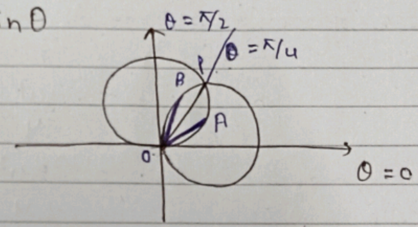
$$= \frac{2a^2\pi}{6} + 2a^2\int_{\pi/3}^{\pi/2} 1+\cos 2\theta \, d\theta$$

$$= \frac{a^2\pi}{6}\cdot 2 + 2a^2\frac{\pi}{6} + \frac{a^2}{2}\left[\sin 2\theta\right]_{\pi/3}^{\pi/2}$$

$$= \frac{a^2\pi}{6}\cdot 2 + 2\cdot a^2\frac{\pi}{6} + \frac{a^2}{2}\left[-\frac{\sqrt{3}}{2}\right]$$

$$= \frac{2a^2\pi}{3} - \frac{a^2\sqrt{3}}{4}$$

Ex - find the area common to the circles $r = a\cos\theta$ & $r = a\sin\theta$



$\Rightarrow$ pt. of intersection P

$a\cos\theta = a\sin\theta$

$\tan\theta = 1$

$$\boxed{\theta = \frac{\pi}{4}}$$

total area = OAPO + OPBO

$$= \int_{\theta=0}^{\pi/4}\int_{r=0}^{a\sin\theta} r\,dr\,d\theta + \int_{\pi/4}^{\pi/2}\int_{r=0}^{a\cos\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi/4} \frac{a^2\sin^2\theta}{2}\,d\theta + \int_{\pi/4}^{\pi/2} \frac{a^2\cos^2\theta}{2}\,d\theta$$

$$= \frac{a^2}{2}\int\left[\frac{1-\cos 2\theta}{2}\right]_0^{\pi/4} + \int\frac{a^2}{2}\left[\frac{1+\cos 2\theta}{2}\right] d\theta$$

$$= \frac{a^2}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/4} + \frac{a^2}{4}\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/4}^{\pi/2}$$

$$= \frac{a^2}{4}\left[\frac{\pi}{4} - \frac{1}{2}\right] + \frac{a^2}{4}\left[\frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{2}\right]$$

$$= \frac{-a^2}{8} + \frac{a^2\pi}{8} - \frac{a^2}{8}$$

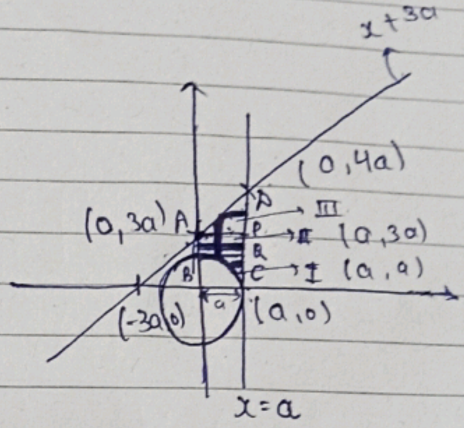
$$= -\frac{a^2}{4} + \frac{a^2\pi}{8}.$$

Q. Change the Order of Integration $\int_0^a \int_{\sqrt{a^2-x^2}}^{x+3a} f(x,y)\,dx\,dy$.

$\Rightarrow$ we have

$x = 0$ to $x = a$

$y = \sqrt{a^2-x^2}$ & $y = x+3a$



pt. of intersection.

$\sqrt{a^2-x^2} = x+3a$

$a^2-x^2 = x^2+9a^2+6ax$

$= 2x^2+8a^2+6ax = 0$

$x^2+4a^2+3ax = 0$

~~$x^2+4ax - ax + 4a^2 = 0$~~

~~$x(x+4a) - a(x-$~~

$x = \frac{-3a \pm \sqrt{9a^2-16a^2}}{2}$

imaginary pt. of Int.

$\Rightarrow$ No inter.

Region of Int. ABCDA

Now, Change the Order of Int.

We draw 3 new strip

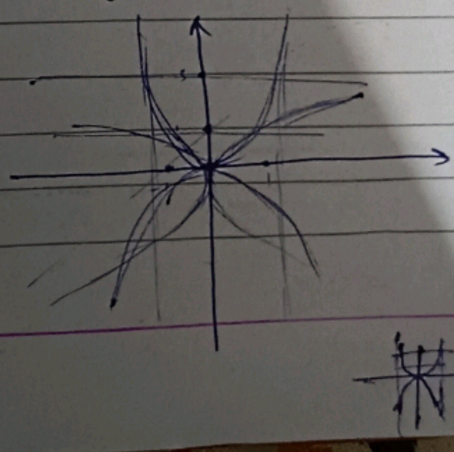
ABCDA = APDA + ABQP + BQCB

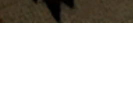
$= \int_{y=3a}^{4a} \int_{y-3a}^{a} f(x,y)\,dx\,dy + \int_a^{3a} \int_0^a f(x,y)\,dx\,dy + \int_0^a \int_0^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy$

Q. Express as a single integral and evaluate.

$I = \int_{y=1}^{3} dy \int_{x=-1}^{1} dx + \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy$

$x = y^2 + 0 \quad x = y^2$





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