(L-1)

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Mathematics

Code: K24AS11

Calculus for Engineers (CFE)

(200 Marks Paper)

- 100 Internal
  - 80
    - Midsem-I (40)
    - Midsem-II (40)
  - 20
    - CT_1 (5) → (Pen Paper)
    - CT_2 (5) → (Quiz)
    - CT_3 (5) → (Assignment)
    - CT_4 (5) → (Attendance)
- 100 External

5 Units

2 - Calculus
2 - Integrals
1 - Vector

UNIT-1

(Calculus-1)

(Differential Calculus-1)

$y = f(x)$ [one variable]

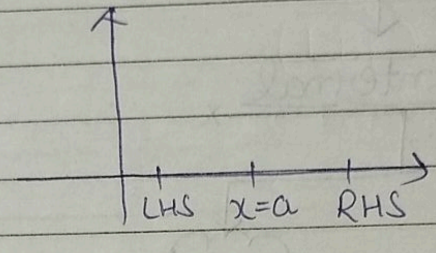
x → independent variable

y → dependent variable

$z = f(x, y)$
↳ independent variables
↳ dependent variables
[two variable]

Limits: $y = f(x)$

$\lim_{x \to a} f(x) = l$



Definition: A function $f(x,y)$ is said to have a limit l in region R as $x \to a$ & $y \to b$ iff (if and only if) corresponding to a tue number $\epsilon > 0$ (epsillion (small quantity))

there exists another number $\delta > 0$ such that

~~$|f(x,y) - l| > \epsilon$ whenever~~

$$0 < |x-a| < \delta$$
$$0 < |y-b| < \delta$$
$$|f(x,y) - l| < \epsilon$$

(OR)

$$\left[\lim_{\substack{x \to a \\ y \to b}} f(x,y) = l \right]$$

NOTE: $\lim_{x \to a} \left[\lim_{y \to b} f(x,y) \right] = \lim_{y \to b} \left[\lim_{x \to a} f(x,y) \right] = l$

(x constant) (y constant)

★ ek ki limit nikalenge dusre ko constant rakhke
★ if LHS = RHS (then limit exists otherwise not)

Que ① → for ex: Evaluate limit : $\lim_{\substack{x \to 1 \\ y \to 1}} \frac{x(y-2)}{y(x-2)}$

Sol: $\lim_{x \to 1} \left[\lim_{y \to 1} \frac{x(y-2)}{y(x-2)} \right] = \lim_{x \to 1} \left[\frac{x(1-2)}{1(x-2)} \right]$

↳ constant

LHS:

$= \lim_{x \to 1} \left(\frac{-x}{x-2} \right) = 1$

RHS: $\lim_{y \to 1} \left[\lim_{x \to 1} \frac{x(y-2)}{y(x-2)} \right]$

$= \lim_{y \to 1} \left[\frac{1(y-2)}{y(-1)} \right] = \frac{1(1-2)}{-1} = 1$

(LHS = RHS)

(limit exists)

Que ② → Evaluate: $\lim_{\substack{x \to \infty \\ y \to 2}} \left(\frac{x^2 y + 4}{x^2 + 2y^2} \right)$

(divide में ले जाओ)

Sol: $= \lim_{\substack{x \to \infty \\ y \to 2}} \left(\frac{x^2 \left(y + \frac{4}{x^2} \right)}{x^2 \left(1 + \frac{2y^2}{x^2} \right)} \right) = \lim_{x \to \infty} \left(\lim_{y \to 2} \frac{\left(y + \frac{4}{x^2} \right)}{\left(1 + \frac{2y^2}{x^2} \right)} \right)$

$= \lim_{x \to \infty} \left(\frac{2 + \frac{4}{x^2}}{1 + \frac{2 \times 4}{x^2}} \right) = 2$

Que ③ → $\lim_{\substack{x \to 0 \\ y \to 0}} \left(\frac{x+y}{2x-y} \right)$

Sol: $\lim_{x \to 0} \left(\lim_{y \to 0} \frac{x+y}{2x-y} \right)$

LHS: $\lim\limits_{x\to 0}\left(\frac{x+0}{2x}\right)$

$= \lim\limits_{x\to 0}\left(\frac{x}{2x}\right) = \frac{1}{2}\ \frac{0\uparrow}{0\uparrow}$

$= \lim\limits_{x\to 0}\left(\frac{1}{2}\right) = \frac{1}{2}$

RHS: $\lim\limits_{y\to 0}\left(\lim\limits_{x\to 0}\frac{0+y}{-y}\right)$

$= -1$

(LHS $\neq$ RHS)

(LDE) ~~(Limit doesn't exist)~~

Check for Origin

☆ put $y = mx$

$= \lim\limits_{x\to 0}\left[\frac{x+mx}{2x-mx}\right]$

$= \lim\limits_{x\to 0}\left[\frac{1+m}{2-m}\right] = \frac{1+m}{2-m}$ if this is dependent on m then limit doesn't exist

if it is independent of m then limit exists

$\therefore$ it depends upon m so limit is not unique

hence $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{x+y}{2x-y}$ does not exist.

Continuity

Continuity of a function of two Variables

A function $f(x,y)$ is said to be continous at point (a,b) if

① $f(x,y)$ is defined at (a,b)

(2) $\lim f(x,y)$ at (a,b) exists

(3) $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$ (value of function at (a,b) = value of limit)

For ex: Discuss the continuity of the function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq 0 \\ 0 & (x,y) = 0 \end{cases} \quad \text{at } (0,0).$$

Sol: put $y = mx$ (as we are checking limit at $x=0, y=0$) (at origin)

$$= \frac{x(mx)}{x^2+(m^2x^2)} = \frac{x^2 m}{x^2(1+m^2)} = \frac{m}{(1+m^2)}$$

Since limit does not exist (not unique) depends upon m hence this function is discontinous at $(0,0)$.

For ex: Discuss the continuity of

$$f(x,y) = \begin{cases} x^2+y^2+2y & (x,y) \neq (1,2) \\ 0 & (x,y) = (1,2) \end{cases}$$

at point $(1,2)$.

Sol: Check limit: $\lim\limits_{\substack{x\to 1 \\ y\to 2}} (x^2+y^2+2y)$

$$= \lim_{x\to 1}\left(\lim_{y\to 2} x^2+y^2+2y\right)$$

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$$= \lim_{x \to 1} \left(x^2 + 4 + 2y \right) = 9$$

Value of $f(x,y)$ at $(1,2) \neq 9$. hence not continous
$0 \neq 9$

for ex: Is the function $f(x,y) = \dfrac{xy(x^2-y^2)}{x^2+y^2}$ when $(x,y) \neq (0,0)$ & $f(0,0) = 4$ is continous at $(0,0)$?

(for checking at Origin limit we put $y = mx$)

Sol: (M-1):

$$= \frac{x(mx)(x^2 - m^2x^2)}{(x^2 + m^2x^2)}$$

$$= \frac{mx^2(x^2 - m^2x^2)}{x^2(m^2+1)} = \frac{mx^2(1-m^2)\cancel{x^2}}{(1+m^2)\cancel{x^2}} = 0$$

(M-2): taking $y^2 = x^2$

$$= \frac{xy(x^2-y^2)}{(x^2+y^2)} = \frac{xy(x^2-x^2)}{(x^2+x^2)} = 0 = 0$$

HW

Basic Differentiation Formulas

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1. $\frac{d}{dx}(x^2) = 2x$
2. $\frac{d}{dx}(\sin x) = \cos x$
3. $\frac{d}{dx}(\cos x) = -\sin x$
4. $\frac{d}{dx}(\sec x) = \sec x \tan x$
5. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$
6. $\frac{d}{dx}(\tan x) = \sec^2 x$
7. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$
8. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
9. $\frac{d}{dx}(a^x) = a^x \log a$
10. $\frac{d}{dx}(x^x) = x^x(1 + \log x)$
11. $\frac{d}{dx}(e^x) = e^x$
12. $\frac{d}{dx}(k) = 0$, k is any constant
13. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
14. $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$
15. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
16. $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$
17. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$
18. $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$
19. **Product Rule**

$$\frac{d}{dx}(uv) = u'v + v'u$$
$$= v\frac{du}{dx} + u\frac{dv}{dx}$$

20. **Quotient Rule**

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - v'u}{v^2}$$
$$= \frac{\frac{du}{dx}\cdot v - \frac{dv}{dx}\cdot u}{v^2}$$

21. $\frac{d}{dx}(x^n) = nx^{n-1}$

Evaluate the following limits:

Que-1 $\lim\limits_{\substack{x\to 1\\ y\to 2}} \dfrac{2x^2+y^2}{2xy}$

Sol: By Reversal method,

LHS: $\lim\limits_{x\to 1}\left[\lim\limits_{y\to 2} \dfrac{2x^2+y^2}{2xy}\right]$

$= \lim\limits_{x\to 1}\left[\dfrac{2x^2+4}{2x\times 2}\right]$

$= \lim\limits_{x\to 1}\left[\dfrac{2x^2+4}{4x}\right]$

$= \dfrac{2\times 1+4}{4} = \dfrac{6}{4} = \dfrac{3}{2}$

RHS: $\lim\limits_{y\to 2}\left[\lim\limits_{x\to 1} \dfrac{2x^2+y^2}{2xy}\right]$

$= \lim\limits_{y\to 2}\left[\dfrac{2\times 1+y^2}{2y}\right]$

$= \lim\limits_{y\to 2}\left[\dfrac{2+y^2}{2y}\right]$

$= \dfrac{2+4}{2\times 2} = \dfrac{6}{4} = \dfrac{3}{2}$

LHS = RHS, Hence limit will exist & the answer is 3/2.

Que-2 $\lim\limits_{\substack{x\to 2\\ y\to 3}} \dfrac{x^3+y^2}{x^2-y}$

Sol: By Reversal method,

LHS: $\lim\limits_{x\to 2}\left[\lim\limits_{y\to 3} \dfrac{x^3+y^2}{x^2-y}\right]$

$= \lim\limits_{x\to 2}\left[\dfrac{x^3+9}{x^2-3}\right]$

$= \dfrac{8+9}{4-3} = 17$

RHS: $\lim\limits_{y\to 3}\left[\lim\limits_{x\to 2} \dfrac{x^3+y^2}{x^2-y}\right]$

$= \lim\limits_{y\to 3}\left[\dfrac{8+y^2}{4-y}\right]$

$= \dfrac{8+9}{4-3} = 17$

LHS = RHS, Hence limit will exist & the answer is 17.

Que 3 → $\lim\limits_{\substack{x\to\infty \\ y\to 3}} \dfrac{2xy-3}{x^3+4y^3}$

Sol: $\lim\limits_{\substack{x\to\infty \\ y\to 3}} \left(\dfrac{x\left(2y-\cancel{\frac{3}{x}}\right)}{x^3\left(1+\cancel{\frac{4y^3}{x^3}}\right)} \right) = \dfrac{1}{x^2} = \dfrac{1}{\infty} = 0$

Que 4 → $\lim\limits_{\substack{x\to 0 \\ y\to 0}} \dfrac{xy}{y-x^2}$; $x \neq 0,\ y \neq 0$.

Sol: (here at origin we will use $y=mx$ vala method)

$= \lim\limits_{x\to 0} \left[\dfrac{x(mx)}{mx-x^2}\right] = \dfrac{mx^2}{mx-x^2} = \dfrac{(mx)\cancel{x}}{\cancel{x}(m-x)}$

$= \dfrac{mx}{m-x} = \dfrac{m \times 0}{m} = 0$

Que 5 → $\lim\limits_{\substack{x\to 0 \\ y\to 0}} \dfrac{x-y}{x^2+y^2}$; $x \neq 0,\ y \neq 0$.

Sol: put $y=mx$ → to check at origin

(M-1)

$= \lim\limits_{x\to 0} \left(\dfrac{x-mx}{x^2+m^2x^2}\right)$

$= \lim\limits_{x\to 0} \left(\dfrac{x(1-m)}{x^2(1+m^2)}\right)$

$= \dfrac{1}{x}\left(\dfrac{1-m}{1+m^2}\right) = \infty$

(LDE)

(M-2) : (LHS) $\lim\limits_{x\to 0}\left[\lim\limits_{y\to 0} \dfrac{x-y}{x^2+y^2}\right]$

$= \lim\limits_{x\to 0}\left[\dfrac{x}{x^2}\right]$

$= \lim\limits_{x\to 0}\left[\dfrac{1}{x}\right] = \dfrac{1}{0} = \infty$

Que 6 → $\lim\limits_{\substack{x\to 1 \\ y\to 1}} \dfrac{xy-2x}{xy-2y}$

Sol: LHS $\lim\limits_{x\to 1}\left[\lim\limits_{y\to 1}\dfrac{xy-2x}{xy-2y}\right]$ RHS : $\lim\limits_{y\to 1}\left[\lim\limits_{x\to 1}\dfrac{xy-2x}{xy-2y}\right]$

$= \lim\limits_{x\to 1}\left[\dfrac{x-2x}{x-2}\right]$ $= \lim\limits_{y\to 1}\left[\dfrac{y-2}{-y}\right]$

$= \dfrac{1-2}{1-2} = \dfrac{-1}{-1} = 1$ $= \dfrac{1-2}{-1} = 1$

(LHS = RHS)

(limit exists) Ans

Que (7) $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{x^3+2y^3}{x^2+4y^2}$, $x \neq 0, y \neq 0$.

Sol: use $y = mx$

$= \dfrac{x^3+2(m^3x^3)}{x^2+4m^2x^2}$

$\dfrac{x(1+2m^3)}{(1+4m^2)} = 0$

$= \dfrac{x^3+2m^3x^3}{x^2+4m^2x^2} = \dfrac{x+2m^3x}{1+4m^2} = 0$ Ans

Que (8) $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{x^2y^3}{x^2+y^2}$, $x \neq 0, y \neq 0$

Sol: use $y = mx$

$= \lim\limits_{x\to 0}\left(\dfrac{x^2m^3x^3}{x^2+m^2x^2}\right)$

$= \lim\limits_{x\to 0}\left(\dfrac{m^3x^3}{1+m^2}\right) = 0$ Ans

Que (9) → $\lim_{\substack{x\to 0\\ y\to 0}} \frac{xy+2}{x^2+y^2}, x\neq 0, y\neq 0.$

Sol: $= \lim_{x\to 0}\left[\lim_{y\to 0} \frac{xy+2}{x^2+y^2}\right]$

LHS $= \lim_{x\to 0}\left[\frac{x\times 0+2}{x^2+0}\right] = \lim_{x\to 0}\left[\frac{2}{x^2}\right] = \frac{1}{0} = \infty$

RHS $= \lim_{y\to 0}\left[\lim_{x\to 0} \frac{xy+2}{x^2+y^2}\right]$

$= \lim_{y\to 0}\left[\frac{2}{y^2}\right] = \infty$ limit $= \infty$ ~~Ans~~ as (LHS = RHS)

Que (10) → $\lim_{\substack{x\to 1\\ y\to 1}} \frac{3x(y-2)}{2y(x-2)}$

Sol: LHS : $\lim_{x\to 1}\left[\lim_{y\to 1} \frac{3x(y-2)}{2y(x-2)}\right]$

$= \lim_{x\to 1}\left[\frac{3x(1-2)}{2(x-2)}\right]$

$= \frac{3(-1)}{2(-1)} = \frac{3}{2}$

RHS : $\lim_{y\to 1}\left[\lim_{x\to 1} \frac{3x(y-2)}{2y(x-2)}\right]$

$= \lim_{y\to 1}\left[\frac{3(y-2)}{2y(-1)}\right]$

$= \frac{3(1-2)}{2(-1)} = \frac{3}{2}$

as LHS = RHS

Hence limit will exist & finite.

TEST FOR CONTINUITY

Que (1) → $f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}, & \text{when } x\neq 0, y\neq 0 \\ 0, & \text{when } x=0, y=0 \end{cases}$

at origin.

Sol: We have to check at origin so we have to use $y = mx$ method

$$= \frac{x(mx)(x^2 - m^2x^2)}{x^2 + m^2x^2} \qquad \lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

$$= \frac{m\cancel{x^2}(x^2 - m^2x^2)}{\cancel{x^2}(1+m^2)}$$

= Value of limit
= value of function at that point

$$= \frac{m(x^2 - m^2x^2)}{(1+m^2)} = 0$$

Hence $f(x,y)$ is continous at origin.

Que-2 $\rightarrow$ $f(x,y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2}, & \text{where } x \neq 0, y \neq 0 \\ 0, & \text{when } x = 0, y = 0 \end{cases}$

at origin.

Sol: by $y = mx$ as we have to check it at origin.

$$= \lim_{x \to 0} \left(\frac{x^2 - m^2x^2}{x^2 + m^2x^2} \right) = \frac{\cancel{x^2}(1-m^2)}{\cancel{x^2}(1+m^2)}$$

$\rightarrow$ limit depends upon m hence limit does not exist.

Since the limit does not exist, hence the $f(x,y)$ is discontinous at origin.

Que-3 $\rightarrow$ $f(x,y) = \begin{cases} \dfrac{x^3 y^3}{x^3 + y^3}, & \text{when } x \neq 0, y \neq 0 \\ 0, & \text{when } x = 0, y = 0 \end{cases}$ at origin

Sol: put $y=mx$.

$$= \lim_{x\to 0}\left(\frac{x^3m^3x^3}{x^3+m^3x^3}\right) = \frac{x^3m^3x^3}{x^3(1+m^3)} = \frac{m^3x^3}{1+m^3} = 0$$

Value of limit = Value of $f(x,y)$ at origin $=0$

Hence $f(x,y)$ is continous at origin.

Que (4): $f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & \text{when } x\neq 0, y\neq 0\\ 0, & \text{when } x=0, y=0\end{cases}$

at origin.

Sol: put $y=mx$

$$= \lim_{x\to 0}\left(\frac{x(mx)}{x^2+m^2x^2}\right) = \frac{mx^2}{x^2(1+m^2)} = \frac{m}{1+m^2}.$$

= (LDE) as limit is depending upon m

Since (LDE) hence
$f(x,y)$ is discontinous at origin.

Que (5): $f(x,y)=\begin{cases}x^3+y^3, & \text{when } x\neq 0, y\neq 0 \text{ at origin}\\ 0, & \text{when } x=0, y=0\end{cases}$

Sol: $\lim_{\substack{x\to 0\\ y\to 0}}(x^3+y^3)$ → not an ~~intermediate~~ (indeterminate) form

So directly put the values.

limit $=0$

Value of $f(x,y)$ at $(0,0)=0$.

Hence $f(x,y)$ is continous at origin.

Que ⑥ $f(x,y) = \begin{cases} \dfrac{x^2+2y}{x+y^2}, & \text{at the point } (1,2). \\ 1, & \text{when } x=1, y=2 \end{cases}$

Sol: By Reversal method,

LHS: $\lim\limits_{\substack{x\to 1 \\ y\to 2}} \left(\dfrac{x^2+2y}{x+y^2}\right) : \lim\limits_{x\to 1}\left[\lim\limits_{y\to 2}\left(\dfrac{x^2+2y}{x+y^2}\right)\right]$

$= \lim\limits_{x\to 1}\left[\dfrac{x^2+4}{x+4}\right] = \dfrac{5}{5} = 1$

RHS: $\lim\limits_{y\to 2}\left[\lim\limits_{x\to 1}\left(\dfrac{x^2+2y}{x+y^2}\right)\right] = \lim\limits_{y\to 2}\left[\dfrac{1+2y}{1+y^2}\right]$

$= \dfrac{1+4}{1+4} = 1$

LHS = RHS (limit exists)

&

value of $f(x,y)$ at $(1,2)$ is also $= 1 =$ limit

Hence $f(x,y)$ is continous at $(1,2)$.

Que ⑦ Show that the function $f(x,y) = \begin{cases} 2x^2+y, & (x,y)\neq(1,2) \\ 0, & (x,y)=(1,2) \end{cases}$ is discontinous at $(1,2)$.

Sol: $2\times 1+2=4$ (limit)

Value of $f(x,y)=0$ at $(1,2)$

Since limit $\neq$ Value of $f(x,y)$ hence $f(x,y)$ is discontinous at $(1,2)$.

Que ⑧: Show that the function $f(x,y) = \begin{cases} (x+y)\sin\left(\frac{1}{x+y}\right), & x+y \neq 0 \\ 0, & x+y=0 \end{cases}$
is continous at $(0,0)$ but its partial derivatives of first order do not exist at $(0,0)$.

Sol: $\lim\limits_{\substack{x\to 0\\ y\to 0}} \left\{(x+y)\sin\left(\frac{1}{x+y}\right)\right\}$

put $y=mx$ or by Reversal method

LHS: $\lim\limits_{x\to 0}\left[\lim\limits_{y\to 0}(x+y)\sin\left(\frac{1}{x+y}\right)\right]$ (using $\lim\limits_{x\to 0}\frac{\sin(x)}{x} = 1$)

(x constant) $= \lim\limits_{x\to 0}\left[x\sin\left(\frac{1}{mx}\right)\right]$

$= \lim\limits_{x\to 0}\left[x\,\frac{\sin\left(\frac{1}{mx}\right)}{\left(\frac{1}{mx}\right)}\times\frac{1}{mx}\right]$ $\to 1$

$= \frac{1}{m} \to$ (LDE)

We have to check limit at origin hence put $y=mx$

$= \lim\limits_{x\to 0}\left[(x+mx)\sin\left(\frac{1}{x+mx}\right)\right]$

$= \lim\limits_{x\to 0}\left[x(1+m)\,\frac{\sin\left(\frac{1}{x+mx}\right)}{\left(\frac{1}{x+mx}\right)}\times\left(\frac{1}{mx+x}\right)\right]$ $\to 1$

$= 1 \to$ limit is finite

Value of $f(x,y)$ at $(0,0) = (0+0)\sin\left(\frac{1}{0}\right) = 0\sin(\infty)$

$= 0 \times$ something

$= 0$

Partial Differentiation

★ $y = f(x) = \frac{dy}{dx}$ = Ordinary Differentiation

$z = f(x, y)$

★ $z_x = f_x = \left[\frac{\partial z}{\partial x}\right]_y = \frac{\partial z}{\partial x} \rightarrow$ 1st Order Partial derivative of z wrt x (keeping y constant)
within
(Curly or dell brackets)

★ $\frac{\partial z}{\partial y} \rightarrow$ 1st Order partial derivative of z wrt y
(keeping x constant)

★ $\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = f_{xx} = \left[\frac{\partial^2 z}{\partial x^2}\right]_y \rightarrow$ 2nd Order partial derivative of z wrt x (keeping y constant)

★ $f_{xy} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}\right] = \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right] = \frac{\partial^2 z}{\partial x \partial y} = f_{yx}$

for ex: $f(x, y) = x^3 + 7x^2y + 8y^3 + 3 + 2y - 2x$ — (1)

find : $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$

Sol: diff (1) partially wrt x
(y constant)

$f_x = \frac{\partial f}{\partial x} = 3x^2 + 14xy + 0 - 2$

$= 3x^2 + 14xy - 2$

$$f_y = \frac{\partial f}{\partial y} = 0 + 7x^2 + 24y^2 + 2 = 7x^2 + 24y^2 + 2$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(3x^2 + 14xy - 2\right) = 6x + 14y$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(7x^2 + 24y^2 + 2\right) = 24 \times 2y = 48y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(7x^2 + 24y^2 + 2\right)$$
$$= 14x$$

Que:- $f = x^3 + y^3 + z^3 + 3xyz$ find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial z^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial z \partial x}$.

Sol: ☆ $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x^3 + y^3 + z^3 + 3xyz\right) = 3x^2 + 0 + 0 + 3yz$
$= 3x^2 + 3yz$

☆ $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^3 + y^3 + z^3 + 3xyz\right) = 0 + 3y^2 + 0 + 3xz$

☆ $\frac{\partial f}{\partial z} = 0 + 0 + 3z^2 + 3xy$

☆ $\frac{\partial}{\partial x}\left(3x^2 + 3yz\right) = 6x$

☆ $\frac{\partial}{\partial y}\left(3y^2 + 3xz\right) = 6y$

☆ $\frac{\partial}{\partial z}\left(3z^2 + 3xy\right) = 6z$

☆ $\frac{\partial}{\partial x}\left(3y^2 + 3xz\right) = 3z$

☆ $\frac{\partial}{\partial z}\left(3x^2 + 3yz\right) = 0 + 3y$

Que. $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$. find all 2nd order derivatives.

Sol:

$$\begin{array}{ccc} & u(x,y) & \\ \swarrow & & \searrow \\ \text{(1st Order):}\ \frac{\partial u}{\partial x}(x,y) & & \frac{\partial u}{\partial y}(x,y) \end{array}$$

(2nd Order) $\frac{\partial^2 u}{\partial x^2}$ (1) $\qquad \frac{\partial^2 u}{\partial x \partial y}$ (3) $\qquad \frac{\partial^2 u}{\partial y^2}$ (2)

★ $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(x^2 \tan^{-1}\left(\frac{y}{x}\right)\right) - \frac{\partial}{\partial x}\left(y^2 \tan^{-1}\left(\frac{x}{y}\right)\right)$

(y → constant) (x → variable)

$$= \left[2x \tan^{-1}\left(\frac{y}{x}\right) + \frac{1}{1+\frac{y^2}{x^2}} \times y \times \left(\frac{-1}{x^2}\right) \times x^2\right] - \left[\cancel{y^2}\, \frac{1}{1+\frac{x^2}{y^2}} \times \frac{1}{\cancel{y}}\right]$$

$$\frac{\partial u}{\partial x} = \left[2x \tan^{-1}\left(\frac{y}{x}\right) + \frac{\cancel{x^2}}{x^2+y^2} \times x^2 y \times \frac{-1}{\cancel{x^2}}\right] - \left[y \times \frac{y^2}{y^2+x^2}\right]$$

$$\frac{\partial u}{\partial x} = 2x \tan^{-1}\left(\frac{y}{x}\right) - \frac{x^2 y}{x^2+y^2} - \frac{y^3}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = 2x \tan^{-1}\left(\frac{y}{x}\right) - \left(\frac{x^2 y + y^3}{x^2+y^2}\right)$$

$$\frac{\partial u}{\partial x} = 2x \tan^{-1}\left(\frac{y}{x}\right) - y\left(\frac{\cancel{x^2+y^2}}{\cancel{x^2+y^2}}\right)$$

$$\boxed{\frac{\partial u}{\partial x} = 2x \tan^{-1}\left(\frac{y}{x}\right) - y} \quad \text{—— (1)}$$

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$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\right] = \frac{\partial}{\partial x}\left[2x\tan^{-1}\left(\frac{y}{x}\right) - y\right]$$

(y → constant)
$$= \frac{\partial}{\partial x}\left[\underbrace{2x\tan^{-1}\left(\frac{y}{x}\right)}_{\text{variable}}\right] - \frac{\partial y}{\partial x}$$

$$= 2\left[\tan^{-1}\left(\frac{y}{x}\right) + \frac{1}{1+\frac{y^2}{x^2}} \times y \times \left(\frac{-1}{x^2}\right) \times x\right] - 0$$

$$\frac{\partial^2 u}{\partial x^2} = 2\left[\tan^{-1}\left(\frac{y}{x}\right) + \frac{x^{\cancel{2}}}{x^2+y^2} \times \frac{-y}{\cancel{x}}\right]$$

Ans (1)
$$\boxed{\frac{\partial^2 u}{\partial x^2} = 2\tan^{-1}\left(\frac{y}{x}\right) - \frac{2xy}{x^2+y^2}}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)\right]$$

y → variable, x → constant

$$= \frac{\partial}{\partial y}\left[x^2\tan^{-1}\left(\frac{y}{x}\right)\right] - \frac{\partial}{\partial y}\left[y^2\tan^{-1}\left(\frac{x}{y}\right)\right]$$

$$= \left(x^2 \frac{1}{\left(1+\frac{y^2}{x^2}\right)} \times \frac{1}{\cancel{x}}\right) - \left(2y\tan^{-1}\left(\frac{x}{y}\right) + \frac{1}{1+\frac{x^2}{y^2}} \times x \times \frac{-1}{\cancel{y^2}} \times \cancel{y^2}\right)$$

$$= \left(\frac{x}{\frac{x^2+y^2}{x^2}}\right) - \left(2y\tan^{-1}\left(\frac{x}{y}\right) - \frac{xy^2}{x^2+y^2}\right)$$

$$= \frac{x^3}{x^2+y^2} - 2y\tan^{-1}\left(\frac{x}{y}\right) + \frac{xy^2}{x^2+y^2}$$

$$= \frac{x^3+xy^2}{x^2+y^2} - 2y\tan^{-1}\left(\frac{x}{y}\right)$$

$$= \frac{x\cancel{(x^2+y^2)}}{\cancel{(x^2+y^2)}} - 2y\tan^{-1}\left(\frac{x}{y}\right)$$

$$\frac{\partial u}{\partial y} = x - 2y\tan^{-1}\left(\frac{x}{y}\right)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left[x - 2y\tan^{-1}\left(\frac{x}{y}\right)\right]$$

x → const
$$= -2\left[\tan^{-1}\left(\frac{x}{y}\right) + \frac{1}{1+\frac{x^2}{y^2}} \times x \times \left(\frac{-1}{y^2}\right) \times y\right]$$

$$\frac{\partial^2 u}{\partial y^2} = -2\tan^{-1}\left(\frac{x}{y}\right) + \frac{x\cancel{y}^2}{\cancel{y}(x^2+y^2)}$$

$$\frac{\partial^2 u}{\partial y^2} = -2\tan^{-1}\left(\frac{x}{y}\right) + \frac{xy}{x^2+y^2}$$

Ans (2) $$\boxed{\frac{\partial^2 u}{\partial y^2} = \frac{xy}{x^2+y^2} - 2\tan^{-1}\left(\frac{x}{y}\right)}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial y}\right] = \frac{\partial}{\partial x}\left[x - 2y\tan^{-1}\left(\frac{x}{y}\right)\right]$$

$$= \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial x}\left[2y\tan^{-1}\left(\frac{x}{y}\right)\right]$$

$$= 1 - 2\cancel{y}\frac{1}{1+\frac{x^2}{y^2}} \times \frac{1}{\cancel{y}}$$

$$\frac{\partial^2 u}{\partial x \partial y} = 1 - \frac{2y^2}{x^2+y^2} = \frac{x^2+y^2-2y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}$$

Ans (3) $$\boxed{\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2-y^2}{x^2+y^2}}$$

Que→ If $x^x y^y z^z = c$. Show that at $x=y=z$

$$\frac{\partial^2 z}{\partial x \partial y} = -[x\log(ex)]^{-1}$$

Sol: $x^x y^y z^z = c$ —— (1)

taking log both side,

$x\log x + y\log y + z\log z = \log c$ —— (11)

(jiske with respect → independent)

$z \rightarrow$ dependent

$x, y \rightarrow$ independent

$z = f(x, y)$

diff (II) partially wrt x keeping y constant

$$[\log x + 1] + 0 + [1 + \log z]\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = -\frac{(1+\log x)}{(1+\log z)} \quad \text{--- (III)}$$

differentiating partially (III) wrt y,

$$\frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}\right] = \frac{-\partial}{\partial y}\left[\frac{(1+\log x)}{(1+\log z)}\right] \quad z, y \to \text{variable}, \; x \to \text{constant}$$

$$\frac{\partial^2 z}{\partial x \partial y} = -(1+\log x)\left[\frac{-1}{(1+\log z)^2} \times \frac{1}{z}\right]\frac{\partial z}{\partial y}$$

$$=$$

$$\frac{\partial z}{\partial y} = \frac{-(1+\log y)}{(1+\log z)} \qquad = \frac{(1+\log x)}{z(1+\log z)^2} \times \frac{-(1+\log y)}{(1+\log z)}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-(1+\log x)(1+\log y)}{z(1+\log z)^3}$$

at $x = y = z$

$$= \frac{-(1+\cancel{\log x})(1+\cancel{\log x})}{x(1+\log x)^3}$$

$$= \frac{-(1+\log x)^2}{x(1+\log x)^3}$$

$$= \frac{-1}{x(1+\log x)}$$

$$= -[x(1+\log x)]^{-1}$$

$$= -[x(\log_e e + \log_e x)]^{-1}$$

$$= -[x \log_e ex]^{-1} \qquad (\text{LHS} = \text{RHS}) \; (\text{H.P})$$

Que: find the value of n so that the equation

$v = r^n(3\cos^2\theta - 1)$

Satisfy $\frac{\partial}{\partial r}\left[r^2 \frac{\partial v}{\partial r}\right] + \frac{1}{\sin\theta} \cdot \frac{\partial}{\partial \theta}\left[\sin\theta \frac{\partial v}{\partial \theta}\right] = 0$

Sol: $\frac{\partial v}{\partial r} = \frac{\partial}{\partial r}\left[r^n(3\cos^2\theta - 1)\right]$ $(n, \theta \to \text{constant})$

variable

$\frac{\partial v}{\partial r} = \frac{\partial}{\partial r}\left[r^n 3\cos^2\theta\right] - \frac{\partial}{\partial r}(r^n)$

$\frac{\partial v}{\partial r} = 3\cos^2\theta \times n r^{n-1} - n r^{n-1}$

$\frac{\partial v}{\partial r} = 3n\cos^2\theta\, r^{n-1} - n r^{n-1}$

$\times r^2$ $= 3n\cos^2\theta\, r^{n-1+2} - n r^{n-1+2}$

$r^2 \frac{\partial v}{\partial r} = 3n\cos^2\theta\, r^{n+1} - n r^{n+1}$

$= \frac{\partial}{\partial r}\left[3n\cos^2\theta\, r^{n+1}\right] - \frac{\partial}{\partial r}\left[n r^{n+1}\right]$ $(n, \theta \to \text{constant})$

$= 3n\cos^2\theta(n+1) r^{n+\cancel{1}-\cancel{1}} - n(n+1) r^{n+\cancel{1}-\cancel{1}}$

$= 3n\cos^2\theta(n+1) r^n - n(n+1) r^n$ —— (1)

$\frac{\partial v}{\partial \theta} = \frac{\partial}{\partial \theta}\left[r^n(3\cos^2\theta - 1)\right] = \frac{\partial}{\partial \theta}\left[3r^n\cos^2\theta\right] - \frac{\partial}{\partial \theta}\left[r^n\right]$

$(n, r \to \text{constant})$ $= 3r^n \times 2\cos\theta(-\sin\theta)$

$= \sin\theta \times 6r^n\cos\theta(-\sin\theta)$

$= \frac{\partial}{\partial \theta}\left[-6r^n\cos\theta\sin^2\theta\right]$

$= -6r^n \frac{\partial}{\partial \theta}\left[\cos\theta\sin^2\theta\right]$

$= -6r^n\left[(-\sin\theta)(\sin^2\theta) + 2\sin\theta\cos\theta\cos\theta\right]$

$= -6r^n\left[-\sin^3\theta + 2\sin\theta\cos^2\theta\right] \times \frac{1}{\sin\theta}$

$= -6 r^n \times \cancel{\sin\theta} \left[-\sin^2\theta + 2\cos^2\theta \right] \times \frac{1}{\cancel{\sin\theta}}$

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$= -6 r^n \left[-\sin^2\theta + 2\cos^2\theta \right]$

$= -6 r^n \left[-\sin^2\theta + 2(1-\sin^2\theta) \right]$

$= -6 r^n \left[-\sin^2\theta + 2 - 2\sin^2\theta \right]$

$= \boxed{-6 r^n \left[2 - 3\sin^2\theta \right]}$ —— (11)

add eqn (1) & (11) & equate it to 0

$3n\cos^2\theta (n+1) r^n - n(n+1) r^n - 6 r^n (2 - 3\sin^2\theta) = 0$

$3n\cos^2\theta (n+1) r^n - n(n+1) r^n - 6 r^n (2 - 3\sin^2\theta) = 0$

$3n\cos^2\theta (n+1) r^n = r^n n(n+1) + 6 r^n (2 - 3\sin^2\theta)$

$3n(1-\sin^2\theta)(n+1)\cancel{r^n} = \cancel{r^n} n(n+1) + 6\cancel{r^n} (2 - 3\sin^2\theta)$

$3n(n+1)(1-\sin^2\theta) - n(n+1) = 6(2 - 3\sin^2\theta)$

$n(n+1)\left[3(1-\sin^2\theta) - 1\right] = 6\left[2 - 3\sin^2\theta\right]$

$n(n+1)\left[3 - 3\sin^2\theta - 1\right] = 6\left[2 - 3\sin^2\theta\right]$

$n(n+1)\left[\cancel{2 - 3\sin^2\theta}\right] = 6\left[\cancel{2 - 3\sin^2\theta}\right]$

$n(n+1) = 6$

$n = 2$ Ans

Ques If $z = e^{ax+by} f(ax-by)$

then prove that $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$.

Sol: $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left[e^{ax+by} f(ax-by) \right]$ ($y \to$ constant)

$= \left(e^{ax+by} (a) f(ax-by) + f'(ax-by)(a) e^{ax+by} \right) b$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left[e^{ax+by} f(ax-by)\right]$$

$$= e^{ax+by}(b) f(ax-by) + f'(ax-by)(-b)e^{ax+by}$$

$$\partial z = b\left(e^{ax+by}(a) f(ax-by)\right) + f'(ax-by)(a) e^{ax+by}))$$

$$+ a\left(e^{ax+by}(b) f(ax-by) + f'(ax-by)(-b) e^{ax+by}\right)$$

$$= 2abz \quad \text{Ans}$$

× HW ×

Que ① → If $z(x+y) = x^2+y^2$, Show that

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) \quad \text{—— (3)}$$

Sol: $z(x+y) = x^2+y^2$

$$z = \frac{(x^2+y^2)}{(x+y)}$$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x^2+y^2}{x+y}\right) \quad (y \to \text{constant})$$

u, v

$$= \frac{2x(x+y) - (x^2+y^2)}{(x+y)^2} = \frac{\cancel{2x^2} + 2xy - \cancel{x^2} - y^2}{(x+y)^2}$$

$$\frac{\partial z}{\partial x} = \frac{x^2 - y^2 + 2xy}{(x+y)^2} \quad \text{—— (1)}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x^2+y^2}{x+y}\right) \quad (x \to \text{constant})$$

u, v

$$= \frac{2y(x+y) - (x^2+y^2)}{(x+y)^2}$$

$$= \frac{2xy + 2y^2 - x^2 - y^2}{(x+y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{y^2 - x^2 + 2xy}{(x+y)^2} \quad \text{—— (2)}$$

LHS: $$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \left(\frac{x^2-y^2+2xy}{(x+y)^2} - \frac{y^2-x^2+2xy}{(x+y)^2}\right)^2$$

$$= \left(\frac{x^2-y^2+\cancel{2xy}-y^2+x^2-\cancel{2xy}}{(x+y)^2}\right)^2$$

$$= \left(\frac{2x^2-2y^2}{(x+y)(x+y)}\right)^2$$

$$= \left(\frac{2\cancel{(x+y)}(x-y)}{\cancel{(x+y)}(x+y)}\right)^2$$

$$= \frac{4(x-y)^2}{(x+y)^2} \quad \text{—— (3)}$$

RHS: $$4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

$$= 4\left(1 - \left(\frac{x^2-y^2+2xy}{(x+y)^2}\right) - \left(\frac{y^2-x^2+2xy}{(x+y)^2}\right)\right)$$

$$= 4\left(\frac{(x+y)^2 - (x^2-y^2+2xy) - (y^2-x^2+2xy)}{(x+y)^2}\right)$$

$$= 4\left(\frac{\cancel{x^2}+y^2+\cancel{2xy}-\cancel{x^2}+\cancel{y^2}-\cancel{2xy}-\cancel{y^2}+x^2-2xy}{(x+y)^2}\right)$$

$$= 4\left(\frac{x^2+y^2-2xy}{(x+y)^2}\right) = \frac{4(x-y)^2}{(x+y)^2} \quad \text{—— (4)}$$

$eq^n (3) = eq^n (4)$

LHS = RHS

(Hence Proved)

Que (2) → If $u = \tan^{-1}\left(\frac{2xy}{x^2-y^2}\right)$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Sol: $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[\tan^{-1}\left(\frac{2xy}{x^2-y^2}\right)\right]$ y → constant, x → variable (denominator)

$$\left[\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}\right]$$

$$\frac{\partial u}{\partial x} = \frac{1}{1+\left(\frac{2xy}{x^2-y^2}\right)^2} \cdot \left[\frac{2y(x^2-y^2) - 2x(2xy)}{(x^2-y^2)^2}\right]$$

$$= \frac{1}{\left(\frac{(x^2-y^2)^2 + 4x^2y^2}{(x^2-y^2)^2}\right)}\left[\frac{2yx^2-2y^3-4x^2y}{(x^2-y^2)^2}\right]$$

$$= \frac{\cancel{(x^2-y^2)^2}}{(x^2-y^2)^2+4x^2y^2} \times \frac{(2yx^2-2y^3-4x^2y)}{\cancel{(x^2-y^2)^2}}$$

$$= \frac{2yx^2-2y^3-4x^2y}{(x^2-y^2)^2+4x^2y^2} = \frac{2y(x^2-y^2-2x^2)}{(x^2-y^2)^2+4x^2y^2}$$

$$\frac{\partial u}{\partial x} = \frac{2y(-x^2-y^2)}{(x^2-y^2)^2+4x^2y^2}$$

$$= \frac{-2y(x^2+y^2)}{x^4+y^4-2x^2y^2+4x^2y^2}$$

$$= \frac{-2y(x^2+y^2)}{x^4+y^4+2x^2y^2} = \frac{-2y\cancel{(x^2+y^2)}}{(x^2+y^2)^{\cancel{2}}}$$

$$\frac{\partial u}{\partial x} = \frac{-2y}{x^2+y^2} \quad \text{--- (1)}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{-2y}{x^2+y^2}\right) \quad y \to \text{constant}$$

$$= \frac{-2x \times -2y}{(x^2+y^2)^2} = \frac{4xy}{(x^2+y^2)^2} \quad \text{--- (2)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\tan^{-1}\left(\frac{2xy}{x^2-y^2}\right)\right] \quad (x \to \text{constant})$$

$$= \frac{1}{1+\left(\frac{2xy}{x^2-y^2}\right)^2} \times \left[\frac{2x(x^2-y^2)-(-2y)(2xy)}{(x^2-y^2)^2}\right]$$

$$= \left(\frac{1}{\frac{(x^2-y^2)^2+4x^2y^2}{\cancel{(x^2-y^2)^2}}}\right) \times \left[\frac{2x^3-2xy^2+4xy^2}{\cancel{(x^2-y^2)^2}}\right]$$

$$= \frac{1}{(x^2-y^2)^2+4x^2y^2} \times \left(2x^3-2xy^2+4xy^2\right)$$

$$= \frac{2x^3-2xy^2+4xy^2}{x^4+y^4-2x^2y^2+4x^2y^2}$$

$$= \frac{2x(x^2-y^2+2y^2)}{x^4+y^4+2x^2y^2}$$

$$= \frac{2x(x^2+y^2)}{(x^2+y^2)^2} = \frac{2x}{(x^2+y^2)}$$

$$\frac{\partial U}{\partial y} = \frac{2x}{x^2+y^2} \quad \text{——} \; (3)$$

$$\frac{\partial^2 U}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{2x}{x^2+y^2}\right) \qquad (x \to \text{constant})$$

$$= \frac{-2y \times 2x}{(x^2+y^2)^2}$$

$$\frac{\partial^2 U}{\partial y^2} = \frac{-4xy}{(x^2+y^2)^2} \quad \text{——} \; (4)$$

from eqn (2) & (4)

$$= \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}$$

$$= \frac{-4xy}{(x^2+y^2)^2} + \frac{4xy}{(x^2+y^2)^2}$$

$$= 0$$

LHS = RHS

Hence Proved.

Que (3) If $u = (x^2+y^2+z^2)^{-1/2}$ prove that

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0.$$

Sol: $u = (x^2+y^2+z^2)^{-1/2} = \dfrac{1}{(x^2+y^2+z^2)^{1/2}}$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{(x^2+y^2+z^2)^{1/2}}\right) \qquad y, z \to \text{constant}$$

∂x ↳ variable

$$(x^n = nx^{n-1})$$

$$= \frac{\partial}{\partial x}\left((x^2+y^2+z^2)^{-1/2}\right) \qquad \frac{-1-2}{2} = \frac{-3}{2}$$

$$= \frac{-1}{2}(x^2+y^2+z^2)^{\frac{-1}{2}-\frac{1}{1}}$$

$$\frac{\partial u}{\partial x} = \frac{-1}{2}(x^2+y^2+z^2)^{\frac{-3}{2}} \cdot 2x$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{-1}{\cancel{2}}(x^2+y^2+z^2)^{\frac{-3}{2}} \cdot \cancel{2}x\right]$$

$$= \frac{\partial}{\partial x}\left[-x(x^2+y^2+z^2)^{-3/2}\right]$$

$$= (-1)(x^2+y^2+z^2)^{-3/2} + \left(\frac{+3}{\cancel{2}}\right)(x^2+y^2+z^2)^{\frac{-3}{2}-1} \cdot \cancel{2}x \qquad \frac{-3-2}{2}$$

$$\times (+x)$$

$$\frac{\partial^2 u}{\partial x^2} = -(x^2+y^2+z^2)^{-3/2} + 3x^2(x^2+y^2+z^2)^{-5/2} \text{ —— (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left((x^2+y^2+z^2)^{-1/2}\right) \qquad (x, z \to \text{constant})$$

$$= \frac{-1}{\cancel{2}}(x^2+y^2+z^2)^{-3/2} \cdot \cancel{2}y$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left[-y(x^2+y^2+z^2)^{-3/2}\right]$$

$$= (-1)(x^2+y^2+z^2)^{-3/2} + \left(\frac{+3}{\cancel{2}}\right)(x^2+y^2+z^2)^{-5/2} \cdot \cancel{2}y(+y)$$

$$\frac{\partial^2 u}{\partial y^2} = -(x^2+y^2+z^2)^{-3/2} + 3y^2(x^2+y^2+z^2)^{-5/2} \text{ —— (2)}$$

by Symmetry,

$$\frac{\partial^2 u}{\partial z^2} = -(x^2+y^2+z^2)^{-3/2} + 3z^2(x^2+y^2+z^2)^{-5/2} \quad \text{--- (3)}$$

add eqn (1) + (2) + (3) (Solving LHS)

$$= \underbrace{-(x^2+y^2+z^2)^{-3/2}} + 3z^2(x^2+y^2+z^2)^{-5/2}$$

$$+ \underbrace{-(x^2+y^2+z^2)^{-3/2}} + 3x^2(x^2+y^2+z^2)^{-5/2}$$

$$+ \underbrace{-(x^2+y^2+z^2)^{-3/2}} + 3y^2(x^2+y^2+z^2)^{-5/2}$$

$$= -3(x^2+y^2+z^2)^{-3/2} + 3(x^2+y^2+z^2)^{-5/2}(x^2+y^2+z^2)$$

$$= -3(x^2+y^2+z^2)^{-3/2} + 3(x^2+y^2+z^2)^{1-\frac{5}{2}}$$

$$= \cancel{-3(x^2+y^2+z^2)^{-3/2}} \cancel{+3(x^2+y^2+z^2)^{-3/2}}$$

$$= 0$$

LHS = RHS

(Hence Proved)

Que (4): If $z = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$, prove that

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{x^2-y^2}{x^2+y^2}.$$

Sol: $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}\right]$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left[x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)\right]$$

$\to$ variable

$y \to$ constant

$$= \frac{\partial}{\partial x}\left(x^2\tan^{-1}\left(\frac{y}{x}\right)\right) - \frac{\partial}{\partial x}\left[y^2\tan^{-1}\left(\frac{x}{y}\right)\right)$$

$$= \left[2x\tan^{-1}\left(\frac{y}{x}\right) + \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-1}{\cancel{x^2}}\right) \times \cancel{x^2} xy\right] -$$

$$\left[y^{\cancel{2}}\frac{1}{1+\frac{x^2}{y^2}} \times \frac{1}{\cancel{y}}\right]$$

$$= 2x\tan^{-1}\left(\frac{y}{x}\right) + \frac{x^2}{x^2+y^2}(-y) - \left[\frac{y(y^2)}{x^2+y^2}\right]$$

$$= 2x\tan^{-1}\left(\frac{y}{x}\right) - \frac{x^2y}{x^2+y^2} - \frac{y^3}{x^2+y^2}$$

$$= 2x\tan^{-1}\left(\frac{y}{x}\right) - \frac{x^2y - y^3}{x^2+y^2}$$

$$\frac{\partial z}{\partial x} = 2x\tan^{-1}\left(\frac{y}{x}\right) - \left(\frac{x^2y+y^3}{x^2+y^2}\right)$$

$$\frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}\right] = \frac{\partial}{\partial y}\left[2x\tan^{-1}\left(\frac{y}{x}\right)\right] - \frac{\partial}{\partial y}\left[\frac{x^2y+y^3}{x^2+y^2}\right]$$

$x \to$ constant

$y \to$ variable

$$= \left[2x \times \frac{1}{x} \times \frac{1}{1+\frac{y^2}{x^2}}\right] - \left[\frac{(x^2+3y^2)(x^2+y^2) - 2y(x^2y+y^3)}{(x^2+y^2)^2}\right]$$

$$= \frac{2x^2}{(x^2+y^2)} - \left[\frac{x^4+x^2y^2+3x^2y^2+3y^4-2x^2y^2-2y^4}{(x^2+y^2)^2}\right]$$

$$= \frac{2x^2(x^2+y^2) - x^4 - 4x^2y^2 - y^4 + 2x^2y^2}{(x^2+y^2)^2}$$

$$= \frac{2x^4+2x^2y^2-x^4-y^4-4x^2y^2+2x^2y^2}{(x^2+y^2)^2}$$

$$= \frac{x^4-y^4-2x^2y^2+2x^2y^2}{(x^2+y^2)^2}$$

$$\cancel{= \frac{x^4-2x^2y^2-y^4-2x^2y^2}{(x^2+y^2)^2}}$$

$$\cancel{= x^2(x^2-2y)-y^2(y^2+2)}$$

$$\cancel{= \frac{x^4-y^4-4x^2y^2}{(x^2+y^2)^2}}$$

$$= \frac{x^4-y^4}{(x^2+y^2)^2} = \frac{(x^2+y^2)(x^2-y^2)}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2}{x^2+y^2}$$

Hence Proved.

Que-⑤ → If $u = e^{xyz}$, find the value of $\frac{\partial^3 u}{\partial x \partial y \partial z}$

Sol: $\frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}\left[\frac{\partial^2 u}{\partial y \partial z}\right] = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial z}\right]\right]$

$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}(e^{xyz})$ $z \to$ variable
$x, y \to$ constant

$= e^{xyz} \cdot xy$

$= \frac{\partial}{\partial y}\left[e^{xyz} \cdot xy\right]$ $x, z \to$ constant

$= x \frac{\partial}{\partial y}\left[e^{xyz} \cdot y\right]$

$= x \cdot \left[e^{xyz} \cdot (xz) \cdot y + e^{xyz}\right]$

$= \frac{\partial}{\partial x}\left[x^2 zy e^{xyz} + x e^{xyz}\right]$ ($y, z \to$ constant)

$= \frac{\partial}{\partial x}(x^2 zy e^{xyz}) + \frac{\partial}{\partial x}(x e^{xyz})$

$= yz \frac{\partial}{\partial x}(x^2 e^{xyz}) + \frac{\partial}{\partial x}(x e^{xyz})$

$= yz\left[2x e^{xyz} + e^{xyz} \cdot yz x^2\right] + \left[e^{xyz} + e^{xyz} \cdot yzx\right]$

$= 2xyz e^{xyz} + x^2 y^2 z^2 e^{xyz} + e^{xyz} + xyz e^{xyz}$

$= \cancel{2xyzu} + x^2 y^2 z^2 u + u + \cancel{xyzu}$

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$\cancel{=(3xyzu + u + x^2+y^2+z^2)}$ $\qquad u = e^{xyz}$

$= 3xyzu + u + x^2y^2z^2u$ $\qquad \ln u = xyz$

$= u(3xyz + 1 + x^2y^2z^2)$

$= e^{xyz}(1 + 3xyz + x^2y^2z^2)$ Ans

Que-6 → Prove that if $f(x,y) = \frac{1}{\sqrt{y}} e^{-\frac{(x-a)^2}{4y}}$,

$f_{xy}(x,y) = f_{yx}(x,y)$ $\qquad$ Let $z = f(x,y)$

Sol: $f_{xy} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right]$

$f_{yx} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}\right]$

$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left[\frac{1}{\sqrt{y}} e^{-\frac{(x-a)^2}{4y}}\right]$ $\quad x, a \to$ constant

$= -\frac{1}{2}(y)^{-3/2} e^{-\frac{(x-a)^2}{4y}} + \frac{1}{\sqrt{y}} \times e^{-\frac{(x-a)^2}{4y}} \times \frac{+(x-a)^2}{4} \times \frac{+1}{y^2}$

$= -\frac{1}{2}(y)^{-3/2} \overbrace{e^{-\frac{(x-a)^2}{4y}}}^{m} + \frac{1}{\sqrt{y}} \times e^{-\frac{(x-a)^2}{4y}} \times \frac{(x-a)^2}{4} \times \frac{1}{y^2}$

$= -\frac{1}{\cancel{2}}(y)^{-3/2}\, m + (y)^{-5/2}\, m\, \frac{(x-a)^2}{\cancel{4}\,2}$

$= m \times \frac{1}{2}\left[-1 + \frac{1}{y}\frac{(x-a)^2}{2}\right] \times (y)^{-3/2}$

$$\frac{\partial z}{\partial y} = m \times \frac{1}{2} \times (y)^{-3/2}\left[-1 + \frac{1}{2y}(x-a)^2\right]$$

$$m = e^{\frac{-(x-a)^2}{4y}}$$

$$= \frac{\partial}{\partial x}\left[\frac{m}{2}(y)^{-3/2}\left[-1 + \frac{1}{2y}(x-a)^2\right]\right]$$

$a, y \rightarrow$ constant

$$= \frac{(y)^{-3/2}}{2}\left[\frac{\partial}{\partial x}(-m) + \frac{\partial}{\partial x}\left[\frac{(x-a)^2}{2y}\right]\right] \rightarrow \frac{1}{\cancel{2}y} \times \cancel{2}(x-a) = \frac{(x-a)}{y}$$

$$= \frac{(y)^{-3/2}}{2}\left[\frac{\partial}{\partial x}\left[-e^{-\frac{(x-a)^2}{4y}}\right]\right.$$

$$= + e^{\frac{-(x-a)^2}{4y}} \times \frac{+1}{\cancel{4}_2 y} \cancel{2}(x-a)$$

$$= e^{\frac{-(x-a)^2}{4y}} \times \frac{(x-a)}{2y}$$

$$f_{xy} = \frac{(y)^{-3/2}}{2}\left[\frac{(x-a)}{2y} e^{-\frac{(x-a)^2}{4y}} + \frac{(x-a)}{y}\right]$$

$$f_{xy} = \frac{(x-a)}{y} \times \frac{(y)^{-3/2}}{2}\left[\frac{e^{-(x-a)^2/4y}}{2} + 1\right]$$

$$f_{xy} = \left(\frac{x-a}{2}\right)(y)^{-5/2}\left[\frac{e^{-(x-a)^2/4y}}{2} + 1\right] \quad —(1)$$

$$f_{yx} = \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}\right]$$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left[\frac{1}{\sqrt{y}} e^{\frac{-(x-a)^2}{4y}}\right] \quad (y, a \to \text{constant})$$

$$= \frac{1}{\sqrt{y}} \frac{\partial}{\partial x}\left[e^{\frac{-(x-a)^2}{4y}}\right]$$

$$= \frac{1}{\sqrt{y}} \times e^{\frac{-(x-a)^2}{4y}} \times \frac{-1}{\cancel{4}_2 y} \times \cancel{2}(x-a)$$

$$\frac{\partial z}{\partial x} = \frac{1}{(y)^{3/2}} \times e^{\frac{-(x-a)^2}{4y}} \times \frac{-1}{2}(x-a)$$

$$= \frac{-(x-a)}{2y^{3/2}} e^{\frac{-(x-a)^2}{4y}}$$

$$= \frac{\partial}{\partial y}\left[\frac{-(x-a)}{2y^{3/2}} e^{\frac{-(x-a)^2}{4y}}\right] \quad (x, a \to \text{constant})$$

$$= \frac{-(x-a)}{2} \frac{\partial}{\partial y}\left[y^{-3/2} e^{\frac{-(x-a)^2}{4y}}\right]$$

$$= \frac{-(x-a)}{2} \times \left[\frac{-3}{2}(y)^{-\frac{3}{2}-1} e^{\frac{-(x-a)^2}{4y}} + e^{\frac{-(x-a)^2}{4y}} \times \frac{+(x-a)^2}{4} \times \frac{+1}{y^2} \times y^{-3/2}\right] \quad \frac{-3}{2} \frac{-2}{1}$$

$$= \frac{-(x-a)}{2}\left[\frac{-3}{\cancel{2}}(y)^{-5/2} \cancel{m} + \cancel{m} \times \frac{(x-a)^2}{\cancel{4}\,2} \times (y)^{-7/2}\right]$$

$$= \frac{-(x-a)}{2} \times m \times \frac{1}{2} \times (y)^{-5/2}\left[-3 + \frac{(x-a)^2}{2} \times \frac{1}{y}\right]$$

$$= \frac{-(x-a)}{2} \times m \times \frac{1}{2} \times (y)^{-5/2} \left[-3 + \frac{(x-a)^2}{2y} \right]$$

$$= \frac{-(x-a)}{4} \times e^{-\frac{(x-a)^2}{4y}} \times (y)^{-5/2} \left[\frac{-6y + (x-a)^2}{2y} \right]$$

(Same as CW)

Que (7) → If $V = r^n(3\cos^2\theta - 1)$, then find the value of n, so that it satisfies the relation:

$$\frac{\partial}{\partial r}\left[r^2 \frac{\partial V}{\partial r}\right] + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta \frac{\partial V}{\partial\theta}\right] = 0.$$

Sol: $V = r^n(3\cos^2\theta - 1)$

$$\frac{\partial V}{\partial r} = \frac{\partial}{\partial r}\left[r^n(3\cos^2\theta - 1)\right]$$

↳ variable

θ & n → constant

$$= \frac{\partial}{\partial r}\left[r^n 3\cos^2\theta\right] - \frac{\partial}{\partial r}\left[r^n\right]$$

$$\frac{\partial V}{\partial r} = 3\cos^2\theta\, n(r)^{n-1} - n r^{n-1}$$

$$= n r^{n-1}(3\cos^2\theta - 1) \times r^2$$

$$\times r^2 = n r^{n+1}(3\cos^2\theta - 1)$$

$$= \frac{\partial}{\partial r}\left[n r^{n+1}(3\cos^2\theta - 1)\right] \quad \theta, n \to \text{constant}$$

$$= n(3\cos^2\theta - 1)\frac{\partial}{\partial r}(r^{n+1})$$

$$= (3\cos^2\theta - 1)\, n \times (n+1)\, r^n$$

$$= r^n(3\cos^2\theta - 1)\, n(n+1) \quad \text{—— (1)}$$

$$\frac{\partial V}{\partial\theta} = \frac{\partial}{\partial\theta}\left[r^n(3\cos^2\theta - 1)\right] \quad r, n \to \text{constant}$$

$$= 3r^n \frac{\partial}{\partial\theta}(\cos^2\theta) - \cancel{\frac{\partial}{\partial\theta}(r^n)}^{0}$$

$$= 3r^n \times 2\cos\theta(-\sin\theta) \times \sin\theta$$

$$= 6r^n \cos\theta(-\sin^2\theta)$$

$$= \frac{\partial}{\partial\theta}\left[-6r^n \cos\theta \sin^2\theta\right]$$

$$= -6r^n \frac{\partial}{\partial\theta}\left(\cos\theta \sin^2\theta\right)$$

$$= -6r^n \left[(-\sin\theta)(\sin^2\theta) + 2\sin\theta\cos\theta\cos\theta\right]$$

$$= -6r^n\left[-\sin^3\theta + 2\sin\theta\cos^2\theta\right]$$

$$= -6r^n\left[\sin\theta\left[-\sin^2\theta + 2\cos^2\theta\right]\right]$$

$$\sin\theta\left(-\sin^2\theta + 2(1-\sin^2\theta)\right)$$

$$\sin\theta\left(-\sin^2\theta + 2 - 2\sin^2\theta\right)$$

$$= -6r^n\left[\cancel{\sin\theta}(2-3\sin^2\theta)\right] \times \frac{1}{\cancel{\sin\theta}}$$

$$= -6r^n(2-3\sin^2\theta) \quad \text{--- (2)}$$

add eqn (1) + (2)

$$0 = -6r^n(2-3\sin^2\theta) + r^n n(n+1)(3\cos^2\theta - 1)$$

$$6\cancel{r^n}(2-3\sin^2\theta) = n(n+1)\cancel{r^n}(3\cos^2\theta - 1)$$

$$3(1-\sin^2\theta) - 1$$

$$\cancel{3}(\quad 3 - 3\sin^2\theta - 1$$

$$6 = n(n+1) \qquad n=2 \quad \text{Ans}$$

Que (8) → If $u = r^m$, where $r^2 = x^2+y^2+z^2$,

find the value of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$

Sol: $\cancel{\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(r^m) =}$

$r^2 = x^2+y^2+z^2$

↳ diff w.r.t x keeping y & z constant

$2r\frac{\partial r}{\partial x} = 2x$

$\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(r^m) = mr^{m-1}\frac{\partial r}{\partial x} = mr^{m-1}\times\frac{x}{r} = mxr^{m-1-1}$

$\frac{\partial u}{\partial x} = mxr^{m-2}$

$\cancel{\frac{\partial}{\partial x}\left[mxr^{m-2}\right] = mr^{m-2}}$ — (1)

$\cancel{\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(r^m) = mr^{m-1}\frac{\partial r}{\partial y} = mr^{m-1}\times\frac{y}{r} = myr^{m-2}}$ — (2)

$\cancel{\frac{\partial}{\partial y}\left[myr^{m-2}\right] = mr^{m-2}}$ — (2)

$\cancel{= 3mr^{m-2} = \frac{3mr^m}{r^2} = \frac{3mu}{r^2}}$

$\frac{\partial u}{\partial x} = m\underset{\sim}{x}\underset{\sim}{r}^{m-2}$ $x \to$ variable
$r \to$ variable (bcoz it is a funct of x)
$m \to$ constant

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(mxr^{m-2}) = m\frac{\partial}{\partial x}(\underset{\sim}{x}\,\underline{r^{m-2}})$

$= m\left[\left(r^{m-2} + (m-2)r^{m-2-1}\frac{\partial r}{\partial x}\times x\right)\right]$

$$\boxed{\frac{\partial^2 u}{\partial x^2} = m\left[r^{m-2} + xr^{m-3}(m-2)\frac{\partial r}{\partial x}\right]} \quad \text{— (1)}$$

$\frac{\partial u}{\partial y} = myr^{m-2}$ — (2)

$y, r \to$ variable
$m \to$ constant

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y}\left[m y r^{m-2}\right] = m\left[r^{m-2} + (m-2) r^{m-2-1}\frac{\partial r}{\partial y} \times y\right]$$

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$$\boxed{\frac{\partial^2 v}{\partial y^2} = m r^{m-2} + m(m-2) y r^{m-3}\frac{\partial r}{\partial y}} \quad (2)$$

$$\frac{\partial v}{\partial z} = \frac{\partial}{\partial z}(r^m) = m r^{m-1}\frac{\partial r}{\partial z} = m r^{m-2} \times \frac{z}{r}$$

$$\frac{\partial v}{\partial z} = m z r^{m-3}$$

$$\frac{\partial^2 v}{\partial z^2} = \frac{\partial}{\partial z}\left(m z r^{m-3}\right) = m\left[r^{m-3} + (m-3) r^{m-4} (z)\frac{\partial r}{\partial z}\right]$$

$z, r \to$ variable

$m \to$ constant

$$\boxed{\frac{\partial^2 v}{\partial z^2} = m r^{m-3} + m(m-3) r^{(m-4)} (z)\frac{\partial r}{\partial z}} \quad (3)$$

Now adding eqn (1), (2), (3)

$$= m r^{m-3} + m(m-3) r^{(m-4)} z\frac{\partial r}{\partial z} + m r^{m-2} + m(m-2) y r^{m-3}\frac{\partial r}{\partial y}$$

$$+ m r^{m-2} + m x r^{m-3} (m-2)\frac{\partial r}{\partial x}$$

$$= m r^{m-3} + m(m-3) r^{(m-4)} z \times \frac{z}{r} + m r^{m-2} + m(m-2) y r^{m-3} \times \frac{y}{r}$$

$$+ m r^{m-2} + m x r^{m-3} (m-2) \times \frac{x}{r}$$

$$= m r^{m-3} + m(m-3) r^{(m-5)} \frac{z^2}{r} + m r^{m-2} + m(m-2) y^2 r^{(m-4)}$$

$$+ m r^{m-2} + m x^2 r^{(m-4)} (m-2)$$

$$= m r^{m-3} + 2 m r^{m-2} + m r^{(m-4)}\left[(m-3)\frac{z^2}{r} + y^2(m-2) + x^2(m-2)\right.$$

$$(m-3)\frac{z^2}{r} + (m-2)(x^2+y^2)$$

$$= m(m+1) r^{m-2} \text{ Ans}$$

Que (9) → If $u = \log(x^3+y^3+z^3-3xyz)$, show that

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}.$$

Sol: $u = \log(x^3+y^3+z^3-3xyz)$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[\log(x^3+y^3+z^3-3xyz)\right] \quad \begin{pmatrix} y, z \to \text{constant} \\ x \to \text{variable} \end{pmatrix}$$

$$= \frac{1}{(x^3+y^3+z^3-3xyz)}\left(3x^2-3yz\right)$$

$$\frac{\partial u}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial y} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u = \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}$$

$$= \frac{3x^2+3y^2+3z^2-3xy-3yz-3zx}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x^3+y^3+z^3-3xyz)}$$

$$= \frac{3\cancel{(x^2+y^2+z^2-xy-yz-zx)}}{(x+y+z)\cancel{(x^2+y^2+z^2-xy-yz-zx)}} = \frac{3}{(x+y+z)}$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= \frac{-9}{(x+y+z)^2} \quad \text{(H.P.)}$$

Que. If $z = x\log(x+r) - r$ where $r^2 = x^2+y^2$

Show that (i) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{x+r}$

(ii) $\frac{\partial^3 z}{\partial x^3} = -\left(\frac{x}{r^3}\right)$

Sol: $z = f(x, r)$

$\cancel{\frac{\partial z}{\partial x} = \frac{1}{x+r}\left[1+\partial\right.}$

$z = x\log(x+r) - r$

$r^2 = x^2+y^2$

$2r\frac{\partial r}{\partial x} = 2x$

$\boxed{\frac{x}{r} = \frac{\partial r}{\partial x}}$ $\boxed{\frac{\partial r}{\partial y} = \frac{y}{r}}$

$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left[x\log(x+r) - r\right]$ ~~r→ root~~

$= \log(x+r) + \frac{1}{(x+r)} \times x - \frac{\partial r}{\partial x}$ (r, x → both are variable)

$= \log(x+r) + \frac{x}{(x+r)} - \frac{x}{r}$

$\frac{\partial^2 z}{\partial x} = \frac{\partial}{\partial x}\left[\log(x+r) + \frac{x}{x+r} - \frac{x}{r}\right]$

$2r\frac{\partial r}{\partial x} = 2x$

$\frac{\partial r}{\partial x} = \frac{x}{r}$

(M-11): $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left[x\log(x+r) - r\right]$

$= \frac{\partial}{\partial x}\left[x\log(x+r)\right] - \frac{\partial r}{\partial x}$ (x, r → variable)

$= \log(x+r) + \frac{1}{(x+r)}\left[1+\frac{\partial r}{\partial x}\right]x - \frac{x}{r}$

$= \log(x+r) + \frac{x}{(x+r)}\left[1+\frac{x}{r}\right] - \frac{x}{r} = \frac{x}{(x+r)}\left[\frac{x+r}{r}\right] - \frac{x}{r}$

$\frac{\partial z}{\partial x} = \log(x+r) + \frac{\cancel{x(x+r) - x(x+r)}}{\cancel{r(x+r)}}$ 0

$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\log(x+r)\right) = \frac{1}{(x+r)}$

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{r}$$

Again $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial y} = \frac{x}{x+r}\left[0 + \frac{\partial r}{\partial y}\right] - \frac{\partial r}{\partial y}$

$$= \frac{x}{x+r} \times \frac{y}{r} - \frac{y}{r}$$

$$= \frac{xy - xy - ry}{r(x+r)} = \frac{-ry}{r(x+r)}$$

$$\left(\frac{\partial z}{\partial y} = \frac{-y}{(x+r)}\right)$$

again diff w.r.t y

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{-y}{x+r}\right] \quad (x \to \text{constant})$$

$$= \frac{(-1)(x+r) - \frac{\partial r}{\partial y}(-y)}{(x+r)^2}$$

$$= \frac{-(x+r) - \frac{y}{r} \times -y}{(x+r)^2}$$

$$= \frac{-(x+r) + \frac{y^2}{r}}{(x+r)^2}$$

$$= \frac{-r(x+r) + y^2}{r(x+r)^2}$$

$$= \frac{-rx - r^2 + y^2}{r(x+r)^2} = \frac{1}{r} - \frac{x}{r(x+r)}$$

$$= \frac{x+r - rx}{r(x+r)}$$

$$= \frac{1}{(x+r)}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{-x}{r(x+r)}$$

$r^{-1} \quad -1r^{-2} = \frac{-1}{r^2}$

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{r}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{-1}{r^2} \times \frac{\partial r}{\partial x}$$

$$= \frac{-1}{r^2} \times \frac{x}{r} = \frac{-x}{r^3} \text{ Ans}$$

(HW)

Que① If $\theta = t^n \cdot e^{\frac{-r^2}{4t}}$ for what value of n will make

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial \theta}{\partial r}\right] = \frac{\partial \theta}{\partial t}$$

Ans $\left(n = -3/2\right)$

Sol:

$\frac{\partial \theta}{\partial r} = \frac{\partial}{\partial r}\left[t^n e^{\frac{r^2}{4t}}\right]$ $r \to$ variable, $n, t \to$ constant

$= t^n \cdot e^{r^2/4t} \cdot \frac{1}{2 4t} \times 2r$

$= e^{r^2/4t} \cdot t^n \times \frac{r}{2t}$

$= \frac{r}{2} e^{r^2/4t} \times t^{n-1} \times r^2$

$= \frac{r^3}{2} t^{n-1} e^{r^2/4t}$

$= \frac{\partial}{\partial r}\left[\frac{r^3}{2} t^{n-1} e^{r^2/4t}\right]$

$= \frac{t^{n-1}}{2} \frac{\partial}{\partial r}\left[r^3 e^{r^2/4t}\right]$

$= \frac{t^{n-1}}{2}\left[3r^2 e^{r^2/4t} + e^{r^2/4t} \cdot \frac{2r}{4t} \cdot r^3\right]$

$= \frac{t^{n-1}}{2}\left[3r^2 e^{r^2/4t} + \frac{r^4}{2t} e^{r^2/4t}\right] \times \frac{1}{r^2}$

$= \frac{t^{n-1}}{2r^2}\left[3r^2 \cancel{e^{r^2/4t}} + \frac{r^4}{2t} \cancel{e^{r^2/4t}}\right]$

$= e^{r^2/4t} \frac{\cdot r^2}{2r^2} \times t^{n-1}\left[3 + \frac{r^2}{t}\right]$

$= \frac{e^{r^2/4t}}{2} t^{n-1}\left[3 + \frac{r^2}{t}\right]$ — ①

$= \frac{\partial}{\partial t}\left[t^n \cdot e^{r^2/4t}\right]$ $n, r \to$ constant

$= nt^{n-1} e^{r^2/4t} + e^{r^2/4t} \times \frac{r^2}{4} \times \frac{-1}{t^2} \times t^n$

$= nt^{n-1} e^{r^2/4t} - \frac{r^2}{4} \times t^{n-2} e^{r^2/4t}$

$= \cancel{t^{n-1}} \cancel{e^{r^2/4t}}\left[n - \frac{r^2}{4} \times \frac{1}{t}\right]$ — ⑪

$\cancel{n-2}\ n+1$

equating eqn ① & ⑪

$$\frac{1}{2}\left[3 + \frac{r^2}{t}\right] = n - \frac{r^2}{4t}$$

$$\frac{3t + r^2}{2\cancel{t}} = \frac{4nt - r^2}{4\cancel{t}}$$

$$6t + \cancel{2r^2} = 4nt - \cancel{r^2}$$

$$3r^2 = 4nt - 6t$$

Que 2 If $z = x^y + y^x$, verify $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$.

Sol:

$$\frac{\partial}{\partial x}(x^y) = y x^{y-1} \quad \text{(variable)}^{\text{constant}}$$

NOTE:

$$\frac{\partial}{\partial y}(x^y) = x^y \log x \quad \text{(constant)}^{\text{variable}}$$

Solving LHS, $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right]$

$$\frac{\partial}{\partial y}(x^y + y^x) = \frac{\partial}{\partial y}(x^y) + \frac{\partial}{\partial y}(y^x)$$

y → variable
x → constant

$$\frac{\partial z}{\partial y} = x^y \log x + x y^{x-1}$$

$$= \frac{\partial}{\partial x}\left[x^y \log x + x y^{x-1}\right]$$

$$= \frac{\partial}{\partial x}\left[x^y \log x\right] + \frac{\partial}{\partial x}\left[x y^{x-1}\right]$$

$$= y x^{y-1} \log x + \frac{1}{x} \times x^y + y^{x-1} + y^{x-1} \log y \; x$$

$$= \left(y x^{y-1} \log x + x^{y-1} + y^{x-1}(1 + \log y) x\right) \quad \text{— (1)}$$

Solving RHS, $\frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}\right] = \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right]$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^y + y^x) = y x^{y-1} + y^x \log y$$

$$= \frac{\partial}{\partial y}\left[y x^{y-1} + y^x \log y\right]$$

$$= x^{y-1} + x^{y-1} \log x \cdot y + \cancel{y}\, x y^{x-1} \log y + \frac{1}{y} y^x$$

$$= x^{y-1} + x^{y-1} y \log x + x y^{x-1} \log y + \frac{y\, y^{x-1}}{\cancel{y}}$$

$$= \left(y x^{y-1} \log x + x^{y-1} + y^{x-1}(1 + \log y) x\right) \quad \text{— (2)}$$

eqn (1) = eqn (2) (LHS = RHS)

(Hence Proved)

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* Homogeneous
* Non homogeneous
* Reducible

Homogeneous function: A function $f(x,y)$ is said to be homogeneous function in which power of each term is the same.

A function $f(x,y)$ is a homogeneous function of order n if the degree of each term is n in x & y.

$$f(x,y) = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_{n-1}xy^{n-1} + a_ny^n$$

$$= x^n\left[a_0 + a_1\left(\frac{x}{y}\right)\left(\frac{y}{x}\right) + a_2\left(\frac{y}{x}\right)^2 + \cdots + a_{n-1}\left(\frac{y}{x}\right)^{n-1} + a_n\left(\frac{y}{x}\right)^n\right)$$

$$= x^n\phi\left(\frac{y}{x}\right)$$

Ex: $f = \frac{x^3+y^3}{x+y}$ put $x \to tx$, $y \to ty$

$$= \frac{t^3x^3 + t^3y^3}{tx+ty}$$

$$= \frac{t^3(x^3+y^3)}{t(x+y)} = \frac{t^2(x^3+y^3)}{(x+y)}$$

$$= t^2f.$$

Yes it is homogeneous of degree 2.

Ex: $f = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$

Sol: put $x \to tx$ & $y \to ty$

$$f = \tan^{-1}(t^2f)$$

$$\tan f = t^2f$$

$\tan f$ is homogeneous function of degree $n = 2$ (homogeneous)

↓

(Reducible to homogeneous function)

HW

PRACTICE QUESTIONS

Que 1 → If $z = x\log(x+r) - r$ where $r^2 = x^2+y^2$

Prove that (i) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{x+r}$

(ii) $\frac{\partial^3 z}{\partial x^3} = \frac{-x}{r^3}$

Sol: (i) $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(x\log(x+r) - r\right)$ $x \to$ variable, $r \to$ variable, $y \to$ constant

$= \log(x+r) + \frac{1}{(x+r)}\left(1+\frac{\partial r}{\partial x}\right)x - \frac{\partial r}{\partial x}$

$r^2 = x^2+y^2$ ($y \to$ constant)

$\cancel{2}r\frac{\partial r}{\partial x} = \cancel{2}x$

$\frac{\partial r}{\partial x} = \frac{x}{r}$

$= \log(x+r) + \frac{1}{(x+r)}\left(1+\frac{x}{r}\right)x - \frac{x}{r}$

$= \log(x+r) + \frac{1}{\cancel{(x+r)}}\left(\frac{\cancel{x+r}}{r}\right)x - \frac{x}{r}$

$= \log(x+r) + \cancel{\frac{x}{r}} - \cancel{\frac{x}{r}} = \log(x+r)$

$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\log(x+r)\right) = \frac{1}{(x+r)}\left(1+\frac{\partial r}{\partial x}\right)$

$= \frac{1}{(x+r)}\left(1+\frac{x}{r}\right) = \frac{1}{\cancel{(x+r)}}\frac{\cancel{(x+r)}}{r} = \frac{1}{r}$

$\frac{\partial^2 z}{\partial x^2} = \frac{1}{r}$ — (1)

$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(x\log(x+r) - r\right)$

$x \to$ constant

$y, r \to$ variable

$$= \frac{1}{(x+r)} \times \frac{\partial r}{\partial y} \times x - \frac{\partial r}{\partial y}$$

$$r^2 = x^2 + y^2$$

$$2r \frac{\partial r}{\partial y} = 2y$$

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial z}{\partial y} = \frac{1}{(x+r)} \times \frac{y}{r} \times x - \frac{y}{r}$$

$$\cancel{= \frac{r}{r(x+r)}} \qquad = \frac{xy}{r(x+r)} - \frac{y}{r}$$

$$= \frac{(xy) - y(x+r)}{r(x+r)}$$

$$= \frac{\cancel{xy} - \cancel{xy} - y\cancel{r}}{\cancel{r}(x+r)} = \frac{-y}{x+r}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{-y}{x+r}\right)$$

$$= \frac{-1(x+r) - \frac{\partial r}{\partial y}(-y)}{(x+r)^2}$$

$$= \frac{-x-r+y \times \frac{y}{r}}{(x+r)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{-x-r+\frac{y^2}{r}}{(x+r)^2} = \frac{-xr-r^2+y^2}{r(x+r)^2} \quad (2)$$

adding eqn (1) & (11)

$$= \frac{-xr-r^2+y^2}{r(x+r)^2} + \frac{1}{r} = \frac{-xr-r^2+y^2+(x+r)^2}{r(x+r)^2}$$

$$= \frac{-xr-\cancel{r^2}+y^2+x^2+\cancel{r^2}+2xr}{r(x+r)^2}$$

$$= \frac{x^2+y^2-xr+2xr}{r(x^2+r^2+2xr)}$$

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$r^2 = x^2+y^2$

$y^2 = r^2 - x^2$

$$= \frac{\cancel{x^2} + r^2 - \cancel{x^2} - \cancel{xr} + 2\cancel{x}r}{r(x^2+r^2+2xr)}$$

$$= \frac{xr + r^2}{r(x+r)^2} = \frac{\cancel{r}(x+r)}{\cancel{r}(x+r)^2}$$

$$= \frac{1}{(x+r)}$$

(Hence Proved)

(iii) $$\frac{\partial^3 z}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x^2}\right) = \frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{-1}{r^2}\frac{\partial r}{\partial x} = \frac{-1}{r^2} \times \frac{x}{r}$$

$$= \frac{-x}{r^3}$$

(Hence Proved)

Que 2 → Show that function $f(x,y) = \begin{cases} \frac{x-y}{x+y} & (x,y) \neq 0 \\ 0 & (x,y) = 0 \end{cases}$ is discontinous at (0,0).

Sol: at (0,0), $f(x,y) = 0$

Check limit : put $y = mx$

$$\lim_{x \to 0} \frac{x - mx}{x + mx} = \frac{x(1-m)}{x(1+m)} = \frac{1-m}{1+m}$$ → Limit depends upon m. Hence Limit does not exist

Hence $f(x,y)$ is discontinous at (0,0).

(Hence Proved)

Que 3 → If $u = \log(x^3+y^3+z^3-3xyz)$ Prove that

$$\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right]^2 u = \frac{-9}{(x+y+z)^2}.$$

Sol: $u = \log(x^3+y^3+z^3-3xyz)$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[\log(x^3+y^3+z^3-3xyz)\right]$$ (y, z → constant) (x → variable)

$$= \frac{1}{(x^3+y^3+z^3-3xyz)} \times 3x^2 - 3yz$$

$$\frac{\partial u}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial y} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u = \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)(u) = \frac{3x^2+3y^2+3z^2-3xy-3yz-3xz}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3(x^2+y^2+z^2-xy-yz-xz)}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3\cancel{(x^2+y^2+z^2-xy-yz-xz)}}{(x+y+z)\cancel{(x^2+y^2+z^2-xy-yz-zx)}}$$

$$= \frac{3}{(x+y+z)}$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= \frac{-3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2}$$

$$= \frac{-9}{(x+y+z)^2}$$

LHS = RHS (Hence proved)

Que (4) → If $u = \frac{y}{z} + \frac{z}{x} + \frac{x}{y}$ Show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0 \quad \text{—— (4)}$$

Sol: $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{y}{z} + \frac{z}{x} + \frac{x}{y}\right)$ $x \to$ variable, $y, z \to$ constant

$$= \frac{\partial}{\partial x}\left(\frac{y}{z}\right) + \frac{\partial}{\partial x}\left(\frac{z}{x}\right) + \frac{\partial}{\partial x}\left(\frac{x}{y}\right)$$

$$= 0 + z \times \frac{-1}{x^2} + \frac{1}{y}$$

$$\frac{\partial u}{\partial x} = \frac{-z}{x^2} + \frac{1}{y} \quad \text{—— (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y}{z} + \frac{z}{x} + \frac{x}{y}\right) = \frac{\partial}{\partial y}\left(\frac{y}{z}\right) + \frac{\partial}{\partial y}\left(\frac{z}{x}\right) + \frac{\partial}{\partial y}\left(\frac{x}{y}\right)$$

($y \to$ variable, $x, z \to$ constant)

$$= \frac{1}{z} + 0 + x \times \frac{-1}{y^2}$$

$$\frac{\partial u}{\partial y} = \frac{-x}{y^2} + \frac{1}{z} \quad \text{—— (2)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}\left(\frac{y}{z} + \frac{z}{x} + \frac{x}{y}\right) = \frac{\partial}{\partial z}\left(\frac{y}{z}\right) + \frac{\partial}{\partial z}\left(\frac{z}{x}\right) + \frac{\partial}{\partial z}\left(\frac{x}{y}\right)$$

$z \to$ variable, $x, y \to$ constant

$$= \frac{-y}{z^2} + \frac{1}{x}$$

$$\frac{\partial u}{\partial z} = \frac{1}{x} - \frac{y}{z^2} \quad \text{—— (3)}$$

Now putting the values of eqn (1), (2), (3) in (4)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

$$x\left(\frac{-z}{x^2} + \frac{1}{y}\right) + y\left(\frac{-x}{y^2} + \frac{1}{z}\right) + z\left(\frac{1}{x} - \frac{y}{z^2}\right) = 0$$

$$\cancel{\frac{-z}{x}} + \cancel{\frac{x}{y}} - \cancel{\frac{x}{y}} + \cancel{\frac{y}{z}} + \cancel{\frac{z}{x}} - \cancel{\frac{y}{z}} = 0$$

$$0 = 0$$

(Hence Proved)

Que (5) If $u = f(r)$ where $r^2 = x^2 + y^2$. Prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r).$$

Sol: $u = f(r)$, $r^2 = x^2 + y^2$

$r^2 = x^2 + y^2$

partially differentiating wrt x,

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} \quad \text{--- (1)}$$

partially differentiating wrt y,

$$2r\frac{\partial r}{\partial y} = 2y$$

$$\frac{\partial r}{\partial y} = \frac{y}{r} \quad \text{--- (2)}$$

$u = f(r)$ (diff wrt x)

$$\frac{\partial u}{\partial x} = f'(r)\frac{\partial r}{\partial x} = f'(r) \times \frac{x}{r}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[f'(r)\frac{x}{r}\right] =$$

$x, r \rightarrow$ both are variables

$$\frac{\partial^2 u}{\partial x^2} = \frac{\left(f''(r)\frac{\partial r}{\partial x}x + f'(r)\right)r - \frac{\partial r}{\partial x}\left(f'(r)x\right)}{r^2}$$

$$= \frac{\left(f''(r)\frac{x}{r}\times x + f'(r)\right)r - \frac{x}{r}\times x f'(r)}{r^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\left(\frac{x^2}{r}f''(r) + f'(r)\right)r - \frac{x^2}{r}f'(r)}{r^2} \quad \text{— } (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(f(r)\right)$$

$$\frac{\partial u}{\partial y} = f'(r)\frac{\partial r}{\partial y} = f'(r)\times\frac{y}{r}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{f'(r)\,y}{r}\right]$$

$$= \frac{\frac{\partial}{\partial y}\left(f'(r)y\right)r - \frac{\partial r}{\partial y}\left(f'(r)y\right)}{r^2}$$

$$= \frac{\left(f''(r)\frac{\partial r}{\partial y}y + f'(r)\right)r - \frac{y}{r}\times f'(r)\times y}{r^2}$$

$$= \frac{\left(f''(r)\times\frac{y}{r}\times y + f'(r)\right)r - \frac{f'(r)y^2}{r}}{r^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\left(f''(r)\frac{\times y^2}{r} + f'(r)\right)r - \frac{y^2}{r}f'(r)}{r^2} \quad \text{— } (2)$$

Adding eqn (1) & (2)

$$= \frac{\frac{x^2}{r}f''(r) + r f'(r) - \frac{x^2}{r}f'(r) + f''(r)y^2 + r f'(r) - \frac{y^2}{r}f'(r)}{r^2}$$

$$= \frac{x^2 f''(r) + r f'(r) - \frac{x^2}{r} f'(r) + y^2 f''(r) + r f'(r) - \frac{y^2}{r} f'(r)}{r^2}$$

$$= \frac{f''(r)(x^2+y^2) + 2r f'(r) - \frac{f'(r)}{r}(x^2+y^2)}{r^2}$$

$$= \frac{r f''(r)(x^2+y^2) + 2r^2 f'(r) - f'(r)(x^2+y^2)}{r^3}$$

$$= \frac{r f''(r)(x^2+y^2) + f'(r)\left[2r^2 - x^2 - y^2\right]}{r^3}$$

M-2: $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial}{\partial x}\left[\dfrac{f'(r)\, x}{r}\right]$

$$= \frac{\frac{\partial}{\partial x}\left[f'(r)\cdot x\right] r - \frac{\partial r}{\partial x}\left[x f'(r)\right]}{r^2}$$

$$= \frac{r\left[f''(r)\frac{\partial r}{\partial x} x + f'(r)\right] - \frac{x}{r} \times x \times f'(r)}{r^2}$$

$$= \frac{r f'(r) + r x \times \frac{x}{r} f''(r) - f'(r)\frac{x^2}{r}}{r^2}$$

$$= \frac{r f'(r) + x^2 f''(r) - f'(r)\frac{x^2}{r}}{r^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{r^2 f'(r) + x^2 r f''(r) - x^2 f'(r)}{r^3} \quad \text{—— (1)}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{r^2 f'(r) + r y^2 f''(r) - f'(r) y^2}{r^3} \quad \text{—— (2)}$$

Now add eqn (1) & (2)

$$= \frac{r^2 f'(r) + r x^2 f''(r) - f'(r) x^2 + r^2 f'(r) + r y^2 f''(r) - f'(r) y^2}{r^3}$$

$$= \frac{1}{r^3}\left[r^2 f'(r) + r x^2 f''(r) - f'(r)x^2 + r^2 f'(r) + r y^2 f''(r) - f'(r) y^2\right]$$

$$= \frac{1}{r^3}\left[(x^2+y^2) f'(r) + r x^2 f''(r) - f'(r) x^2 + (x^2+y^2) f'(r) + r y^2 f''(r) - f'(r) y^2\right]$$

$$= \frac{1}{r^3}\left[f'(r)\left[x^2+y^2 - x^2 + x^2+y^2 - y^2\right] + r f''(r)(x^2+y^2)\right]$$

$$= \frac{1}{r^3}\left[f'(r)\left[x^2+y^2\right] + r f''(r)(x^2+y^2)\right]$$

$$= \frac{1}{r^3}\left[f'(r)\left[r^2\right] + r f''(r) r^2\right]$$

$$= \frac{f'(r)\, r^2}{r^3} + \frac{r^3 f''(r)}{r^3}$$

$$= \frac{f'(r)}{r} + f''(r)$$

(LHS = RHS) (Hence Proved)

Ques 6 $\Rightarrow$ If $u(x,y,z) = \log(\tan x + \tan y + \tan z)$

Show that

$$\sin(2x)\frac{\partial u}{\partial x} + \sin(2y)\frac{\partial u}{\partial y} + \sin(2z)\frac{\partial u}{\partial z} = 2.$$

Sol: $u = \log(\tan x + \tan y + \tan z)$

$$\frac{\partial u}{\partial x} = \frac{\sec^2 x}{\tan x + \tan y + \tan z}$$

$$\frac{\partial u}{\partial y} = \frac{\sec^2 y}{\tan x + \tan y + \tan z}$$

$$\frac{\partial u}{\partial z} = \frac{\sec^2 z}{\tan x + \tan y + \tan z}$$

$$= \frac{\sin(2x)\sec^2 x + \sin(2y)\sec^2 y + \sin(2z)\sec^2 z}{\tan x + \tan y + \tan z}$$

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$$= \frac{\frac{2\sin x\cos x}{\cos^2 x} + \frac{2\sin y\cos y}{\cos^2 y} + \frac{2\sin z\cos z}{\cos^2 z}}{\tan x + \tan y + \tan z}$$

$$= \frac{2\tan x + 2\tan y + 2\tan z}{\tan x + \tan y + \tan z}$$

$$= \frac{2\cancel{(\tan x + \tan y + \tan z)}}{\cancel{(\tan x + \tan y + \tan z)}}$$

$$= 2$$

(LHS = RHS)

(Hence Proved)

(L-4) 25/09/2024

Euler Theorem: Statement: If u is homogeneous function of deg n in x & y. Then

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$

(ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$.

For ex: If $u = \frac{x^3+y^3}{x+y}$. evaluate $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

& also $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$

Sol: $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x^3+y^3}{x+y}\right)$ ($y \to$ constant)

$$= \frac{3x^2(x+y) - (x^3+y^3)}{(x+y)^2}$$

$$= \frac{3x^3 + 3x^2y - x^3 - y^3}{(x+y)^2} = \frac{3x^2 - (x^2+y^2-xy)}{(x+y)} \quad \text{— (1)}$$

$x \to tx$, $y \to ty$

$u = \frac{t^3x^3 + t^3y^3}{t(x+y)} = \frac{t^2(x^3+y^3)}{(x+y)}$ (degree = 2)

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x^3+y^3}{x+y}\right) \quad x \to \text{constant}$$

$$= 3y^2(x+y)$$

M-2: u is homogeneous function of degree $\boxed{n=2}$
by Euler's theorem,

(i) $= 2u$

(ii) $n(n-1)u = 2(2-1)u = 2u$

for ex: If $u = x\sin^{-1}\left(\frac{x}{y}\right) + y\tan^{-1}\left(\frac{y}{x}\right)$

(i) Evaluate $xU_x + yU_y$.

(ii) $x^2U_{xx} + y^2U_{yy} + 2xyU_{xy}$

Sol: $u = tx\sin^{-1}\left(\frac{tx}{ty}\right) + ty\tan^{-1}\left(\frac{ty}{tx}\right)$

$u = t\,u$ degree $n=1$

(i) u

(ii) $(1-1)1u = 0$

for ex: $u = \sin^{-1}\left(\frac{x^2+y^2}{xy}\right)$ $\qquad \sin^{-1}\left(\frac{t^2x^2+t^2y^2}{t^2xy}\right)$

$xU_x + yU_y = ?$ $\qquad$ (degree $=0$)

Sol: $= 0$

Euler Deduction: If $f(u)$ is homogeneous function of deg n in x & y then

(i) $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = n\cdot\frac{f(u)}{f'(u)} \to \phi(u)$

(ii) $x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\partial y} + y^2\frac{\partial^2 v}{\partial y^2} = \phi(u)[\phi'(u)-1]$

where $\phi(u) = \frac{n\cdot f(u)}{f'(u)}$ $\qquad = \phi(u)[\phi'(u)-1]$

Ex: if $u = \left(\sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

Sol: $\sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$

$\sin u = \frac{tx+ty}{\sqrt{tx}+\sqrt{ty}} = \frac{t(x+y)}{\sqrt{t}(\sqrt{x}+\sqrt{y})}$

$= \frac{\sqrt{t}(x+y)}{(\sqrt{x}+\sqrt{y})}$

$\sin u = f(u)$ is homogeneous function of deg $n = \frac{1}{2}$

$f(u) = \sin u$, $f'(u) = \cos u$

By Euler deduction

$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = \frac{n f(u)}{f'(u)} = \frac{1}{2}\frac{\sin u}{\cos u}$

$= \frac{1}{2}\tan u$

Now, $\phi(u) = \frac{1}{2}\tan u$

diff $\phi'(u) = \frac{1}{2}\sec^2 u$.

$x^2\frac{\partial^2 v}{\partial x^2} + y^2\frac{\partial^2 v}{\partial y^2} + 2xy\frac{\partial^2 v}{\partial x \partial y} = \phi(u)[\phi'(u)-1]$

$= \frac{1}{2}\tan u\left[\frac{1}{2}\sec^2 u - 1\right]$

Ques: Verify Euler's theorem,

If $u = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$. Verify.

Sol: $u = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$

u is homogeneous function of degree $n=0$

Now, acc to euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$$

Que. Verify Euler's theorem,

(i) $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$

Sol: $u = \sin^{-1}\left(\frac{tx}{ty}\right) + \tan^{-1}\left(\frac{ty}{tx}\right)$

$u = t^0$ $\quad n=0$

LHS=0

$\frac{1}{\sqrt{1-x^2}}$

by partial differ.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[\sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left[\frac{y}{x}\right]\right]$$

$$= \frac{1}{\sqrt{1-\frac{x^2}{y^2}}} \times \frac{1}{y} + \frac{1}{1+\frac{y^2}{x^2}} \times y \times \frac{-1}{x^2}$$

$$= \frac{1}{\sqrt{\frac{y^2-x^2}{\cancel{y^2}}}} \times \frac{1}{\cancel{y}} + \frac{\cancel{x^2}}{x^2+y^2} \times \frac{-y}{\cancel{x^2}}$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{y^2-x^2}} \quad \frac{-y}{x^2+y^2} \quad \text{— ①}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left[\frac{y}{x}\right]\right]$$

$$= \frac{1}{\sqrt{1-\frac{x^2}{y^2}}} \times \frac{-1}{y^2} \times x + \frac{1}{1+\frac{y^2}{x^2}} \times \frac{1}{x}$$

$$\frac{\partial u}{\partial y} = \frac{-xy}{\sqrt{y^2-x^2}} + \frac{x}{1+y^2} \quad \text{— ②}$$

HW: Verify the Euler:

$$u=(x^{1/2}+y^{1/2})(x^n+y^n).$$

Sol: Check homogeneous

$x \to tx,\ y \to ty$

$$u=((tx)^{1/2}+(ty)^{1/2})((tx)^n+(ty)^n)$$

$$u=t^{1/2}(x^{1/2}+y^{1/2})\,t^n(x^n+y^n)$$

$$u=t^{n+1/2}\,u$$

$$n=n+\frac{1}{2}$$

degree

Solving LHS: $x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}$

$$\frac{\partial}{\partial x}\left[(x^{1/2}+y^{1/2})(x^n+y^n)\right]=\frac{\partial}{\partial x}\left[x^{n+\frac{1}{2}}+x^n y^{1/2}+x^{1/2}y^n+y^{n+1/2}\right]$$

y → constant

$$=\left(n+\frac{1}{2}\right)x^{n+\frac{1}{2}-1}+y^{1/2}\times n x^{n-1}+y^n\times\frac{1}{2}x^{\frac{1}{2}-1}$$

$n-\frac{1}{2}+1 \quad n+1$

$-\frac{1}{2}+1$

$$\frac{\partial v}{\partial x}=\left(n+\frac{1}{2}\right)x^{(n-1/2)}+nx^{n-1}y^{1/2}+\frac{x^{-1/2}y^n}{2}$$

$\times x$

$$x\frac{\partial v}{\partial x}=\left(n+\frac{1}{2}\right)x^{(n+\frac{1}{2})}+ny^{1/2}x^n+\frac{y^n}{2}x^{1/2} \quad \text{— (1)}$$

$$\frac{\partial v}{\partial y}=\frac{\partial}{\partial y}\left[(x^{n+1/2}+x^n y^{1/2}+x^{1/2}y^n+y^{n+1/2})\right]$$

$$=x^n\times\frac{1}{2}(y)^{\frac{1}{2}-1}+x^{1/2}\times n y^{n-1}+\left(n+\frac{1}{2}\right)y^{(n+\frac{1}{2}-1)}$$

$n-\frac{1}{2}+1 \quad n+1$

$$=\left(\frac{x^n}{2}(y)^{-1/2}+nx^{1/2}y^{n-1}+ny^{n-\frac{1}{2}}+\frac{1}{2}y^{n-1/2}\right)\times y$$

$n-\frac{1}{2}+1=n+\frac{1}{2}$

$$=\left(\frac{x^n}{2}(y)^{1/2}+nx^{1/2}y^n+ny^{n+1/2}+\frac{1}{2}y^{n+1/2}\right) \quad \text{— (2)}$$

= add eqn (1) & (2)

$$\left(n+\frac{1}{2}\right)u=\left(n+\frac{1}{2}\right)u$$

LHS = RHS

(Hence Proved)

HW

Que ①: If $z = x^4 y^2 \sin^{-1}\left(\frac{x}{y}\right) + \log x - \log y$

Evaluate $xU_x + yU_y$.

Que ②: If $u = \sin^{-1}\left(\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right)$

Show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + 3\tan u = 0.$$

Sol ①: $z = x^4 y^2 \sin^{-1}\left(\frac{x}{y}\right) + \log\left(\frac{x}{y}\right)$

$\left\{\begin{matrix} x \to tx \\ y \to ty \end{matrix}\right\}$ $z = t^4 x^4 t^2 y^2 \sin^{-1}\left(\frac{tx}{ty}\right) + \log\left(\frac{tx}{ty}\right)$

$$z = \underbrace{t^6\left(x^4 y^2 \sin^{-1}\frac{x}{y}\right)}_{t^6 = n=6} + \underbrace{t^0 \log\left(\frac{x}{y}\right)}_{n=0}$$

$z = A + B$

$A =$ by eulers theorem

$= 6u + 0u = 6u$ Ans

Sol ②: $u = \sin^{-1}\left(\frac{tx+2ty+3tz}{\sqrt{t^8x^8+t^8y^8+t^8z^8}}\right)$

$u = \sin^{-1}\left(\frac{t(x+2y+3z)}{t^4\sqrt{x^8+y^8+z^8}}\right)$

$\sin u = t^{-3}\left(\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right)$

is homogeneous

$$= -3\frac{f(u)}{f'(u)} = \frac{-3\sin u}{\cos u} = -3\tan u$$

$-3\tan u + 3\tan u = 0$

$0 = 0$

(H.P)

(L-5)

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for ex: If $u = \sin^{-1}\left[\frac{x^{1/3}+y^{1/3}}{\sqrt{x}-\sqrt{y}}\right]^{1/2}$

Show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{12}\tan u = 0.$

Sol: Reducible,

$x \to tx,\ y \to ty$

$u = \sin^{-1}\left[\frac{t^{1/3}(x^{1/3}+y^{1/3})}{t^{1/2}(\sqrt{x}-\sqrt{y})}\right]^{1/2}$

$\frac{1}{3}-\frac{1}{2}$

$\overline{\ \ 6}$

$\sin u = t^{-\frac{1}{6}\times\frac{1}{2}} u \qquad (n = -1/6) \qquad f(u) = \sin u$

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n \cdot \frac{f(u)}{f'(u)}$ (by Euler's)

$= \frac{-1}{6\times 2} \times \frac{\sin u}{\cos u} = \frac{-1}{12}\tan u$

$= \frac{-1}{12}\tan u + \frac{1}{12}\tan u = 0$

$0 = 0$

(LHS = RHS)

(Hence Proved)

for ex: If $u = \log\left(\frac{x^5+y^5+z^5}{x^2+y^2+z^2}\right)$. Prove that

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 3.$

Sol: Reducible karenge → Numerator & denominator में Alag Alag power hai.

$e^u = t^3\left[\frac{x^5+y^5+z^5}{x^2+y^2+z^2}\right]$ Put $x = tx$

$y = ty$

$z = tz$

Let $f(u) = e^u$ is homogeneous function of degree $\boxed{n=3}$

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$

$$x\frac{\partial}{\partial x}e^u + y\frac{\partial}{\partial y}e^u + z\frac{\partial}{\partial z}e^u = ne^u$$

$$x\cancel{e^u}\frac{\partial u}{\partial x} + y\cancel{e^u}\frac{\partial u}{\partial y} + z\cancel{e^u}\frac{\partial u}{\partial z} = 3\cancel{e^u}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 3$$

(Hence Proved)

Composite function: $y = f(x)$

(1) ordinary differentiation

(2) $z = f(x, y)$

Partial differentiation

(3) $z = f\left(\frac{x}{y}, x^2 - 2y\right)$

$z = f(r, s)$

[where $r = \frac{x}{y}$, $s = x^2 - 2y$]

$$\left\{ z \xrightarrow[\text{of}]{\text{is a function}} r, s \xrightarrow[\text{is a fuction of}]{} x, y \right\}$$

z is composite function of x & y.

Case-1: $u = f(x, y)$ & $x = \phi(t)$

$y = \psi(t)$

Sol: $u \to x, y \to t$ (one variable)

(two variables)

$$\begin{tikzcd} & x \arrow[dr, "\text{diff ordinary}"] & \\ u \arrow[ur, "\text{diff partial}"] \arrow[dr] & & t \\ & y \arrow[ur] & \end{tikzcd}$$

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}}$$

~~Case-2: $u = f(x, y)$, $x = \phi(r, s)$~~
~~$y = \psi(r, s)$~~

~~$u \longrightarrow x, y \longrightarrow r, s$~~

Case-2: $u = f(r, s)$ $\quad r = \phi(x, y)$
$s = \psi(x, y)$

$$u \longrightarrow r, s \longrightarrow x, y$$

(partial)

$$\begin{tikzcd} & r \arrow[dr] & \\ u \arrow[ur] \arrow[dr] & & x\,y \\ & s \arrow[ur] & \end{tikzcd}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial y}$$

$$\begin{tikzcd} & r \arrow[dr, "\text{partial}"] & \\ u \arrow[ur, "\text{partial}"] \arrow[dr] & & x, y \\ & s \arrow[ur] & \end{tikzcd}$$

(partial $\frac{\partial u}{\partial x}$)

for ex: If $u = x \log(xy)$ where $\underline{x^3 + y^3 + 3xy = 1}$ find $\frac{du}{dx}$

we have

$$\begin{tikzcd} & x \arrow[dr] & \\ u \arrow[ur] \arrow[dr] & & x \\ & y \arrow[ur] & \end{tikzcd}$$

$u = x \log x + x \log y$

diff n partially wrt x & y

$$\frac{\partial v}{\partial x} = 1 + \log x + \log y = \log e + \log(xy)$$

$$\frac{\partial v}{\partial x} = \log(exy)$$

$$\frac{\partial v}{\partial y} = 0 + \frac{x}{y}$$

Now $x^3 + y^3 + 3xy + 1 = 0$

diff w.r.to x:-

$$3x^2 + 3y^2\frac{dy}{dx} + 3x\frac{dy}{dx} + 3y = 0$$

$$\boxed{\frac{dy}{dx} = \frac{x^2+y}{y^2+x}}$$

$$\frac{du}{dx} = \frac{\partial v}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial v}{\partial y}\cdot\frac{dy}{dx} \quad \text{--- (1)}$$

put $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ & $\frac{dy}{dx}$

$$\frac{du}{dx} = \log exy + \frac{x}{y}\left[\frac{x^2+y}{y^2+x}\right]$$

$$\frac{du}{dx} = \log exy - \frac{x(x^2+y)}{y(y^2+x)}$$

(HW)-1:

Ques If $u = \sin^{-1}(x-y)$ $x = 3t$, $y = 4t^3$, prove that

$$\frac{du}{dt} = \frac{3}{\sqrt{1-t^2}}$$

HW-2:

Ques If $u = u\left(\frac{y-x}{xy}, \frac{z-x}{zx}\right)$

Show that, $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$

$u = u[r, s]$ where $r = \frac{y-x}{xy} = \frac{1}{x} - \frac{1}{y}$

$$s = \frac{z-x}{zx} = \frac{1}{x} - \frac{1}{z}$$

$$(u) \to (r,s) \to (x,y,z)$$

(partial)

① $\frac{\partial u}{\partial x}$

$$\begin{tikzcd} & r \arrow[dr] & \\ u \arrow[ur] \arrow[dr] & & (x,y,z) \\ & s \arrow[ur] & \end{tikzcd}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}$$

$$r = \frac{y-x}{xy} = \frac{1}{x} - \frac{1}{y} \qquad \frac{\partial r}{\partial x} = \frac{-1}{x^2}, \quad \frac{\partial r}{\partial y} = \frac{1}{y^2}$$

$$\frac{\partial r}{\partial z} = 0$$

$$s = \frac{z-x}{zx} = \frac{1}{x} - \frac{1}{z}$$

$$\frac{\partial s}{\partial x} = \frac{-1}{x^2}, \quad \frac{\partial s}{\partial y} = 0, \quad \frac{\partial s}{\partial z} = \frac{1}{z^2}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\left[\frac{-1}{x^2}\right] + \frac{\partial u}{\partial s}\left[\frac{-1}{x^2}\right]$$

$$\cancel{= x^2}.$$

$$x^2\frac{\partial u}{\partial x} = \frac{-\partial u}{\partial r} - \frac{\partial u}{\partial s}$$

② $\frac{\partial u}{\partial y} =$

$\frac{\partial u}{\partial z} =$

HW.

HW(Q-1): If $u = \sin^{-1}(x-y)$, $x = 3t$, $y = 4t^3$, prove that $\frac{du}{dt} = \frac{3}{\sqrt{1-t^2}}$

Sol:

(M-1)

$$u \xrightarrow{\text{partial}} x, y \xrightarrow{\text{ordinary}} t$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[\sin^{-1}(x-y)\right] = \frac{1}{\sqrt{1-(x-y)^2}}$$

variable
$y \to$ constant

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\sin^{-1}(x-y)\right] = \frac{1}{\sqrt{1-(x-y)^2}} \times -1$$

variable
$x \to$ constant

$$\frac{dx}{dt} = \frac{d}{dt}(3t) = 3$$

$$\frac{dy}{dt} = \frac{d}{dt}(4t^3) = 12t^2$$

$$\frac{du}{dt} = \frac{1}{\sqrt{1-(x-y)^2}} \times 3 + \frac{-1}{\sqrt{1-(x-y)^2}} \times 12t^2$$

$$\frac{du}{dt} = \frac{3}{\sqrt{1-(x-y)^2}} - \frac{12t^2}{\sqrt{1-(x-y)^2}}$$

$$\frac{du}{dt} = \frac{3-12t^2}{\sqrt{1-(3t-4t^3)^2}}$$

(M-2) by substitution,

$$u = \sin^{-1}(3t-4t^3)$$

$$t \to \sin\theta$$

$$u = \sin^{-1}(3\sin\theta - 4\sin^3\theta)$$

$$u = 3\theta$$

$$\frac{du}{dt} = \frac{3d\theta}{dt}$$

$$= 3 \times \frac{d}{dt}(\sin^{-1}t)$$

$$= \frac{3 \times 1}{\sqrt{1-t^2}}$$

(H.P)

(partial differentiability ↳ not in syllabus)

HW Q(2) If $u = u\left(\frac{y-x}{xy}, \frac{z-x}{zx}\right)$

Show that, $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0.$

Sol: $u = u[r, s]$

where $r = \frac{y-x}{xy}$, $s = \frac{z-x}{zx}$

$r = \frac{y}{xy} - \frac{x}{xy}$, $s = \frac{z}{zx} - \frac{x}{zx}$

$r = \frac{1}{x} - \frac{1}{y}$ $\quad$ $s = \frac{1}{x} - \frac{1}{z}$

$$(u) \longrightarrow (r, s) \longrightarrow (x, y, z)$$

(partial)

(1) $\frac{\partial u}{\partial x} =$

$$u \to r, s \to (x, y, z)$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}}$$

$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{x} - \frac{1}{y}\right) = \frac{-1}{x^2}$

x → variable

$y \to$ constant

$\frac{\partial s}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{x} - \frac{1}{z}\right) = \frac{-1}{x^2}$

$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\left(\frac{-1}{x^2}\right) + \frac{\partial u}{\partial s}\left(\frac{-1}{x^2}\right).$

$$x^2\frac{\partial u}{\partial x} = \frac{-\partial u}{\partial r} - \frac{\partial u}{\partial s} \quad \text{— (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \cancel{\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}}^{0}$$

$$= \frac{\partial u}{\partial r} \times \frac{1}{y^2}$$

$$y^2\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \quad \text{—— (2)}$$

$$\frac{\partial r}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{x} - \frac{1}{y}\right)$$

$$= -\left(\frac{-1}{y^2}\right) = \frac{1}{y^2}$$

$$\frac{\partial s}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{x} - \frac{1}{z}\right)$$

$$= 0$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z}$$

$$\frac{\partial r}{\partial z} = \frac{\partial}{\partial z}\left(\frac{1}{x} - \frac{1}{y}\right) = 0$$

z variable, (x, y → constant)

$$\frac{\partial s}{\partial z} = \frac{\partial}{\partial z}\left(\frac{1}{x} - \frac{1}{z}\right) = -\left(\frac{-1}{z^2}\right) = \frac{1}{z^2}$$

$$z^2\frac{\partial u}{\partial z} = z^2 \cancel{\times \frac{\partial u}{\partial r} \times 0}^{0} + \frac{\partial u}{\partial s} \times \frac{1}{\cancel{z^2}} \times \cancel{z^2}$$

$$= \frac{\partial u}{\partial s} \quad \text{—— (3)}$$

add eqn (1), (2) & (3)

LHS = RHS = 0 Ans

(H.P.)

(L-6)

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Que 1. Prove that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}$

where $x = u\cos\alpha - v\sin\alpha$

$y = u\sin\alpha + v\cos\alpha$.

Sol: z is composite function of u and v

$z \longrightarrow x, y \longrightarrow u, v$ $\quad z < \begin{matrix} x \\ y \end{matrix} > u, v$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\times\cos\alpha + \frac{\partial z}{\partial y}\times\sin\alpha$$

$\cancel{\frac{\partial x}{\partial u}} =$ $\boxed{\frac{\partial}{\partial u} = \cos\alpha\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\sin\alpha}$ (Operator)

Now, $\frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\left[\frac{\partial z}{\partial u}\right]$

$$\frac{\partial^2 z}{\partial u^2} = \left[\cos\alpha\frac{\partial}{\partial x} + \sin\alpha\frac{\partial}{\partial y}\right]\left[\cos\alpha\frac{\partial z}{\partial x} + \sin\alpha\frac{\partial z}{\partial y}\right]$$

$$= \cos^2\alpha\frac{\partial^2 z}{\partial x^2} + \cos\alpha\sin\alpha\frac{\partial^2 z}{\partial x\partial y} + \sin\alpha\cos\alpha\frac{\partial^2 z}{\partial x\partial y} + \sin^2\alpha\frac{\partial^2 z}{\partial y^2}$$

$$\frac{\partial^2 z}{\partial u^2} = \cos^2\alpha\frac{\partial^2 z}{\partial u^2} + \sin 2\alpha\frac{\partial^2 z}{\partial x\partial y} + \sin^2\alpha\frac{\partial^2 z}{\partial y^2} \quad \text{— (1)}$$

Simplify, (Similarly),

$$\frac{\partial^2 z}{\partial v^2} = \cos^2\alpha\frac{\partial^2 z}{\partial y^2} \mp \sin(2\alpha)\frac{\partial^2 z}{\partial x\partial y} + \sin^2\alpha\frac{\partial^2 z}{\partial x^2} \quad \text{— (2)}$$

(1) + (2)

$$\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = (\cos^2\alpha + \sin^2\alpha)\frac{\partial^2 z}{\partial x^2} + (\cos^2\alpha + \sin^2\alpha)\frac{\partial^2 z}{\partial y^2}$$

Que 2

Sol:

$$\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}=\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}$$

(LHS = RHS)

(Hence Proved)

Que ② If z is a function of x & y where $x=e^{u}+e^{-v}$, $y=e^{-u}-e^{v}$

Show that $\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}$.

Sol:

$$z \to x, y \to u, v$$

z is a composite function of u & v

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\times\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\times\frac{\partial y}{\partial u}$$

$$\frac{\partial x}{\partial u}=\frac{\partial}{\partial u}(e^{u}+e^{-v})=e^{u}+\cancel{e^{-v}}\to 0$$

$$\frac{\partial y}{\partial u}=\frac{\partial}{\partial u}(e^{-u}-\cancel{e^{v}})=-e^{-u}$$

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}e^{u}-\frac{\partial z}{\partial y}e^{-u}$$

$$\boxed{\frac{\partial}{\partial u}=e^{u}\frac{\partial}{\partial x}-\frac{\partial}{\partial y}e^{-u}}$$

$$\frac{\partial x}{\partial v}=\frac{\partial}{\partial v}(e^{u}+e^{-v})=-e^{-v}$$

$$\frac{\partial y}{\partial v}=\frac{\partial}{\partial v}(\cancel{e^{-u}}-e^{v})=-e^{v}$$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\times\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\times\frac{\partial y}{\partial v}$$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\times -e^{-v}+\frac{\partial z}{\partial y}\times -e^{v}$$

$$\boxed{\frac{\partial z}{\partial v}=-e^{-v}\frac{\partial z}{\partial x}-e^{v}\frac{\partial z}{\partial y}}$$

$$\frac{\partial z}{\partial x}(e^{u}+e^{-v})+\frac{\partial z}{\partial y}(e^{-v}-e^{-u})$$

$$x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}$$

$$=\frac{\partial z}{\partial x}e^{u}-\frac{\partial z}{\partial y}e^{-u}+e^{-v}\frac{\partial z}{\partial x}+e^{v}\frac{\partial z}{\partial y}=x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}$$

(HP)

Que (3) Show that function

$$f(x,y) = \begin{cases} x^2+4y, & (x,y) \neq (1,2) \\ 0, & (x,y) = (1,2) \end{cases}$$

is not continous at (1,2).

(4) What is degree of $u(x,y) = x^{1/3} y^{-4/3} \tan^{-1}\left(\frac{y}{x}\right)$

(5) If $z = \log(x^3+y^3-x^2y-y^2x)$ find $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$

(6) If $\cos u = \frac{x^5-2y^5-6z^5}{\sqrt{ax^3+by^3+cz^3}}$ evaluate

$xU_x + yU_y + zU_z$.

(7) If $u = f(y-z, z-x, x-y)$ Prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Sol: (3) $\lim_{\substack{x\to 1 \\ y\to 2}} (x^2+4y) = 1+4\times 2 = 1+8 = 9$

$f(x,y) = 0$ at $x=1$ & $y=2$

Value of function $\neq$ limit

$2 \neq 9$

hence $f(x,y)$ is not continous at (1,2).

(4) $u = t^{1/3} x^{1/3} \cdot t^{-4/3} y^{-4/3} \tan^{-1}\left(\frac{ty}{tx}\right)$

$x \to tx$
$y \to ty$

$= t^{\frac{1}{3}-\frac{4}{3}} = \frac{-3}{3} = -1$

$n = -1$

(5) $z = \log(x^3 + y^3 - x^2y - y^2x)$

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$$

$$\frac{\partial}{\partial x}\left[\log(x^3+y^3-x^2y-y^2x)\right]$$

$$\frac{\partial z}{\partial x} = \frac{1}{(x^3+y^3-x^2y-y^2x)} \times (3x^2-2xy-y^2) \quad \text{--- (1)}$$

$$\frac{\partial z}{\partial y} = \frac{1}{(x^3+y^3-x^2y-y^2x)} (3y^2 - x^2 - \cancel{y^2}\ 2xy) \quad \text{--- (2)}$$

$$= \frac{\cancel{3x^2 - 2xy - y^2 + 3y^2 - x^2 - y^2x} - 2xy}{x^3+y^3-x^2y-y^2x}$$

$$= \cancel{2x^2 - 2y^2 - 2xy + 3y^2}$$

$$= \cancel{\frac{2x^2 + y^2 - 2xy}{x^3+y^3-x^2y-y^2x}}$$

$$= \frac{\cancel{3x^2 - 4xy - y^2 + 3y^2 - x^2}}{x^3+y^3-x^2y-y^2x}$$

$$= \frac{2x^2 + 2y^2 - 4xy}{x^3+y^3-x^2y-y^2x} \cancel{= 2(x+y)} = \frac{2(x-y)^2}{x^3+y^3-x^2y-y^2x}$$

Ans

(6) $\cos u = \dfrac{x^5 - 2y^5 - 6z^5}{\sqrt{ax^3 + by^3 + cz^3}}$

$xu_x + yu_y + zu_z = ?$

$x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = ?$

$\dfrac{t^5}{t^{3/2}}$

$5 - \dfrac{3}{2} = \dfrac{10-3}{2} = \dfrac{7}{2}$

nu

$\dfrac{\partial u}{\partial x} =$ (y, z → constant)

by euler's

$= \dfrac{7}{2} \cancel{\tan} \dfrac{\cos u}{-\sin u}$

$= -\dfrac{7}{2} \cot u$ Ans

(7) $u = f(\overset{a}{y-z}, \overset{b}{z-x}, \overset{c}{x-y})$

$\dfrac{\partial x}{\partial a} = \dfrac{\partial}{\partial a}$

$\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$

$u \to x \to a$
$u \to y \to b$
$u \to z \to c$

$\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial x} \times \dfrac{\partial x}{\partial a} + \dfrac{\partial u}{\partial y} \times \dfrac{\partial y}{\partial b} + \dfrac{\partial u}{\partial z} \times \dfrac{\partial z}{\partial c}$

Differentiability of Real valued function:

Def: A function $f(x,y)$ is said to be differentiable at pt (a,b) if there exists two numbers A and B independent of h & k

Such that $f(a+h, b+k) - f(a,b) = Ah + Bk + \sqrt{h^2+k^2}$

$\phi(h,k) \rightarrow$ function
$\rightarrow$ value of phi at (h,k)

where $\lim\limits_{(h,k)\to(a,b)} \phi(h,k) = 0$

NOTE: ① $(Ah + Bk)$ is called derivative of f at (a,b)

② The distance b/w fixed point $(\cancel{A,B}\ a,b)$ and variable point $(\cancel{Ah, B}\ (a+h, b+k)$ is h^2+k^2

③ A function which is differentiable at a point possess partial derivative at that point but converse first order need not be true.

For ex: Show that function f is differentiable at origin $(0,0)$

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{(x^2+y^2)} , & (x,y) \neq (0,0) \\ 0 , & (x,y) = (0,0) \end{cases}$$

Sol: Let $f(0+h, 0+k) - \cancel{f(0,0)}^{0} = \dfrac{hk(h^2-k^2)}{(h^2+k^2)}$ (division method not applicable)

$$= 0h + 0k + \sqrt{h^2+k^2} \cdot \frac{hk(h^2-k^2)}{(h^2+k^2)^{3/2}}$$

Equating with $f(0+h, 0+k) - f(0,0)$

$= A\cdot h + B\cdot k + \sqrt{h^2+k^2}\,\phi(h,k)$

$\boxed{A=0}$ $\boxed{B=0}$

$$\phi(x,y) = \frac{hk[h^2-k^2]}{(h^2+k^2)^{3/2}}$$

$(h = mk)$

$$= \frac{mk^2[m^2k^2-k^2]}{(m^2k^2+k^2)^{3/2}}$$

$$= \frac{mk^2 \times k^2(m^2-1)}{(m^2+1)^{3/2} k^{2\times 3/2}}$$

$$\lim_{k\to 0} = \frac{mk^4(m^2-1)}{(m^2+1)^{3/2}\cancel{k^3}}$$

$$\lim_{k\to 0} = \frac{m(m^2-1)k}{(m^2+1)^{3/2}} = 0$$

(doesn't depends ~~DNE~~ upon m)

(limit exists & = 0)

$$\Rightarrow \lim_{(h,k)\to(0,0)} \phi(h,k) = 0$$

$\Rightarrow f(x,y)$ is differentiable at $(0,0)$

And derivative at $(0,0) = hA + kB$

$= 0 + 0 = 0$

Que.

Sol:

(i)

(ii)

h^2

Que: Discuss the differentiability of $f(x,y) = \dfrac{x^3-y^3}{x^2+y^2}$, $(x,y) \neq (0,0)$. $f(0,0)=0$. Also find $f_x(0,0)$ & $f_y(0,0)$

Sol:

$y = mx$

$\cancel{\lim_{x\to 0} \dfrac{x^3-m^3x^3}{x^2+m^2x^2} = \dfrac{x^3(1-m^3)}{x^2(1+m^2)} = \dfrac{x(1-m^3)}{(1+m^2)}}$

(I) $f_x(0,0) = \lim_{h\to 0} \dfrac{f(0+h,0) - f(0,0)}{h}$

$= \lim_{h\to 0} \dfrac{f(h,0)-0}{h} = \lim_{h\to 0} \dfrac{\frac{h^3-0}{h^2+0} - 0}{h} = 1$

(II) $f_y(0,0) = \lim_{k\to 0} \dfrac{f(0,0+k) - f(0,0)}{k}$

$= \lim_{k\to 0} \dfrac{f(0,k)-0}{h} = \lim_{k\to 0} \dfrac{1}{k} \times \dfrac{0-k^3}{0+k^2} = -1$

(use of first principle)

1st order partial derivative exists at origin
$f_x(0,0)$ & $f_y(0,0)$ exists.

Differentiability: $f(0+h, 0+k) - f(0,0)$

$= f(h,k) - \cancel{f(0,0)}^{0}$

$= \dfrac{h^3-k^3}{h^2+k^2}$

$(h^2+k^2)\,)\, h^3-k^3 \,(\, h$
$h^3 \pm hk^2$
$(-)$
$-k^3 - hk^2$
$-k^3 - h^2k$
$+ \quad +$
$h^2k - hk^2$

$h^2+k^2 \,)\, h^3-k^3 \,(\, h$
$h^3 + hk^2$
$(-)\ (-)$

HW Ques

$$\frac{h^3-k^3}{h^2+k^2} = h-k + \frac{h^2k-hk^2}{h^2+k^2}$$

$$= 1h + (-1)k + \sqrt{h^2+k^2}\,\frac{h^2k-k^2h}{(h^2+k^2)^{3/2}}$$

$A=1$
$B=-1$ $\quad \phi(x,y) = \frac{h^2k-k^2h}{(h^2+k^2)^{3/2}}$

after equating with $Ah+Bk+\sqrt{h^2+k^2}\,\phi(h,k)$

$$\lim_{(x,y)\to(0,0)} \phi(x,y) = \lim_{\substack{x\to 0\\ y\to 0}} \phi(x,y)$$

Sol

$y=mx$

$k=mh$

$$= \lim_{x\to 0} \frac{h^2\times mh - m^2h^2h}{(h^2+m^2h^2)^{3/2}}$$

$$= \lim_{x\to 0} \frac{h^3m-m^2h^3}{(h^2+m^2h^2)^{3/2}} \rightarrow \frac{h^3m(1-m)}{h^{2\times\frac{3}{2}}(1+m^2)}$$

put $h=mk$

$$= \lim_{k\to 0} (mk)^2k - k^2(mk)$$

(value depends upon m)

$f(x,y)$ is not differentiable at $(0,0)$ as its limit depends upon m

(HW)

Ques $f(x,y) = \begin{cases} \dfrac{x^2y}{x^4+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Show that ~~f(x~~ f_x, f_y exits everywhere in $(-1 \le x \le 1)$ and $(-1 \le y \le 1)$ although discontinous in x &y plane. and also discuss differentiability.

(Check differentiability)

Sol:

$$f_x(-1,1) = \lim_{h\to 0} \frac{f(a+h, b+k) - f(a,b)}{h}$$

$a=-1$, $b=1$

$$= \lim_{h\to 0} \frac{f(-1+h, 1+k) - f(-1,1)}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{(h-1)^2(k+1)}{(h-1)^4+(k+1)^2} - \left(\dfrac{1\times 1}{1+1}\right)}{h}$$

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HW

Q-29 If $z = u^2 + v^2$ and $u = at^2$, $v = 2at$, then find $\frac{dz}{dt}$.

Sol:

$$\begin{array}{ccc} & \nearrow u \searrow & \\ z & & a, t \\ & \searrow v \nearrow & \end{array}$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial t} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial t} \quad \text{—(v)}$$

$$\frac{\partial z}{\partial u} = \frac{\partial}{\partial u}(u^2+v^2) = 2u \quad \text{—(i)}$$

variable, v = constant

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(at^2) = 2at \quad \text{—(ii)}$$

$$\frac{\partial z}{\partial v} = \frac{\partial}{\partial v}(u^2+v^2) = 2v \quad \text{—(iii)}$$

variable
u → constant

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial t}(2at) = 2a \quad \text{—(iv)}$$

Now putting the values of eqn (i), (ii), (iii) & (iv) in (v)

$$\frac{dz}{dt} = 2u \times 2at + 2v \times 2a$$
$$= 4uat + 4av$$
$$= 4(at^2)at + 4a(2at)$$
$$= 4a^2t^3 + 8a^2t$$
$$= 4a^2t(t^2+2) \quad \text{Ans}$$

Q-30 If $u(x,y) = (\sqrt{x}+\sqrt{y})^5$, find the value of
$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}.$$

Sol: $u = (\sqrt{x}+\sqrt{y})^5$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[(\sqrt{x}+\sqrt{y})^5\right] = 5(\sqrt{x}+\sqrt{y})^4 \times \frac{1}{2}(x)^{\frac{1}{2}-1}$$

(y → constant) $= \frac{5}{2}(x)^{-1/2}(\sqrt{x}+\sqrt{y})^4$

$$= \frac{5}{2\sqrt{x}}(\sqrt{x}+\sqrt{y})^4 \quad \text{—(i)}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\right] = \frac{\partial}{\partial x}\left[\frac{5}{2\sqrt{x}}(\sqrt{x}+\sqrt{y})^4\right] \quad (y \to \text{constant})$$

$$= \frac{5}{2}\frac{\partial}{\partial x}\left[\frac{1}{\sqrt{x}}(\sqrt{x}+\sqrt{y})^4\right]$$

$x^{-1/2}$

$$= \frac{5}{2}\left[\frac{-1}{2}(x)^{-\frac{1}{2}-1}(\sqrt{x}+\sqrt{y})^4 + \cancel{4}^{\,2}(\sqrt{x}+\sqrt{y})^3 \times \frac{1}{\cancel{2}}(x)^{\frac{1}{2}-1} \times \frac{1}{\sqrt{x}}\right]$$

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$$= \frac{5}{2}\left[-\frac{1}{2}(x)^{-3/2}(\sqrt{x}+\sqrt{y})^4 + 2(\sqrt{x}+\sqrt{y})^3 (x)^{-1/2} \times \frac{1}{\sqrt{x}} \right] \quad \frac{1}{\sqrt{x}}$$

$$= \frac{5}{2}\left[-\frac{1}{2}(x)^{-3/2}(\sqrt{x}+\sqrt{y})^4 + \frac{2(\sqrt{x}+\sqrt{y})^3}{x} \right] \} \times x^2$$

$$2 - \frac{3}{2} = \frac{1}{2}$$

$$= \frac{5}{2} x^2 \times -\frac{1}{2} x^{-3/2} (\sqrt{x}+\sqrt{y})^4 + \frac{2 \times 5}{2} \frac{(\sqrt{x}+\sqrt{y})^3}{x} \times x^2$$

$$= -\frac{5}{4} x^{1/2} (\sqrt{x}+\sqrt{y})^4 + 5x(\sqrt{x}+\sqrt{y})^3$$

$$= 5x^{1/2}(\sqrt{x}+\sqrt{y})^3 \left[-\frac{1}{4}(\sqrt{x}+\sqrt{y}) + x^{1/2} \right] \quad (1)$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left[(\sqrt{x}+\sqrt{y})^5 \right] = 5(\sqrt{x}+\sqrt{y})^4 \left[\frac{1}{2}(y)^{\frac{1}{2}-1} \right]$$

$(x \to \text{constant})$ $$= \frac{5}{2}(\sqrt{x}+\sqrt{y})^4 (y)^{-1/2}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{5}{2}(y)^{-1/2}(\sqrt{x}+\sqrt{y})^4 \right] \quad (x \to \text{constant})$$

$$= \frac{5}{2}\frac{\partial}{\partial y}\left[(y)^{-1/2}(\sqrt{x}+\sqrt{y})^4 \right] \qquad -\frac{1}{2} - \frac{1}{2}$$

$$-\frac{1-2}{2} \quad = \frac{5}{2}\left[-\frac{1}{2}(y)^{-\frac{1}{2}-1}(\sqrt{x}+\sqrt{y})^4 + \cancel{4}^{2}(\sqrt{x}+\sqrt{y})^3 \times \frac{1}{\cancel{2}}(y)^{\frac{1}{2}-1} \times y^{-1/2} \right]$$

$$= \frac{5}{2}\left[-\frac{1}{2}(y)^{-3/2}(\sqrt{x}+\sqrt{y})^4 + 2(\sqrt{x}+\sqrt{y})^3 (y)^{-1} \right]$$

$$= \frac{5}{2} \times -\frac{1}{2}(y)^{-\frac{3}{2}+2}(\sqrt{x}+\sqrt{y})^4 + \frac{5}{2} \times 2(\sqrt{x}+\sqrt{y})^3 (y)^{-1+2}$$

$$= -\frac{5}{4}(y)^{1/2}(\sqrt{x}+\sqrt{y})^4 + 5(\sqrt{x}+\sqrt{y})^3 (y)^1 \quad (2)$$

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial v}{\partial y} \right]$$

$$= \frac{\partial}{\partial x}\left[\frac{5}{2}(\sqrt{x}+\sqrt{y})^4 (y)^{-1/2} \right] \quad (y \to \text{const})$$

$$= \frac{5}{\cancel{2}}(y)^{-1/2} \times \cancel{4}(\sqrt{x}+\sqrt{y})^3 \times \frac{1}{\cancel{2}}(x)^{\frac{1}{2}-1}$$

$$= 5(y)^{-1/2}(\sqrt{x}+\sqrt{y})^3 (x)^{-1/2}$$

$$= \cancel{5\sqrt{x}\sqrt{y}} = 5 \times \frac{1}{\sqrt{x}\sqrt{y}}(\sqrt{x}+\sqrt{y})^3 \times 2\sqrt{xy}$$

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$$= \underline{5(\sqrt{x}+\sqrt{y})^3 \times 2\sqrt{x}\sqrt{y}} \quad \text{—— (3)}$$

Now add eqn (1), (2) & (3)

$$= 10\sqrt{x}\sqrt{y}(\sqrt{x}+\sqrt{y})^3 + 5y(\sqrt{x}+\sqrt{y})^3 - \frac{5}{4}(y)^{1/2}(\sqrt{x}+\sqrt{y})^4$$
$$+ 5\sqrt{x}(\sqrt{x}+\sqrt{y})^3\left[-\frac{(\sqrt{x}+\sqrt{y})}{4} + \sqrt{x}\right]$$

$$= 10\sqrt{x}\sqrt{y}(\sqrt{x}+\sqrt{y})^3 + 5y(\sqrt{x}+\sqrt{y})^3 - \frac{5}{4}\sqrt{y}(\sqrt{x}+\sqrt{y})^4$$
$$+ 5\sqrt{x}(\sqrt{x}+\sqrt{y})^3 \times -\frac{(\sqrt{x}+\sqrt{y})}{4}$$
$$+ 5\sqrt{x}(\sqrt{x}+\sqrt{y})^3 \times \sqrt{x}$$

$$= 10\sqrt{x}\sqrt{y}(\sqrt{x}+\sqrt{y})^3 + 5y(\sqrt{x}+\sqrt{y})^3 - \frac{5}{4}\sqrt{y}(\sqrt{x}+\sqrt{y})^4$$
$$- \frac{5}{4}\sqrt{x}(\sqrt{x}+\sqrt{y})^4 + 5x(\sqrt{x}+\sqrt{y})^3$$

$$= (\sqrt{x}+\sqrt{y})^3\left[10\sqrt{x}\sqrt{y} + 5y - \frac{5}{4}\sqrt{y}(\sqrt{x}+\sqrt{y}) - \frac{5}{4}\sqrt{x}(\sqrt{x}+\sqrt{y}) + 5x\right]$$

$$= -\frac{5}{4}(\sqrt{x}+\sqrt{y})(\sqrt{y}-\sqrt{x})$$
$$+ \frac{5}{4}(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})$$
$$x-y$$

$$= (\sqrt{x}+\sqrt{y})^3\left[10\sqrt{x}\sqrt{y} + 5y + 5x + \frac{5}{4}(x-y)\right]$$

$$= (\sqrt{x}+\sqrt{y})^3\left[10\sqrt{x}\sqrt{y} + 5y + 5x + \frac{5}{4}x - \frac{5}{4}y\right]$$

$$5 - \frac{5}{4} = \frac{20-5}{4}$$

$$= (\sqrt{x}+\sqrt{y})^3\left[10\sqrt{x}\sqrt{y} + \frac{15}{4}y + \frac{25}{4}x\right]$$

$$\frac{15}{4}$$
$$\frac{5+20}{4}$$
$$\frac{25}{4}$$

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Q-(31) If $u = x^2yz - 4y^2z^2 + 2xz^3$, then find

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}.$$

Sol: $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[x^2yz - 4y^2z^2 + 2xz^3\right]$

$(y, z \to \text{constant})$

$$\frac{\partial u}{\partial x} = 2xyz + 0 + 2z^3$$

$$x\frac{\partial u}{\partial x} = 2x^2yz + 2xz^3 \quad \text{—— (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(x^2yz - 4y^2z^2 + 2xz^3\right) \quad (x, z \to \text{const})$$

$$= x^2z - 4z^2 \times 2y + 0$$

$$y\frac{\partial u}{\partial y} = y\left(x^2z - 8z^2y\right)$$

$$= x^2yz - 8z^2y^2 \quad \text{—— (2)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}\left(x^2yz - 4y^2z^2 + 2xz^3\right)$$

$(x, y \to \text{const})$

$$= x^2y - 4y^2 \times 2z + 2x \times 3z^2$$

$$z\frac{\partial u}{\partial z} = zx^2y - 8y^2z^2 + 6xz^3 \quad \text{—— (3)}$$

add eqn (1), (2) (3)

$$= 2x^2yz + 2xz^3 + x^2yz - 8z^2y^2 + zx^2y - 8y^2z^2 + 6xz^3$$

$$= 8xz^3 + 3x^2yz - 16z^2y^2 + x^2yz$$

$$= 8xz^3 + 4x^2yz - 16z^2y^2$$

$$= 4\left(2xz^3 + x^2yz - 4z^2y^2\right)$$

$$= 4u \text{ Ans}$$

Q-(32)

Sol:

Q-(33)

Sol

Q-32 → If $u = \log\left(\frac{x^2}{y}\right)$, then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

Sol: $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[\log\left(\frac{x^2}{y}\right)\right]$ ($y \to$ const)

$= \frac{1}{\left(\frac{x^2}{y}\right)} \times \frac{1}{y} \times 2x = \frac{y}{x^2} \times \frac{2x}{y} = \frac{2}{x}$

$x\frac{\partial u}{\partial x} = \cancel{x} \times \frac{2}{\cancel{x}} = 2$ — (1)

$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\log\left(\frac{x^2}{y}\right)\right) = \frac{1}{\left(\frac{x^2}{y}\right)} \times x^2 \times \frac{-1}{y^2}$

($x \to$ const) $= \frac{y}{\cancel{x^2}} \times \cancel{x^2} \times \frac{-1}{y^2} = \frac{-1}{y}$

$\cancel{y} \times \frac{-1}{\cancel{y}} = -1$ $\quad y\frac{\partial u}{\partial y} = -1$ — (2)

adding eqn (1) + (II)

$= 2 - 1 = 1$ Ans

Q-33 → If $u = x^2 + 2xy + y^2 + x + y$, then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

Sol: $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^2 + 2xy + y^2 + x + y)$ ($y \to$ const)

$= 2x + 2y + 0 + 1 + 0$

$x\frac{\partial u}{\partial x} = x(2x + 2y + 1) = 2x^2 + 2xy + x$ — (1)

$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^2 + 2xy + y^2 + x + y)$ ($x \to$ const)

$= 0 + 2x + 2y + 0 + 1 = (2x + 2y + 1)$

$y\frac{\partial u}{\partial y} = 2xy + 2y^2 + y$ — (2)

add eqn (1) & (2)

$= 2x^2 + x + 2xy + 2xy + 2y^2 + y$

$= 2x^2 + 2y^2 + x + y + 4xy$ Ans

Que (34) → What is the degree of homogeneous function

$$u(x,y) = \frac{x^2(x^2-y^2)^{1/3}}{(x^2+y^2)^{2/3}} ?$$

Sol: $x \to tx$
$y \to ty$

$$2+\frac{2}{3}-\frac{4}{3} \qquad \frac{2-\frac{2}{3}}{} \qquad \frac{6-2}{3}$$

$$= \frac{t^2x^2\left[t^2x^2-t^2y^2\right]^{1/3}}{\left[t^2x^2+t^2y^2\right]^{2/3}}$$

$$= \frac{t^2x^2\, t^{2/3}\left[x^2-y^2\right]^{1/3}}{t^{2\times\frac{2}{3}}\left[x^2+y^2\right]^{2/3}}$$

$$= t^{2+2/3-4/3} = t^{4/3} \qquad n = 4/3 \text{ Ans}$$

Que (35) → What is the degree of the homogeneous function

$$u(x,y) = \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}} ?$$

Sol: $x \to tx$
$y \to ty$

$$= \frac{\sqrt{tx}-\sqrt{ty}}{\sqrt{tx}+\sqrt{ty}} = \frac{\cancel{t^{1/2}}(\sqrt{x}-\sqrt{y})}{\cancel{t^{1/2}}(\sqrt{x}+\sqrt{y})}$$

$$= t^0 \qquad n = 0 \text{ Ans}$$

Que (36) → What is the degree of the homogeneous function $u(x,y) = \cancel{x}\, x^{1/3} y^{-4/3} \tan^{-1}\left(\frac{y}{x}\right)$?

Sol: $x \to tx,\ y \to ty$

$$= t^{1/3}x^{1/3}\, t^{-4/3} y^{-4/3} \tan^{-1}\left(\frac{ty}{tx}\right)$$

$$= (t)^{\frac{1}{3}-\frac{4}{3}} = -1$$

$$n = -1 \text{ Ans}$$

Que (37) →

Sol:

Que (37) → If $z = \log(x^3 + y^3 - x^2y - xy^2)$, then find $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$.

Sol: $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left[\log(x^3+y^3-x^2y-xy^2)\right]$ ($y, z \to$ const)

$$= \frac{1}{(x^3+y^3-x^2y-xy^2)} \times 3x^2 - 2xy - y^2$$

$$\frac{\partial z}{\partial x} = \frac{3x^2 - y^2 - 2xy}{x^3+y^3-x^2y-xy^2} \quad \text{—— (1)}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\log(x^3+y^3-x^2y-xy^2)\right) \quad (x \to \text{const})$$

$$= \frac{1}{(x^3+y^3-x^2y-xy^2)} \times 3y^2 - x^2 - 2xy$$

$$\frac{\partial z}{\partial y} = \frac{3y^2 - x^2 - 2xy}{(x^3+y^3-x^2y-xy^2)} \longrightarrow (2)$$

add eqn (1) & (2)

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{\cancel{3}x^2 - y^2 - 2xy + \cancel{3}y^2 - x^2 - 2xy}{(x^3+y^3-x^2y-xy^2)}$$

$$= \frac{2x^2 + 2y^2 - 4xy}{x^3+y^3-x^2y-xy^2} = \frac{(x-y)^2 + (x-y)^2}{x^2(x-y) + y^2(y-x)}$$

$$= \frac{(x-y)^2 + (x-y)^2}{(x^2-y^2)(x-y)}$$

$$= \frac{2(x-y)^{\cancel{2}}}{(x^2-y^2)\cancel{(x-y)}} = \frac{2(x-y)}{(x+y)(x-y)}$$

$$= \frac{2}{(x+y)} \quad \text{Ans}$$

Que (38) → If $w = (y-z)(z-x)(x-y)$, then find

$$\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z}.$$

Sol: $\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\left[(y-z)(z-x)(x-y)\right]$ $\boxed{(y, z \to \text{const})}$

$w = (y-z)(z-x)(x-y)$

$w = \cancel{(yz - yx)}$

$w = (yz - yx - z^2 + zx)(x-y)$

$w = \cancel{xyz} - x^2y - xz^2 + x^2z - y^2z + y^2x + yz^2 - \cancel{xyz}$

$w = -x^2y - xz^2 + x^2z - y^2z + y^2x + yz^2$

$\frac{\partial w}{\partial x} = -y(2x) - z^2 + 2xz - 0 + y^2 + 0$

$\frac{\partial w}{\partial x} = -2xy - z^2 + y^2 + 2xz$ —— (1)

$\frac{\partial w}{\partial y} = -x^2 + 0 + 0 - z \times 2y + x \times 2y + z^2$

$= -x^2 - 2zy + 2xy + z^2$ —— (2)

$\frac{\partial w}{\partial z} = \frac{\partial w}{\partial z} = 0 - x \times 2z + x^2 - y^2 + 0 + y \times 2z$

$= -2xz + x^2 - y^2 + 2yz$ —— (3)

add eqn (1) + (2) + (3)

$= \cancel{-2xy} \cancel{-z^2} \cancel{+y^2} \cancel{+2xz} \cancel{-x^2} \cancel{-2zy} \cancel{+2xy} \cancel{+z^2}$
$\quad \cancel{-2xz} \cancel{+x^2} \cancel{-y^2} \cancel{+2yz}$

$= 0$ Ans

Que (39) =

Sol:

Que (40)

Sol:

(y, h, z → const

Que 39 → If $x^y = y^x$ then find the value of $\frac{dy}{dx}$ at $(2,4)$.

Sol: $x^y = y^x$ taking log both side

$\boxed{y \ln x = x \ln y}$ diff both side wrt x.

$\frac{d}{dx}(y \ln x) = \frac{d}{dx}(x \ln y)$ ($y \to$ const)

$\cancel{y \times \frac{1}{x} = \ln y}$

$\cancel{\frac{y}{x} = \ln y}$

$y = x \ln y$

$\cancel{\frac{dy}{dx} = \ln y \, x = \ln(4) \cdot 2}$

$= 4 \ln 2$ Ans

$\frac{dy}{dx} \ln x + \frac{1}{x} y = \ln y + \frac{1}{y} x \times \frac{dy}{dx}$

$\frac{dy}{dx}\left(\ln x - \frac{x}{y}\right) = \left(\ln y - \frac{y}{x}\right)$

$$\frac{dy}{dx} = \frac{\ln y - \frac{y}{x}}{\ln x - \frac{x}{y}} = \frac{\ln 4 - \frac{4}{2}}{\ln 2 - \frac{2}{4}}$$

$$= \frac{\ln 4 - 2}{\ln 2 - \frac{1}{2}} = \frac{2\ln 2 - 2}{\ln 2 - 2^{-1}}$$

$$= 4\left(\frac{\log 2 - 1}{2\log 2 - 1}\right) \text{ Ans}$$

Que 40 → If $v(x,y,z) = \cos(3x)\cos(4y)\sin(h)\,5z$ then find $V_{xx} + V_{yy} + V_{zz}$.

Sol: $V_{xx} = \frac{\partial^2 v}{\partial x^2}$, $V_{yy} = \frac{\partial^2 v}{\partial y^2}$, $V_{zz} = \frac{\partial^2 v}{\partial z^2}$

$v = \cos(3x)\cos(4y)\sin(h)\,5z$

($y, h, z \to$ const) $\frac{\partial v}{\partial x} = \cos(4y)\sin(h)\,5z \cdot -\sin(3x) \times 3$

$\frac{\partial v}{\partial x} = -3 \times 5z \cos(4y)\sin(h)\sin(3x)$

$\frac{\partial v}{\partial x} = -15z\cos(4y)\sin(h)\sin(3x)$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial v}{\partial x}\right] = -15z\cos(4y)\sin(h) \times \cos(3x) \times 3$$

$$= -45z\cos(4y)\sin(h)\cos(3x)$$

— ①

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial v}{\partial y}\right] = \frac{\partial}{\partial y}\left[5z\cos(3x)\sin(h) - \sin(4y) \times 4\right]$$

$$= \frac{\partial}{\partial y}\left[-20z\cos(3x)\sin(h)\sin(4y)\right] \quad (ii)$$

$$= -20z\cos(3x)\sin(h) \times \cos(4y) \times 4$$

$$= -80z\cos(3x)\sin(h)\cos(4y) \quad (iii)$$

$$\frac{\partial^2 v}{\partial z^2} = \frac{\partial}{\partial z}\left[\frac{\partial v}{\partial z}\right] = \frac{\partial}{\partial z}\left[\cos(3x)\cos(4y)5\sin(h)\right] \qquad \begin{array}{r} -45 \\ -80 \\ \hline -125 \end{array}$$

$$= 0$$

$$= \frac{-125z\cos(3x)\sin(h)\cos(4y)}{5z}$$

$$= -25 \text{ Ans}$$

Ques (4) If $u = x^3y^2\sin^{-1}\left(\frac{y}{x}\right)$, then find (AKTU 2019)

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

Sol: $u = x^3y^2\sin^{-1}\left(\frac{y}{x}\right)$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[x^3y^2\sin^{-1}\left(\frac{y}{x}\right)\right] \quad (y \to \text{const})$$

$$= y^2\frac{\partial}{\partial x}\left[x^3\sin^{-1}\left(\frac{y}{x}\right)\right]$$

$$= y^2\left[3x^2\sin^{-1}\left(\frac{y}{x}\right) + \frac{1}{\sqrt{1-\frac{y^2}{x^2}}} \times y \times \frac{-1}{x^2} \times x^3\right]$$

$$= y^2\left[3x^2\sin^{-1}\left(\frac{y}{x}\right) + \frac{1}{\sqrt{\frac{x^2-y^2}{x^2}}} \times -xy\right]$$

$$(1) \quad xy^2\left[3x^2\sin^{-1}\left(\frac{y}{x}\right) - \frac{x^2y}{\sqrt{x^2-y^2}}\right] \quad (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[x^3y^2\sin^{-1}\left(\frac{y}{x}\right)\right] \quad (x \to \text{const})$$

$$= x^3\frac{\partial}{\partial y}\left[y^2\sin^{-1}\left(\frac{y}{x}\right)\right]$$

(ii)

Sol:

$y\frac{\partial u}{\partial y}$

$$y\frac{\partial v}{\partial y} = yx^3\left[2y\sin^{-1}\left(\frac{y}{x}\right) + \frac{1}{\sqrt{1-\frac{y^2}{x^2}}} \times \frac{1}{x} \times y^2\right] + \frac{1}{\frac{\sqrt{x^2-y^2}}{x}}$$

$$= yx^3\left[2y\sin^{-1}\left(\frac{y}{x}\right) + \frac{\cancel{x}}{\sqrt{x^2-y^2}} \times \frac{y^2}{\cancel{x}}\right]$$

$$= x^3y\left[2y\sin^{-1}\left(\frac{y}{x}\right) + \frac{y^2}{\sqrt{x^2-y^2}}\right] \text{ --- (ii)}$$

add eqn (i) + (ii)

$$= 2x^3y^2\sin^{-1}\left(\frac{y}{x}\right) + \cancel{\frac{x^3y^3}{\sqrt{x^2-y^2}}} + 3x^3y^2\sin^{-1}\left(\frac{y}{x}\right) \cancel{-\frac{x^3y^3}{\sqrt{x^2-y^2}}}$$

$$= 5x^3y^2\sin^{-1}\left(\frac{y}{x}\right) = 5u$$ Ans

(ii) $x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x \partial y} + y^2\frac{\partial^2 v}{\partial y^2}$

Sol: $\frac{\partial v}{\partial x} = y^2\left[3x^2\sin^{-1}\left(\frac{y}{x}\right) - \frac{x^2y}{\sqrt{x^2-y^2}}\right]$

$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial v}{\partial x}\right]$ $y \to$ w.r.t

$$= \frac{\partial}{\partial x}\left[3x^2y^2\sin^{-1}\left(\frac{y}{x}\right)\right] - \frac{\partial}{\partial x}\left[\frac{x^2y}{\sqrt{x^2-y^2}}\right]$$

$$= 3y^2\frac{\partial}{\partial x}\left[x^2\sin^{-1}\left(\frac{y}{x}\right)\right] - y\frac{\partial}{\partial x}\left[\frac{x^2}{\sqrt{x^2-y^2}}\right]$$

$$= 3y^2\left[2x\sin^{-1}\left(\frac{y}{x}\right) + \frac{1}{\sqrt{1-\frac{y^2}{x^2}}} \times y \times \frac{-1}{x^2} \times x^2\right]$$

$$-y\left[\frac{2x\sqrt{x^2-y^2} - \frac{1}{2} \times \frac{1}{\sqrt{x^2-y^2}} \times 2x \times x^2}{(x^2-y^2)}\right]$$

Que (42) If $\log u = \frac{x^2y^2}{x+y}$, then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

Sol: Check for homogeneous

$$\log u = \frac{t^2x^2 t^2y^2}{tx+ty} = \frac{t^4(x^2y^2)}{t(x+y)} \rightarrow \log u$$

$n=3$ (by Eulers)

$$= \frac{3\log u}{\frac{1}{u}} = 3u\log u \text{ Ans}$$

Que (43) If $(\sqrt{x}+\sqrt{y})\cot u = x+y$, then find $4x\frac{\partial u}{\partial x} + 4y\frac{\partial u}{\partial y}$

Sol: $\cot u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$

Check for homogeneous,

$$\cot u = \frac{tx+ty}{t^{1/2}(\sqrt{x}+\sqrt{y})} = t^{1-1/2} = \frac{1}{2} \quad n=\frac{1}{2}$$

(by Euler's) $= \frac{1}{2} \times \frac{\cot u}{-\operatorname{cosec}^2 u} = \frac{1}{2} \times \frac{\frac{\cos u}{\sin u}}{-\frac{1}{\sin^2 u}}$

$$= -\frac{1}{2}\,\frac{\cos u \sin^2 u}{\sin u} = -\frac{1}{2}\,\frac{\cos u \sin u \times 2}{\times 2}$$

$$= -\frac{\sin 2u}{4} \times 4 = -\sin(2u) \text{ Ans}$$

Que (44): If $\cos u = \frac{x^5 - 2y^5 + 6z^5}{\sqrt{ax^3+by^3+cz^3}}$, then find

$xu_x + yu_y + zu_z$

Sol: Check for homogeneous,

$$\cos u = \frac{t^5(x^5 - 2y^5 + 6z^5)}{t^{3/2}\sqrt{ax^3+by^3+cz^3}}$$

$$n = \frac{5}{1} - \frac{3}{2} = \frac{7}{2} \quad n=\frac{7}{2}$$

by Euler's as $\cos u$ is homogeneous of degree $n = 7/2$

$$= \frac{7}{2} \times \frac{\cos u}{-\sin u}$$

$$= -\frac{7}{2} \cot u \quad \text{Ans}$$

Que (45) If $u = \sin^{-1}(x^2+y^2)^{1/5}$, then find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

Sol: $\sin u = (x^2+y^2)^{1/5}$ = as $\sin u$ is homogeneous of degree $n = 2/5$

$x \to tx \quad y \to ty$

$n = 2/5$

$$= \frac{2}{5}\frac{\sin u}{\cos u} = \frac{2}{5}\tan u \quad \text{Ans}$$

Que (46) (i) If $u = x^2 + 2xy - y\log z$, where $x = s+t^2$, $y = s-t^2$ and $z = 2t$, then find the $\frac{\partial u}{\partial t}$ at $(1, 2)$

Sol:

$$\begin{tikzcd} & x \arrow[r] & s, t \\ u \arrow[ur] \arrow[dr] & & \\ & y \arrow[uur] & \end{tikzcd}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \times \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \times \frac{\partial y}{\partial t}$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^2 + 2xy - y\log z) \quad (y, z \to \text{const})$$

$$= 2x + 2y + 0$$

$$\frac{\partial u}{\partial x} = 2(x+y) \quad \text{— (1)}$$

$$\frac{\partial x}{\partial t} = \frac{\partial}{\partial t}(s+t^2) = 2t \quad \text{— (2)}$$

$(s \to \text{const})$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^2 + 2xy - y\log z) \quad (x, z \to \text{const})$$

$$= 0 + 2x - \log z$$

$$= 2x - \log z \quad \text{— (3)}$$

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial t}(s - t^2) = -2t \quad \text{— (4)}$$

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$$\frac{\partial v}{\partial t} = (2x+2y)(2t) + (2x-\log z)(-2t)$$

$\log_b A = c$

$\log_e a = c$

$a = b^c$

$1 = e^0$

$=$ at $x=1$ $y=2$ $z=1$

$$= (2\times1+2\times2)(2t)+(2-\log 1)(-2t)$$

$$= (2+4)(2t)+(2-0)(-2t)$$

$$= 6\times2t + 2\times -2t$$

$$= 12t - 4t = 8t$$

$$= 8 \text{ Ans}$$

$z = 2t$

$1 = 2t$

(ii) if $u = x^2 - y^2 + \sin(yz)$, where $y = e^x$ & $z = \log x$, then find $\frac{\partial u}{\partial x}$.

$$\frac{\partial y}{\partial x} = \frac{\partial}{\partial x} e^x = e^x$$

$$\frac{\partial z}{\partial x} = \frac{1}{x}$$

Sol:

$u \to x$, $u \to y$, $u \to z$

$$\frac{\partial}{\partial x}\left(x^2 - y^2 + \sin(yz)\right)$$

$x \to$ variable

$y \to$ variable

$z \to$ variable

$$= 2x - 2y\frac{\partial y}{\partial x} + \cos(yz)\left[\frac{\partial y}{\partial x} z + \frac{\partial z}{\partial x} xy\right]$$

$$= 2x - 2ye^x + \cos(yz)\left[e^x \log x + \frac{y}{x}\right]$$

$$= 2x - 2ye^x + \cos(e^x \log x)\left[e^x \log x + \frac{y}{x}\right]$$

$$= \cancel{2(x - ye^x) + e^x \cos}$$

$$= 2(x - e^{2x}) + e^x \cos(e^x \log x)\left(\log x + \frac{1}{x}\right)$$

Ans

Q-(49) If $u = e^{x^2+y^2+z^2}$, then find $\frac{\partial^3 u}{\partial x \partial y \partial z}$

Sol: $u = e^{x^2+y^2+z^2}$

$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}\left(e^{x^2+y^2+z^2}\right)$ $(x, y \to \text{const})$

$\frac{\partial u}{\partial z} = e^{x^2+y^2+z^2} \cdot 2z$

$\frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial z}\right]\right]$

$= \frac{\partial}{\partial y}\left[2z e^{x^2+y^2+z^2}\right] = 2z e^{x^2+y^2+z^2} \times 2y$

$(x, z \to \text{const})$ $= 4yz e^{x^2+y^2+z^2}$

$= \frac{\partial}{\partial x}\left[4yz e^{x^2+y^2+z^2}\right]$ $y, z \to \text{const}$

$= 4yz\, e^{x^2+y^2+z^2} \times 2x$

$= 8xyz\, e^{x^2+y^2+z^2}$

$= 8xyzu$ Ans

Que (50) (i) Evaluate $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

Sol: we have to check limit at $x=0$ & $y=0$
at origin
put $y = mx$

$= \lim_{x\to 0} \frac{x(mx)}{x^2+m^2x^2} = \frac{m\cancel{x^2}}{\cancel{x^2}(1+m^2)} = \frac{m}{(1+m^2)}$

limit is depending upon m hence it doesn't exist

(ii) Show that the function

$$f(x,y) = \begin{cases} x^2+4y, & \text{when } (x,y) \neq (1,2) \\ 0, & \text{when } (x,y) = (1,2) \end{cases}$$

is not continuous at the point $(1,2)$.

Sol: limit at $(1,2) = 1+8 = 9$

$f(x,y \text{ at } 1,2) = 0$

Value of func $\neq$ limit

Hence $f(x,y)$ is not continous at $(1,2)$.

Que→ If $u = \tan^{-1}\left(\frac{y^2}{x}\right)$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\sin(2u)\sin^2(u).$$

Sol: Check homogenous for Applying Euler's.

$x \to tx$ & $y \to ty$

$u = \tan^{-1}\left(\frac{t^2y^2}{tx}\right)$

$\tan u = \frac{t^2y^2}{tx}$

$\tan u = \frac{ty^2}{x}$

tanu is homogeneous of degree 1

by euler's,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \phi(u)[\phi'(u)-1] \quad (HP)$$

$\phi(u) = n \cdot \frac{f(u)}{f'(u)} = \frac{\tan u}{\sec^2 u}$

$f(u) = \tan u$

$f'(u) = \sec^2 u$

$= \frac{\frac{\sin u}{\cos u}}{\frac{1}{\cos^2 u}}$

$= \frac{\sin u \cos^2 u}{\cancel{\cos u}}$

$\sin u \cos u$

$= \frac{1 \times \tan u}{\sec^2 u}\left[\frac{\tan^2 u}{\sec^4 u} - \frac{1}{1}\right]$

$= \frac{\frac{\sin u}{\cos u}}{\frac{1}{\cos^2 u}}\left[\frac{\frac{\sin^2 u}{\cos^2 u}}{\frac{1}{\cos^4 u}} - \frac{1}{1}\right]$

$= \frac{\sin u \cancel{\cos^2 u}}{\cancel{\cos u}}\left[\frac{\sin^2 u \cancel{\cos^4 u}}{\cancel{\cos^2 u}} - \frac{1}{1}\right]$

$= \sin u \cos u\left[\sin^2 u \cos^2 u - 1\right]$

$\phi(u) = \sin u \cos u$

$\phi'(u) = \cancel{\cos u \cos u + (-\sin^2 u)}$

$= \cos^2 u - \sin^2 u$

$= \sin u \cos u\left[\cos^2 u - \sin^2 u - 1\right]$

$= \frac{\sin u \cos u \times 2}{2}\left[\cancel{1} - \sin^2 u - \cancel{1}\right]$

$= \frac{2\sin(2u)}{2} \times -\cancel{2}\sin^2 u$

$= -\sin(2u)\sin^2 u$

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Ques: if $u = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$

evaluate $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$.

Sol: Check homogeneous for applying Euler's

$x \to tx$ & $y \to ty$

$u = t^2x^2\tan^{-1}\left(\frac{ty}{tx}\right) - t^2y^2\tan^{-1}\left(\frac{tx}{ty}\right)$

$u = t^2\left(x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)\right)$

$n=2$

by Euler's,

$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$

$= 2(2-1)u$

$= 2u$

Ques: if $u = x\sin^{-1}\left(\frac{x}{y}\right) + y\sin^{-1}\left(\frac{y}{x}\right)$, evaluate

$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$.

Sol: Check homogeneous,

$x \to tx$ & $y \to ty$

$u = tx\sin^{-1}\left(\frac{tx}{ty}\right) + ty\sin^{-1}\left(\frac{ty}{tx}\right)$

$u = t\left[x\sin^{-1}\left(\frac{x}{y}\right) + y\sin^{-1}\left(\frac{y}{x}\right)\right]$

$n=1$

$= n(n-1)u$

$= 1(1-1) = 0$ Ans

Ques: If $u=$

Show th

Sol: sin

(Solving LHS) sin

Ques: If

x^2

Sol:

$-\frac{1}{12}$

Ques If $u = \sin^{-1}\left(\dfrac{x^{1/3}+y^{1/3}}{x^{1/2}-y^{1/2}}\right)^{1/2}$ (MTU 2012)

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Show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{-1}{12}\tan u$.

Sol: (Solving LHS)

$$\sin u = \left(\frac{t^{1/3}x^{1/3} + t^{1/3}y^{1/3}}{t^{1/2}x^{1/2} - t^{1/2}y^{1/2}}\right)^{1/2}$$

$$\sin u = \left[\frac{t^{1/3}}{t^{1/2}}\left(\frac{x^{1/3}+y^{1/3}}{x^{1/2}-y^{1/2}}\right)\right]^{1/2}$$

$$n = \left(\frac{1}{3}-\frac{1}{2}\right)\times\frac{1}{2} = \frac{2-3}{6}\times\frac{1}{2} = \frac{-1}{12}$$

$$n = \frac{-1}{12}$$

$$= nu = \frac{-1}{12}\times\frac{\sin u}{\cos u} = \frac{-1}{12}\tan u$$

LHS = RHS (H.P.)

Ques If $u = \operatorname{cosec}^{-1}\left(\dfrac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)^{1/2}$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}(13+\tan^2 u)$$

Sol:

$$\operatorname{cosec} u = \left(\frac{t^{1/2}x^{1/2} + t^{1/2}y^{1/2}}{t^{1/3}x^{1/3} + t^{1/3}y^{1/3}}\right)^{1/2}$$

$$\operatorname{cosec} u = \left(t^{1/2 - 1/3}\right)\times\frac{1}{2}$$

$$= \left(\frac{1}{2}-\frac{1}{3}\right)\times\frac{1}{2} = \frac{3-2}{6}\times\frac{1}{2} = \frac{1}{12}$$

$$n = \frac{1}{12}$$

$$= \frac{1}{12}\times\frac{\operatorname{cosec} u}{-\operatorname{cosec} u\cot u}$$

$$= \frac{-1}{12}\tan u\left[\frac{-1}{12}\sec^2 u - \frac{1}{1}\right]$$

$-\frac{1}{12}$

$$= \frac{1}{144}\tan u\sec^2 u + \frac{\tan u}{12}$$

$$= \frac{\tan u\sec^2 u + 12\tan u}{144} = \frac{\tan u}{144}(\sec^2 u + \tan$$