KIET GROUP OF INSTITUTIONS, DELHI – NCR, GHAZIABAD

B. Tech. (I SEM), 2024-25

CALCULAS FOR ENGINEERS (K24AS11)

MODULE 2 (Differential Calculus II)

Syllabus: Taylor and Maclaurin's Theorem for function of two variables, Jacobians, properties of Jacobian (without proof), Hessian Matrix, Maxima and Minima of functions of two variables.

course Outcomes:

S.NO.	Course Outcome	BL
CO 2	Apply knowledge of partial differentiation in extrema, series expansion of function and Jacobians.	2,3

CONTENT

2.1.Taylor's and Maclaurin's Theorem for functions of two variables

2.1.1Taylor's Theorem for function of two variables: If $f(x, y)$ be any function in two variables x and y such that f(x, y) and all its partial derivatives up to desired order are finite and continuous for any point (x, y) subject to $a \le x \le a+h,\ \ b \le y \le b+k$, then

$$f(a+h, b+k) = f(a,b) + \frac{1}{1!}\left\{h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right\} + \frac{1}{2!}\left\{h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right\}$$
$$+\frac{1}{3!}\left\{h^3\frac{\partial^3 f}{\partial x^3} + +3h^2k\frac{\partial^3 f}{\partial x^2 \partial y} + 3hk^2\frac{\partial^3 f}{\partial x \partial y^2} + +k^3\frac{\partial^3 f}{\partial y^3}\right\} + \ldots\ldots\ldots\ldots\ldots. (1)$$

Here equation (1) represent Taylor series expansion of $f(a+h, b+k)$ in powers of h and k. Also, in this series values of all order partial derivatives are determined at point (a, b).

Taking $a+h = x\ \ or\ h = x-a\ \ \&\ b+k = y\ \ or\ k = y-b$ in equation (1), we get

$$f(x,y) = f(a,b) + \frac{1}{1!}\left\{(x-a)\frac{\partial f}{\partial x} + (y-b)\frac{\partial f}{\partial y}\right\} + \frac{1}{2!}\left\{(x-a)^2\frac{\partial^2 f}{\partial x^2} + 2(x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y} + (y-b)^2\frac{\partial^2 f}{\partial y^2}\right\} + \frac{1}{3!}\left\{(x-a)^3\frac{\partial^3 f}{\partial x^3} + +3(x-a)^2(y-b)\frac{\partial^3 f}{\partial x^2 \partial y} + 3(x-a)(y-b)^2\frac{\partial^3 f}{\partial x \partial y^2} + +(y-b)^3\frac{\partial^3 f}{\partial y^3}\right\} + \ldots. (2)$$

Here equation (2) represents Taylor's series expansion of a function $f(x, y)$ about point (a, b). Also, it is said to represent Taylor's series expansion in powers of (x $-$ a)and$(y-b)$.

Note: For simplicity sometime we represent first order derivatives $\frac{\partial f}{\partial x}\ by\ f_x, \frac{\partial f}{\partial y}\ by\ f_y$ and second order derivatives $\frac{\partial^2 f}{\partial x^2}\ by\ f_{x^2}, \frac{\partial^2 f}{\partial x \partial y}\ by\ f_{xy}, \frac{\partial^2 f}{\partial y^2}\ by\ f_{y^2}$ and so on other higher order derivatives terms are represented.

By replacing a by x and b by y in equation (1), we obtain Taylor's series expansion of function $f(x+h, y+k)$ in powers of h and k as

$$f(x+h, y+k) = f(x,y) + \frac{1}{1!}\left\{h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right\} + \frac{1}{2!}\left\{h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right\}$$
$$+\frac{1}{3!}\left\{h^3\frac{\partial^3 f}{\partial x^3} + +3h^2k\frac{\partial^3 f}{\partial x^2 \partial y} + 3hk^2\frac{\partial^3 f}{\partial x \partial y^2} + +k^3\frac{\partial^3 f}{\partial y^3}\right\} + \ldots\ldots\ldots\ldots\ldots.\ (3)$$

Note: In Taylor's series expansion represented by equation (3), values of all order partial derivatives are determined at point (x, y).

2.1.2. Maclaurin's Theorem for function of two variables:

Taking $a = \ b = 0$ and replacing h by x & k by y respectively in Taylor's series expansion given by equation (1), we get

$$f(x,y) = f(0,0) + \frac{1}{1!}\left\{x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right\} + \frac{1}{2!}\left\{x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2}\right\} + \frac{1}{3!}\left\{x^3\frac{\partial^3 f}{\partial x^3} + +3x^2y\frac{\partial^3 f}{\partial x^2 \partial y} + 3xy^2\frac{\partial^3 f}{\partial x \partial y^2} + +y^3\frac{\partial^3 f}{\partial y^3}\right\} + \ldots$$

This is known as Maclaurin's theorem.

The above series is known as Maclaurin's series expansion of given function $f(x,y)$. This will be important to note here that to find Maclaurin's series expansion of given function we shall require to find out values of all partial derivatives at point (0, 0).
In other words, we can say that Maclaurin's series is nothing but Taylor's series expansion about the point (0,0).

Example-1: Expand $x^2y+3y-2$ in powers of $(x-1)$& $(y+2)$ by using Taylor's series.
Solution: We know that Taylor's series expansion of any function $f(x,y)$ in powers of

$(x-a)$& $(y-b)$ is given by

$$f(x,y)=f(a,b)+\frac{1}{1!}\left\{(x-a)\frac{\partial f}{\partial x}+(y-b)\frac{\partial f}{\partial y}\right\}+\frac{1}{2!}\left\{(x-a)^2\frac{\partial^2 f}{\partial x^2}+2(x-a)(y-b)\frac{\partial^2 f}{\partial x\partial y}+(y-b)^2\frac{\partial^2 f}{\partial y^2}\right\}+\frac{1}{3!}\left\{(x-a)^3\frac{\partial^3 f}{\partial x^3}++3(x-a)^2(y-b)\frac{\partial^3 f}{\partial x^2\partial y}+3(x-a)(y-b)^2\frac{\partial^3 f}{\partial x\partial y^2}++(y-b)^3\frac{\partial^3 f}{\partial y^3}\right\}$$
+... (1)

Let $f(x,y)=x^2y+3y-2$ … … … … … … … … … … … (2)

To obtain required Taylor's series we take a $=1$ and b $=-2$ and find all values at point (1, -2). Now from equation (2),

$f(x,y)=x^2y+3y-2\ \ or\ \ f(1,-2)=-10$

$\frac{\partial f}{\partial x}=2xy\ \ or\ \ \frac{\partial f}{\partial x}(1,-2)=-4$

$\frac{\partial f}{\partial y}=x^2+3\ \ or\ \ \frac{\partial f}{\partial y}(1,-2)=4$

$$\frac{\partial^2 f}{\partial x^2}=2y\ \ or\ \ \frac{\partial^2 f}{\partial x^2}(1,-2)=-4,\frac{\partial^2 f}{\partial x\partial y}=2x\ \ or\ \ \frac{\partial^2 f}{\partial x\partial y}=2,\frac{\partial^2 f}{\partial y^2}=0\ or\ \frac{\partial^2 f}{\partial y^2}(1,-2)=0$$

$$\frac{\partial^3 f}{\partial x^3}(1,-2)=0,\frac{\partial^3 f}{\partial x^2\partial y}=2\ or\ \frac{\partial^3 f}{\partial x^2\partial y}(1,-2)=2,\frac{\partial^3 f}{\partial x\partial y^2}(1,-2)=0,\frac{\partial^2 f}{\partial y^3}(1,-2)=0$$

Putting all these values in equation (1), we get

$$x^2y+3y-2=\ -10+\frac{1}{1!}\{(x-1)(-4)+(y+2)(4)\}+\frac{1}{2!}\{(x-1)^2(-4)+2(x-1)(y+2)(2)+(y+2)^2(0)\}+\frac{1}{3!}\{(x-1)^3(0)+3(x-1)^2(y+2)(2)+3(x-1)(y+2)^2(0)+(y+2)^3(0)\}$$

$$\boldsymbol{x^2y+3y-2=-10-4(x-1)+4(y+2)-2(x-1)^2+2(x-1)(y+2)+(x-1)^2(y+2).}$$

Example-2: Expand x^y in powers of $(x-1)$& $(y-1)$ up to and inclusive of 3rd degree term.

Solution: By Taylor's series we have

$$f(x,y)=f(a,b)+\frac{1}{1!}\left\{(x-a)\frac{\partial f}{\partial x}+(y-b)\frac{\partial f}{\partial y}\right\}+\frac{1}{2!}\left\{(x-a)^2\frac{\partial^2 f}{\partial x^2}+2(x-a)(y-b)\frac{\partial^2 f}{\partial x\partial y}+(y-b)^2\frac{\partial^2 f}{\partial y^2}\right\}+\frac{1}{3!}\left\{(x-a)^3\frac{\partial^3 f}{\partial x^3}++3(x-a)^2(y-b)\frac{\partial^3 f}{\partial x^2\partial y}+3(x-a)(y-b)^2\frac{\partial^3 f}{\partial x\partial y^2}++(y-b)^3\frac{\partial^3 f}{\partial y^3}\right\}+..\ (1)$$

Here let $f(x,y)=x^y$ … … … … … … … … … … … .. (2), To get required series expansion, we take a=b=1and find all values at (1, 1)

Now from equation (2), $f(x,y) = x^y \;\; or\; f(1,1) = 1$. Also differentiating equation partially, we get

$$\frac{\partial f}{\partial x} = yx^{y-1} \;\; or \;\; \frac{\partial f}{\partial x}(1,1) = 1 \;\; , \frac{\partial f}{\partial y} = x^y logx \;\; or \;\; \frac{\partial f}{\partial y}(1,1) = 0$$

$$\frac{\partial^2 f}{\partial x^2} = y(y-1)x^{y-2}\; or \;\; \frac{\partial^2 f}{\partial x^2}(1,1) = 0, \frac{\partial^2 f}{\partial x \partial y} = yx^{y-1}logx + x^{y-1}\; or \;\; \frac{\partial^2 f}{\partial x \partial y}(1,1) = 1,$$

$$\frac{\partial^2 f}{\partial y^2} = x^y (logx)^2 \;\; or \frac{\partial^2 f}{\partial y^2}(1,1) = 0 \& \frac{\partial^3 f}{\partial x^3} = y(y-1)(y-2)x^{y-2}\; or \; \frac{\partial^3 f}{\partial x^3} = 0$$

$$\frac{\partial^3 f}{\partial x^2 \partial y} = y(y-1)x^{y-2}logx + yx^{y-2} + (y-1)x^{y-2}\; or \;\; \frac{\partial^3 f}{\partial x^2 \partial y}(1,-1) = 1,$$

$$\frac{\partial^3 f}{\partial x \partial y^2} = yx^{y-1}(logx)^2 + 2logx.x^{y-1}\; or \;\; \frac{\partial^3 f}{\partial x \partial y^2}(1,1) = 0$$

$$\frac{\partial^3 f}{\partial y^3} = \; x^y (logx)^3 \;\; or \;\frac{\partial^3 f}{\partial y^3}(1,1) = 0$$

Using all these values in equation (1) we get

$$\boldsymbol{x^y = 1 + (x-1) + (x-1)(y-1) + \frac{1}{2}(x-1)^2(y-1) + \cdots} \text{ ... ...}$$

Example-3: Obtain Taylor's series expansion off$(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$ about $(1,1)$ upto and including the second-degree terms. Hence, computef(1.1,0.9).

By Taylor's theorem know that

$$f(x,y) = f(a,b) + \frac{1}{1!}\left\{(x-a)\frac{\partial f}{\partial x}(a,b) + (y-b)\frac{\partial f}{\partial y}(a,b)\right\} + \frac{1}{2!}\left\{(x-a)^2\frac{\partial^2 f}{\partial x^2}(a,b) + 2(x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y}(a,b) + (y-b)^2\frac{\partial^2 f}{\partial y^2}(a,b)\right\} + \text{ .... } (1)$$

Therefore, in this problem we select $a = 1, b = 1$. Also, we have

$$\mathrm{f(x,y)} = \tan^{-1}\left(\frac{y}{x}\right) \;\; or\; f(1,1) = \tan^{-1}(1) = \frac{\pi}{4}$$

Differentiating f(x, y) partially with respect to x and y we get

$$\frac{\partial f}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right) \;\; or \;\; \frac{\partial f}{\partial x} = \frac{-y}{x^2+y^2} \;\;\; or \; \frac{\partial f}{\partial x}(1,1) = \frac{-1}{2}$$

$$\frac{\partial f}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) \;\; or \;\; \frac{\partial f}{\partial y} = \frac{x}{x^2+y^2} \;\;\; or \; \frac{\partial f}{\partial y}(1,1) = \frac{1}{2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{y(2x)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2} \;\;\; or \; \frac{\partial^2 f}{\partial x^2}(1,1) = \frac{1}{2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{(x^2+y^2).1 - x.2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \quad or \quad \frac{\partial^2 f}{\partial x \partial y}(1,1) = 0$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{-x(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2} \quad or \quad \frac{\partial^2 f}{\partial y^2}(1,1) = \frac{-1}{2}$$

Putting all these values in equation (1) we get

$$\mathrm{f(x,y)} = \tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4} + \frac{1}{1!}\left\{(x-1)\left(\frac{-1}{2}\right) + (y-1)\left(\frac{1}{2}\right)\right\} + \frac{1}{2!}\left\{(x-1)^2\left(\frac{1}{2}\right) + 2(x-1)(y-1)\frac{\partial^2 f}{\partial x \partial y}(0) + (y-1)^2\left(\frac{-1}{2}\right)\right\} + \ldots\ldots\ldots\ldots\ldots$$

$$\mathbf{f(x,y) = \tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4} - \frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{1}{4}(x-1)^2 - \frac{1}{4}(y-1)^2 + \ldots\ldots\ldots\ldots\ldots\ (2)}$$

Putting $x = 1.1$ & $y = 0.9$ in equation (2), we get

$$\mathrm{f}(1.1, 0.9) = \frac{\pi}{4} - \frac{1}{2}(1.1-1) + \frac{1}{2}(0.9-1) + \frac{1}{4}(1.1-1)^2 - \frac{1}{4}(0.9-1)^2$$

$\mathbf{f(1.1, 0.9) = 0.6857.}$

Example-4: Expand $\frac{(x+h)(y+k)}{(x+h)+(y+k)}$ in powers of h and k up to and inclusive second-degree terms.

Solution: By Taylor's theorem we have

$$f(x+h, y+k) = f(x,y) + \frac{1}{1!}\left\{h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right\} + \frac{1}{2!}\left\{h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right\} + \ldots\ldots\ldots\ldots\ldots\ (1)$$

Here let us take $f(x+h, y+k) = \frac{(x+h)(y+k)}{(x+h)+(y+k)}$ such that $f(x,y) = \frac{xy}{x+y}$ ……… (2)

Differentiating equation (2) partially with respect to x and y we get

$$\frac{\partial f}{\partial x} = \frac{(x+y)y - xy.1}{(x+y)^2} \quad or \quad \frac{\partial f}{\partial x} = \frac{y^2}{(x+y)^2}$$

$$\frac{\partial f}{\partial y} = \frac{(x+y)x - xy.1}{(x+y)^2} \quad or \quad \frac{\partial f}{\partial y} = \frac{x^2}{(x+y)^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{y^2\{-2\}}{(x+y)^3} = -\frac{2y^2}{(x+y)^3}, \frac{\partial^2 f}{\partial y^2} = \frac{x^2\{-2\}}{(x+y)^3} = -\frac{2x^2}{(x+y)^3}$$

and $\frac{\partial^2 f}{\partial x \partial y} = \frac{(x+y)^2(2x) - x^2\{2(x+y)\}}{(x+y)^4}$ or $\frac{\partial^2 f}{\partial x \partial y} = \frac{2xy}{(x+y)^3}$

Putting all the values in equation (1), we get

$$\frac{(x+h)(y+k)}{(x+h)+(y+k)} = \frac{xy}{x+y} + \frac{1}{1!}\left\{h\frac{y^2}{(x+y)^2} + k\frac{x^2}{(x+y)^2}\right\}$$

$$+\frac{1}{2!}\left\{h^2.\frac{-2y^2}{(x+y)^3} + 2hk\frac{2xy}{(x+y)^3} + k^2\frac{-2x^2}{(x+y)^3}\right\} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{(x+h)(y+k)}{(x+h)+(y+k)} = \frac{xy}{x+y} + \frac{y^2}{(x+y)^2}h + \frac{x^2}{(x+y)^2}k - \frac{y^2}{(x+y)^3}h^2 + \frac{2xy}{(x+y)^3}hk - \frac{x^2}{(x+y)^3}k^2 + \cdots$$

Example5: Expand $e^{ax}\sin by$ in powers of x and y as far as terms of third degree.

Solution: By Maclaurin's theorem we have

$$f(x,y) = f(0,0) + \tfrac{1}{1!}\left\{x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right\} + \tfrac{1}{2!}\left\{x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x\partial y} + y^2\frac{\partial^2 f}{\partial y^2}\right\} + \tfrac{1}{3!}\left\{x^3\frac{\partial^3 f}{\partial x^3} + +3x^2y\frac{\partial^3 f}{\partial x^2\partial y} + 3xy^2\frac{\partial^3 f}{\partial x\partial y^2} + +y^3\frac{\partial^3 f}{\partial y^3}\right\} + \ldots\ldots\ldots\ldots \quad (1).$$

In this equation (1), value of all order partial derivatives is evaluated at$(0,0)$.

So, let $f(x,y) = e^{ax}\sin by \qquad or\ f(0,0) = 0$

On differentiating $f(x,y)$ partially with respect to x and y we get

$$\frac{\partial f}{\partial x} = ae^{ax}\sin by \quad or\ f(0,0) = 0$$

$$\frac{\partial f}{\partial y} = be^{ax}\cos by \quad or\ \frac{\partial f}{\partial y}(0,0) = b$$

$$\frac{\partial^2 f}{\partial x^2} = a^2e^{ax}\sin by \quad or\ \frac{\partial^2 f}{\partial x^2}(0,0) = 0$$

$$\frac{\partial^2 f}{\partial y^2} = -b^2e^{ax}\sin by \quad or\ \frac{\partial^2 f}{\partial y^2}(0,0) = 0$$

$$\frac{\partial^2 f}{\partial x\partial y} = abe^{ax}\cos by \quad or\ \frac{\partial^2 f}{\partial x\partial y} = ab$$

$$\frac{\partial^3 f}{\partial x^3} = a^3e^{ax}\sin by \quad or\ \frac{\partial^3 f}{\partial x^3}(0,0) = 0$$

$$\frac{\partial^3 f}{\partial y^3} = -b^3e^{ax}\cos by \quad or\ \frac{\partial^3 f}{\partial y^3}(0,0) = -b^3$$

$$\frac{\partial^3 f}{\partial x^2\partial y} = a^2be^{ax}\cos by \quad or\ \frac{\partial^3 f}{\partial x^2\partial y}(0,0) = a^2b$$

$$\frac{\partial^3 f}{\partial x\partial y^2} = -ab^2e^{ax}\sin by \quad or\ \frac{\partial^3 f}{\partial x\partial y^2}(0,0) = 0$$

Putting all these values in equation (1), we get

$$e^{ax}\sin by = 0 + \frac{1}{1!}\{0 + by\} + \frac{1}{2!}\{0 + 2abxy + 0)\} + \frac{1}{3!}\left\{0 + +3a^2b.x^2y\frac{\partial^3 f}{\partial x^2\partial y} - -b^3y^3\right\} + \cdots$$

$$e^{ax}\sin by = by + ab\,xy + \frac{a^2b}{2}.x^2y - \frac{b^3}{6}y^3 + \cdots$$

Practice Exercise:

Q-1: Expand $e^x cosy$ near the point $\left(1, \frac{\pi}{4}\right)$ by Taylor's theorem.

Ans: $e^x cosy = \frac{e}{\sqrt{2}}\left[1 + (x-1) - \left(y - \frac{\pi}{4}\right) + \frac{(x-1)^2}{2} - (x-1)\left(y - \frac{\pi}{4}\right) - \left(y - \frac{\pi}{4}\right)^2 + \dots\right]$

Q-2: Expand y^x about $(1, 1)$ up to second degree terms and hence evaluate$(1.02)^{1.03}$.

Ans: $y^x = 1 + (y-1) + (x-1)(y-1) + \dots\dots\dots$&$(1.02)^{1.03} = 1.0206$

Q-3: Expand $cosx\ cosy$ in powers of x & y up to 4th order terms.

Ans: $cosx\ cosy = 1 - \frac{x^2}{2} - \frac{y^2}{2} + \frac{x^4}{24} + \frac{x^2y^2}{4} + \frac{y^4}{24} + \dots\dots\dots$

Q-4: Obtain linearized form $T(x, y)$ of the function $f(x, y) = x^2 - xy + \frac{y^2}{2} + 3$ at the point $(3, 2)$using Taylor's series expansion.

Ans: $T(x, y) = 8 + 4(x-3) - (y-2)$.

Q-5: Expand $e^{-(x^2+y^2)} \cos xy$ about the point $(0, 0)$ up to second order derivative terms.

Ans: $e^{-(x^2+y^2)} \cos xy = 1 - x^2 - y^2 +$......................

2.2.Jacobian

Jacobian is a functional determinant which is very useful to transform of variables from Cartesian to polar, cylindrical, and spherical co-ordinate in multiple integral.

(1) If $u(x,y)$ and $v(x,y)$ are two functions then the jacobian of u and v is denoted by $J(u,v)$ or $\frac{\partial(u,v)}{\partial(x,y)}$ and its value is $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

(2) If $u_1, u_2, u_3, \dots, u_n$ are function $x_1, x_2, x_3, \dots, x_n$ then $\frac{\partial(u_1,u_2,u_3,\dots,u_n)}{\partial(x_1,x_2,x_3,\dots,x_n)}$ is $\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial u_n}{\partial x_1} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$

2.2.1. Properties of Jacobians

1. Chain Rule Property

If $u(r,s)$ and $v(r,s)$ are two functions i.e. u and v are two function of r and s and $r(x,y)$ and $s(x,y)$ are two functions i.e. r and s are two function of x and y then$u$ and v becomes the functions of x and y

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} * \frac{\partial(r,s)}{\partial(x,y)}$$

2. If $\frac{\partial(u,v)}{\partial(x,y)} = J_1$ and $\frac{\partial(x,y)}{\partial(r,s)} = J_2$ then $J_1 J_2 = 1$

3. If u_i and x_i are related as following

$$u_1 = f(x_1)$$

$$u_2 = f(x_1, x_2)$$

$$\dots\dots\dots\dots\dots\dots..$$

$$u_n = f(x_1, x_2, x_3, \dots, x_n)$$

Then, $\frac{\partial(u_1,u_2,u_3,\dots,u_n)}{\partial(x_1,x_2,x_3,\dots,x_n)} = \frac{\partial u_1}{\partial x_1} \cdot \frac{\partial u_2}{\partial x_2} \dots \frac{\partial u_n}{\partial x_n}$

4. If $u(x,y)$ and $v(x,y)$ are two dependent functions then $\frac{\partial(u,v)}{\partial(x,y)} = 0$

5. If $u_1, u_2, u_3$are implicit function x_1, x_2, x_3 i.e.

$$F_1(\text{If } u_1, u_2, u_3, x_1, x_2, x_3) = 0$$
$$F_2(\text{If } u_1, u_2, u_3, x_1, x_2, x_3) = 0$$
$$F_3(\text{If } u_1, u_2, u_3, x_1, x_2, x_3) = 0$$

$$\frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)} = (-1)^3 \left[\frac{\partial(F_1,F_2,F_3)}{\partial(x_1,x_2,x_3)} \middle/ \frac{\partial(F_1,F_2,F_3)}{\partial(u_1,u_2,u_3)} \right]$$

2.1.2. Functional Relationship

If $u_1, u_2, u_3, \dots, u_n$ are functions $x_1, x_2, x_3, \dots, x_n$. Then the necessary condition for the existence of a relation of the form $F(\text{If } u_1, u_2, \dots, u_n) = 0$ is that $\frac{\partial(u_1,u_2,u_3,\dots,u_n)}{\partial(x_1,x_2,x_3,\dots,x_n)} = 0$

Example 1: If $u = x + 2y + z, v = x + 2y + 3z$ and $w = 2x + 3y + 5z$ then find the Jacobian $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

Solution: According to the definition of Jacobian $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{vmatrix} = 2$

Example 2: If $u = xyz, v = xy + yz + zx \; and \; w = x + y + z$ then find the jacobian $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

Solution: According to the definition of jacobian $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} yz & zx & xy \\ y+z & z+x & x+y \\ 1 & 1 & 1 \end{vmatrix}$

To solve the above determinant, we want to reduce the matrix in simplest form so we will use the property (elementary operation) of matrix.

By applying the elementary operation $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_2$, we get

$$\begin{vmatrix} yz & z(x-y) & y(x-z) \\ y+z & x-y & x-z \\ 1 & 0 & 0 \end{vmatrix}$$

which can be reduced to $\begin{vmatrix} z(x-y) & y(x-z) \\ x-y & x-z \end{vmatrix} = (x-y)(y-z)(z-x)$

Example 3. If $u = rcos\theta, v = rsin\theta$ find $\frac{\partial(u,v)}{\partial(r,\theta)}$ and $\frac{\partial(r,\theta)}{\partial(u,v)}$. Also $J_1J_2 = 1$

Solution:$J_1 = \frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix} = \begin{vmatrix} cos\theta & -rsin\theta \\ sin\theta & cos\theta \end{vmatrix} = r$

$r = \sqrt{u^2 + v^2}$ and $\theta = tan^{-1}\frac{v}{u}$

$$J_2 = \frac{\partial(r,\theta)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial r}{\partial u} & \frac{\partial r}{\partial v} \\ \frac{\partial \theta}{\partial u} & \frac{\partial \theta}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{u}{\sqrt{u^2+v^2}} & \frac{v}{\sqrt{u^2+v^2}} \\ \frac{-v}{\sqrt{u^2+v^2}} & \frac{u}{\sqrt{u^2+v^2}} \end{vmatrix} = \frac{1}{\sqrt{u^2+v^2}} = \frac{1}{r}$$

Hence $J_1 J_2 = 1$

Example 4: Verify the chain rule for Jacobian if $u = x, v = \ x\, tan\, y, w = z$

Solution: Let$J_1 = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ tany & xsec^2 y & 0 \\ 0 & 0 & 1 \end{vmatrix} = xsec^2 y$

Solving x, y, z in terms of u, v and w

$x = u, y = tan^{-1}\frac{v}{u}$and $z = w$

$$J_2 = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ \frac{-v}{u^2+v^2} & \frac{u}{u^2+v^2} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{u}{u^2+v^2} = \frac{1}{xsec^2 y}$$

So $J_1 J_2 = 1$. Therefore, chain rule is verified.

Example 5: If $x + y + z = u, y + z = uv, z = uvw$, then show that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2 v$

Solution: reduce the given value of x, y and z in terms of $u, v\ and\ w$

$$x = u(1 - v)$$

$$y = uv(1 - w)$$

$$z = uvw$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}$$

To solve the above determinant, we want to reduce the matrix in simplest form so we will use the property (elementary operation) of matrix.

By applying the elementary operation$R_1 \to R_1 + R_2 + R_3$, we get

$$\begin{vmatrix} 1 & 0 & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix} = u^2 v$$

Example 6: If u, v and w are the roots of the cubic equation $(x-a)^3+(y-b)^3+(z-c)^3=0$ then find the value of $\frac{\partial(u,v,w)}{\partial(a,b,c)}$

Solution: The cubic equation$(x-a)^3+(y-b)^3+(z-c)^3=0$

$$3x^3-3x^2(a+b+c)+3x(a^2+b^2+c^2)-(a^3+b^3+c^3)=0$$

Given that $u, v\ and\ w$ are the roots of the cubic equation. So

$$u+v+w=a+b+c$$

$$uv+vw+wu=a^2+b^2+c^2$$

$$uvw=\frac{a^3+b^3+c^3}{3}$$

Since u, v and w are the implicit function of x, y and z. so use the property no. 5 for the implicit function.

$$F_1=u+v+w-a-b-c=0$$

$$F_2=uv+vw+wu-a^2-b^2-c^2=0$$

$$F_3=uvw-\frac{a^3+b^3+c^3}{3}=0$$

$$\frac{\partial(u,v,w)}{\partial(a,b,c)}=(-1)^3\left[\frac{\partial(F_1,F_2,F_3)}{\partial(a,b,c)}\Big/\frac{\partial(F_1,F_2,F_3)}{\partial(u,v,w)}\right]$$

$$\frac{\partial(F_1,F_2,F_3)}{\partial(a,b,c)}=\begin{vmatrix}-1 & -1 & -1\\ -2a & -2b & -2c\\ -a^2 & -b^2 & -c^2\end{vmatrix}$$

By applying the elementary operation $C_2\to C_2-C_1$ and $C_3\to C_3-C_2$, we get

$$\begin{vmatrix}-1 & 0 & 0\\ -2a & -2b+2a & -2c+2a\\ -a^2 & -b^2+a^2 & -c^2+a^2\end{vmatrix}$$

$$=-2(a-b)(b-c)(c-a)$$

$$\frac{\partial(F_1,F_2,F_3)}{\partial(u,v,w)}=\begin{vmatrix}1 & 1 & 1\\ v+w & u+w & u+v\\ vw & uw & uv\end{vmatrix}$$

By applying the elementary operation $C_2\to C_2-C_1$ and $C_3\to C_3-C_2$, we get

$$\begin{vmatrix}1 & 0 & 0\\ v+w & u-v & u-w\\ vw & w(u-v) & v(u-w)\end{vmatrix}$$

$$= -(u-v)(v-w)(w-u)$$

So

$$\frac{\partial(u,v,w)}{\partial(a,b,c)} = (-1)^3 \frac{\frac{\partial(F_1,F_2,F_3)}{\partial(a,b,c)}}{\frac{\partial(F_1,F_2,F_3)}{\partial(u,v,w)}} = -\frac{2(\mathrm{a}-\mathrm{b})(\mathrm{b}-\mathrm{c})(\mathrm{c}-\mathrm{a})}{(\mathrm{u}-\mathrm{v})(\mathrm{v}-\mathrm{w})(\mathrm{w}-\mathrm{u})}$$

Example 7: Prove that $u = x + 2y + z, v = x - 2y + 3z \text{ } and \text{ } w = 2xy - xz + 4yz - 2z^2$ are not independent and find the relation between them.

Solution: To check the dependency of u, v and w we have to calculate $J(u, v, w)$if $J(u, v, w)$ is zero it means the functions are not independent i.e. they are dependent.

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 1 & -2 & 3 \\ 2y - z & 2x + 4z & -x + 4y - 4z \end{vmatrix}$$

By applying the elementary operation $C_2 \to C_2 - 2C_1$ and $C_3 \to C_3 - C_2$, we get

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & -4 & 2 \\ 2y - z & 2x + 6z - 4y & -x + 2y - 3z \end{vmatrix} = 0$$

i.e. $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$

so the function u, v and w are not independent i.e. dependent. If they are dependent then there exists a relation between them.

$u + v = 2x + 4z \ldots(1)$

$u - v = 4y - 2z \ldots(2)$

By multiplying both equations, we get

$$u^2 - v^2 = 4(2xy - xz + 4yz - 2z^2)$$

$$u^2 - v^2 = 4w$$

is the relation between u, v and w.

Example 8: Prove that $u = \frac{x+y}{1-xy}$ and $v = tan^{-1}x + tan^{-1}y$ are functionally related. Find the relation between them.

Solution: As we discussed above question to check the dependency of u, v and w we have to calculate $J(u, v)$ if $J(u, v)$ is zero it means the functions are not independent i.e. they are dependent.

$$J(u,v) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix} = 0$$

So u *and* v are functionally related

$$tan^{-1}x + tan^{-1}y = tan^{-1}\frac{x+y}{1-xy}$$

$$v = tan^{-1}u$$

So u *and* v are functionally related with the relation $v = tan^{-1}u$.

Practice Exercise:

Q-1: If $u_1 = x_1 + x_2 + x_3 + x_4$, $u_1 u_2 = x_2 + x_3 + x_4$, $u_1 u_2 u_3 = x_3 + x_4$, $u_1 u_2 u_3 u_4 = x_4$ the show that $\frac{\partial(x_1,x_2,x_3,x_4)}{\partial(u_1,u_2,u_3,u_4)} = {u_1}^3 {u_2}^2 u_3$.

Q-2: If $u^3 + v^3 = x + y$, $u^2 + v^2 = x^3 + y^3$ then show that $\frac{\partial(u,v)}{\partial(x,y)} = \frac{y^2 - x^2}{2uv(u-v)}$.

Q-3: If $u_1 = x_1 + x_2 + x_3 + x_4$, ${u_1}^2 u_2 = x_2 + x_3$ and ${u_1}^3 u_3 = x_3$then find the value of $\frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)}$

Ans: $\boldsymbol{{u_1}^{-5}}$

Q-4: If $x = v^2 + w^2, y = w^2 + u^2 z = u^2 + v^2$ then show that $\frac{\partial(x,y,z)}{\partial(u,v,w)} \cdot \frac{\partial(u,v,w)}{\partial(x,y,z)} = 1$

Q-5: Use the Jacobian to prove that the function $u = x + y - z, v = x - y + z$ and$w = x^2 + y^2 + z^2 - 2yz$ are not independent of one another.

Q-6: Show that $u = x + y + z$, $v = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$ and $w = x^3 + y^3 + z^3 - 3xyz$ are functionally related.

Q-7: If $u = x + y + z$, $v = x^2 + y^2 + z^2$ and $w = x^3 + y^3 + z^3 - 3xyz$, prove that $u, v \; and \; w$ are not independent and hence find the relation between them. **Ans:** $\boldsymbol{2w = u(3v - u^2)}$

Q-8: Show that the functions: $u = x + y + z$, $v = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$ and $w = x^3 + y^3 + z^3 - 3xyz$ are functionally related. Find the relation between them. **Ans:** $\boldsymbol{w = \frac{u(u^2 + 3v)}{4}}$

Q-9: If $u = x^2 - y^2$, $v = 2xy$ and If $x = rcos\theta, y = rsin\theta$ then $\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^3$.

Q-10: If $u = sin^{-1}x + sin^{-1}y$ and $v = x\sqrt{1 - y^2} + y\sqrt{1 - x^2}$, find $\frac{\partial(u,v)}{\partial(x,y)}$. Prove that $u \; and \; v$ functionally related, find the relation between them. **Ans:** $\boldsymbol{v = \sin u}$

2.3.Maxima and minima of function of two variables

A function $f(x, y)$ is said to have a maximum value at $x = a, y = b$ if

$f(a, b) > f(a + h, b + k)$, for small and independent values of h and k, positive or negative.

A function $f(x, y)$ is said to have a minimum value at $x = a, y = b$ if

$f(a, b) < f(a + h, b + k)$, for small and independent values of h and k, positive or negative.

Thus, f (x, y) has a maximum or minimum value at a point (a, b) according as

$$F = f(a + h, b + k) - f(a, b) < or > 0$$

The maximum and minimum value of a function is called its extreme value.

2.3.1. Rule to find the extreme values of a function $z = f(x, y)$

1. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$
2. Solve $\frac{\partial z}{\partial x} = 0$ by $\frac{\partial z}{\partial y} = 0$ simultanously
   Let $(a, b), (c, d)$ .... Be the solution of these equations
3. For each solution in step 2 find$r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$ $t = \frac{\partial^2 z}{\partial y^2}$
4. (a) If $rt - s^2 > 0$ and $r < 0$ for a particular (a, b) of step 2, then z has maximum value at(a, b).
   (b) If $rt - s^2 > 0$ and $r > 0$ for a particular (a, b) of step 2, then z has minimum value at (a, b)
   (c) If $rt - s^2 < 0$ for a particular (a, b) of step 2, then z has no extreme value at (a, b)

Example 1. Find the extreme value for the function $x^2 + y^2 + 6x + 12$

Solution. Let $f(x, y) = x^2 + y^2 + 6x + 12$

By taking the partial derivative with respect to x and y respectively

$\frac{\partial f}{\partial x} = 2x + 6, \frac{\partial f}{\partial y} = 2y, r = \frac{\partial^2 f}{\partial^2 x} = 2, s = \frac{\partial^2 f}{\partial x \partial y} = 0$ and $t = \frac{\partial^2 f}{\partial^2 y} = 2$

By equating the first order partial derivative to 0, we get

$2x + 6 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$ and

$$2y = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots . (2)$$

After solving equation (1) and (2)

We get $x = -3$ and $y = 0$

The stationary point is $(-3,0)$

At $(-3,0)$

$rt - s^2 = 4 > 0 \; and \; r = 2 > 0$

According to working rule, $f(x,y)$ has minimum value at $(-3,0)$ and the minimum value is 3

Example 2: Find the extreme values of function $x^3 + y^3 - 3axy$.

Solution: Here $f(x,y) = x^3 + y^3 - 3axy$

$$f_x = 3x^2 - 3ay, f_y = 3y^2 - 3ax, r = f_{xx} = 6x, s = f_{xy} = -3a, t = f_{yy} = 6y$$

By equating the first order partial derivative to zero

$x^2 - ay = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots.(1)$ and

$y^2 - ax = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots.(2)$

From equation (1) $y = \frac{x^2}{a}$

Put the value of y in equation (2) $\frac{x^4}{a^2} - ax = 0 \; or \; x(x^3 - a^3) \; or \; x = 0, a$

When $x = 0, y = 0$; when $x = a, y = a$

There are two stationary points $(0,0)$ and (a,a)

Now, $rt - s^2 = 36xy - 9a^2$

At $(\mathbf{0},\mathbf{0})$ $rt - s^2 = -9a^2 < 0$

It means $f(x,y)$ has no extreme value at $(0,0)$

At $(\boldsymbol{a},\boldsymbol{a})$ $rt - s^2 = 27a^2 > 0$

$f(x,y)$ has extreme value at $(0,0)$

$$r = 6a$$

if $a > 0, r > 0$ so that $f(x,y)$ has a minimum value at (a,a) and minimum value = $-a^3$

if $a > 0, r < 0$ so that $f(x,y)$ has a maximum value at (a,a) and maximum value = a^3

Example 3: Examine for minimum and maximum values $sin\,x + sin\,y + sin\,(x + y)$

Solution: Here $f(x,y) = sin\,x + sin\,y + sin\,(x + y)$

$$f_x = \cos x + \cos(x + y)$$

$$f_y = \cos y + \cos(x + y)$$

$$r = f_{xx} = -\sin x - \sin(x + y)$$

$$s = f_{xy} = \cos x - \sin(x + y)$$

$$t = f_{yy} = -\sin y - \sin(x + y)$$

By equating the first order partial derivative to zero

$\cos x + \cos(x + y) = 0$ ....(1)

$\cos y + \cos(x + y) = 0$...............................(2)

By subtracting equation (2) from (1)

$cos\ x = \ cos\ y\ or\ \ x = y$

put the value in equ (1) $cos\ 2x \ = \ -cosx \ = \ cos(\pi - x)$

$$x = \frac{\pi}{3} = y$$

The stationary point is $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right) r = -\sqrt{3}, s = \frac{\sqrt{3}}{2}, t = -\sqrt{3}$

$$rt - s^2 = \frac{9}{4} > 0\ and\ r < 0$$

So $f(x, y)$ has a maximum value at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ and maximum value is$\frac{3\sqrt{3}}{2}$.

Example 4. In a plane triangle ABC, find the maximum value of $cosA\ cosB\ cosC$.

Solution. In a triangle $A + B + C = \pi$

Convert the given function into two variables A and B

So $\cos A \cos B \cos C = -cosA \cos B\ cos\ (\pi - A - B) = f(A, B)$

Taking the partial derivative of f

$$\frac{\partial f}{\partial A} = -\cos B[-\sin A \cos(A + B) - \cos A\ sin(A + B)] = \cos B \sin(2A + B)]$$

$$\frac{\partial f}{\partial B} = -\cos A[-\sin B \cos(A + B) - \cos B\ sin(A + B)] = \cos A \sin(A + 2B)]$$

$$r = 2\cos B\ \cos\ (2A + B), s = \square \cos\ (2A + 2B), t = 2\cos A\ \cos(A + 2B)$$

By equating the first order partial derivative to zero

$\frac{\partial f}{\partial A} = 0$ and $\quad \frac{\partial f}{\partial B} = 0$

$\cos B \sin(2A + B) = 0$ ……………………………..(1) and

$$\cos A \sin(A + 2B) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Solving the equation (1) and (2)

if $\cos B = 0$ then $B = \frac{\pi}{2}$ then $\cos A \sin(A + \pi) = 0$ or $-\cos A \sin A = 0$

either $\cos A = 0$ this implies $A = \frac{\pi}{2}$ it means $C = 0$ which is not possible in triangle

or $\sin A = 0$ this implies $A = 0$ or π not possible in triangle

so $\cos A \neq 0$ similarly $\cos B \neq 0$

$\sin(2A + B) = 0$ or $2A + B = \pi$…............(3)

$\sin(A + 2B) = 0$ or $A + 2B = \pi$…(4)

solving equation (3) and (4) $A = B = \frac{\pi}{3}$

so $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ is the stationary point

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

$$r = -1, s = -\frac{1}{2}, t = -1$$

$$rt - s^2 = \frac{3}{4} > 0 \ and\ r = -1 < 0$$

So $f(A, B)$ is maximum at $A = B = \frac{\pi}{3}$ and maximum value is $\frac{1}{8}$.

Example 5. A rectangular box, open at the top, is to have a given capacity. Find the dimensions of the box requiring last material for its construction.

Solution. Let $x, y, and\ z$ be the length, breadth, and height of a rectangular box respectively. Let V be the given capacity and S is surface

Capacity is given so V is constant

$V = xyz$ or $z = \frac{V}{xy}$

$$S = xy + 2xz + 2yz = xy + \frac{2V}{y} + \frac{2V}{x} = f(x, y)$$

By taking the partial derivative

$f_x = y - \frac{2V}{x^2}, f_y = x - \frac{2V}{y^2}, r = f_{xx} = \frac{4V}{x^3}, s = f_{xy} = 1, t = f_{yy} = \frac{4V}{y3}$

By equating the first order partial derivative to zero

$f_x = y - \frac{2V}{x^2} = 0$..........................(1) and

$f_y = x - \frac{2V}{y^2} = 0$ ....(2)

From equation (1) $y = \frac{2V}{x^2}$

Put the value of y in equation (2) we get $x - 2V\frac{x^4}{4V^2} = 0$

$x\left(1 - \frac{x^3}{2V}\right) = 0$

$x = (2V)^{1/3}$ and $y = \frac{2V}{x^2}$ so $y = (2V)^{1/3}$

Hence $\left((2V)^{1/3}, (2V)^{1/3}\right)$ is the stationary point

At $\left((2V)^{1/3}, (2V)^{1/3}\right)$

$r = 2, s = 1$ and $t = 2$

$rt - s^2 = 3 > 0 \; and \; r = 2 > 0$

So $f(x, y)$ has minimum value at $x = y = (2V)^{1/3}$and$z = \frac{V}{xy} = \frac{y}{2}$

Example 6. Find the shortest distance between the lines $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$ and

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$$

Solution: Let $P(x, y, z)$ and $Q(x, y, z)$ be the points on the given line respectively.

$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} = \delta$ and $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1} = \mu$

So $P(\delta + 3, 5 - 2\delta, 7 + \delta), Q(7\mu - 1, -1 - 6\mu, \; \mu - 1)$are two points on the first and second line respectively.

The distance between points P and Q is

$$PQ = \sqrt{(\delta + 3 - 7\mu + 1)^2 + (5 - 2\delta + 6\mu + 1)^2 + (\delta + 7 - \mu + 1)^2}$$

$$= \sqrt{6\delta^2 + 86\mu^2 - 40\delta\mu + 116}$$

If the distance is max or min so it will be the square of the distances.

$$f(\delta,\mu) = PQ^2 = 6\delta^2 + 86\mu^2 - 40\delta\mu + 116$$

Taking the partial derivatives

$$\frac{\partial f}{\partial \delta} = 12\delta - 40\mu$$

$$\frac{\partial f}{\partial \mu} = 172\mu - 40\delta$$

$$r = \frac{\partial^2 f}{\partial \delta^2} = 12$$

$$s = \frac{\partial^2 f}{\partial \delta \partial \mu} = -40$$

$$t = \frac{\partial^2 f}{\partial \mu^2} = 172$$

Equate the first order partial derivative to zero

$$\frac{\partial f}{\partial \delta} = 12\delta - 40\mu = 0 \ldots \ldots . .(1)$$

$$\frac{\partial f}{\partial \mu} = 172\mu - 40\delta = 0 \ldots \ldots (2)$$

By solving equation (1) and (2) , we get $\delta = 0, \mu = 0$

The stationary point is $(0,0)$

At $(0,0)$ $\quad rt - s^2 = 12 * 172 - (-40)^2 > 0 \; and \; r = 12 > 0$

So $f(x,y)$ is minimum at $(0,0)$ and the shortest distance is $PQ = \sqrt{116} = 2\sqrt{29}$

Practice Exercise:

Q-1: Examine for extreme values $f(x,y) = x^3 + y^3 - 3xy$**Ans: Min value = -1 at $(\mathbf{1}, \mathbf{1})$**

Q-2: Examine for extreme values $f(x,y) = 3x^2 - y^2 + x^3$**Ans: Max value = 4 at $(\mathbf{-2}, \mathbf{0})$**

Q-3: Examine for extreme values $f(x,y) = x^3y^2(1 - x - y)$**Ans: Max value = $\mathbf{\frac{1}{432}}$ at $\left(\mathbf{\frac{1}{2}}, \mathbf{\frac{1}{3}}\right)$**

Q-4: Determine the points where the function $f(x,y) = x^2 + y^2 + 6x + 12$ has a maximum or minimum.

Ans: Min value = 3 at $(\mathbf{-2}, \mathbf{0})$

Q-5: Find the maximum value of the function $f(x,y) = 1 - x^2 - y^2$.**Ans:1**

Q-6: Prove that if the perimeter of a triangle is constant, its area is maximum when the triangle is equilateral

Q-8: A rectangular box, open at the top, is to have a volume of 32c.c. Find the dimensions of the box requiring least material for its construction. **Ans:4,4,2**

2.4. Hessian matrix

In mathematics, the Hessian matrix, Hessian or (less commonly) Hess matrix is **a square matrix of second-order partial derivatives of a scalar-valued function**.

If $f: R^n \rightarrow R$ is a function, then Hessian matrix of f is denoted by H_f or $H_{F(x,y)}$

Hessian matrix is often used in machine learning and data science algorithms for optimizing a function of interest.

Example1: If $f(x, y) = x^3 + 2y^2 + 3xy^2$ Find H_f

$$H_f = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 6x & 6y \\ 6y & 4+6x \end{bmatrix}.$$

E- Link for more understanding

1. https://www.youtube.com/watch?v=XzaeYnZdK5o&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=1
2. https://www.youtube.com/watch?v=9-tir2V3vYY&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=14
3. https://www.youtube.com/watch?v=aqfSOOiO2kI&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=15
4. https://www.youtube.com/watch?v=GoyeNUaSW08&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=16
5. https://www.youtube.com/watch?v=jiEaKYI0ATY&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=17
6. https://www.youtube.com/watch?v=G0V_yp0jz5c&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=18
7. https://www.youtube.com/watch?v=G0V_yp0jz5c&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=18
8. https://www.youtube.com/watch?v=McT-UsFx1Es&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=9
9. https://www.youtube.com/watch?v=XzaeYnZdK5o&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=1
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19. https://www.youtube.com/watch?v=6tQTRlbkbc8
20. https://www.youtube.com/watch?v=8ZAucbZscNA