UNIT-4
Multiple Integral

Syllabus: Evaluation of double integrals, change of order of integration, Change of variable (double -integral). Application of double integrals to find area of a region.

Employ the concept of multiple integral to find area of bounded region.

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4.2 INTRODUCTION TO MULTIPLE INTEGRAL

Multiple integral is a natural extension of a definite integral to a function of two variables (Double integral) or more variables.

4.2.1 APPLICATIONS

- Double integrals are useful in finding **Area**.
- Double integrals are useful in finding **Volume**.
- Double integrals are useful in finding **Mass**.
- Double integrals are useful in finding **Centroid**.
- In finding **Average value of a function**.
- In finding **Distance, Velocity, Acceleration.**
- Useful in calculating **Kinetic energy** and **Improper Integrals**.
- In finding **Arc Length** of a curve.
- The most important application of Multiple Integrals involves finding **areas bounded by a curve** and **coordinate axes** and **area between two curves.**
- It includes finding solutions to various complicated problems of **work** and **energy.**
- Multiple integrals are used in many applications in physics. The **gravitational potential** associated with a mass distribution given by a mass **measure on three-dimensional Euclidean space R3** is calculated by **multiple integration.**
- In **electromagnetism, Maxwell's equations** can be written using multiple integrals to calculate the total **magnetic** and **electric fields**.
- We can determine **the probability of an event** if we know the **probability density function** using **double integration.**

4.3 MULTIPLE INTEGRATION

4.3.1 Double integral

A double integral is its counterpart in two dimensions. Let a single valued and bounded function f(x, y) of two independent variables x, y defined in a closed region R.

Then double integral of f(x, y) over the region R is denoted by,

$$\boxed{\iint_R f(x,y)dA}$$

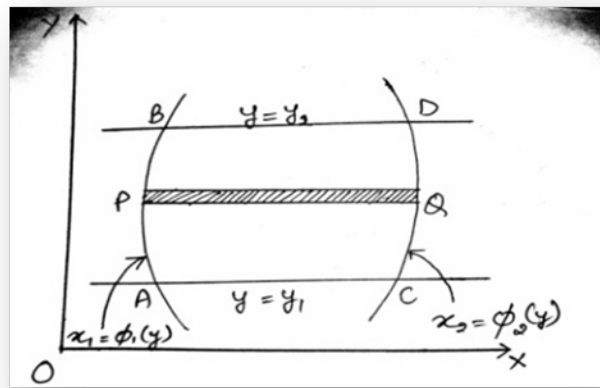
Also we can express as $\iint_R f(x,y)dx\,dy$ or $\iint_R f(x,y)dy\,dx$

4.3.1.2 Evaluation of Double Integral in Cartesian Coordinates

The method of evaluating the double integrals depends upon the nature of the curves bounding the region R. Let the region be bounded by the curves $x = x_1, x = x_2$ and $y = y_1, y = y_2$.

(i) ***When x_1, x_2 are functions of y and y_1, y_2 are constants***: If we have functional limits of x in terms of dependent variable y [$x_1 = \emptyset_1(y), x_2 = \emptyset_2(y)$] and constant limits of variable y then we will first integrate with respect to variable x in case of double integral, as follows:

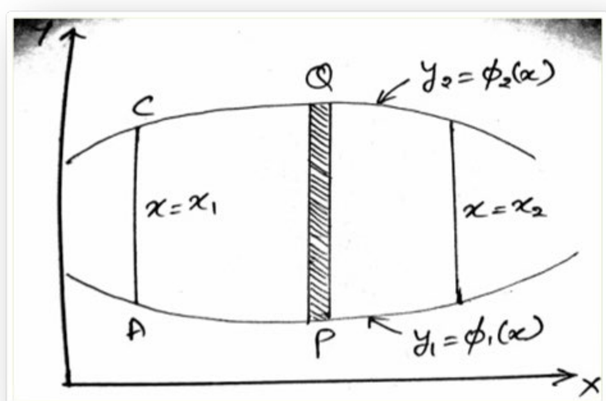
$$\boxed{\iint_R f(x,y)\,dx\,dy = \int_{y_1}^{y_2}\left\{\int_{x_1=\emptyset_1(y)}^{x_2=\emptyset_2(y)} f(x,y)\,dx\right\} dy}$$



(Here we have drawn the strip parallel to x axis, because variable limits [$x_1 = \emptyset_1(y), x_2 = \emptyset_2(y)$] are provided.)

(ii) When y_1, y_2 are functions of x and x_1, x_2 are constants: If we have functional limits of y in terms of dependent variable x [$y = \emptyset_1(x), y_2 = \emptyset_2(x)$] and constant limits of variable x then we will first integrate with respect to variable y in case of double integral, as follows:

$$\boxed{\iint_R f(x,y)\,dx\,dy = \int_{x_1}^{x}\left\{\int_{y_1=\emptyset_1(x)}^{y_2=\emptyset_2(x)} f(x,y)\,dy\right\} dx}$$

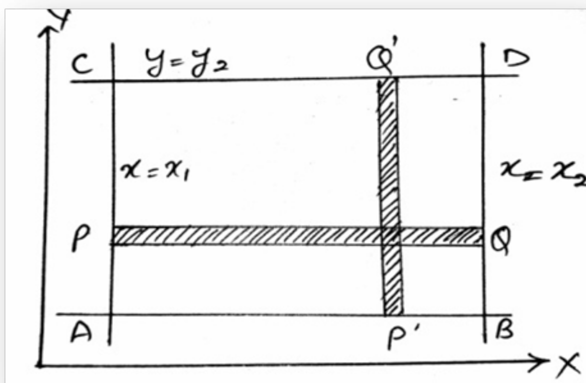


(Here we have drawn the strip parallel to y axis, because variable limits [$y = \emptyset_1(x), y_2 = \emptyset_2(x)$] are provided.)

(iii) ***When x_1, x_2, y_1, y_2 are constants:*** If we have both the variables x and y with constant limits x_1, x_2, y_1, y_2 then we can first integrate with respect to any variable x or y in case of double integral, as follows:

$$\boxed{\iint_R f(x,y)dx\,dy = \int_{x_1}^{x2}\{\int_{y_1}^{y_2} f(x,y)dy\}\,dx}$$

$$\boxed{\iint_R f(x,y)dx\,dy = \int_{y_1}^{y_2}\{\int_{x_1}^{x_2} f(x,y)dx\}\,dy}$$



(Here we can draw the strip parallel to any of the axes, because both x and y are having constant limits.)

From case no. (i) and (ii) discussed above, we observe that integration is to be performed w.r.t. the variable limits first and then w.r.t. the variable with constant limits.

4.3.1.3 Solved examples

Example1: Evaluate $\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)(1-y^2)}}$.

Solution: $\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)(1-y^2)}} = \int_0^1 \left[\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-y^2)}}\right] dy$ (Here we have constant limits for both x and y variables, so we may integrate w.r.t. any variable the first)

$$= \int_0^1 \frac{1}{\sqrt{(1-y^2)}} [sin^{-1}x]_0^1 dy$$

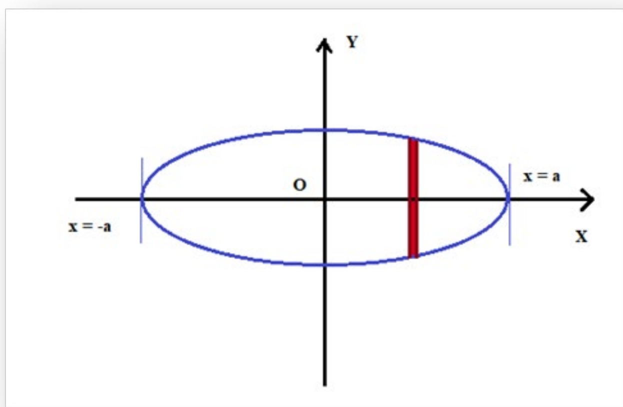
$$= \int_0^1 \frac{1}{\sqrt{(1-y^2)}} \frac{\pi}{2} dy$$

$$= \frac{\pi}{2} [sin^{-1}y]_0^1 = \frac{\pi}{2} . \frac{\pi}{2} = \frac{\pi^2}{4}$$

Example 2: Evaluate $\iint (x+y)^2\, dxdy$ over the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution: (Here we have area bounded by the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, depending on variables x and y so we have to construct a strip parallel to any one axis to observe variable limits of one variable.)

For the ellipse we may write $\frac{y}{b} = \pm\sqrt{1 - \frac{x^2}{a^2}}$ or $y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$



∴ The region of integration R can be expressed as

$-a \le x \le a,\ -\frac{b}{a}\sqrt{a^2 - x^2} \le y \le \frac{b}{a}\sqrt{a^2 - x^2}$,

where we have chosen variable limits of y and constant limits of x.

So first we will integrate w.r.t. y,

$$\therefore \iint (x+y)^2\, dxdy = \iint_R (x^2 + y^2 + 2xy) dxdy$$

$$= \int_{-a}^{a} \int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2 + y^2 + 2xy) dxdy$$

$$= \int_{-a}^{a}\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}}(x^2+y^2)dxdy + \int_{-a}^{a}\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}}(2xy)dxdy$$

$$= \int_{-a}^{a}\int_{0}^{\frac{b}{a}\sqrt{a^2-x^2}} 2(x^2+y^2)dxdy + 0$$ [using the property of even and odd functions.]

$$= \int_{-a}^{a}\left[2\left(x^2y+\frac{y^3}{3}\right)\right]_0^{\left(\frac{b}{a}\right)\sqrt{a^2-x^2}} dx$$

$$= 2\int_{-a}^{a}\left[x^2\frac{b}{a}\sqrt{a^2-x^2}+\frac{1}{3}\frac{b^3}{a^3}(a^2-x^2)^{3/2}\right]dx$$

$$= 4\int_{0}^{a}\left[\frac{b}{a}x^2\sqrt{a^2-x^2}+\frac{b^3}{3a^3}(a^2-x^2)^{3/2}\right]dx$$ Putting x = a sinθ

$\Rightarrow$ dx = a cos $\theta d\theta$

$$= 4\int_{0}^{\pi/2}\left[\frac{b}{a}a^2\sin^2\theta . a\cos\theta+\frac{b^3}{3a^3}.a^3\cos^3\theta\right]\times a\cos\theta d\theta$$

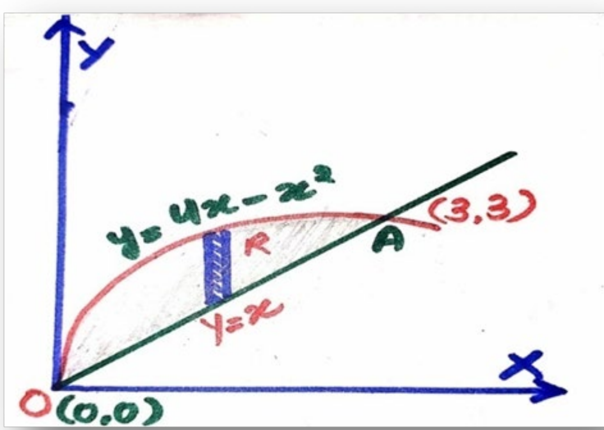
$$= 4\int_{0}^{\pi/2}\left[a^3b\sin^2\theta.\cos^2\theta+\frac{ab^3}{3}.\cos^4\theta\right]d\theta$$

$$= 4\ [a^3b.\frac{1}{4}.\frac{1}{2}.\frac{\pi}{2}+\frac{ab^3}{3}\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} = \frac{\pi}{4}(a^3b+ab^3) = \frac{\pi}{4}ab(a^2+b^2)$$

Example3: Evaluate $\iint y\ dxdy$ over the part of the plane bounded by the lines y= x and the parabola y = 4x - x^2.

Solution: The line y =x and the parabola y = 4x - x^2 intersect each other at two distinct points O (0,0) and A (3,3). Now in the intersected area we will construct a strip suitably parallel to x-axis or y-axis to have variable limit of one variable in terms of other variable.

$$\iint_R y\ dxdy = \int_0^3\int_x^{4x-x^2} y\ dydx$$



$$= \int_0^3 \left(\frac{y^2}{2}\right)_x^{4x-x^2} dx$$

$$= \frac{1}{2}\int_0^3 [(4x-x^2)^2 - x^2]\, dx$$

$$= \frac{1}{2}\int_0^3 [15x^2 + x^4 - 8x^3]\, dx$$

$$= \frac{1}{2}\left(5x^3 + \frac{x^5}{5} - 2x^4\right)_0^3 \quad = \frac{54}{5}$$

Example 4: Evaluate $\iint y\, dxdy$ over the region R bounded by the parabolas $y^2 = 4x\; x^2 = 4y$.

Solution: Solving $y^2 = 4x \text{ and } x^2 = 4y$, we have

$$\left(\frac{x^2}{4}\right)^2 = 4x \;\; or \;\; x(x^3 - 64) = 0$$

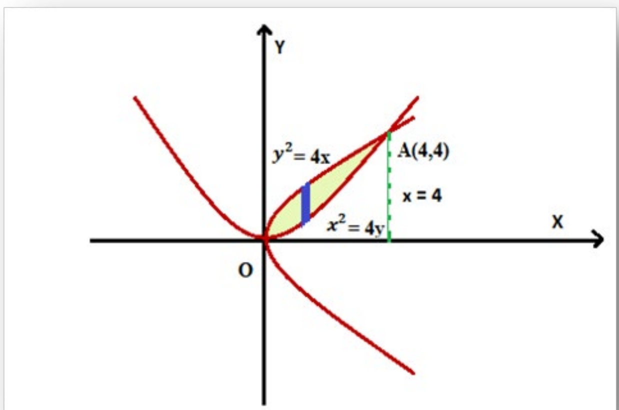
∴ $\quad$ x = 0, 4

When x = 4, y = 4

∴ Co-ordinates of A (intersection point of parabolas) are (4, 4)

The region R can be expressed as

$0 \leq x \leq 4, \; \frac{x^2}{4} \leq y \leq 2\sqrt{x}$



$$\therefore \iint_R y\, dx\, dy \;\; \text{y dx} = \int_0^4 \int_{x^2/4}^{2\sqrt{x}} y\, dydx$$

$$= \int_0^4 \frac{1}{2}[y^2]_{x^2/4}^{2\sqrt{x}} dx = \frac{1}{2}\int_0^4 \left(4x - \frac{x^4}{16}\right) dx$$

$$= \frac{1}{2}\left[2x^2 - \frac{x^5}{80}\right]_0^4 = \frac{1}{2}\left[32 - \frac{1024}{80}\right] = \frac{48}{5}$$

Example 5: Evaluate $\iint_S \sqrt{xy - y^2\, dxdy}$, where S is a triangle with vertices (0, 0), (10, 1) and (1, 1).

Solution: Let OAB be the triangle formed by given vertices (0, 0), (10, 1) and (1, 1) as shown in the figure through shaded area.

The equation of the line joining O (0, 0) and A (1, 1) can be find as follows,

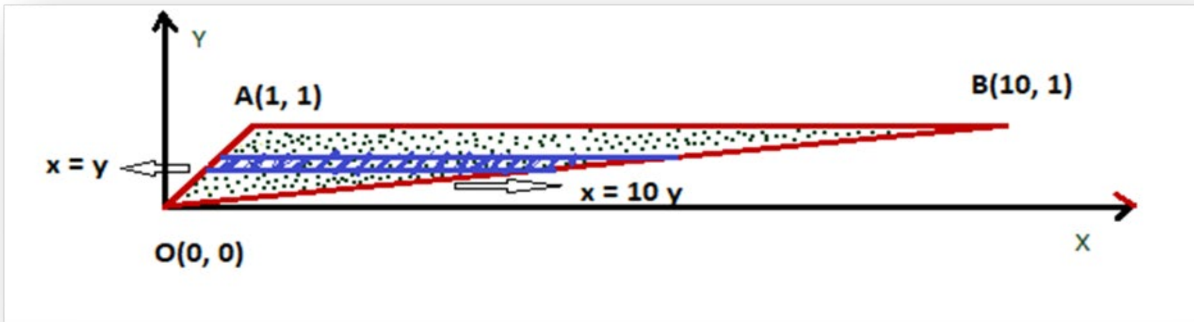
$y - 0 = \frac{1-0}{1-0}(x-0) \quad \Rightarrow y = x$

The equation of the line joining O (0, 0) and B (10, 1) can be calculated as follows,

$y - 0 = \frac{1-0}{10-0}(x-0) \quad \Rightarrow x = 10\,y$

Here we have taken strip intentionally parallel to x-axis, so that the strip bounded by x = y and x = 10 y may cover the complete shaded area from y = 0 to y = 1.

Hence the region of integration can be expressed as $y \le x \le 10y, 0 \le y \le 1$



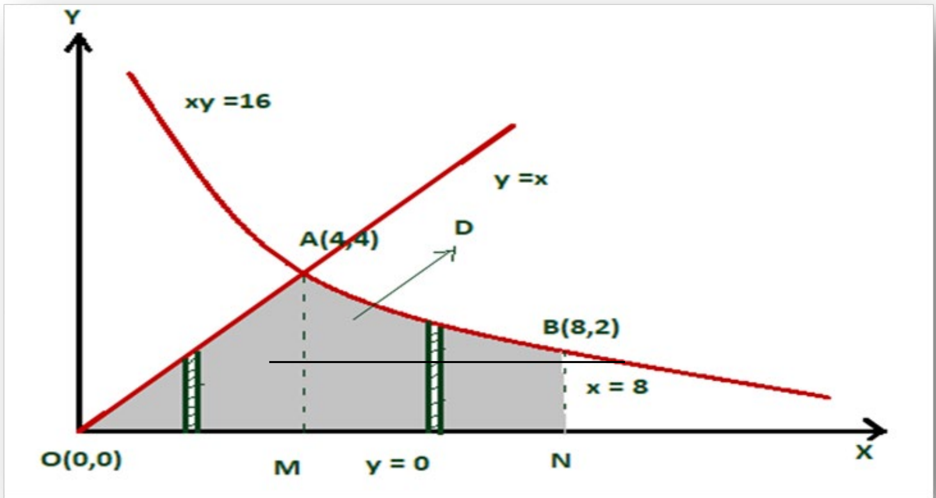
$$\iint_S \sqrt{xy - y^2}\,\text{dxdy} = \int_0^1 \int_y^{10y} \sqrt{xy - y^2}\,\text{dxdy}$$

$$= \int_0^1 \left[\frac{2}{3}\frac{(xy-y^2)^{3/2}}{y}\right]_y^{10y} \text{dy} = \int_0^1 \frac{2}{3y}(9y^2)^{3/2} dy$$

$$= 18\int_0^1 y^2 dy = 18\left(\frac{y^3}{3}\right)_0^1 = 6$$

Example 6: Let D be the region in the first quadrant bounded by the curves xy = 16, x = y, y = 0 and x = 8.Sketch the region of integration of the following integral $\iint_D x^2\,dxdy$ and evaluate it by expressing it as an appropriate repeated integral.

Solution: In this question we have to integrate the given function within the region bounded by the straight line x = y, hyperbola xy = 16, y = 0 and x = 8, so first we will draw the figure for clarity finding all intersection points of curves provided in question.



Here we can see the equations x = y and xy = 16, on solving give intersection point at A (4, 4)

Similarly on solving xy =16 and x = 8, we get the intersection point at B (8, 2)

Drawing the curves we get the intersection area as shown in figure.

Now we are to decide with respect to which variable we should first integrate, we construct strips in such a manner that the complete area may be covered.

Here we cannot cover the whole shaded area using single strip (Neither parallel to x- axis nor parallel to y- axis).

Because area is changing from dotted lines, if we plot strip parallel to y-axis and also area is changing from lines drawn, if we plot the strip parallel to x-axis. So in both cases we need to draw two strips.

Here we are splitting the area OABNO in two parts by AM as shown in figure and plotted strips parallel to y- axis from x = 0 to x = 4 and from x= 4 to x = 8

Then, $\iint_D x^2\,dxdy = \int_{x=0}^{x=4}\int_{y=0}^{y=x} x^2\,dydx + \int_{x=4}^{x=8}\int_{y=0}^{y=16/x} x^2\,dydx$

$$= \int_{x=0}^{x=4} x^2\,dx \int_{y=0}^{y=x} dy + \int_{x=4}^{x=8} x^2\,dx \int_{y=0}^{y=16/x} dy$$

$$= \int_{x=0}^{x=4} x^2\,(y)_0^x\,dx + \int_{x=4}^{x=8} x^2\,(y)_0^{16/x}\,dx$$

$$= \int_0^4 x^3 dx + \int_4^8 16x\,dx = \left(\frac{x^4}{4}\right)_0^4 + (8x^2)_4^8 = 64 + 8(64 - 16) = 64 + 384 = 448$$

4.3.1.4 Practice problems

1. Evaluate $\int_1^2 \int_0^x \frac{dydx}{x^2+y^2}$ Ans : $\pi/_2 \log 2$

2. Evaluate $\int_1^{\log 8} \int_0^{\log y} e^{x+y}\, dx\, dy$. Ans. 8(log8-2)+e

3. Evaluate:

(i) $\int_0^1 dx \int_0^x e^{y/x}\ dy$ Ans. ½

(ii) $\int_0^1 \int_{y^2}^{y} (1+xy^2) dxdy$ Ans. 41/210

(iii) $\int_1^a \int_1^b \frac{1}{xy} dxdy$ Ans. log a log b

(iv) $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2) dydx$ Ans.3/ 35

4. Evaluate $\iint_R \left(1-\frac{x^2}{a^2}+\frac{y^2}{b^2}\right) dxdy$ over the first quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=0$. Ans. $\frac{\pi ab}{4}$

5. Evaluate $\iint xy(x+y)dxdy$ over the area between $y=x^2\ and\ y=x$. Ans. 3/56

6. Evaluate $\iint_A\ xy\ dxdy$, where A is the domain bounded by x-axis, ordinate x = 2a and the curve $x^2=4ay$.

Ans. $\frac{a^4}{3}$

4.3.1.5 Evaluation of Double Integrals in Polar Coordinates

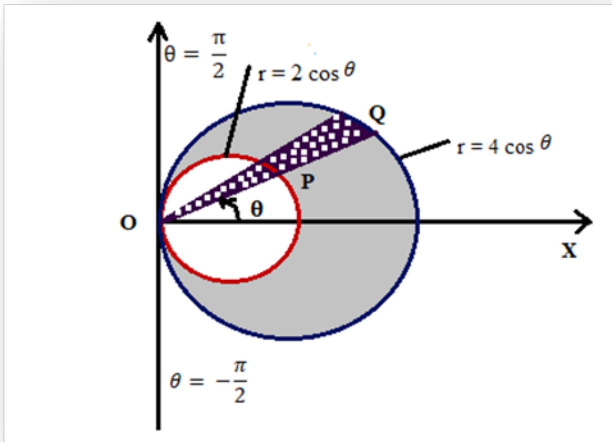
In polar coordinates we know that, x = r cos θ and y = r sin θ

Sometimes integration can be easier by converting Cartesian form to polar form. In such cases we may evaluate integral by polar coordinates using variable r and θ in the same manner as we done earlier. Here we draw radial strip to decide the limit in order to cover the whole area.

4.3.1.6 Solved Examples

Example 1: Evaluate $\iint r^3\, drd\theta$, over the area bounded between the circles r = 2 cos θ and r = 4 cos θ.

Solution: The region of integration R is shown shaded. Here r varies from r = 2 cos θ to r = 4 cos θ while θ varies from $-\frac{\pi}{2}\ to \frac{\pi}{2}$.



$$\iint_R r^3 \, dr \, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{2\cos\theta}^{4\cos\theta} r^3 \, dr d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{r^4}{4}\right]_{2\cos\theta}^{4\cos\theta} d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{4}(256\cos^4\theta - 16\cos^4\theta) \, d\theta$$

$$= 60 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta \, d\theta = 120 \int_0^{\frac{\pi}{2}} \cos^4\theta d\theta = 120 \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} = \frac{45}{2}\pi$$

Reference formula: $\int_0^{\pi/2} \sin^n\theta \, d\theta = \int_0^{\pi/2} \cos^n\theta \, d\theta = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \ldots\ldots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} & if\ n\ is\ even \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \ldots\ldots \frac{4}{5} \cdot \frac{2}{3} & if\ n\ is\ odd \end{cases}$

Example 2: Evaluate $\int_0^{\frac{\pi}{2}} \left[\int_0^{a\cos\theta} r\sqrt{a^2 - r^2} \, dr\right] d\theta$.

Solution: $I = \int_0^{\pi/2} \left[\int_0^{a\cos\theta} -\frac{1}{2}(a^2 - r^2)^{1/2}(-2r) dr\right] d\theta$

$$= \int_0^{\pi/2} \left[-\frac{1}{2} \cdot \frac{(a^2-r^2)^{3/2}}{3/2}\right]_0^{a\cos\theta} d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2} (a^3\sin^3\theta - a^3) d\theta = \frac{a^3}{3}\left[\frac{2}{3} - \frac{\pi}{2}\right] = \frac{a^3}{3}(3\pi - 4).$$

4.3.1.7 Practice problems

1. Evaluate $\int_0^{\pi/2} \int_0^{a\cos\theta} r\sin\theta \quad dr d\theta$ Ans: $\frac{a^2}{6}$

2. Evaluate $\int_0^{\pi} \int_0^{a(1-cos\theta)} r^2 \sin\theta \, dr \, d\theta$ Ans: $\frac{4}{3}a^3$

3. Evaluate $\iint r^3 \, dr d\theta$, over the area bounded between the circles r = 2 sinθ and r = 4 sin θ. Ans: $\frac{45\pi}{2}$

4.3.2 CHANGE OF ORDER OF INTEGRATION

In double integral, if the limits of integration are constant, then the order of integration does not matter, provided the limits of integration are changed accordingly. Thus,

$$\int_c^d \int_a^b f(x,y) dxdy = \int_a^b \int_c^d f(x,y) dxdy$$

But if the limits of integration are variable, then in order to change the order of limits of integration we have to construct the rough figure of given region of integration and re construct the strip parallel to that axis with respect to which we want to first integrate.

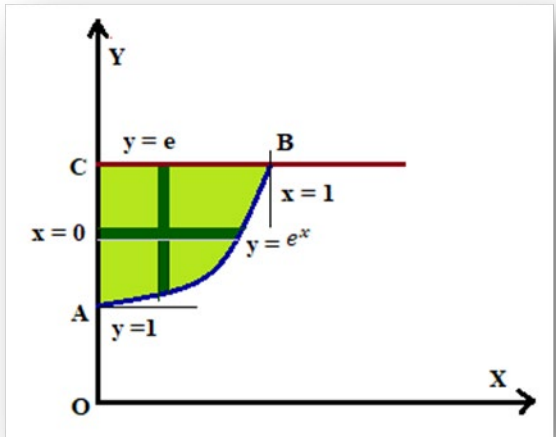
Mostly we need this process to make our integration simpler if possible.

4.3.2.1 Solved examples

Example 1: Evaluate the following integral by changing the order of integration

$$\int_0^1 \int_{e^x}^{e} \frac{dxdy}{\log y} \ .$$

Solution: The given limits show that the region of integration is bounded by the curves y = e^x, y = e, x = 0, x = 1. Plotting these curves we have the shaded region of integration as shown in figure.



In given problem we had variable limits of y in terms of x, so we had to integrate w.r.t. y the first. But we are instructed to solve this problem changing the order of integration.

Now to integrate first w.r.t. x we have to find variable limits of x in terms of y. So we have to construct a strip parallel to x- axis in order to find variable limits of x.

From the strip we can see lower limit lies on x = 0 and y = $e^x \Rightarrow x = \log y$ in between the constant limits of y from y = 1 to y = e.

Hence $\int_0^1 \int_{e^x}^{e} \frac{dxdy}{\log y} = \int_1^e \int_0^{\log y} \frac{dxdy}{\log y}$

$$= \int_1^e \left(\frac{x}{\log y}\right)_0^{\log y} dy$$

$$= \int_1^e 1.\, dy = (y)_1^e = \text{(e-1)}.$$

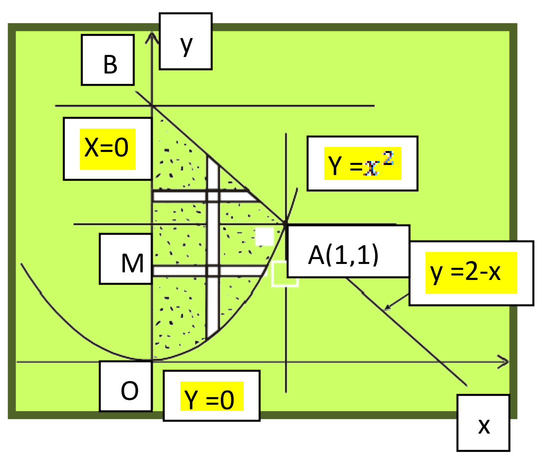
Example 2: Change the order of integration in I = $\int_0^1 \int_{x^2}^{2-x} xy\, dydx$ and hence evaluate the same

Solution: From the variable limits of integration, it is clear that we have to integrate first w.r. to y which varies from y $=x^2$ to y = 2-x and then with respect to x which varies from x = 0 to x = 1. The region of integration is divided into vertical strips.

For changing the order of integration, we divide the region of integration into horizontal strips.

Solving y = x^2 and y = 2-x, the co-ordinates of A are (1,1).

Draw AM ⊥ OY. The region of integration is divided into two parts, OAM and MAB.



For the region OAM, x varies from 0 to 2-y and y varies from 1 to 2.

$$\therefore \int_0^1 \int_{x^2}^{2-x} xy\,dydx = \int_0^1 \int_0^{\sqrt{y}} xy\,dydx + \int_1^2 \int_0^{2-y} xy\,dydx$$

$$= \int_0^1 y\left(\frac{x^2}{2}\right)_0^{\sqrt{y}} dy + \int_1^2 \left(\frac{x^2}{2}\right)_0^{2-y} dy = \frac{1}{2}\int_0^1 y^2 dy + \frac{1}{2}\int_1^2 y(2-y)^2 dy$$

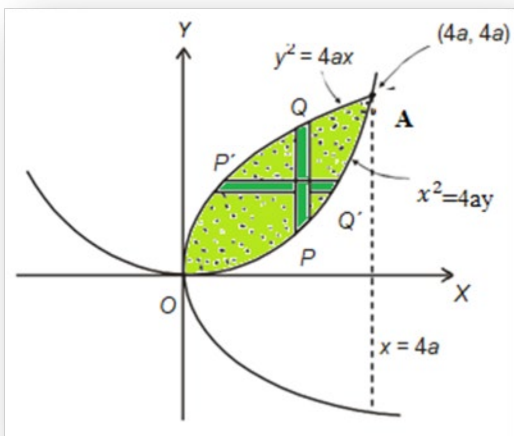
$$= \frac{1}{2}\left(\frac{y^3}{3}\right)_0^1 + \frac{1}{2}\int_1^2 (4y - 4y^2 + y^3)dy = \frac{1}{6} + \frac{1}{2}\left[2y^2 - \frac{4y^3}{3} + \frac{y^4}{4}\right]_1^2$$

$$= \frac{1}{6} + \frac{1}{2}\left[\left(8 - \frac{32}{3} + 4\right) - \left(2 - \frac{4}{3} + \frac{1}{4}\right)\right] = \frac{3}{8}$$

Example 3: Change the order of integration in the following integral and evaluate:

$$\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dydx\,.$$

Solution: From the limit of integral, it is clear that we have to first integrate with respect to y, having variable limits of y in terms of x (y = $x^2/4a$ to y = $2\sqrt{ax}$). So integration is performed along the strip PQ shown in figure.



In above figure we drawn both the parabolas intersecting at A(4a,4a).

In order to change the order of integration we draw a strip P'Q', which extend from P' on parabola x = $y^2/4a$ to Q' on parabola x= $2\sqrt{ay}$ and varying from y = 0 to y = 4a.

$$\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dydx = \int_0^{4a} \int_{y^2/4a}^{2\sqrt{ay}} dxdy$$

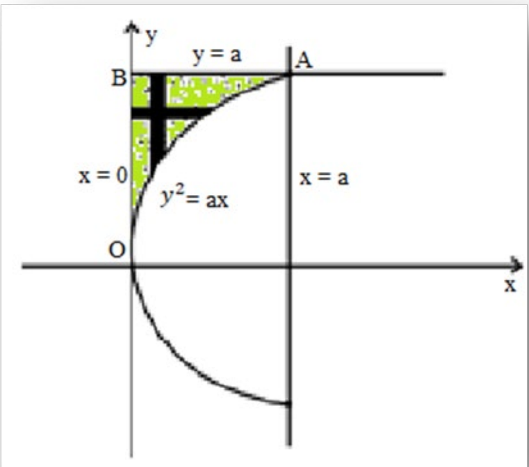
$$= \int_0^{4a} (x)_{y^2/4a}^{2\sqrt{ay}}\, dy = \int_0^{4a} (2\sqrt{ay} - y^2/_{4a})dy$$

$$= \left[2\sqrt{a}.\frac{y^{3/2}}{3/2} - \frac{y^3}{12a}\right]_0^{4a} = \frac{4}{3}\sqrt{a}(4a)^{\,3/2} - \frac{64a^3}{12a} = \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}$$

Example 4: Change the order of integration and hence evaluate $\int_0^a \int_{\sqrt{ax}}^a \frac{y^2 dydx}{\sqrt{y^4 - a^2x^2}}$.

Solution: The given limits shows that the area of integration lies between $y^2 = ax$, y = a, x = 0, x = a.

We can consider it as lying between y =0, y = a, x = 0, x = $\frac{y^2}{a}$



by changing the order of integration. Hence the given integral,

$$\int_{x=0}^{a} \int_{y=\sqrt{ax}}^{a} \frac{y^2 dydx}{\sqrt{y^4 - a^2x^2}} = \int_{y=0}^{a} \int_{x=0}^{\frac{y^2}{a}} \frac{y^2 dydx}{\sqrt{y^4 - a^2x^2}}$$

$$= \frac{1}{a}\int_{y=0}^{a} \int_{x=0}^{\frac{y^2}{a}} \frac{y^2 dydx}{\sqrt{\left(\frac{y^2}{a}\right)^2 - x^2}}$$

$$= \frac{1}{a}\int_0^a y^2 \left[sin^{-1}\left(\frac{ax}{y^2}\right)\right]_0^{y^2/_a} dy$$

$$= \frac{1}{a}\int_0^a y^2[sin^{-1}(1) - sin^{-1}(0)]dy$$

$$= \frac{\pi}{2a}\int_0^a y^2 dy = \frac{\pi}{2a}\left(\frac{y^3}{3}\right)_0^a = \frac{\pi}{6a}(a^3) = \frac{\pi a^2}{6}$$

Example5: Evaluate the following integral by changing the order of integration:

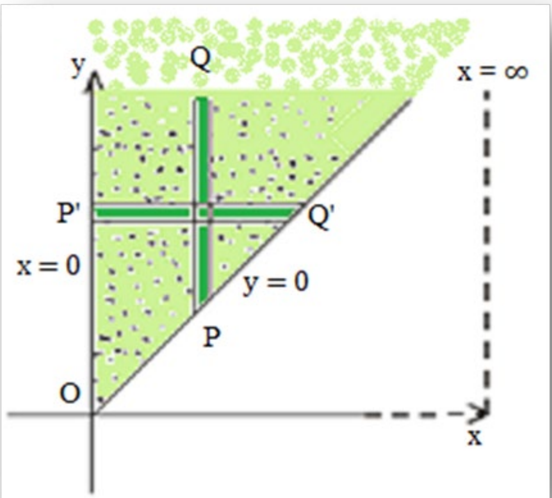
$$\int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\, dydx$$

Solution: The given limits shows that the area of integration lies between y = x, y = ∞, x = 0 and x = ∞.

We can consider it as lying between x = 0, x = y, y = 0 and y = ∞ by changing the order of integration.

Hence the given integral,

$$\int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\, dydx = \int_0^{\infty}\int_0^{y} \frac{e^{-y}}{y}\, dxdy$$



$=\int_0^\infty \frac{e^{-y}}{y}(x)_0^y\,dy$

$=\int_0^\infty e^{-y}\,dy$

$=(-e^{-y})_0^\infty$

= 1-0 = 1

4.3.2.2 Practice problems

1. Evaluate the integrals by changing the order of integration:

(i)	$\int_0^1\int_x^{\sqrt{2-x^2}} \frac{x\,dydx}{\sqrt{x^2-y^2}}$	Ans: $1-\frac{1}{\sqrt{2}}$
(ii)	$\int_0^a\int_{x^2/a}^{2a-x} xy\,dydx$	Ans: $\frac{3a^4}{8}$
(iii)	$\int_0^1\int_{y^2}^{2-y} xy\,dydx$	Ans: 3/8
(iv)	$\int_0^1\int_x^1 \sin y^2\,dydx$	Ans: ½(1- cos 1)
(v)	$\int_0^\infty\int_0^x x e^{\frac{-x^2}{y}}\,dydx$	Ans: $\frac{1}{2}$
(vi)	$\int_0^2\int_{\frac{x^2}{4}}^{3-x} xy\,dydx$	Ans: $\frac{8}{3}$
(vii)	$\int_0^{2a}\int_{\frac{x^2}{4a}}^{3a-x}(x^2+y^2)\,dydx$	Ans: $\frac{314}{35}a^4$
(viii)	$\int_0^1\int_0^{\sqrt{1-x^2}} \frac{e^y\,dydx}{(e^y+1)\sqrt{x^2-y^2}}$	Ans: $\frac{\pi}{2}\log\frac{(e+1)}{2}$
(ix)	$\int_0^3\int_0^{6/x} x^2\,dydx$	Ans: 27
(x)	$\int_0^1\int_{x^2}^{2-x} f(x,y)\,dydx$	Ans: $\int_0^1\int_0^{2\sqrt{y}} f(x,y)dxdy+\int_1^2\int_0^{2-y} f(x,y)\,dxdy$

4.3.3 CHANGE OF VARIABLES

In simple integration, we use substitution to make our integration simpler than before. Similarly, in double or triple integration we use suitable change of variables to make the evaluation of integration simple.

In general, there are following four types of transformation:

i. To change Cartesian co-ordinates (x, y) to some given co-ordinates (u, v).

ii. To change Cartesian co-ordinates (x, y) to polar co-ordinates (r, θ).

iii. To change Cartesian co-ordinates (x, y, z) to spherical polar co-ordinates (r, θ, ϕ).

iv. To change Cartesian co-ordinates (x, y, z) to cylindrical co-ordinates (r, ϕ, z).

4.3.3.1 To change Cartesian co-ordinates (x, y) to some given co-ordinates (u, v):

Let there be two variables (x, y) in the double integral $\iint_R f(x, y)dxdy$ .We are to change these variables to some variables (u, v) under the transformation $x = \phi(u, v), y = \varphi(u, v)$. Under this transformation the given integral takes the form $\iint_{R'} f\{\phi(u, v), \varphi(u, v)\}|J|dudv$ where $J = \dfrac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$ is called the Jacobian of given transformation from (x, y) to (u, v). Also R' is the region in the uv-plane corresponding to the region R in *xy-plane*.

4.3.3.2 ***To change Cartesian co-ordinates (x, y) to polar co-ordinates (r, θ):***

We know that $x = r\cos\theta, y = r\sin\theta$ and $x^2 + y^2 = r^2$,so

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left(\cos^2\theta + \sin^2\theta\right) = r.$$

$$\Rightarrow \iint_R f(x, y)dxdy = \iint_{R'} f(r\cos\theta, r\sin\theta)r\,drd\theta.$$

4.3.3.3 Solved examples

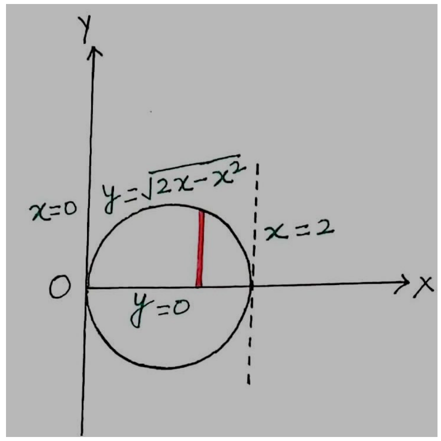
Example 1: Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} \dfrac{xdydx}{\sqrt{x^2 + y^2}}$ by changing into polar co-ordinates.

Solution: In the given integral,

x varies from 0 to 2

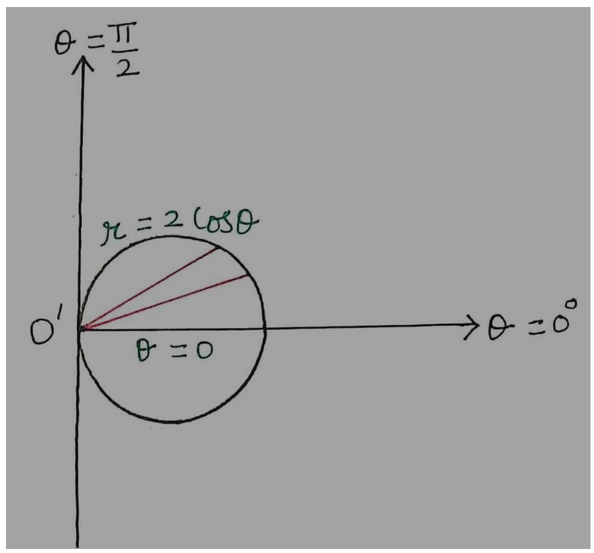
y varies from 0 to $\sqrt{2x - x^2}$

Now, $y = \sqrt{2x - x^2} \Rightarrow y^2 = 2x - x^2$ *or* $x^2 + y^2 = 2x$.



In polar co-ordinates, we have $x = r\cos\theta; y = r\sin\theta$,

Therefore, in polar coordinates $x^2 + y^2 = 2x$ becomes $r^2 = 2r\cos\theta$ or $r = 2\cos\theta$.



For this region of integration r varies from 0 to $2\cos\theta$ and θ varies from 0 to $\frac{\pi}{2}$.

So for polar coordinates, put $x = r\cos\theta; y = r\sin\theta$ and $dxdy = rdrd\theta$ in the given integral,

$$I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{2\cos\theta}\frac{r\cos\theta}{r}rdrd\theta = \int_{\theta=0}^{\pi/2}\int_{r=0}^{2\cos\theta} r\cos\theta drd\theta$$

$$= \int_{\theta=0}^{\pi/2}\cos\theta\left(\frac{r^2}{2}\right)_0^{2\cos\theta} d\theta = \int_{\theta=0}^{\pi/2} 2\cos^3\theta d\theta = 2.\frac{2}{3} = \frac{4}{3}$$

Example 2: Evaluate $\iiint z(x^2 + y^2)dxdydz$ over the volume of the cylinder $x^2 + y^2 = 1$ intercepted by the planes $z = 2$ and $z = 3$.

Solution: Here,

$I = \iiint z\left(x^2 + y^2\right)dxdydz$

Now using cylindrical polar coordinates,

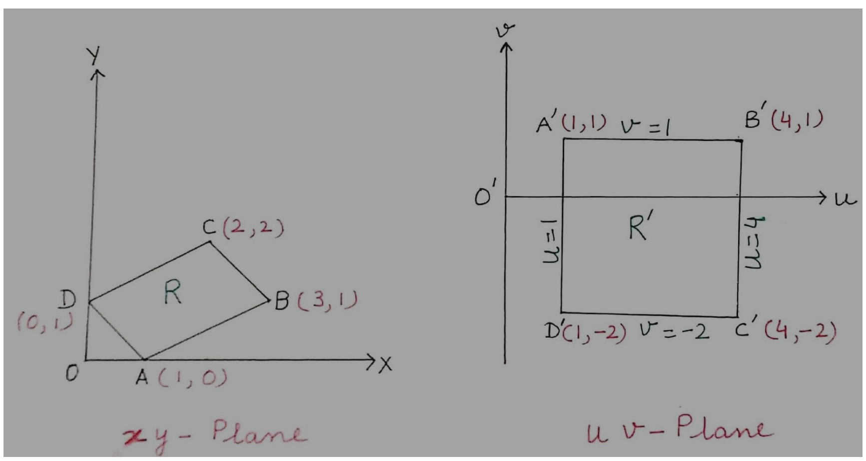
$$I = \int_{z=2}^{3}\int_{\phi=0}^{2\pi}\int_{r=0}^{1} z.r^2.rdrd\phi dz$$

$$I = \int_{z=2}^{3}\int_{\phi=0}^{2\pi} z\left(\frac{r^4}{4}\right)_0^1 d\phi dz$$

$$I = \frac{1}{4}\int_{z=2}^{3} z(\phi)_0^{2\pi} dz = \frac{1}{4}.2\pi\left(\frac{z^2}{2}\right)_2^3 = \frac{\pi}{4}(9-4) = \frac{5\pi}{4}$$

Example 3: Evaluate $\iint_R (x+y)^2 dx\,dy$, where R is the parallelogram in the *xy*-plane with vertices (1, 0), (3, 1), (2, 2), (0, 1) using the transformation $u = x + y$ and $v = x - 2y$.

Solution: The region R in *xy*-plane i.e., parallelogram ABCD with vertices A (1,0), B (3,1), C (2,2), D (0,1) becomes region R' in *uv*-plane i.e., rectangle $A'B'C'D'$ with vertices A' (1,1), B' (4,1), C' (4,-2), D' (1,-2).



Solving the given equations for x and y, we get $x = \frac{1}{3}(2u+v), y = \frac{1}{3}(u-v)$.

Here

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{3}.$$

$$\therefore \iint_R (x+y)^2 dx\,dy = \iint_{R'} u^2 |J| du\,dv = \int_{-2}^{1}\int_{1}^{4} u^2 \frac{1}{3} du\,dv = \int_{-2}^{1} 7dv = 21.$$

Example 4: Using the transformation $x-y=u, x+y=v,$ show that $\iint_R \sin\left(\frac{x-y}{x+y}\right) dx\,dy = 0,$ where R is the region bounded by the co-ordinate axes and $x+y=1$ in first quadrant.

Solution: Here, region R is a triangle OAB in *xy-plane* having sides $x=0, y=0$ and $x+y=1$.

Also

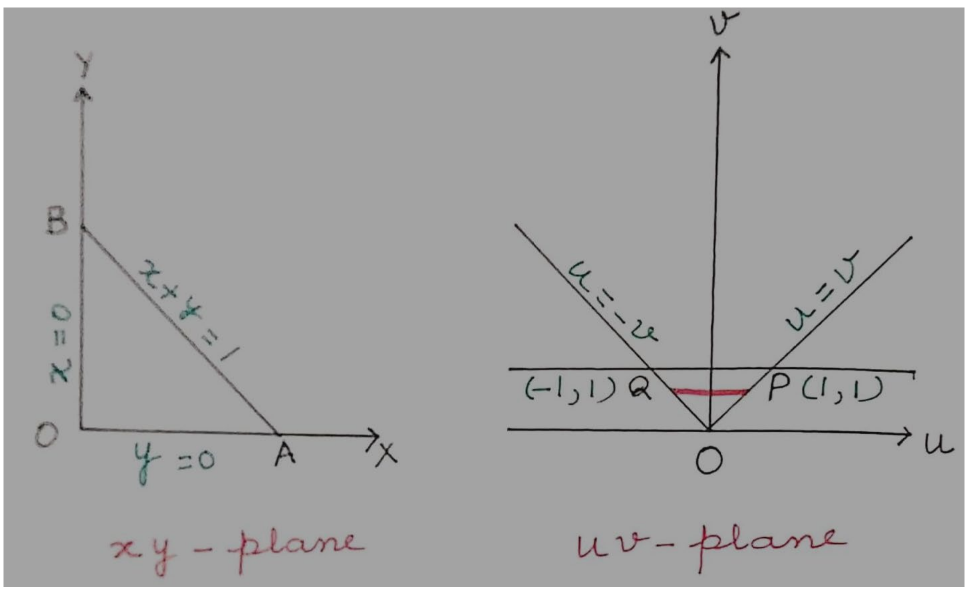
$$J=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}1 & -1\\ 1 & 1\end{vmatrix}=2.$$

Using given transformation, we get

If $x=0, y=0$ then $u=-v, u=v.$

If $x+y=1$ then $v=1$.

Thus corresponding region $R^{'}$ in *uv*-plane is a triangle $O^{'}A^{'}B^{'}$ bounded by $u=-v, u=v, v=1$.



Therefore, in *uv*-plane the given integral becomes

$$I=\int_{0}^{1}\int_{-v}^{v}\sin\left(\frac{u}{v}\right)\frac{1}{2}\,du\,dv$$

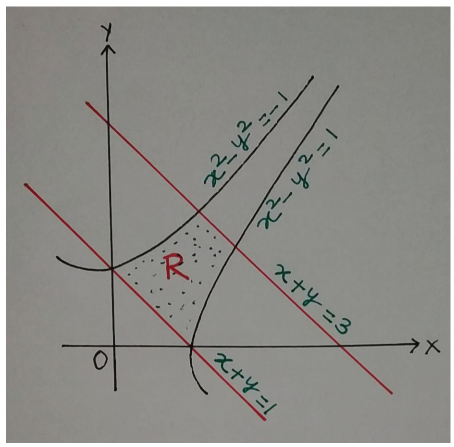
$$I=\frac{1}{2}\int_{0}^{1}\left[\frac{-\cos\left(\frac{u}{v}\right)}{\left(\frac{1}{v}\right)}\right]_{-v}^{v}dv$$

$$I=\frac{1}{2}\int_{0}^{1}v\left[-\cos 1+\cos(-1)\right]dv$$

$$I=0.$$

Example 5: Using the transformation $u=x-y$ and $v=x+y$, evaluate the integral $\iint_R (x-y)e^{x^2-y^2}\,dx\,dy$ where R is the region bounded by the lines $x+y=1$ and $x+y=3$ and the curves $x^2-y^2=-1$ and $x^2-y^2=1$.

Solution: Region R in $xy-plane$ is shown below:



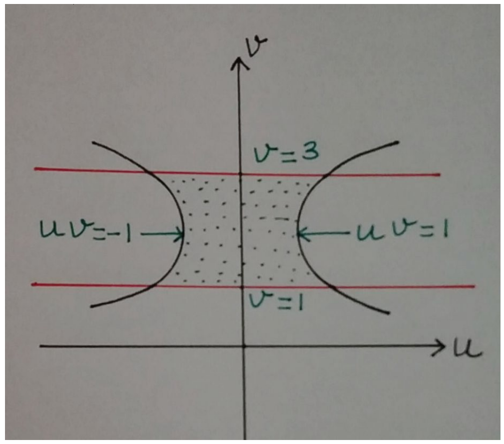
From given transformation,

$$u=x-y;v=x+y$$

$$\Rightarrow x=\frac{u+v}{2};y=\frac{v-u}{2}$$

Using this, $x+y=1;x+y=3$ become $v=1;v=3$. Also the curves $x^2-y^2=-1;x^2-y^2=1$ become $uv=1;uv=-1$.

Thus, under the given transformation region R in $xy-plane$ becomes region R' in $uv-plane$.



Also $J(x,y)=\dfrac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{1}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{vmatrix}=\dfrac{1}{2}$

Therefore, under the given transformation

$$\iint_R (x-y)e^{x^2-y^2}\,dx\,dy=\frac{1}{2}\int_{v=1}^{3}\int_{u=-1/v}^{1/v} ue^{uv}\,du\,dv$$

$$\iint_R (x-y)e^{x^2-y^2}\,dx\,dy=\frac{2}{3e}.$$

4.3.3.6 Practice problems

1. Evaluate the following by changing into polar coordinates:

$$\int_0^a\int_0^{\sqrt{a^2-y^2}} y^2\sqrt{x^2+y^2}\,dx\,dy \qquad \text{Ans: } \frac{\pi a^5}{20}$$

2. Change into polar coordinates and evaluate $\int_0^\infty\int_0^\infty e^{-\left(x^2+y^2\right)}\,dy\,dx.$ Ans: $\dfrac{\pi}{4}$

3. Let D be the region in the first quadrant bounded by $x=0, y=0$ and $x+y=1$. Change the variables x, y to u, v where $x+y=u, y=uv$ and evaluate

$$\iint_D xy(1-x-y)^{1/2}\,dx\,dy. \qquad \text{Ans: } 16/945$$

4. Prove that the area in the positive quadrant bounded by the curves $y^2=4ax, y^2=4bx, xy=c^2$ and $xy=d^2$ is $\frac{1}{3}\left(d^2-c^2\right)\log\left(\frac{b}{a}\right); d>c, b>a.$

5. Determine the value of the integral $\iiint_D e^{\sqrt{x^2+y^2+z^2}}\,dV$ where D is the region bounded by the planes

$y = 0, z = 0, y = x$ and the sphere $x^2 + y^2 + z^2 = 9.$ Ans: $\frac{\pi\left(5e^3 - 2\right)}{4}$

Hint: $0 \le r \le 3; 0 \le \theta \le \pi/4; 0 \le \phi \le \pi/2$

4.4 APPLICATIONS

4.4.1 AREA

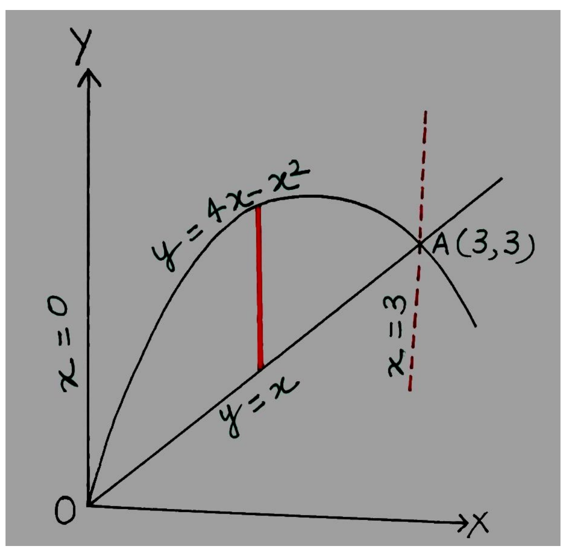
4.4.1.1 Cartesian Co-ordinates: The area A of the region bounded by two curves $y = f_1(x), y = f_2(x)$ and the lines $x = a, x = b$ is given by $A = \int_a^b \int_{f_1(x)}^{f_2(x)} dy\,dx$.

4.4.1.2 Polar Co-ordinates: The area A of the region bounded by two curves $r = f_1(\theta), r = f_2(\theta)$ and the lines $\theta = \alpha, \theta = \beta$ is given by $A = \int_\alpha^\beta \int_{f_1(\theta)}^{f_2(\theta)} r\,dr\,d\theta$.

4.4.1.3 Solved Examples

Example 1: Find the area lying between the parabola $y = 4x - x^2$ and the line $y = x$.

Solution: Solving the equations of given curves, we get $x = 0,3.$



Selecting the vertical strip, the required area lies between $x = 0, x = 3$ and $y = x, y = 4x - x^2$.

Therefore, required area

$$A=\int_0^3\int_x^{4x-x^2}dy\,dx$$

$$A=\int_0^3[y]_x^{4x-x^2}dx$$

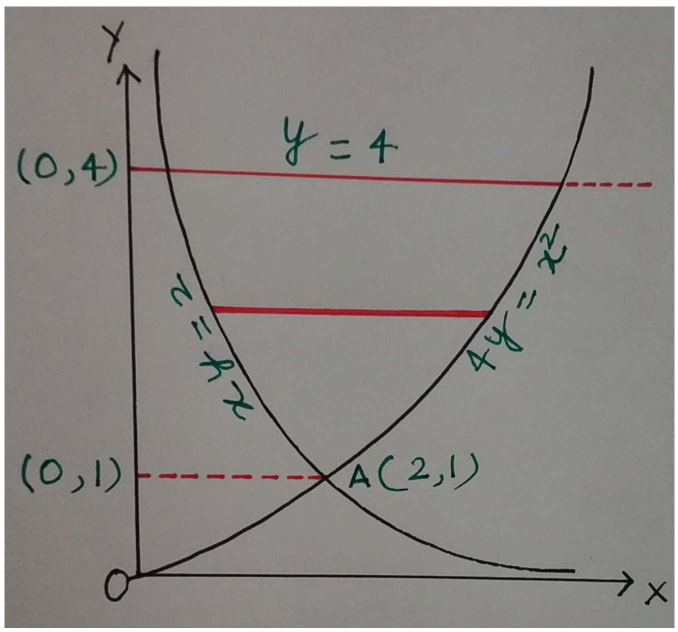
$$A=\int_0^3\left(3x-x^2\right)dx$$

$$A=\left[\frac{3x^2}{2}-\frac{x^3}{3}\right]_0^3=\frac{27}{2}-9=\frac{9}{2}.$$

Example 2: Determine the area of region bounded by the curves $xy=2, 4y=x^2, y=4$.

Solution: Selecting horizontal strip, the required area lies between

$xy=2, 4y=x^2, y=4$ alongwith point of intersection of curves $xy=2, 4y=x^2$, *i.e.*, $x=2, y=1$.



Therefore, required area

$$A=\int_{y=1}^{4}\int_{x=2/y}^{2\sqrt{y}}dx\,dy$$

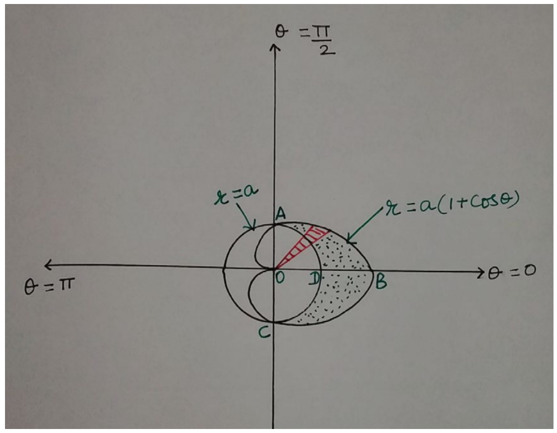
$$A=\int_1^4\left(2\sqrt{y}-2/y\right)dy$$

$$A=2\left(\frac{2}{3}y^{3/2}-\log y\right)_1^4$$

$$A=2\left[\left(\frac{16}{3}-2\log 2\right)-\frac{2}{3}\right]=\frac{28}{3}-4\log 2$$

Example 3: Find, by double integration, the area lying inside the cardioid $r=a(1+\cos\theta)$ and outside the circle $r=a$.

Solution: Since the bounded region is symmetric about initial line, we calculate the area lying above the initial line only.



Required area

$$A = 2\int_{\theta=0}^{\pi/2}\int_{r(circle)}^{r(cardioid)} r\,d\theta\,dr$$

$$A = 2\int_{\theta=0}^{\pi/2}\int_{a}^{a(1+\cos\theta)} r\,d\theta\,dr$$

$$A = 2\int_{\theta=0}^{\pi/2}\left[\frac{r^2}{2}\right]_a^{a(1+\cos\theta)} d\theta$$

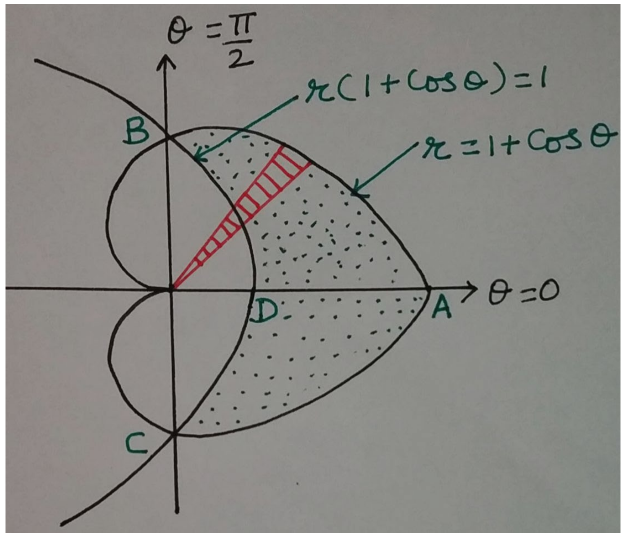
$$A = a^2\int_{\theta=0}^{\pi/2}\left[(1+\cos\theta)^2-1\right]d\theta$$

$$A = a^2\int_{\theta=0}^{\pi/2}\left[\cos^2\theta+2\cos\theta\right]d\theta$$

$$A = a^2\left(\frac{1}{2}.\frac{\pi}{2}+2\right) = \frac{a^2}{4}(\pi+8)\left(\text{using Walli's formula}\right)$$

Example 4: Find by the double integration, the area lying inside a cardioid $r = (1+\cos\theta)$ and outside the parabola $r(1+\cos\theta) = 1$.

Solution: Since bounded region is symmetric about initial line,



Therefore, required area=2area above the initial line

$$A = 2\int_{\theta=0}^{\pi/2}\int_{r=parabola}^{r=cardioid} r\,dr\,d\theta$$

$$A = 2\int_{0}^{\pi/2}\left(\frac{r^2}{2}\right)_{\frac{1}{1+\cos\theta}}^{1+\cos\theta} d\theta$$

$$A = \int_{0}^{\pi/2}\left[(1+\cos\theta)^2 - \frac{1}{(1+\cos\theta)^2}\right]d\theta$$

$$A = \int_{0}^{\pi/2}(1+\cos\theta)^2\,d\theta - \int_{0}^{\pi/2}\frac{1}{(1+\cos\theta)^2}\,d\theta$$

$$A = \int_{0}^{\pi/2}\left(1+\cos^2\theta+2\cos\theta\right)d\theta - \frac{1}{4}\int_{0}^{\pi/2}\sec^4\frac{\theta}{2}\,d\theta$$

Let $I_1 = \int_0^{\pi/2} \left(1 + \cos^2\theta + 2\cos\theta\right) d\theta$

and $I_2 = \frac{1}{4}\int_0^{\pi/2} \sec^4 \frac{\theta}{2} d\theta$, then

$$I_1 = \int_0^{\pi/2} \left(1 + \cos^2\theta + 2\cos\theta\right) d\theta$$

$$I_1 = \frac{\pi}{2} + \frac{1}{2}.\frac{\pi}{2} + 2 = \frac{3\pi}{4} + 2$$

$$I_2 = \frac{1}{4}\int_0^{\pi/2} \sec^4 \frac{\theta}{2} d\theta$$

$$I_2 = \frac{1}{4}\int_0^{\pi/4} \sec^4 \phi .2d\phi \qquad let \frac{\theta}{2} = \phi$$

$$\Rightarrow d\theta = 2d\phi$$

$$I_2 = \frac{1}{2}\int_0^{\pi/4} \left(1 + \tan^2\phi\right)\sec^2\phi \, d\phi \qquad let\, t = \tan\phi$$

$$\Rightarrow dt = \sec^2\phi d\phi$$

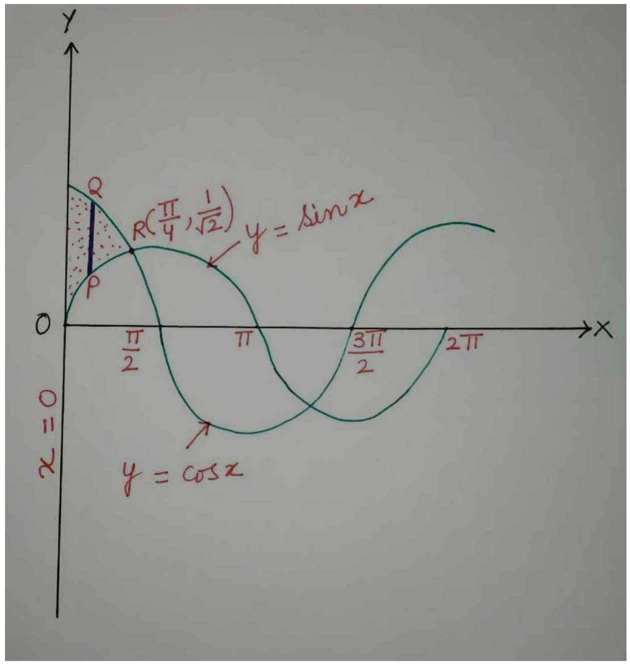
$$I_2 = \frac{1}{2}\int_0^{1} \left(1 + t^2\right) dt$$

$$I_2 = \frac{1}{2}\left(t + \frac{t^3}{3}\right)_0^1 = \frac{2}{3}$$

Hence required area= $\frac{3\pi}{4} + 2 - \frac{2}{3} = \frac{3\pi}{4} + \frac{4}{3}$.

Example 5: Find the area bounded by the lines $y = \sin x, y = \cos x, x = 0$.

Solution: Clearly, dotted portion is the required area.



Therefore,

$$A = \int_0^{\pi/4} \int_{\sin x}^{\cos x} dy\, dx$$

$$A = \int_0^{\pi/4} (\cos x - \sin x) dx$$

$$A = (\sin x + \cos x)_0^{\pi/4}$$

$$A = \left(\frac{1}{\sqrt{2}} - 0\right) + \left(\frac{1}{\sqrt{2}} + 1\right) = \left(\sqrt{2} - 1\right)$$

4.4.1.4 Practice problems

1. Find the area between the curves $y = x^2$ and $y = x^3$. Ans:1/12

2. Using double integral find the area bounded by the curves $y = 1 - x^2$ and $y = x^2 - 3$. Ans: $\frac{16\sqrt{2}}{3}$

Hint: $-\sqrt{2} \le x \le \sqrt{2}; x^2 - 3 \le y \le 1 - x^2$

3. Find, by double integration, the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Ans: πab

4. Evaluate the area enclosed between the parabola $y = x^2$ and the straight line $y = x$. Ans: $\frac{1}{6}$

5. Find, by double integration, the area of the region enclosed by the curves $x^2 + y^2 = a^2; x + y = a$ in the first quadrant. Ans: $\dfrac{a^2(\pi - 2)}{4}$

6. Show, by double integration, that the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\dfrac{16}{3}a^2$.

7. Find the area bounded by the circles $r = 2\sin\theta$ and $r = 4\sin\theta$. Ans: 3π

8. Find the area of one loop of the lemniscate $r^2 = a^2 \cos 2\theta$. Ans: $\dfrac{a^2}{2}$

9. Find the area of the loop of the curve $x^3 + y^3 = 3xy$. Ans: $\dfrac{3a^2}{2}$

4.5 RELATED LINKS

1. https://www.youtube.com/watch?time_continue=1&v=4rc3w1sGoNU&feature=emb_logo
2. https://www.youtube.com/watch?time_continue=1&v=wtY5fx6VMGQ&feature=emb_logo
3. https://www.youtube.com/watch?v=6ntZ1KQL04A&feature=emb_logo
4. https://www.khanacademy.org/math/multivariable-calculus/integrating-multivariable-functions/double-integrals-topic/v/double-integral-1
5. https://www.khanacademy.org/math/multivariable-calculus/integrating-multivariable-functions/double-integrals-topic/v/double-integrals-2
6. https://www.khanacademy.org/math/multivariable-calculus/integrating-multivariable-functions/double-integrals-topic/v/double-integrals-3
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8. https://www.khanacademy.org/math/multivariable-calculus/integrating-multivariable-functions/double-integrals-topic/v/double-integrals-5
9. https://www.khanacademy.org/math/multivariable-calculus/integrating-multivariable-functions/double-integrals-topic/v/double-integrals-6
10. https://www.khanacademy.org/math/multivariable-calculus/integrating-multivariable-functions/double-integrals-a/a/double-integrals
11. https://www.khanacademy.org/math/multivariable-calculus/integrating-multivariable-functions/double-integrals-a/a/double-integrals-over-non-rectangular-regions
12. https://www.khanacademy.org/math/multivariable-calculus/integrating-multivariable-functions/double-integrals-a/a/double-integrals-beyond-volume
13. https://www.khanacademy.org/math/multivariable-calculus/integrating-multivariable-functions/double-integrals-a/v/polar-coordinates-1
14. https://www.khanacademy.org/math/multivariable-calculus/integrating-multivariable-functions/double-integrals-a/a/double-integrals-in-polar-coordinates