KIET GROUP OF INSTITUTIONS, DELHI – NCR, GHAZIABAD

B. Tech. (I SEM), 2023-24

Calculus for Engineers (K24AS11)

UNIT1(Differential Calculus I)

Syllabus: Introduction to Limits, continuity and differentiability for function of two variables, Higher order Partial derivatives, Euler's Theorem for homogeneous functions, Total derivative of composite functions.

Course Outcomes:

S.NO.	Course Outcome	BL
1	Apply the concept of partial differentiation in application of homogeneous and composite functions	2,3

CONTENT

UNIT I. Differential Calculus - I

Introduction of limits, Continuity and differentiability for function of two variables

LIMIT

A function $f(x,y)$ is said to have a limit L as the point (x,y) approaches (a,b) and is denoted as

$$\lim_{(x,y)\to(a,b)} f(x,y) = L. \quad \text{or} \quad \lim_{\substack{x\to a \\ y\to b}} f(x,y) = L.$$

Method of Obtaining limit

1. Evaluate $\lim f(x,y)$ along path I: $x\to a$ and $y\to b$
2. Evaluate $\lim f(x,y)$ along path II : $y\to b$ and $x\to a$

If the limit values along path I and II are same, then the limit exist. Otherwise not

3. If $a=0$, $b=0$, evaluate limit along say path $y=mx$ and $y=mx^n$.

Q1 Evaluate $\lim_{(x,y)\to(1,2)} \frac{x^2+2y}{x^2+y^2}$

Sol $\lim_{\substack{x\to 1 \\ y\to 2}} \frac{x^2+2y}{x^2+y^2}$.

Case 1 Put $x=1$

$\lim_{y\to 2} \frac{1+2y}{1+y^2} = \frac{1+4}{1+4} = \frac{5}{5} = 1 = f_1$.

Case 2 Put $y=2$

$\lim_{x\to 1} \frac{x^2+2\times 2}{x^2+4} = \frac{1+4}{1+4} = \frac{5}{5} = 1 = f_2$.

~~Case 3~~ $\Rightarrow f_1 = f_2 = 1$

$\Rightarrow$ limit exist

$\therefore \lim_{(x,y)\to(1,2)} \frac{x^2+2y}{x^2+y^2} = 1$

Q2. Evaluate $\lim_{\substack{x\to 0 \\ y\to 0}} \frac{x^2 y}{x^4+y^2}$.

Sol Case 1 Put $x=0$

$\lim_{y\to 0} \frac{0\cdot y}{0+y^2} = 0 = f_1$

Case 2 Put $y=0$

$\lim_{x\to 0} \frac{x^2\cdot 0}{x^4+0} = 0 = f_2$

Case 3 Put $y=mx$

$\lim_{x\to 0} \frac{x^2(mx)}{x^4+(mx)^2} = \lim_{x\to 0} \frac{m x^3}{x^4+m^2x^2} = 0 = f_3$

Case 4 Put $y=mx^2$.

$\lim_{x\to 0} \frac{x^2(mx^2)}{x^4+(mx^2)^2} = \lim_{x\to 0} \frac{m \cancel{x^4}}{\cancel{x^4}(1+m^2)} = \frac{m}{1+m^2} = f_4$

$\Rightarrow f_1 = f_2 = f_3 \neq f_4$

$\Rightarrow$ limit does not exist

Q3 Evaluate

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)$$

Sol Put $x=0$

Case 1 $\lim_{y\to 0} (0+y^2) = 0 = f_1$

Case 2. Put $y=0$

$\lim_{x\to 0} (x^2+0) = 0 = f_2$

Case 3 Along $y=mx$

$\lim_{x\to 0} x^2+(mx)^2$

$= \lim_{x\to 0} x^2(1+m^2) = 0 = f_3$

Case 4 Along $y=mx^2$

$\lim_{x\to 0} x^2+(mx^2)^2$

$= \lim_{x\to 0} x^2(1+m^2x^2) = 0 = f_4$

$\Rightarrow f_1 = f_2 = f_3 = f_4.$

$\Rightarrow$ limit exists

$\therefore \lim_{(x,y)\to(0,0)} x^2+y^2 = 0$

Q4. Evaluate $\lim_{\substack{x\to 1\\ y\to 2}} \frac{3x^2y}{x^2+y^2+5}$

Sol Case I Put $x=1$

$\lim_{y\to 2} \frac{3(1^2)y}{(1)^2+y^2+5} = \frac{3\times 2}{1+4+5} = \frac{\cancel{6}\,3}{\cancel{10}\,5} = \frac{3}{5} = f_1$

Case 2 Put $y=2$.

$\lim_{x\to 1} \frac{3x^2(2)}{x^2+(2)^2+5} = \frac{6(1)^2}{4+(1^2)+5} = \frac{\cancel{6}\,3}{\cancel{10}\,5} = \frac{3}{5} = f_2$

$\Rightarrow f_1 = f_2 \Rightarrow$ limit exists & value is $\frac{3}{5}$.

Q5 Evaluate $\lim\limits_{\substack{x\to\infty \\ y\to 3}} \frac{2x-3}{x^3+4y^3}$

Sol. $\lim\limits_{\substack{x\to\infty \\ y\to 3}} \frac{2x-3}{x^3+4y^3}$.

Case 1 Put $x \to \infty$.

$$\lim_{y\to 3}\left[\lim_{x\to\infty} \frac{2x-3}{x^3+4y^3}\right]$$

$$\lim_{y\to 3}\left[\lim_{x\to\infty} \frac{\frac{2}{x^2}-\frac{3}{x^3}}{1+4\left(\frac{y}{x}\right)^3}\right]$$

$$= \lim_{y\to 3}\left(\frac{0-0}{1+4(0)}\right) = 0 = f_1$$

Case 2 Put $y = 3$.

$$\lim_{x\to\infty}\left[\lim_{y\to 3} \frac{2x-3}{x^3+4y^3}\right]$$

$$= \lim_{x\to\infty}\left[\frac{2x-3}{x^3+108}\right]$$

$$= \lim_{x\to\infty} \frac{\frac{2}{x^2}-\frac{3}{x^3}}{1+\frac{108}{x^3}} = \frac{0-0}{1+0} = 0 = f_2.$$

$\Rightarrow f_1 = f_2$

Hence limit exist with value zero.

Continuity of two variables

A function $f(x,y)$ is said to be continuous at a point (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

Test of Continuity at a Point (a,b)

1. $f(a,b)$ should be well defined
2. $\lim f(x,y)$ as $(x,y)\to(a,b)$ should exists (must be unique and same along any path)

Q1 If $f(x,y) = \begin{cases} \dfrac{x^2+2y}{x+y^2} \\ 1 \end{cases}$ when $x=1, y=2$.

Sol ① $f(x,y) \Rightarrow f(a,b) = f(1,2) = \dfrac{1^2+2(2)}{1+(2)^2}$

$$= \frac{1+4}{1+4} = \frac{5}{5} = 1$$

$\therefore f(a,b) = 1 = f_1$

② $\displaystyle\lim_{\substack{x\to 1\\ y\to 2}} \frac{x^2+2y}{x+y^2}$

Case 1 Put $x=1$ $\displaystyle\lim_{y\to 2} \frac{1+2y}{1+y^2} = \frac{1+4}{1+4} = \frac{5}{5} = 1 = f_1$

Case 2 Put $y=2$ $\displaystyle\lim_{x\to 1} \frac{x^2+4}{x+4} = \frac{1+4}{1+4} = \frac{5}{5} = 1 = f_2$.

$\Rightarrow f_1 = f_2$ $\therefore$ limit exist

$\therefore \displaystyle\lim_{\substack{x\to 1\\ y\to 2}} \frac{x^2+2y}{x+y^2} = 1 = f(1,2)$

$\Rightarrow f(x,y)$ is Continuous

Q.2 If $f(x,y) = \dfrac{x^3-y^3}{x^2+y^2}$ when $x \neq 0, y \neq 0$

and $f(x,y) = 0$ when $x=0, y=0$,

find out whether the function $f(x,y)$ is continuous at Origin

Sol. Case I First calculate the limit of the function

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3-y^3}{x^2+y^2} = \lim_{y\to 0}\left(\lim_{x\to 0}\frac{x^3-y^3}{x^2+y^2}\right)$$

$$= \lim_{y\to 0}\left(\frac{-y^3}{y^2}\right) = \lim_{y\to 0}(-y) = 0 = f_1$$

Case 2

$$\lim_{\substack{y\to 0\\ x\to 0}} \frac{x^3-y^3}{x^2+y^2} = \lim_{x\to 0}\left(\lim_{y\to 0}\frac{x^3-y^3}{x^2+y^2}\right) = \lim_{x\to 0}\left(\frac{x^3}{x^2}\right)$$

$$= \lim_{x\to 0} x = 0 = f_2$$

Case 3

$$\lim_{\substack{y=mx\\ x\to 0}} \frac{x^3-y^3}{x^2+y^2} = \lim_{x\to 0}\left[\lim_{y=mx}\frac{x^3-y^3}{x^2+y^2}\right]$$

$$= \lim_{x\to 0}\left[\frac{x^3-m^3x^3}{x^2+m^2x^2}\right] = \lim_{x\to 0}\frac{x^3(1-m^3)}{x^2(1+m^2)}$$

$$= \lim_{x\to 0}\frac{x(1-m^3)}{(1+m^2)} = 0 = f_3$$

Case 4

$$\lim_{\substack{y=mx^2\\ x\to 0}} \frac{x^3-y^3}{x^2+y^2} = \lim_{x\to 0}\left[\lim_{y=mx^2}\frac{x^3-y^3}{x^2+y^2}\right]$$

$$= \lim_{x\to 0}\left[\frac{x^3-m^3x^6}{x^2+m^2x^4}\right] = \lim_{x\to 0}\frac{x^3(1-m^3x^3)}{x^2(1+m^2x^2)}$$

$$= \lim_{x\to 0}\frac{x(1-m^3x^3)}{1+m^2x^2} = 0 = f_4$$

$\Rightarrow$ $f_1 = f_2 = f_3 = f_4$

$\Rightarrow$ limit exist and equal to zero

$\Rightarrow$ $\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = 0 = f(0,0)$

$\Rightarrow$ function f is Continuous at the Origin

Q3 Discuss the Continuity of the function

$$f(x,y) = \frac{x}{\sqrt{x^2+y^2}} \quad \text{when } (x,y) \neq (0,0)$$

and $f(x,y) = 2$ when $(x,y) = (0,0)$

Sol: At first find the limit

(1) $$\lim_{\substack{x\to 0 \\ y\to 0}} \frac{x}{\sqrt{x^2+y^2}} = \lim_{y\to 0}\left(\lim_{x\to 0} \frac{x}{\sqrt{x^2+y^2}}\right) = \lim_{y\to 0} 0 = 0 = f_1$$

(2) $$\lim_{\substack{y\to 0 \\ x\to 0}} \frac{x}{\sqrt{x^2+y^2}} = \lim_{x\to 0}\left(\lim_{y\to 0} \frac{x}{\sqrt{x^2+y^2}}\right) = \lim_{x\to 0} 1 = 1 = f_2$$

$\Rightarrow f_1 \neq f_2$

$\Rightarrow$ The function is discontinuous at the origin

Differentiability in two Variables

~~If~~ The function $f(x,y)$ is differentiable at point (a,b) if

$$\lim_{\substack{h\to 0\\ K\to 0}} \frac{f(a+h,b+K)-f(a,b)-h\,f_x(a,b)-K\,f_y(a,b)}{\sqrt{h^2+K^2}}$$ exis

* If $f(x,y)$ is differentiable, then the Partial derivatives f_x and f_y both exist and are finite

$$f_x = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{K\to 0} \frac{f(x,y+K)-f(x,y)}{K}.$$

Q1 Let $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

Show that $f(x,y)$ is Continuous but not differentiable at $(0,0)$

Sol To show $\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = f(0,0)$

$$f_1 = \lim_{\substack{x\to 0\\ y\to 0}} = \lim_{y\to 0}\left(\lim_{x\to 0} \frac{xy}{\sqrt{x^2+y^2}}\right) = 0$$

$$f_2 = \lim_{\substack{x\to 0\\ y\to 0}} = \lim_{x\to 0}\left(\lim_{y\to 0} \frac{xy}{\sqrt{x^2+y^2}}\right) = 0$$

$$f_3 = \lim_{\substack{x\to 0\\ y=mx.}} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{x\to 0} \frac{x(mx)}{\sqrt{x^2+(mx)^2}} = 0.$$

$$f_4 = \lim_{\substack{x\to 0\\ y=mx^2}} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{x\to 0} \frac{x(mx^2)}{\sqrt{x^2+(mx^2)^2}}$$

$$= \lim_{x\to 0} \frac{mx^3}{\sqrt{x^2+mx^4}} = 0$$

$\Rightarrow$ $f_1 = f_2 = f_3 = f_4$

$\therefore$ limit exists and equal to zero

$\therefore$ $\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = (0,0)$

$\Rightarrow$ $f(x,y)$ is Continuous at $(0,0)$

Test of differentiability

$$f_x(0,0) = \lim_{x\to 0} \frac{f(0+h,0) - f(0,0)}{h} = 0$$

$$f_y(0,0) = \lim_{K\to 0} \frac{f(0,0+K) - f(0,0)}{K} = 0$$

$$\lim_{(h,K)\to(0,0)} \frac{f(0+h, 0+K) - f(0,0) - h f_x(0,0) - K f_y(0,0)}{\sqrt{h^2+K^2}}$$

$$\Rightarrow \lim_{(h,K)\to(0,0)} \frac{f(h,K)}{\sqrt{h^2+K^2}}$$

$$= \lim_{(h,K)\to(0,0)} \frac{\frac{hK}{\sqrt{h^2+K^2}}}{\sqrt{h^2+K^2}} = \lim_{(h,K)\to(0,0)} \frac{hK}{h^2+K^2}$$

let $K = mh$

$$\lim_{h\to 0} \frac{mh^2}{h^2+m^2h^2} = \lim_{h\to 0} \frac{m}{1+m^2}$$

$$= \frac{m}{1+m^2}$$

Limit depends on m

$\Rightarrow$ $f(x,y)$ is not Unique

$\Rightarrow$ function is not differentiable but it is Continuous.

Q2 Show that the function

$$f(x,y) = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is not differentiable at Origin although Partial derivatives $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ exist at $(0,0)$.

Sol. $f_x(0,0) = \lim_{h\to 0} \dfrac{f(0+h,0) - f(0,0)}{h}$

$$= \lim_{h\to 0} \frac{\frac{h^3-0}{h^2-0}}{h} = \lim_{h\to 0} \frac{h^3}{h^3} = 1$$

$$f_y(0,0) = \lim_{K\to 0} \frac{f(0,0+K) - f(0,0)}{K} = -1.$$

Using formula

$$\lim_{(h,K)\to(0,0)} \frac{f(0+h,0+K) - f(0,0) - h f_x(0,0) - K f_y(0,0)}{\sqrt{h^2+K^2}}$$

$$\lim_{(h,K)\to(0,0)} \frac{\frac{h^3-K^3}{h^2+K^2} - 0 - h + K.}{\sqrt{h^2+K^2}}$$

At Origin, Along $K = mh$.

$$= \lim_{h\to 0} \frac{\frac{h^3 - m^3h^3}{h^2+m^2h^2} - h + mh}{\sqrt{h^2+m^2h^2}}$$

$$= \lim_{h\to 0} \frac{h\frac{(1-m^3)}{1+m^2} - h + mh}{h\sqrt{1+m^2}} = \lim_{h\to 0} \frac{\frac{1-m^3}{1+m^2} - 1 + m}{\sqrt{1+m^2}}$$

$\Rightarrow$ limit depends on m
$\Rightarrow$ limit is not Unique

$\Rightarrow f(x,y)$ is not differentiable at $(0,0)$

1.2 Partial Derivatives and its Applications: The concept of partial differentiation is applied for function of two or more independent variables. Partial differentiation is the process of finding various orders partial derivatives of given function. Partial derivatives are very useful in determining Maxima and Minima, Jacobian, Series expansion of function of several variables, Error and Approximation, In solution of wave and heat equations etc.

1.2.1 First order partial derivatives:

If$u = f(x, y)$ be any function of two independent variables x, yhen first order partial derivative of u with respect to $'x'$ is obtained by differentiating u with respect to 'x' treating y and function of 'y' as constant and is denoted by $\frac{\partial u}{\partial x}$

Similarly fist order partial derivative of u with respect to 'y' is obtained by differentiating u with respect to 'y' treating 'x' and functions of 'x' as constant and is denoted by$\frac{\partial u}{\partial y}$.

Example-1: If $u = x^2 + 2xy - y^2 + 2x - 3y + 5$ then find $\frac{\partial u}{\partial x} and \frac{\partial u}{\partial y}$.

Solution: Here we have given $u = x^2 + 2xy - y^2 + 2x - 3y + 5$

Therefore, to get value of $\frac{\partial u}{\partial x}$ we shall differentiate u with respect to 'x' treating 'y' as constant

$$\boldsymbol{\frac{\partial u}{\partial x} = 2x + 2y + 2}$$

Also, to find $\frac{\partial u}{\partial y}$ we shall differentiate u with respect to 'y' treating 'x' as constant

$$\boldsymbol{\frac{\partial u}{\partial y} = 2x - 2y - 3}.$$

1.2.2 Second and higher order partial derivatives:

If $u = f(x, y)$ be any function of two independent variables x, y then second order partial derivative of u with respect to $'x'$ is obtained by differentiating u with respect to 'x' twice in succession treating 'y' and function of 'y' as constant each time and is denoted by $\frac{\partial^2 u}{\partial x^2}$.

Similarly second order partial derivative of u with respect to 'y' is obtained by differentiating u with respect to 'y' twice treating 'x' and functions of 'x' as constant each time and is denoted by$\frac{\partial^2 u}{\partial y^2}$.

Here if we differentiate $\frac{\partial u}{\partial y}$ with respect to 'x' treating 'y' as constant then we get second order derivative of u denoted as$\frac{\partial^2 u}{\partial x \partial y}$. Similarly differentiating $\frac{\partial u}{\partial x}$ with respect to 'y' treating 'x' and function of 'x' as constant we get second order derivative of u denoted as $\frac{\partial^2 u}{\partial y \partial x}$. It is interesting to see that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$. In the similar way third and other higher order partial derivatives can be evaluated.

Example-2: Find all first and second order partial derivatives of$u(x, y) = x^y$. Hence show that$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

Solution: Here we have $u(x, y) = x^y \dots \dots \dots \dots (1)$

Differentiating equation (1) partially with respect to ‘x’ and ‘y’ we get

$$\frac{\partial u}{\partial x} = yx^{y-1} \; \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots. \; (2)$$

$$\frac{\partial u}{\partial y} = x^y logx \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.. (3)$$

Differentiating equation (2) partially with respect to ‘x’ and ‘y’ we get

$$\frac{\partial^2 u}{\partial x^2} = y(y-1)x^{y-2} \; \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots. \; (4)$$

$$\boldsymbol{\frac{\partial^2 u}{\partial y \partial x} = yx^{y-1}\, logx + x^{y-1}} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots. \; (5)$$

Now differentiating equation (3) partially with respect to ‘y’ and ‘x’ we get

$$\frac{\partial^2 u}{\partial y^2} = x^y (logx)^2 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots. \; (6)$$

$$\frac{\partial^2 u}{\partial x \partial y} = yx^{y-1}\, logx + x^y \left(\frac{1}{x}\right)$$

$$\boldsymbol{\frac{\partial^2 u}{\partial x \partial y} = yx^{y-1}\, logx + x^{y-1}} \dots\dots\dots\dots\dots\dots\dots. (7)$$

From equations (5) & (7), it is clear that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

Example-3: If $u = \log(tanx + tany + tanz)$, then prove that $sin2x \frac{\partial u}{\partial x} + sin2y \frac{\partial u}{\partial y} + sin2z \frac{\partial u}{\partial z} = 2$.

Solution: In this problem, we have $u = \log(tanx + tany + tanz)$ … … … … … … … (1)

Differentiating equation (1) partially with respect to ‘x’ we get

$$\frac{\partial u}{\partial x} = \frac{1}{tanx + tany + tanz}(sec^2 x) \dots\dots\dots\dots\dots\dots\dots\dots.. (2)$$

Similarly differentiating (1) partially with respect to ‘y’ and ‘z’ we get

$$\frac{\partial u}{\partial y} = \frac{1}{tanx + tany + tanz}(sec^2 y) \dots\dots\dots\dots\dots\dots\dots\dots.. (3)$$

$$\frac{\partial u}{\partial z} = \frac{1}{tanx + tany + tanz}(sec^2 z) \dots\dots\dots\dots\dots\dots\dots\dots.. (4)$$

Multiplying equations (2), (3) and (4) with $sin2x, sin2y \; and \; sin2z$ respectively and then adding, we get

$$sin2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + sin2z \frac{\partial u}{\partial z} = sin2x \left\{\frac{1}{tanx+tany+tanz}(sec^2 x)\right\} +$$

$$sin2y \left\{\frac{1}{tanx+tany+tanz}(sec^2 y)\right\} + sin2z \left\{\frac{1}{tanx+tany+tanz}(sec^2 z)\right\}$$

$sin2x\ \frac{\partial u}{\partial x}$+sin2y$\frac{\partial u}{\partial y} + sin2z\ \frac{\partial u}{\partial z} = \frac{2tanx}{tanx+tany+tanz} + \frac{2tany}{tanx+tany+tanz} + \frac{2tanz}{tanx+tany+tanz} +$

$$\boldsymbol{sin2x\ \frac{\partial u}{\partial x} + \mathrm{sin2y}\frac{\partial u}{\partial y} + sin2z\ \frac{\partial u}{\partial z} = 2.}$$

Example-4: If $x = rcos\theta, y = rsin\theta$, then find (a) $\left(\frac{\partial x}{\partial r}\right)_\theta\ and\ \left(\frac{\partial y}{\partial \theta}\right)_r$, and (b) $\left(\frac{\partial r}{\partial x}\right)_y\ and\ \left(\frac{\partial \theta}{\partial y}\right)_x$.

Solution: Here we have

$x = rcos\theta\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$,

and $y = rsin\theta\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2)$

Differentiating equation (1) partially with respect to ‘r’ treating ‘θ’ as constant, we get

$$\boldsymbol{\left(\frac{\partial x}{\partial r}\right)_\theta = cos\theta}$$

Similarly differentiating (2) partially with respect to ‘θ’ treating ‘r’ as constant we get

$$\boldsymbol{\left(\frac{\partial y}{\partial \theta}\right)_r = rcos\theta}$$

Again, from equations (1) and (2), we get

$r^2 = x^2 + y^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ (3)

and, $\theta = \tan^{-1}\left(\frac{y}{x}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ (4)

Differentiating equation (3) partially with respect to ‘x’ treating ‘y’ as constant, we get

$$2r\left(\frac{\partial r}{\partial x}\right)_y = 2x \quad \boldsymbol{or}\ \boldsymbol{\left(\frac{\partial r}{\partial x}\right)_y = \frac{x}{r}}$$

Similarly Differentiating equation (3) partially with respect to ‘y’ treating ‘x’ as constant, we get

$$\left(\frac{\partial \theta}{\partial y}\right)_x = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right)$$

$$\boldsymbol{\left(\frac{\partial \theta}{\partial y}\right)_x = \frac{x}{x^2+y^2}}\ .$$

Example-5: If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that

a). $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$, and b) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$.

Solution: Here we have, $u = \log(x^3 + y^3 + z^3 - 3xyz) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$

Differentiating equation partially with respect to ‘x’ we get

$$\frac{\partial u}{\partial x}=\frac{1}{(x^3+y^3+z^3-3xyz)}(3x^2-3yz)\dots\dots\dots\dots\dots\dots\dots\dots(2)$$

Similarly differentiating equation (1) partially with respect to 'y' and 'z' we get

$$\frac{\partial u}{\partial y}=\frac{1}{(x^3+y^3+z^3-3xyz)}(3y^2-3zx)\dots\dots\dots\dots\dots\dots\dots\dots(3)$$

$$\frac{\partial u}{\partial z}=\frac{1}{(x^3+y^3+z^3-3xyz)}(3z^2-3xy)\dots\dots\dots\dots\dots\dots\dots\dots(4)$$

Adding equations (2), (3) & (4) we get

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{1}{(x^3+y^3+z^3-3xyz)}(3x^2+3y^2+3z^2-3xy-3yz-3zx)$$

$$\boldsymbol{\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3}{x+y+z}}\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (5)$$

Now,

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=-\frac{3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}$$

$$\boldsymbol{\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=-\frac{9}{(x+y+z)^2}}.$$

Example-6:Find $\frac{\partial u}{\partial r}$ *and* $\frac{\partial u}{\partial \theta}$ if $u=e^{rcos\theta}.\cos(rsin\theta)$

Solution: We have $u=e^{rcos\theta}.\cos(rsin\theta)\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(1)$

Differentiating equation (1) partially with respect to 'r' we get

$$\frac{\partial u}{\partial r}=cos\theta.e^{rcos\theta}.\cos(rsin\theta)+e^{rcos\theta}.\{-\sin(rsin\theta)\}.sin\theta$$

$$\frac{\partial u}{\partial r}=e^{rcos\theta}\{cos\theta.\cos(rsin\theta)-\sin\theta.\sin(rsin\theta\}$$

$$\frac{\partial u}{\partial r}=e^{rcos\theta}.cos(rsin\theta+\theta)\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(2)$$

Now, differentiating equation (1) partially with respect to 'θ' we get

$$\frac{\partial u}{\partial \theta} = -rsin\theta . e^{rcos\theta} . \cos(rsin\theta) + e^{rcos\theta} . \{-\sin(rsin\theta)\} . rcos\theta$$

$$\frac{\partial u}{\partial \theta} = -re^{rcos\theta}\{sin\theta . . \cos(rsin\theta) + \sin(rsin\theta) . cos\theta\}$$

$$\frac{\partial u}{\partial \theta} = -re^{rcos\theta} . \sin(rsin\theta + \theta) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \quad (3)$$

1.3Total Derivative:

If $u = f(x, y)$ be any function in two variables x, y and if $\delta x, \delta y$ are small increment in values of variables x and y respectively. Then δu corresponding change in value of dependent variable u is given by

$$\delta u = f(x + \delta x, y + \delta y) - f(x, y)$$

$$\delta u = f(x + \delta x, y + \delta y) - f(x, y + \delta y) + f(x, y + \delta y) - f(x, y)$$

$$\delta u = \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x} \delta x + \frac{f(x, y + \delta y) - f(x, y)}{\delta y}$$

Taking limit$\delta x \to 0$ and $\delta y \to 0$, we get

$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$, which is called total derivative of $u = f(x, y)$.

Similarly if $u = f(x, y, z)$.then$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$

1.3.1Composite function and its Derivative: If$u = f(x, y)$, where $x = \varphi(t), y = \omega(t)$ then u iscalled a composite function of single variable t,then the value of total derivative of u with respect to t is given by

$$\frac{du}{dt} = \frac{\partial u}{\partial x} . \frac{dx}{dt} + \frac{\partial u}{\partial y} . \frac{dy}{dt}$$

Also if $z = f(x, y)$, where $x = \varphi(u, v), y = \omega(u, v)$ then z is called a composite function of two variables u , v and the value of total derivative of z with respect to u, v are given by

$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} . \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} . \frac{\partial y}{\partial u}$ and $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} . \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} . \frac{\partial y}{\partial v}$respectively.

Example-7: If $u = x^3 + y^3$where $x = acost$, $y = asint$. Find the value of $\frac{du}{dt}$ and hence verify the result.

Solution: Here we have, $u = x^3 + y^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..$ (1)

Where, $x = acost$, $y = asint$ $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ (2)

Therefore, u is a composite function in single variable t. So we shall have

$$\frac{du}{dt} = \frac{\partial u}{\partial x} . \frac{dx}{dt} + \frac{\partial u}{\partial y} . \frac{dy}{dt} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Now from equations (1) & (2)

$$\frac{\partial u}{\partial x} = 3x^2 \ \ \& \ \ \frac{\partial u}{\partial y} = 3y^2 \ \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (4)$$

$$\frac{dx}{dt} = -asint \quad \& \quad \frac{dy}{dt} = acost \ \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Using equations (4) & (5) in equation (3), we get

$$\frac{du}{dt} = 3x^2(-asint) + 3y^2(acost)$$

$$\frac{du}{dt} = 3a^3(sin^2t.cost - cos^2t.sint) \ldots\ldots.... \qquad (6)$$

{Putting values of x and y from equation (2)}

Again, from equations (1) and (2), we get

$$u = a^3cos^3t + a^3sin^3t$$

So, differentiating both sides with respect to t we get

$$\frac{du}{dt} = a^3[(3cos^2t).(-sint) + (3sint^2t).cost$$

$$\frac{du}{dt} = 3a^3(sin^2t.cost - cos^2t.sint) \ \ \ldots\ldots\ldots\ldots\ldots \quad (7)$$

From equations (6) & (7) result is verified

Example-8: If $u = x^2 - y^2 + sinyz$, where $y = e^x, z = logx$, then find $\frac{du}{dx}$.

Solution: Here we have, $u = x^2 - y^2 + sinyz \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$

Where $y = e^x, z = logx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (2)$

Here u will be composite function in single variable x. So we shall have

$$\frac{du}{dx} = \frac{\partial u}{\partial x}.\frac{dx}{dx} + \frac{\partial u}{\partial y}.\frac{dy}{dx} + \frac{\partial u}{\partial z}.\frac{dz}{dx} \ \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Now, differentiating equations (1) and (2), we get

$$\frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = -2y + zcosyz \,\&\, \frac{\partial u}{\partial z} = ycosyz \ \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (4)$$

$$\frac{dy}{dx} = e^x \& \frac{dz}{dx} = \frac{1}{x} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Using equations (4) and (5) in equation (3), we get

$$\frac{du}{dx} = 2x + (-2y + zcosyz)e^x + ycosyz\left(\frac{1}{x}\right)$$

Putting values of y & z from equation (2), we get

$$\frac{du}{dx} = 2x + (-2e^x + logx.cosyz)e^x + e^x\left(\frac{1}{x}\right)cosyz$$

Example-9: If $v = f(2x - 3y, 3y - 4z, 4z - 2x)$ compute the value of $6v_x + 4v_y + 3v_z$.

Solution: Given that $v = f(2x - 3y, 3y - 4z, 4z - 2x)$

Let $v = f(r, s, t)$ ……………………………………………………….(1)

Where $r = 2x - 3y, s = 3y - 4z, t = 4z - 2x$ ……………….(2)

So, from equation (2), we get

$$\frac{\partial r}{\partial x} = 2, \frac{\partial r}{\partial y} = -3, \frac{\partial r}{\partial z} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$\frac{\partial s}{\partial x} = 0, \frac{\partial s}{\partial y} = 3, \frac{\partial s}{\partial z} = -4 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$\frac{\partial t}{\partial x} = -2, \frac{\partial t}{\partial y} = 0, \frac{\partial t}{\partial z} = 4 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Now, $v_x = \frac{\partial v}{\partial x} = v_x = \frac{\partial v}{\partial r}.\frac{\partial r}{\partial x} + \frac{\partial v}{\partial s}.\frac{\partial s}{\partial x} + \frac{\partial v}{\partial t}.\frac{\partial t}{\partial x}$

$v_x = \frac{\partial f}{\partial r}(2) + \frac{\partial f}{\partial s}(0) + \frac{\partial f}{\partial t}(-2)$ Or $v_x = 2\left(\frac{\partial f}{\partial r} - \frac{\partial f}{\partial t}\right)$ ……………….(6)

$$v_y = \frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}.\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}.\frac{\partial s}{\partial y} + \frac{\partial v}{\partial t}.\frac{\partial t}{\partial y}$$

$v_y = \frac{\partial f}{\partial r}(-3) + \frac{\partial f}{\partial s}(3) + \frac{\partial f}{\partial t}(0)$ Or $v_y = 3\left(-\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s}\right)$ ……………….(7)

$$v_z = \frac{\partial v}{\partial z} = \frac{\partial v}{\partial r}.\frac{\partial r}{\partial z} + \frac{\partial v}{\partial s}.\frac{\partial s}{\partial z} + \frac{\partial v}{\partial t}.\frac{\partial t}{\partial z}$$

$v_z = \frac{\partial f}{\partial r}(0) + \frac{\partial f}{\partial s}(-4) + \frac{\partial f}{\partial t}(4)$ Or $v_z = 4\left(-\frac{\partial f}{\partial s} + \frac{\partial f}{\partial t}\right)$ ……………….(8)

From equations (6). (7) and (8), we get

$6v_x + 4v_y + 3v_z == 12\left\{\left(\frac{\partial f}{\partial r} - \frac{\partial f}{\partial t}\right) + \left(-\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s}\right) + \left(-\frac{\partial f}{\partial s} + \frac{\partial f}{\partial t}\right)\right\}$or $6v_x + 4v_y + 3v_z = 0.$

Example-10: If $u = u\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, then show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$

Solution: Here we have $u = u\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$ ………………..(1)

Let $u = u(r, s)$ ……………………………………………………(2)

$where\ r = \frac{y-x}{xy}$ or $r = \frac{1}{x} - \frac{1}{y}$ & s= $\frac{z-x}{xz}$ or $s = \frac{1}{x} - \frac{1}{z}$ … (3)

From equation(3), we get $\frac{\partial r}{\partial x} = -\frac{1}{x^2}, \frac{\partial r}{\partial y} = \frac{1}{y^2}, \frac{\partial r}{\partial z} = 0$ and $\frac{\partial s}{\partial x} = -\frac{1}{x^2}, \frac{\partial s}{\partial y} = 0, \frac{\partial s}{\partial z} = \frac{1}{z^2}$

Also, $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}.\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}.\frac{\partial s}{\partial x}$

$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\left(-\frac{1}{x^2}\right) + \frac{\partial u}{\partial s}\left(-\frac{1}{x^2}\right)$ or $x^2\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial r} - \frac{\partial u}{\partial s}$ ………………………….. (4)

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}.\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}.\frac{\partial s}{\partial y}$$

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\left(\frac{1}{y^2}\right) + \frac{\partial u}{\partial s}(0)$ or $y^2\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}$ ……………………….. (5)

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}.\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}.\frac{\partial s}{\partial z}$$

$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}\left(\frac{1}{z^2}\right)$ or $z^2\frac{\partial u}{\partial z} = \frac{\partial u}{\partial s}$ ……………………….. (6)

Adding equations (4), (5) and (6), we get

$x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = \left(-\frac{\partial u}{\partial r} - \frac{\partial u}{\partial s}\right) + \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}$ or $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0.$

Example-11: If $u = f(r), where\ x = r cos\theta, y = r sin\theta$. Prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r)$.

Solution: Here we have, $u = f(r)$ …………………………. (1)

Also, since $x = r cos\theta, y = r sin\theta$

Therefore $r^2 = x^2 + y^2$ ………………………… (2)

Differentiating equation (2) partially with respect to 'x' and 'y' we get

$\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}$.

Also, from equations (1) and (2)

$\frac{\partial u}{\partial x} = \frac{df}{dr}.\frac{\partial r}{\partial x}$ or $\frac{\partial u}{\partial x} = \frac{df}{dr}.\left(\frac{x}{r}\right)$ ………………………… (3)

and

$\frac{\partial u}{\partial y} = \frac{df}{dr}.\frac{\partial r}{\partial y}$ or $\frac{\partial u}{\partial y} = \frac{df}{dr}.\left(\frac{y}{r}\right)$ ………………………… (4)

Differentiating equation (3) partially with respect to 'x' we get

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left\{\frac{df}{dr}.\left(\frac{x}{r}\right)\right\}$ or $\frac{\partial^2 u}{\partial x^2} = \frac{df}{dr}\frac{\partial}{\partial x}\left\{\left(\frac{x}{r}\right)\right\} + \frac{\partial}{\partial x}\left(\frac{df}{dr}\right).\left(\frac{x}{r}\right)$

$\frac{\partial^2 u}{\partial x^2} = \frac{df}{dr}.\frac{r.1 - x.\frac{\partial r}{\partial x}}{r^2} + \frac{d^2 f}{dr^2}\frac{\partial r}{\partial x}\left(\frac{x}{r}\right)$ or $\frac{\partial^2 u}{\partial x^2} = \frac{r.1 - x.\frac{x}{r}}{r^2} + \frac{d^2 f}{dr^2}.\left(\frac{x}{r}\right).\left(\frac{x}{r}\right)$

$$\frac{\partial^2 u}{\partial x^2} = \frac{df}{dr}.\left(\frac{r^2 - x^2}{r^3}\right) + \frac{d^2 f}{dr^2}\left(\frac{x^2}{r^2}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Similarly differentiating equation (4) partially with respect to 'y' we get

$$\frac{\partial^2 u}{\partial y^2} = \frac{df}{dr}.\left(\frac{r^2 - y^2}{r^3}\right) + \frac{d^2 f}{dr^2}\left(\frac{y^2}{r^2}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(6)$$

Adding equations (5) and (6) we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{df}{dr}.\left\{\frac{r^2 - (x^2 + y^2)}{r^3}\right\} + \frac{d^2 f}{dr^2}\left(\frac{x^2 + y^2}{r^2}\right)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{df}{dr}.\left(\frac{1}{r}\right) + \frac{d^2 f}{dr^2}$$

Or $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r)$. Where $\frac{df}{dr}$=$f'(r)$&$\frac{d^2 f}{dr^2} = f''(r)$

1.3.2Differentiation of an implicit function:

For any implicitly defined function $f(x, y) = 0$

We have, $\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$

$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$ or $\frac{dy}{dx} = -\frac{f_x}{f_x}$

Also second order derivative is given by $\frac{d^2 y}{dx^2} = \frac{(f_y)^2 f_{xx} - 2f_x f_y\ f_{xy} + f_{yy}(f_x)^2}{(f_y)^3}$

Example-13: If $x^3 + 3x^2 y + 6xy^2 + y^3 = 1, Find\ \frac{dy}{dx}$.

Solution: Let $f(x, y) \equiv x^3 + 3x^2 y + 6xy^2 + y^3 - 1 = 0$ ………………………..(1)

Therefore from equation (1), $f_x = \frac{\partial f}{\partial x} = 3x^2 + 6xy + 6y^2$&$f_y = \frac{\partial f}{\partial y} = 3x^2 + 12xy + 3y^2$

$\therefore\ \frac{dy}{dx} = -\frac{f_x}{f_x}$ will implies that $\frac{dy}{dx} = -\frac{3x^2 + 6xy + 6y^2}{3x^2 + 12xy + 3y^2}$

Or $\frac{dy}{dx} = -\frac{x^2 + 2xy + 2y^2}{x^2 + 4xy + y^2}$.

Example-14: Find$\frac{du}{dx}$, if $u = xlogxy, where\ x^3 + y^3 + 3xy = 1$.

Solution: Here we have $u = xlogxy$ ……………………..(1)

And let $f(x, y) \equiv x^3 + y^3 + 3xy - 1 = 0$ ………………(2)

From equations (1) & (2), we get

$$\frac{\partial u}{\partial x} = logxy + x\left(\frac{1}{xy}\right)y\ or\ \frac{\partial u}{\partial x} = 1 + logxy\ \&\frac{\partial u}{\partial y} = x\left(\frac{1}{xy}\right)x\ or\ \frac{\partial u}{\partial y} = \frac{x}{y} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$f_x = \frac{\partial f}{\partial x} = 3x^2 + 3y\ \&f_y = \frac{\partial f}{\partial y} = 3y^2 + 3x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Also from equation (2), $f(x,y)$ is an implicitly defined function

So $\frac{dy}{dx} = -\frac{f_x}{f_x} = -\left(\frac{x^2+y}{y^2+x}\right)$ putting values from equation (4)

Now, value of $\frac{du}{dx}$ can be determine as

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Putting values from equations (3) & (4) in equation (5), we get

$\frac{du}{dx} = 1 + logxy + \frac{x}{y}\left\{-\left(\frac{x^2+y}{y^2+x}\right)\right\}$ Or $\boldsymbol{\frac{du}{dx} = 1 + logxy - \frac{x}{y}\left(\frac{x^2+y}{y^2+x}\right)}$.

Example-15: If $\varphi(x,y,z) = 0$, show that $\left(\frac{\partial y}{\partial z}\right)_x \cdot \left(\frac{\partial z}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial y}\right)_z = -1$.

Solution: Here we have $f(x,y,z) = 0$ $\ldots\ldots\ldots\ldots\ldots\ldots (1)$

From equation (1),

$$\left(\frac{\partial y}{\partial z}\right)_x = -\frac{\frac{\partial \varphi}{\partial z}}{\frac{\partial \varphi}{\partial y}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\left(\frac{\partial z}{\partial x}\right)_y = -\frac{\frac{\partial \varphi}{\partial x}}{\frac{\partial \varphi}{\partial z}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$\left(\frac{\partial x}{\partial y}\right)_z = -\frac{\frac{\partial \varphi}{\partial y}}{\frac{\partial \varphi}{\partial x}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

From equations (2), (3) & (4), we get

$$\left(\frac{\partial y}{\partial z}\right)_x \cdot \left(\frac{\partial z}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial y}\right)_z = \left(-\frac{\frac{\partial \varphi}{\partial z}}{\frac{\partial \varphi}{\partial y}}\right) \cdot \left(-\frac{\frac{\partial \varphi}{\partial x}}{\frac{\partial \varphi}{\partial z}}\right) \cdot \left(-\frac{\frac{\partial \varphi}{\partial y}}{\frac{\partial \varphi}{\partial x}}\right)$$

Or $\boldsymbol{\left(\frac{\partial y}{\partial z}\right)_x \cdot \left(\frac{\partial z}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial y}\right)_z = -1}$.

Practice Exercise:

Q-1: Find all first order derivatives of function $u = \sin^{-1}\frac{x}{y}$.

Ans: $\boldsymbol{\frac{\partial u}{\partial x} = \frac{y}{\sqrt{y^2-x^2}}}$ & $\boldsymbol{\frac{\partial u}{\partial y} = \frac{-x}{\sqrt{y^2-x^2}}}$.

Q-2: If $z = f(x+ct) + \varphi(x-ct)$, show that $\frac{\partial^2 z}{\partial t^2} = c^2\frac{\partial^2 z}{\partial x^2}$.

Q-3: If $x = e^{r\cos\theta}.\cos(r\sin\theta)\,, y = e^{r\cos\theta}.\sin(r\sin\theta)$, prove that $\frac{\partial x}{\partial r} = \frac{1}{r}\frac{\partial y}{\partial \theta}\,,\ \frac{\partial y}{\partial r} = \frac{1}{r}\frac{\partial x}{\partial \theta}$.

Q-4: If $u = u(y - z, z - x, x - y)$ show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

Q-5: $u = f(r, s, t), r = \frac{x}{y}\,, s = \frac{y}{z}\,, t = \frac{z}{x}$. Show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.

Q-6: If $x + y = 2e^{\theta}\cos\varphi\,, x - y = 2i\, e^{\theta}\sin\varphi$, then show that $\frac{\partial^2 v}{\partial \theta^2} + \frac{\partial^2 v}{\partial \varphi^2} = 4xy\frac{\partial^2 v}{\partial x \partial y}$.

Q-7: If $z = u^2 + v^2, u = r\cos\theta, v = r\sin\theta$. Find values of $\frac{\partial z}{\partial r}$ & $\frac{\partial z}{\partial \theta}$.

Ans: $\boldsymbol{\frac{\partial z}{\partial r} = 2r}$ **&** $\boldsymbol{\frac{\partial z}{\partial \theta} = 0}$ **.**

Q-8: If three thermodynamic variables P, V, T are connected by a relation $f(P, V, T) = 0$. Then show that $\left(\frac{\partial P}{\partial T}\right)_V \left(\frac{\partial T}{\partial V}\right)_P \left(\frac{\partial V}{\partial P}\right)_T = -1$.

Q-9: If $u = f(x^2 + 2yz, y^2 + 2zx)$, then prove that $(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0$.

Q-10: If $f(x, y) = 0, \varphi(y, z) = 0$. Show that $\frac{\partial f}{\partial y} . \frac{\partial \varphi}{\partial z} . \frac{dz}{dx} = \frac{\partial f}{\partial x} . \frac{\partial \varphi}{\partial y}$.

1.4. Euler's Theorem for Homogeneous function:

1.4.1. Homogeneous Function:

. A function $u = f(x, y)$ is said to be a homogeneous function in variables x, y of degree n if it is expressible in any one of the following form $u = x^n\varphi\left(\frac{y}{x}\right)$ or $u = y^n\omega\left(\frac{x}{y}\right)$.

Important points about homogeneous function:

1. Any polynomial function in two variables x and y is said to be homogeneous If all its terms are of the same degree.

For example the function $f(x, y) = x^3 + 3xy^2 - 5x^2y + y^3$ is a homogeneous function in x, y of degree 3 as degree of each term present is same and is equal to 3.

Also, it can be expressed in form $f(x, y) = x^3\left\{1 - 5\left(\frac{y}{x}\right) + 3\left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right)^3\right\}$ or $f(x, y) = x^3\varphi\left(\frac{y}{x}\right)$

2. In case of non-polynomials, a function $u = f(x, y)$ in two variables x and y is said to be a Homogeneous function of degree n if for any positive number μ , $f(\mu x, \mu y) = \mu^n.f(x, y)$.

Ex: Function $f(x, y) = \cos\left(\frac{x^2+y^2}{2xy}\right)$ will be an homogeneous function in two variables x, y of degree 0 as we have

$$f(\mu x, \mu y) = \cos\left(\frac{\mu^2x^2 + \mu^2y^2}{2\mu^2xy}\right) \quad or \quad f(\mu x, \mu y) = \mu^0\cos\left(\frac{x^2 + y^2}{2xy}\right)$$

For example the function $f(x, y) = x^2\tan^{-1}\frac{y}{x}$ will be a homogeneous function of degree 2 in variables x, y as it is identical with form $f(x, y) = x^n\varphi\left(\frac{y}{x}\right)$ but the function $\cos^{-1}\frac{x^2}{y}$ is not homogeneous function of x, y.

3. If $u = f(x, y, z)$ is a homogeneous function in three variables x, y, z of degree n then it can be expressed in any of these three forms, $u = x^n\varphi\left(\frac{y}{x}, \frac{z}{x}\right)$ $or\ u = y^n\varphi\left(\frac{x}{y}, \frac{z}{y}\right)$ $or\ u = z^n\varphi\left(\frac{x}{z}, \frac{y}{z}\right)$

1.4.2. Euler's Theorem for Homogeneous Function:

Statement: If $u = f(x,y)$ be any homogeneous function in two variables x and y of degree n then

$$\boldsymbol{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.}$$

Proof: Since $u = f(x,y)$ is homogeneous function in two variables x, y of degree n therefore it can be expressed as

$$u(x,y) = x^n\varphi\left(\frac{y}{x}\right) \quad \dots\dots\dots\dots \quad (1)$$

Differentiating equation (1) partially with respect to x and y we get

$$\frac{\partial u}{\partial x} = nx^{n-1}\varphi\left(\frac{y}{x}\right) + x^n\varphi'\left(\frac{y}{x}\right).\left(\frac{-y}{x^2}\right) \quad \dots\dots\dots\dots \quad (2)$$

$$\frac{\partial u}{\partial y} = x^n\varphi'\left(\frac{y}{x}\right).\left(\frac{1}{x}\right) \quad \dots\dots\dots\dots \quad (3)$$

Multiplying equation (2) & (3) by x & y respectively and then adding we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n\varphi\left(\frac{y}{x}\right) - yx^{n-1}\varphi'\left(\frac{y}{x}\right) + yx^{n-1}\varphi'\left(\frac{y}{x}\right)$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n\varphi\left(\frac{y}{x}\right) \quad or \ \boldsymbol{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu}$$

Note: Similarly, if $u = f(x,y,z)$ is a homogeneous function in three variables x, y, z of degree n then by Euler's theorem we shall have,

$$\boldsymbol{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu}$$

Example-1: If $u = (\sqrt{x} + \sqrt{y})^5$, then find value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

Solution: Here we have $u = (\sqrt{x} + \sqrt{y})^5 \quad \dots\dots\dots\dots \quad (1)$

From (1) we have $u = (\sqrt{x})^5\left(1 + \sqrt{\frac{y}{x}}\right)^5$ or $u = x^{\frac{5}{2}}\left(1 + \sqrt{\frac{y}{x}}\right)^5$

This is of form $u = x^n\varphi\left(\frac{y}{x}\right)$, so given function u is a homogeneous function in variables x, y of degree $n = \frac{5}{2}$. Therefore from Euler's theorem for homogeneous function of degree n, we shall have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \quad \boldsymbol{or} \quad \boldsymbol{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{5}{2}u}$$

Example-2: If $u = f\left(\frac{y}{x}\right)$, then find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

Solution: Here $u = f\left(\frac{y}{x}\right)$, so it is expressible as $u = x^0 f\left(\frac{y}{x}\right)$

Therefore, u is a homogeneous function in variables x, y of degree $n = 0$.

Hence by Euler's theorem for homogeneous functions

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\,u \quad \textbf{or} \quad \boldsymbol{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0}$$

Example-3: Verify Euler's theorem for homogeneous function for $u=x^{\frac{1}{3}}y^{-\frac{4}{3}}\tan^{-1}\left(\frac{y}{x}\right)$.

Solution: Here we have $u=x^{\frac{1}{3}}y^{-\frac{4}{3}}\tan^{-1}\left(\frac{y}{x}\right)$ ……………………………… (1)

Replacing $x\ by\ \mu x$ & $y\ by\ \mu y$ in equation (1) we get

$$u=\mu^{\left(\frac{1}{3}-\frac{4}{3}\right)}x^{\frac{1}{3}}y^{-\frac{4}{3}}\tan^{-1}\left(\frac{y}{x}\right) \quad or \quad u=\mu^{(-1)}x^{\frac{1}{3}}y^{-\frac{4}{3}}\tan^{-1}\left(\frac{y}{x}\right)$$

Clearly u is a homogeneous function in variables x, y of degree -1.

Therefore, from Euler's theorem on homogeneous function, we must have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-u \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

Now differentiating equation (1) partially with respect to x we get

$$\frac{\partial u}{\partial x}=\frac{1}{3}x^{\left(\frac{1}{3}-1\right)}y^{-\frac{4}{3}}\tan^{-1}\left(\frac{y}{x}\right)+x^{\frac{1}{3}}y^{-\frac{4}{3}}\frac{1}{\left\{1+\left(\frac{y}{x}\right)^2\right\}}\left(\frac{-y}{x^2}\right)\ldots\ldots\ldots\ldots\ldots (3)$$

$$\frac{\partial u}{\partial y}=\frac{-4}{3}x^{\frac{1}{3}}y^{\left(\frac{-4}{3}-1\right)}\tan^{-1}\left(\frac{y}{x}\right)+x^{\frac{1}{3}}y^{-\frac{4}{3}}\frac{1}{\left\{1+\left(\frac{y}{x}\right)^2\right\}}\left(\frac{1}{x}\right)\ldots\ldots\ldots\ldots.\ (4)$$

Multiplying equations (3) & (4) by x & y respectively and then adding we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{3}x^{\frac{1}{3}}y^{-\frac{4}{3}}\tan^{-1}\left(\frac{y}{x}\right)+x^{\frac{1}{3}}y^{-\frac{4}{3}}\frac{1}{\left\{1+\left(\frac{y}{x}\right)^2\right\}}\left(\frac{-y}{x}\right)-\frac{4}{3}x^{\frac{1}{3}}y^{-\frac{4}{3}}\tan^{-1}\left(\frac{y}{x}\right)+x^{\frac{1}{3}}y^{-\frac{4}{3}}\frac{1}{\left\{1+\left(\frac{y}{x}\right)^2\right\}}\left(\frac{y}{x}\right)$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=(-1)x^{\frac{1}{3}}y^{-\frac{4}{3}}\tan^{-1}\frac{y}{x} \quad or\ \boldsymbol{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-u} \quad \ldots\ldots\ldots\ldots \quad (5)$$

From equations (2) & (5) Euler theorem is verified.

Example-4: If u be a homogeneous function in two variables x, y of degree n, then show that

i) $x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=(n-1)\frac{\partial u}{\partial x}$

ii) $x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}=(n-1)\frac{\partial u}{\partial y}$

iii) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u$

Solution: Here u is a homogenous function in two variables x, y of degree n

∴By Euler's theorem for homogeneous function, we shall have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\,u \quad \dots\dots\dots\dots\dots\dots (1)$$

Differentiating equation (1) partially with respect to x we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=n\frac{\partial u}{\partial x} \quad or \quad x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=(n-1)\frac{\partial u}{\partial x} \quad \dots\dots\dots\dots (2)$$

Again, differentiating equation (1) partially with respect to y, we get

$$x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=n\frac{\partial u}{\partial y} \quad or \quad x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}=(n-1)\frac{\partial u}{\partial y} \quad \dots\dots\dots\dots (3)$$

Multiplying equation (2) and (3) by x & y respectively and then adding, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(n-1)x\frac{\partial u}{\partial x}+(n-1)y\frac{\partial u}{\partial y}$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(n-1)\left\{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right\} \quad \dots\dots\dots\dots\dots\dots (4)$$

Using equation (1) in equation (4), we get

$$\boldsymbol{x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u.}$$

Note: For a homogeneous u of two variables x and y we have, $\boldsymbol{x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u}$

This is known as Euler's theorem for second order derivative.

Example-5: If $u=x^n f\left(\frac{y}{x}\right)+y^{-n}\varphi\left(\frac{x}{y}\right)$, then show that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n^2u.$$

Solution: Given that $u=x^n f\left(\frac{y}{x}\right)+y^{-n}\varphi\left(\frac{x}{y}\right)$

Let $u_1=x^n f\left(\frac{y}{x}\right) \quad \dots\dots\dots\dots\dots\dots (1)$

$u_2=y^{-n}\varphi\left(\frac{x}{y}\right) \quad \dots\dots\dots\dots\dots\dots (2)$

Such that, $u=u_1+u_2$ .... (3)

From equation (1) it is clear that u_1 is a homogenous function in x, y of degree n, therefore from Euler's theorem

$$x\frac{\partial u_1}{\partial x}+y\frac{\partial u_1}{\partial y}=nu_1 \quad \dots\dots\dots\dots\dots\dots (4)$$

Also $x^2\frac{\partial^2 u_1}{\partial x^2}+2xy\frac{\partial^2 u_1}{\partial x\partial y}+y^2\frac{\partial^2 u_1}{\partial y^2}=n(n-1)u_1 \quad \dots\dots\dots (5)$

Similarly, u_2 is a homogenous function in x, y of degree –n, Therefore

$$x\frac{\partial u_2}{\partial x}+y\frac{\partial u_2}{\partial y}=-nu_2 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(6)$$

Also $x^2\frac{\partial^2 u_2}{\partial x^2}+2xy\frac{\partial^2 u_2}{\partial x\partial y}+y^2\frac{\partial^2 u_2}{\partial y^2}=n(n+1)u_2 \quad \dots\dots\dots.(7)$

Adding equations (4) and (6) we get

$$x\frac{\partial}{\partial x}(u_1+u_2)+y\frac{\partial}{\partial y}(u_1+u_2)=n(u_1-u_2) \quad Or\ x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n(u_1-u_2) \quad \dots\dots \quad (8)$$

Again, adding equations (5) & (7), we get

$$x^2\frac{\partial^2}{\partial x^2}(u_1+u_2)+2xy\frac{\partial^2}{\partial x\partial y}(u_1+u_2)+y^2\frac{\partial^2}{\partial y^2}(u_1+u_2)=n(n-1)u_1+n(n+1)u_2$$

Or $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n^2$u- n $((u_1-u_2)$ ……………………………………. (9)

At last adding equations (8) & (9), we get

$$\boldsymbol{x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n^2u.} \qquad \{\text{Using equation (8)}$$

Hence the result.

1.4.3Deduction on Euler's Theorem for homogeneous function:

If $u=u(x,y)$ is not homogeneous but $f(u)$ be the homogenous function in variables x, y of degree n, then

$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\ \frac{f(u)}{f'(u)}$.

Proof: Since $v=f(u)$ is given to be homogeneous function in variables x, y of degree n,

Therefore, by Euler's theorem

$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=nv$ Or $x\frac{\partial}{\partial x}f(u)+y\frac{\partial}{\partial y}f(u)=nf(u)$ {since$v=f(u)$

$$f'(u)\left\{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right\}=nf(u)\ Or \quad \boldsymbol{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\ \frac{f(u)}{f'(u)}}$$

Note: (i) If$f(u)$ be homogeneous function in three variables x, y, z. then above result can be written as

$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=n\ \frac{f(u)}{f'(u)}$.

Example-6: If $f(u)$ be the homogenous function in variables x, y of degree n, then show that $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)\{g'(u)-1\}$.

Solution: Since$f(u)$ is homogenous function in variables x, y of degree n, so we shall have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\,\frac{f(u)}{f'(u)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Let $g(u)=n\,\frac{f(u)}{f'(u)}$, then from equation (1), we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=g(u) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \quad (2)$$

Differentiating equation (2) partially with respect to x we get,

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=g'(u)\frac{\partial u}{\partial x} \quad or\; x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=\{g'(u)-1\}\frac{\partial u}{\partial x} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \quad (3)$$

Again, differentiating equation (2) partially with respect y we get

$$x\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial u}{\partial y}+y\frac{\partial^2 u}{\partial y^2}=g'(u)\frac{\partial u}{\partial y} \quad or\; x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}=\{g'(u)-1\}\frac{\partial u}{\partial y} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(4)$$

Multiplying equation (3) & (4) by x & y respectively, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\{g'(u)-1\}\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

Using equation (2) in equation (5) we get

$\boldsymbol{x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)\{g'(u)-1\}}$. Hence **the result**

Example-7: If$u=log\left(\frac{x^4+y^4}{x+y}\right)$, then find the value of $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$.

Solution: Here $u=log\left(\frac{x^4+y^4}{x+y}\right)$, then u is not homogenous function but

$e^u=\left(\frac{x^4+y^4}{x+y}\right)$ $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.$ (1),

Here $f(u)=e^u$ is homogeneous function in x, y of degree $n=3$. So, from deduction to Euler's theorem

$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\,\frac{f(u)}{f'(u)}$, Now since $n=3$ & $f(u)=e^u$

Therefore $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3\,\frac{e^u}{e^u}$ $\quad or \quad$ $\boldsymbol{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3}$.

Example-8: If $u=\sin^{-1}\left(\frac{x^2+y^2+z^2}{ax+by+cz}\right)$, then prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=tanu$.

Solution: We have $u=\sin^{-1}\left(\frac{x^2+y^2+z^2}{ax+by+cz}\right)$, clearly u is not homogeneous function

But $\sin u=\left(\frac{x^2+y^2+z^2}{ax+by+cz}\right)$ $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.$ (1)

is a homogeneous function in three variables x, y, z of degree $n=1$. Therefore buy deduction to Euler's theorem we have $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=n\,\frac{f(u)}{f'(u)}$ $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2)$

In this problem, f(u) = $\sin u$, so from equation (2) we shall have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=1.\frac{sinu}{cosu}$$

$$\boldsymbol{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=tanu.}$$

Example-9: If $u=\sin^{-1}\left\{\frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{2}}-y^{\frac{1}{2}}}\right\}^{\frac{1}{2}}$**,** then show that

(i) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{12}tanu$

(ii) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{1}{144}tanu(\,sec^2u+12)$

Solution: We have $u=\sin^{-1}\left\{\frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{2}}-y^{\frac{1}{2}}}\right\}^{\frac{1}{2}}$ … … … … … … … … … … …. (1)

Here u is not homogeneous but $f(u)=sinu$ will be homogeneous function in x, y of degree $n=-\frac{1}{12}$.

Therefore, by deduction to Euler's theorem

$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\,\frac{f(u)}{f'(u)}$ Or $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{12}\frac{sinu}{cosu}$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{12}tanu$$

Again we have $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)\{g'(u)-1\}$, where $g(u)=n\,\frac{f(u)}{f'(u)}=-\frac{1}{12}\frac{sinu}{cosu}=--\frac{1}{12}tanu$

Therefore $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-\frac{1}{12}tanu\left(-\frac{1}{12}sec^2u-1\right)$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{1}{144}tanu(\,sec^2u+12)$$

Example-10: If $u=\tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$**,** then prove that

(i) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\,sin2u$

(ii) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=sin4u-sin2u.$

Solution: Here$u=\tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, is not homogenous but $f(u)=tanu$ will be homogeneous function in x, y of degree $n=2$. Therefore, from deduction to Euler's theorem

$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\,\frac{f(u)}{f'(u)}$ or $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\frac{tanu}{sec^2u}$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\frac{sinu}{cosu}cos^2u\quad or\;\; x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2sinu.cosuu$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = sin2u \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots..(1)$$

From equation (1), $g(u) = n\,\frac{f(u)}{f'(u)} = sin2u$

Also we have $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)\{g'(u) - 1\}$

Therefore $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = sin2u(2cos2u - 1)$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2sin2ucos2u - sin2u$$

$$\boldsymbol{x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = sin4u - sin2u}$$

Practice Exercise:

Q-1: State Euler's theorem for homogeneous function and verify it for $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$.

Q-2: Verify Euler's theorem for $z = \frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{2}}+y^{\frac{1}{2}}}$.

Q-3: If $u = \sin^{-1}\left(\frac{x+2y+3z}{\sqrt{x^8++y^8+z^8}}\right)$. show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + 3\,tanu = 0$.

Q-4: Apply Euler's theorem for homogeneous function for $u = (x^2 - 2xy + y^2)^{\frac{3}{2}}$ to evaluate values of (i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ and (ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$.

Ans: (i) 3u and (ii) 6u.

Q-5: If $u = \cos^{-1}\left(\frac{x^3+y^3+z^3}{ax+by+cz}\right)$, then show that $xu_x + yu_y + zu_z = -2\,cotu.$

Q-6: If $u = \csc^{-1}\sqrt{\frac{x^{\frac{1}{2}} + y^{\frac{1}{2}}}{x^{\frac{1}{3}} + y^{\frac{1}{3}}}}$, then evaluate $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

Ans: $\boldsymbol{-\frac{1}{12}tanu}$.

Q-7: If $u = \tan^{-1}\frac{y^2}{x}$, show that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -sin2u\,sin^2u$.

1. https://www.youtube.com/watch?v=XzaeYnZdK5o&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=1
2. https://www.youtube.com/watch?v=9-tir2V3vYY&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=14
3. https://www.youtube.com/watch?v=aqfSOOiO2kI&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=15
4. https://www.youtube.com/watch?v=GoyeNUaSW08&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=16
5. https://www.youtube.com/watch?v=jiEaKYI0ATY&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=17
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