(L-1)

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Complex Variables

$z = x + iy$ $\quad i = \sqrt{-1}$

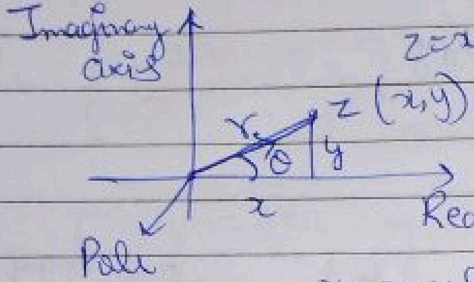
x → real part, iy → imaginary part

$z = [x, y]$ → Ordered pair

$z = x + iy$

$x, y \in \mathbb{R}$

$z = x + iy$



(Argand Plane)

$x = r\cos\theta$
$y = r\sin\theta$
$z = re^{i\theta}$

Circle: $|z-a| = r$ centre $= (a,0)$
radius $= r$

ex: $|z+i| = 2$
$C: (0,-1)$
$R = 2$

Open Disk: The set of points which satisfy the equation $|z-a| < r$. This set consists of all points which lie inside circle C.

Closed disk: $|z-a| \leq r$

Annulus: $r_2 \leq |z-a| \leq r_1$

where C_1 & C_2 are two concentric circles having radius r_1 & r_2.

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Finite Complex Plane: A Complex plane which does not include $z=\infty$.

Extended Plane: The complex plane ~~to~~ in which $z=\infty$ has been added is called extended plane

Function of a Complex Variable :-

$z=x+iy \rightarrow$ Complex variable where $x,y \in R$
if corresponding to each value of z (complex variable) their corresponds one or more value of another complex variable w (u+iv) then w is called the funtion of complex variable z.
i.e. $w=f(z)$.
$u+iv=f(z)$

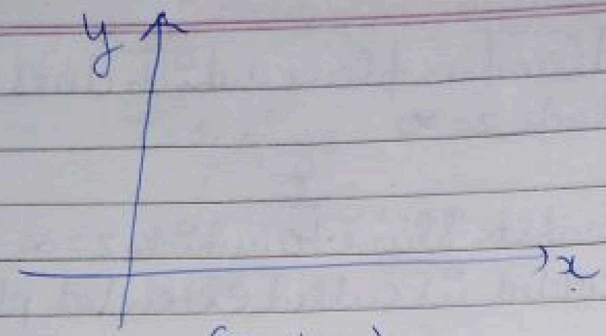
for ex: $w=z^2$
$u+iv=(x+iy)^2=(x^2-y^2)+2ixy$
Equating real & imaginary part

$u=x^2-y^2$
$v=2xy$

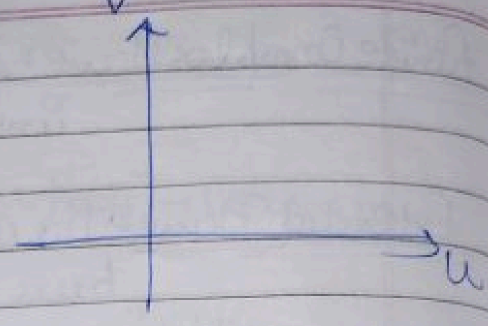
NOTE: if ~~two~~ to each value of z there corresponds one and only one value of w then ~~if~~ w is called single value fuction of z & if more than one value of w then w is multivalued fuction

for ex: $w=\pm\sqrt{z}$

NOTE: to represent $w=f(z)$ graphically we need two argand planes namely z plane and w-plane



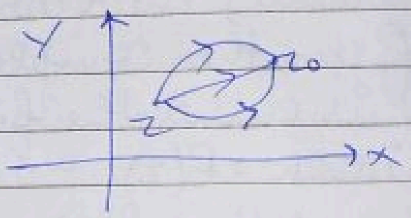
(Z-plane)



(W-plane)

Limit of f(z): A fuction $f(z)$ is said to have limit l as $z \to z_0$ if for $\epsilon > 0$ there exists $\delta > 0$ such that $|f(z) - l| < \epsilon$ whenever

$0 < |z - z_0| < \delta$



Continuity of f(z):

A single valued fuction $f(z)$ is said to be continous at $z = z_0$ if $\lim_{z \to z_0} f(z)$ exists, $f(z_0)$ exists &

$$\lim_{z \to z_0} f(z) = f(z_0)$$

Derivative of f(z): if $w = f(z)$

$$\text{Then } f'(z) = \frac{dw}{dz} = \lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z}$$

NOTE: $f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$

(Imp)

single valued

Analytic function: A function $f(z)$ is said to be analytic at point $z = z_0$ if $f(z)$ is differentiable not only at z_0 but some neighbourhood of z_0

also differentiable

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~~Entire~~ function:

NOTE: (i) holomorphic, regular, monogenic all are similar words for analytic.

(ii) A function $f(z)$ is said to be analytic at $z=\infty$ if $f(\frac{1}{z})$ is analytic at $z=0$

(iii) Analytic $\Rightarrow$ differentiability but converse need not be true

(iv) A function $f(z)$ may be differentiable in a domain except for a finite no. of points. These points known as regular point of $f(z)$ in that domain

Entire function: A function $f(z)$ is said to be entire function if it is analytic in a finite complex plane. at every point

NOTE: Entire function $\Rightarrow$ Analytic $\Rightarrow$ differentiable $\Rightarrow$ Continous $\Rightarrow$ limit exists

Imp Necessary & Sufficient Conditions for $f(z)$ to be Analytic

Let $w=f(z)$ be a complex function then $u+iv=w=f(z)$ is analytic if and only iff

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ & $\frac{\partial v}{\partial y}$ are continous funtions of x & y

(2-Marker CR eqns)

(ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ [Cauchy ~~Riemens~~ Reimann Equations or C-R equations]

for ex: Show that function $e^x[\cos y + i\sin y]$ is holomorphic. Also find derivative of $f(z)$.

Sol: we have, $w = u+iv = f(z)$

$u+iv = e^x\cos y + ie^x\sin y$

equating real & imag. parts

$u = e^x\cos y \quad v = e^x\sin y$

$\left.\frac{\partial u}{\partial x}\right|_{y\text{-const}} = e^x\cos y$

$\frac{\partial u}{\partial y} = -e^x\sin y$

$\frac{\partial v}{\partial x} = e^x\sin y$

$\frac{\partial v}{\partial y} = e^x\cos y$

Acc. to algebra of continuous functions

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous functions in x & y being product of continuous functions (Sufficient condn)

(ii) from ①, ②, ③ & ④

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\Rightarrow$ CR equations satisfied (necessary condn)

hence by necessary & sufficient conditions $f(z)$ is analytic.

NOTE

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$= e^x[\cos y + i\sin y] = e^x e^{iy} = e^{x+iy}$

Que: Verify function $f(z) = \dfrac{xy^2(x+iy)}{x^2+y^4}$, $z \neq 0$; $f(0) = 0$ is analytic or not.

Sol: Let $f(z) = w = u + iv = \dfrac{x^2y^2}{x^2+y^4} + i\,\dfrac{xy^2y}{x^2+y^4}$

$$u = \frac{x^2y^2}{x^2+y^4}, \quad v = \frac{xy^3}{x^2+y^4}$$

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0 \quad \text{---} (1)$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0 \quad \text{---} (2)$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0 \quad \text{---} (3)$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0 \quad \text{---} (4)$$

from eqn (1), (2), (3), (4)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

~~C-R eqns satisfied at origin~~

$f(0) = 0 \rightarrow$ implies that $u = 0, v = 0$

To find $f'(z)$ at $z = 0$:-

$$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z}$$

$$= \lim_{z\to 0} \frac{\frac{xy^2(x+iy)}{x^2+y^4} - 0}{z}$$

$\neq$

$$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{xy^2}{x^2+y^4} \quad (z = x+iy)$$

put $y = mx$

$$= \lim_{x\to 0} \frac{x(m^2x^2)}{x^2+m^4x^4} = \frac{m^2x^3}{m^4x^4+x^2} = \frac{x^3 m^2}{x^2(m^4x^2+1)}$$

$$= \lim_{x\to 0} \frac{m^2 x}{x^2m^4+1} = 0$$

as value of limit is finite & not depending upon m hence it is differentiable

$z \to 0$ as $x^2 = y^4 \Rightarrow x = y^2$

Now $\lim_{\substack{x\to 0\\ y\to 0}} \frac{xy^2}{x^2+y^4}$ put $x = y^2$

$$f'(0) = \lim_{y\to 0} \frac{y^2y^2}{y^4+y^4} = \frac{1}{2}$$

$\Rightarrow f'(0)$ is not unique

$f(z)$ is not analytic at origin although the CR eqns are satisfied.

Que: Examine the nature of function (check analyticity) in the region including origin.

$$f(z) = \begin{cases} \dfrac{x^2y^5(x+iy)}{x^4+y^{10}}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

Sol: $w = f(z) = u + iv = \dfrac{x^3y^5}{x^4+y^{10}} + i\,\dfrac{x^2y^6}{x^4+y^{10}}$

$$u = \frac{x^3y^5}{x^4+y^{10}}, \quad v = \frac{x^2y^6}{x^4+y^{10}}$$

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

To find $f'(z)$ at $z=0$:

$$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z} \quad (z = x+iy)$$

$$= \lim_{z\to 0} \frac{\dfrac{x^2y^5(x+iy)}{x^4+y^{10}} - 0}{z}$$

$$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2y^5}{x^4+y^{10}} = \lim_{x\to 0} \frac{x^2 m^5 x^5}{x^4+m^{10}x^{10}} = \frac{m^5x^7}{m^{10}x^{10}+x^4}$$

$y = mx$

$$= \frac{m^5x^7}{x^4(m x^6+1)}$$

$$= \frac{m^5x^3}{1+mx^6} = 0$$

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put $x^2 \to y^5$

$$f'(0) = \lim_{y\to 0} \frac{x^2y^5}{x^4+y^{10}} = \frac{y^5y^5}{y^{10}+y^{10}} = \frac{y^{10}}{2y^{10}} = \frac{1}{2}$$

$f'(0)$ is not unique

$f(z)$ is not analytic at origin

(although CR eqns are satisfied)

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Ques Prove that function $f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$, $z \neq 0$, $f(0)=0$

Prove that function $f(z)$ is continuous & C-R eqns are satisfied at Origin yet $f'(0)$ does not exists.

Sol: $w = f(z) = u+iv = \dfrac{x^3 + ix^3 - y^3 + y^3 i}{x^2+y^2}$

$$u+iv = \left(\frac{x^3-y^3}{x^2+y^2}\right) + i\left(\frac{x^3+y^3}{x^2+y^2}\right)$$

$$u = \frac{x^3-y^3}{x^2+y^2}, \quad v = \frac{x^3+y^3}{x^2+y^2}$$

(1) Continuity check:

Change in polar coordinates:

put $x = r\cos\theta$, $y = r\sin\theta$

$$u = \frac{r^3[\cos^3\theta - \sin^3\theta]}{r^2[\cos^2\theta + \sin^2\theta]} = r[\cos^3\theta - \sin^3\theta]$$

$$v = \frac{r^3[\cos^3\theta + \sin^3\theta]}{r^2[\cos^2\theta + \sin^2\theta]} = r[\cos^3\theta + \sin^3\theta]$$

as $z \to 0$, $r \to 0$ at origin value of $r = 0$

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$$\lim_{z\to 0} u = \lim_{r\to 0} r[\cos^3\theta - \sin^3\theta] = 0$$

$$\lim_{z\to 0} v = \lim_{r\to 0} r[\cos^3\theta + \sin^3\theta] = 0$$

$$\Rightarrow \lim_{z\to 0} f(z) = 0 = f(0)$$

f(z) is continous at origin

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0} \frac{\frac{x^3}{x^2}-0}{x}$$

$$= \frac{x^3}{x^3} = 1$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0} \frac{\frac{0-y^3}{0+y^2}-0}{y}$$

$$= \frac{-y^3}{y^3} = -1$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x} = \lim_{x\to 0} \frac{\frac{x^3}{x^2}-0}{x} = 1$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y} = \lim_{y\to 0} \frac{\frac{0+y^3}{0+y^2}-0}{y} = 1$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$$

$$1 = 1 \qquad -1 = -1$$

CR eqns satisfied. as $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are constant funtions hence they are contirous.

f(z) is analytic

$f'(z)$ at $(0,0)$ $\qquad z = x+iy$

$$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z}$$

$$= \lim_{z\to 0} \frac{\left(\frac{x^3-y^3}{x^2+y^2}\right) + i\left(\frac{x^3+y^3}{x^2+y^2}\right) - 0}{(x+iy)}$$

put $y = mx$

$$= \lim_{x\to 0} \frac{\left(\frac{x^3-m^3x^3}{x^2+m^2x^2}\right) + i\left(\frac{x^3+m^3x^3}{x^2+m^2x^2}\right) - 0}{(x+imx)}$$

$$= \frac{\frac{x^3(1-m^3)}{\cancel{x^2}(1+m^2)} + i\,\frac{x^3(1+m^3)}{x^2(1+m^2)}}{x(1+im)}$$

$$= \frac{\cancel{x}\left(\frac{1-m^3}{1+m^2}\right) + i\cancel{x}\left(\frac{1+m^3}{1+m^2}\right)}{\cancel{x}(1+im)}$$

$$= \frac{\left(\frac{1-m^3}{1+m^2}\right) + i\left(\frac{1+m^3}{1+m^2}\right)}{(1+im)}$$

This depends upon m hence it is not differentiable & not analytic.

Ptz $\quad$ $f'(0)$ depends upon m
$f'(0)$ is not unique
$f'(0)$ DNE, $f(z)$ is not unique.

Harmonic functions:-

Laplace eqn:- Let $u = u(x,y)$

u is a function of x & y

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

A function of x & y which ~~causes~~ powers continous partial have Ist & IInd order ~~partial~~ continous partial derivatives and satisfy Laplace eqn is known as Harmonic functions.

i.e. If $u(x,y)$, Then u will be harmonic if $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

NOTE: If $f(z) = u + iv$ is analytic function then u & v will be harmonic and called Harmonic conjugates of each other. but converse need not be true.

If $f(z) = u+iv$ is analytic

~~for ex: Show that $f(z) = \frac{1}{2}\log(x^2+y^2)$ is harmonic find its harmonic conjugates~~

~~Sol:~~

for ex: If $f(z) = u + iv$ is analytic and $u = \frac{1}{2}\log(x^2+y^2)$ Show that u is harmonic. find its harmonic conjugate.

Sol: $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[\frac{1}{2}\log(x^2+y^2)\right] = \frac{1}{\cancel{2}} \times \frac{1}{x^2+y^2} \times \cancel{2}x = \frac{x}{x^2+y^2}$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{x}{x^2+y^2}\right] = \frac{(x^2+y^2) - (2x)(x)}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$

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$$\frac{\partial^2 u}{\partial x^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{2}\log(x^2+y^2)\right) = \frac{1}{\cancel{2}} \times \frac{1}{x^2+y^2} \times \cancel{2}y$$

$$= \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2) - 2y(y)}{(x^2+y^2)^2} = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2} = 0$$

u is harmonic function as $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

To find v (Harmonic conjugate) :-

$$v(x,y)$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$v = \int \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= \int -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

$\because$ f(z) is analytic

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

by CR eqns

$$V = \int \frac{-y\,dx + x\,dy}{(x^2+y^2)}$$

NOTE: $\int M\,dx + N\,dy = \int M\,dx + \int N\,dy$

y constant

Integrating terms which are free from x

$$\int \frac{-y\,dx + x\,dy}{(x^2+y^2)} = -\int \frac{y\,dx}{x^2+y^2} + \int \frac{x\,dy}{x^2+y^2}$$

y(const)

$N = \frac{x}{x^2+y^2}$

$$= -y \int \frac{dx}{x^2+y^2} + 0 \text{ (No term free from x)}$$

$$= \frac{-y}{y}\tan^{-1}\left(\frac{x}{y}\right)$$

$$\boxed{V = \tan^{-1}\left(\frac{y}{x}\right) + c}$$

Ex: $\int 2xy\,dy + \int (y^2 + \cancel{x})\,dx$

Now it has 2 terms

$$= 2y\left(\frac{x^2}{2}\right) + \frac{y^3}{3}$$

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Que→ Discuss analyticity of log z also find its derivative

Sol: $f(z) = \log z$

$u+iv = \log(x+iy)$

$u+iv = \log(r\cos\theta + ir\sin\theta)$

$u+iv = \log(r \cdot e^{i\theta})$

$u+iv = \log(r) + \log(e)^{i\theta}$

$u+iv = \log r + i\theta$

Equating real & imaginary parts

$u = \log r,\ v = \theta$

$\frac{\partial u}{\partial r} = \frac{1}{r}$, $\frac{\partial u}{\partial \theta} = 0$, $\frac{\partial v}{\partial r} = 0$, $\frac{\partial v}{\partial \theta} = 1$

$\frac{\partial u}{\partial r} = r\frac{\partial v}{\partial \theta}$ and $\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$

CR eqns satisfied everywhere except at origin

$\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$ are continous functions of r and θ except at origin.

To find $f'(z)$;

$\therefore f(z)$ is analytic

$f'(z) = \frac{1}{z}$

or $f'(z) = (\cos\theta - i\sin\theta)\frac{\partial w}{\partial r}$

$= (\cos\theta - i\sin\theta)\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$

$= (\cos\theta - i\sin\theta)\left(\frac{1}{r} + i\times 0\right)$

$= \frac{1}{r}(\cos\theta - i\sin\theta) = \frac{e^{-i\theta}}{r} = \frac{1}{z}$ $\left[\because z = re^{i\theta}\right.$

Ques Show that $r^n[\cos(n\theta)+i\sin(n\theta)]$ is analytic everywhere except at $r=0$, also find $f'(z)$.

Sol: $u+iv = r^n\cos(n\theta) + i r^n \sin(n\theta)$

equating real & imaginary parts,

$u = r^n\cos(n\theta)$ and $v = r^n\sin(n\theta)$

$\frac{\partial u}{\partial r} = n r^{n-1}\cos(n\theta)$ — (1) $\quad \frac{\partial v}{\partial r} = n r^{n-1}\sin(n\theta)$ — (3)

$\frac{\partial u}{\partial \theta} = -n r^n \sin(n\theta)$ — (2) $\quad \frac{\partial v}{\partial \theta} = n r^n \cos(n\theta)$ — (4)

from (1), (2), (3), (4),

$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$

Hence CR eqns are satisfied

$\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$ are continuous functions everywhere except at $r=0$

Hence by necessary and sufficient conditions function is analytic everywhere except at $r=0$.

Ques Show that $u = x^3 - 3xy^2$ is harmonic. Also find analytic function of which it is real part.

Sol: $\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \qquad \frac{\partial u}{\partial y} = -6xy$

$\frac{\partial^2 u}{\partial x^2} = 6x \qquad \frac{\partial^2 u}{\partial y^2} = -6x$

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$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0$$

$\Rightarrow$ u is a harmonic function

To find v:-

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \quad [\text{By CR eqns}]$$

$$dv = 6xy\, dx + 3(x^2 - y^2) dy$$

$$\int dv = \int 6xy\, dx + 3\int (x^2 - y^2) dy$$

$$v = 6y\frac{x^2}{2} - y^3 + c$$

$$v = 3yx^2 - y^3 + c$$

$$f(z) = u + iv$$

$$= x^3 - 3xy^2 + i(3yx^2 - y^3)$$

$$= (x+iy)^3$$

$$= z^3$$

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Ques ①: Show that $e^x \cos y$ is harmonic, find corresponding analytic function $f(z)$.

Sol: $u = u(x,y)$

u is a function of x, y

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial}{\partial x}(e^x \cos y) = e^x \cos y$$

$$\frac{\partial}{\partial x}(e^x \cos y) = e^x \cos y$$

$$\frac{\partial}{\partial y}(e^x \cos y) = -e^x \sin y$$

$$\frac{\partial}{\partial y}(-e^x \sin y) = -e^x \cos y$$

$$-e^x \cos y + e^x \cos y = 0$$

$$0 = 0$$

Hence it is harmonic

as $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$f(z) = u + iv$

$= e^x \cos y + i\, e^x \sin y$

$= e^x[\cos y + i \sin y]$

$= e^x e^{iy}$

$= e^{x+iy}$

$= e^z$

we will take help of CR eqns & total differentiation

Now, $dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$

$dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$ [by CR eqns f(z) being analytic]

$dv = +e^x \cancel{\cos y}\ \sin y\, dx + e^x \cos y\, dy$

Integrating both side,

$$\int dv = \int e^x \sin y\, dx + e^x \cos y\, dy$$

$$v = \int d[\sin y \cdot e^x) = e^x \sin y$$

(jo change hora usko hatao)

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$f(z) \to u+iv \to$ analytic function

~~Que ②~~ NOTE: if u is given (Real part given)

① $$f(z) = \int \left[\left(\frac{\partial u}{\partial x}\right)_{(z,0)} - i\left(\frac{\partial u}{\partial y}\right)_{(z,0)} \right] dz + c$$

② if v (imaginary part) is given

$$f(z) = \int \left[\left(\frac{\partial v}{\partial y}\right)_{(z,0)} + i\left(\frac{\partial v}{\partial x}\right)_{(z,0)} \right] dz + c$$

we can also use this method to directly find $f(z)$

$$f(z) = \int (e^x \cos y)_{(z,0)} + i(-e^x \sin y)_{(z,0)} dz + c$$

$$= \int e^z \cos 0 + i(-e^z \sin 0) dz + c$$

$$f(z) = \int e^z dz + c = e^z + c$$

Que ② If $u = e^x[x\cos y - y\sin y]$ is harmonic ~~function whose real part is $e^{-x}[x\sin y - y\cos y]$~~ find analytic function $f(z) = u+iv$ such that $f(1) = e$.

Sol: $$f(z) = \int \left[\left(\frac{\partial u}{\partial x}\right)_{(z,0)} + i\left(\frac{\partial u}{\partial y}\right)_{(z,0)} \right] dz + c$$

$u = e^x[x\cos y - y\sin y]$

$$\frac{\partial u}{\partial x} = \int \left[e^x(x\cos y - y\sin y) + e^x \cos y \right]_{(z,0)}$$
$$+ i\left[e^x(-x\sin y - \sin y - y\cos y \right]_{(z,0)} dz$$
$$+ c$$

$$\left.\frac{\partial u}{\partial x}\right|_{(z,0)} = \int \Big[e^z(z\cos 0 - 0) + e^z\cos 0 + i\,\big(e^z(\cancel{-z\sin 0} - \cancel{\sin 0} - \cancel{0})\big)\Big]dz + c$$

$$= \int \left[ze^z + e^z\right]dz + c$$

$$f(z) = \left[ze^z + \cancel{e^z} - \cancel{e^z}\right] + c \qquad f(z) = ze^z + c \quad \text{—(1)}$$

put $z=1$, $f(1) = e$ in eqn (1)

$$e = e + c$$

$$\boxed{c = 0}$$

$$\boxed{f(z) = ze^z + 0}$$

Que (3): let $f(z) = u(r,\theta) + iv(r,\theta)$ be an analytic function. If $u = -r^3\sin(3\theta)$. Then construct its corresponding analytic function $f(z)$.

Sol: To find $v = \frac{\partial v}{\partial r}dr + \frac{\partial v}{\partial \theta}d\theta$

C-R eqns $\boxed{\begin{aligned} \frac{\partial u}{\partial \theta} &= -r\frac{\partial v}{\partial r} \\ \frac{\partial v}{\partial \theta} &= r\frac{\partial u}{\partial r} \end{aligned}}$

$$= -\frac{1}{r}\frac{\partial u}{\partial \theta}dr + r\frac{\partial u}{\partial r}d\theta$$

$$dv = -\frac{1}{r}\left[-3r^3\cos 3\theta\right]dr + r\left[-3r^2\sin 3\theta\right]d\theta$$

$$v = \int +3r^2\cos 3\theta\, dr - 3r^3\sin 3\theta\, d\theta$$

$$v = \int d(\cos 3\theta \cdot r^3)$$

$$\boxed{v = r^3\cos(3\theta)}$$

$$f(z) = u + iv$$
$$= -r^3 \sin 3\theta + i r^3 \cos(3\theta)$$
$$f(z) = i^2 r^3 \sin(3\theta) + i r^3 \cos(3\theta)$$
$$= i r^3 \left(i \sin(3\theta) + \cos(3\theta) \right)$$
$$= i r^3 e^{3i\theta}$$
$$\left(z = e^{i\theta} r \right)$$
$$\boxed{f(z) = i z^3}$$

HW

Que(4): find analytic function $f(z) = u + iv$, given that $v = \left(r - \frac{1}{r} \right) \sin\theta$, $r \neq 0$

Ans: $\left(z + \frac{1}{z} \right) + c$.

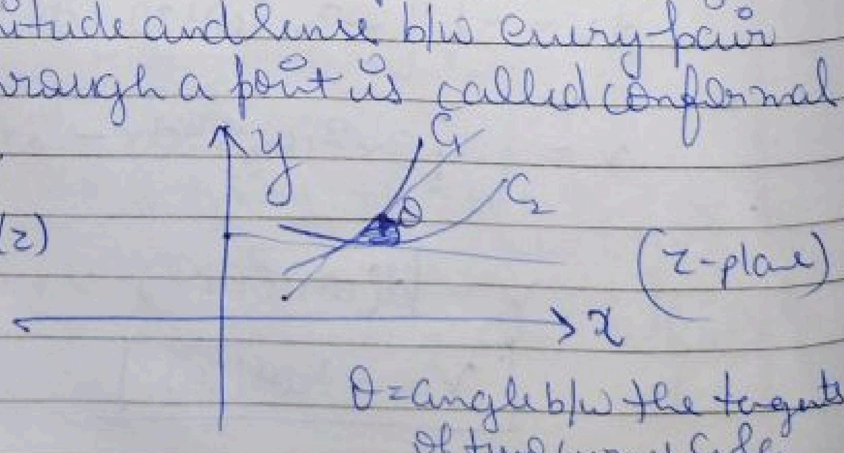
Transformation or Mapping

The function $w = f(z)$ define a mapping or transformation from z-plane to w-plane

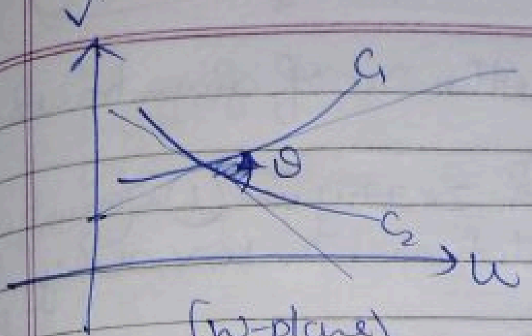
$z = x + iy$
$w = u + iv$

Conformal Mapping: A transformation which ~~results~~ preserves angles in both magnitude and sense b/w every pair of curve through a point is called conformal mapping.

let $w = f(z)$



θ = angle b/w the tangents of two curves C_1 & C_2



(w-plane)

NOTE: if $f(z)$ is Analytic and $f'(z) \neq 0$ in region R of z plane, then
① mapping $w = f(z)$ is conformal
but converse need not be true

② A point at which $f'(z) = 0$ called critical points of the transformation.

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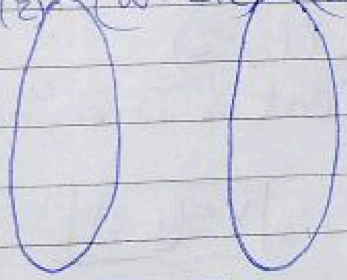
Some basic Transformations

① Translation: $w = z + c$ where c is complex constant

Input $f(z)$ $(w = z + c)$ Output $(w = f(z))$

★ Shape & size preserve
★ origin changes



② Rotation: $\boxed{w = ze^{i\theta}}$

$\theta > 0 \rightarrow$ anticlockwise rotation
$\theta < 0 \rightarrow$ clockwise rotation

③ Magnification: $\boxed{w = az}$ $a \rightarrow$ Real
(figure gets magnified or shrinked by 'a' times)

④ Inversion: $\boxed{w = \frac{1}{z}}$

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~~for ex~~: Step ①: Let $w = f(z)$ be the eqn of given transformation

Step ②: put $w = u+iv$ & $z = x+iy$ in ① & (equate real & imaginary parts) and find relation b/w u, v, x & y.

Step ③: Put values of x & y in these relations and find values of u & v

Step ④: draw graph of z plane & w-plane if required.

for ex: Let a rectangular domain are bounded by $x=0, y=0, x=2, y=1$. Determine the region R' of w-plane into which R is mapped under the transformation $w = z+(1-2i)$.

Sol. $w = z + [1-2i]$

$u+iv = (x+iy) + [1-2i]$

$u+iv = (x+1) + i(y-2)$

equating Real & Imaginary parts

$z=0$ $(0,0)$ $\quad \boxed{u = (x+1)}$, $\boxed{v = (y-2)}$

$z=1$ $(1,0)$

$z=1+i$ $(1,1)$ $\quad$ ($x=0, y=0, x=2$ & $y=1$ → are not the points they are the boundary conditions)

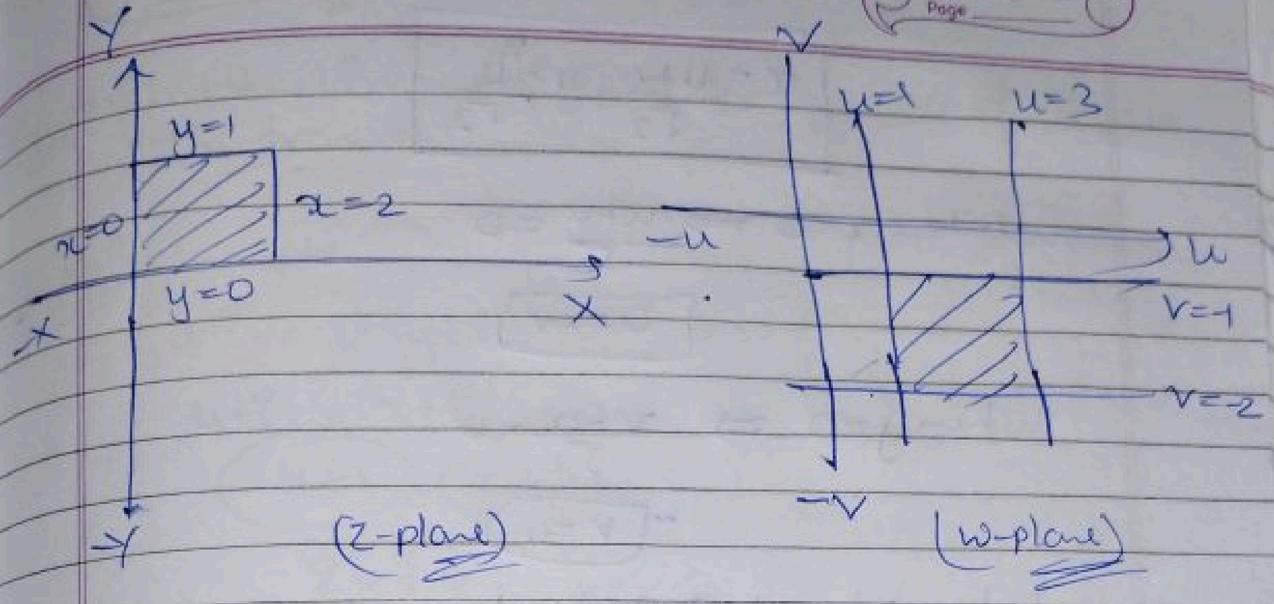
$z=i$ $(0,1)$

when $x=0$ $\quad$ Also, if $y=0$

$\boxed{u=1}$ $\quad$ $\boxed{v=-2}$ → (these are also the lines not the ~~plan~~ points)

when $x=2$ $\quad$ $y=1$

$\boxed{u=3}$ $\quad$ $\boxed{v=-1}$



Ques Determine the region in w-plane in which ~~the rectangle~~ a triangular region bounded by $x=0, y=0$ & $x+y=1$ transformed under the transformation. $w = ze^{i\pi/4}$.

Sol: $w = ze^{i\theta}$ $\theta = \frac{\pi}{4} > 0$ anticlockwise

$w = ze^{i\pi/4}$

$(u+iv) = (x+iy)e^{i\pi/4}$

$(u+iv) = (x+iy)\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$(u+iv) = \frac{x+ix+iy-y}{\sqrt{2}}$

$u+iv = \frac{x+ix+iy-y}{\sqrt{2}}$

$u = \frac{x-y}{\sqrt{2}}$ $v = \frac{x+y}{\sqrt{2}}$

when $x=0$

Adding eqn (1) & (2)

$u+v = \frac{x-y+x+y}{\sqrt{2}}$

$u+v = \frac{2x}{\sqrt{2}} = x\sqrt{2}$

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$$\boxed{x = \frac{u+v}{\sqrt{2}},\ y = \frac{v-u}{\sqrt{2}}}$$

when $x=0 \Rightarrow \frac{u+v}{\sqrt{2}} = 0$

$$\boxed{u=-v}$$

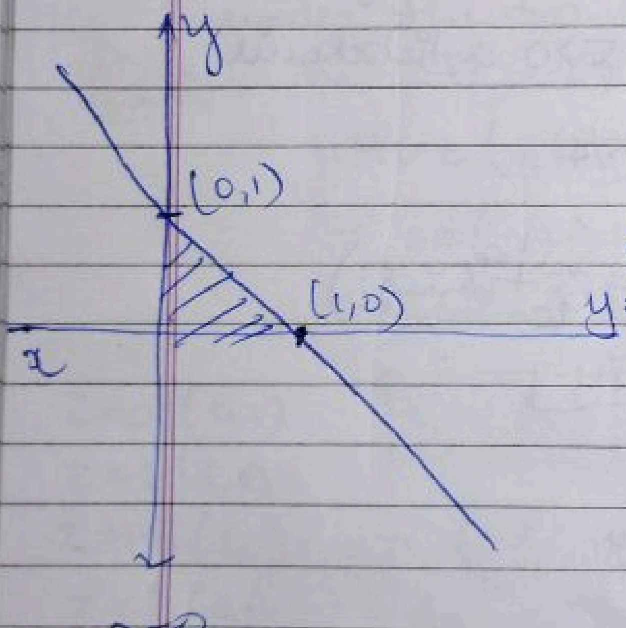
when $y=0 \Rightarrow \frac{v-u}{\sqrt{2}} = 0$

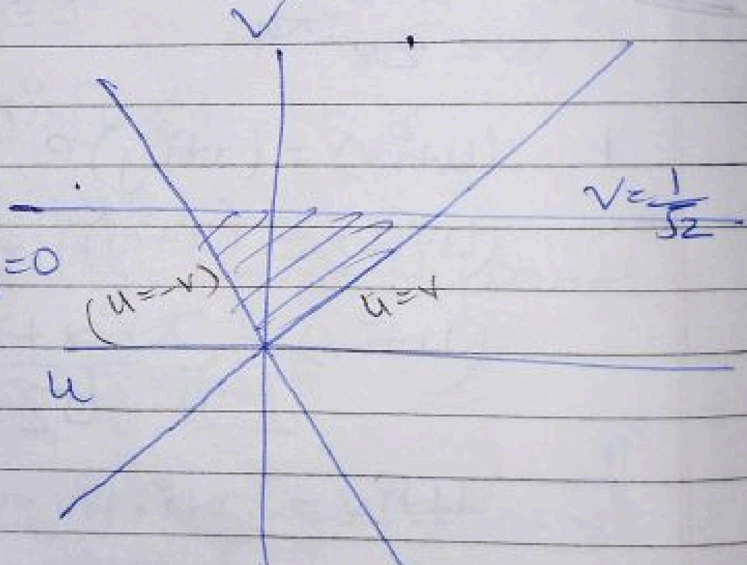
$$\boxed{v=u}$$

when $x+y=1 \Rightarrow \frac{\cancel{u}+v+v-\cancel{u}}{\sqrt{2}} = \frac{2v}{\sqrt{2}} = \sqrt{2}v$

$$v = \frac{1}{\sqrt{2}}$$

★ 3 inputs & 3 outputs





$x+y=1$

$x=0$
$y=1$

$y=0$
$x=1$

Ques: find the image of line $2x+y-3=0$. under the transformation

$$w = z+2i.$$

Sol: $w = z+c$ (eqn formula input here)

$u+iv = x+iy+2i$

$u+iv = x+i(2+y)$

$u = x$, $v = 2+y$

$2u+v-2-3=0$

$$\boxed{2u+v=5}$$

Ques: Draw the image of square whose vertices are at (0,0), (1,0), (1,1) & (0,1) in the z plane under the transformation $w=(1+i)z$ what has transformation done with original figure. (square).

Sol: $w = (1+i)z$

$w = az$ (magnification)

(a = real number)

$u+iv = (1+i)(x+iy)$

$u+iv = x+iy+xi-y$

$u+iv = (x-y)+i(x+y)$

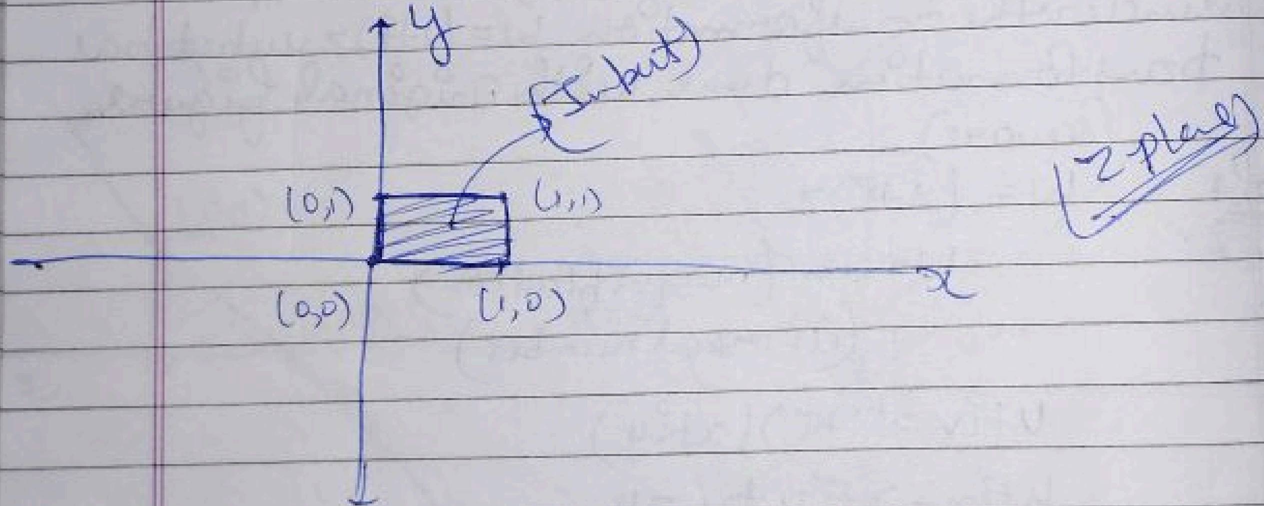
$u = x-y$ $\quad$ $v = x+y$

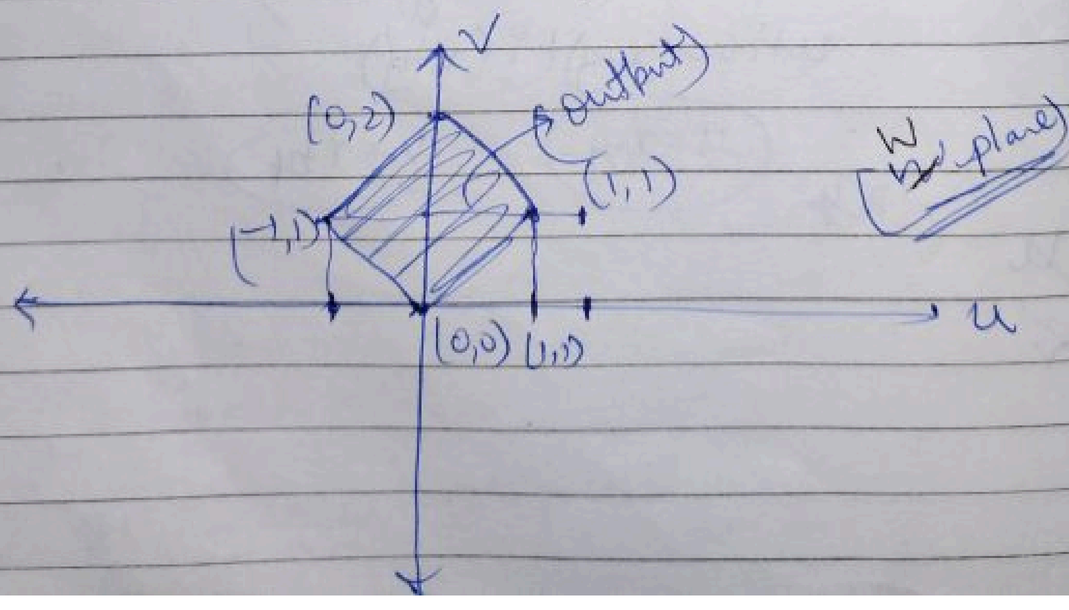
xy

u

v

(x,y)	(mediator) $u=x-y$, $v=x+y$	(u,v)
(0,0)	$u=0-0=0$, $v=0+0=0$	(0,0)
(1,0)	$1-0=1=u$, $v=1$	(1,1)
(1,1)	$u=0$, $v=2$	(0,2)
(0,1)	$u=-1$, $v=1$	(-1,1)





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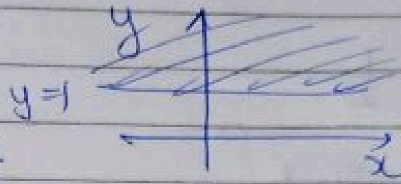
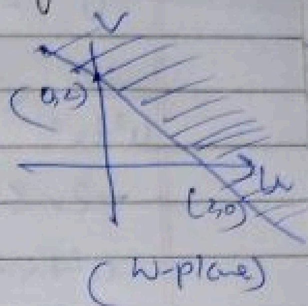
Que: find the image of $\boxed{y>1}$ (input) under the transformation $w=(1-i)z$. ~~let~~

Sol: ~~let w = (i~~





let $w=(1-i)z$

$u+iv = (1-i)(x+iy)$ (z-plane)

$u+iv = x+iy-ix+y$

$u+iv = (x+y)+i(y-x)$

equating real & Im parts

$u = x+y$ —(i), $v = y-x$ —(ii)

Solving eqn (i) & (ii)

$u+v = 2y$

$y = \frac{u+v}{2}$ Now, $y>1 \Rightarrow \frac{u+v}{2}>1$

image ← $\boxed{u+v>2}$

Que: find the image of infinite strip $\frac{1}{4} \leq y \leq \frac{1}{2}$ under the transformation $w=\frac{1}{z}$.

Sol: we have $w=\frac{1}{z}$

$z=\frac{1}{w}$

$x+iy = \frac{1}{u+iv} \times \frac{u-iv}{u-iv} = \frac{u-iv}{u^2+v^2}$

equating real & img parts

$x = \frac{u}{u^2+v^2}$, $y = \frac{-v}{u^2+v^2}$

$y \geq \frac{1}{4}$

$y \leq \frac{1}{2}$

$\frac{-v}{u^2+v^2} \leq \frac{1}{2}$

$-2v \leq u^2+v^2$

$u^2+v^2+2v \geq 0$

$u^2+v^2+2v+1 \geq 1$

$u^2+(v+1)^2 \geq 1$

Exterior of circle

C = (0,-1)

Radius = 1

$\frac{-v}{u^2+v^2} \geq \frac{1}{4}$

$-4v \geq u^2+v^2$

$u^2+v^2+4v \leq 0$

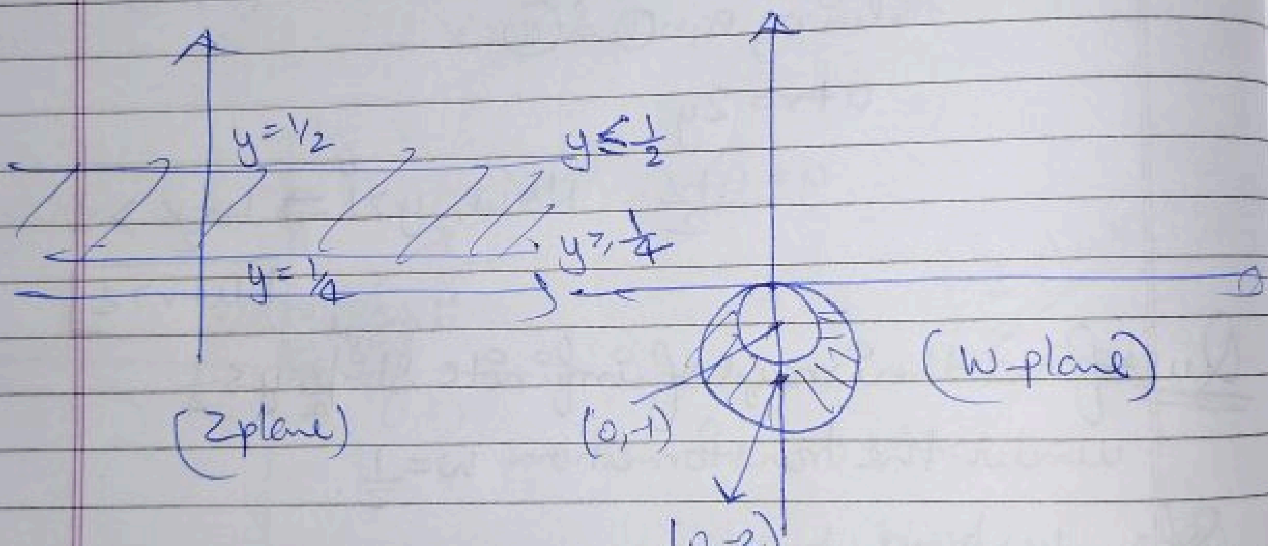
$u^2+v^2+4v+4 \leq 4$

$u^2+(v+2)^2 \leq 4$

Interior of circle

C = (0,-2)

Radius = 2



NOTE: Straight line is also known as circle of infinite radius

Ques: Find the image of $|z-3i| = 3$ under transformation $w = \frac{1}{z}$.

Sol: $w = \frac{1}{z}$

$z = \frac{1}{w}$

$x+iy = \frac{1}{u+iv}$

$x+iy = \frac{u-iv}{u^2+v^2}$

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$x = \frac{u}{u^2+v^2}$, $y = \frac{-v}{u^2+v^2}$

$|z-3i| = 3$

$|x+iy-3i| = 3$

$\left|\frac{u}{u^2+v^2} + \frac{i(-v)}{(u^2+v^2)} - 3i\right| = 3$

$\left|\frac{u-iv}{u^2+v^2} - 3i\right| = 3$

$\frac{u-iv-3i(u^2+v^2)}{u^2+v^2}$

$\sqrt{(x)^2+(y-3)^2} = \cancel{0}\,3$

$x^2+(y-3)^2 = 9$

$x = \frac{u}{u^2+v^2}$, $y = \frac{-v}{u^2+v^2}$

$\frac{u^2}{(u^2+v^2)^2} + \left(\frac{-v}{u^2+v^2} - \frac{3}{1}\right)^2 = 9$

$\frac{u^2}{(u^2+v^2)^2} + \left(\frac{(+v+3(u^2+v^2))}{(u^2+v^2)^2}\right)^2 = 9$

$u^2+v^2+\cancel{9(u^2+v^2)^2}+6v(u^2+v^2) = \cancel{9(u^2+v^2)^2}$

$(u^2+v^2)+6v(u^2+v^2) = 0$

$(u^2+v^2)(6v+1) = 0$

$v = \frac{-1}{6}$

$u^2+v^2 \neq 0$

$6v+1=0$ Ans

Que → find the image of circle $|z-1|=1$ under the transformation $w=\frac{1}{z}$.

Sol: $w=\frac{1}{z}$

$z=\frac{1}{w}$

$x=\frac{u}{u^2+v^2}$ $\quad y=\frac{-v}{u^2+v^2}$

$|x+iy-1|=1$

$\sqrt{(x-1)^2+(y)^2}=1$

$(x-1)^2+(y)^2=1$

$\left(\frac{u}{u^2+v^2}-1\right)^2+\left(\frac{-v}{u^2+v^2}\right)^2=1$

$\left(\frac{u-u^2-v^2}{u^2+v^2}\right)^2+\left(\frac{-v}{u^2+v^2}\right)^2=1$

$(u-u^2-v^2)^2+v^2=u^2+v^2$

$(u-(u^2+v^2))^2+v^2=(u^2+v^2)$

$\cancel{(u)^2}+(u^2+v^2)^2-2u(u^2+v^2)+\cancel{v^2}=\cancel{(u^2+v^2)}^2$

$(u^2+v^2)^{\cancel{2}}=2u\,\cancel{(u^2+v^2)}^2$

$\cancel{u^2+v^2}=2u$

$1=2u$

$u=\frac{1}{2}$

$1-2u=0$

Ques find the image of $|z|=3$ under mapping $w=3z$.

Sol. $w=3z$

$u+iv=3(x+iy)$

$u+iv=3x+3yi$

$u=3x, \quad v=3y$

$|x+iy|=3$

$\sqrt{x^2+y^2}=3$

$x^2+y^2=9$

$\left(\frac{u}{3}\right)^2+\left(\frac{v}{3}\right)^2=9$

$u^2+v^2=9\times 9$

$u^2+v^2=81$ $\quad r=9$

Bilinear Transformation :- (Mobius Transformation)

A transformation of the form $w=\frac{az+b}{cz+d}$ where a, b, c, d are complex constants and $ad-bc \neq 0$ (determinate of $f(z)$)

NOTE: ① Mobius transformation is conformal as well as it's inverse transformation is also bilinear

② every bilinear transformation is combination of 4 basic transformations such as rotation, magnification, translation & conversion.

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Cross Ratio Property: if there are four points z_1, z_2, z_3 & z_4 then the ratio

$\frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$ is called cross Ratio of $(z_1, z_2, z_3$ & $z_4)$

2nd

NOTE: Every Bilinear transformation preserves cross ratio property.

Invariant or fixed Points (self mapped) **:** The points which coincides with their transformation are ~~is~~ called fixed or invariant points.

i.e. $z = f(z)$

~~Ques If f(z) = log z~~

Working rule to evaluate the eqn of Bilinear Transformation

If we have given $(z_1, z_2$ & $z_3)$ & $(w_1, w_2$ & $w_3)$ three points in z plane & three points in w-plane then we have 2 methods to evaluate:

(2) Methods

① **Cross Ratio:**

z_1	z
z_2	a
z_3	b
z_4	c

apna of ardinge

w_1	w
w_2	e
w_3	f
w_4	g

By cross Ratio property:

$$\frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)} = \frac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)}$$

2nd Method: Let $w = \frac{az+b}{cz+d}$

$$w = \frac{(a/d)z+(b/d)}{(c/d)z+1}$$

$$\boxed{w = \frac{pz+q}{rz+1}} \quad \text{---(1)} \qquad \begin{array}{l} c/d \to r \\ a/d \to p \\ b/d \to q \end{array}$$

Put z_1, z_2, z_3
& w_1, w_2, w_3
we get 3 equations in 3 variables p, q & r
Solve these eqns for p, q & r & put these value of p, q & r in eqn (1) $w = \frac{pz+q}{rz+1}$

Ex: find the bilinear transformation, which maps the points $z = 1, -i, -1$ to the points $w = i, 0, -i$ respectively. Also show that transformation maps the region outside the circle $|z| = 1$ into half plane $R(w) \geq 0$.

Sol: Let $w = \frac{az+b}{cz+d} = \frac{(a/d)z+(b/d)}{(c/d)z+1}$

$$w = \frac{pz+q}{rz+c} \quad \text{---(1)} \qquad \text{where } \frac{a}{d} = p, \frac{b}{d} = q, \frac{c}{d} = r$$

$w = \frac{pz+q}{rz+1}$, $i = p+q$

z	w
$-i$	i

Que: find the bilinear transformation which maps the points $z = 1, -i, -1$ to the points $w = i, 0, -i$ respectively. Also show that the transformation maps the region outside the circle $|z| = 1$ into half plane $R(w) \geq 0$.

Sol: Let $w = \dfrac{az+b}{cz+d} = \dfrac{(a/d)z + (b/d)}{(c/d)z + 1}$

$$w = \frac{pz+q}{rz+1} \longrightarrow (1)$$

where $a/d = p$, $b/d = q$, $c/d = r$

z	w	$w = \dfrac{pz+q}{rz+1}$
1	i	$i = \dfrac{p+q}{r+1} \rightarrow ir + i = p + q$ — (2)
$-i$	0	$0 = \dfrac{-ip+q}{-ir+1} \rightarrow q = ip$ — (3)
-1	$-i$	$-i = \dfrac{-p+q}{-r+1} \Rightarrow ir - i = -p + q$ (4)

putting $q = ip$ in (2) & (4)

$$ir + i = p + ip$$

$$ir - ip + i - p = 0 \longrightarrow (5)$$

$$ir - i + p - ip = 0 \longrightarrow (6)$$

(5) - (6)

$$\cancel{ir} - \cancel{ip} + i - p = 0$$
$$\cancel{ir} - \cancel{ip} - i + p = 0$$
$$- \quad + \quad + \quad -$$

$$2i - 2p = 0$$
$$p = i$$

$$q = -1$$

Substituting the values in (2)

$$ir + i = i - 1$$
$$ir = -1$$
$$r = \frac{-1}{i} \times \frac{i}{i} = i$$

Put these values in (1)

$$w = \frac{pz+q}{rz+1}$$
$$w = \frac{iz-1}{iz+1}$$

$$a = i,\ b = -1,\ c = i,\ d = 1$$
$$ad - bc = i + i = 2i \neq 0$$

This is bilinear transformation.

Now,

$$w(iz+1) = iz - 1$$
$$z(iw - i) = -1 - w$$
$$z = \frac{1+w}{i(1-w)}$$

To find image :-

outside $|z| = 1$

$$|z| \geq 1$$
$$\left|\frac{1+w}{i(1-w)}\right| \geq 1$$

$$|1+w| \geq |i|\,|1-w|$$

$$|1+u+iv| \geq |1-u-iv|$$

$$\sqrt{(1+u)^2+v^2} \geq \sqrt{(1-u)^2+v^2}$$

$$(1+u)^2+v^2 \geq (1-u)^2+v^2$$

$$1+u^2+2u+v^2 \geq 1+u^2-2u+v^2$$

$$4u \geq 0$$

$$Re(w) \geq 0$$

Ques Determine the bilinear transformation which maps the points $z=0,-1,i$ onto $w=i,0,\infty$. Also find the image of the unit circle.

Sol: Let $w = \dfrac{az+b}{cz+d}$

$$w = \frac{(a/d)z + b/d}{(c/d)z + d/d}$$

$$w = \frac{pz+q}{rz+1} \quad \text{— (1)} \quad \text{where } p = a/d,\ q = b/d,\ r = c/d$$

z	w	$w = \frac{pz+q}{rz+1}$
0	i	$i = \frac{0+q}{0+1} \Rightarrow q = i$
-1	0	$0 = \frac{-p+q}{-r+1} \Rightarrow q = p \Rightarrow p = i$
i	∞	$\infty = \frac{pi+q}{ri+1} \Rightarrow \frac{1}{0} = \frac{ip+q}{ir+1}$

$$\Rightarrow ir+1 = 0 \Rightarrow$$

$$r = \frac{-1}{i} \times \frac{i}{i}$$

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Putting values in (1),

$$w = \frac{iz+i}{iz+1}$$

$$w(iz+1) = iz+i$$

$$wiz + w = iz + i$$

$$iz(w-1) = i - w$$

$$z = \frac{i-w}{i(w-1)}$$

$$ad - bc = i \times 1 - (i)(i) = i+1 \neq 0$$

This is required bilinear transformation

Now, $|z| = 1$

$$\left|\frac{(i-w)}{i(w-1)}\right| = 1$$

$$\frac{|i-w|}{|i|\,|(w-1)|} = 1$$

$$|i-w| = |i|\,|w-1|$$

$$\cancel{|i-(u+iv)| = |u+iv-1|}$$

$$|i-u-iv| = |(u-1)+iv|$$

$$\sqrt{u^2+(1-v)^2} = \sqrt{(u-1)^2+v^2}$$

$$u^2+1+v^2-2v = u^2+1-2u+v^2$$

$$-2v = -2u$$

$$\boxed{u = v}$$

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Ques Find the bilinear transformation which transforms the point $z = \infty, i, 0$ into $w = 0, i, \infty$ respectively.

Sol: By cross ratio property.

$$\frac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$$

z_1	z
z_2	∞
z_3	i
z_4	0

w_1	w
w_2	0
w_3	i
w_4	∞

$$\frac{(w_1-w_2)w_4\left(\frac{w_3}{w_4}-1\right)}{(w_2-w_3)w_4\left(1-\frac{w_1}{w_4}\right)} = \frac{z_2\left(\frac{z_1}{z_2}-1\right)(z_3-z_4)}{z_2\left(1-\frac{z_3}{z_2}\right)(z_4-z_1)}$$

$$\frac{(w-0)\left(\frac{i}{\infty}-1\right)}{(0-i)\left(1-\frac{w}{\infty}\right)} = \frac{\left(\frac{z}{\infty}-1\right)(i-0)}{\left(1-\frac{i}{\infty}\right)(0-z)}$$

$$\frac{-w}{-i} = \frac{-i}{-z}$$

$$wz = i^2$$

$$wz = -1$$

$$w = \frac{-1}{z}$$