Unit - 3 Vector Spaces.

A well-defined collection of objects is called set.

Binary Operation

Let G be a non-empty set

$$G \times G = \{(a,b) : a \in G, b \in G\}$$

If $f : G \times G \rightarrow G$

Then f is called binary operation on G.

Algebraic Structure

A non-empty set G with one of or more binary operations is called algebraic structure. It is denoted by $\langle G, * \rangle$, $\langle G, +, \cdot \rangle$

Group

Let $\langle G, * \rangle$ be any algebraic structure, then it will be group if G satisfies following properties w.r.t. $*$.

- Closure property:- $a * b \in G$, $\forall a, b \in G$.
- Associative property:- Let $a, b, c \in G$

  $a * (b * c) = (a * b) * c$, $\forall a, b, c \in G$.
- Existence of Identity:- e will be the identity of G if

  $a * e = e * a$, $\forall a \in G$.
- Existence of Inverse:

  Every element posses inverse in G.

  if $a \in G$, $a * a^{-1} = e = a^{-1} * a$

Abelian Group: Group which also holds commutative property.

Field: $\langle F, +, \cdot \rangle$, If it follows following properties :-

Group under addition.
- Closure
- Associative
- Identity
- Inverse

Abelian group under multiplication
- C.P.
- A.P.
- Identity
- Inverse
- Commutative

* Multiplication is distributive w.r.t. addition.

$$a\cdot(b+c) = (a\cdot b)+(a\cdot c).$$

Internal Composition

If A be any non-empty set then $a*b \in A \;\forall\; a,b \in A$

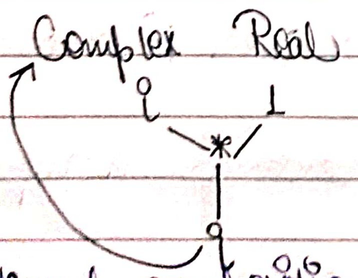
External composition

let V and F be two non-empty sets.

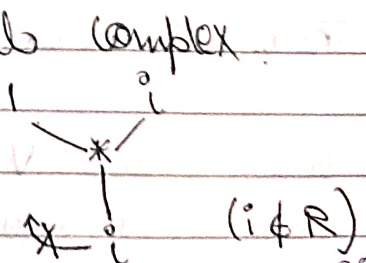
Then, $\alpha \in V$ and $a \in F$

$\Rightarrow \alpha * a \in V$

$\Rightarrow a\,\forall \alpha$ is unique.



X is external composition of C over R.



Not an external composition of R over C.

Vector Space :-

Let $<F, +, \cdot>$ be a field of scalars and V be a non-empty set (set of vectors).

Then, V is vector space over field F if

- V is abelian group w.r.t. addition
- Closure property : $a+b \in V \;\; \forall\, a, b \in V$
- associative property, $a+(b+c) = (a+b)+c, \; \forall\, a, b, c \in V$
- existence of identity : If $a \in V$, then
  $a+0 = a = 0+a, \; \forall\, a \in V$
- existence of inverse : Each element of V posess inverse in V
  i.e. $a+(-a) = 0 = (-a)+a$.
- commutative property $\; a+b = b+a \;\; \forall\, a, b \in V$

Two composition i.e. multiplication and addition of vectors satisfy :

$a\alpha \in V \;\;,\qquad \forall\, a \in F, \alpha \in V$

- $a(\alpha+\beta) = a\alpha + a\beta \qquad \forall\, a \in F, \alpha, \beta \in V$
- $(a+b)\alpha = a\alpha + b\alpha \qquad \forall\, a, b \in F, \alpha \in V$
- $(ab)\alpha = a(b\alpha) \qquad \forall\, a, b \in F$ and $\alpha \in V$
- $1.\alpha = \alpha = \alpha.1 \qquad \forall\, \alpha \in V$

Then, V is called vector space over F denoted by $V(F)$.

Q. Show that V the set of ordered n-tuples over F is a vector space.

$$V = \{(a_1, a_2, a_3, \dots a_n) : a_1, a_2, a_3 \dots a_n \in F\}$$

or

$$V = \{\alpha, \beta, \gamma, \dots\} \text{ where } \alpha = (a_1, a_2, \dots a_n), \; \beta = (b_1, b_2 \dots b_n)$$

$a_1, a_2, \dots b_1, b_2$ are scalars

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(A) V is abelian Group w.r.t. addition.

- Closure, let $\alpha \in V$, $\beta \in V$, $\alpha = (a_1, a_2, \cdots a_n)$: $\{a_1, a_2 \cdots b_1, b_2$ $\beta = (b_1, b_2, \cdots b_n)$: are in F

$$\alpha + \beta = [a_1, a_2, \cdots a_n] + [b_1, b_2, \cdots, b_n]$$
$$= [a_1+b_1, a_2+b_2, \cdots, a_n+b_n]$$ [$\because a_1+b_1, a_2+b_2 \cdots a_n+b_n$ are scalars]

V is closed w.r.t. addition.

- Associative. Let $\alpha, \beta, \gamma \in V$. Then,

$$\alpha + (\beta + \gamma) = [a_1, a_2, \cdots a_n] + [(b_1, b_2, \cdots b_n) + (c_1, c_2 \cdots c_n)]$$
$$= [a_1, a_2, \cdots a_n] + [(b_1+c_1), (b_2+c_2), (b_3+c_3) \cdots (b_n+c_n)]$$
$$= [a_1 + (b_1+c_1)], [a_2 + (b_2+c_2) \cdots a_n + (b_n+c_n)]$$
$$= [a_1 + (b_1+c_1)], [a_2 + (b_2+c_2)], \cdots, [a_n + (b_n+c_n)]$$
$$= [(a_1+b_1)+c_1], [(a_2+b_2)+c_2], \cdots [(a_n+b_n)+c_n]$$
$$= [(a_1+b_1), (a_2+b_2), \cdots (a_n+b_n)] + [c_1, c_2 \cdots c_n]$$
$$= [(a_1, a_2, \cdots a_n) + (b_1, b_2, \cdots b_n)] + [c_1, c_2, \cdots c_n]$$
$$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma \quad , \forall \alpha, \beta, \gamma \in V.$$

- Existence of identity. 0 is additive identity.
  Let $\alpha \in V$.

$$\alpha + 0 = \alpha = 0 + \alpha.$$

L.H.S. $= \alpha + 0$
$= (a_1, a_2, \cdots a_n) + (0, 0, \cdots, 0)$
$= (a_1+0, a_2+0, \cdots, a_n+0)$
$= (a_1, a_2, \cdots, a_n)$
$= \alpha$
$=$ R.H.S.

$\therefore$ Existence of identity property holds $\forall \alpha \in V$.

Existence of inverse.
Let $\alpha \in V$. The $-\alpha = (-a_1, -a_2, \ldots -a_n) \in V$ will be the inverse of α in V if,
$$\alpha + (-\alpha) = 0 = (-\alpha) + \alpha$$

$$\begin{aligned} L.H.S = & \alpha + (-\alpha) \\ = & [a_1, a_2, \ldots a_n] + [-a_1, -a_2, \ldots (-a_n)] \\ = & [a_1 + (-a_1), a_2 + (-a_2), \ldots a_n + (-a_n)] \\ = & [0, 0, \ldots 0] \in V \end{aligned}$$ such that $0 \in F$

Commutative property Let $\alpha, \beta \in V$

$$\begin{aligned} \alpha + \beta = & (a_1, a_2, \ldots a_n) + (b_1, b_2, \ldots b_n) \\ = & (a_1 + b_1, a_2 + b_2, \ldots a_n + b_n) \\ = & (b_1 + a_1, b_2 + a_2, \ldots, b_n + a_n) \quad [\text{Commutativity holds in scalar}] \\ = & (b_1, b_2, \ldots b_n) + (a_1, a_2, \ldots a_n) \\ \alpha + \beta = & \beta + \alpha, \quad \forall \alpha, \beta \in V \end{aligned}$$

$\Rightarrow$ Commutative property holds in V.
$\langle V, + \rangle$ is an abelian group w.r.t. addition.

- Let $a \in F$, $\alpha \in V$. Then,
$$\begin{aligned} a.\alpha = & a.[a_1, a_2, \ldots a_n] \\ = & [a.a_1, a.a_2, \ldots, a.a_n] \in V \end{aligned}$$ since $a.a_1, a.a_2 \ldots$ all scalars.

$\Rightarrow a.\alpha \in V$, $\forall a \in F$ and $\alpha \in V$

- $$\begin{aligned} a.(\alpha + \beta) = & a.[(a_1, a_2, \ldots a_n) + (b_1, b_2, \ldots b_n)] \\ = & a.[(a_1 + b_1), (a_2 + b_2), \ldots (a_n + b_n)] \\ = & [a.(a_1 + b_1), a.(a_2 + b_2) \ldots, a.(a_n + b_n)] \\ = & [(a.a_1 + a.b_1), (a.a_2 + a.b_2) \ldots, (a.a_n + a.b_n)] \\ = & [(a.a_1, a.a_2, \ldots a.a_n) + (a.b_1, a.b_2, \ldots a.b_n)] \\ = & [a.(a_1, a_2, \ldots a_n) + a(b_1, b_2, \ldots b_n)] \\ a.(\alpha + \beta) = & a.\alpha + a.\beta \quad \forall a \in F \;\&\; \alpha, \beta \in V \end{aligned}$$

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- Now, $(a+b).\alpha = (a+b).(a_1, a_2, \ldots a_n)$
$= a.(a_1, a_2, \ldots a_n) + b.(a_1, a_2, \ldots a_n)$
$= (a.a_1, a.a_2, \ldots a.a_n) + (b.a_1, b.a_2, \ldots b.a_n)$
$= a.\alpha + b.\alpha$

- $(a.b).\alpha = (a.b)[a_1, a_2 \ldots a_n]$
$= a[ba_1, ba_2, \ldots ba_n]$
$= a.[b(a_1, a_2, \ldots a_n)]$
$= a.[b\alpha]$

- $1.\alpha = 1.(a_1, a_2, \ldots a_n)$
$= (1.a_1, 1.a_2, \ldots 1.a_n)$
$= (a_1, a_2, \ldots a_n)$
$1.\alpha = \alpha$

Hence V is vector space w.r.t. addition and multiplication.

Q) Let V be the set of all ordered pairs of real numbers with vector addition defined as $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2 + 1, y_1 + y_2 + 1)$. Show that first five properties for vector addition satisfied?

$V = \{(x_1, y_1), (x_2, y_2), (x_3, y_3) \ldots \ldots x_1, x_2, y_1, y_2 \ldots \in R\}$
let $\alpha = (x_1, y_1)$, $\beta = (x_2, y_2)$

1) Closure property: We have,
$\alpha + \beta = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2 + 1, y_1 + y_2 + 1) \in V$
$\forall\ \alpha, \beta \in V$.
$\Rightarrow V$ is closed w.r.t. vector addition.

2) Associative property: let $\gamma = (x_3, y_3)$.
Now,
$(\alpha + \beta) + \gamma = [(x_1, y_1) + (x_2, y_2)] + (x_3, y_3)$
$= [(x_1 + x_2 + 1), (y_1 + y_2 + 1)] + (x_3, y_3)$

$= [(x_1+x_2+1+x_3+1), (y_1+y_2+1+y_3+1)]$

$= [x_1+(x_2+x_3+1)+1, \ y_1+(y_2+y_3+1)+1]$

$= (x_1, y_1) + [(x_2, y_2) + (x_3, y_3)]$

$= \alpha + (\beta + \gamma)$

Associativity holds in V.

- Existence of identity :- let (e_1, e_2) be the identity element in V.

$(x_1, y_1) + (e_1, e_2) = [x_1+e_1+1, y_1+e_2+1]$

$= (x_1, y_1)$ [By def.]

Now,

$x_1 + e_1 + 1 = x_1$ and $y_1 + e_2 + 1 = y_1$

$e_1 = -1$ $e_2 = -1$.

$\therefore (-1, -1)$ is the identity.

- Existence of inverse : let (a, b) be the inverse of (x_1, y_1) in V.

We have,

$(x_1, y_1) + (a, b) = (-1, -1)$

$[x_1+a+1, y_1+b+1] = (-1, -1)$

Equating,

$x_1 + a + 1 = -1$ and $y_1 + b + 1 = -1$.

$a = -2 - x_1$ and. $b = -2 - y_1$

Thus, $(a, b) = (-2-x_1, -2-y_1)$ is inverse of (x_1, y_1) in V.

- Commutative property :-

We show $\alpha + \beta = \beta + \alpha$.

Let $\alpha + \beta = (x_1, y_1) + (x_2, y_2)$

$= [x_1+x_2+1, y_1+y_2+1]$

$= [x_2+x_1+1, y_2+y_1+1]$

$= (x_2, y_2) + (x_1, y_1)$

Commutative prop. holds in V.

Q: Check whether $V = R^2$ is a vector space w.r.t. operation
$(u_1, u_2) + (v_1, v_2) = (u_1+v_1-2,\ u_2+v_2-3)$ and
$\alpha(u_1, u_2) = (\alpha u_1 + 2\alpha - 2,\ \alpha u_2 - 3\alpha + 3)$ $\alpha \in R$.

Let $\alpha \in R$, $(u_1, u_2) \in V \Rightarrow \alpha(u_1, u_2)$
$= [\alpha u_1 + 2\alpha - 2,\ \alpha u_2 - 3\alpha + 3] \in V$.

Again we prove,
$$a\cdot(\alpha+\beta) = a\cdot\alpha + a\cdot\beta$$
$$\begin{aligned}
L.H.S. &= a\cdot[(u_1, u_2) + (v_1, v_2)] \\
&= a[u_1+v_1-2,\ u_2+v_2-3] \\
&= [a\cdot(u_1+v_1-2)+2a-2,\ a\cdot(u_2+v_2-3)-3a+3] \\
&= [a\cdot u_1 + a\cdot v_1 - 2a + 2a - 2,\ a\cdot u_2 + a\cdot v_2 - 3a - 3a + 3]
\end{aligned}$$

$$\begin{aligned}
R.H.S. &= a\cdot\alpha + a\cdot\beta \\
&= a\cdot(u_1, u_2) + a(v_1, v_2) \\
&= [au_1 + 2a - 2,\ au_2 - 3a + 3] + [av_1 + 2a - 2,\ av_2 - 3a + 3] \\
&= [au_1 + 2a - 2 + av_1 + 2a - 2,\ au_2 - 3a + 3 + av_2 - 3a + 3] \\
&= [au_1 + av_1 + 4a - 4,\ au_2 + av_2 - 6a + 6] \\
&\neq L.H.S.
\end{aligned}$$

$\Rightarrow$ V is not a vector space.

Q. Let V be the set of all 2x3 matrices with their elements as rational numbers and F is the field of real numbers then show that V(F) is not vector space.

$$V = \left\{ \begin{bmatrix} 2 & 1 & 0 \\ 3 & 0 & 2 \end{bmatrix}, \ldots\ldots \right\}$$

$$F = \{2, \sqrt{2}, \ldots\ldots\}$$

$a\alpha = \sqrt{2}\cdot\begin{bmatrix} 2 & 1 & 0 \\ 3 & 0 & 2 \end{bmatrix} \notin V$ $[\because 2\sqrt{2}$ is irrational$]$

$\Rightarrow$ V is not vector space $\left[\because a \in F, \alpha \in V \Rightarrow a\cdot\alpha \notin V\right]$

Q. Let V be the set of all pairs (x, y) of real numbers and R be the field of real numbers. Examine in each case whether V is vector space over F or not.

i) $(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$

$c.(x, y) = (0, cy)$

ii) $(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$

$c.(x, y) = (c^2 x, c^2 y)$

$(5+1)(x, y) + (36x, 36y)$

$6x + 6y$

iii) $(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$

$c.(x, y) = (|c| x, |c| y)$

i) We have to show

$$1.\alpha = \alpha.$$

$$1.\alpha = 1.(x, y)$$
$$= (0, y)$$
$$\neq \alpha$$

$\Rightarrow$ V is not vector space over R.

ii) We have,

$$(a+b)\alpha = a\alpha + b\alpha$$

$$\text{L.H.S.} = (a+b).\alpha$$
$$= (a+b).(x, y)$$
$$= [(a+b)^2 x, (a+b)^2 y]$$

$$\text{R.H.S.} = a.\alpha + b.\alpha$$
$$= a.(x, y) + b(x, y)$$
$$= (a^2 x, a^2 y) + (b^2 x, b^2 y)$$
$$\neq \text{L.H.S}$$

$\Rightarrow$ V is not vector space over R.

Vector Subspaces

Let V be the vector space over field F, then a non-empty subset W of V is called vector subspace of V if W itself a vector space w.r.t. vector addition and scaler multiplication in V.

To prove W is vector subspace of V, we use necessary and sufficient conditions.

(i) $\alpha + \beta \in W \quad , \forall \alpha, \beta \in W$

$a.\alpha \in W \quad , \forall \alpha \in W \ \& \ a \in F$

OR

(ii) $a.\alpha + b\beta \in W \quad , \forall \alpha, \beta \in W$ and $a, b \in F$

OR

(iii) $\alpha - \beta \in W \quad , \forall \alpha, \beta \in W$

$a.\alpha \in W \quad , a \in F, \alpha \in W$

Q: Consider the following subset of 3-tuples space $R^3(R)$. Check they are vector subspace or not.

(i) $W_1 = \{(a_1, a_2, 0) : a_1, a_2 \in R\}$

(ii) $W_2 = \{(a_1, 0, a_3) : a_1, a_3 \in R\}$

(iii) $W_3 = \{(0, 0, a_3) : a_3 \in R\}$

(i) Let $\alpha, \beta \in W$ and $a, b \in R$

$\alpha = (a_1, a_2, 0) \quad : a_1, a_2 \in R$

$\beta = (b_1, b_2, 0) \quad : b_1, b_2 \in R$

Now,

$$a.\alpha + b.\beta = a(a_1, a_2, 0) + b(b_1, b_2, 0)$$
$$= (aa_1, aa_2, 0) + (bb_1, bb_2, 0)$$
$$= (aa_1 + bb_1, aa_2 + bb_2, 0) \in W$$

$\Rightarrow a\alpha + b\beta \in W \quad \forall\, a, b \in R,\ \alpha, \beta \in W.$

Hence by necessary and sufficient conditions W is vector subspace of $R^3(R)$.

(ii) Let $\alpha, \beta \in W$ and $a, b \in R$

$\alpha = (a_1, 0, a_3) \quad : a_1, a_3 \in R$

$\beta = (b_1, 0, b_3) \quad : b_1, b_3 \in R$

Now,

$$a.\alpha + b\beta = a(a_1, 0, a_3) + b(b_1, 0, b_3)$$
$$= (aa_1, 0, aa_3) + (bb_1, 0, bb_3)$$
$$= (aa_1 + bb_1, 0, aa_3 + bb_3) \in W$$

$\Rightarrow$ ~~$a\alpha + b$~~ $a\alpha + b\beta \in W \quad \forall\, a, b \in R$ and $\alpha, \beta \in W$.

Hence by necessary and sufficient conditions W is vector subspace of $R^3(R)$.

(iii) Let $\alpha, \beta \in W$ and $a, b \in R$

$\alpha = (0, 0, a_3) \quad : a_3 \in R$

$\beta = (0, 0, b_3) \quad : b_3 \in R.$

Now,

$$a.\alpha + b\beta = a(0, 0, a_3) + b(0, 0, b_3)$$
$$= (0, 0, aa_3) + (0, 0, bb_3)$$
$$= (0, 0, aa_3 + bb_3) \in W$$

$\Rightarrow a\alpha + b\beta \in W \quad \forall\, a, b \in R$ and $\alpha, \beta \in W$

Hence by necessary and sufficient conditions W is vector subspace of $R^3(R)$

Q. Show that subset $W = \{(a,b,c) : a+b+c = 0\}$ is subspace of R^3.

Let $\alpha, \beta \in W$

$$\alpha = (a_1, b_1, c_1) : a_1+b_1+c_1 = 0 \quad \text{——} ①$$
$$\beta = (a_2, b_2, c_2) : a_2+b_2+c_2 = 0 \quad \text{——} ②$$

For $a, b \in R$,

Now, $a.\alpha + b.\beta = a(a_1, b_1, c_1) + b.(a_2, b_2, c_2)$

$$= (aa_1, ab_1, ac_1) + (ba_2, bb_2, bc_2)$$
$$= (\underbrace{aa_1 + ba_2}_{A}, \underbrace{ab_1+bb_2}_{B}, \underbrace{ac_1+bc_2}_{C})$$

Now, we have to show, $A+B+C=0$

$$A+B+C = (aa_1+ba_2) + (ab_1+bb_2) + (ac_1+bc_2)$$
$$= a(a_1+b_1+c_1) + b(a_2+b_2+c_2)$$
$$= a.0 + b.0 \quad [\text{From ① and ②}]$$
$$= 0$$

$\Rightarrow a.\alpha + b.\beta \in W, \ \forall \ a, b \in R$ and $\alpha, \beta \in W$

By necessary and sufficient conditions W is vector subspace.

Q. $W = \{(x,y,z) : x-3y+4z = 0\}$. Show that W is vector subspace of R^3.

Let $\alpha, \beta \in W$

$$\alpha = (x_1, y_1, z_1) : x_1-3y_1+4z_1 = 0 \quad \text{——} ①$$
$$\beta = (x_2, y_2, z_2) : x_2-3y_2+4z_2 = 0 \quad \text{——} ②$$

For $a, b \in R$,

Now, $a.\alpha + b.\beta = a(x_1, y_1, z_1) + b(x_2, y_2, z_2)$

$$= (ax_1, ay_1, az_1) + (bx_2, by_2, bz_2)$$
$$= (\underbrace{ax_1+bx_2}_{X}, \underbrace{ay_1+by_2}_{Y}, \underbrace{az_1+bz_2}_{Z})$$

Now, we have to show, $\cancel{X+Y+Z=0}$ $X-3Y+4Z=0$

$$X-3Y+4Z = (ax_1+bx_2) - 3(ay_1+by_2) + 4(az_1+bz_2)$$
$$= a(x_1-3y_1+4z_1) + b(x_2-3y_2+4z_2)$$
$$= a.0 + b.0$$

$$X-3Y+4Z = 0$$

$\Rightarrow a.\alpha + b.\beta \in W$, $\forall\, a,b \in R$ and $\alpha, \beta \in W$.

Therefore, by necessary and sufficient condition W is subspace.

Q) Let V be the vector space of 2×2 matrices over field R. Show that W is not a subspace of V where W contains all 2×2 matrices with zero determinate.

Let $\alpha, \beta \in W$ $\quad \alpha = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}_{2\times2}$, $\beta = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}_{2\times2}$ $\quad |\alpha| = 0, |\beta| = 0$

$$\alpha+\beta = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}_{2\times2}$$

but $|\alpha+\beta| = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 \neq 0$

Thus, $\alpha \in W$, $\beta \in W$ but $\alpha+\beta \notin W$

$\Rightarrow$ W is not a subspace of V.

Q. Let $V_3(R)$ be the vector space. Check ~~Show that~~ following subsets of $V_3(R)$ are subspace or not.

(i) $W_1 = \{(x, x, x) : x \in R\}$

(ii) $W_2 = \{(x, 2y, 3z) : x, y, z \in R\}$

(iii) $W_3 = \{(x_1, x_2, x_3) : a_1x_1 + a_2x_2 + a_3x_3 = 0\}$, where a_1, a_2 & a_3 are fixed elements of R and $x_1, x_2, x_3 \in R$.

(i) Let $\alpha, \beta \in W_1$, $\alpha = (a_1, b_1, c_1) = (a_1, a_1, a_1)$
$\beta = (a_2, b_2, c_2) = (a_2, a_2, a_2)$.

Let $a, b \in R$,
Now, $a.\alpha + b\beta = a.(a_1, a_1, a_1) + b(a_2, a_2, a_2)$
$= (aa_1, aa_1, aa_1) + (ba_2, ba_2, ba_2)$
$a\alpha + b\beta = (aa_1 + ba_2, aa_1 + ba_2, aa_1 + ba_2) \in W_1$; $aa_1 + ba_2 \in R$

Therefore by necessary and sufficient conditions W_1 is subspace $\forall a, b \in R$, $\alpha, \beta \in W$.

(ii) Let $\alpha, \beta \in W_2$, $\alpha = (x_1, 2y_1, 3z_1)$, $\forall x_1, 2y_1, z_1 \in R$
$\beta = (x_2, 2y_2, 3z_2)$, $\forall x_2, y_2, z_2 \in R$.

Let $a, b \in R$.
Now, $a.\alpha + b\beta = a.(x_1, 2y_1, 3z_1) + b(x_2, 2y_2, 3z_2)$
$= (ax_1, a2y_1, a.3z_1) + (bx_2, 2by_2, 3bz_2)$
$= (ax_1 + bx_2, 2ay_1 + 2by_2, 3az_1 + 3bz_2)$
$= (ax_1 + bx_2, 2(ay_1 + by_2), 3(az_1 + bz_2))$

Hence $ax_1 + bx_2 \in R$, $ay_1 + by_2 \in R$, $az_1 + bz_2 \in R$.
Therefore, $a\alpha + b\beta \in W_2$

$\Rightarrow$ By necessary and sufficient conditions W_2 is subspace, $\forall a, b \in R$ and $\alpha, \beta \in W_2$.

(iii) Let $\alpha, \beta \in W_3$, $\alpha = (x_1, x_2, x_3)$: $a_1x_1 + a_2x_2 + a_3x_3 = 0$ — (1)
$\beta = (y_1, y_2, y_3)$: $a_1y_1 + a_2y_2 + a_3y_3 = 0$ — (2)

Let $a, b \in R$.
Now, $a\alpha + b\beta = a.(x_1, x_2, x_3) + b(y_1, y_2, y_3)$
$= (ax_1, ax_2, ax_3) + (by_1, by_2, by_3)$
$= (ax_1 + by_1, ax_2 + by_2, ax_3 + by_3)$

Now, we have to show $a\alpha + b\beta \in W_3 \Rightarrow a_1(ax_1 + by_1) + a_2(ax_2 + by_2) + a_3(ax_3 + by_3) = 0$
$a\alpha + b\beta$: $a_1(ax_1 + by_1) + a_2(ax_2 + by_2) + a_3(ax_3 + by_3)$
$= aa_1x_1 + ba_1y_1 + aa_2x_2 + ba_2y_2 + aa_3x_3 + ba_3y_3$

$= a(a_1x_1 + a_2x_2 + a_3x_3) + b(a_1y_1 + a_2y_2 + a_3y_3)$

$= a\cdot 0 + b\cdot 0$ [From ① and ②]

$= 0$

$\Rightarrow a\alpha + b\beta \in W_3$, $\forall\, a, b \in R$ and $\alpha, \beta \in W_3$.

Therefore by necessary and sufficient conditions W_3 is subspace.

Q. Let V be the set of all ordered pairs $(x, y) : x, y \in R$

Examine V is vector space or not for composition.

$$(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$$

$$c.(x, y) = (0, cy).$$

We have to show $1.\alpha = \alpha$.

Let $\alpha = (x, y)$.

$$1.(x, y) = (0, y)$$
$$\neq \alpha$$

$\Rightarrow$ V is not vector space for composition.

Basis and Dimension

Let $V(F)$ be a vector space over field F and $S \subset V$ such that $S = \{v_1, v_2, v_3, \dots v_n\}$

Then S is said to be basis of V if

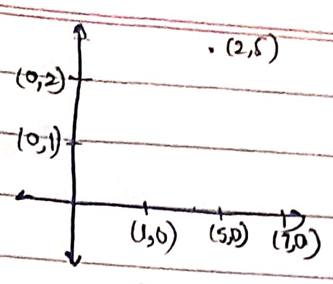
- $v_1, v_2, \dots v_n$ all are linearly independent vectors.
- $V = L(S)$ i.e. S generate V.

eg. $S = \{(1,0), (0,1)\}$. Show that S is basis of R^2.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\rho(A) = 2 =$ no. of vectors

$\Rightarrow$ S is set of L.I. vectors.



Hence S is basis of R^2

$$\dim(R^2) = 2$$

Dimension

The number of elements in the basis is known as dimension of vector space and it is denoted by $\dim(V)$.

Note • There exists a basis for finite dimensional vector space.

• If V is finite dim vector space then two basis of V have same number of elements.

• Every linearly independent subset of finite dimensional vector space $V(F)$ can be extended to form the basis of $V(F)$.

eg: Determine whether the given vectors $V_1 = (1,-1,1)$, $V_2 = (0,1,2)$, $V_3 = (3,0,-1)$ form a basis of R^3.

Let c_1, c_2, c_3 be the scalars such that

$$c_1V_1 + c_2V_2 + c_3V_3 = 0 \quad \text{(Linear combination)}.$$

$$c_1(1,-1,1) + c_2(0,1,2) + c_3(3,0,-1) = 0$$

$$\left.\begin{aligned} c_1 + 0.c_2 + 3c_3 &= 0 \\ -c_1 + c_2 + 0.c_3 &= 0 \\ c_1 + 2c_2 + (-c_3) &= 0 \end{aligned}\right\} \text{Homogeneous system of eq}^n.$$

$$A = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

$$R_2 \to R_2 + R_1, \quad R_3 \to R_3 - R_1$$

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 2 & -4 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & -10 \end{bmatrix}$$

$\rho(A) = 3 =$ no. of vectors.

$\Rightarrow$ Vectors are L.I.

Also, $R^3 = L(S)$

$\Rightarrow S$ form basis of R^3.

Ques. Show that $S = \{(1,2,1), (2,1,0), (1,-1,2)\}$ forms a basis of $V_3(F)$. Express each of standard basis vector e_i, $i = 1,2,3$ as linear combination of above vectors.

Let c_1, c_2, c_3 be the scalars such that

$$c_1V_1 + c_2V_2 + c_3V_3 = 0 \quad \text{——} \quad (1)$$

$$c_1(1,2,1) + c_2(2,1,0) + c_3(1,-1,2) = 0$$

$$c_1 + 2c_2 + c_3 = 0$$

$$2c_1 + c_2 - c_3 = 0$$

$$c_1 + 0.c_2 + 2c_3 = 0$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -3 \\ 0 & -2 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_3$

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$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -4 \\ 0 & -2 & 1 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -4 \\ 0 & 0 & 9 \end{bmatrix}$$

$\rho(A) = 3 =$ no. of vectors.

$\Rightarrow$ S is L.I. ; $c_1 = c_2 = c_3 = 0$.

Also, $V_3(F) = L(S)$

$\Rightarrow$ S form basis of $V_3(F)$

~~Show that $S = \{$~~

Now, $e_1 = k_1 V_1 + k_2 V_2 + k_3 V_3$ —— (2)

$(1, 0, 0) = k_1 (1, 2, 1) + k_2 (2, 1, 0) + k_3 (1, -1, 2)$ → Non-homogeneous system

$$[A \,\vdots\, B] = \left[\begin{array}{ccc:c} 1 & 2 & 1 & 1 \\ 2 & 1 & -1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & 1 \\ 0 & -3 & -3 & -2 \\ 0 & -2 & 1 & -1 \end{array}\right]$$

$R_2 \to R_2 - R_3$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & 1 \\ 0 & -1 & -4 & -1 \\ 0 & -2 & 1 & -1 \end{array}\right]$$

$R_3 \to R_3 - 2R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -1 & -4 & -1 \\ 0 & 0 & 9 & 1 \end{array}\right]$$

$\rho(A) = 3 =$ no. of unknowns.

$\Rightarrow$ unique soln. exists.

$$A'X = B'$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -4 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$K_1 + 2K_2 + K_3 = 1$$
$$-K_2 - 4K_3 = -1$$
$$9K_3 = 1$$
$$\Rightarrow K_3 = 1/9$$

$$-K_2 - 4\left(\frac{1}{9}\right) = -1$$

$$K_2 = 1 - 4/9 = 5/9$$

$$K_1 + \frac{10}{9} + \frac{1}{9} = 1$$

$$K_1 = 1 - 11/9 = -2/9$$

Put these values in eqn ②, $K_1V_1 + K_2V_2 + K_3V_3 = e_1$

$$-\frac{2}{9}(1,2,1) + \frac{5}{9}(2,1,0) + \frac{1}{9}(1,-1,2) = (1,0,0)$$

Eg. Let V be the vector space of 2x2 matrices over field F and

$$S = \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right], \text{ then}$$

Show that S is basis of V(F).

Let K_1, K_2, K_3 and K_4 are scalars such that

$$K_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + K_2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + K_3 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + K_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} K_1 & K_3 \\ K_2 & K_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow$ Vectors are linearly independent.

Also, $\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \alpha a_1 + \beta a_2 + \gamma a_3 + \delta a_4$, where a_1, a_2, a_3 & a_4 are scalars

e.g. $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} = 2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 1 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 3 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Hence S form a basis of V(F).

Ques. Show that $S = \{(1,0), (i,0), (0,1), (0,i)\}$ form a basis of C(R).

$$A = \begin{bmatrix} 1 & i & 0 & 0 \\ 0 & 0 & 1 & i \end{bmatrix}$$

Ques. Let V be the set of ordered pair of complex number over the real field R i.e. V be the vector space C(R). Show that $S = \{(1,0), (i,0), (0,1), (0,i)\}$ is a basis of V.

To show S is linearly independent:-

$$a(1,0) + b(i,0) + c(0,1) + d(0,i) = (0,0).$$

$$(a+ib, c+id) = (0,0)$$

$$\Rightarrow a=0, \; b=0, \; c=0, \; d=0$$

$\Rightarrow$ S is linearly independent.

Also, $C(R) = L(S)$ or $(K_1, K_2) = a(1,0) + b(i,0) + c(0,1) + d(0,i)$

$\Rightarrow$ S form basis of $C(R)$.

Ques. $(x,y) + (x_1, y_1) = (3y + 3y_1, -x - x_1)$

$c(x,y) = (3cy, -cx)$

Not a vector space.

Let $\alpha = (1,2)$, $\beta = (3,4)$, $\gamma = (5,6)$

From associative property,

$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$

L.H.S. $= (\alpha + \beta) + \gamma$

$= [(1,2) + (3,4)] + (5,6)$

$= (18, -4) + (5,6)$

$= (6, -23)$

R.H.S. $= \alpha + (\beta + \gamma)$

$= (1,2) + [(3,4) + (5,6)]$

$= (1,2) + [(30, -8)]$

$= (-18, -31)$

$\Rightarrow$ L.H.S $\neq$ R.H.S.

Hence property of associativity does not holds

Therefore given vector is not a vector space.

Q. Find the coordinate vector of $(1,0,-1)$ with respect to the basis of $e_1(0,1,-1)$, $e_2 = (1,1,0)$, $e_3(1,0,2)$

let x_1, x_2, x_3 be coordinate vector of V

$(1,0,-1) = x_1(0,1,-1) + x_2(1,1,0) + x_3(1,0,2)$.

D) $x_1 + x_2 + x_3 = 1 \Rightarrow x_2 + x_3 = 1$.

$x_1 + x_2 = 0$

$-x_1 + 2x_3 = -1$.

$$[A:B] = \left[\begin{array}{ccc:c} 0 & 1 & -1 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 2 & -1 \end{array}\right]$$

$R_1 \longleftrightarrow R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ -1 & 0 & 2 & -1 \end{array}\right]$$

$R_3 \to R_3 + R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

$R_3 \longrightarrow R_3 - R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 3 & -2 \end{array}\right]$$

$\rho(A:B) = \rho(A) = 3 =$ no. of unknowns.

$\Rightarrow$ Vectors are linearly independent and unique soln. exist.

$A'X = B'$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

$x_1 + x_2 = 0$

$x_2 - x_3 = 1$

$3x_3 = -2$

Q. Is the vector $(3,-1,0,-1)$ in the subspace of R^4 spanned by the vectors $(2,-1,3,2)$, $(-1,1,1,-3)$, $(1,1,9,-5)$?

$\alpha = (3,-1,0,-1)$, $\alpha_1 = (2,-1,3,2)$, $\alpha_2 = (-1,1,1,-3)$, $\alpha_3 = (1,1,9,-5)$.

If α can be expressed as linear combination of vectors $\alpha_1, \alpha_2, \alpha_3$ then it will be in the subspace of R^4 spanned by these vectors otherwise not.

$$\alpha = a\alpha_1 + b\alpha_2 + c\alpha_3$$

$$(3,-1,0,-1) = a(2,-1,3,2) + b(-1,1,1,-3) + c(1,1,9,-5)$$

$2a - b + c = 3$ —— ①

$-a + b + c = -1$ —— ②

$3a + b + 9c = 0$ —— ③

$2a - 3b - 5c = -1$ —— ④

~~On solving ① and ②,~~

$$\cancel{\begin{aligned} 2a - b + c &= 3 \\ -a + b + c &= -1 \\ \hline a + 2c &= 2 \end{aligned}}$$

~~On solving ② and ③,~~

$$\cancel{\begin{aligned} 3a + b + 9c &= 0 \\ -a + b + c &= -1 \\ (+) \quad (-) \quad (-) &\quad (+) \\ \hline 4a + 8c &= 1 \end{aligned}}$$

$$[A \vdots B] = \left[\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ -1 & 1 & 1 & -1 \\ 3 & 1 & 9 & 0 \\ 2 & -3 & -5 & -1 \end{array}\right]$$

$R_4 \to R_4 - R_1$ and $R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ -1 & 1 & 1 & -1 \\ 1 & 2 & 8 & -3 \\ 0 & -2 & -6 & -4 \end{array}\right]$$

$R_1 \longleftrightarrow R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 8 & -3 \\ -1 & 1 & 1 & -1 \\ 2 & -1 & 1 & 3 \\ 0 & -2 & -6 & -4 \end{array}\right]$$

$R_2 \longrightarrow R_2 + R_1$ and $R_3 \rightarrow R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 8 & -3 \\ 0 & 3 & 9 & -4 \\ 0 & -5 & -15 & 9 \\ 0 & -2 & -6 & -4 \end{array}\right]$$

$R_3 \rightarrow R_3 + R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 8 & -3 \\ 0 & 3 & 9 & -4 \\ 0 & -2 & -6 & 5 \\ 0 & -2 & -6 & -4 \end{array}\right]$$

$R_4 \longrightarrow R_4 - R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 8 & -3 \\ 0 & 3 & 9 & -4 \\ 0 & -2 & -6 & 5 \\ 0 & 0 & 0 & -9 \end{array}\right]$$

$R_2 \longrightarrow R_2 + R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 8 & -3 \\ 0 & 1 & 3 & 1 \\ 0 & -2 & -6 & 5 \\ 0 & 0 & 0 & -9 \end{array}\right]$$

$R_3 \rightarrow R_3 + 2R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 8 & -3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & -9 \end{array}\right]$$

$\rho(A \vdots B) = 4$

$\rho(A) = 2$

$\Rightarrow$ No soln exists.

Hence given vector is not a subspace of R^4 spanned by the given vectors.

Q. $V = \{$ set of all $n \times n$ matrices $\}$
$W = \{$ set of all symmetric matrices of order $n \}$

$a\alpha + b\beta \in W$

Let $\alpha, \beta \in W$ Then, $\alpha^T = \alpha$ and $\beta^T = \beta$
Now we show $(a\alpha + b\beta)^T = a\alpha + b\beta$
We have, $(a\alpha + b\beta)^T = (a\alpha)^T + (b\beta)^T$
$= a\alpha^T + b\beta^T$
$(a\alpha + b\beta)^T = a\alpha + b\beta$
$\Rightarrow (a\alpha + b\beta) \in W$, $\forall\, a, b \in F$ and $\alpha, \beta \in W$.
Hence W is a subspace of $V(F)$.

Q. Show that $S = \left\{ \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix} \right\}$.
S form a basis for $M_{2\times2}$.

Soln To show S is the set of L.I. set of vectors.

let $K_1V_1 + K_2V_2 + K_3V_3 + K_4V_4 = 0$

$$K_1\begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix} + K_2\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + K_3\begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix} + K_4\begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} K_1 & 2K_1 - K_2 + 2K_3 \\ K_1 - K_2 + 3K_3 - K_4 & -2K_1 + K_3 + 2K_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$K_1 = 0$
$2K_1 - K_2 + 2K_3 = 0$
$K_1 - K_2 + 3K_3 - K_4 = 0$
$-2K_1 + K_3 + 2K_4 = 0$

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$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 0 \\ 1 & -1 & 3 & -1 \\ -2 & 0 & 1 & 2 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_4 \to R_4 + 2R_1$

$R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$R_4 \to R_4 - R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$\rho(A) = 4 =$ no. of unknowns. i.e. $K_1 = K_2 = K_3 = K_4 = 0$.

$\therefore$ linearly independent.

Also, $\begin{bmatrix} a & c \\ b & d \end{bmatrix} = a_1 \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix} + a_2 \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix} + a_4 \begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix}$

$\Rightarrow M_{2\times 2} = L(S)$

$\Rightarrow S$ is basis for $M_{2\times 2}$.

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Q. Show that set $S = [(1, i, 0), (2i, 1, 1), (0, 1+i, 1-i)]$ form a basis of $V_3(C)$.

Let K_1, K_2, K_3 be scalars such that $K_1V_1 + K_2V_2 + K_3V_3 = 0$

$$A = \begin{bmatrix} 1 & 2i & 0 \\ i & 1 & 1+i \\ 0 & 1 & 1-i \end{bmatrix}$$

$R_2 \to R_2 - iR_1$

$$\sim \begin{bmatrix} 1 & 2i & 0 \\ 0 & 3 & 1+i \\ 0 & 1 & 1-i \end{bmatrix}$$

$R_2 \to R_2 - 3R_3$

$$\sim \begin{bmatrix} 1 & 2i & 0 \\ 0 & 0 & -2+4i \\ 0 & 1 & 1-i \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 2i & 0 \\ 0 & 1 & 1-i \\ 0 & 0 & -2+4i \end{bmatrix}$$

$\rho(A) = 3 =$ no. of unknowns ie $K_1 = K_2 = K_3 = 0$

$\Rightarrow$ S is linearly independent.

Also, $C(R) = L(S)$

$\Rightarrow$ S form a basis of $C(R)$.

Coordinate of a vector w.r.t. basis :-

Let $V(F)$ be the finite dimensional vector space.
Let $B = (\alpha_1, \alpha_2 \cdots \alpha_n)$ be the ordered basis of $V(F)$
Let $\alpha \in V$, then there exists $(x_1, x_2 \cdots x_n)$ n tuple set.

$$\alpha = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n$$

Then $(x_1, x_2, \cdots x_n)$ is called coordinates of α w.r.t. basis.

Ques. Let $B = \{\alpha_1, \alpha_2, \alpha_3\}$ be a ordered basis of $R^3(R)$ where $\alpha_1 = (1,0,-1)$, $\alpha_2 = (1,1,1)$ and $\alpha_3 = (1,0,0)$. Obtain the coordinates of the vector (a,b,c) in a ordered basis B.

Soln. To find coordinates :-

Let (K_1, K_2, K_3) be the required coordinate of (a,b,c) in B, then

$$K_1\alpha_1 + K_2\alpha_2 + K_3\alpha_3 = (a,b,c)$$

$$K_1(1,0,-1) + K_2(1,1,1) + K_3(1,0,0) = (a,b,c)$$

$$\left.\begin{aligned} K_1 + K_2 + K_3 &= a \\ K_2 &= b \\ -K_1 + K_2 + 0.K_3 &= c \end{aligned}\right\} \text{Non-homogeneous.}$$

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 0 & b \\ -1 & 1 & 0 & c \end{array}\right]$$

$R_3 \to R_3 + R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 0 & b \\ 0 & 2 & 1 & a+c \end{array}\right]$$

$R_3 \to R_3 - 2R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & a+c-2b \end{array}\right]$$

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$\rho(A \vdots B) = \rho(A) = 3$

To find soln

$$A'X = B'$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ a+c-2b \end{bmatrix}$$

$$\begin{bmatrix} K_1+K_2+K_3 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ a+c-2b \end{bmatrix}$$

$$K_3 = a+c-2b$$

$$K_2 = b$$

$$K_1+K_2+K_3 = a$$

$$K_1 + b + \cancel{a} + c - 2b = \cancel{a}$$

$$K_1 = -c+b$$

Coordinates of (a, b, c) is (K_1, K_2, K_3) i.e. $(b-c, b, a-2b+c)$.