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Unit - 2 Matrix Factorization.

LU Decomposition Method.

L → lower Triangular matrix

U → Upper Triangular matrix

This method is also known as triangulization method.

In this method, coefficient matrix A of system of equation $AX = B$ is decomposed into product of lower triangular and upper triangular matrix.

so that, $A = LU$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n\times n} = \begin{bmatrix} l_{11} & 0 & 0 & \cdots \\ l_{21} & l_{22} & 0 & 0 \cdots \\ l_{31} & \cdots & & \\ l_{n1} & l_{n2} & l_{n3} \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ 0 & 0 & \cdots & u_{3n} \\ - & - & \cdots & u_{nn} \end{bmatrix}$$

Using matrix multiplication, multiply L and U and equate with A.

Find L and U.

Now, let $AX = B$

$LUX = B$ [Put $A = LU$]

$LY = B$ [let $UX = Y$]

where $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

Find Y and put in the eq^n.

This method will be failed if any of the diagonal element l_{ii} or u_{ii} is zero.

We can choose $l_{ii} = 1$ or $u_{ii} = 1$ but not both.

Ques Solve by LU Decomposition method

$$2x+3y+z=9$$
$$x+2y+3z=6$$
$$3x+y+2z=8$$

Let $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$

$$A = LU$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}\cdot u_{11} & l_{21}u_{12}+u_{22} & u_{13}\cdot l_{21}+u_{23} \\ l_{31}\cdot u_{11} & l_{31}\cdot u_{12}+l_{32}\cdot u_{22} & l_{31}\cdot u_{13}+l_{32}\cdot u_{23}+u_{33} \end{bmatrix}$$

Equating,

Ist row, $u_{11}=2$, $u_{12}=3$, $u_{13}=1$

Ist column, $l_{21}\cdot u_{11}=1$, $l_{31}\cdot u_{11}=3$

$\Rightarrow l_{21}=1/2$, $l_{31}=3/2$

IInd column, $l_{21}\cdot u_{12}+u_{22}=2$

$$1/2 \cdot 3 + u_{22} = 2$$
$$u_{22} = 2 - 3/2 = \underline{1/2}$$

$$l_{31}\cdot u_{12}+l_{32}\cdot u_{22} = 1$$
$$\frac{3}{2}\cdot 3 + l_{32}\cdot\frac{1}{2} = 1$$

$$9 + l_{32} = 2$$
$$\boxed{l_{32} = -7}$$

IIIrd column, $u_{13}\cdot l_{21}+u_{23} = 3$

$$1\cdot 1/2 + u_{23} = 3$$
$$u_{23} = 3 - 1/2 = \underline{5/2}$$

$l_{31} \cdot u_{13} + l_{32} \cdot u_{23} + u_{33} = 2$

$\frac{3}{2} \cdot 1 + (-7)\left(\frac{5}{2}\right) + u_{33} = 2$

$u_{33} = 2 - \frac{3}{2} + \frac{35}{2}$

$u_{33} = \frac{4-3+35}{2} = \frac{36}{2} = 18.$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix}$$

To solve equations.

$AX = B$

$LUX = B$ [Put $A = LU$]

$LY = B$ [Put $UX = Y$]

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix} Y = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_1/2 + y_2 \\ \frac{3}{2}y_1 - 7y_2 + y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

$y_1 = 9$

$\frac{9}{2} + y_2 = 6$

$\Rightarrow \quad y_2 = 6 - 9/2 = 3/2$

$\frac{3}{2}(9) - 7\left(\frac{3}{2}\right) + y_3 = 8$

$y_3 = 5.$

$$Y = \begin{bmatrix} 9 \\ 3/2 \\ 5 \end{bmatrix}$$

Now using eq^n $UX = Y$

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 3/2 \\ 5 \end{bmatrix}$$

$$2x + 3y + z = 9 \quad \text{—— (1)}$$

$$\frac{y}{2} + \frac{5z}{2} = \frac{3}{2} \quad \text{—— (2)}$$

$$18z = 5 \quad \text{—— (3)}$$

$$z = 5/18$$

Putting value of z in (2),

$$\frac{y}{2} + \frac{5}{2}\left(\frac{5}{18}\right) = \frac{3}{2}$$

$$y = 3 - \frac{25}{18} = \frac{29}{18}$$

Putting values in (1),

$$x = \frac{35}{18}$$

Q

$$x_1 + x_2 - x_3 = 2$$
$$2x_1 + 2x_2 + 5x_3 = -3$$
$$3x_1 + 2x_2 - 3x_3 = 6$$

Let $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & 5 \\ 3 & 2 & -3 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}$

$$A = LU$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & 5 \\ 3 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & 5 \\ 3 & 2 & -3 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}\cdot u_{11} & l_{21}\cdot u_{12} + u_{22} & l_{21}\cdot u_{13} + u_{23} \\ l_{31}\cdot u_{11} & l_{31}\cdot u_{12} + l_{32}\cdot u_{22} & l_{31}\cdot u_{13} + l_{32}\cdot u_{23} + u_{33} \end{bmatrix}$$

On Equating,

$u_{11} = 1$, $u_{12} = 1$, $u_{13} = -1$.

Ist column,

$l_{21}\cdot u_{11} = 2$, $l_{31}\cdot u_{11} = 3$

$\Rightarrow l_{21} = 2$, $l_{31} = 3$

IInd column,

$l_{21}\cdot u_{12} + u_{22} = 2$, $l_{31}\cdot u_{12} + l_{32}\cdot u_{22} = 2$.

$2\cdot 1 + u_{22} = 2$

$u_{22} = 0$

Now,

$$A = LU$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & 5 \\ 3 & 2 & -3 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}\cdot u_{12} + l_{22} & l_{21}\cdot u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}\cdot u_{12} + l_{32} & l_{31}\cdot u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$$

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Characterization Equation

If A is a square matrix of order n, then,
$|A - \lambda I| = 0$ is a characteristic equation of A
denoted by ch. Eqⁿ.

$$\boxed{(-1)^n \lambda^n + K_1 \lambda^{n-1} + K_2 \lambda^{n-2} + \dots + K_n = 0} \rightarrow \text{ch eq}^n \text{ of } A.$$

roots of ch. eqⁿ say $\lambda_1, \lambda_2, \dots \lambda_n$
Eigen values of A / characteristic value / latent root

Que: $A = \begin{bmatrix} 4 & 1 \\ 0 & 2 \end{bmatrix}$, find eigen values

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(2-\lambda) = 0$$

$\boxed{\lambda = 2, 4}$ - eigen value of A

Properties of Eigen values :-

Let $\lambda_1, \lambda_2 \dots \lambda_n$ be the eigen values of A.

- Spectrum → set of all eigen values
- Trace → $a_{11} + a_{22} + a_{33} + \dots + a_{nn} = \lambda_1 + \lambda_2 + \dots + \lambda_n$
  sum of diagonal el = sum of all eigen values
- $\lambda_1 . \lambda_2 . \lambda_3 . \dots . \lambda_n = |A|$
- eigen values of A^n are $\lambda_1^n, \lambda_2^n, \lambda_3^n, \dots \lambda_n^n$.
- eigen values of A^{-1} = $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \dots \frac{1}{\lambda_n}$ $(|A| \neq 0)$

- Eigen values of A and A^T are same.
- Eigen values of $A+KI$ are $\lambda_1+K, \lambda_2+K, \dots, \lambda_n+K$
- zero is one of the eigen value of A if $|A|=0$.
- Eigen values of real symmetric matrix are all real.
- Eigen value of adj(A) are $\frac{|A|}{\lambda}$ $[|A| \neq 0]$
- Eigen values of a triangular matrix are its principle diagonal elements i.e. $a_{11}, a_{22}, a_{33} \dots$

Eigen Vectors:-

Consider the linear transformation $Y = AX$ — (1)

If λ be any scalar then $Y = \lambda X$ — (2)

From (1) and (2),

$$AX = \lambda X$$

$$[A - \lambda I]X = 0 \rightarrow (3) \text{ Homogeneous System.}$$

Corresponding to each λ(eigen value) we get a non-zero soln of eqn (3) which is known as eigen vector.

Note

- Eigen vector is always non-zero.
- Corresponding to each eigen value there exists infinite many eigen vectors.

$$\lambda \begin{cases} X_1 \\ X_2 \\ X_3 \end{cases}$$

Que. Find eigen values and corresponding eigen vectors of

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Characteristic eqn. $|A - \lambda I| = 0$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - [a_{11} + a_{22} + a_{33}]\lambda^2 + [\text{cofactors of } a_{11}, a_{22}, a_{33}]\lambda - |A| = 0$$

$$\lambda^3 - 12\lambda^2 + \left[\begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix} \right]\lambda - 32 = 0$$

$$\lambda^3 - 12\lambda^2 + [8 + 14 + 14]\lambda - 32 = 0$$

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

$$(\lambda - 2)(\lambda^2 - 10\lambda + 16) = 0$$

$$(\lambda - 2)(\lambda - 2)(\lambda - 8) = 0$$

eigen values → $\lambda = 2, 2, 8.$

To find eigen vectors :-

Let us consider matrix $[A - \lambda I]X = 0$

Put $\lambda = 8$,

$$[A - 8I]X = 0$$

$$\begin{bmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{\begin{vmatrix} -5 & -1 \\ -1 & -5 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & -1 \\ 2 & -5 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & -5 \\ 2 & -1 \end{vmatrix}} = K_1$$

$$\frac{x_1}{24} = \frac{-x_2}{12} = \frac{x_3}{12} = K_1$$

$$\frac{x_1}{2} = \frac{-x_2}{1} = \frac{x_3}{1} = K_1$$

eigen vector $\lambda = 8$, $\lambda = K_1 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

For $\lambda = 2$, $[A - 2I]X = 0$

$$\begin{bmatrix} 6-2 & -2 & 2 \\ -2 & 3-2 & -1 \\ 2 & -1 & 3-2 \end{bmatrix} X = 0$$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + \frac{R_1}{2}$, $R_3 \to R_3 - \frac{R_1}{2}$

$$\sim \begin{bmatrix} 4 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$4x_1 - 2x_2 + 2x_3 = 0$$

let $x_2 = K_2$, $x_3 = K_3$

$$x_1 = \frac{x_2 - x_3}{2} = \frac{K_2 - K_3}{2}$$

$$X = \begin{bmatrix} \frac{K_2 - K_3}{2} \\ K_2 \\ K_3 \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{K_2}{2} - \frac{K_3}{2} \\ K_2 + 0 \cdot K_3 \\ 0 \cdot K_2 + K_3 \end{bmatrix}$$

$\lambda_1 \neq \lambda_2 \neq \lambda_3 \rightarrow$ Possible

$\lambda_1 = \lambda_2 \neq \lambda_3 <$ Possible / Not possible

$\lambda_1 = \lambda_2 = \lambda_3 \rightarrow$ Not possible

$$X = K_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + K_3 \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$$

$$X = K_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + K_3 \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

Diagonalisation of a matrix

Diagonalisation is a process by which we can reduce matrix in diagonal form by using similarity transform i.e. $D = M^{-1}AM$, where $M \rightarrow$ modal matrix (Matrix of vectors)

Note For diagonalisation it is necessary, that in modal matrix M we have three linearly independent columns

Steps:- Let A be any square matrix

(i) Find ch-eq^n $|A - \lambda I| = 0$

(ii) Find eigen values

(iii) Find eigen vectors corresponding to each eigen value.

(iv) Find Modal matrix.

(v) Find inverse of modal matrix

(vi) Find diagonal matrix by $D = M^{-1}AM$

Q. Reduce matrix A into diagonal form by similarity transformation.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$

Ch. Eq^n $|A-\lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & -1 & 2 \\ 0 & 2-\lambda & -1 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(2-\lambda)(3-\lambda) = 0$$

$\boxed{\lambda = 1, 2, 3}$ eigen values.

Eigen vectors: Consider matrix

$$[A-\lambda I](X) = 0$$

where, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \rightarrow$ is eigen vector.

For $\lambda = 1$,

$$[A-I]X = 0$$

$$\begin{bmatrix} 1-1 & -1 & 2 \\ 0 & 2-1 & -1 \\ 0 & 0 & 3-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\frac{x_1}{\begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 0 & -1 \\ 0 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}} = K_1$$

$$\frac{x_1}{2} = \frac{x_2}{0} = \frac{x_3}{0} = K_1$$

$$X = K_1 \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad X = K_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda = 2$,

$$\begin{bmatrix} 1-2 & -1 & 2 \\ 0 & 2-2 & -1 \\ 0 & 0 & 3-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -1 & -1 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\frac{x_1}{\begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -1 & -1 \\ 0 & 0 \end{vmatrix}} = K_2$$

$$\frac{x_1}{1} = \frac{-x_2}{1} = \frac{x_3}{0} = K_2$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = K_2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

For $\lambda = 3$,

$$\begin{bmatrix} 1-3 & -1 & 2 \\ 0 & 2-3 & -1 \\ 0 & 0 & 3-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -2 & -1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\frac{x_1}{\begin{vmatrix} -1 & 2 \\ -1 & -1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & 2 \\ 0 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & -1 \\ 0 & -1 \end{vmatrix}} = K_3$$

$$\frac{x_1}{3} = \frac{-x_2}{2} = \frac{x_3}{2} = K_3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = K_3 \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$$

Now,

$$M = \begin{bmatrix} 1 & -1 & 3 \\ 0 & +1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$M^{-1} = \frac{1}{|M|} \operatorname{adj} M$$

$$= \frac{1}{|M|} \begin{bmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{bmatrix}^T$$

$$= \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ -1 & 2 & 1 \end{bmatrix}^T$$

$$= \frac{1}{2} \begin{bmatrix} 2 & 2 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Now,

$$D = M^{-1} A M$$

$$= \frac{1}{2} \begin{bmatrix} 2 & 2 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \\ 0 & +1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$D = \frac{1}{2} \begin{bmatrix} 2 & -2+4 & 4-2-3 \\ 0 & 4 & -2+6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$D = \frac{1}{2} \begin{bmatrix} 2 & 2 & -1 \\ 0 & 4 & 4 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Q. Find modal matrix M and hence diagonalize the matrix

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}.$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$(6-\lambda)\begin{vmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix} + 2\begin{vmatrix} -2 & -1 \\ 2 & 3-\lambda \end{vmatrix} + 2\begin{vmatrix} -2 & 3-\lambda \\ 2 & -1 \end{vmatrix} = 0$$

$$(6-\lambda)\left((3-\lambda)^2 - 1\right) + 2\left(-6+2\lambda+2\right) + 2\left(2-(6-2\lambda)\right) = 0$$

$$\lambda = 2, 2, 8.$$

For $\lambda = 2$,

$$X = k_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

For $\lambda = 8$,

$$X = k_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Modal matrix

$$M = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & -1 \\ 0 & 2 & 1 \end{bmatrix}$$

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$$|M| = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 0 & -1 \\ 0 & 2 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 0 & -1 \\ 2 & 1 \end{vmatrix} - 2\begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix}$$

$$= 2 - 2(-1-4)$$

$$= 2 + 10$$

$$= 12$$

$$M^{-1} = \frac{1}{|M|} \text{adj } M$$

$$= \frac{1}{12}\begin{bmatrix} 2 & -2 & 4 \\ 5 & 1 & -2 \\ 1 & 5 & 2 \end{bmatrix}^T$$

$$= \frac{1}{12}\begin{bmatrix} 2 & 5 & 1 \\ -2 & 1 & 5 \\ 4 & -2 & 2 \end{bmatrix}$$

$$D = M^{-1}AM$$

$$D = \frac{1}{12}\begin{bmatrix} 2 & 5 & 1 \\ -2 & 1 & 5 \\ 4 & -2 & 2 \end{bmatrix}\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & -1 \\ 0 & 2 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

Q. Diagonalise the matrix $A = \begin{bmatrix} 2 & -3 \\ 4 & -2 \end{bmatrix}$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & -3 \\ 4 & -2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(-2-\lambda)+12=0$$
$$-4-\cancel{2\lambda}+\cancel{2\lambda}+\lambda^2+12=0$$
$$\lambda^2+8=0$$
$$\lambda^2=-8$$
$$\lambda=\pm\sqrt{-8}=\pm 2\sqrt{2}\,i$$

$$[A-\lambda I]X=0$$

$$\begin{bmatrix} 2-2\sqrt{2}i & -3 \\ 4 & -2-2\sqrt{2}i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=0$$

$\cancel{x}y$

$$(2-2\sqrt{2}i)x_1-3x_2=0$$
$$4x_1-(2+2\sqrt{2}i)x_2=0$$

$$4(2-2\sqrt{2}i)x_1-12x_2=0$$
$$4(2-2\sqrt{2})x_1-(4+8)x_2=0$$

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Eigen Value Decomposition Method

- Find characteristic eqn of A i.e. $|A-\lambda I| = 0$
- Find eigen values say $\lambda_1, \lambda_2, \lambda_3$
- Find eigen vectors corresponding to each eigen value.
- Find modal matrix M and its inverse M^{-1}.
- Find spectral matrix (D) i.e. diagonal matrix of A say $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$
- Write $A = MDM^{-1}$ $\rightarrow$ EVD

Note: Eigen Value decomposition is widely used because it makes complex tasks simpler. It helps in calculating power matrix in a easy. It is used in technique like PCA to reduce large data set into smallest dimensions making them easier to analyse. It is used in tasks like image enhancement and image compression. In quantum mechanics it helps in understanding how system change.
For eigen value decomposition and matrix must be square and eigen values should be linearly independent.

Q. Reduce in eigen value decomposition.

$$A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

ch. Eqn:

$$|A-\lambda I| = 0$$

$$\begin{vmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(3-\lambda) - 2 = 0$$

$$12-4\lambda-3\lambda+\lambda^2-2 = 0$$

$$\lambda^2-7\lambda+10 = 0$$

For eigen vector, $[A-\lambda I]X=0 \Rightarrow [A-5I]X=0$

$$\begin{bmatrix} 4-5 & 1 \\ 2 & 3-5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=0$$

$$-x_1+x_2=0$$

$$x_2=x_1$$

let $x_1=K_1$

$\Rightarrow x_2=K_1$

$X=K_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $K_1 \neq 0$

For $\lambda=2$,

$$\begin{bmatrix} 4-2 & 1 \\ 2 & 3-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=0$$

$$2x_1+x_2=0$$

let $x_2=K_2$

$$x_1=\frac{-K_2}{2}$$

$X=K_2\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$, $K_2 \neq 0$ or $K_2\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Spectral matrix

$$D=\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}=\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$$

Modal matrix, $M=\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$

$$M^{-1}=\frac{1}{|M|}\text{adj}M=\frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

By EVD,

$$A=MDM^{-1}$$

$$=\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}\frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 5 & -2 \\ 5 & 4 \end{bmatrix}\cdot\frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

$$=\frac{1}{3}\begin{bmatrix} 10+2 & 5-2 \\ 10-4 & 5+4 \end{bmatrix}=\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

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Q. If $A = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ 4 & -1 & 4 \end{bmatrix}$, find its eigen decomposition.

ch. Eq^n → $|A - \lambda I| = 0$

$$\begin{vmatrix} 5-\lambda & 2 & 0 \\ 2 & 5-\lambda & 0 \\ 4 & -1 & 4-\lambda \end{vmatrix} = 0$$

$$(4-\lambda) \begin{vmatrix} 5-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)\left[(5-\lambda)^2 - 4\right] = 0$$

$4 - \lambda = 0$ or $(5-\lambda)^2 - 4 = 0$

$\lambda = 4$ or $25 + \lambda^2 - 10\lambda - 4 = 0$

$$\lambda^2 - 10\lambda + 21 = 0$$

$$\lambda^2 - 7\lambda - 3\lambda + 21 = 0$$

$$\lambda(\lambda - 7) - 3(\lambda - 7) = 0$$

$$\lambda' = 7, 3$$

$$|A| = \begin{vmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ 4 & -1 & 4 \end{vmatrix} = 4 \begin{vmatrix} 5 & 2 \\ 2 & 5 \end{vmatrix} = 4(25-4) = 84$$

For $\lambda = 3$,

$$[A - 3I] X = 0$$

$$\begin{bmatrix} 5-3 & 2 & 0 \\ 2 & 5-3 & 0 \\ 4 & -1 & 4-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 4 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\frac{x_1}{2} = \frac{-x_2}{2} = \frac{x_3}{-2-8} = K_1$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = K_1 \begin{bmatrix} 2 \\ -2 \\ -10 \end{bmatrix} = K_1 \begin{bmatrix} 1 \\ -1 \\ -5 \end{bmatrix}, K_1 \neq 0$$

For $\lambda = 4$,

$$[A - 4I] X = 0$$

$$\begin{bmatrix} 5-4 & 2 & 0 \\ 2 & 5-4 & 0 \\ 4 & -1 & 4-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 4 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\frac{x_1}{0} = \frac{-x_2}{0} = \frac{x_3}{-3} = K_2 \text{ (say)}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = K_2 \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix} = K_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, K_2 \neq 0$$

For $\lambda = 7$,

$$[A - 7I] X = 0$$

$$\begin{bmatrix} 5-7 & 2 & 0 \\ 2 & 5-7 & 0 \\ 4 & -1 & 4-7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 4 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\frac{x_1}{6} = \frac{-x_2}{-6} = \frac{x_3}{-2+8} = K_3$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1} = K_3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = K_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & -5 \end{bmatrix}$$

$$|M| = -1 \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 2$$

$$M^{-1} = \frac{1}{|M|} \operatorname{adj} M$$

$$= \frac{1}{2} \begin{bmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{bmatrix}^T$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 4 & 1 \\ 1 & -6 & -1 \\ 0 & 2 & 0 \end{bmatrix}^T$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 4 & -6 & 2 \\ 1 & -1 & 0 \end{bmatrix}$$

By EVD,

$$A = MDM^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & -5 \end{bmatrix} \begin{bmatrix} 7 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 4 & -6 & 2 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 7 & 0 & 3 \\ 7 & 0 & -3 \\ 7 & 4 & -15 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 4 & -6 & 2 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ 4 & -1 & 4 \end{bmatrix}$$

Singular Value Decomposition

SVD of a matrix is a factorization of matrix into three parts/matrices i.e. $A = U \Sigma V^T$

where, U : modal matrix by AA^T

V^T : Transpose of modal matrix A^TA

Σ : diagonal matrix of singular values.

where singular value $= \sqrt{\text{eigen value}}$

$$\sigma_1 = \sqrt{\lambda_1}$$

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$$

Working Rule:

- Find $B = AA^T$ and $C = A^TA$
- Find eigen values and corresponding eigen vectors of B and C.
- Let eigen vectors of B, X_1 and X_2 find $U = [N(X_1) \cdot N(X_2)]$
  $N \rightarrow$ normalised
- Let eigen vectors of C are Y_1, Y_2 find $V = [N(Y_1), N(Y_2)]$
  Then find V^T.
- By SVD, $A = U \Sigma V^T$

<u>Note.</u> SVD is applicable for both square matrix and rectangular matrix.

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Q. Decompose the matrix by singular values

$$A = \begin{bmatrix} -4 & -7 \\ 1 & 4 \end{bmatrix}$$

$$A^T = \begin{bmatrix} -4 & 1 \\ -7 & 4 \end{bmatrix}$$

$$B = AA^T = \begin{bmatrix} -4 & -7 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -4 & 1 \\ -7 & 4 \end{bmatrix} = \begin{bmatrix} 16+49 & -4-28 \\ -4-28 & 1+16 \end{bmatrix} = \begin{bmatrix} 65 & -32 \\ -32 & 17 \end{bmatrix}$$

$$C = A^TA = \begin{bmatrix} -4 & 1 \\ -7 & 4 \end{bmatrix}\begin{bmatrix} -4 & -7 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 16+1 & 28+4 \\ 28+4 & 49+16 \end{bmatrix} = \begin{bmatrix} 17 & 32 \\ 32 & 65 \end{bmatrix}$$

For eigen values,

$$|B-\lambda I| = 0$$

$$\begin{vmatrix} 65-\lambda & -32 \\ -32 & 17-\lambda \end{vmatrix} = 0$$

$$(65-\lambda)(17-\lambda) - 1024 = 0$$

$$1105 - 65\lambda - 17\lambda + \lambda^2 - 1024 = 0$$

$$\lambda^2 - 82\lambda + 81 = 0$$

$$\lambda^2 - \lambda - 81\lambda + 81 = 0$$

$$\lambda(\lambda-1) - 81(\lambda-1) = 0$$

$$\lambda = 1, 81$$

$$|C-\lambda I| = 0$$

$$\begin{vmatrix} 17-\lambda & 32 \\ 32 & 65-\lambda \end{vmatrix} = 0$$

$$(17-\lambda)(65-\lambda) - 1024 = 0$$

$$\lambda = 1, 81.$$

$$\sigma_1 = \sqrt{\lambda_1} , \quad \sigma_2 = \sqrt{\lambda_2}$$
$$= \sqrt{81} , \quad = \sqrt{1}$$
$$= 9 \qquad = 1$$

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$$

For eigen vectors, For B,

$\lambda = 81$

$$[B - \lambda I] X = 0$$

$$\begin{bmatrix} 65-\lambda & -32 \\ -32 & 17-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 65-81 & -32 \\ -32 & 17-81 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -16 & -32 \\ -32 & -64 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow x_1 + 2x_2 = 0$$

let $x_2 = K_1$

$x_1 = -2K_1$

$$X_1 = K_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} , K_1 \neq 0$$

$$N(X_1) = K_1 \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} , K_1 \neq 0$$

For $\lambda = 1$,

$$\begin{bmatrix} 65-1 & -32 \\ -32 & 17-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 64 & -32 \\ -32 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$2x_1 - x_2 = 0$

let $x_1 = K_2$

$x_2 = 2K_2$

$$X_2 = K_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}, K_2 \neq 0$$

$$N(X_2) = K_2 \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}, K_2 \neq 0$$

$$U = (N(X_1), N(X_2))$$

$$U = \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

For C,

$\lambda = 81$,

$$\begin{bmatrix} 17-81 & 32 \\ 32 & 65-81 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -64 & 32 \\ 32 & -16 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0$$

$$-2y_1 + y_2 = 0$$

Let $y_1 = K_1$

$y_2 = 2K_1$

$$Y_1 = K_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}, K_1 \neq 0 \qquad N(Y_1) = K_1 \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}, K_1 \neq 0$$

$\lambda = 1$,

$$\begin{bmatrix} 17-1 & 32 \\ 32 & 65-1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 16 & 32 \\ 32 & 64 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0$$

$$y_1 + 2y_2 = 0$$

Let $y_2 = K_2$,
$$y_1 = -2K_2$$

$$Y_2 = K_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}, K_2 \neq 0$$

$$N(Y_2) = K_2 \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$V = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

$$V^T = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

Now by SVD,

$$A = U \Sigma V^T$$

$$= \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-18}{\sqrt{5}} & 1/\sqrt{5} \\ \frac{9}{\sqrt{5}} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-18-2}{5} & \frac{-36+1}{5} \\ \frac{9-4}{5} & \frac{18+2}{5} \end{bmatrix}$$

$$= \begin{bmatrix} -4 & -7 \\ 1 & 4 \end{bmatrix}$$

Q. Find SVD of $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$

$$B = AA^T = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}_{2\times3} \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}_{3\times2}$$

$$= \begin{bmatrix} 9+1+1 & -3+3+1 \\ -3+3+1 & 1+9+1 \end{bmatrix} = \begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix}$$

$$C = A^TA = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}_{3\times2} \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}_{2\times3} = \begin{bmatrix} 9+1 & 3-3 & 3-1 \\ 3-3 & 1+9 & 1+3 \\ 3-1 & 1+3 & 1+1 \end{bmatrix}$$

$$C = \begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix}$$

For eigen values,

$$|B - \lambda I| = 0$$

$$\begin{vmatrix} 11-\lambda & 1 \\ 1 & 11-\lambda \end{vmatrix} = 0$$

$$(11-\lambda)^2 - 1 = 0$$

$$121 + \lambda^2 - 22\lambda - 1 = 0$$

$$\lambda^2 - 22\lambda + 120 = 0$$

$$\lambda^2 - 12\lambda - 10\lambda + 120 = 0$$

$$\lambda = 12, 10$$

$$|C - \lambda I| = 0$$

$$\begin{vmatrix} 10-\lambda & 0 & 2 \\ 0 & 10-\lambda & 4 \\ 2 & 4 & 2-\lambda \end{vmatrix} = 0$$

$$(10-\lambda)\begin{vmatrix} 10-\lambda & 4 \\ 4 & 2-\lambda \end{vmatrix} + 2\begin{vmatrix} 0 & 10-\lambda \\ 2 & 4 \end{vmatrix} = 0$$

$$(10-\lambda)\left[(10-\lambda)(2-\lambda)-16\right]+2\left[0-2(10-\lambda)\right]=0$$

$$(10-\lambda)\left[20-10\lambda-2\lambda+\lambda^2-16\right]+2\left[-20+2\lambda\right]=0$$

$$(10-\lambda)(\lambda^2-12\lambda+4)-40+4\lambda=0$$

$$\lambda = 10, 12, 0$$

For eigen vectors, $\lambda = 12$,

$$[B-\lambda I]X=0$$

$$\begin{bmatrix} 11-12 & 1 \\ 1 & 11-12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=0$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=0$$

$$-x_1+x_2=0$$

Let $x_2 = K_1$,

$x_1 = K_1$

$$X_1 = K_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}, K_1 \neq 0 \qquad N(X_1) = K_1\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, K_1 \neq 0$$

$\lambda = 10$,

$$\begin{bmatrix} 11-10 & 1 \\ 1 & 11-10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=0$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=0$$

$$x_1+x_2=0$$

let $x_2 = K_2$,

$x_1 = -K_2$

$$X_2 = K_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}, K_2 \neq 0$$

$$N(X_2) = K_2\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, K \neq 0$$

$$U = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

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$\lambda=12$,

$$[C-\lambda I]Y = 0$$

$$\begin{bmatrix} 10-12 & 0 & 2 \\ 0 & 10-12 & 4 \\ 2 & 4 & 2-12 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -2 & 0 & 2 \\ 0 & -2 & 4 \\ 2 & 4 & -10 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 0$$

$$\frac{y_1}{20-16} = \frac{-y_2}{0-8} = \frac{y_3}{0+4} = K_1$$

$$\frac{y_1}{4} = \frac{y_2}{8} = \frac{y_3}{4} = K_1$$

$$Y_1 = K_1\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, K_1 \neq 0 \qquad N(Y_1) = K_1\begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, K_1 \neq 0$$

$\lambda=10$,

$$[C-10I]Y = 0$$

$$\begin{bmatrix} 10-10 & 0 & 2 \\ 0 & 10-10 & 4 \\ 2 & 4 & 2-10 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 4 \\ 2 & 4 & -8 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 0$$

$$\frac{y_1}{-16} = \frac{-y_2}{0-8} = \frac{y_3}{0} = K_2$$

$$\frac{y_1}{-16} = \frac{y_2}{8} = \frac{y_3}{0} = K_2$$

$$Y_2 = K_2\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, K_2 \neq 0 \qquad N(Y_2) = K_2\begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}, K_2 \neq 0$$

$\lambda = 0$,
$$\begin{bmatrix} 10-0 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 0$$

$$\frac{y_1}{-20} = \frac{-y_2}{40} = \frac{y_3}{100} = K_3$$
$$\frac{y_1}{2} = \frac{y_2}{4} = \frac{y_3}{-10} = K_3$$

$$Y_3 = K_3 \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}, K_3 \neq 0$$

$$N(Y_3) = K_3 \begin{bmatrix} 1/\sqrt{30} \\ 2/\sqrt{30} \\ -5/\sqrt{30} \end{bmatrix}, K_3 \neq 0$$

$$V = \begin{bmatrix} 1/\sqrt{6} & -2/\sqrt{5} & 1/\sqrt{30} \\ 2/\sqrt{6} & 1/\sqrt{5} & 2/\sqrt{30} \\ 1/\sqrt{6} & 0 & -5/\sqrt{30} \end{bmatrix}$$

$$V^T = \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 1/\sqrt{30} & 2/\sqrt{30} & -5/\sqrt{30} \end{bmatrix}$$

By SVD,

$$A = U \Sigma V^T$$
$$= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 12 & 0 & 0 \\ 0 & 10 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ 2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 1/\sqrt{30} & 2/\sqrt{30} & -5/\sqrt{30} \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$$

Q. If possible, diagonalise.

$$A = \begin{bmatrix} -1 & 4 \\ 0 & 3 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -1-\lambda & 4 \\ 0 & 3-\lambda \end{vmatrix} = 0$$

$$(-1-\lambda)(3-\lambda) = 0$$

$$\lambda = -1, 3$$

For $\lambda = 3$,

$$[A - \lambda I] X = 0$$

$$\begin{bmatrix} -1-3 & 4 \\ 0 & 3-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -4 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$-4x_1 + 4x_2 = 0$$

$$x_1 - x_2 = 0$$

Let $x_2 = K_1$

$x_1 = K_1$

$$X_1 = K_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, K_1 \neq 0$$

For $\lambda = -1$,

$$\begin{bmatrix} -1+1 & 4 \\ 0 & 3+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 4 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$x_2 = 0$$

$$x_1 = K_2$$

$$X_2 = K_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}, K_2 \neq 0$$

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$