★ ((Unit-1) → Maths Syllabus of Prect)

07/Oct/2024

(L-1)

Date 03/10/24

Differential Calculus-II

Taylor Series / Expansion for the function of two variables

★ for one variable

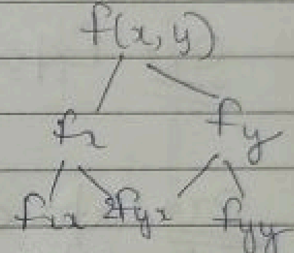
$$f(x) = f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a)$$

$$+ \cdots \frac{h^n}{n!} f^n(a) + \cdots$$

Taylor's Expansion

★ for two variables the function $f(x,y)$

$$f(x,y) \rightarrow f_x, \quad f(x,y) \rightarrow f_y$$



$$f(x,y) = f(a+h, b+k) =$$

$$= f(a,b) + \left[h f_x(a,b) + k f_y(a,b)\right] + \frac{1}{2!}\left[h^2 f_{xx}(a,b) + k^2 f_{yy}(a,b)\right.$$

$$\left. + 2hk f_{xy}(a,b)\right] + \frac{1}{3!}\left[h^3 f_{xxx} + k^3 f_{yyy} + 3h^2k f_{xxy} + 3k^2h f_{yyx}\right] \cdots$$

$$\left[(a+b)^3 = a^3 + b^3 + 3a^2b + 3ab^2\right]$$

CASE-1: Taylor's expansion about point (a,b) OR in powers of $(x-a)$ & $(y-b)$.

$$\left\{\begin{array}{l} \text{replace } \overset{h}{x} \rightarrow (x-a) \rightarrow h \\ k\ y \rightarrow (y-b) \rightarrow k \end{array}\right\}$$

$$f(x,y) = f(a,b) + \left[(x-a) f_x(a,b) + (y-b) f_y(a,b)\right] + \frac{1}{2!}$$

$$\left[(x-a)^2 f_{xx}(a,b) + (y-b)^2 f_{yy}(a,b) + 2(x-a)(y-b) f_{xy}(a,b)\right]$$

CASE-2: (Maclaurian Series) expansion of Taylor's about $(0,0)$ OR in powers of x & y.

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$$f(x,y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)] + \frac{1}{2!}[x^2 f_x(0,0) + y^2 f_y(0,0) + 2xy f_{xy}(0,0)] + \cdots$$

Ex: Expand in powers of $(x-1)$ & $(y+2)$
↳ Point $(1,-2)$.

Ex: Expand in powers of x & y
↳ Point $(0,0)$

Ex: Expand in powers of $(x+2)$ & $(y-4)$.
↳ Point $(-2, 4)$

Ex: Expand $e^x \sin y$ in powers of x & y upto 3rd degree

Sol: Let $f(x,y) = e^x \sin y$

Taylor's Series about origin $(0,0)$ i.e. in power of x and y :-

$$f(x,y) = f(0,0) + [x f_x(0,0) + y f_y(0,0)] + \frac{1}{\underline{|2}}[x^2 f_{xx}(0,0) + y^2 f_{yy}(0,0) + 2xy f_{xy}(0,0)] + \frac{1}{\underline{|3}}[x^3 f_{xxx}(0,0) + y^3 f_{yyy}(0,0), + 3x^2y f_{xxy}(0,0) + 3y^2x f_{yyx}(0,0)] + \cdots \quad -(1)$$

Now, put $(x=0, y=0)$

$f(x,y) = e^x \sin y = 0$

$f_x = e^x \sin y = 0$

$f_y = e^x \cos y = 1$

$f_{xy} = e^x \cos y = 1$

$f_{xx} = e^x \sin y = 0$

$f_{yy} = e^x(-\sin y) = 0$

$f_{xxx} = e^x \sin y = 0$

$f_{xxy} = e^x \cos y = 1$

$f_{yyx} = -e^x \sin y = 0$

$f_{yyy} = -e^x \cos y = -1$

put $x=0, y=0$ in derivatives and functions

put these values in eqn ①

$$f(x,y) = 0 + [x \times 0 + y \times 1] + \frac{1}{2!}[\cancel{x^2 \times 0} + \cancel{y^2 \times 0} + 2xy \times 1]$$

$$+ \frac{1}{3!}[\cancel{x^3 \times 0} + y^3 \times -1 + 3x^2 y \times 1 + 3y^2 x \times 0] - -$$

$$= 0 + [y] + \frac{1}{2}[2xy] + \frac{1}{6}[-y^3 + 3x^2 y]$$

$$= y + xy + \frac{1}{6}[3x^2 y - y^3] \text{ Ans}$$

Ex: Expand $\tan^{-1}\left(\frac{y}{x}\right)$ in the neighbourhood of $(1,1)$ upto 2nd deg terms hence compute $f(1.1, 0.9)$ approximately.

Sol: Taylor's series expansion in powers of $(x-1)$ & $(y-1)$

$$f(x,y) = f(1,1) + [(x-1) f_x(1,1) + (y-1) f_y(1,1)] + \frac{1}{\lfloor 2}$$

$$[(x-1)^2 f_{xx}(1,1) + (y-1)^2 f_{yy}(1,1) + 2(x-1)(y-1)$$

$$f_{xy}(1,1)] + - - \quad ①$$

Now let

$$f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$$

$$f_x = \frac{1 \times y}{1 + \frac{y^2}{x^2}} \times \frac{-1}{x^2} = \frac{x^2}{x^2+y^2} \times \frac{-1}{\cancel{x^2}} = \frac{-1 \times y}{x^2+y^2} = \frac{-y}{x^2+y^2}$$

$$f_y = \frac{1}{1 + \frac{y^2}{x^2}} \times \frac{1}{x} = \frac{x}{x^2+y^2}$$

$$f_{xx} = \frac{\partial}{\partial x}\left[\frac{-y}{x^2+y^2}\right] = \frac{2xy}{(x^2+y^2)^2}$$

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$f_{yy} = \frac{\partial}{\partial y}\left[\frac{x}{x^2+y^2}\right] = \frac{-2xy}{(x^2+y^2)^2}$

$f_{xy} = \frac{\partial}{\partial x}\left[f_y\right] = \frac{\partial}{\partial x}\left[\frac{x}{x^2+y^2}\right] = \frac{-x^2+y^2}{(x^2+y^2)^2}$

at $(1,1)$

$f_x = -1/2$ $\quad f_{yy} = \frac{-2\times 1}{4} = \frac{-1}{2}$ $\quad f(1,1) = \tan^{-1}(1)$
$= \pi/4$

$f_y = 1/2$ $\quad f_{xy} = -1+1 = 0$

$f_{xx} = \frac{1}{2}$

Now put all these values in eqn (1)

$f(x,y) = \frac{\pi}{4} + \left[\frac{(x-1)}{-2} + \frac{(y-1)}{2}\right] + \frac{1}{2!}\left[\frac{(x-1)^2}{2} + \frac{(y-1)^2}{-2}\right] \ldots$

Put $x=1.1$, $y=0.9$

$f(x,y)\Big|_{x=1.1,\, y=0.9} = \frac{\pi}{4} + \left[\frac{(1.1-1)}{-2} + \frac{(0.9-1)}{2}\right] + \frac{1}{2}\left[\frac{(1.1-1)^2}{2} + \frac{(0.9-1)^2}{-2}\right] \ldots$

$= 0.6857$ Ans
(approx)

(Imp)
Ex: Expand x^y in powers of $(x-1)$ & $(y-1)$ upto 3rd degree. Hence evaluate $(1.1)^{1.02}$ or $f(1.1, 1.02)$.

Sol: $\frac{d}{dx}(x^n) = nx^{n-1}$ (variable)$^{\text{constant}}$

$\frac{d}{dx}(a^x) = a^x \log a$ (constant)$^{\text{variable}}$

Let $f(x,y) = x^y$

$f_x = yx^{y-1}$

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$f_{xx} = y(y-1)x^{y-2}$

$f_{xxx} = y(y-1)(y-2)x^{y-3}$

$f_y = x^y \log x$

$f_{yy} = x^y \cdot (\log x)^2$

$f_{yyy} = x^y (\log x)^3$

$f_{xy} = \frac{\partial}{\partial x} f_y = \frac{\partial}{\partial x}(x^y \log x)$
x variable, y constant

$= y x^{y-1} \log x + \frac{1}{x} x^y = y x^{y-1} \log x + x^{y-1}$

$f_{xxy} = \frac{\partial}{\partial y} f_{xx} = \frac{\partial}{\partial y}\left[y(y-1)x^{y-2}\right]$

$= [2y-1]x^{y-2} + (y^2-y)\frac{1}{x^2} \cdot x^y \log x$

$f_{yyx} = \frac{\partial}{\partial x} f_{yy} = \frac{\partial}{\partial x}\left[x^y (\log x)^2\right]$

$= y x^{y-1} (\log x)^2 + x^y \cdot \frac{2 \log x}{x}$

HW. Expand y^x about $(1,1)$

& Taylor's Series expansion in powers of $(x-1)$ and $(y-1)$

$$f(x,y) = f(1,1) + [(x-1)f_x(1,1) + (y-1)f_y(1,1)]$$
$$+ \frac{1}{2!}[(x-1)^2 f_{xx}(1,1) + (y-1)^2 f_{yy}(1,1) + 2(x-1)(y-1)$$
$$f_{xy}(1,1)] + \frac{1}{3!}[(x-1)^3 f_{xxx}(1,1) + (y-1)^3 f_{yyy}(1,1)$$
$$+ 3(x-1)^2(y-1) f_{xxy} + 3(x-1)(y-1)^2 f_{xyy}] + \cdots$$

Now let $f(x,y) = y^x \to f(1,1) = 1$

$f_x = y^x \log y = f_x(1,1) = 0$

$f_y = xy^{x-1} = f_y(1,1) = 1$

$f_{xx} = y^x (\log y)^2 = f_{xx}(1,1) = 0$

$f_{yy} = x(x-1)y^{x-2} \to f_{xy}(1,1) = 0$

$f_{xy} = y^x\left(\frac{1}{y}\right) + \log y\,(xy^{x-1}) \to f_{xy}(1,1) = 1$

$f_{xxx} = y^x(\log y)^3 \to f_{xxx}(1,1) = 0$

$f_{yyy} = x(x-1)(x-2)y^{x-3} \to f_{yyy}(1,1) = 0$

$f_{xxy} = y^x\left(2\frac{\log y}{y}\right) + (\log y)^2\, xy^{x-1}$

$\to f_{xxy}(1,1) = 0$

$f_{xyy} = (x-1)y^{x-2} + x\left[\log y\,(x-1)y^{x-2} + \frac{y^{x-1}}{y}\right]$

$\to f_{xyy}(1,1) = 1$

Put values in eqn ①

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$$f(x,y) = 1 + [(x-1)\cdot 0 + (y-1)1] + \frac{1}{2}[(x-1)^2\cdot 0$$
$$+ (y-1)^2\cdot 0 + 2(x-1)(y-1)\cdot 1] + \frac{1}{6}[(x-1)^3\cdot 0$$
$$+ (y-1)^3\cdot 0 + 3(x-1)^2(y-1)\cdot 0 + 3(x-1)(y-1)^2\cdot 1]$$
$$f(x,y) = \cancel{1} + y - \cancel{1} + (x-1)(y-1) + \frac{(x-1)(y-1)^2}{2}$$
$$= y + (x-1)(y-1) + \frac{(x-1)(y-1)^2}{2} \text{ Ans}$$

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Jacobian: Jacobian is a functional determinate which is used for transformation of variables from cartesian to Polar cylindrical, Spherical in multiple integral

$$\int dx\,dy = \int dr\,d\theta$$

Def: if u & v are functions of two independent variables x & y

$$u = f(x,y), \quad v = f(x,y)$$

Then $J(u,v) = \dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$ or $\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

for ex: If $x = r\cos\theta$, $y = r\sin\theta$ then find Jacobian of find $\dfrac{\partial(x,y)}{\partial(r,\theta)}$.

$$\frac{\partial x}{\partial r} = \cos\theta$$

Sol: $\begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\left[\cos^2\theta + \sin^2\theta\right]$

$$\boxed{J = \frac{\partial(x,y)}{\partial(r,\theta)} = r} \;**$$

NOTE: $\iint f(x,y)\,dx\,dy = \iint f(r,\theta)\,|J|\,dr\,d\theta$

where $J = \dfrac{\partial(x,y)}{\partial(r,\theta)}$ Jacobian of (x,y) wrt (r,θ)

((x,y) से जाते हैं (r,θ) में)

Properties of Jacobian

① **Chain Rule:** if $u = f(r,s)$
$v = f(r,s)$
$r = \psi(x,y)$
$s = \psi_2(x,y)$

$$\begin{tikzcd} u,v \arrow[r] \arrow[rr, bend right] & r,s \arrow[r] & x,y \end{tikzcd}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}$$

for ex: If $u = x^2 - y^2$, $x = r\cos\theta$ then show that
$v = 2xy$, $y = r\sin\theta$, $\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^3$

Sol: $\cancel{\frac{\partial(x)}{\partial x} = 2x}$ $\cancel{r}$

$\cancel{\frac{\partial y}{\partial x} = 2y}$

Given, $u,v \longrightarrow x,y \longrightarrow r,\theta$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} \times \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= 4(x^2+y^2) \times r = 4r^3$$

LHS = RHS (H.P)

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NOTE: If J_1 is Jacobian of u, v wrt x & y and J_2 is Jacobian of x, y wrt u & v

Then $\boxed{J_1 J_2 = 1}$

i.e. $\dfrac{\partial(u,v)}{\partial(x,y)} \times \dfrac{\partial(x,y)}{\partial(u,v)} = 1$

NOTE: $\dfrac{\partial(x,y)}{\partial(r,\theta)} \times \dfrac{\partial(r,\theta)}{\partial(x,y)} = r \times \dfrac{\partial(r,\theta)}{\partial(x,y)} = 1$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r}$$

(2) Jacobian of Implicit functions

If $f_1(u, v, x, y) = 0$
$f_2(u, v, x, y) = 0$

NOTE: (-1) ki power utni multiply hogi jitne functions ho.

ex: $f_1 = u^2 + v^2 + x + y^3 = 0$
$f_2 = u^3 + v^3 + x + y = 0$

$$\frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\dfrac{\partial(f_1 f_2)}{\partial(x,y)}}{\dfrac{\partial(f_1 f_2)}{\partial(u,v)}}$$

ex: if $y_1(x_1 - x_2) = 0 = f_1$
$y_2(x_1^2 + x_1 x_2 + x_2^2) = 0 = f_2$

find $\dfrac{\partial(y_1 y_2)}{\partial(x_1 x_2)} = (-1)^2 \dfrac{\dfrac{\partial(f_1 f_2)}{\partial(x_1 x_2)}}{\dfrac{\partial(f_1 f_2)}{\partial(y_1 y_2)}}$

Sol:

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Sol: $\frac{\partial f_1}{\partial x_1} = y_1 \qquad \frac{\partial f_2}{\partial x_1} = 2x_1y_2 + y_2x_2$

$\frac{\partial f_1}{\partial x_2} = -y_1 \qquad \frac{\partial f_2}{\partial x_2} = y_2x_1 + 2x_2y_2$

$\frac{\partial f_1}{\partial y_1} = x_1 - x_2$

$\frac{\partial f_1}{\partial y_2} = 0$

$\frac{\partial f_2}{\partial y_1} = 0$

$\frac{\partial f_2}{\partial y_2} = x_1^2 + x_1x_2 + x_2^2$

$$\frac{\partial(f_1 f_2)}{\partial(x_1 x_2)} = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} y_1 & -y_1 \\ 2x_1y_2 + y_2x_2 & y_2x_1 + 2x_2y_2 \end{vmatrix}$$

$= y_1y_2x_1 + 2x_2y_1y_2 + 2x_1y_1y_2 + x_2y_1y_2$

$= 3x_1y_1y_2 + 3x_2y_1y_2$

$= 3(x_1y_1y_2 + x_2y_1y_2)$

$= \underline{3y_1y_2(x_1+x_2)}$

Now,

$$\frac{\partial(f_1 f_2)}{\partial(y_1, y_2)} = \begin{vmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} x_1 - x_2 & 0 \\ 0 & x_1^2 + x_2^2 + x_1x_2 \end{vmatrix}$$

$= (x_1 - x_2)(x_1^2 + x_2^2 + x_1x_2)$

$= x_1^3 + \cancel{x_1x_2^2} + \cancel{x_1^2x_2} - \cancel{x_1^2x_2} - x_2^3 - \cancel{x_1x_2^2}$

$= \underline{(x_1^3 - x_2^3)}$

$$\frac{\partial(y_1,y_2)}{\partial(x_1,x_2)} = \frac{3y_1y_2(x_1+x_2)}{x_1^3-x_2^3}$$

Q: If $u^3+v^3+w^3 = x+y+z = f_1$

$u^2+v^2+w^2 = x^3+y^3+z^3 = f_2$

and $u+v+w = \cancel{x^3}\ x^2+y^2+z^2 = f_3$

Show that $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \dfrac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$

Sol: $f_1 = u^3+v^3+w^3-x-y-z$

$f_2 = u^2+v^2+w^2-x^3-y^3-z^3$

$f_3 = u+v+w-x^2-y^2-z^2$

$$\frac{\partial f_1}{\partial u} = 3u^2,\ \frac{\partial f_1}{\partial v} = 3v^2,\ \frac{\partial f_1}{\partial w} = 3w^2$$

$$\frac{\partial f_1}{\partial x} = -1,\ \frac{\partial f_1}{\partial y} = -1,\ \frac{\partial f_1}{\partial z} = -1$$

$$\frac{\partial f_2}{\partial u} = 2u^2$$

Now, $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\dfrac{\partial(f_1f_2f_3)}{\partial(x,y,z)}}{\dfrac{\partial(f_1f_2f_3)}{\partial(u,v,w)}}$$

$$= - \frac{\begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix}} = + \frac{6 \begin{vmatrix} +1 & +1 & +1 \\ +\cancel{3}x^2 & +\cancel{3}y^2 & +\cancel{3}z^2 \\ +\cancel{2}x & +\cancel{2}y & +\cancel{2}z \end{vmatrix}}{6 \begin{vmatrix} \cancel{3}u^2 & \cancel{3}v^2 & \cancel{3}w^2 \\ \cancel{2}u & \cancel{2}v & \cancel{2}w \\ 1 & 1 & 1 \end{vmatrix}}$$

$C_2 \to C_2 - C_1$
$C_3 \to C_3 - C_1$

$$= \cancel{\begin{vmatrix} +1 \\ 3x^2 \\ 2x \end{vmatrix}} = \frac{\cancel{6}\begin{vmatrix} +1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x & y & z \end{vmatrix}}{\cancel{6}\begin{vmatrix} u^2 & v^2 & w^2 \\ u & v & w \\ 1 & 1 & 1 \end{vmatrix}}$$

$$= \frac{\begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} 1 & 0 & 0 \\ x^2 & y^2-x^2 & z^2-x^2 \\ x & y-x & z-x \end{vmatrix} \end{array}}{\begin{array}{c} \begin{vmatrix} u^2 & v^2-u^2 & w^2-u^2 \\ u & v-u & w-u \\ 1 & 0 & 0 \end{vmatrix} \\ \begin{array}{ccc} + & - & + \end{array} \end{array}}$$

$$= \frac{(y^2-x^2)(z-x)-(y-x)(z^2-x^2)}{(v^2-u^2)(w-u)-(v-u)(w^2-u^2)}$$

$$= \frac{(y^2-x^2)(z-x)+(x-y)(z^2-x^2)}{(v^2-u^2)(w-u)+(u-v)(w^2-u^2)}$$

$$= \frac{(y-x)(y+x)(z-x)+(x-y)(z-x)(z+x)}{(v-u)(v+u)(w-u)+(u-v)(w-u)(w+u)}$$

$$= \frac{-(x-y)(x+y)(z-x)+(x-y)(z-x)(x+z)}{-(u-v)(v+u)(w-u)+(u-v)(w-u)(w+u)}$$

$$= \frac{(x-y)(z-x)\left[-(x+y)+(x+z)\right]}{(u-v)(w-u)\left[-(v+u)+(w+u)\right]}$$

$$= \frac{(x-y)(z-x)\left[\cancel{-x}-y+\cancel{x}+z\right]}{(u-v)(w-u)\left[-v-\cancel{u}+w+\cancel{u}\right]} = \frac{(x-y)(z-x)(z-y)}{(u-v)(w-u)(w-v)}$$

$$\to \frac{(x-y)(z-x)(y-z)}{(u-v)(v-w)(w-u)}$$

on taking -ve common from both numerator & denominator

Hence Proved

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Ex:

Sol:

(3) Jacobian of the function if u_i's and x_i's are related by relation :-

$$u_1 = f_1(x_1)$$
$$u_2 = f_2(x_1, x_2)$$
$$u_3 = f_3(x_1, x_2, x_3 \text{———})$$
$$u_x = f_x(x_1, x_2, x_3 - - x_n)$$

$$= \frac{\partial(u_1 u_2 \cdots u_n)}{\partial(x_1, x_2, \cdots x_n}$$

$$= \frac{\partial u_1}{\partial x_1} \times \frac{\partial u_2}{\partial x_2} \times \frac{\partial u_3}{\partial x_3} \times \frac{\partial u_4}{\partial x_4} \text{———}$$

ex: If $y_1 = 1 - x_1$ find $\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)}$

$y_2 = x_1(1 - x_2)$

$y_3 = x_1 x_2 (1 - x_3)$

Sol: $= \frac{\partial y_1}{\partial x_1} \times \frac{\partial y_2}{\partial x_2} \times \frac{\partial y_3}{\partial x_3} = -1(-x_1)(-x_1 x_2) = x_1^2 x_2$

Imp: Functional Relationship (2 marks condition)

If $u_1, u_2, \text{———} u_n$ are functions of $(x_1, x_2 \text{——} x_n)$

then the necessary condition for functional relation b/w $u_1, u_2, \text{——} u_n$ is $J = 0$.

NOTE: if $J = 0 \Rightarrow u_1, u_2, \text{——} u_n$ are dependent i.e. there exists a relation.

if $J \neq 0 \Rightarrow$ They are independent

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Ex: If $u = \sin^{-1}x + \sin^{-1}y$

$v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

find $\frac{\partial(u,v)}{\partial(x,y)}$. Is u & v funtionally related if so find relation.

Sol: $\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-x^2}}, \quad \frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-y^2}}$

$\frac{\partial v}{\partial x} = \sqrt{1-y^2} - \frac{yx}{\sqrt{1-x^2}}$

$\frac{\partial v}{\partial y} = \frac{-xy}{\sqrt{1-y^2}} + \sqrt{1-x^2}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}} \\ \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}} & \frac{-xy}{\sqrt{1-y^2}} + \sqrt{1-x^2} \end{vmatrix}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{-xy}{\sqrt{1-x^2}\sqrt{1-y^2}} + 1 - 1 + \frac{xy}{\sqrt{1-x^2}\sqrt{1-y^2}}$$

$$= 0 \Rightarrow \boxed{J = 0}$$

u & v are functionally related.

To find Relation: $u = \sin^{-1}x + \sin^{-1}y$

$= \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$

i.e. $u = \sin^{-1}v$

$\boxed{v = \sin u}$ Ans

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Ex: Are the functions $u = \frac{x-y}{x+z}$, $v = \frac{x+z}{y+z}$ functionally dependent, if so find relation b/w them.

Sol: $u = \frac{x-y}{x+z}$, $\cancel{v = \frac{x+z}{y+z}}$

$$\cancel{\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x-y}{x+z}\right) = \frac{1(x+z) - (1)(x-y)}{(x+z)^2} = \frac{(x+z) - (x-y)}{(x+z)^2}}$$

$J_1 = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$ (z = constant)

$J_2 = \frac{\partial(u,v)}{\partial(y,z)} = \begin{vmatrix} u_y & u_z \\ v_y & v_z \end{vmatrix}$ (x = constant)

$J_3 = \frac{\partial(u,v)}{\partial(x,z)} = \begin{vmatrix} u_x & u_z \\ v_x & v_z \end{vmatrix}$ (y = constant)

$u = \frac{x}{x+z} - \frac{y}{x+z}$

$v = \frac{x}{y+z} + \frac{z}{y+z}$

$u_x = \frac{(x+z) - x}{(x+z)^2} = \frac{z}{(x+z)^2} + \frac{y}{(x+z)^2}$

$u_x = \frac{z+y}{(z+x)^2}$ — (1)

$u_y = \frac{-1}{x+z}$ — (2)

$u_z = \frac{-x}{(x+z)^2} + \frac{y}{(x+z)^2} = \frac{y-x}{(x+z)^2}$ — (3)

$v_x = \frac{x}{y+z} + \frac{z}{y+z} = \frac{1}{y+z}$

$v_y = \frac{-(x+z)}{(y+z)^2}$.

$v_z = \frac{-x}{(y+z)^2} + \frac{y}{(y+z)^2} = \frac{y-x}{(y+z)^2}$

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$$J_1 = \begin{vmatrix} \frac{z+y}{(z+x)^2} & \frac{-1}{(x+z)} \\ \frac{1}{y+z} & \frac{-(x+z)}{(y+z)^2} \end{vmatrix} = \frac{-(z+y)(x+z)}{(z+x)^2(y+z)^2} + \frac{1}{(x+z)(y+z)}$$

$$= 0$$

$$J_2 = \begin{vmatrix} \frac{-1}{x+z} & \frac{y-x}{(x+z)^2} \\ \frac{-(z+z)}{(y+z)^2} & \frac{y-x}{(y+z)^2} \end{vmatrix} = \frac{-1}{(x+z)}\frac{(y-x)}{(y+z)^2} + \frac{(y-x)(x+z)}{(x+z)^2(y+z)^2}$$

$$= \frac{-(x+z)(y-x) + (y-x)(x+z)}{(x+z)^2(y+z)^2}$$

$$= (x-y)(x+z) + (y-x)(x+z)$$

$$= (x+z)\left[\cancel{x}-\cancel{y}+\cancel{y}-\cancel{x}\right] = 0$$

$$J_3 = \begin{vmatrix} \frac{y}{(x+z)^2} & \frac{y-x}{(x+z)^2} \\ \frac{\cdot 1}{(y+z)} & \frac{y-x}{(y+z)^2} \end{vmatrix} = 0$$

$$\therefore J_1 = J_2 = J_3 = 0$$

u & v are functionally related

to find relation, $1-v = 1-\left(\frac{x+z}{y+z}\right) = \frac{y-x}{y+z}$

$(1-u) = 1 - \frac{(x-y)}{x+z} \quad x+z-x+y \quad = \frac{y+z}{x+y}$

$$\boxed{1-u = \frac{1}{v}}$$

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Imp ex: If u, v, w are roots of cubic equation $(\lambda-x)^3+(\lambda-y)^3+(\lambda-z)^3=0$ in λ find

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}.$$

Sol: $\lambda^3-x^3-3\lambda^2x+3\lambda x^2$

using $(a-b)^3=a^3-b^3-3a^2b+3ab^2$

$=3\lambda^3-(x^3+y^3+z^3)-3\lambda^2(x+y+z)+3\lambda(x^2+y^2+z^2)=0$

$3\lambda^3-3(x+y+z)\lambda^2+3(x^2+y^2+z^2)\lambda-(x^3+y^3+z^3)=0$

$a=3,\ b=-3(x+y+z)$

$c=3(x^2+y^2+z^2)$

$d=-(x^3+y^3+z^3)$

u, v, w are roots,

Sum of roots $=-b/a$

product of roots $=-d/a$

product of doublets $=c/a$

★ $u+v+w=x+y+z$

$$\boxed{f_1=u+v+w-x-y-z}$$

★ $uv+vw+wu=x^2+y^2+z^2$

$$\boxed{f_2=uv+vw+wu-x^2-y^2-z^2}$$

★ $uvw=\dfrac{x^3+y^3+z^3}{3}=f_3$

By implicit function,

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}=(-1)^3\cdot\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}$$

★ (for unit-2: choose any topic)

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\dfrac{\partial(f_1 f_2 f_3)}{\partial(x,y,z)}}{\dfrac{\partial(f_1 f_2 f_3)}{\partial(u,v,w)}} \quad \text{—} ①$$

$$\frac{\partial f_1}{\partial x} = \frac{\partial}{\partial x}(u+v+w-x-y-z) = -1$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix}$$

$$= -2\begin{vmatrix} +1 & +1 & +1 \\ -2x & -2y & +2z \\ +x^2 & +y^2 & +z^2 \end{vmatrix}$$

$C_2 \to C_2 - C_1$

$C_3 \to C_3 - C_1$

$$= \begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^2 & y^2-x^2 & z^2-x^2 \end{vmatrix}$$

$$= 1\left((y-x)(z^2-x^2) - (z-x)(y^2-x^2)\right)$$

$$= (yz^2 - yx^2 - xz^2 + x^3) - (zy^2 - zx^2 - xy^2 + x^3)$$

$$= yz^2 - yx^2 - xz^2 + \cancel{x^3} - zy^2 + zx^2 + xy^2 - \cancel{x^3}$$

$$= yz^2 - yx^2 - xz^2 - zy^2 + zx^2 + xy^2$$

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$$= {}^{2}(y-x)(z-x)\left[z+\cancel{x}-y-\cancel{x}\right] = -2(y-x)(z-x)(z-y)$$

Now. $$\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ v+w & u+w & u+v \\ vw & uw & uv \end{vmatrix} \quad \begin{matrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{matrix}$$

$$u+\cancel{w}-v-\cancel{w}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ v+w & u-v & u-w \\ vw & w(u-v) & v(u-w) \end{vmatrix} \quad \begin{matrix} uw-vw \\ u+\cancel{v}-\cancel{v}-w \\ uv-vw \\ v(u-w) \end{matrix}$$

$$= 1\left[(u-v)v(u-w) - w(u-v)(v-w)\right]$$

$$= v(u-v)(u-w) - w(u-v)(u-w)$$

$$= (u-v)\left[vu - vw - wu + w^2\right]$$

$$= (u-v) \qquad = (u-v)(u-w)(v-w)$$

(L-4)

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Que-1: Verify chain rule for Jacobian if $x=u$, $y=u\tan v$, $z=w$.

Sol: Inverse property ko chain rule maan lia

$JJ' = 1 \rightarrow$ we have to verify this

$$\cancel{\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1} \qquad (J_1 J_2 = 1)$$

$$\cancel{\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = 1}$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} \times \frac{\partial(x,y,z)}{\partial(u,v,w)} = 1$$

$$\underbrace{\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}}_{J_2} \times \underbrace{\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}}_{J_1}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ \tan v & u\sec^2 v & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$u = x$

$v = \tan^{-1}\left(\frac{y}{u}\right) = \tan^{-1}\left(\frac{y}{x}\right)$

$w = z$

$= u\sec^2 v$ — (1)

$$= \begin{vmatrix} 1 & 0 & 0 \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$= \frac{x}{x^2+y^2}$ — (2)

Now $JJ' = \frac{u\sec^2 v}{1} \times \frac{x}{x^2+y^2}$

$x^2+y^2 = u^2 + u^2\tan^2 v$

$= u^2(1+\tan^2 v)$

$= u^2\sec^2 v$

$= \frac{\cancel{u\sec^2 v}}{1} \times \frac{x=u}{\cancel{u^2\sec^2 v}}$

$= 1$ (Hence Proved)

Que ② If $x=\sqrt{vw},\ y=\sqrt{wu},\ z=\sqrt{uv}$

$u=r\sin\theta\cos\phi,\ v=r\sin\theta\sin\phi,\ w=r\cos\theta$

then calculate $\dfrac{\partial(x,y,z)}{\partial(\cancel{u,v,w})}$.
$\qquad\qquad\qquad\qquad r,\theta,\phi$

Sol: $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$

$x,y,z \longrightarrow u,v,w \longrightarrow r,\theta,\phi$

use composite.

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}\times\frac{\partial(u,v,w)}{\partial(r,\theta,\phi)}=?$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{vmatrix}=\begin{vmatrix}0 & \frac{1}{2}\sqrt{\frac{w}{v}} & \frac{1}{2}\sqrt{\frac{v}{w}}\\ \frac{1}{2}\sqrt{\frac{w}{u}} & 0 & \frac{1}{2}\sqrt{\frac{u}{w}}\\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} & 0\end{vmatrix} \quad —①$$

$\frac{\partial x}{\partial u}=\frac{\partial}{\partial u}(\sqrt{vw})=0$

$\frac{\partial x}{\partial v}=\frac{\partial}{\partial v}(\sqrt{vw})=\sqrt{w}\times\frac{1}{2\sqrt{v}}=\frac{\sqrt{w}}{2\sqrt{v}}$

$\frac{\partial x}{\partial w}=\frac{\partial}{\partial w}(\sqrt{vw})=\sqrt{v}\times\frac{1}{2\sqrt{w}}$ $\qquad \frac{\partial z}{\partial w}=\frac{\partial}{\partial w}(\sqrt{uv})=0$

$\frac{\partial y}{\partial u}=\frac{\partial}{\partial u}(\sqrt{wu})=\sqrt{w}\times\frac{1}{2\sqrt{u}}$

$\frac{\partial y}{\partial v}=\frac{\partial}{\partial v}(\sqrt{wu})=0$

$\frac{\partial y}{\partial w}=\frac{\partial}{\partial w}(\sqrt{wu})=\sqrt{u}\times\frac{1}{2\sqrt{w}}$

$\frac{\partial z}{\partial u}=\frac{\partial}{\partial u}(\sqrt{uv})=\sqrt{v}\times\frac{1}{2\sqrt{u}}$

$\frac{\partial z}{\partial v}=\frac{\partial}{\partial v}(\sqrt{uv})=\sqrt{u}\times\frac{1}{2\sqrt{v}}$

Now solving determinant,

by taking common, $\frac{1}{2}, \frac{1}{\sqrt{u}}, \frac{1}{\sqrt{v}}, \frac{1}{\sqrt{w}}$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{\sqrt{u}\sqrt{v}\sqrt{w}} \begin{array}{ccc} + & - & + \\ \end{array} \begin{vmatrix} 0 & \sqrt{w} & \sqrt{v} \\ \sqrt{w} & 0 & \sqrt{u} \\ \sqrt{v} & \sqrt{u} & 0 \end{vmatrix}$$

$$= -\sqrt{w}\left[0 - \sqrt{u}\sqrt{v}\right] + \sqrt{v}\left[\sqrt{w}\sqrt{u} - 0\right]$$

$$= \left(\sqrt{u}\sqrt{v}\sqrt{w} + \sqrt{v}\sqrt{w}\sqrt{u}\right) \times \frac{1}{8} \times \frac{1}{\sqrt{u}\sqrt{v}\sqrt{w}}$$

$$= 2\sqrt{u}\cancel{\sqrt{v}\sqrt{w}} \times \frac{1}{8\sqrt{u}\cancel{\sqrt{v}\sqrt{w}}}$$

$= \frac{1}{4}$ Ans

$$\frac{\partial(u,v,w)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} & \frac{\partial u}{\partial \phi} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} & \frac{\partial v}{\partial \phi} \\ \frac{\partial w}{\partial r} & \frac{\partial w}{\partial \theta} & \frac{\partial w}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\phi\cos\theta & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\sin\phi\cos\theta & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} \quad \text{—②}$$

$\frac{\partial u}{\partial r} = \frac{\partial}{\partial r}(r\sin\theta\cos\phi) = \sin\theta\cos\phi$

$\frac{\partial u}{\partial \theta} = \frac{\partial}{\partial \theta}(r\sin\theta\cos\phi) = r\cos\phi\cos\theta$ $\quad \frac{\partial}{\partial r}(r\cos\theta) = \cos\theta$

$\quad \frac{\partial}{\partial \theta}(r\cos\theta) = r(-\sin\theta$

$\frac{\partial u}{\partial \phi} = r\sin\theta(-\sin\phi)$

$\frac{\partial}{\partial r}(r\sin\theta\sin\phi) = \sin\theta\sin\phi$ $\quad \frac{\partial}{\partial \phi}(r\cos\theta) = 0$

$\frac{\partial}{\partial \theta}(r\sin\theta\sin\phi) = r\sin\phi\cos\theta$

$\cancel{\frac{\partial}{\partial \theta}}\ \frac{\partial}{\partial \phi}(r\sin\theta\sin\phi) = r\sin\theta\cos\phi$

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Now Solving determinant ②

$$= r^2 \sin\theta \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix}$$

$\oplus \quad \ominus \quad \oplus$

$$= r^2\sin\theta \left[\cos\theta \underbrace{\begin{vmatrix} \cos\theta\cos\phi & -\sin\phi \\ \cos\theta\sin\phi & \cos\phi \end{vmatrix}} + \sin\theta \underbrace{\begin{vmatrix} \sin\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\phi \end{vmatrix}} \right]$$

$\sin\theta\cos^2\phi + \sin^2\phi\sin\theta$

$\sin\theta$

$\cos\theta\cos^2\phi + \cos\theta\sin^2\phi$

$\cos\theta$

$$= r^2\sin\theta \left[\cos^2\theta + \sin^2\theta \right] = r^2\sin\theta$$

$$= \frac{r^2\sin\theta}{4} \text{ Ans}$$

NOTE: Cartesian $\rightarrow$ Spherical me jate waqt
~~use Jacobian - $r^2\sin\theta$ vala determinant~~

Que ③ :- Determine functional Relationship between
$u = \frac{x+y}{x-y}$, $v = \frac{xy}{(x-y)^2}$.

Que ④ :- If $u = \frac{x+y}{1-xy}$ and $v = \tan^{-1}x + \tan^{-1}y$ find $\frac{\partial(u,v)}{\partial(x,y)}$. Are they functionally dependent if yes find relation between them.

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Ques⑧ If $u^2+v+w = x+y^2+z^2$

$u+v^3+w = x^2+y+z^2$

$u+v+w^3 = x^2+y^2+z$

Show that,

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{1-4(xy+yz+zx)}{2-3(u^2+v^2+w^2)+27u^2v^2w^2}$$

Sol-③ $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x+y}{x-y}\right) = \frac{1(x-y)-(x+y)}{(x-y)^2} = \frac{\cancel{x}-y-\cancel{x}-y}{(x-y)^2} = \frac{-2y}{(x-y)^2} \quad ①$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(\frac{xy}{(x-y)^2}\right) = \frac{y(x-y)^2-2(x-y)xy}{(x-y)^{2\times 2}}$$

$$= \frac{y(x^2+y^2-2xy)-2xy(x-y)}{(x-y)^4}$$

$$= \frac{\cancel{x^2y}+y^3-\cancel{2xy^2}-2x^2y+\cancel{2xy^2}}{(x-y)^4}$$

$$= \frac{-x^2y+y^3}{(x-y)^4} = \frac{y(-x^2+y^2)}{(x-y)^4}$$

$$= \frac{y\cancel{(y-x)}(y+x)}{(y-x)^4} = \frac{y(x+y)}{(y-x)^3} \quad ②$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x+y}{x-y}\right) = \frac{(x-y)-(-1)(x+y)}{(x-y)^2} = \frac{(x-y)+x+y}{(x-y)^2}$$

$$= \frac{x-\cancel{y}+x+\cancel{y}}{(x-y)^2} = \frac{2x}{(x-y)^2} \quad ③$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left(\frac{xy}{(x-y)^2}\right) = \frac{x(x-y)^2 \mp 2(x-y)(xy)}{(x-y)^2} = \frac{x(x^2+y^2-2xy) \mp 2xy(xy)}{(x-y)^2}$$

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$$= \frac{x(x^2+y^2-2xy) \mp 2xy(xy)}{(x-y)^2}$$

$$= \frac{x^3+xy^2-2x^2y \mp 2x^2y^2}{(x-y)^2}$$

$$= x(x^2$$

$$\frac{\partial v}{\partial y} = \frac{x(x+y)}{(x-y)^3}$$

$$J = \begin{vmatrix} \frac{-2y}{(x-y)^2} & \frac{2x}{(x-y)^2} \\ \frac{y(x+y)}{(y-x)^3} & \frac{x(x+y)}{(x-y)^3} \end{vmatrix} = 0$$

Jacobian=0, hence the functional relationship exists:

$$u = \frac{x+y}{x-y}, \quad v = \frac{xy}{(x-y)^2}$$

$$u^2 = \frac{(x+y)^2}{(x-y)^2} - \frac{1}{1} = \frac{(x+y)^2-(x-y)^2}{(x-y)^2}$$

$$= \frac{x^2+y^2+2xy-(x^2+y^2-2xy)}{(x-y)^2}$$

$$= \frac{\cancel{x^2}+\cancel{y^2}+2xy-\cancel{x^2}-\cancel{y^2}+2xy}{(x-y)^2}$$

$$= \frac{4xy}{(x-y)^2}$$

$$\boxed{u^2-1 = 4v}$$ Ans

Sol (4)

$\frac{\partial v}{\partial x} =$

$\frac{\partial v}{\partial y} =$

$\frac{\partial v}{\partial x} =$

$\frac{\partial x}{\partial y} =$

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Sol-4) $u = \frac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$

$$\frac{\partial(u,v)}{\partial(x,y)} = ? = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x+y}{1-xy}\right) = \frac{(1-xy) - (-(y)(x+y))}{(1-xy)^2} = \frac{1-xy+y(x+y)}{(1-xy)^2}$$

$$= \frac{1-\cancel{xy}+\cancel{xy}+y^2}{(1-xy)^2} = \frac{y^2+1}{(1-xy)^2}$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left(\tan^{-1}x + \tan^{-1}y\right) = \frac{1}{1+y^2}$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(\tan^{-1}x + \tan^{-1}y\right) = \frac{1}{1+x^2}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x+y}{1-xy}\right) = \frac{(1-xy) - (-x)(x+y)}{(1-xy)^2}$$

(x ~~const~~)

$$= \frac{(1-xy)+x(x+y)}{(1-xy)^2} = \frac{1-\cancel{xy}+x^2+\cancel{xy}}{(1-xy)^2} = \frac{x^2+1}{(1-xy)^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{y^2+1}{(1-xy)^2} & \frac{x^2+1}{(1-xy)^2} \\ \frac{1}{x^2+1} & \frac{1}{y^2+1} \end{vmatrix} = \frac{(\cancel{y^2+1})}{(1-xy)^2} \times \frac{1}{(\cancel{y^2+1})} - \frac{(\cancel{x^2+1})}{(1-xy)^2} \times \left(\frac{1}{\cancel{x^2+1}}\right)$$

$$= \cancel{\frac{1}{(1-xy)^2}} - \cancel{\frac{1}{(1-xy)^2}} = 0$$

$J = 0$, hence They are functionally dependent

$\tan^{-1}x + \tan^{-1}y = \left(\frac{x+y}{1-xy}\right)$ → using this formula.

$\boxed{v = u}$

↳ This is the relation b/w u & v.

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Sol (8) $f_1 = u^2+v+w-x-y^2-z^2$

$f_2 = u+v^3+w-x^2-y-z^2$

$f_3 = u+v+w^3-x^2-y^2-z$

To find: $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$

using Implicit function relation,

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\;\frac{\partial(f_1 f_2 f_3)}{\partial(x,y,z)}\;}{\frac{\partial(f_1 f_2 f_3)}{\partial(u,v,w)}}$$

$$= (-1) \frac{\begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix}} = (-1) \begin{vmatrix} -1 & -2z & -2x \\ -2y & -1 & -2z \\ -2x & -2y & -1 \end{vmatrix}$$

$\dfrac{\partial f_1}{\partial x_1} = \dfrac{\partial}{\partial x}\left(u^2+v+w-x-y^2-z^2\right) = -1$

$\dfrac{\partial f_1}{\partial y} = -2y$, $\dfrac{\partial f_2}{\partial z} = -2z$, $\dfrac{\partial f_1}{\partial u} = 2u$, $\dfrac{\partial f_2}{\partial v} = 3v^2$

$\dfrac{\partial f_1}{\partial z} = -2z$, $\dfrac{\partial f_3}{\partial x} = -2x$, $\dfrac{\partial f_1}{\partial v} = 1$, $\dfrac{\partial f_2}{\partial w} = 1$

$\dfrac{\partial f_2}{\partial x} = -2x$, $\dfrac{\partial f_3}{\partial y} = -2y$, $\dfrac{\partial f_1}{\partial w} = 1$, $\dfrac{\partial f_3}{\partial u} = 1$

$\dfrac{\partial f_2}{\partial y} = -1$, $\dfrac{\partial f_3}{\partial z} = -1$, $\dfrac{\partial f_2}{\partial u} = 1$, $\dfrac{\partial f_3}{\partial v} = 1$

$\dfrac{\partial f_3}{\partial w} = 3w^2$

Now putting the values in Jacobian

$$= (-1)\begin{vmatrix} -1 & -2z & -2x \\ -2y & -1 & -2z \\ -2x & -2y & -1 \end{vmatrix} = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} 1 & 2z & 2x \\ 2y & 1 & 2z \\ 2x & 2y & 1 \end{vmatrix} \end{array}$$

$$\begin{vmatrix} 2w & 1 & 1 \\ 1 & 3v^2 & 1 \\ 1 & 1 & 3w^2 \end{vmatrix} \qquad \begin{vmatrix} 2w & 1 & 1 \\ 1 & 3v^2 & 1 \\ 1 & 1 & 3w^2 \end{vmatrix}$$

$$= 1(1-4yz) - 2z(2y-4xz) + 2x(4y^2-2x)$$

$$\begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} 2w & 0 & 1 \\ 1 & 3v^2-1 & 1 \\ 1 & 1-3w^2 & 3w^2 \end{vmatrix} \end{array}$$

$$= 2w\left(3w^2(3v^2-1) + (3w^2-1)\right) + 1\left(1-3w^2-3v^2+1\right)$$

$$= 2w\left[9v^2w^2 - \cancel{3w^2} + \cancel{3w^2} - 1\right] + \left[2-3w^2-3v^2\right]$$

$$= 2w\left[9v^2w^2-1\right] + \left[2-3w^2-3v^2\right]$$

$$= 18v^2w^3 - 2w + 2 - 3w^2 - 3v^2$$

$$= \frac{1-4yz-4yz+8xz^2+8xy^2-4x^2}{18v^2w^3-2w+2-3w^2-3v^2}$$

$$= 1-8yz+8xz^2+8xy^2-4x^2$$