6. A classic unsolved problem in number theory asks if there are infnitely many pairs of 'twin primes' ,pairs of primes separated by 2, such as 3 and 5, 11 and 13, or 71 and 73.
Prove that the only prime triple (i.e. three primes, each 2 from the next) is 3, 5, 7

(idea.) Establish a fact that 3 divides one of the elements of a sequence of three 2-seperated integers.
Use that fact to prove given statement.

Lemma 1 : For all integers a, the sequence $(a, a+2, a+4)$ contains one element that is divisible by 3.

Proof:

There exists 2 unique integers k and r such that $a=3k+r$ where $0 \leq r < 3$
($\because$ Euclids division lemma on dividend a and divisor 3).

Case 0: $r=0$

$a=3k$

3 divides a, which is an element of the given sequence.

Case 1: $r=1$

$a=3k+1$

$a+2=3k+3$

$a+2=3(k+1)$

3 divides $a+2$, which is an element of the given sequence.

Case 2: $r=2$

$a=3k+2$

$a+4=3k+6$

$a+4=3(k+2)$

3 divides $a+4$, which is an element of the given sequence.

$\therefore$ 3 divides any one of the elements in given sequence.
[*Q.E.D*]

Statement: The only prime triple (i.e. three primes, each 2 from the next) is 3, 5, 7.

Proof:

A triple of primes is a sequence of the form $(p, p+2, p+4)$.
From Lemma 1, one of the primes in such a sequence has to be divisible by 3.
Since 3 is the only prime number that is divisible by 3, a prime triple has to include 3.
We have three potential prime triples: $(3, 5, 7)$, $(1, 3, 5)$ and $(-1, 1, 3)$.
Out of the three, $(3, 5, 7)$ is the only one that contains only primes. (-1 and 1 are not prime numbers)
$\therefore$ The only possible prime triple is $(3, 5, 7)$.

[Q.E.D]