Question : Given an infnite collection A_n , $n=1,2,3,\ldots$ of intervals of the real line, their intersection is defned to be $\bigcap_{n=1}^{\infty} A_n = \{x | (\forall n \in \mathbb{N})[x \in A_n]\}$.

Give an example of a family of intervals A_n, n = 1, 2, . . ., such that $A_{n+1} \subset A_n$ for all n and $\bigcap_{n=1}^{\infty} A_n$ contains a single real number.

(Idea:) Define a family of intervals A such that $A_n = \left[\frac{-1}{n}, \frac{+1}{n}\right]$

Answer:

Let A be family of sets, such that $A_n = \left\{x \in \mathbb{R} : |x| \leq \frac{1}{n}\right\}$ where $n \geq 1$

Statement 1: $(\forall n \geq 1)[A_{n+1} \subset A_n]$
proof:

for any positive integer n and real number m, it is obvious that if $|m| \leq \frac{1}{n+1}$ then $|m| \leq \frac{1}{n}$. ($\because$ $\frac{1}{n+1} < \frac{1}{n}$)

From the definition of family A, we can rephrase above statement as
$(\forall m, n \in \mathbb{N})[m \in A_{n+1} \Rightarrow m \in A_n]$
which is equivalent to saying that $A_{n+1} \subset A_n$

$\therefore$ Every set is a superset of it's succesor.

Statement 2: $\bigcap_{n=1}^{\infty} A_n = \{\}$
The real number 0 belongs to every interval in family A.
proof:

For every natural number n ,0 lies in between $\frac{-1}{n}$ and $\frac{+1}{n}$.

In other words, $|0| < \frac{1}{n}$ which means $0 \in A_n$ ($\because$ definition of family A)

Q.E.D

No other real number belongs to the intersection of sets in A.
proof (via contradiction):

Negation statement: there exists real number $y > 0$ such that for any natural number n, $y \in A_n$.

Let $m \in \mathbb{N}$ such that $\frac{1}{m} < |y|$.

$|y| > \frac{1}{m}$

$\neg\left(|y| \leq \frac{1}{m}\right)$

$y \notin A_m$ $\because$ definition of family

Contradiction

$\therefore$ There is no real number other than 0 that is common to all sets in family A

$\therefore$ Intersection of all sets in family A is an empty set.

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