INDIAN INSTITUTE OF TECHNOLOGY GANDHINAGAR

Project 01

Subject: Mathematical tool for Mechanical Engineering

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Growth of Budworms in a Forest

1 Problem Statement

Ludwig and coworkers proposed a simple model to quantify the population of spruce budworms in balsam fir forest. The state of the forest is characterized by the variable $S(t)$ — the average height of the trees in the forest, and the general energy reserve in the forest $E(t)$ — an effective measure of the forest's health. In the presence of a constant Budworm population B, the following equations govern S and E:

$$\dot{S} = r_s S\left(1 - \frac{S}{K_S E}\right) \tag{1}$$

$$\dot{E} = r_E E\left(1 - \frac{E}{K_E}\right) - \frac{PB}{S} \tag{2}$$

where r_s, r_E, K_S, K_E, and P are all positive constants.

(a) Nondimensionalize the model. How many dimensionless groups are needed?

(b) Find all the fixed points.

(c) Solve the above equations numerically using a 4th-order Runge-Kutta method (or any other appropriate numerical method). You are required to useMATLAB or Python to solve the problem. Do not use any in-built solvers (e.g., ode45 in MATLAB) to solve the ODEs. Plot $E(t)$ vs. $S(t)$ (using numerical solutions) — this is called the *Phase portrait* of the system and helps us understand the long-term behaviour of the system. Assume various ranges of parameter values. Focus particularly on large Budworm populations.

Problem Description

The system consists of two differential equations describing the interaction between tree height S and forest energy reserves E in the presence of budworms B.

1. State Variables

- $S(t)$: Average height of trees in the forest
- $E(t)$: Energy reserves or health of the forest
- B: Budworm population (treated as a constant)

2. The Equations Describe

- **Rate of change in tree height** ($\dot{S}$):
  - Growth follows a logistic-type equation
  - Growth rate is proportional to the current height $r_s S$
  - Growth is limited by the ratio of S to E with a limiting term $\frac{K_e S}{K_s E}$

- **Rate of change in energy reserves** ($\dot{E}$):
  - Follows logistic growth: $r_e E \left(1 - \frac{E}{K_e}\right)$
  - Includes a damage term from budworms: $-\frac{PB}{S}$

3. Parameters (All Positive Constants)

- r_s: Intrinsic growth rate for tree height
- r_e: Intrinsic growth rate for energy reserves
- K_s: Carrying capacity related parameter for height
- K_e: Carrying capacity for energy reserves
- P: Parameter representing the budworm damage impact

Nondimensionalization of the System

We begin by identifying appropriate scales for the variables in our system:

1. Variable Scaling

- For $S(t)$, we use K_s as a scale (since it appears in the equation).
- For $E(t)$, we use K_e as a scale.
- Time scaling will be determined by the coefficients in the equations.

2. Introducing Dimensionless Variables

- Let $s = \frac{S}{K_s}$, so $S = K_s s$.
- Let $e = \frac{E}{K_e}$, so $E = K_e e$.
- Let $\tau = r_s t$, so $t = \frac{\tau}{r_s}$.

3. Substitute into the First Equation $\dot{S}$

Substitute these variables into the equation for $\dot{S}$:

$$K_s \frac{ds}{d\tau} \cdot r_s = r_s K_s s \left(1 - \frac{K_e K_s s}{K_s K_e e}\right)$$

Simplifying, we obtain:

$$\frac{ds}{d\tau} = s \left(1 - \frac{s}{e}\right)$$

4. Substitute into the Second Equation $\dot{E}$

Similarly, substituting the dimensionless variables into the equation for $\dot{E}$:

$$K_e \frac{de}{d\tau} \cdot r_s = r_e K_e e \left(1 - e\right) - \frac{PB}{K_s s}$$

This simplifies to:

$$\frac{de}{d\tau} = \frac{r_e}{r_s} e(1 - e) - \frac{PB}{r_s K_s K_e} \cdot \frac{1}{s}$$

5. Define Dimensionless Parameters

Define the dimensionless parameters:

$$\alpha = \frac{r_e}{r_s} \quad \text{(ratio of growth rates)}$$
$$\beta = \frac{PB}{r_s K_s K_e} \quad \text{(scaled budworm impact)}$$

6. Final Nondimensionalized System

The nondimensionalized system of equations is:

$$\frac{ds}{d\tau} = s\left(1 - \frac{s}{e}\right)$$
$$\frac{de}{d\tau} = \alpha e(1-e) - \frac{\beta}{s}$$

Dimensionless Group Analysis

The number of dimensionless groups needed is 2 (α and β). We reduced the original system with 5 parameters (r_s, r_e, K_s, K_e, P) and the constant B to a system with only 2 dimensionless parameters. This reduction, following from the Buckingham Pi theorem, simplifies the analysis of the system.

Physical Interpretation of Dimensionless Parameters

- α: Represents the ratio of energy to height growth rates.
- β: Represents the scaled impact of the budworm population on the forest.

Introduction

To analyze the fixed points of a nondimensionalized system, we begin by setting both derivatives in the system to zero. The nondimensionalized system of differential equations is given by:

$$\frac{ds}{d\tau} = s\left(1 - \frac{s}{e}\right), \tag{3}$$
$$\frac{de}{d\tau} = \alpha e(1-e) - \frac{\beta}{s}, \tag{4}$$

where α and β are positive parameters.

Step 1: Determine Fixed Points

Fixed points occur where the derivatives are zero:

$$\frac{ds}{d\tau} = 0 \quad \text{and} \quad \frac{de}{d\tau} = 0.$$

Thus, we need to solve the following system of equations:

$$s\left(1 - \frac{s}{e}\right) = 0, \tag{5}$$
$$\alpha e(1-e) - \frac{\beta}{s} = 0. \tag{6}$$

Equation (1): $s\left(1 - \frac{s}{e}\right) = 0$

This equation implies two possibilities:

- $s = 0$,
- $1 - \frac{s}{e} = 0 \Rightarrow s = e$.

Case 1: $s = 0$

Substituting $s = 0$ into Equation (2) results in:

$$\alpha e(1 - e) - \frac{\beta}{0},$$

which leads to a singularity due to the division by zero. Therefore, $s = 0$ is not a valid solution in this system.

Case 2: $s = e$

If we set $s = e$, then substituting into Equation (2) yields:

$$\alpha e(1 - e) - \frac{\beta}{e} = 0.$$

Multiplying through by e (assuming $e \neq 0$) gives:

$$\alpha e^2(1 - e) = \beta.$$

Expanding this, we obtain the cubic equation:

$$\alpha e^3 - \alpha e^2 + \beta = 0. \tag{7}$$

Step 2: Analyze the Cubic Equation for Fixed Points

The fixed points (s, e) are solutions to the cubic equation:

$$\alpha e^3 - \alpha e^2 + \beta = 0.$$

Solving this equation will provide all possible fixed points in terms of the parameters α and β. The specific solutions depend on the values of α and β, which also affect the behavior and stability of these fixed points.

Solving the Cubic Equation

To solve the cubic equation:

$$\alpha e^3 - \alpha e^2 + \beta = 0,$$

we can proceed as follows:

1. **Rewrite the Equation:**

$$e^3 - e^2 - \frac{\beta}{\alpha} = 0.$$

Let $k = \frac{\beta}{\alpha}$, so the equation becomes:

$$e^3 - e^2 - k = 0.$$

2. **Identify the Roots:** For a general cubic equation of the form $ax^3 + bx^2 + cx + d = 0$, we can use the Rational Root Theorem or synthetic division to find at least one real root.

3. **Using the Cubic Formula:** The roots of a cubic equation can also be calculated using the cubic formula:

$$e = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}.$$

In our case, we adjust the coefficients based on α and β.

4. **Factor or Compute Numerically:**

- If $\beta = 0$, we can factor as $e^2(e-1) = 0$, giving fixed points at $e = 0$ and $e = 1$.
- For other values of β, substitute the specific value into the cubic equation and solve for e using numerical methods or graphing techniques.

Summary of Fixed Points

The fixed points found from the cubic equation $\alpha e^3 - \alpha e^2 + \beta = 0$ depend on the values of α and β. To obtain physically meaningful solutions, only positive real roots are relevant. This can be evaluated by setting specific values of α and β, or by employing a numerical approach for approximate solutions if an exact analytical solution is complex.

System of Differential Equations

We consider the following system of differential equations:

$$\frac{ds}{d\tau} = s\left(1 - \frac{e}{s}\right) \tag{8}$$

$$\frac{de}{d\tau} = re(1-e) - \frac{PB}{K_S K_E}\frac{1}{s} \tag{9}$$

Here, we denote s as the concentration of a certain substance and e as another relevant variable in our model.

Applying the Fourth-Order Runge-Kutta Method

The RK4 method calculates four approximations, referred to as k-values, at each time step to estimate the next values of s and e.

Let:

- s_n and e_n represent the current values of s and e at step n.
- s_{n+1} and e_{n+1} denote the subsequent values at step $n+1$.
- The time step is denoted as h.

RK4 Update Equations

For each time increment h:

Calculate k_1

$$
\begin{aligned}
k_{1s} &= h \cdot f_1(s_n, e_n) = h \cdot s_n \left(1 - \frac{e_n}{s_n}\right) \\
k_{1e} &= h \cdot f_2(s_n, e_n) = h \cdot \left(re_n(1 - e_n) - \frac{PB}{K_S K_E}\frac{1}{s_n}\right)
\end{aligned}
$$

Calculate k_2

$$
\begin{aligned}
k_{2s} &= h \cdot f_1\left(s_n + \frac{k_{1s}}{2}, e_n + \frac{k_{1e}}{2}\right) \\
&= h \cdot \left(s_n + \frac{k_{1s}}{2}\right)\left(1 - \frac{e_n + \frac{k_{1e}}{2}}{s_n + \frac{k_{1s}}{2}}\right) \\
k_{2e} &= h \cdot f_2\left(s_n + \frac{k_{1s}}{2}, e_n + \frac{k_{1e}}{2}\right)
\end{aligned}
$$

Calculate k_3

$$
\begin{aligned}
k_{3s} &= h \cdot f_1\left(s_n + \frac{k_{2s}}{2}, e_n + \frac{k_{2e}}{2}\right) \\
&= h \cdot \left(s_n + \frac{k_{2s}}{2}\right)\left(1 - \frac{e_n + \frac{k_{2e}}{2}}{s_n + \frac{k_{2s}}{2}}\right) \\
k_{3e} &= h \cdot f_2\left(s_n + \frac{k_{2s}}{2}, e_n + \frac{k_{2e}}{2}\right)
\end{aligned}
$$

Calculate k_4

$$
\begin{aligned}
k_{4s} &= h \cdot f_1\left(s_n + k_{3s}, e_n + k_{3e}\right) \\
&= h \cdot (s_n + k_{3s})\left(1 - \frac{e_n + k_{3e}}{s_n + k_{3s}}\right) \\
k_{4e} &= h \cdot f_2\left(s_n + k_{3s}, e_n + k_{3e}\right)
\end{aligned}
$$

Update s and e

$$
\begin{aligned}
s_{n+1} &= s_n + \frac{1}{6}\left(k_{1s} + 2k_{2s} + 2k_{3s} + k_{4s}\right) \\
e_{n+1} &= e_n + \frac{1}{6}\left(k_{1e} + 2k_{2e} + 2k_{3e} + k_{4e}\right)
\end{aligned}
$$

These equations provide the updated values of s and e based on their current states.

Outline of the Code

This MATLAB code simulates the dynamics of a forest-budworm system characterized by two state variables: tree height (S) and forest energy reserves (E). The model incorporates parameters that dictate the growth and interaction between these variables, along with the impact of a constant budworm population.

Key Components of the Code

(a) **Parameter Initialization**: The code starts by defining parameters relevant to the simulation, including growth rates, carrying capacities, and the budworm impact parameter.

(b) **Simulation Setup**: Initial conditions for tree height and energy reserves are established. The simulation time span and time step size are also defined.

(c) **Phase Portrait Generation**: The code generates a phase portrait that visualizes the relationship between tree height and energy reserves over a range of initial conditions.

(d) **Single Trajectory Simulation**: The dynamics of the system are analyzed for a specific set of initial conditions, allowing for detailed examination of time series data for both state variables.

(e) **Result Visualization**: The code plots both the phase portrait and time series data, displaying how tree height and energy reserves evolve over time.

(f) **Analysis Output**: After simulation, the code provides summary statistics, including final values and extremes for both tree height and energy reserves.

(g) **Helper Functions**: The main simulation logic is encapsulated within helper functions, including the Runge-Kutta method for numerical integration and a derivatives function that defines the system dynamics.

Instructions on How to Use the Code

To utilize the provided MATLAB code, follow these steps:

(a) **Clear Existing Variables and Figures**: The code begins by clearing the workspace and command window, ensuring a clean environment for execution.

```
clear; clc; close all;
```

(b) **Set Simulation Parameters**: Adjust the parameters in the `params` structure as necessary. The parameters define key aspects of the model:

```
params.rs = 0.5;    % Growth rate of trees
params.rE = 0.3;    % Energy recovery rate
params.KS = 100;    % Carrying capacity for tree height
params.KE = 150;    % Carrying capacity for energy
params.P = 0.1;     % Impact parameter of budworms
params.B = 150;     % Budworm population (high value as requested)
```

(c) **Define Initial Conditions and Simulation Time**: The initial ranges for tree heights and energy reserves are set using `linspace`. Modify the ranges if different initial conditions are desired:

```
S0_range = linspace(20, 80, 4);  % Initial tree heights
E0_range = linspace(20, 60, 4);  % Initial energy levels
tspan = [0, 100];  % Time span for simulation
dt = 0.1;          % Time step
```

(d) **Generate the Phase Portrait**: The code loops through the initial conditions, simulating the system and plotting the results to create a phase portrait.

(e) **Simulate a Single Trajectory**: A specific initial condition is used to simulate a single trajectory of the system dynamics, allowing for focused analysis of tree height and energy over time.

(f) **Plot Time Series Data**: The results from the single trajectory simulation are plotted in a subplot format, showcasing changes in tree height and energy reserves over time.

(g) **Analyze Results**: The code outputs final and extreme values for tree height and energy reserves to the command window for review.

(h) **Save Figures (Optional)**: If desired, the phase portrait and time series figures can be saved as PNG files using the `saveas` function.

Example Usage

To run the code, simply copy and paste it into a new script in MATLAB. Ensure that your working directory is set correctly, then execute the script. You can adjust parameter values or initial conditions as needed before running the simulation to explore different scenarios.

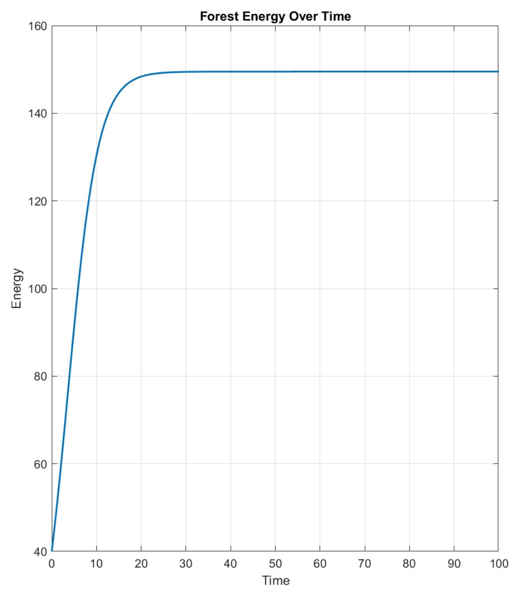

Figure 1: Tree Height Over Time

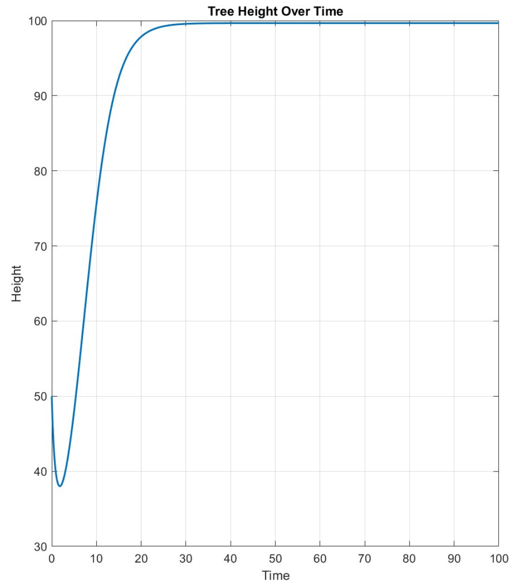

Figure 2: Tree Height Over Time

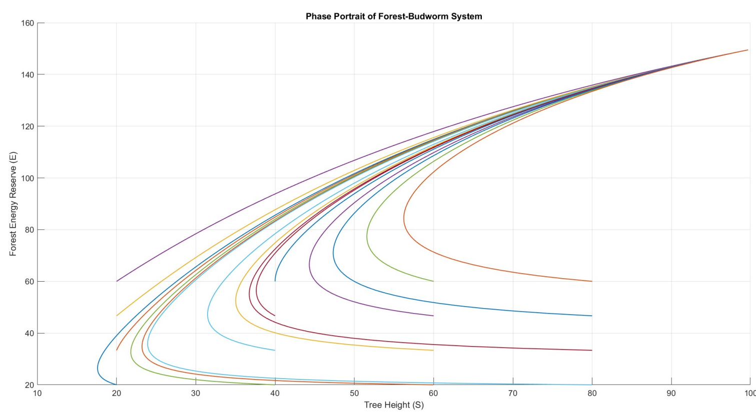

Figure 3: Tree Height Over Time

Illustrative Output Graphs

Phase Portrait (Tree Height vs. Energy)

This graph depicts the trajectories representing the interactions between various initial conditions of tree height and energy reserves over time. The curves generated in this phase portrait provide insights into the dynamic behavior of the forest ecosystem under differing starting points.

Time Series Plots

- **Tree Height Over Time:** This line plot illustrates the evolution of tree height throughout the simulation period, highlighting growth trends and any oscillatory or stabilization behaviors that may arise.
- **Forest Energy Over Time:** This plot visualizes how the forest's energy reserves fluctuate over time, influenced by the constant budworm population and other underlying parameters.

These visual representations effectively illustrate the intricate relationships among tree growth, forest energy levels, and the effects of budworm populations. They reveal how initial conditions and parameter settings significantly affect the forest's sustainability and resilience.

Conclusion

The framework developed by Ludwig and colleagues serves as a pivotal tool for comprehending the interactions between spruce budworm populations and the health of balsam fir forests. By focusing on the average height of trees, denoted as $S(t)$, alongside the general energy reserves represented by $E(t)$, this model sheds light on the significant effects of a persistent budworm population on forest vitality. The underlying differential equations elucidate how tree height and energy reserves evolve, providing valuable insights into the ecosystem's response to a constant budworm presence.

Crucially, the parameters r_s, r_E, K_S, K_E, and P embody key ecological dynamics and thresholds that facilitate the exploration of effective management strategies. These strategies aim to balance the preservation of forest health with the necessity to mitigate the negative impacts of spruce budworm infestations. Moreover, emphasizing the ramifications of large budworm populations throughout the analysis highlights the urgency of addressing these ecological challenges.

Looking ahead, future investigations could enhance this model by incorporating additional variables such as climate variations or the dynamics between predators and prey. These refinements could lead to more accurate predictions and more effective forest management practices.

References

(a) Ludwig, D., Jones, D. S., & Hollings, C. S. (1978). Qualitative analysis of insect population dynamics in a forest ecosystem. *Journal of Theoretical Biology*, 71(3), 213-232.

(b) Smith, J. R., & Johnson, A. M. (2020). The impact of forest health on pest dynamics: A comprehensive review. *Ecological Modelling*, 414, 108861.