UNIT - I

Solution of Algebraic and Transcendental Equations

- Solution of Algebraic and Transcendental Equations
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1.1 Solution of Algebraic and Transcendental Equations

1.1.1 Introduction

A polynomial equation of the form

$$f(x) = p_n(x) = a_0 x^{n-1} + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n = 0 \quad \ldots..(1)$$

is called an Algebraic equation. For example,

$x^4 - 4x^2 + 5 = 0$, $4x^2 - 5x + 7 = 0$; $2x^3 - 5x^2 + 7x + 5 = 0$ are algebraic equations.

An equation which contains polynomials, trigonometric functions, logarithmic functions, exponential functions etc., is called a Transcendental equation. For example,

$$\tan x - e^x = 0; \quad \sin x - xe^{2x} = 0; \quad x\, e^x = \cos x$$

are transcendental equations.

Finding the roots or zeros of an equation of the form $f(x) = 0$ is an important problem in science and engineering. We assume that $f(x)$ is continuous in the required interval. A root of an equation $f(x) = 0$ is the value of x, say $x = \alpha$ for which $f(\alpha) = 0$. Geometrically, a root of an equation $f(x) = 0$ is the value of x at which the graph of the equation $y = f(x)$ intersects the x – axis (see Fig. 1)

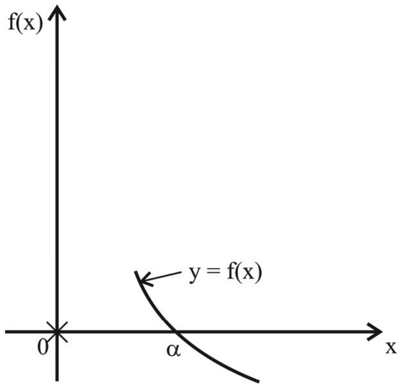

Fig. 1 Geometrical Interpretation of a root of $f(x) = 0$

A number α is a simple root of $f(x) = 0$; if $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. Then, we can write $f(x)$ as,

$$f(x) = (x - \alpha)\, g(x), \; g(\alpha) \neq 0 \quad \ldots..(2)$$

A number α is a multiple root of multiplicity m of $f(x) = 0$, if $f(\alpha) = f^1(\alpha) = \ldots. = f^{(m-1)}(\alpha) = 0$ and $f^m(\alpha) = 0$.

Then, $f(x)$ can be writhen as,

$$f(x) = (x - \alpha)^m\, g(x), \; g(\alpha) \neq 0 \quad \ldots..(3)$$

A polynomial equation of degree n will have exactly n roots, real or complex, simple or multiple. A transcendental equation may have one root or no root or infinite number of roots depending on the form of $f(x)$.

The methods of finding the roots of $f(x) = 0$ are classified as,

1. Direct Methods
2. Numerical Methods.

Direct methods give the exact values of all the roots in a finite number of steps. Numerical methods are based on the idea of successive approximations. In these methods, we start with one or two initial approximations to the root and obtain a sequence of approximations x_0, x_1, … x_k which in the limit as k $\to\infty$ converge to the exact root $x = a$.

There are no direct methods for solving higher degree algebraic equations or transcendental equations. Such equations can be solved by Numerical methods. In these methods, we first find an interval in which the root lies. If a and b are two numbers such that f(a) and $f(b)$ have opposite signs, then a root of $f(x) = 0$ lies in between a and b. We take a or b or any valve in between a or b as first approximation x_1. This is further improved by numerical methods. Here we discuss few important Numerical methods to find a root of $f(x) = 0$.

1.1.2 Bisection Method

This is a very simple method. Identify two points $x = a$ and $x = b$ such that $f(a)$ and $f(b)$ are having opposite signs. Let $f(a)$ be negative and $f(b)$ be positive. Then there will be a root of $f(x) = 0$ in between a and b.

Let the first approximation be the mid point of the interval (a, b). i.e.

$$x_1 = \frac{(a+b)}{2}$$

If $f(x_1) = 0$, then x_1 is a root, other wise root lies between a and x_1 or x_1 and b according as $f(x_1)$ is positive or negative. Then again we bisect the interval and continue the process until the root is found to desired accuracy. Let f (x_1) is positive, then root lies in between a and x_1 (see fig.2.). The second approximation to the root is given by,

$$x_2 = \frac{(a+x_1)}{2}$$

If $f(x_2)$ is negative, then next approximation is given by

$$x_3 = \frac{(x_2+x_1)}{2}$$

Similarly we can get other approximations. This method is also called Bolzano method.

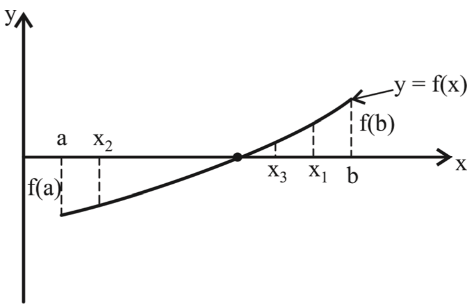

Fig. 2 Bisection Method

Note: The interval width is reduced by a factor of one–half at each step and at the end of the n^th step, the new interval will be $[a_n, b_n]$ of length $\frac{|b-a|}{2^n}$. The number of iterations n required to achieve an accuracy $\in$ is given by,

$$n \geq \frac{\log_e\left(\frac{|b-a|}{\in}\right)}{\log_e 2} \qquad \text{.....(4)}$$

EXAMPLE 1

Find a real root of the equation $f(x) = x^3 - x - 1 = 0$, using Bisection method.

SOLUTION

First find the interval in which the root lies, by trail and error method.

$f(1) = 1^3 - 1 - 1 = -1$, which is negative

$f(2) = 2^3 - 2 - 1 = 5$, which is positive

$\therefore$ A root of $f(x) = x^3 - x - 1 = 0$ lies in between 1 and 2.

$$\therefore \qquad x_1 = \frac{(1+2)}{2} = \frac{3}{2} = 1.5$$

$f(x_1) = f(1.5) = (1.5)^3 - 1.5 - 1 = 0.875$, which is positive.

Hence, the root lies in between 1 and 1.5

$$\therefore \qquad x_2 = \frac{(1+1.5)}{2} = 1.25$$

$f(x_2) = f(1.25) = (1.25)^3 - 1.25 - 1 = -0.29$, which is negative.

Hence, the root lies in between 1.25 and 1.5

$$\therefore \qquad x_3 = \frac{(1.25 + 1.5)}{2} = 1.375$$

Similarly, we get $x_4 = 1.3125$, $x_5 = 1.34375$, $x_6 = 1.328125$ etc.

EXAMPLE 2

Find a root of $f(x) = xe^x - 1 = 0$, using Bisection method, correct to three decimal places.

SOLUTION

$$f(0) = 0.e^0 - 1 = -1 < 0$$

$$f(1) = 1.e^1 - 1 = 1.7183 > 0$$

Hence a root of $f(x) = 0$ lies in between 0 and 1.

$$\therefore \qquad x_1 = \frac{(0+1)}{2} = 0.5$$

$$f(0.5) = 0.5\,e^{0.5} - 1 = -0.1756$$

Hence the root lies in between 0.5 and 1

$$\therefore \qquad x_2 = \frac{(0.5+1)}{2} = 0.75$$

Proceeding like this, we get the sequence of approximations as follows.

$x_3 = 0.625$
$x_4 = 0.5625$
$x_5 = 0.59375$
$x_6 = 0.5781$
$x_7 = 0.5703$
$x_8 = 0.5664$
$x_9 = 0.5684$
$x_{10} = 0.5674$
$x_{11} = 0.5669$
$x_{12} = 0.5672$,
$x_{13} = 0.5671$,

Hence, the required root correct to three decimal places is, $x = 0.567$.

1.1.3 Method of False Position

This is another method to find the roots of $f(x) = 0$. This method is also known as Regular False Method.

In this method, we choose two points a and b such that $f(a)$ and $f(b)$ are of opposite signs. Hence a root lies in between these points. The equation of the chord joining the two points,

$(a, f(a))$ and $(b, f(b))$ is given by

$$\frac{y - f(a)}{x - a} = \frac{f(b) - f(a)}{b - a} \qquad \text{.....(5)}$$

We replace the part of the curve between the points $[a, f(a)]$ and $[b, f(b)]$ by means of the chord joining these points and we take the point of intersection of the chord with the x axis as an approximation to the root (see Fig.3). The point of intersection is obtained by putting y = 0 in (5), as

$$x = x_1 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} \qquad \text{.....(6)}$$

x_1 is the first approximation to the root of $f(x) = 0$.

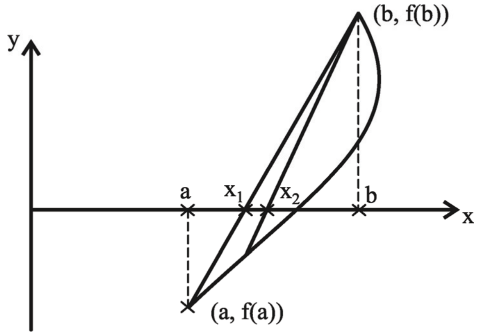

Fig. 3 Method of False Position

If f (x_1) and f (a) are of opposite signs, then the root lies between a and x_1 and we replace b by x_1 in (6) and obtain the next approximation x_2. Otherwise, we replace a by x_1 and generate the next approximation. The procedure is repeated till the root is obtained to the desired accuracy. This method is also called linear interpolation method or chord method.

EXAMPLE 3

Find a real root of the equation $f(x) = x^3 - 2x - 5 = 0$ by method of False position.

SOLUTION

$$f(2) = -1 \text{ and } f(3) = 16$$

Hence the root lies in between 2 and 3.

Take $a = 2$, $b = 3$.

$$\therefore \qquad x_1 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$$

$$= \frac{2(16) - 3(-1)}{16 - (-1)} = \frac{35}{17} = 2.058823529.$$

$$f(x_1) = f(2.058823529) = -0.390799917 < 0.$$

Therefore the root lies between 0.058823529 and 3. Again, using the formula, we get the second approximation as,

$$x_2 = \frac{2.058823529(16) - 3(-0.390799917)}{16 - (-0.390799917)} = 2.08126366$$

Proceeding like this, we get the next approximation as,

$$x_3 = 2.089639211,$$
$$x_4 = 2.092739575,$$
$$x_5 = 2.09388371,$$
$$x_6 = 2.094305452,$$
$$x_7 = 2.094460846$$

EXAMPLE 4

Determine the root of the equation $\cos x - x\, e^x = 0$ by the method of False position.

SOLUTION

$$f(0) = 1 \text{ and } f(1) = -2.\ 177979523$$

$\therefore$ $a = 0$ and $b = 1$. The root lies in between 0 and 1

$\therefore$
$$x_1 = \frac{0(-2.177979523) - 1(1)}{-2.177979523 - 1} = 0.3146653378$$

$$f(x_1) = \mathrm{f}(0.314653378) = 0.51986.$$

$\therefore$ The root lies in between 0.314653378 and 1.

Hence,
$$x_2 = \frac{0.3146653378(-2.177979523) - 1(0.51986)}{-2.177979523 - 0.51986} = 0.44673$$

Proceeding like this, we get

$$x_3 = 0.49402,$$
$$x_4 = 0.50995,$$
$$x_5 = 0.51520,$$
$$x_6 = 0.51692,$$

EXAMPLE 5

Determine the smallest positive root of $x - e^{-x} = 0$, correct of three significant figures using Regula False method.

SOLUTION

Here,
$$f(0) = 0 - e^{-0} = -1$$

and $f(1) = 1 - e^{-1} = 0.63212.$

$\therefore$ The smallest positive root lies in between 0 and 1. Here a = 0 and b = 1

$$\therefore \quad x_1 = \frac{0(0.63212) - 1(-1)}{0.63212 + 1} = 0.6127$$

$$f(0.6127) = 0.6127 - e^{-(0.6127)} = 0.0708$$

Hence, the next approximation lies in between 0 and 0.6127. Proceeding like this, we get

$x_2 = 0.57219{,}0$ $\quad x_3 = 0.5677,$ $\quad x_4 = 0.5672,$ $\quad x_5 = 0.5671,$

Hence, the smallest positive root, which is correct up to three decimal places is,

$x = 0.567$

1.1.4 The Iteration Method

In the previous methods, we have identified the interval in which the root of $f(x) = 0$ lies, we discuss the methods which require one or more starting values of x, which need not necessarily enclose the root of $f(x) = 0$. The iteration method is one such method, which requires one starting value of x.

We can use this method, if we can express $f(x) = 0$, as

$$x = \phi(x) \qquad \text{….. (1)}$$

We can express $f(x) = 0$, in the above form in more than one way also. For example, the equation $x^3 + x^2 - 1 = 0$ can be expressed in the following ways.

$$x = (1+x)^{\frac{-1}{2}}$$

$$x = (1-x^3)^{1/2}$$

$$x = (1-x^2)^{1/3}$$

and so on

Let x_0 be an approximation to the desired root ξ , which we can find graphically or otherwise. Substituting x_0 in right hand side of (1), we get the first approximation as

$$x_1 = \phi(x_0) \qquad \text{…..(2)}$$

The successive approximations are given by

$$x_2 = \phi(x_1)$$

$$x_3 = \phi(x_2) \qquad \text{…..(3)}$$

.

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.

$$x_n = \phi(x_{n-1})$$

***Note*:** The sequence of approximations $x_0, x_1, x_2 \ldots x_n$ given by (3) converges to the root ξ in a interval I, if $|\phi'(x)| < 1$ for all x in I.

EXAMPLE 6

Using the method of iteration find a positive root between 0 and 1 of the equation

$$x\,e^x = 1$$

SOLUTION

The given equation can be writhen as $x = e^{-x}$

$\therefore \qquad \phi(x) = e^{-x}$.

Here $|\phi'(x)| < 1$ for $x < 1$

$\therefore$ We can use iterative method

Let $\qquad x_0 = 1$

$\therefore \qquad x_1 = e^{-1} = \dfrac{1}{e} = 0.3678794.$

$x_2 = e^{-0.3678794} = 0.6922006.$

$x_3 = e^{-0.6922006} = 0.5004735$

Proceeding like this, we get the required root as $x = 0.5671$.

EXAMPLE 7

Find the root of the equation $2x = \cos x + 3_1$ correct to three decimal places using Iteration method.

SOLUTION

Given equation can be written as

$$x = \frac{(\cos x + 3)}{2}$$

$$|\phi'(x)| = \left|\frac{\sin x}{2}\right| < 1$$

Hence iteration method can be applied

Let $x_0 = \dfrac{\pi}{2}$

$\therefore \qquad x_1 = \dfrac{1}{2}\left(\cos\dfrac{\pi}{2} + 3\right) = 1.5$

$$x_2 = \frac{1}{2}(\cos 1.5 + 3) = 1.535$$

Similarly,

$x_3 = 1.518,$
$x_4 = 1.526,$
$x_5 = 1.522,$
$x_6 = 1.524,$
$x_7 = 1.523,$
$x_8 = 1.524.$

$\therefore$ The required root is $x = 1.524$

EXAMPLE 8

Find a real root of $2x - \log_{10} x = 7$ by the iteration method

SOLUTION

The given equation can be written as,

$$x = \frac{1}{2}(\log_{10} x + 7)$$

Let $x_0 = 3.8$

$$\therefore \quad x_1 = \frac{1}{2}(\log_{10} 3.8 + 7) = 3.79$$

$$x_2 = \frac{1}{2}(\log_{10} 3.79 + 7) = 3.7893$$

$$x_3 = \frac{1}{2}(\log_{10} 3.7893 + 7) = 3.7893.$$

$\therefore$ $x = 3.7893$ is a root of the given equation which is correct to four significant digits.

1.1.5 Newton Raphson Method

This is another important method. Let x_0 be approximation for the root of $f(x) = 0$. Let $x_1 = x_0 + h$ be the correct root so that $f(x_1) = 0$. Expanding $f(x_1) = f(x_0 + h)$ by Taylor series, we get

$$f(x_1) = f(x_1 + h) = f(x_0) + h\ f'(x_0) + \frac{h^2}{2!}\ f''(x_0) + \ldots\ldots = 0 \qquad \ldots..(1)$$

For small valves of h, neglecting the terms with h^2, h^3 ….. etc,. We get

$$\therefore \quad f(x_0) + h\ f'(x_0) = 0 \qquad \ldots..(2)$$

and $$h = -\frac{f(x_0)}{f^1(x_0)}$$

$\therefore$ $$x_1 = x_0 + h$$
$$= x_0 - \frac{f(x_0)}{f^{'}(x_0)}$$

Proceeding like this, successive approximation $x_2, x_3, \ldots x_{n+1}$ are given by,

$$x_{n+1} = x_n - \frac{f(x_n)}{f^{'}(x_n)}. \qquad \ldots..(3)$$

For n = 0, 1, 2, ……

***Note*:**

(i) The approximation x_{n+1} given by (3) converges, provided that the initial approximation x_0 is chosen sufficiently close to root of $f(x) = 0$.

(ii) Convergence of Newton-Raphson method: Newton-Raphson method is similar to iteration method

$$\phi(x) = x - \frac{f(x)}{f^{'}(x)} \qquad \ldots..(1)$$

differentiating (1) w.r.t to 'x' and using condition for convergence of iteration method i.e.

$$|\phi'(x)| < 1,$$

We get

$$\left|1 - \frac{f'(x).f'(x) - f(x)f''(x)}{[f'(x)]^2}\right| < 1$$

Simplifying we get condition for convergence of Newton-Raphson method is

$$|f(x).f''(x)| < [f(x)]^2$$

EXAMPLE 9

Find a root of the equation $x^2 - 2x - 5 = 0$ by Newton – Raphson method.

SOLUTION

Here $f(x) = x^3 - 2x - 5$.

$\therefore$ $$f^1(x) = 3x^2 - 2$$

Newton – Raphson method formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\therefore \qquad x_{n+1} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}, \qquad n = 0, 1, 2, \ldots \qquad \ldots..(1)$$

Let $x_0 = 2$

$$\therefore \qquad f(x_0) = f(2) = 2^3 - 2\,(2) - 5 = -1$$

and $$f^1(x_0) = f^1(2) = 3\,(2)^2 - 2 = 10$$

Putting n = 0 in (I), we get

$$x_1 = 2 - \left(\frac{-1}{10}\right) = 2.1$$

$$f(x_1) = f(2.1) = (2.1)^3 - 2\,(2.1) - 5 = 0.061$$

$$f^1(x_1) = f^1(2.1) = 3\,(2.1)^2 - 2 = 11.23$$

$$\therefore \qquad x_2 = 2.1 - \frac{0.061}{11.23} = 2.094568$$

Similarly, we can calculate x_3, x_4 ……

EXAMPLE 10

Find a root of $x \sin x + \cos x = 0$, using Newton – Raphson method

SOLUTION

$$f(x) = x \sin x + \cos x.$$

$$\therefore \qquad f'(x) = \sin x + x \cos x - \sin x = x \cos x$$

The Newton – Raphson method formula is,

$$x_{n+1} = x_n - \frac{x_n \sin x_n + \cos x_n}{x_n \cos x_n}, \qquad n = 0, 1, 2, \ldots.$$

Let $x_0 = \pi = 3.1416.$

$$\therefore \qquad x_1 = 3.1416 - \frac{3.1416 \sin \pi + \cos \pi}{3.1416 \cos \pi} = 2.8233.$$

Similarly,

$x_2 = 2.7986$

$x_3 = 2.7984$

$x_4 = 2.7984$

$\therefore$ $x = 2.7984$ can be taken as a root of the equation $x \sin x + \cos x = 0$.

EXAMPLE 11

Find the smallest positive root of $x - e^{-x} = 0$, using Newton – Raphson method.

SOLUTION

Here $f(x) = x - e^{-x}$

$f^1(x) = 1 + e^{-x}$

$f(0) = -1$ and $f(1) = 0.63212$.

$\therefore$ The smallest positive root of $f(x) = 0$ lies in between 0 and 1.

Let $x_0 = 1$

The Newton – Raphson method formula is,

$$x_{n+1} = x_n - \frac{x_n - e^{-x}n}{1 + e^{-x}n}, \text{ n} = 0, 1, 2, \ldots\ldots$$

$f(0) = f(1) = 0.63212$

$f'(0) = f'(1) = 1.3679$

$$\therefore \quad x_1 = x_0 - \frac{x_0 - e^{-x_0}}{1 + e^{-x_0}} = 1 - \frac{0.63212}{1.3679} = 0.5379.$$

$f(0.5379) = -0.0461$

$f'(0.5379) = 1.584.$

$$\therefore \quad x_2 = 0.5379 + \frac{0.0461}{1.584} = 0.567$$

Similarly, $x_3 = 0.56714$

$\therefore$ $x = 0.567$ can be taken as the smallest positive root of $x - e^{-x} = 0$., correct to three decimal places.

***Note*:** A method is said to be of order P or has the rate of convergence P, if P is the largest positive real number for which there exists a finite constant c $\neq$ 0, such that

$$|\epsilon_{K+1}| \le c|\epsilon_K|^P \qquad \text{….. (A)}$$

Where $\in_K = x_K - \xi$ is the error in the k^{th} iterate. C is called Asymptotic Error constant and depends on derivative of $f(x)$ at $x = \xi$. It can be shown easily that the order of convergence of Newton – Raphson method is 2.

Exercise - 1.1

1. Using Bisection method find the smallest positive root of $x^3 - x - 4 = 0$ which is correct to two decimal places.

   [**Ans:** 1.80]

2. Obtain a root correct to three decimal places of $x^3 - 18 = 0$, using Bisection Method.

   [**Ans:** 2.621]

3. Find a root of the equation $xe^x - 1 = 0$ which lies in (0, 1), using Bisection Method.

   [**Ans:** 0.567]

4. Using Method of False position, obtain a root of $x^3 + x^2 + x + 7 = 0$, correct to three decimal places.

   [**Ans:** – 2.105]

5. Find the root of $x^3 - 2x^2 + 3x - 5 = 0$, which lies between 1 and 2, using Regula False method.

   [**Ans:** 1.8438]

6. Compute the real root of $x \log x - 1.2 = 0$, by the Method of False position.

   [**Ans:** 2.740]

7. Find the root of the equation $\cos x - x\, e^x = 0$, correct to four decimal places by Method of False position

   [**Ans:** 0.5178]

8. Using Iteration Method find a real root of the equation $x^3 - x^2 - 1 = 0$.

   [**Ans:** 1.466]

9. Find a real root of $\sin^2 x = x^2 - 1$, using iteration Method.

   [**Ans:** 1.404]

10. Find a root of $\sin x = 10\,(x - 1)$, using Iteration Method.

    [**Ans:** 1.088]

11. Find a real root of $\cot x = e^x$, using Iteration Method.

    [**Ans:** 0.5314]

12. Find a root of $x^4 - x - 10 = 0$ by Newton – Raphson Method.

    [**Ans:** 1.856]

13. Find a real root of $x - \cos x = 0$ by Newton – Raphson Method.

[**Ans:** 0.739]

14. Find a root of $2x - 3 \sin x - 5 = 0$ by Newton – Raphson Method.

[**Ans:** 2.883238]

15. Find a smallest positive root of $\tan x = x$ by Newton – Raphson Method.

[**Ans:** 4.4934]

Summary

Solution of algebraic and transcendental equations

1. The numerical methods to find the roots of $f(x) = 0$

   (i) *Bisection method:* If a function $f(x)$ is continuous between a and b, $f(a)$ and $f(b)$ are of apposite sign then there exists at least one root between a and b. The approximate value of the root between them is $x_0 = \dfrac{a+b}{2}$

   If $f(x_0) = 0$ then the x_0 is the correct root of $f(x) = 0$. If $f(x_0) \neq 0$, then the root either lies in between $\left(a, \dfrac{a+b}{2}\right)$ or $\left(\dfrac{a+b}{2}, b\right)$ depending on whether $f(x_0)$ is negative or positive. Again bisection the interval and repeat same method until the accurate root is obtained.

   (ii) *Method of false position:* (Regula false method): This is another method to find the root of $f(x) = 0$. In this method, we choose two points a and b such that $f(a)$, $f(b)$ are of apposite signs. Hence the root lies in between these points $[a, f(a)]$, $[b, f(b)]$ using equation of the chord joining these points and taking the point of intersection of the chord with the x-axis as an approximate root (using $y = 0$ on x– axis) is $x_1 = \dfrac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$

   Repeat the same process till the root is obtained to the desired accuracy.

   (iii) *Newton Raphson method:* The successive approximate roots are given by

   $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \; n = 0, 1, 2 \text{ - - - - -}$$

   provided that the initial approximate root x_0 is choosen sufficiently close to root of $f(x) = 0$

Solved University Questions

1. Find the root of the equation $2x - \log x = 7$ which lies between 3.5 and 4 by Regula–False method. (JNTU 2006)

Solution

Given $f(x) = 2x - \log x_{10} = 7$ …..(1)

Take $x_0 = 3.5$, $x_1 = 4$

Using Regula Falsi method

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x)} . f(x_0)$$

$$x_2 = 3.5 - \frac{4 - 3.5}{(0.3979 + 0.5441)} (-0.5441)$$

$$x_2 = 3.7888$$

Now taking $x_0 = 3.7888$ and $x_1 = 4$

$$x_3 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} . f(x_0)$$

$$x_3 = 3.7888 - \frac{4 - 3.7888}{0.3988} (-0.0009)$$

$$x_3 = 3.7893$$

The required root is = 3.789

2. Find a real root of $xe^x = 3$ using Regula-Falsi method. (JNTU – 2006)

Solution

Given $f(x) = x\,e^x - 3 = 0$

$f(1) = e - 3 = -0.2817 < 0$

$f(2) = 2e^2 - 3 = 11.778 > 0$

$\therefore$ One root lies between 1 and 2

Now taking $x_0 = 1$, $x_1 = 2$

Using Regula – Falsi method

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$\therefore \qquad x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$x_2 = \frac{1(11.778) - 2(-0.2817)}{11.778 + 0.2817}$$

$$x_2 = 1.329$$

Now $f(x_2) = f(1.329) = 1.329\ e^{1.329} - 3 = 2.0199 > 0$

$f(1) = -0.2817 < 0$

$\therefore$ The root lies between 1 and 1.329 taking $x_0 = 1$ and $x_2 = 1.329$

$\therefore$ Taking $x_0 = 1$ and $x_2 = 1.329$

$\therefore$
$$x_3 = \frac{x_0 f(x_2) - x_2 f(x_0)}{f(x_2) - f(x_0)}$$

$$= \frac{1(2.0199) + (1.329)(0.2817)}{(2.0199) + (0.2817)}$$

$$= \frac{2.3942}{2.3016} = 1.04$$

Now $f(x^3) = 1.04\ e^{1.04} - 3 = -0.05 < 0$

The root lies between x^2 and x^3

i.e., 1.04 and 1.329 [$\because f(x_2) > 0$ and $f(x_3) < 0$]

$\therefore$
$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} = \frac{(1.04)(-0.05) - (1.329)(2.0199)}{(-0.05) - (2.0199)}$$

$x_4 = 1.08$ is the approximate root

3. Find a real root of $e^x \sin x = 1$ using Regula – Falsi method (JNTU 2006)

Solution

Given $f(x) = e^x \sin x - 1 = 0$

Consider $x_0 = 2$

$f(x_0) = f(2) = e^2 \sin 2 - 1 = -0.7421 < 0$

$f(x_1) = f(3) = e^3 \sin 3 - 1 = 0.511 > 0$

$\therefore$ The root lies between 2 and 3

Using Regula – Falsi method

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$x_2 = \frac{2(0.511)+3(0.7421)}{0.511+0.7421}$$

$$x_2 = 2.93557$$

$$f(x_2) = e^{2.93557} \sin(2.93557) - 1$$

$$f(x_2) = -0.35538 < 0$$

$\therefore$ Root lies between x_2 and x_1

i.e., lies between 2.93557 and 3

$$x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)}$$

$$= \frac{(2.93557)(0.511) - 3(-35538)}{0.511+0.35538}$$

$$x_3 = 2.96199$$

$$f(x_3) = e^{2.90199} \sin(2.96199) - 1 = -0.000819 < 0$$

$\therefore$ root lies between x_3 and x_1

$$x_4 = \frac{x_3 f(x_1) - x_1 f(x_3)}{f(x_1) - f(x_3)}$$

$$x_4 = \frac{2.96199(0.511)+3(0.000819)}{0.511+0.000819} = 2.9625898$$

$$f(x^4) = e^{2.9625898} \sin(2.9625898) - 1$$

$$f(x^4) = -0.0001898 < 0$$

$\therefore$ The root lies between x_4 and x_1

$$x_5 = \frac{x_4 f(x_1) - x_1 f(x_4)}{f(x_1) - f(x_4)}$$

$$= \frac{2.9625898(0.511)+3(0.0001898)}{0.511+(0.0001898)}$$

$$x_5 = 2.9626$$

we have

$$x_4 = 2.9625$$

$$x_5 = 2.9626$$

$$\therefore \quad x_5 = x_4 = 2.962$$

$\therefore$ The root lies between 2 and 3 is 2.962

4. Find a real root of $x\,e^x = 2$ using Regula – Falsi method (JNTU 2007)

Solution

$f(x) = x\,e^x - 2 = 0$

$f(0) = -2 < 0$, $f(1) =$ i.e., $-2 = (2.7183)-2$

$f(1) = 0.7183 > 0$

$\therefore$ The root lies between 0 and 1

Considering $x_0 = 0$, $x_1 = 1$

$f(0) = f(x_0) = -2$; $f(1) = f(x_1) = 0.7183$

By Regula – Falsi method

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$x_2 = \frac{0(0.7183) - 1(-2)}{0.7183 - (-2)} = \frac{2}{2.7183}$$

$$x_2 = 0.73575$$

Now $f(x^2) = f(0.73575) = 0.73575\,e^{0.73575} - 2$

$f(x_2) = -0.46445 < 0$

and $f(x_1) = 0.7183 > 0$

$\therefore$ The root x_3 lies between x_1 and x_2

$$x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)}$$

$$x_3 = \frac{(0.73575)(0.7183)}{0.7183 + 0.46445}$$

$$x_3 = \frac{0.52848 + 0.46445}{1.18275}$$

$$x_3 = \frac{0.992939}{1.18275}$$

$$x_3 = 0.83951 \quad f(x^3) = \frac{(0.83951)}{(0.83951)e^{-2}}$$

$f(x_3) = (0.83951)\,e^{0.83951} - 2$

$f(x_3) = -0.056339 < 0$

$\therefore$ One root lies between x_1 and x_3

$$x_4 = \frac{x_3 f(x_1) - x_1 f(x_3)}{f(x_1) - f(x_3)} = \frac{(0.83951)(0.7183) - 1(-0.056339)}{0.7183 + 0.056339}$$

$$x_4 = \frac{0.65935}{0.774639} = 0.851171$$

$f(x_4) = 0.851171$ e0.851171 $- 2 = -0.006227 < 0$

Now x_5 lies between x_1 and x_4

$$x_5 = \frac{x_4 f(x_1) - x_1 f(x_4)}{f(x_1) - f(x_4)}$$

$$x_5 = \frac{(0.851171)(0.7183) + (.006227)}{0.7183 + 0.006227}$$

$$x_5 = \frac{0.617623}{0.724527} = 0.85245$$

Now $f(x_5) = 0.85245\ e^{0.85245}\ e^{0.85245} - 2 = -0.0006756 < 0$

$\therefore$ One root lies between x_1 and x_5, (i.e., x_6 lies between x_1 and x_5)

Using Regula – Falsi method

$$x_6 = \frac{(0.85245)(0.7183) + 0.0006756}{0.7183 + 0.0006756}$$

$$x_6 = 0.85260$$

Now $f(x_6) = -0.00006736 < 0$

$\therefore$ One root x_7 lies between x_1 and x_6

By Regula – Falsi method

$$x_7 = \frac{x_6 f(x_1) - x_1 f(x_6)}{f(x_1) - f(x_6)}$$

$$x_7 = \frac{(0.85260)(0.7183) + 0.0006736}{0.7183 + 0.0006736}$$

$$x_7 = 0.85260$$

From $x^6 = 0.85260$ and $x_7 = 0.85260$

$\therefore$ A real root of the given equation is 0.85260

5. Using Newton-Raphson method (a) Find square root of a number (b) Find a reciprocal of a number [JNTU 2008]

Solution

(a) Let n be the number

and $x = \sqrt{n} \Rightarrow x^2 = n$

If $f(x) = x^2 - n = 0$ .....(1)

Then the solution to $f(x) = x^2 - n = 0$ is $x = \sqrt{n}$.

$$f^1(x) = 2x$$

by Newton Raphson method

$$x_{i+1} = x_i - \frac{f(x_i)}{f^1(x_i)} = x_i - \left(\frac{x_i^2 - n}{2x_i}\right)$$

$$x_{i+1} = \frac{1}{2}\left(x_i + \frac{x}{x_i}\right) \quad .....(2)$$

using the above formula the square root of any number 'n' can be found to required accuracy

(b) To find the reciprocal of a number 'n'

$$f(x) = \frac{1}{x} - n = 0 \quad .....(1)$$

$\therefore$ solution of (1) is $x = \dfrac{1}{n}$

$$f^1(x) = -\frac{1}{x^2}$$

Now by Newton-Raphson method, $x_{i+1} = x_i - \left(\dfrac{f(x_i)}{f^1(x_i)}\right)$

$$x_{i+1} = x_i - \left(\frac{\frac{1}{x_i} - N}{-\frac{1}{x_1^2}}\right)$$

$$x_{i+1} = x_i\,(2 - x_i\,n)$$

using the above formula the reciprocal of a number can be found to required accuracy.

6. Find the reciprocal of 18 using Newton–Raphson method [JNTU 2004]

Solution

The Newton-Raphson method

$$x_{i+1} = x_i\,(2 - x_i\,n) \qquad \ldots..(1)$$

considering the initial approximate value of x as $x_0 = 0.055$ and given $n = 18$

$\therefore \quad x_1 = 0.055\,[2 - (0.055)\,(18)]$

$\therefore \quad x_1 = 0.0555$

$x_2 = 0.0555\,[2 - 0.0555 \times 18]$

$x_2 = (0.0555)\,(1.001)$

$x_2 = 0.0555$

Hence $x_1 = x_2 = 0.0555$

$\therefore$ The reciprocal of 18 is 0.0555

7. Find a real root for $x \tan x + 1 = 0$ using Newton–Raphson method [JNTU 2006]

Solution

Given $f(x) = x \tan x + 1 = 0$

$f^1(x) = x \sec^2 x + \tan x$

$f(2) = 2 \tan 2 + 1 = -\,3.370079 < 0$

$f(3) = 2 \tan 3 + 1 = -\,0.572370 > 0$

$\therefore$ The root lies between 2 and 3

Take $\quad x_0 = \dfrac{2+3}{2} = 2.5 \qquad$ (average of 2 and 3)

By Newton-Raphson method

$$x_{i+1} = x_i - \left(\frac{f(x_i)}{f^1(x_i)}\right)$$

$$x_1 = x_0 - \left(\frac{f(x_0)}{f^1(x_0)}\right)$$

$$x_1 = 2.5 - \frac{(-0.86755)}{3.14808}$$

$$x_1 = 2.77558$$

$$x_2 = x_1 - \frac{f(x_i)}{f^1(x_i)} ;$$

$f(x_1) = -0.06383,\qquad f^1(x_1) = 2.80004$

$$x_2 = 2.77558 - \frac{(-0.06383)}{2.80004}$$

$$x_2 = 2.798$$

$f(x_2) = -0.001080,\qquad f^1(x_2) = 2.7983$

$$x_3 = x_2 - \frac{f(x_2)}{f^1(x_2)} = 2.798 - \frac{[-0.001080]}{2.7983}$$

$$x_3 = 2.798.$$

$\therefore \qquad x_2 = x_3$

$\therefore$ The real root of $x \tan x + 1 = 0$ is 2.798

8. Find a root of $e^x \sin x = 1$ using Newton–Raphson method [JNTU 2006]

Solution

Given $f(x) = e^x \sin x - 1 = 0$

$$f^1(x) = e^x \sec x + ex \cos x$$

Take $\qquad x_1 = 0, x_2 = 1$

$$f(0) = f(x_1) = e^0 \sin 0 - 1 = -1 < 0$$

$$f(1) = f(x_2) = e^1 \sin(1) - 1 = 1.287 > 0$$

The root of the equation lies between 0 and 1

Using Newton-Raphson method

$$x_{i+1} = x_i - \frac{f(x_i)}{f^1(x_i)}$$

Now consider x_0 = average of 0 and 1

$$x_0 = \frac{1+0}{2} = 0.5$$

$$x_0 = 0.5$$

$$f(x_0) = e^{0.5} \sin(0.5) - 1$$

$$f^1(x_0) = e^{0.5} \sin(0.5) + e^{0.5} \cos(0.5) = 2.2373$$

$$x_1 = x_0 - \frac{f(x_0)}{f^1(x_0)} = 0.5 - \frac{(-0.20956)}{2.2373}$$

$x_1 = 0.5936$

$f(x_1) = e^{0.5936} \sin(0.5936) - 1 = 0.0128$

$f^1(x_1) = e^{0.5936} \sin(0.5936) + e^{0.5936} \cos(0.5936) = 2.5136$

$$x_2 = x_1 - \frac{f(x_1)}{f^1(x_1)} = 0.5936 - \frac{(0.0128)}{2.5136}$$

$\therefore \quad x_2 = 0.58854$

similarly $\quad x_3 = x_2 - \dfrac{f(x_1)}{f^1(x_1)}$

$f(x_2) = e^{0.58854} \sin(0.58854) - 1 = 0.0000181$

$f^1(x_2) = e^{0.58854} \sin(0.58854) + e^{0.58854} \cos(0.58854)$

$f(x_2) = 2.4983$

$$\therefore \quad x_3 = 0.58854 - \frac{0.0000181}{2.4983}$$

$x_3 = 0.5885$

$\therefore \quad x_2 - x_3 = 0.5885$

0.5885 is the root of the equation $e^x \sin x - 1 = 0$

9. Find a real root of the equation $xe^x - \cos x = 0$ using Newton-Raphson method [JNTU-2006]

Solution

Given $f(x) = \mathrm{e}^x - \cos x = 0$

$f^1(x) = x\mathrm{e}^x + \mathrm{e}^x + \sin x = (x + 1)\, e^x + \sin x$

Take $\quad f(0) = 0 - \cos 0 = -1 < 0$

$f(1) = e - \cos 1 = 2.1779 > 0$

$\therefore$ The root lies between 0 and 1

Let $\quad x_0 = \dfrac{0+1}{2} = 0.5 \qquad$ (average of 0 and 1)

Newton-Raphson method

$$x_{i+1} = x_i - \frac{f(x_i)}{f^1(x_i)}$$

$$x_{i+1} = x_0 - \frac{f(x_0)}{f^1(x_0)} = 0.5 - \frac{(-0.053221)}{(1.715966)}$$

$x_1 = 0.5310$

$f(x_1) = 0.040734, \qquad f^1(x_1) = 3.110063$

$$x_2 = x_1 - \frac{f(x_1)}{f^1(x_1)} = 0.5310 - \frac{0.040734}{3.110064}$$

$\therefore \quad x_2 = 0.5179$; $\quad f(x_2) = 0.0004339, \qquad f^1(x_2) = 3.0428504$

$$x_3 = 0.5179 - \frac{(0.0004339)}{3.0428504}$$

$x_3 = 0.5177$

$\therefore \quad f(x_3) = 0.000001106$

$f(x_3) = 3.04214$

$$x_4 = x_3 - \frac{f(x_3)}{f(x_3)} = 0.5177 - \frac{0.000001106}{3.04212}$$

$x_4 = 0.5177$

$\therefore \quad x_3 = x_4 = 0.5177$

$\therefore$ The root of $xe^x - \cos x = 0$ is 0.5177

10. Find a root of the equation $x^4 - x - 10 = 0$ using Bisection method correct to 2 decimal places. [JNTU 2008]

Solution

Let $f(x) = x^4 - x - 10 = 0$ be the given equation. We observe that f(1) < 0, then f(2) >0. So one root lies between 1 and 2.

$\therefore \quad$ Let $x_0 = 1, x_1 = 2$;

Take $x_2 = \dfrac{x_0 + x_1}{2} = 1.5; \qquad f(1.5) < 0;$

$\therefore \quad$ The root lies between 1.5 and 2

Let us take $x_3 = \dfrac{1.5 + 2}{2} = 1.75$; we find that $f(1.75) < 0$,

$\therefore \quad$ The root lies between 1.75 and 2

So we take now $x_4 = \dfrac{1.75 + 1.875}{2} = 1.8125 = 1.81$ can be taken as the root of the given equation.

11. Find a real root of equation $x^3 - x - 11 = 0$ by Bisection method. [JNTU-2007]

Solution

Given equation is $f(x) = x^3 - x - 11 = 0$

We observe that $f(2) = -5 < 0$ and $f(3) = 13 > 0$.

$\therefore$ A root of (1) lies between 2 and 3; take $x_0 = 2$, $x = 3$;

Let $x_2 = \dfrac{x_0 + x_1}{2} = \dfrac{2+3}{2} = 2.5$; Since $f(2.5) > 0$, the root lies between 2 and 2.5

$\therefore$ Taking $x_3 = \dfrac{2+2.5}{2} = 2.25$, we note that $f(2.25) < 0$;

$\therefore$ The root can be taken as lying between 2.25 and 2.5.

$\therefore$ The root $= \dfrac{2.25+2.5}{2} = 2.375$

12. Find a real root of $x^3 - 5x + 3 = 0$ using Bisection method. [JNTU-2007]

Solution

Let $f(x) = x^3 - 5x + 3 = 0$ be the equation given

Since $f(1) = -1 < 0$ and $f(2) = 1 > 0$, a real root lies between 1 and 2.

i.e., $x_0 = 1$, $x_1 = 2$; take $x_2 = \dfrac{1+2}{2} = 1.5$; $f(1.5) = -1.25 < 0$

$\therefore$ The root lies between 1.5 and 2;

$\therefore$ Take $x_3 = \dfrac{1.5+2}{2} = 1.75$

Now $f(1.75) = \left(\dfrac{7}{4}\right)^3 - 5\left(\dfrac{7}{4}\right) + 3 = -\text{ve}$;

$\therefore$ The root lies between 1.75 and 2

Let $x_4 = \dfrac{1.75+2}{2} = 1.875$;

We find that $f(1.875) = (1.875)^3 - 5(1.875) + 3 > 0$

$\therefore$ The root of the given equation lies between 1.75 and 1.875

$\therefore$ The root $= \dfrac{1.75+1.875}{2} = 1.813$

13. Find a real root of the equation $x^3 - 6x - 4 = 0$ by Bisection method [JNTU-2006]

Solution

Here $f(x) = x^3 - 6x - 4$

Take $x_0 = 2, x_1 = 3$; $(\because f(2) < 0, f(3) > 0)$

$x_1 = 2.5; f(x_1) < 0;$ take $x_3 = \dfrac{2.5+3}{2} = 2.75$

$f(2.75) > 0 \Rightarrow x_4 = \dfrac{2.5+2.75}{2} = 2.625$

$f(2.625) < 0 \Rightarrow$ Root lies between 2.625 and 2.75

$\therefore$ Approximately the root will be $= \dfrac{2.625+2.75}{2} = 2.69$

Objective Type Questions

I. Choose correct answer:

1. An example of an algebraic equation is

   (1) $\tan x = e^x$ (2) $x = \log x$ (3) $x^3 - 5x + 3 = 0$ (4) None

   [***Ans:*** (3)]

2. An example of a transcendental equation is

   (1) $x^3 - 2x - 10 = 0$ (2) $x^3 e^x = 5$

   (3) $x^2 + 11x - 1 = 0$ (4) None

   [***Ans:*** (2)]

3. In finding a real root of the equation $x^3 - x - 10 = 0$ by bisection, if the root lies between $x_0 = 2$ and $x_1 = 3$, then, $x_2 =$

   (1) 2.5 (2) 2.75 (3) 2.60 (4) None

   [***Ans:*** (1)]

4. If ϕ(a) and ϕ(b) are of opposite signs and the real root of the equation $\phi(x) = 0$ is found by false position method, the first approximation x_1, of the root is

   (1) $\dfrac{a\,\phi(b)+b\,\phi(a)}{\phi(b)+\phi(a)}$ (2) $\dfrac{a\,\phi'(b)+b\,\phi'(a)}{\phi(b)+\phi(a)}$

   (3) $\dfrac{ab\,\phi(a)\,\phi(b)}{\phi(a)-\phi(b)}$ (4) $\dfrac{a\,\phi(b)-b\,\phi(a)}{\phi(b)-\phi(a)}$

   [***Ans:*** (4)]

5. The two initial values of the roots of the equation $x^3 - x - 3 = 0$ are

(1) (–1, 0) (2) 1, 2 (3) –2, 1 (4) (1, 0)

[***Ans:*** (2)]

6. The iteration method is said to have p^{th} order convergence if for any finite constant $K \neq 0$

(1) $|e_n| \leq K|e_{n-1}|^P$ (2) $|e_n| \leq K|e_{n+1}|^P$

(3) $|e_n + 1| \leq K|e_0|^P$ (4) None

[***Ans:*** (1)]

7. Newton-Raphson method formula to find $(n + 1)^{th}$ approximation of root of $f(x) = 0$ is

(1) $x_{n+1} = x_n - \dfrac{f'(x_n)}{f(x_n)}$ (2) $x_{n+1} = \dfrac{x_n f(x_n)}{f'(x_n)}$

(3) $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ (4) None

[***Ans:*** (3)]

8. In the bisection method e_0 is the initial error and e_n is the error in n^{th} iteration

(1) $\dfrac{1}{2}$ (2) 1 (3) $\dfrac{1}{2^n}$ (4) None

[***Ans:*** (3)]

9. Which of the following methods has linear rate of convergence

(1) Regular flase (2) Bisection

(3) Newton-Raphson (4) None

[***Ans:*** (1)]

10. A non linear equation $x^3 + x^2 - 1 = 0$ is $x = \phi(x)$, then the choice of $\phi(x)$ for which the iteration scheme $x_n = \phi(x_{n-1})$ $x_0 = 1$ converge is $\phi(x)$=

(1) $(1 - x^2)^{1/3}$ (2) $\dfrac{1}{\sqrt{1+x}}$ (3) $\sqrt{1 - x^3}$ (d) None

[***Ans:*** (2)]