VBD

Based on R.T.M. Nagpur University New Syllabus

Based on New Syllabus

DISCRETE MATHEMATICS AND GRAPH THEORY

V. R. Manthena

SOLVED QUESTION BANK + UNIVERSITY PAPER SOLUTIONS

B.TECH. IV SEMESTER

COMP. SCI. & ENGG., INFO. TECH., COMP. ENGG., COMP. TECH., (CSE,IT,CE,CT)

B.TECH. III SEMESTER

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VBD

DISCRETE MATHEMATICS AND GRAPH THEORY

FOR

B.Tech. III SEMESTER

Artificial Intelligence

(AI)

B.Tech. IV SEMESTER

Comp. Science & Engg., Info. Tech., Comp. Engg., Comp. Tech.

(CS/IT/CE/CT)

This book is the property of Chaitanyaa Vijay

By

V. R. Manthena

M.Sc., B.Ed.

Assistant Professor

Based on R.T.M. Nagpur University New Syllabus

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Statistical Analysis

Exam. / Unit No.	S-14	W-14	S-15	W-15	S-16	W-16	S-17	W-17	S-18
Unit - I	20 M	20 M	20 M	20 M	20 M	20 M	20 M	20 M	17 M
Unit-II	36 M	36 M	36 M	36 M	36 M	36 M	39 M	36 M	39 M
Unit-III	30 M	24 M	24 M	24 M	24 M	24 M	24 M	24 M	25 M
Unit-IV	18 M	24 M	24 M	24 M	24 M	24 M	21 M	24 M	24 M
Unit-V	36 M	36 M	36 M	36 M	36 M	36 M	36 M	36 M	36 M
Unit-VI	20 M	20 M	20 M	20 M	20 M	20 M	20 M	20 M	20 M
Questions covered in VBD	160 M (100 %)	160 M (100 %)	160 M (100 %)	160 M (100 %)	160 M (100 %)	160 M (100 %)	160 M (100 %)	160 M (100 %)	160 M (100 %)

Exam. / Unit No.	W-18	S-19	W-19
Unit - I	20 M	20 M	20 M
Unit-II	36 M	36 M	36 M
Unit-III	24 M	24 M	24 M
Unit-IV	24 M	24 M	24 M
Unit-V	36 M	36 M	36 M
Unit-VI	20 M	20 M	20 M
Questions covered in VBD	160 M (100 %)	160 M (100 %)	160 M (100 %)

Note:- Due to Covid - 19 Summer - 2020 to Winter - 2021 exams were conducted online by the respective Colleges.

PREFACE

This *VBD* on **Discrete Mathematics & Graph Theory** has been prepared for the students of **B. Tech. Fourth Semester Bachelor Technology (CBCS)- CS,CT, CE, IT and B.Tech. III Semester AI** as per the New Syllabus prescribed by the RTM-Nagpur University.

The subject matter has been thoroughly dealt with, covering 100 % syllabus.

Each unit begins with its syllabus. The previously asked University questions are fully covered in the order as per the syllabus.

In each unit, sufficient theory is given then followed by Important Useful Formulae and Examples based on them.

Every type of example is included alongwith previously asked.

Fully solved University Question Papers are included at the end with solutions reference inside.

I am grateful to the publishers and the staff of *VBD* Publications, Nagpur, who have taken considerable pains and shown extreme co-operation during the preparation of this book.

I will appreciate suggestions from teachers and students for the improvement of the book in the future editions.

Author

SYLLABUS

[B. Tech. 4 SEM CS,CT,CE,IT, 3 SEM AI]

UNIT - I
SET THEORY, RELATIONS AND FUNCTIONS

Sets: Review of propositions and logical operations, Principle of mathematical induction, Review of sets, Types and operations on sets.

Relations: Ordered pairs and n-tuples, Types of relations, Composite relation, Transitive closure of a relation, Partially ordered set, Hasse diagrams.

Functions: Definition, Composition of functions, Types of functions, Characteristics function and its properties.

UNIT - II
FUZZY SET AND FUZZY LOGIC

Fuzzy sets and systems, Crisp set, Operations and combinations on Fuzzy sets, Relation between Crisp set and Fuzzy set, Fuzzy relations, Overview of Fuzzy logic and classical logic.

UNIT - III
GROUP THEORY AND RING THEORY

Binary operation, Algebraic structure, Groupoid, Semigroup, Monoid, Group, Subgroup, Normal subgroup (Only definitions and examples), Ring, Commutative ring, Ring with unity, Zero divisor, Integral domain, Field (Only definitions and simple examples).

UNIT - IV
GRAPH THEORY

Basic concepts of graph theory, Digraphs, Basic definitions, Matrix representation of graphs, Subgraphs and quotient graphs, Isomorphic graphs, Paths and circuits, Reachability and connectedness, Node base, Euler's path & Hamilton's path, Tree, Binary tree, Undirected tree, Spanning tree, Weighted graphs (Only definitions and examples), Minimal spanning tree by Prim's algorithm & Kruskal's algorithm, Representation of algebraic expressions by Venn diagram and binary tree.

UNIT - V
COMBINATORICS

Permutations and combinations, Pigeonhole principle with simple applications, Recurrence relations (Concept and definition only), Generating functions, Solution of recurrence relations using generating functions.

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[B.Tech. 4 SEM CS, CT, CE, IT, 3 SEM AI]

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UNIT - II
FUZZY SET AND FUZZY LOGIC

- Fuzzy sets and systems
- Crisp set
- Operations and combinations on Fuzzy sets
- Relation between Crisp set and Fuzzy set
- Fuzzy relations
- Overview of Fuzzy logic and classical logic

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Note:- Due to Covid - 19 Summer - 2020 to Winter - 2021 exams were conducted online by the respective Colleges.

UNIT - I
MATHEMATICAL LOGIC AND SET THEORY

SYLLABUS

Propositions and Logical Operations, Quantifiers, Conditional Statements and Tautologies, Methods of Proof, Principle of Mathematical Induction, Basic concepts of set theory, Operations on Sets, The power set.

UNIVERSITY PAPER SOLUTIONS
SINCE SUMMER - 2006

Note to the students :
- The questions given below are from previous Nagpur University examination papers.
- Though these questions are from old university papers, they have been included in the new syllabus.
- Questions which are out of new syllabus are not included here.

SUMMER-06 (CS/CT/CE)

Q.1. Prove that:

$A-(B \cap C) = (A-B) \cup (A-C)$ **5M**

Ans. P.1-37, Ex.61

Q.2. Show that:

$$1^2+2^2+3^2+.....+n^2 = \frac{n(n+1)(2n+1)}{6}, \ n \geq 1$$

by mathematical induction. **4M**

Ans. P.1-45, Ex.86

Q.3. Write the negation of each statement as simply as possible.

(i) If he works, he will earn money.

(ii) She swims if and only if the water is warm.

(iii) If it snows, then they do not drive the car. **3M**

Ans. P.1-14, Ex.13

Q.4. Determine the validity of the following argument:

If 7 is less than 4, then 7 is not a prime number. 7 is not less than 4.

Therefore, 7 is a prime number. **5M**

Ans. P.1-23, Ex.36

Q.5. Show that: $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$ is a tautology. **5M**

Ans. P.1-16, Ex.21

SUMMER - 06 (IT)

Q.1. Write the logical negation of the following statement using symbolic notations.

"For every man there exists a woman who loves him". **3M**

Ans. P.1-30, Ex.57

Q.2. State whether the argument given is valid or not. If it is valid, identify the tautology or tautoliogies on which it is based.

If my plumbing plans do not meet the construction code, then I cannot build my house. If I hire a licensed contractor, then my plumbing plans will meet the construction code. I hired a licensed contractor.

$\therefore$ I can build my house. **6M**

Ans. P.1-23, Ex.37

Q.3. Prove or disprove that 3 divides (n^3-n) for every positive integer n. **5M**

Ans. P.1-46, Ex.87

Q.4. If $A \cup B = A \cup C$, must $B = C$? Explain. **3M**

Ans. P.1-40, Ex.69(a)

Q.5. A survey of 500 television watchers produced the following information. 285 watch football games; 115 watch basketball games, 45 watch football and basketball games, 70 watch football and hockey games, 50 watch hockey and basketball games and 50 do not watch any of the three kinds of games.

i) How many people in the survey watch all three kinds of games?

ii) How many people watch exactly one of the sports? **6M**

Ans. P.1-39, Ex.67

WINTER - 06 (CS/CT/CE)

Q.1. State principle of Mathematical Induction. **2M**

Ans. P.1-43, Definition

Q.2. Show that :

$((P \vee Q) \wedge (\rceil(\rceil P \wedge \rceil Q \vee \rceil R))) \vee (\rceil P \wedge \rceil Q) \vee (\rceil P \wedge \rceil R)$ is a tautology. **6M**

Ans. P.1-17, Ex.23

Q.3. Check the validity of the following argument using truth table:

Smoking is healthy.

If smoking is healthy, then Cigarettes are prescribed by physicians.

$\therefore$ Cigarettes are prescribed by physicians. **5M**

Ans. P.1-23, Ex.38

Q.4. Show that $\sqrt{2}$ is irrational. **3M**

Ans. P.1-28, Ex.50

Q.5. Write the negation of:

i) "If two lines are each perpendicular to a third line in the plane, then the two lines are parallel."

ii) "The dish ran away with the spoon, but the grass is wet." **2M**

Ans. P.1-14, Ex.14

WINTER - 06 (IT)

Q.1. Write the following statement in terms of primary statements p, q, r and logical connectives. Also write the negation:

If TOM stole a pie and ran away, then the three pigs do not have any supper. **4M**

Ans. P.1-14, Ex.15

Q.2. If $p \rightarrow q$ is true, can you determine the truth value of $\sim(p \rightarrow q) \wedge \sim p$? Explain. **4M**

Ans. P.1-11, Ex.3

Q.3. State whether the argument given is valid or not. If it is valid, identify the tautology or tautologies on which it is based.

If I graduate this semester, then I will have passed the Discrete Mathematics course.

If I do not study Discrete Mathematics for 10 hours a week, then I will not Pass Discrete Mathematics. If I study Discrete Mathematics for 10 hours a week, then I can not play cricket.

If I play cricket, I will not graduate this semester. **6M**

Ans. P.1-22, Ex.35

Q.4. Define : Symmetric difference of two sets. **2M**

Ans. P.1-33, Definition.

Q.5. Suppose that $A \oplus B = A \oplus C$.

Does this guarantee that $B = C$?

Justify your conclusion. **4M**

Ans. P.1-40, Ex.69(b)

Q.6. A survey of 500 television watchers produced the following information. 285 watch football games; 115 watch basketball games, 45 watch football and basketball games, 70 watch football and hockey games, 50 watch hockey and basketball games and 50 do not watch any of the three kinds of games.

(i) How many people in the survey watch all three kinds of games?

(i) How many people watch exactly one of the sports? **7M**

Ans. P.1-39, Ex.67

SUMMER - 07(CS/CT/CE)

Q.1. For two sets A and B, prove that :

(i) $A - (B \cup C) = (A - B) \cap (A - C)$

(ii) $A - B = A - (A \cap B)$ **6M**

Ans. P.1-37, 40, Ex.60 & 70

Q.2. Verify that the proposition $(p \vee q) \wedge \sim(p \vee q)$ is a contradiction. **4M**

Ans. P.1-17, Ex.24

Q.3. Show that : $P \rightarrow Q \Leftrightarrow (\sim P \vee Q)$. **3M**

Ans. P.1-12, Ex.5

Q.4. Determine the validity of the argument :

If I study then I will not fail in Maths.

If I do not play hockey then I will study.

But I failed in Mathematics.

$\therefore$ Therefore I must have played hockey. **5M**

Ans. P.1-21, Ex.33

Q.5. Consider the conditional proposition $p \rightarrow q$. Write converse, inverse and contrapositive of it. Is any of them logically equivalence to $p \rightarrow q$? Verify by truth table. **5M**

Ans. P.1-13, Ex.

Q.6. Negate the statement : **3M**

(i) If she works, she will earn money.

(ii) He swims if and only if the water is warm.

(iii) If it snows, then they do not drive a car.

Ans. P.1-14, Ex.13

SUMMER - 07 (IT)

Q.1. State and prove De-Morgans's Laws for any two sets A and B. **4M**

Ans. P.1-36

Q.2. Prove that:
$(A \cap B) \cup (A \cap B') = A$. **4M**

Ans. P.1-38, Ex.62

Q.3. A sample of 80 people have revealed that 24 like cinema and 62 like television programmes. Find the number of people who like both cinema and television programmes. **6M**

Ans. P.1-40, Ex.68

Q.4. Write the contrapositive of each of the following statement :

(i) If I have enough money then I will buy a car and I will buy a house.

(ii) If I have time and I am not too tired than I will go to the store. **6M**

Ans. P.1-15, Ex.17

Q.5. Construct the truth table for
$[(P \wedge \sim Q) \rightarrow R] \rightarrow [P \rightarrow (Q \vee R)]$ and verify that it is a tautology. **5M**

Ans. P.1-17, Ex.25

Q.6. Is the following argument valid?
"If the labour market is perfect, then the wages of all persons in a particular employment will be equal. But it is always the case that wages for such persons are not equal.
$\therefore$ The labour market is not perfect." **7M**

Ans. P.1-23, Ex.39

WINTER - 07(CS/CT/CE)

Q.1. Prove that:
If $A \cap (B - C) = (A \cap B) - (A \cap C)$,
then $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$. **3M**

Ans. P.1-41, Ex.72

Q.2. Let $U = \{a, b, c, d, e, f\}$
$A = \{a, b, c\}$ and $B = \{b, e, f\}$
Find $B' - A'$, $(A \cap B)'$, $A' \cup B'$. **3M**

Ans. P.1-41, Ex.73

Q.3. Prove by Mathematical induction that the sum of the cubes of three consecutive integers are divisible by 9. **3M**

Ans. P.1-46, Ex.88

Q.4. Determine the validity of the argument :
If I study, then I will not fail in Mathematics.
If I do not play basketball, then I will study.
But I failed in Mathematics
Therefore I must have played basketball. **6M**

Ans. P.1-21, Ex.33

Q.5. Negate the statements :
(1) It is daylight and all the people have arisen.
(2) He swims if and only if the water is warm.
(3) "2 is even or (– 3) is negative". **3M**

Ans. P.1-14, 15, Ex.13 & 18

Q.6. Consider the conditional proposition $p \rightarrow q$. Write converse, inverse and contrapositive of it. Is any of them logically equivalence to $p \rightarrow q$? Verify by truth table. **4M**

Ans. P.1-13.

WINTER - 07 (IT)

Q.1. Prove by contradiction, that $\sqrt{2}$ is irrational. **4M**

Ans. P.1-28, Ex.50

Q.2. State whether the argument given is valid or not. If it is valid, identify the tautology or tautologies on which it is based.
If my plumbing plans do not meet the construction code, then I cannot build my house. If I hire a licensed contractor, then my plumbing plans will meet the construction code. I hired a licensed contractor.
$\therefore$ I can build my house. **6M**

Ans. P.1-23, Ex.37

Q.3. Write converse, inverse, contrapositive and negation of "Oxygen in air decreases as you go up". **4M**

Ans. P.1-14, Ex.16

Q.4. In a class of 25 students, 12 have taken Mathematics 8 have taken Mathematics but not Biology. Find the number of students who have taken Mathematics and Biology and those who have taken Biology but not Mathematics. **5M**

Ans. P.1-40, Ex.71

Q.5. Prove that the following statements are equivalent:
(i) $A \subset B$
(ii) $A \cap B' = \phi$
(iii) $A \cap B = A$
(iv) $A \cup B = B$
(v) $B' \subset A'$. **8M**

Ans. P.1-39, Ex.66

SUMMER - 08 (CS/CT/CE)

Q.1. Prove or disprove $\sqrt{2}$ is an irrational number. **6M**

Ans. P.1-28, Ex.50

Q.2. Show that:
$((P \vee Q) \wedge (\neg(\neg P \wedge \neg Q \vee \neg R))) \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$ is a tautology. **7M**

Ans. P.1-17, Ex.23

Q.3. Determine the validity of the following arguments:
If it rains, the street gets wet
If it rains, accident happens
The street gets wet.
∴ Accidents happen. **6M**

Ans. P.1-24, Ex.40

SUMMER - 08 (IT)

Q.1. Define the terms:
(i) Tautology
(ii) Contradiction
(iii) Contingency. **3M**

Ans. P.1-16.

Q.2. Show that the proposition $\neg(p \wedge q)$ and $\neg p \vee q$ are logically equivalent. **3M**

Ans. P.1-18, Ex.28

Q.3. Represent the argument symbolically and check whether it is valid or not:
If it rains today, then we will not have a party today.
If we do not have party today, then we will have a party tomorrow.
∴ If it rains today, then we will have a party tomorrow. **7M**

Ans. P.1-24, Ex.41

Q.4. Without using truth table, show that
$P \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$ **6M**

Ans. P.1-19, Ex.29(vi)

Q.5. Prove or disprove: sum of any five consecutive integers is divisible by 5. **3M**

Ans. P.1-29, Ex.53

WINTER - 08 (CS/CT/CE)

Q.1. Prove by principle of mathematical induction:

$$\frac{1}{1.2}+\frac{1}{2.3}+....+\frac{1}{n(n+1)}=\frac{n}{n+1}.$$ **4M**

Ans. P.1-44, Ex.85

Q.2. Define the term:
(i) Tautology
(ii) Contradiction
(iii) Contingency. **3M**

Ans. P.1-16.

Q.3. Check the validity of the following argument:
If it rains today, then we will not have a party today.
If we do not have party today, then we will have a party tomorrow.
∴ If it rains today, then we will have a party tomorrow. **7M**

Ans. P.1-24, Ex.41

Q.4. Prove or disprove:
$x^3 > x^2, \forall x.$ **4M**

Ans. P.1-46, Ex.89

WINTER - 08 (IT)

Q.1. Show that:
$[((P \vee Q) \wedge \sim(\sim P \wedge (\sim Q \vee \sim R))) \vee (\sim P \wedge \sim Q) \vee (\sim P \wedge \sim R)]$ is a tautology. **7M**

Ans. P.1-18, Ex.26

Q.2. Determine the validity of the following arguments using truth table:
If you invest in the stock market, then you will get rich. If you get rich then you will be happy.
You get happy.
∴ You invest in the stock market. **6M**

Ans. P.1-24, Ex.42

Q.3. Let A be a finite set containing n elements, then prove that the power set of A has exactly 2^n elements. **6M**

Ans. P.1-41, Ex.74

Q.4. An investigator interviewed 100 students to determine their preferences for the three drinks- Milk (M), Coffee (C), Tea (T). He reported the following: 10 students had all the three drinks, 20 had 'M' and 'C', 30 had 'C' and 'T', 25 had 'M' and 'T', 12 had 'M' only, 5 had 'C' only and 8 had 'T' only.
(i) How many did not take any of the three drinks?
(ii) How many take milk but not coffee?
(iii) How many take tea and coffee but not milk? **7M**

Ans. P.1-41, Ex.75

SUMMER - 09 (CS/CT/CE)

Q.1. Prove that :
(i) $(A \cap B)' = A' \cup B'$
(ii) $A - (B \cap C) = (A - B) \cup (A - C)$ **6M**

Ans. P.1-37, 36, Ex.61 & De-Morgan's Laws.

Q.2. Show that :

$$1^2+2^2+3^2+.....+n^2=\frac{n(n+1)(2n+1)}{6},\ n \geq 1$$

by mathematical induction. **5M**

Ans. P.1-45, Ex.86

Q.3. Determine the validity of the argument if 7 is less than 5, then 7 is not a prime number. 7 is not less than 5.
∴ 7 is a prime number. **5M**

Ans. P.1-23, Ex.36

Q.4. Verify that the proposition $(P \vee G) \wedge \sim(P \vee G)$ is a contradiction. **5M**

Ans. P.1-17, Ex.24

Q.5. Write the contrapositive of each of the following statement :

(i) If I have enough money then I will buy a car and I will buy a house.

(ii) If I have time and I am not too tired then I will go to market. **4M**

Ans. P.1-15, Ex.17

SUMMER - 09 (IT)

Q.1. Determine the validity of the following argument :
"If I get the job and work hard, then I will get promoted. If I get promoted, then I will be happy. I will not be happy. Therefore, either I will not get the job or I will not work hard". **8M**

Ans. P.1-25, Ex.43

Q.2. Prove by contradiction, that $\sqrt{2}$ is irrational. **3M**

Ans. P.1-28, Ex. 50

Q.3. Translate each of the following statement into symbolic, using quantifiers :

(i) There is a student who can speak Tamil and who knows C^{++}.

(ii) There is a student who can speak Tamil but does not know C^{++}.

(iii) Every student either can speak Tamil or knows C^{++}.

(iv) No student can speak Tamil or knows C^{++}. **4M**

Ans. P.1-31, Ex.59(a)

Q.4. Out of 250 candidates who failed in an examination, it was revealed that 128 failed in Mathematics, 87 in Physics, and 134 in aggregate. 31 failed in Mathematics and in Physics, 54 failed in aggregate and in Mathematics, 30 failed in aggregate and in Physics. Find how many candidates failed :

(i) In all the three subject.

(ii) In Physics but not in aggregate or in Mathematics.

(iii) In the aggregate or in Mathematics, but not in Physics. **6M**

Ans. P.1-41, Ex.76

Q.5. Prove by mathematical Induction that $6^{n+2} + 7^{2n+1}$ is divisible by 43 for each positive integer n. **4M**

Ans. P.1-46, Ex.90

WINTER - 09 (CS/CT/CE)

Q.1. Prove that:

(i) $A - B = A - (A \cap B)$

(ii) $A \cup (B \cup C) = (A \cup B) \cup C$. **6M**

Ans. P.1-40, 35, Ex.70 & Laws of algebra 6(1)

Q.2. Prove, using truth table:

(i) $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

(ii) $p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$. **6M**

Ans. P.1-20 Ex.30

Q.3. State the converse, inverse and contrapositive of the following:

(i) If today is Easter, then tomorrow is Monday.

(ii) If P is a square, then P is a rectangle. **3M**

Ans. P.1-15, Ex.19

Q.3. Check the validity of the following argument:
If Roli has completed MCA or MBA, then she is assured of a good job.
If Roli is assured of a good job, she is happy. Roli is not happy. So Roli has not completed MBA. **5M**

Ans. P.1-28, Ex.48

WINTER - 09 (IT)

Q.1. Let, $U = \{x : x \in N, 1 \le x \le 12\}$ be the univer sal set and $A = \{1, 9, 10\}$, $B = \{3, 4, 6, 11, 12\}$ and $C = \{2, 5, 6\}$ are subsets of U. Find the sets.

(i) $(A \cup B) \cap (A \cup C)$

(ii) $A \cup (B \cap C)$

(iii) $(A \cup B \cup C)^c$. **6M**

Ans. P.1-42, Ex.77

Q.2. A TV survey shows that 60% people see programme A, 50% see programme B, 50% see programme C, 30% see programme A and B, 20% see programme B and C, 30% see programme A and C, 10% do not see any programme. Find:

(i) What per cent see programme A, B, C?

(ii) What per cent see exactly two programmes?

(iii) What per cent see programme A only? **7M**

Ans. P.1-42, Ex.78

Q.3. State whether the following arguments are valid or not. If it is valid, identify the tautology on which it is based.
If I drive to work then I will arrived tired.
If I am not arrived tired then I drive to work.
$\therefore$ I do not drive to work. **6M**

Ans. P.1-26, Ex.44

Q.4. Let A and B be two arbitrary sets:

(i) Show that $P(A \cap B) = P(A) \cap P(B)$ or give a counter example.

(ii) Show that $P(A \cup B) = P(A) \cup P(B)$ or give a counter example.
where P denotes power set. **7M**

Ans. P.1-42, Ex.79

SUMMER - 10 (CS/CT/CE)

Q.1. For any two sets A and B, prove that:

(i) $(A \cap B) \cup (A \cap B') = A$

(ii) $A \cap (B - C) = (A \cap B) - (A \cap C)$. **7M**

Ans. P.1-38, Ex.62 , 63

Q.2. Prove by mathematical induction

$$1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{n(2n+1)(2n-1)}{3}.$$

7M

Ans. P.1-46, Ex.91

Q.3. Write down the negation of the following proposition 'for every number x there is a number y such that $y < x$'. **3M**

Ans. P.1-31, Ex.59

Q.4. If p and q are false proposition, state whether $[(p \vee q) \wedge (-p)] \vee (-q)$ is true or false. **3M**

Ans. P.1-12, Ex.6

Q.5. Show that $p \Leftrightarrow q \equiv (p \vee q) \Rightarrow (p \wedge q)$ using

(i) Truth table.

(ii) Algebra of proposition. **7M**

Ans. P.1-21, Ex.31

Q.6. Consider the following:

p : You take a course in Discrete Mathematics

q : You understand Logic

r : You get an A grade on the final exam.

Write in simple sentences the meaning of the following:

(i) $(p \wedge q) \Rightarrow r$

(ii) $(p \wedge \sim q) \Rightarrow -r$

(iii) $\sim p \Rightarrow \sim q$. **6M**

Ans. P.1-12, Ex.7

SUMMER - 10 (IT)

Q.1. Show that:

(i) $\sim(p \vee q) \leftrightarrow \sim p \wedge \sim q$

(ii) $\sim(p \wedge q) \leftrightarrow \sim p \vee \sim q$. **6M**

Ans. P.1-18, Ex.28

Q.2. Show that: $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$ is a tautology. **7M**

Ans. P.1-16, Ex.21

Q.3. Prove that:

(i) $A' \subset B$ if $A \cup B = U$.

(ii) $A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$. **6M**

Ans. P.1-43, Ex.80

Q.4. State whether the following arguments are valid or not. If it is valid, identify the tautology on which it is based.

If I drive to work then I will arrived tired.

If I am not arrived tired then I drive to work.

$\therefore$ I do not drive to work. **6M**

Ans. P.1-26, Ex.44

WINTER - 10(CS/CT/CE)

Q.1. Prove that :

(i) $A - (B \cap C) = (A - B) \cup (A - C)$

(ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. **7M**

Ans. P.1-37, 35, Ex.61 & Laws of algebra 7(ii)

Q.2. State Peano Axioms. **2M**

Ans. P.1-43.

Q.3. Check the validity of the following argument :

If I try hard and I have a talent, then I will become a scientist.

If I become scientist, then I will be happy. Therefore, if I will not be happy, then I did not try hard or I do not have talent. **5M**

Ans. P.1-26, Ex.45

Q.4. Prove that $\sqrt{3}$ is irrational by contradiction. **3M**

Ans. P.1-28, Ex.50

Q.5. Show that: $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$. **5M**

Ans. P.1-12, Ex.8

WINTER - 10 (IT)

Q.1. If p and q are two statements, then prove that :

(i) $(\sim p \vee q) \wedge (\sim q \vee p) \equiv p \leftrightarrow q$

(ii) $\sim (p \leftrightarrow q) \equiv q \leftrightarrow \sim p$. **7M**

Ans. P.1-18, Ex.27(1) (3)

Q.2. Write the converse, inverse, negative and contrapositive of

"If I have time and I am not too tired then I will go to the store". **6M**

Ans. P.1-15, Ex.17

Q.3. State whether the argument given is valid or not. If it is valid, identify the tautology or tautoliogies on which it is based.

If my plumbing plans do not meet the construction code, then I cannot build my house. If I hire a licensed contractor, then my plumbing plans will meet the construction code. I hired a licensed contractor.

$\therefore$ I can build my house. **7M**

Ans. P.1-23, Ex.37

Q.4. Prove that :

(i) $(A - B) \cap B = \varphi$

(ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. **6M**

Ans. P.1-43, 35 , Ex.81 & Laws of algebra 7(ii)

SUMMER - 11(CS/CT/CE)

Q.1. State Peano's axioms. **2M**

Ans. P.1-43

Q.2. By the principle of induction, show that

$3^{4n+2} + 5^{2n+1}$ is a multiple of 14 $\forall\ n \geq 0$. **5M**

Ans. P.1-44, Ex.85

Q.3. Define:

(i) Proposition

(ii) Logical equivalence

(iii) Tautology

(iv) Contradiction

(v) Contingency

(vi) Logical operator. **6M**

Ans. P.1-10, 16, 18

Q.4. Determine validity of the following argument using truth table. If I like Discrete Mathematics then I will study. Either I don't study or I pass Discrete Mathematics.

If don't graduate then I didn't pass mathematics.

$\therefore$ If like Discrete Mathematics then I will graduate. **7M**

Ans. P.1-27, Ex.46

SUMMER - 11 (IT)

Q.1. A computer company receives 40 applications for a job of programmers. Among them 25 knew JAVA, 28 knew ORACLE and 7 did not know any of the Languages.

How many of them knew both the languages? **6M**

Ans. P.1-43, Ex.82

Q.2. Show that: $(p \to q) \wedge (r \to q) \Leftrightarrow (p \vee r) \to q$ **6M**

Ans. P.1-12, Ex.9

Q.3. Test the validity of the following arguments. If the initialization is correct and the loop terminates, then the required output is obtained. The output has been obtained Therefore, if the initialization is correct, the loop must terminate. **7M**

Ans. P.1-27, Ex.47

WINTER - 11(CS/CT/CE)

Q.1. Prove that for two sets A and B

(i) $A - (B \cup C) = (A - B) \cap (A - C)$.

(ii) $(A \cap B)' = A' \cup B'$. **6M**

Ans. P.1-37, 36, Ex. 60 & De-Morgan's Laws

Q.2. Prove that

$$1 + 2 + 3 + \ldots\ldots + n = \frac{n(n+1)}{2}$$

by principle of Mathematical induction. **4M**

Ans. P.1-44, Ex.85

Q.3. Test the validity of the following arguments:

If I study, then I will not fail in Mathematics.

If I do not play basketball, then I will study.

But I failed in Mathematics

Therefore I must have played basketball. **5M**

Ans. P.1-21, Ex.33

Q.4. Show that :

$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$ **5M**

Ans. P.1-12, Ex.8

Q.5. Negate the statements: **3M**

(i) It is daylight and all the people have arisen.

(ii) He scrims if and only if water is warm.

(iii) 2 is even or – 3 is negative.

Ans. P.1-14, 15, Ex.13 & 18

WINTER - 11 (IT)

Q.1. State and prove De-Morgan's laws for any two sets A and B. **6M**

Ans. P.1-36.

Q.2. Let, $U = \{x : x \in N,\ 1 \leq x \leq 12\}$

be the universal set and $A = \{1, 9, 10\}$, $B = \{3, 4, 6, 11, 12\}$ and $C = \{2, 5, 6\}$ are subsets of U. Find the sets.

(i) $(A \cup B) \cap (A \cup C)$

(ii) $A \cup (B \cap C)$

(iii) $(A \cup B \cup C)^c$. **7M**

Ans. P.1-42, Ex.77

Q.3. State whether the given argument is valid or not.If my plumbing plans do not meet the construction code, then I cannot build my house. If I hire a licensed contractor, then my plumbing plans will meet the construction code.

I hired a licensed contractor.

$\therefore$ I can build my house. **6M**

Ans. P.1-23, Ex.37

Q.4. Construct the truth table for

$[(p \wedge \sim q) \to r] \to [p \to (q \vee r)]$

and verify that it is a tautology. **4M**

Ans. P.1-17, Ex.25

Q.5. Prove by contradiction that $\sqrt{2}$ is irrational. **4M**

Ans. P.1-28, Ex.50

SUMMER - 12 (CS/CT/CE)

Q.1. Prove that for two sets A and B :

(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(ii) $(A \cup B)' = A' \cap B'$. **6M**

Ans. P.1-35 (Laws of algebra 7(i) & P. 1-36 (De-Morgan's laws)

Q.2. Show that:

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}, n \geq 1,$$ **3M**

Ans. P.1-45, Ex.86

Q.3. Show that the following statements are logically equivalent :

(i) $(p \leftrightarrow q) \equiv (\sim p \vee q) \wedge (\sim q \vee p)$

(ii) $p \rightarrow (q \wedge r) \equiv (p \rightarrow q) \wedge (p \rightarrow r)$ **6M**

Ans. P.1-18, Ex.27

Q.4. Test the validity of the following arguments:

If I study, then I will not fail in Mathematics.

If I do not play basketball, then I will study.

But I failed in Mathematics

Therefore I must have played basketball. **5M**

Ans. P.1-21, Ex.33

Q.5. Write the negation of the following statements :

(i) Ram is in Class XI or Arun is in Class XII.

(ii) If Ram is good, then Shyam is king. **3M**

Ans. P.1-13, Ex.12

SUMMER - 12 (IT)

Q.1. Write converse, inverse, contrapositive and negation of the statement :

"Oxygen in air decreases as you go up." **4M**

Ans. P.1-14, Ex.16

Q.2. Determine the validity of the following arguments :

If you invest in the stock market, then you will get rich.

If you get rich, then you will be happy.

You get happy.

$\therefore$ You invest in the stock market. **4M**

Ans. P.1-24, Ex.42

Q.3. Prove that $\sqrt{2}$ is irrational. **4M**

Ans. P.1-28, Ex.50

Q.4. Prove that the following statements are equivalent:

(i) $A \subset B$

(ii) $A \cap B' = \phi$

(iii) $A \cap B = A$

(iv) $A \cup B = B$ and

(v) $B' \subset A'$. **8M**

Ans. P.1-39, Ex.66

WINTER - 12 (CS/CT/CE)

Q.1. Show that:

$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$ **4M**

Ans. P.1-12, Ex.10

Q.2. Write negation and contrapositive of the statement "If I have time and I am not tired, then I will go to the market". **4M**

Ans. P.1-15, Ex.17

Q.3. If p : today is Monday

q : The grass is wet

r : The dish ran away with the spoon. Write each of the following in terms of p.q.r. and logical connectivities.

(i) Today is Monday and the dish did not run away with the spoon.

(ii) Either the grass is wet or today is Monday. **5M**

Ans. P.1-15, Ex.20

Q.4. State whether the argument given is valid or not. If it is valid, identify the tautology or tautologies on which it is based :

If I graduate this semester, then I will have passed the Discrete Mathematics for 10 hours a week, then I will not pass Discrete Mathematics for 10 hours a week, then I can not play cricket.

$\therefore$ If I play cricket; I will not graduate this semester. **7M**

Ans. P.1-22, Ex.35

Q.5. Prove by mathematical induction :

$$1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{n(2n+1)(2n-1)}{3}$$ **6M**

Ans. P.1-46, Ex.91

WINTER - 12 (IT)

Q.1. Check the validity of the following argument- "If Roli has completed MCA or MBA, then she is assured of a good job. If Roli is assured of a good job, she is happy. Roli is not happy. So Roli has not completed MBA." **6M**

Ans. P.1-28, Ex.48

Q.2. Write the converse, inverse, contrapositive and negation of the statement :

"Oxygen in air decreases as you go up". **3M**

Ans. P.1-14, Ex.16

Q.3. Prove that $\sqrt{2}$ is irrational. **4M**

Ans. P.1-28, Ex.50

Q.4. Using principle of mathematical induction establish the following formula for the integers $n \geq 1$

$$1+2+3+.....+n=\frac{n(n+1)}{2}$$ **6M**

Ans. P.1-44, Ex.85

Q.5. Prove that :

(i) $(A \cap B) \cup (A \cap \sim B) = A$

(ii) $A \cap (\sim A \cup B) = A \cap B$. **7M**

Ans. P.1-38, 43, Ex.62, Ex.83.

SUMMER - 13(CS/CT/CE)

Q.1. For two sets A and B prove that

(i) $A-(B \cup C)=(A-B) \cap (A-C)$

(ii) $A-(B \cap C)=(A-B) \cup (A-C)$ **6M**

Ans. P.1-37, Ex.60, 61

Q.2. Prove by mathematical induction

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+.....+\frac{1}{n(n+1)}=\frac{n}{n+1}$$ **7M**

Ans. P.1-44, Ex.85

Q.3. Show that the following statements are logically equivalent :

(i) $(p \leftrightarrow q) \equiv (\sim p \vee q) \wedge (\sim q \vee p)$.

(ii) $p \rightarrow (q \wedge r) \equiv (p \rightarrow q) \wedge (p \rightarrow r)$. **7M**

Ans. P.1-18, Ex.27

Q.4. Test the validity of the following arguments:

If I study, then I will not fail in Mathematics.

If I do not play basketball, then I will study.

But I failed in Mathematics

Therefore I must have played basketball. **6M**

Ans. P.1-21, Ex.33

SUMMER - 13 (IT)

Q.1. Define: Tautology, contradiction, contingency. **3M**

Ans. P.1-16.

Q.2. Prove that there is no rational number $\frac{p}{q}$, whose square is 2. **4M**

Ans. P.1-28, Ex.50

Q.3. Prove that: $A \oplus B = (A \cup B) - (A \cap B)$. **6M**

Ans. P.1-43, Ex. 84

Q.4. Show that: $\sim(p \wedge q)$ and $\sim p \vee \sim q$ are logically equivalent. **3M**

Ans. P.1-18, Ex.28

Q.5. Write the converse, inverse, negation and contra-positive.

If I have time and I am not too tired than I will go to the store. **4M**

Ans. P.1-15, Ex.17

Q.6. A survey of 500 television watchers produced the following information. 285 watch football games; 115 watch basketball games, 45 watch football and basketball games, 70 watch football and hockey games, 50 watch hockey and basketball games and 50 do not watch any of the three kinds of games.

(i) How many people in the survey watch all three kinds of games?

(i) How many people watch exactly one of the sports? **6M**

Ans. P.1-39, Ex.67

SOLVED QUESTION BANK

[Sequence given as per syllabus]

INTRODUCTION

Mathematical logic developed in India, China, Europe during 19th century. George Boole and Augustus De-Morgan represented systematic mathematical treatments of logic. Mathematical logic is a science, which deals with the method of reasoning. It uses symbolic language to express its principles. It is a part of Discrete Mathematics, as it is always useful for the verification of algorithm in a programming language. It is also useful in verifying the validity of arguments.

PROPOSITIONS AND LOGICAL OPERATIONS, CONDITIONAL STATEMENTS

Statement or proposition: *CS/CT/CE: S-11*

All declarative sentences whose truth value can be determined is called a statement or proposition. All declarative sentences are either true or false. These possibilities are known as truth values.

If the statement is true, then use denote it by T and if the statement is false, then we denote it by F.

Logical connectives (Logical Operators): Logical connectives are logical tools which are required to connect two or more statements.

Connectives	Symbol	Name
NOT	$\sim$ or $\rceil$	Negation
AND	$\wedge$	Conjunction
OR	$\vee$	Disjunction
IF...THEN	$\rightarrow$ or $\Rightarrow$	Implication
IF AND ONLY IF (IFF)	$\leftrightarrow$ or $\Leftrightarrow$	Bi-Implication

(1) **Negation:** If p is any statement then negation of p is denoted by $\sim p$ or $\rceil p$. Moreover if p has a truth value 'T' then $\sim p$ has a truth value 'F' and vice versa.

Truth Table:

p	$\sim p$
T	F
F	T

(2) **Conjunction:** If p and q are statements, then the conjunction of p and q is denoted by $p \wedge q$ (read as p and q) and is a statement which has a truth value 'T' if both p and q are true, otherwise it has a truth value 'F'.

For example: p - Delhi is the capital of India.

q - Mumbai is the capital of Maharashtra.

$p \wedge q$: Delhi is the capital of India and Mumbai is of Maharashtra.

Truth Table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

(3) **Disjunction:** If p and q are two statements, then the disjunction of p and q is denoted by $p \vee q$ (read as p or q) and is a statement which has a truth value 'F' if both p and q are 'F', otherwise it has a truth value 'T'.

If we consider p and q from the above example, then $p \vee q$: Delhi is the capital of India or London is the capital Great Britain.

Truth Table:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

(4) **Conditional or Implication:** If p and q are two statements, then the conditional statement is denoted by $p \rightarrow q$ or $p \Rightarrow q$ (read as if p then q) and has a truth value 'F' if p is 'T' and q is 'F', otherwise it has a truth value 'T'.

If we consider p and q from the above example, then $p \rightarrow q$: If Delhi is the capital of India then London is the capital of Great Britain.

Truth Table:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

(5) **Bi-implication or Bi-conditional:** If p and q are two statements, then the Bi-conditional statement is

denoted by $p \leftrightarrow q$ or $p \Leftrightarrow q$ (read as p if and only if q) and has a truth value 'T' if both p and q are T and F.

If we consider p and q from the above example, then $p \leftrightarrow q$: Delhi is the capital of Indian if and only if London is the capital of Great Britain.

Truth Table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

It is to be noted that:

(i) $p \leftrightarrow q = (p \to q) \wedge (q \to p)$.

(ii) $T \wedge F = F$ and $T \vee F = T$.

(iii) $p \wedge F = F$ and $p \vee F = p$.

(iv) $T \wedge p = p$ and $T \vee p = T$.

IMPORTANT FORMULAE:

(i) $p \leftrightarrow q = (p \to q) \wedge (q \to p)$.

(ii) $T \wedge F = F$ and $T \vee F = T$.

(iii) $p \wedge F = F$ and $p \vee F = p$.

(iv) $T \wedge p = p$ and $T \vee p = T$.

Ex.1. Write the following statements in symbolic form :

(i) If Jack studies hard then he passes the exam.

(ii) x is an even number iff it is divisible by 2.

(iii) Either Shyam is successful or he is a peon.

Soln.

(i) p : Jack studies hard.

q : He passes the exam.

Hence, the given statement "If Jack studies hard then he passes the exam" in symbolic form is

$p \to q$

(ii) p : x is an even number.

q : It is divisible by 2

Hence, the given statement "x is an even number iff it is divisible by 2" in symbolic form is

$p \leftrightarrow q$

(iii) p : Shyam is successful.

q : He is a peon

Hence, the given statement "Either Shyam is successful or he is a peon" in symbolic form is

$p \vee q$

..... Ans.

Ex.2. Construct the truth table for the following formula.

(1) $(\sim q \to \sim p) \vee q$

(2) $(p \wedge q) \vee (p \to r) \vee (\sim q \to r)$.

Soln. Given:

(1) $(\sim q \to \sim p) \vee q$

The truth table is constructed below.

p	q	$\sim p$	$\sim q$	$\sim q \to \sim p$	$(\sim q \to \sim p) \vee q$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

(2) $(p \wedge q) \vee (p \to r) \vee (\sim q \to r)$

The truth table is constructed below.

1	2	3	4	5	6	7	8	9
p	q	r	$p \wedge q$	$p \to r$	$(p \wedge q) \vee (p \to r)$	$\sim q$	$\sim q \to r$	$6 \vee 8$
T	T	T	T	T	T	F	T	T
T	T	F	T	F	T	F	T	T
T	F	T	F	T	T	T	T	T
T	F	F	F	F	F	T	F	F
F	T	T	F	T	T	F	T	T
F	T	F	F	T	T	F	T	T
F	F	T	F	T	T	T	T	T
F	F	F	F	T	T	T	F	T

Ex.3. If $p \to q$ is true, can you determine the truth value of $\sim(p \to q) \wedge \sim p$. ***IT : W-06***

Soln. Since $p \to q$ is true $\Rightarrow p \to q \equiv T$

Now $\sim(p \to q) \equiv F$

Also $\sim(p \to q) \wedge \sim p = F \wedge \sim p = F$

$\therefore$ $\boxed{\text{The truth value of } \sim(p \to q) \wedge \sim p \text{ is false}}$ **Ans.**

Ex.4. If statements p and q are true and r and s are false, find the truth values of

(i) $(\sim(\sim p \wedge r) \vee (r \to q)) \leftrightarrow (p \wedge r)$

(ii) $((\sim p \wedge q) \wedge \sim r) \vee ((q \to p) \vee (\sim s \vee r))$.

Soln. $p = T \Rightarrow \sim p = F$

$q = T \Rightarrow \sim q = F$

$r = F \Rightarrow \sim r = T$

$s = F \Rightarrow \sim s = T$

(i) $(\sim(\sim p \wedge r) \vee (r \to q)) \leftrightarrow (p \wedge r)$

$= [\sim F \wedge F) \vee (F \to T)] \leftrightarrow (T \wedge F)$

$= [\sim F \vee T] \leftrightarrow F$

$= (T \vee T) \leftrightarrow F$

$= T \leftrightarrow F = F$ ..... **Ans.**

ii) $((\sim p \wedge q) \wedge \sim r) \vee ((q \rightarrow p) \vee (\sim s \vee r))$

$= [(F \wedge T) \wedge T] \wedge [(T \rightarrow T) \vee (T \vee F)]$

$= [F \wedge T] \vee [T \vee T]$

$= F \vee T = T$ ..... **Ans.**

Ex.5. Show that:

$P \rightarrow Q \Leftrightarrow (\sim P \vee Q)$. **CS/CT/CE : S-07**

Soln. We prove this by TT

P	Q	(1) $P \rightarrow Q$	$\sim P$	(2) $\sim P \vee Q$	(3) (1) $\Leftrightarrow$ (2)
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

From the above table, we observe that

$\boxed{P \rightarrow Q \Leftrightarrow (\sim P \vee Q)}$ **Ans.**

Ex.6. If p and q are false propositions, state whether $[(p \vee q) \wedge (\sim p) \vee (\sim q)$ is true or false.

CS/CT/CE : S-10

Soln. Since p and q are false

$\therefore$ $\sim p$ and $\sim q$ are true

$\therefore$ $p \vee q$ is false

$(p \vee q) \wedge (\sim p)$ i.e. $F \wedge T$ is false

$\therefore$ $[(p \vee q) \wedge (\sim p)] \vee (\sim q)$ i.e. $F \vee T$ is true

$\therefore$ $\boxed{[(p \vee q) \wedge (\sim p)] \vee (\sim q) \text{ is true}}$ **Ans.**

Ex.7. Consider the following:

p : You take a course in DMGT.

q : You understand logic.

r : You get an A grade in the final exam.

Write in simple sentences the meaning of the following:

(i) $(p \wedge q) \Rightarrow r$

(ii) $(p \wedge \sim q) \Rightarrow \sim r$

(iii) $\sim p \Rightarrow \sim q$. **CS/CT/CE : S-10**

Ans.

(i) . $(p \wedge q) \Rightarrow r$

If $p \wedge q$, then r

If you take a course in DMGT and understand logic, then you get an A grade in the final exam.

..... **Ans.**

(ii) $(p \wedge \sim q) \Rightarrow \sim r$

If $p \wedge \sim q$, then $\sim r$

If you take a course in DMGT and do not understand logic, then you do not get an A grade in the final exam.

..... **Ans.**

(iii) $\sim p \Rightarrow \sim q$

If $\sim p$, then $\sim q$

If you do not take a course in DMGT, then you do not get an A grade in the final exam. ..... **Ans.**

Ex.8. Show that: $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$.

CS/CT/CE : W-10,11

Soln. We prove this by using TT

p	q	r	(1) $q \vee r$	✓ $p \wedge (1)$	(2) $p \wedge q$	(3) $p \wedge r$	✓ (2) $\vee$ (3)
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

From the above TT, we observe that

$\therefore$ $\boxed{p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)}$ **Ans.**

Ex.9. Show that: $(p \rightarrow q) \wedge (r \rightarrow q) \Leftrightarrow (p \vee r) \rightarrow q$

IT : S-11

Soln. We prove this by TT

p	q	r	(1) $p \rightarrow q$	(2) $r \rightarrow q$	✓ (1) $\wedge$ (2)	(3) $p \vee r$	✓ (3) $\rightarrow q$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	F	F	F	T	F
T	F	F	F	T	F	T	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	F	T
F	F	T	T	F	F	T	F
F	F	F	T	T	T	F	T

From the above TT, we observe that

$\therefore$ $\boxed{(p \rightarrow q) \wedge (r \rightarrow q) \Leftrightarrow (p \vee r) \rightarrow q}$ **Ans.**

Ex.10. Show that: $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

CS/CT/CE : W-12

Soln. We prove this by TT

p	q	r	(1) $p \to q$	(2) $q \to r$	(3) (1) $\wedge$ (2)	(4) $p \to r$	✓ (3)→(4)
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

From the above TT, we observe that

$\therefore$ $\boxed{[(p \to q) \wedge (q \to r)] \to (p \to r)}$ **Ans.**

Converse, Inverse and Contrapositive:

CS/CT/CE : S-07, W-07

If p and q are two statements and $p \to q$ is an implication, then

(i) $q \to p$ is called its converse.

(ii) $\sim p \to \sim q$ is called its inverse.

(iii) $\sim q \to \sim p$ is called its contrapositive.

It is to be noted that every implication is equivalent to its contrapositive.

i.e. $p \to q \equiv \sim q \to \sim p$ and

$q \to p \equiv \sim p \to \sim q$.

We can prove above results by truth table as shown below.

1	2	3	4	5	6	7	8
p	q	$\sim p$	$\sim q$	$p \to q$	$q \to p$	$\sim p \to \sim q$	$\sim q \to \sim p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

From columns (5) and (8) it is clear that

$p \to q \equiv \sim q \to \sim p$.

From columns (6) and (7) it is clear that

$q \to p \equiv \sim p \to \sim q$.

Ex.11. Write the converse, inverse and contrapositive of the following statements.

(1) If a triangle is equilateral, then it is equiangular.

(2) If x is divisible by 2, then it is even.

Soln.

(1) p : A triangle is equilateral

q : It is equiangular

$p \to q$: If a triangle is equilateral, then it is equiangular.

Converse : $q \to p$: If a triangle is equiangular, then it is equilateral.

Inverse : $\sim q \to p$: If a triangle is not equilateral, then it is not equiangular.

Contra positive: $\sim q \to \sim p$: If triangle is not equiangular, then it is not equilateral.

(2) p : x is divisible by 2

q : x is even.

$p \to q$: If x is divisible by 2, then it is even.

Converse : $q \to p$: If x is even, then it is divisible by 2.

Inverse : $\sim p \to \sim q$: If x is not divisible by 2, then it is not even. ..... **Ans.**

Contrapositive : $\sim q \to \sim p$: If x is not even, then it is not divisible by 2. ..... **Ans.**

Negation of compound statement:

(i) $\sim(p \wedge q) \equiv \sim p \vee \sim q$.

(ii) $\sim(p \vee q) \equiv \sim p \wedge \sim q$.

(iii) $\sim(p \to q) \equiv p \wedge \sim q$.

(iv) $\sim(p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \vee \sim p)$.

IMPORTANT FORMULAE:

i) $\sim(p \wedge q) \equiv \sim p \vee \sim q$

ii) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

iii) $\sim(p \to q) \equiv p \wedge \sim q$

iv) $\sim(p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \vee \sim p)$

Ex. 12. Write the negation of the following statements.

(1) If Ram is good, then Shyam is king.

(2) He is handsome and She is beautiful.

(3) If carol is not sick, she goes to picnic and she will have good time.

(4) Ram is good, but he helps everyone.

(5) Either Ram is guilty or Shyam is not telling the truth.

CS/CT/CE: S-12

Soln.

(1) Let, p : Ram is good

q : Shyam is king

Then the given statement is $p \to q$

Its negation is $\sim(p \to q) \equiv p \wedge \vee q$

i.e. Ram is good & Shyam is not king.

..... **Ans.**

(2) Let, p : He is handsome

q : She is beautiful

Then the given statement is $p \wedge q$

Its negative is $\sim(p \wedge q) \equiv \sim p \vee \sim q$

i.e. He is not handsome or She is not beautiful.

..... **Ans.**

(3) Let, p : Carol is not sick

q : She goes to picnic.

r : She will have good time.

Then the given statement is $p \to (q \wedge r)$

Its negation is $\sim [p \to (q \wedge r)] \equiv p \wedge \sim (q \wedge r)$

$\equiv p \wedge (\sim q \vee \sim r)$

i.e. Carol is not sick and she does not go to picnic or she will not have a good time. ..... **Ans.**

(4) Let, p : Ram is good

q : He helps every one

Then the given statement is $p \wedge q$

Its negation is $\sim (p \wedge q) \equiv \sim p \vee \sim q$

i.e. Ram is not good or he does not help everyone. ..... **Ans.**

(5) Let, p : Ram is guilty

q : Shyam is not telling truth

Then the given statement is $p \vee q$

Its negation is $\sim (p \vee q) \equiv \sim p \wedge \sim q$

i.e. Ram is not guilty and Shyam is telling the truth. ..... **Ans.**

Ex.13. Write the negation of each of the following statements.

(i) If he works, he will earn money.

(ii) She swims iff the water is warm.

(iii) If it snows, then they do not drive the car.

CS/CT/CE : W-07,11, S-06,07

Soln.

(i) Let, p : He work

q : He will earn money

$\therefore$ Given statements is $p \to q$

We have $\sim (p \to q) \equiv p \wedge \sim q$

$\therefore$ The negation is

He works and he will not earn money ..... **Ans.**

(ii) Let, p : She swims

q : The water is warm

$\therefore$ Given statement is $p \leftrightarrow q$

We have $\sim (p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$

$\therefore$ The negation is

She swims and the water is not warm or

The water is warm and she does not swim. ... **Ans.**

(iii) Let, p : It snows

q : They do not drive the car

$\therefore$ Given statement is $(p \to q)$

We have $\sim (p \to q) \equiv p \wedge \sim q$

$\therefore$ The negation is

It snows and they drive the car. ..... **Ans.**

Ex.14. Write the negation of:

(i) If two lines are perpendicular to a third line in a plane, then the two lines are parallel.

(ii) The dish ran away with the spoon, but the grass is wet.

CS/CT/CE: W-06

Soln.

(i) Let, p : Two lines are perpendicular to a third line in a plane.

q : The two lines are parallel

$\therefore$ Given statement is $p \to q$

We have $\sim (p \to q) \equiv p \wedge \sim q$

$\therefore$ The negation is

The lines are perpendicular to a third line in a plane and are not parallel. ..... **Ans.**

(ii) Let, p : The dish ran away with the spoon

q : The grass is wet

$\therefore$ Given statement is $p \vee q$

We have $\sim (p \vee q) \equiv \sim p \wedge \sim q$

$\therefore$ The negation is

The dish did not ran away with the spoon and the grass is not wet. ..... **Ans.**

Ex.15. Write the negation of "If Tom stole a pie and ran away, then the three pigs do not have any supper."

IT : W-06

Soln. Let, p : Tom stole a pie

q : Tom ran away

r : Three pigs do not have any supper

$\therefore$ Given statement is $(p \wedge q) \to r$

We have $\sim ((p \wedge q) \to r) \equiv (p \wedge q) \wedge \sim r$

$\therefore$ The negation is

Tom stole a pie and ran away and the three pigs have supper. ..... **Ans.**

Ex.16. Write the converse, inverse, contrapositive and negation of "Oxygen in air decreases as you go up."

IT : W-07,12, S-12

Soln. Let, p : Oxygen in air decreases

q : You go up

$\therefore$ Given statements is $p \to q$

(i) **Converse :** If you go up, then oxygen in air decreases $(q \to p)$

(ii) **Inverse :** If oxygen in air is not decreased, then you have not gone up. $(\sim p \to \sim q)$

(iii) **Contrapositive :** If you do not go up, then oxygen in air do not decrease. $(\sim q \to \sim p)$

(iv) **Negation :** Oxygen in air increases as you go down. $(p \wedge (\sim q))$ ..... **Ans.**

Ex.17. Write the contrapositive of each of the following statements.

(i) If I have enough money, then I will buy a car and a house.

(ii) If I have time and I am not too tired, then I will go to the store. **IT: S-07,13, W-10, CS/CT/CE : S-09, W-12**

Soln.

(i) Let, p : I have enough money

q : I will buy a car

r : I will buy a house

$\therefore$ Given statement is $p \to (q \wedge r)$

Its contrapositive is $\sim (q \wedge r) \to \sim p$

i.e. $(\sim q \vee \sim r) \to \sim p$

It I won't buy a car or house, then I have no money.

..... **Ans.**

(ii) Let, p : I have time

q : I am not too tired

r : I will go to the store

$\therefore$ Given statement is $(p \wedge q) \to r$

(a) **Converse:** $r \to (p \wedge q)$

If I go to the store, then I have time and I am not too tired.

(b) **Inverse :** $\sim (p \wedge q) \to \sim r$ i.e. $(\sim p \vee \sim q) \to \sim r$

If I don't have time or I am too tired, then I will not go to the store. ..... **Ans.**

(c) **Contrapositive:** $\sim r \to \sim (p \wedge q)$ i.e. $\sim r \to (\sim p \vee \sim q)$

If I do not go to the store, then I don't have time or I am too tired.

(d) **Negation :** $\sim [(p \wedge q) \to r] \equiv (p \wedge q) \wedge \sim r$

I have time and I am not too tired and I will not go to the store. ..... **Ans.**

Ex.18. Negate the statements:

(i) It is daylight and all the people have a risen.

(ii) 2 is even or – 3 is negative. **CS/CT/CE: W-07,11**

Soln.

(i) Let, p : It is daylight

q : All people have arisen

We have $\sim (p \wedge q) \equiv \sim p \vee \sim q$

$\therefore$ The negation is

Its not daylight or all the people have not arisen.

..... **Ans.**

(ii) Let, p : 2 is even

q : – 3 is negative

We have $\sim (p \vee q) \equiv \sim p \wedge \sim q$

$\therefore$ The negation is

2 in not even and – 3 is not negative. ..... **Ans.**

Ex.19. State the converse, inverse and contrapositive of:

(i) If today is Easter, then tomorrow is Monday.

(ii) If p is a square, then p is a rectangle. **CS/CT/CE: W-09**

Soln.

i) Let p : Today is Easter

q : Tomorrow is Monday

$\therefore$ Given statement is $p \to q$

(a) **Converse :** If tomorrow is Monday, then today is Easter.

$(q \to p)$

(b) **Inverse :** If today is not Easter, then tomorrow is not Monday.

$(\sim p \to \sim q)$

(c) **Contrapositive :** If tomorrow is not Monday, then today is not Easter.

$(\sim q \to \sim p)$ ..... **Ans.**

ii) Let, p : p is a square

q : p is a rectangle

$\therefore$ Given statement is $p \to q$

(a) **Converse :** If p is a rectangle, then p is a square.

$(q \to p)$

(b) **Inverse :** If p is not a square, then p is not a rectangle.

$(\sim p \to \sim q)$

(c) **Contrapositive :** If p is not a rectangle, then p is not a square.

$(\sim q \to \sim p)$ ..... **Ans.**

Ex.20. If p : Today is monday

q : The grass is wet

r : The dish ran away with the spoon.

Write each of the following in terms of p, q, r and logical connectives.

(i) Today is Monday and the dish did not run away with the spoon.

(ii) Either the grass is wet or today is Monday. **CS/CT/CE: W-12**

Soln. (i) $p \wedge \sim q$

(ii) $p \vee q$ ..... **Ans.**

STATEMENT PATTERN

IT: S-08,13, CS/CT/CE: W-08, S-11

A pattern obtained by combining two or more statements using logical connectives is called statement pattern.

Following are the statement patterns:

(i) Tautology

(ii) Contradiction

(iii) Contingency

(i) Tautology: In a statement pattern, if all truth values are true then it is called a tautology.

(ii) Contradiction : In a statement pattern, if all truth values are false then it is called a contradiction.

iii) Contingency: If a statement pattern is neither a tautology nor a contradiction, then it is called a contingency.

Propositions like:

(i) The professor is either a woman or a man.

(ii) 3 is a prime number and 4 is even number are always true and are called tautologies.

Proposition like: Sun revolves around the earth is always false and is called contradiction.

Ex.21. Check which of the following are tautology, contradiction and contingency.

(i) $(p \to q) \leftrightarrow (\sim p \vee q)$

(ii) $((p \vee q) \wedge r) \vee (p \vee q)$

(iii) $((p \to q) \wedge (q \to r)) \to (p \to r)$

(iv) $p \wedge (q \wedge \sim p)$.

CS/CT/CE: S-06, IT: S-10

Soln.

(i) Given statement pattern is $(p \to q) \leftrightarrow (\sim p \vee q)$

(i) p	(ii) q	(iii) $\sim p$	(iv) $p \to q$	(v) $\sim p \vee q$	(vi) (iv) $\leftrightarrow$ v
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Since all the entries in column (vi) are true, hence given statement pattern is a tautology. ..... **Ans.**

(ii) Given statement pattern is $((p \vee q) \wedge r) \vee (p \vee q)$

(i) p	(ii) q	(iii) r	(iv) $p \vee q$	(v) $(p \vee q) \wedge r$	(vi) $((p \vee q) \wedge r) \vee (p \vee q)$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	T	T	T
T	F	F	T	F	T
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	F	F	F
F	F	F	F	F	F

Since some entries is column (vi) are true and some are false, hence given statement pattern is a contingency. ..... **Ans.**

(iii) Given statement pattern is

$((p \to q) \wedge (q \to r)) \to (p \to r)$

(i) p	(ii) q	(iii) r	(iv) $p \to q$	(v) $q \to r$	(vi) $p \to r$	(vii) $(p \to q) \wedge (q \to r)$	(viii) $[(p \to q) \wedge (q \to r)] \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	T	T	T	T	T	T

Since all the entries in the column (vii) are true, hence given statement pattern is a tautology. ..... **Ans.**

(iv) Given statement pattern is $p \wedge (q \wedge \sim p)$

(i) p	(ii) q	(iii) $\sim p$	(iv) $(q \wedge \sim p)$	(v) $p \wedge (q \wedge \sim p)$
T	T	F	F	F
T	F	F	F	F
F	T	T	T	F
F	F	T	F	F

Since all the entries in the column (v) are false, hence the given statement pattern is a contradiction.

..... **Ans.**

Note: To show $A \leftrightarrow B$, it is enough to prove $A \leftrightarrow B$ is a tautology.

Ex.22. Show that:

$$[p \to (q \to r)] \leftrightarrow [P \to (\sim q \vee r)].$$

Soln.

(1) p	(2) q	(3) r	(4) ~q	(5) q→r	(6) p→(q→r)	(7) ~q∨r	(8) 1→7	(9) 6↔8
T	T	T	F	T	T	T	T	T
T	T	F	F	F	F	F	F	T
T	F	T	T	T	T	T	T	T
T	F	F	T	T	T	T	T	T
F	T	T	F	T	T	T	T	T
F	T	F	F	F	T	F	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T

From column 9, we conclude that the given expression is a tautology.

$\therefore [p \to (q \to r)] \leftrightarrow [p \to (\sim q \vee r)]$ ..... **Ans.**

Ex.23. Show that: $((P \vee Q) \wedge (\rceil(\rceil P \wedge \rceil Q \vee \rceil R))) \vee (\rceil P \wedge \rceil Q) \vee (\rceil P \wedge \rceil R)$ is a tautology. **CS/CT/CE: W-06, S-08**

Soln.

P	Q	R	⌉P	⌉Q	⌉R	(1) P∨Q	(2) ⌉P∧⌉Q∨⌉R	(3) ⌉(2)	(4) (1)∧(3)	(5) ⌉P∧⌉Q	(6) ⌉P∧⌉R	(7) (5)∨(6)	(4)∨(7)
T	T	T	F	F	F	T	F	T	T	F	F	F	T
T	T	F	F	F	T	T	T	F	F	F	F	F	F
T	F	T	F	T	F	T	F	T	T	F	F	F	T
T	F	F	F	T	T	T	T	F	F	F	F	F	F
F	T	T	T	F	F	T	F	T	T	F	F	F	T
F	T	F	T	F	T	T	T	F	F	F	T	T	T
F	F	T	T	T	F	F	T	F	F	T	F	T	T
F	F	F	T	T	T	F	T	F	F	T	T	T	T

From the above table, we observe that given proposition is a contingency and not a tautology. ..... **Ans.**

Ex.24. Verity that the proposition $(p \vee q) \wedge \sim (p \vee q)$ is a contradiction. **CS/CT/CE: S-07,09**

Soln.

p	q	(i) p∨q	(ii) ~(p∨q)	(1)∧(2)
T	T	T	F	F
T	F	T	F	F
F	T	T	F	F
F	F	F	T	F

From the above table, we observe that

$\therefore$ $\boxed{(p \vee q) \wedge \sim (p \vee q) \text{ is a contradiction}}$ **Ans.**

Ex.25. Construct the truth table for

$[(P \wedge \sim Q) \to R] \to [P \to (Q \vee R)]$ and verity that it is a tautology. **IT: S-07, W-11**

Soln.

P	Q	R	~Q	(1) P∧~Q	(2) (1)→(R)	(3) Q∨R	(4) P→(3)	(2)→(4)
T	T	T	F	F	T	T	T	T
T	T	F	F	F	T	T	T	T
T	F	T	T	T	T	T	T	T
T	F	F	T	T	F	F	F	T
F	T	T	F	F	T	T	T	T
F	T	F	F	F	T	T	T	T
F	F	T	T	F	T	T	T	T
F	F	F	T	F	T	F	T	T

From the above table, we observe that given proposition is a tautology. ..... **Ans.**

Ex.26. Show that: $[((P \vee Q) \wedge \sim(\sim P \wedge (\sim Q \vee \sim R)))] \vee (\sim P \wedge \sim Q) \vee (\sim P \wedge \sim R)]$ is a tautology. **IT: W-08**

Soln.

P	Q	R	~P	~Q	~R	(1) P∨Q	(2) ~Q ∨ ~R	(3) ~P ∧ (2)	(4) ~ (3)	(5) (1) ∧ (4)	(6) ~P ∧ ~Q	(7) ~P ∧ ~R	(8) (6) ∨ (7)	(5) ∨ (8)
T	T	T	F	F	F	T	F	F	T	T	F	F	F	T
T	T	F	F	F	T	T	T	F	T	T	F	F	F	T
T	F	T	F	T	F	T	T	F	T	T	F	F	F	T
T	F	F	F	T	T	T	T	F	T	T	F	F	F	T
F	T	T	T	F	F	T	F	F	T	T	F	F	F	T
F	T	F	T	F	T	T	T	T	F	F	F	T	T	T
F	F	T	T	T	F	F	T	T	F	F	T	F	T	T
F	F	F	T	T	T	F	T	T	F	F	T	T	T	T

From the above table, we observe that the given proposition is a tautology. ..... **Ans.**

LOGICAL EQUIVALENCE

CS/CT/CE : S-11

Two statements say A and B are said to be logically equivalent if the truth values of A and truth values of B are same.

Ex.27. Show that following statements are logically equivalent: **IT: W-10, CS/CT/CE : S-12,13**

(1) $(p \leftrightarrow q) \equiv (\sim p \vee q) \wedge (\sim q \vee p)$.

(2) $p \rightarrow (q \wedge r) \equiv (p \rightarrow q) \wedge (p \rightarrow r)$.

(3) $\sim(p \leftrightarrow q) \equiv q \leftrightarrow \sim p$.

Soln. (1)

(i) p	(ii) q	(iii) ~p	(iv) ~q	(v) p↔q	(vi) ~p ∨ q	(vii) ~q ∨ p	(viii) (vi)∧(vii)
T	T	F	F	T	T	T	T
T	F	F	T	F	F	T	F
F	T	T	F	F	T	F	F
F	F	T	T	T	T	T	T

From columns (v) and (viii), we observe that the two given statements are equivalent.

$\therefore$ $\boxed{p \leftrightarrow q \equiv (\sim p \vee q) \wedge (\sim q \vee p)}$ **Ans.**

(2)

(i) p	(ii) q	(iii) r	(iv) q ∧ r	(v) p→(q∧r)	(vi) p→q	(vii) p → r	(viii) (vi)∧(vii)
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	F
T	F	T	F	F	F	T	F
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

From columns (v) and (viii), we observe that the two given statements are equivalent.

$\therefore$ $\boxed{p \rightarrow (q \wedge r) \equiv (p \rightarrow q) \wedge (p \rightarrow r)}$ **Ans.**

(3)

(i) p	(ii) q	(iii) ~p	(iv) p↔q	(v) ~(p↔q)	(vi) q↔~p
T	T	F	T	F	F
T	F	F	F	T	T
F	T	T	F	T	T
F	F	T	T	F	F

As columns (v) and (vi) are same

$\therefore$ $\boxed{\sim(p \leftrightarrow q) \equiv q \leftrightarrow \sim p}$ **Ans.**

Ex.28. State and prove De-Morgan's laws. **IT: S-08,10,13**

Soln. First law: The negation of conjunction of two or more statements is logically equivalent to the disjunction of negation of the statements.

If p and q are two statements, then

$\sim(p \wedge q) \equiv \sim p \vee \sim q$

(i) p	(ii) q	(iii) p∧q	(iv) ~(p∧q)	(v) ~p	(vi) ~q	(vii) ~p ∨ ~q
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

From columns (iv) and (vii), we observe that

$\sim(p \wedge q) \equiv \sim p \vee \sim q$

Second Law: The negation of disjunction of two or more statements is logically equivalent to conjuction of negation of the statements.

If p and q are two statements, then

$\sim(p \vee q) \equiv \sim p \wedge \sim q$

(i) p	(ii) q	(iii) ~p	(iv) ~q	(v) p ∨ q	(vi) ~ (p∨q)	(vii) ~p ∧ ~q
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

From columns (vi) and (vii), we observe that

$\sim(p \vee q) \equiv \sim p \wedge \sim q$

LAWS OF ALGEBRA IN LOGIC

Propositions satisfy various laws in logic. These laws are useful to simplify expressions. Note that the exception of the innovation law, all laws come in pairs, called dual pairs. For each expression, one finds the dual by replacing all 'T' by 'F' and all 'F' by 'T' and replacing all $\wedge$ by $\vee$ and all $\vee$ by $\wedge$.

(i) Idempotent laws:

(a) $p \vee p \equiv p$

(b) $p \wedge p \equiv p$

(ii) Associative laws:

(a) $p \wedge (q \wedge r) = (p \wedge q) \wedge r$

(b) $p \vee (q \vee r) = (p \vee q) \vee r$

(iii) Commutative laws:

(a) $p \wedge q \equiv q \wedge p$

(b) $p \vee q = q \vee p$

(iv) Distributive laws:

(a) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

(b) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(v) Identity laws:

(a) $p \vee F \equiv p$

(b) $p \wedge T \equiv p$

(c) $T \vee p \equiv T$

(d) $F \wedge p \equiv F$

(vi) Complement laws:

(a) $p \vee \sim p \equiv T$

(b) $p \wedge \sim p = F$

(c) $\sim T = F$ and $\sim F = T$

vii) Absorption laws:

(a) $p \wedge (p \vee q) \equiv p$

(b) $p \vee (p \wedge q) = p$

viii) De-Morgan's laws:

(a) $\sim(p \wedge q) \equiv \sim p \vee \sim q$

(b) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

IMPORTANT FORMULAE:

(i) Idempotent laws:

(a) $p \vee p \equiv p$

(b) $p \wedge p \equiv p$

(ii) Associative laws:

(a) $p \wedge (q \wedge r) = (p \wedge q) \wedge r$

(b) $p \vee (q \vee r) = (p \vee q) \vee r$

(iii) Commutative laws:

(a) $p \wedge q \equiv q \wedge p$

(b) $p \vee q = q \vee p$

(iv) Distributive laws:

(a) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

(b) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(v) Identity laws:

(a) $p \vee F \equiv p$

(b) $p \wedge T \equiv p$

(c) $T \vee p \equiv T$

(d) $F \wedge p \equiv F$

(vi) Complement laws:

(a) $p \vee \sim p \equiv T$

(b) $p \wedge \sim p = F$

(c) $\sim T = F$ and $\sim F = T$

(vii) Absorption laws:

(a) $p \wedge (p \vee q) \equiv p$

(b) $p \vee (p \wedge q) = p$

(viii) De-Morgan's laws:

(a) $\sim(p \wedge q) \equiv \sim p \vee \sim q$

(b) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

Ex.29. Without using truth table, show that:

(i) $p \vee (p \wedge q) \equiv p$

(ii) $p \wedge (p \vee q) \equiv p$

(iii) $(p \wedge (\sim p \vee q)) \vee (q \wedge \sim(p \wedge q)) \equiv q$.

(iv) $\{(p \wedge \sim q) \wedge (\sim p \vee \sim q)\} \vee q \equiv p \vee q$

(v) $\{(p \vee \sim q) \wedge (\sim p \vee \sim q)\} \vee q \equiv T$

(vi) $p \leftrightarrow q \equiv (p \wedge q) \vee (\sim p \wedge \sim q)$. **IT: S-08**

Soln.

(i) Consider L.H.S. $\equiv p \vee (p \wedge q)$

$\equiv (p \wedge T) \vee (p \wedge q)$ ....(Identity law)

$\equiv p \wedge (T \vee q)$ ....(Distributive law)

$\equiv p \wedge T$ ....(Identity law)

$\equiv p$ ....(Identity law)

$\equiv$ R.H.S.

$\therefore$ $\boxed{p \vee (p \wedge q) \equiv p}$ **Ans.**

(ii) Consider L.H.S. $\equiv p \wedge (p \vee q)$

$\equiv (p \vee F) \wedge (p \vee q)$ ....(Identity law)

$\equiv p \vee (F \wedge q)$ ....(Distributive law)

$\equiv p \vee F$ ....(Identity law)

$\equiv p$ ....(Identity law)

$\equiv$ R.H.S.

$\therefore$ $\boxed{p \wedge (p \vee q) \equiv p}$ **Ans.**

(iii) Consider L.H.S. $\equiv [p \wedge (\sim p \vee q)] \vee [q \wedge \sim(p \wedge q)]$

$\equiv [(p \vee \sim p) \vee (p \wedge q)] \vee [q \wedge (\sim p \vee \sim q)]$

(Distributive law and De-Morgan's law)

$\equiv [F \vee (p \wedge q)] \vee [(q \wedge \sim p) \vee (q \wedge \sim q)]$

(Distributive law and complement law)

$\equiv (p \wedge q) \vee [(q \wedge \sim p) \vee F]$

$\equiv (p \wedge q) \vee (q \wedge \sim p)$

$\equiv [(p \wedge q) \vee q] \wedge [(p \wedge q) \vee \sim p]$

$\equiv q \wedge [(p \vee \sim p) \wedge (q \vee \sim p)]$

$\equiv q \wedge [T \wedge (q \vee \sim p)]$

$\equiv q \wedge (q \vee \sim p)$ ....(Identity law)

$\equiv q \equiv$ R.H.S. ....(Absorption law)

$\therefore$ $\boxed{(p \wedge (\sim p \vee q)) \vee (q \wedge \sim (p \wedge q)) \equiv q}$ **Ans.**

(iv) Consider L.H.S. $\equiv [(p \wedge \sim q) \wedge (\sim p \vee \sim q)] \vee q$

$\equiv \{[(p \wedge \sim q) \wedge \sim p] \vee [(p \wedge \sim q)] \wedge \sim q\} \vee q$

....(Distributive law)

$\equiv \{[(p \wedge \sim p) \wedge \sim q] \vee [p \wedge (\sim q \wedge \sim q)]\} \vee q$

....(Associative law)

$\equiv \{(F \wedge \sim q) \vee [p \wedge \sim q]\} \vee q$

....(Complement law)

$\equiv \{F \vee [p \wedge \sim q]\} \vee q$ ....(Identity law)

$\equiv (p \wedge \sim q) \vee q$

$\equiv (p \vee q) \wedge (\sim q \vee q)$ ....(Distributive law)

$\equiv (p \vee q) \wedge T \equiv p \vee q$ ....(Complement law)

$\equiv$ R.H.S.

$\therefore$ $\boxed{\{(p \wedge \sim q) \wedge (\sim p \vee \sim q)\} \vee q \equiv p \vee q}$ **Ans.**

(v) Consider L.H.S. $\equiv \{(p \vee \sim q) \wedge (\sim p \vee \sim q)\} \vee q$

$\equiv [(p \vee \sim q) \vee q] \wedge [(\sim p \vee \sim q) \vee q]$

....(Distributive law)

$\equiv [p \vee (\sim q \vee q)] \wedge [\sim p \vee (\sim q \vee q)]$

....(Associative law)

$\equiv (p \vee T) \wedge (\sim p \vee T)$ ....(Complement law)

$\equiv T \wedge T$ ....(Identity law)

$\equiv T \equiv$ R.H.S.

$\therefore$ $\boxed{(p \vee \sim q)) \wedge (\sim p \vee \sim q) \vee q \equiv T}$ **Ans.**

(vi) Consider L.H.S. $\equiv (p \leftrightarrow q)$

$\equiv (p \rightarrow q) \wedge (q \rightarrow p)$

$\equiv (\sim p \vee q) \wedge (\sim q \vee p)$ .... (Logical equivalence)

$\equiv (\sim p \wedge (\sim q \vee p)) \vee (q \wedge (\sim q \vee p))$

$\equiv (\sim p \wedge \sim q) \vee (\sim p \wedge p) \vee (q \wedge \sim q) \vee (q \wedge p)$

$\equiv (\sim p \wedge \sim q) \vee (\sim p \wedge p) \vee (q \wedge \sim q) \vee (q \wedge p)$

$\equiv (\sim p \wedge \sim q) \vee F \vee F \vee (q \wedge p)$

..... (Complement law)

$\equiv (q \wedge p) \vee (\sim p \wedge \sim q)$ .... (Identity law)

$\equiv (p \wedge q) \vee (\sim p \wedge \sim q)$

$\equiv$ R.H.S.

$\therefore$ $\boxed{p \leftrightarrow q = (p \wedge q) \vee (\sim p \wedge \sim q)}$ **Ans.**

Ex.30. Prove using TT

(i) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.

(ii) $p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$.

CS/CT/CE: W-09

Soln.(i)

p	q	r	(1) $q \wedge r$	(2) $p \vee (1)$	(3) $p \vee q$	(4) $p \vee r$	(5) $(3) \wedge (4)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

From the above table, we observe that

$\boxed{p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)}$ **Ans.**

(ii)

p	q	(1) $p \Leftrightarrow q$	(2) $p \Rightarrow q$	(3) $q \Rightarrow p$	✓ $(2) \wedge (3)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

From the above table, we observe that

$\boxed{p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)}$ **Ans.**

Ex.31. Show that:

$p \Leftrightarrow q \equiv (p \vee q) \Rightarrow (p \wedge q)$. **CS/CT/CE : S-10**

Soln.

p	q	✓ $p \Leftrightarrow q$	(1) $p \vee q$	(2) $p \wedge q$	✓ $(1) \Rightarrow (2)$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	F	F
F	F	T	F	F	T

From the above table, we observe that

$\boxed{p \Leftrightarrow q \equiv (p \vee q) \Rightarrow (p \wedge q)}$ **Ans.**

ARGUMENTS AND VALIDITY

An argument is an assertion, that a given set of propositions $S_1, S_2, S_3 \ldots S_n$, is denoted by

$S_1, S_2, S_3 \ldots S_n \vdash S$.

An argument $S_1, S_2, S_3 \ldots S_n \vdash S$ is valid if and only if $S_1 \wedge S_2 \wedge S_3 \wedge S_4 \ldots S_n \vdash S$ is a tautology.

An argument which is not valid is called fallacy.

Note: (i) The following argument is valid

$p, p \to q \vdash q$ (law of detachment)

(ii) The following argument is fallacy

$p \to q, q \vdash p$

(iii) The following argument is valid

$p \to q, q \to r \vdash p \to r$ (law of syllogism).

For example consider the following argument

S_1 : If Sachin plays cricket, then we will win the match.

S_2 : If we win the match, we will be happy.

$\therefore$ S : If Sachin plays cricket, we will be happy.

Here, S_1, S_2 denote the premises of the argument and S is conclusion of the argument.

Let, p : Sachin plays cricket

q : We will win the match

r : We will be happy

$\therefore S : p \to q$

$S_2 : q \to r$

$S : p \to r$

Thus, $S_1, S_2 \vdash S \Rightarrow p \to q$,

$q \to r \vdash p \to r$.

$\therefore$ By law of syllogism, the given argument is valid.

Ex.32. Prove that the following argument is valid

$p \to \sim q, r \to q, r$

$\therefore \sim p$.

Soln. Let, $S_1 : p \to \sim q$

$S_2 : r \to q$

$S_3 : r$

$S : \sim p$

To prove that the above argument is valid, we prove that $S_1 \wedge S_2 \wedge S_3 \to S$

i.e. $[(p \to \sim q) \wedge (r \to q) \wedge r] \to \sim p$ is a tautology.

1 p	2 q	3 r	4 ~p	5 ~q	6 $p \to \sim q$	7 $r \to q$	8 $6 \wedge 7$	9 $8 \wedge 3$	10 $9 \to 4$
T	T	T	F	F	T	T	F	F	T
T	T	F	F	F	F	T	F	F	T
T	F	T	F	T	T	F	F	F	T
T	F	F	F	T	T	T	T	F	T
F	T	T	T	F	T	T	T	T	T
F	T	F	T	F	T	T	T	F	T
F	F	T	T	T	T	F	F	F	T
F	F	F	T	T	T	T	T	F	T

Since all the truth values in the column 10 are 'T', hence the given argument is a tautology and hence valid.

.... **Ans.**

Ex.33. Test the validity of the following argument

If I study, then I will not fail in maths.

If I do not play basketball then I will study.

But I failed in Mathematics.

$\therefore$ I must have played basketball.

CS/CT/CE: S-07,12,13, W-07,11

Soln. Let p : I study

q : I will fail in Mathematics

r : I play basket ball

Then the given argument

$S_1 : p \to \sim q$, $S_2 : \sim r \to p$, $S_3 : q$

$\therefore$ S : r

$\therefore S_1, S_2, S_3 \vdash S$ i.e. $p \to \sim q, \sim r \to p, q \vdash r$

To prove that the above argument is valid, we show that $S_1 \wedge S_2 \wedge S_3 \to S$

i.e. $[(p \to \sim q) \wedge (\sim r \to p) \wedge q] \to r$ is a tautology.

1	2	3	4	5	6	7	8	9	10
p	q	r	~q	~r	p→ ~q	~r→q	6 ∧ 7	8 ∧2	9→3
T	T	T	F	F	F	T	F	F	T
T	T	F	F	T	F	T	F	F	T
T	F	T	T	F	T	T	T	F	T
T	F	F	T	T	T	T	T	F	T
F	T	T	F	F	T	T	T	T	T
F	T	F	F	T	T	F	F	F	T
F	F	T	T	F	T	F	T	F	T
F	F	F	T	T	T	F	F	F	T

Since all the truth values in the column 10 are 'T', hence the given argument is a tautology and hence valid. ...**Ans.**

Ex.34. Show that the following argument is invalid.

If Siddharth solved this problem then he obtains the answer. Siddharth obtained the answer.

Therefore Siddharth solved this problem correctly.

Soln. Let p : Siddharth solved this problem

q : He obtained the answer

Then the given argument is

$S_1 : p \to q$

$S_2 : q$

$\therefore S : p$

$\therefore S_1, S_2 \vdash S$ i.e. $p \to q, q \vdash p$.

To prove that the given argument is invalid, we show that $S_1 \wedge S_2 \not\to S$ i.e. $(p \to q) \wedge q \to p$ is not a tautology.

p	q	p→q	(p→q)∧q	(p→q)∧q→p
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

From the last column, it is clear that the given argument is invalid. ..... **Ans.**

Ex.35. State whether the given argument is valid or not.

If I graduate this semester, then I will have passed DMGT.

If I do not study DMGT for 10 hrs a week, then I will not pass DMGT.

If I study DMGT for 10 hrs a week, then I can not play cricket.

Therefore, if I play cricket, I will not graduate this semester. ***IT: W-06, CS/CT/CE:W-12***

Soln. Let p : I graduate this semester, q : I pass DMGT

r : I study DMGT for 10 hrs a week, S : I play cricket

$\therefore$ Then the given argument is

$S_1 : p \to q$, $S_2 : \sim r \to \sim q$, $S_3 : r \to \sim s$

$\therefore S : s \to \sim p$

$\therefore S_1, S_2, S_3 \vdash S$

i.e. $p \to q, \sim r \to \sim q, r \to \sim s \vdash s \to \sim p$

To show that the above argument is valid, we show that $S_1 \wedge S_2 \wedge S_3 \to S$

i.e. $(p \to q) \wedge (\sim r \to \sim q) \wedge (r \to \sim s) \to (s \to \sim p)$ is a tautology.

1	2	3	4	5	6	7	8	9	10	11	12	13	14
p	q	r	s	~p	~q	~r	~s	p→q	~ r → ~p	r→~s	s→~p	9 ∧ 10 ∧ 11	13 → 12
T	T	T	T	F	F	F	F	T	T	F	F	F	T
T	T	T	F	F	F	F	T	T	T	T	T	T	T
T	T	F	T	F	F	T	F	T	F	T	F	F	T
T	T	F	F	F	F	T	T	T	F	T	T	F	T
T	F	T	T	F	T	F	F	F	T	F	F	F	T
T	F	T	F	F	T	F	T	F	T	T	T	F	T
T	F	F	T	F	T	T	F	F	T	T	F	F	T
T	F	F	F	F	T	T	T	F	T	T	T	F	T
F	T	T	T	T	F	F	F	T	T	F	T	F	T
F	T	T	F	T	F	F	T	T	T	T	T	T	T
F	T	F	T	T	F	T	F	T	F	T	T	F	T
F	T	F	F	T	F	T	T	T	F	T	T	F	T
F	F	T	T	T	T	F	F	T	T	F	T	F	T
F	F	T	F	T	T	F	T	T	T	T	T	T	T
F	F	F	T	T	T	T	F	T	T	T	T	T	T
F	F	F	F	T	T	T	T	T	T	T	T	T	T

Since all the truth values in column (14) are 'T', hence given argument is a tautology and hence valid. ..... **Ans.**

Ex.36. Determine the validity of the following argument.

If 7 is less than 4, then 7 is not a prime number

7 is not less than 4.

∴ 7 is a prime number. ***CS/CT/CE: S-06,09***

Soln. Let, p : 7 is less than 4

q : 7 is a prime number

$\therefore S_1 : p \rightarrow \sim q$

$S_2 : \sim p$

$\therefore S : q$

The given argument will be valid if $(S_1 \wedge S_2) \rightarrow S$ is a tautology i.e. $[(p \rightarrow \sim q) \wedge \sim p] \rightarrow q$ is a tautology.

p	q	~q	~p	(1) $p \rightarrow \sim q$	(2) $(1) \wedge \sim p$	$(2) \rightarrow q$
T	T	F	F	F	F	T
T	F	T	F	T	F	T
F	T	F	T	T	T	T
F	F	T	T	T	T	F

Since $[(p \rightarrow \sim q) \wedge \sim p] \rightarrow q$ is not a tautology, hence the given statement is invalid. **.... Ans.**

Ex.37. Whether the given argument is valid or not.

If my plumbing plans do not meet the construction code, then I cannot build my house.

If I hire a licensed contractor, them my plumbing plans will meet the construction code.

I hired a licensed contractor.

∴ I can build my house. ***IT: S-06, W-07,10,11***

Soln. Let p : My plumbing plans meet the construction code.

q : I can build my house.

r : I hired a licensed contractor.

$S_1 : \sim p \rightarrow \sim q$

$S_2 : r \rightarrow p$

$S_3 : r$

$\therefore S : q$

The given argument will be valid if $(S_1 \wedge S_2 \wedge S_3) \rightarrow S$ is a tautology.

i.e. $[(\sim p \rightarrow \sim q) \wedge (r \rightarrow p) \wedge r] \rightarrow q$ is a tautology.

p	q	r	~p	~q	(1) $\sim p \rightarrow \sim q$	(2) $r \rightarrow p$	(3) $(1) \wedge (2) \wedge r$	$(3) \rightarrow q$
T	T	T	F	F	T	T	T	T
T	T	F	F	F	T	T	F	T
T	F	T	F	T	T	T	T	F
T	F	F	F	T	T	T	F	T
F	T	T	T	F	F	F	F	T
F	T	F	T	F	F	T	F	T
F	F	T	T	T	T	F	F	T
F	F	F	T	T	T	T	F	T

From the above table, we observe that

$(S_1 \wedge S_2 \wedge S_3) \rightarrow S$ is not a tautology, hence the given argument is invalid. **..... Ans.**

Ex.38. Check the validity of the following argument

Smoking is healthy.

If smoking is healthy, then Cigarettes are prescribed by physicians.

∴ Cigarettes are prescribed by physicians.

CS/CT/CE : W-06

Soln. Let, p : Smoking is healthy

q : Cigarettes are prescribed by physicians

$\therefore S_1 : p$

$S_2 : p \rightarrow q$

$\therefore S : q$

The given argument will be valid if $(S_1 \wedge S_2) \rightarrow S$ is a tautology i.e. $[p \wedge (p \rightarrow q)] \rightarrow q$ is a tautology.

p	q	(1) $p \rightarrow q$	(2) $p \wedge (1)$	$(2) \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

From the above table, we observe that

$[p \wedge (p \rightarrow q)] \rightarrow q$ is a tautology, hence the given argument is valid. **..... Ans.**

Ex.39. Is the following argument valid?

If the labour market is perfect, then the wages of all persons in a particular employment will be equal.

But it is always the case that wages for such persons are no equal.

∴ The labour market is not perfect. ***IT: S-07***

Soln. Let p : The labour market is perfect

q : The wages of all persons are equal

r : I hired a licensed contractor.

$S_1 : p \rightarrow q$

$S_2 : \sim q$

$\therefore S : \sim p$

The given argument will be valid if $(S_1 \wedge S_2) \rightarrow S$ is a tautology i.e. $[(p \rightarrow q) \wedge \sim q] \rightarrow p$ is a tautology.

p	q	~p	~q	(1) $p \rightarrow q$	(2) $(1) \wedge \sim q$	$(2) \rightarrow \sim p$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

From the above table, we observe that

$[(p \rightarrow q) \wedge \sim q] \rightarrow \sim p$ is a tautology, hence the given argument is valid. ..... **Ans.**

Ex.40. Determine the validity of the following arguments:

If it rains, the street gets wet,

If it rains, accident happens

The street gets wet.

∴ Accidents happen. ***CS/CT/CE: S-08***

Soln. Let p : It rains

q : The streets get wet

r : Accidents happen

$\therefore S_1 : p \rightarrow q$

$S_2 : p \rightarrow r$

$S_3 : q$

$\therefore S : r$

The given argument will be valid if

$(S_1 \wedge S_2 \wedge S_3) \rightarrow S$ is a tautology

i.e. $[(p \rightarrow q) \wedge (p \rightarrow r) \wedge q] \rightarrow r$ is a tautology

p	q	r	(1) $p \rightarrow q$	(2) $p \rightarrow r$	(3) $(1) \wedge (2) \wedge q$	$(3) \rightarrow r$
T	T	T	T	T	T	T
T	T	F	T	F	F	T
T	F	T	F	T	F	T
T	F	F	F	F	F	T
F	T	T	T	T	T	T
F	T	F	T	T	T	F
F	F	T	T	T	F	T
F	F	F	T	T	F	T

Since $[(p \rightarrow q) \wedge (p \rightarrow r) \wedge q] \rightarrow r$ is not a tautology, hence the given argument is invalid. ..... **Ans.**

Ex.41. Check the validity of the following argument.

If it rains today, then we will not have a party today.

If we do not have a party today, then we will have a party tomorrow.

∴ If it rains today, then we will have a party tomorrow. ***IT: S-08, CS/CT/CE: W-08***

Soln. Let p : It rains today

q : We will have a party today

r : We will have a party tomorrow

$\therefore S_1 : p \rightarrow \sim q$

$S_2 : \sim q \rightarrow r$

$\therefore S_3 : p \rightarrow r$

The given argument will be valid if

$(S_1 \wedge S_2) \rightarrow S_3$ is a tautology

i.e. $[(p \rightarrow \sim q) \wedge (\sim q \rightarrow r)] \rightarrow (p \rightarrow r)$ is a tautology

p	q	r	~q	(1) $p \rightarrow \sim q$	(2) $\sim q \rightarrow r$	(3) $(1) \wedge (2)$	(4) $p \rightarrow r$	$(3) \rightarrow (4)$
T	T	T	F	F	T	F	T	T
T	T	F	F	F	T	F	F	T
T	F	T	T	T	T	T	T	T
T	F	F	T	T	F	F	F	T
F	T	T	F	T	T	T	T	T
F	T	F	F	T	T	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	F	F	T	T

Since $[(p \rightarrow \sim q) \wedge (\sim q \rightarrow r)] \rightarrow (p \rightarrow r)$ is a tautology, hence the given argument is valid. ..... **Ans.**

Ex.42. Determine the validity of the following arguments using truth table:

If you invest in the stock market, then you will get rich. If you get rich then you will be happy.

You get happy.

∴ You invest in the stock market. ***IT : W-08, S-12***

Soln. Let p : You invest in the stock market

q : You get rich

r : You will be happy

$\therefore S_1 : p \rightarrow q$

$S_2 : q \rightarrow r$

$S_3 : q$

$\therefore S_4 : p$

The given argument will be valid if

$(S_1 \wedge S_2 \wedge S_3) \rightarrow S_4$ is a tautology

i.e. $[(p \rightarrow q) \wedge (q \rightarrow r) \wedge q] \rightarrow p$ is a tautology

p	q	r	(1) $p \rightarrow q$	(2) $q \rightarrow r$	(3) $(1) \wedge (2) \wedge q$	$(3) \rightarrow p$
T	T	T	T	T	T	T
T	T	F	T	F	F	T
T	F	T	F	T	F	T
T	F	F	F	T	F	T
F	T	T	T	T	T	F
F	T	F	T	F	F	T
F	F	T	T	T	F	T
F	F	F	T	T	F	T

Since $[(p \rightarrow q) \wedge (q \rightarrow r) \wedge q] \rightarrow p$ is not a tautology, hence the given argument is invalid.

..... **Ans.**

Ex.43. Determine the validity of the following argument :

If I get the job and work hard, then I will get promoted.

If I get promoted, then I will be happy.

I will not be happy.

$\therefore$ Either I will not get the job or I will not work hard. **IT: S-09**

Soln. Let p : I get the job

q : I work hard

r : I will get promoted

s : I will be happy

$\therefore S_1 : (p \wedge q) \rightarrow r$

$S_2 : r \rightarrow s$

$S_3 : \sim s$

$\therefore S_4 : \sim p \vee \sim q$

The given argument will be valid if

$(S_1 \wedge S_2 \wedge S_3) \rightarrow S_4$ is a tautology

i.e. $[((p \wedge q) \rightarrow r) \wedge (r \rightarrow s) \wedge \sim s] \rightarrow (\sim p \vee \sim q)$ is a tautology

p	q	r	s	~p	~q	~s	(1) $p \wedge q$	(2) $(1) \rightarrow r$	(3) $r \rightarrow s$	(4) $(2) \wedge (3) \wedge \sim s$	(5) $\sim p \vee \sim q$	$(4) \rightarrow (5)$
T	T	T	T	F	F	F	T	T	T	F	F	T
T	T	T	F	F	F	T	T	T	F	F	F	T
T	T	F	T	F	F	F	T	F	T	F	F	T
T	T	F	F	F	F	T	T	F	T	F	F	T
T	F	T	T	F	T	F	F	T	T	F	T	T
T	F	T	F	F	T	T	F	T	F	F	T	T
T	F	F	T	F	T	F	F	T	T	F	T	T
T	F	F	F	F	T	T	F	T	T	T	T	T
F	T	T	T	T	F	F	F	T	T	F	T	T
F	T	T	F	T	F	T	F	T	F	F	T	T
F	T	F	T	T	F	F	F	T	T	F	T	T
F	T	F	F	T	F	T	F	T	T	T	T	T
F	F	T	T	T	T	F	F	T	T	F	T	T
F	F	T	F	T	T	T	F	T	F	F	T	T
F	F	F	T	T	T	F	F	T	T	F	T	T
F	F	F	F	T	T	T	F	T	T	T	T	T

Since $(S_1 \wedge S_2 \wedge S_3) \rightarrow S_4$ is a tautology, hence the given argument is valid. ..... **Ans.**

Ex.44. State whether the following arguments are valid or not. If it is valid, identify the tautology on which it is based.

If I drive to work then I will arrived tired.

If I am not arrived tired then I drive to work.

Therefore I do not drive to work. **IT: W-09, S-10**

Soln. Let p : I drive to work

q : I will arrived tired

$\therefore S_1 : p \to q$

$S_2 : \sim q \to p$

$\therefore S : \sim p$

The given argument will be valid if $(S_1 \wedge S_2) \to S$ is a tautology.

i.e. $[(p \to q) \wedge (\sim q \to p)] \to \sim p$ is a tautology

p	q	~p	~q	(1) $p \to q$	(2) $\sim q \to p$	(3) (1)∧(2)	(3)→~p
T	T	F	F	T	T	T	F
T	F	F	T	F	T	F	T
F	T	T	F	T	T	T	T
F	F	T	T	T	F	F	T

Since $(S_1 \wedge S_2) \to S$ is not a tautology, hence the given argument is invalid. ..... **Ans.**

Ex.45. Check the validity of the following argument :

If I try hard and I have a talent, then I will become a scientist.

If I become scientist, then I will be happy. Therefore, if I will not be happy, then I did not try hard or I do not have talent. **CS/CT/CE : W-10**

Soln. Let p : I try hard

q : I have talent

r : I will become a scientist

s : I will be happy.

$\therefore S_1 : (p \wedge q) \to r$

$S_2 : r \to s$

$S_3 : \sim s \to (\sim p \vee \sim q)$

The given argument will be valid if $(S_1 \wedge S_2) \to S_3$ is a tautology.

i.e. $[((p \wedge q) \to r) \wedge (r \to s)] \to [(\sim s \to (\sim p \vee \sim q))]$ is a tautology

p	q	r	s	~p	~q	~s	(1) $p \wedge q$	(2) (1)→r	(3) r→s	(4) (2)∧(3)	(5) ~p∨~q	(6) ~s→(5)	(7) (4)→(6)
T	T	T	T	F	F	F	T	T	T	T	F	T	T
T	T	T	F	F	F	T	T	T	F	F	F	F	T
T	T	F	T	F	F	F	T	F	T	F	F	T	T
T	T	F	F	F	F	T	T	F	T	F	F	F	T
T	F	T	T	F	T	F	F	T	T	T	T	T	T
T	F	T	F	F	T	T	F	T	F	F	T	T	T
T	F	F	T	F	T	F	F	T	T	T	T	T	T
T	F	F	F	F	T	T	F	T	T	T	T	T	T
F	T	T	T	T	F	F	F	T	T	T	T	T	T
F	T	T	F	T	F	T	F	T	F	F	T	T	T
F	T	F	T	T	F	F	F	T	T	T	T	T	T
F	T	F	F	T	F	T	F	T	T	T	T	T	T
F	F	T	T	T	T	F	F	T	T	T	T	T	T
F	F	T	F	T	T	T	F	T	F	F	T	T	T
F	F	F	T	T	T	F	F	T	T	T	T	T	T
F	F	F	F	T	T	T	F	T	T	T	T	T	T

Since $(S_1 \wedge S_2) \to S_3$ is a tautology, hence the given statement is valid. ..... **Ans.**

Ex.46. Determine validity of the following argument using truth table. If I like Discrete Mathematics then I will study. Either I don't study or I pass Discrete Mathematics.

If don't graduate then I didn't pass mathematics.

∴ If I like Discrete Mathematics then I will graduate.

CS/CT/CE :S-11

Soln. Let p : I like DMGT

q : I will study

r : I pass DMGT

s : I will graduate

$\therefore S_1 : p \to q$

$S_2 : \sim q \vee r$

$S_3 : \sim s \to \sim r$

$\therefore S : p \to s$

The given argument will be valid if

$(S_1 \wedge S_2 \wedge S_3) \to S$ is a tautology.

i.e. $[(p \to q) \wedge (\sim q \vee r) \wedge (\sim s \to \sim r)] \to (p \to s)$ is a tautology.

p	q	r	s	~q	~r	~s	(1) p→q	(2) ~q∨r	(3) ~s→~r	(4) (1)∧(2) ∧(3)	(5) p→s	(6) (4)→(5)
T	T	T	T	F	F	F	T	T	T	T	T	T
T	T	T	F	F	F	T	T	T	F	F	F	T
T	T	F	T	F	T	F	T	F	T	F	T	T
T	T	F	F	F	T	T	T	F	T	F	F	T
T	F	T	T	T	F	F	F	T	T	F	T	T
T	F	T	F	T	F	T	F	T	F	F	F	T
T	F	F	T	T	T	F	F	T	T	F	T	T
T	F	F	F	T	T	T	F	T	T	F	F	T
F	T	T	T	F	F	F	T	T	T	T	T	T
F	T	T	F	F	F	T	T	T	F	F	T	T
F	T	F	T	F	T	F	T	F	T	F	T	T
F	T	F	F	F	T	T	T	F	T	F	T	T
F	F	T	T	T	F	F	T	T	T	T	T	T
F	F	T	F	T	F	T	T	T	F	F	T	T
F	F	F	T	T	T	F	T	T	T	T	T	T
F	F	F	F	T	T	T	T	T	T	T	T	T

Since $(S_1 \wedge S_2 \wedge S_3) \to S_3$ is a tautology, hence the given statement is valid. **..... Ans.**

Ex.47. Test the validity of the following arguments.

If the initialization is correct and the loop terminates, then the required output is obtained. The output has been obtained Therefore, if the initialization is correct. The loop must terminate.

IT : S-11

Soln. Let p : the initialization is correct

p : The loop terminates

r : he output has been obtained

$\therefore S_1 : (p \wedge q) \to r$

$S_2 : r$

$\therefore S : p \to q$

The given argument will be valid if

$(S_1 \wedge S_2) \to S$ is a tautology.

i.e. $[((p \wedge q) \to r) \wedge r] \to (p \to q)$ is a tautology.

p	q	r	(1) p∧q	(2) (1)→r	(3) (2)∧r	(4) p→q	(3)→(4)
T	T	T	T	T	T	T	T
T	T	F	T	F	F	T	T
T	F	T	F	T	T	F	F
T	F	F	F	T	F	F	T
F	T	T	F	T	T	T	T
F	T	F	F	T	F	T	T
F	F	T	F	T	T	T	T
F	F	F	F	T	F	T	T

Since $(S_1 \wedge S_2) \to S$ is not a tautology, hence the given argument is invalid. **.... Ans.**

Ex.48. Check the validity of the following argument

If Roli has completed MCA or MBA, then she is assured of a good job.

If Roli is assured of a good job, she is happy.

Roli is not happy.

So Roli has not completed MBA.

CS/CT/CE: W-09, IT: W-12

Soln. Let p : Roli has completed MCA or MBA

q : She is assured a good job

r : She is happy

$\therefore S_1 : p \to q$

$S_2 : q \to r$

$S_3 : \sim r$

$\therefore S : \sim p$

The given argument will be valid, if

$(S_1 \wedge S_2 \wedge S_3) \to S$ is a tautology

i.e. $[(p \to q) \wedge (q \to r) \wedge \sim r] \to \sim p$ is a tautology

p	q	r	~p	~r	(1) p→q	(2) q→r	(3) (1)∧(2)∧~r	(3)→~p
T	T	T	F	F	T	T	F	T
T	T	F	F	T	T	F	F	T
T	F	T	F	F	F	T	F	T
T	F	F	F	T	F	T	F	T
F	T	T	T	F	T	T	F	T
F	T	F	T	T	T	F	F	T
F	F	T	T	F	T	T	F	T
F	F	F	T	T	T	T	F	T

Since $(S_1 \wedge S_2 \wedge S_3) \to S$ is a tautology, hence the given argument is valid. **..... Ans.**

METHODS OF PROOF

Ex.49. Prove that the sum of two odd integers is even.

Soln. Let m and n are two odd number

$\therefore \exists$ integers p and q such that

$m = 2p + 1$ and $n = 2q + 1$

$\therefore m + n = (2p + 1) + (2q + 1)$

$= 2p + 2q + 2$

$= 2(p + q + 1)$

$= 2$ (integer)

$\therefore$ Sum of two odd integers is even **Ans.**

Ex.50. Prove that $\sqrt{2}$ or $\sqrt{3}$ is an irrati ıal number.

CS/CT/CE : W-06,10, S-08

IT : W-07, 11, 12, S-09, 12, 13

OR Prove that there is no rational number $\frac{p}{q}$ whose square is 2.

Soln. Suppose $\sqrt{2}$ is a rational number. Then there exists integers p and q such that $\sqrt{2} = p/q$, $q \neq 0$ and p and q can not have a common factor.

Squaring both side, we get

$2 = \frac{p^2}{q^2}$

$\Rightarrow p^2 = 2q^2$ ...(1)

$\Rightarrow p^2$ is an even number

$\Rightarrow$ p is an even number

$\therefore \exists$ an integer m such that $p = 2m$, then $p^2 = 4m^2$ ...(2)

From (1) and (2), we get

$2q^2 = 4m^2$

$\Rightarrow q^2 = 2m^2$

$\Rightarrow q^2$ is an even number

$\Rightarrow$ q is an even number

Thus p and q both are even numbers. Hence p and q both are multiples of 2. Thus p and q have a common factor 2, which leads to a contradiction.

$\therefore$ Our assumption that $\sqrt{2}$ is a rational number is wrong.

$\therefore \sqrt{2}$ is an irrational number.

On similar grounds, we can show that $\sqrt{3}$ is also an irrational number. **.....Ans.**

Ex.51. Show by contrapositive method :

x^2 is an even number if and only if x is an even number.

Soln. Let p: x^2 is an even number.

q : x is an even number

To show : $p \leftrightarrow q$

We show that $(p \to q) \wedge (q \to p)$ $(p \to q)$

If part :

To show : $p \to q$

i.e. **to prove:** $\sim q \to \sim p$

Given : $\sim q$: x is an odd number

To show : $\sim p$: x^2 is an odd number

Now,

Since x is an odd number, then $\exists m \in Z$ such that

$x = 2m+1 \Rightarrow x^2 = (2m+1)^2$

$x^2 = 4m^2 + 4m + 1$

$x^2 = 2(2m^2 + 2m) + 1$

$= 2$ (integer) $+ 1$

$\Rightarrow x^2$ is an odd number

$\Rightarrow \sim p$

$\therefore \sim q \rightarrow \sim p = p \rightarrow q$

Only if part :

To show: $q \rightarrow p$

Given : x is an even number

To show : x^2 is an even number

Since x is an even number then $\exists n \in Z$ such that

$x = 2n \Rightarrow x^2 = 4n^2$

$\Rightarrow x^2 = 2(2n^2)$

$\Rightarrow x^2$ is an even number. ..... **Ans.**

Ex.52. Show that the product of two odd numbers is odd.

Soln. Let m and n are two odd numbers. Then there exists p and $q \in Z$ such that $m = 2p + 1$ and $n = 2q + 1$

Now $mn = (2p+1)(2q+1)$

$= 4pq + 2pq) + 1$

$= 2[2pq + (p+q)] + 1$

$= 2$ (integer) $+ 1$

= odd number.

Thus product of two odd numbers is odd. .... **Ans.**

Ex.53. Prove that sum of five consecutive numbers is divisible by 5. ***IT : S-08***

Soln. Let x, x + 1, x + 2, x + 3 and x + 4 be five consecutive number. Then its

Sum $= x + x + 1 + x + 2 + x + 3 + x + 4$

$= 5x + 10$

$= 5(x + 2)$

$\therefore$ Sum of 5 consecutive numbers is a multiple of 5 and, hence the sum is divisible by 5. .....**Ans.**

QUANTIFIELS

Predicate Calculus :

The part of declarative sentence describing the properties of an object relation among objects is called predicate. In logic, predicate can be obtained by removing any nouns from a statement. Capital letters are used to denote the predicate and small letters used to denote the name of the individuals or objects.

For example : Manish is an Engineer

Here x is a Manish and P(x) denotes that the he is Engineer.

Thus x P(x) denotes that Manish is an engineer. So x is known as predicate variable and P(x) is known as propositional function, because each choice of x produces a proposition P(x) that is either true or false.

Quantifier :

Quantifiers are words that refer the quantities such as 'some' or 'all' or there exists' and indicate how frequently a certain statement is true.

Following are the types of Quantifiers :

(i) Universal quantifier

(ii) Existential quantifier.

(i) Universal Quantifier :

The phrase 'for all' or 'for every' (denoted by $\forall$) is called universal quantifier.

For example : All teachers are scholars.

Let P(x) : x is a teacher

Then above sentence can written as $\forall \times P(x)$

(ii) Existential Quantifier :

The pharse "There exist" or 'some' (denoted by $\exists$) is called existential quantifiers.

For example : There exists a student such that he is clever.

Let p(x) : x is a student

Then above statement can be written as $\exists x : P(x)$.

Ex.54. Let P(x) : x is a rational number

Q(x) : x is an even number

S(x) : x is divisible by 2

Express the following using quantifiers.

(a) All rational number are real numbers.

(b) Every even number is divisible by 2.

Soln. The given sentences can be written as

(a) For all x, if x is rational number then it is real number, $\forall x [P(x) \rightarrow Q(x)]$.

(b) For all x, if x is an even number, then it is divisible by 2, $\forall x [R(x) \rightarrow S(x)]$. ..... **Ans.**

Ex.48. Check the validity of the following argument

If Roli has completed MCA or MBA, then she is assured of a good job.

If Roli is assured of a good job, she is happy.

Roli is not happy.

So Roli has not completed MBA.

CS/CT/CE: W-09, IT: W-12

Soln. Let p : Roli has completed MCA or MBA

q : She is assured a good job

r : She is happy

$\therefore S_1 : p \to q$

$S_2 : q \to r$

$S_3 : \sim r$

$\therefore S : \sim p$

The given argument will be valid, if

$(S_1 \wedge S_2 \wedge S_3) \to S$ is a tautology

i.e. $[(p \to q) \wedge (q \to r) \wedge \sim r] \to \sim p$ is a tautology

p	q	r	~p	~r	(1) p→q	(2) q→r	(3) (1)∧(2)∧~r	(3)→~p
T	T	T	F	F	T	T	F	T
T	T	F	F	T	T	F	F	T
T	F	T	F	F	F	T	F	T
T	F	F	F	T	F	T	F	T
F	T	T	T	F	T	T	F	T
F	T	F	T	T	T	F	F	T
F	F	T	T	F	T	T	F	T
F	F	F	T	T	T	T	F	T

Since $(S_1 \wedge S_2 \wedge S_3) \to S$ is a tautology, hence the given argument is valid. **..... Ans.**

METHODS OF PROOF

Ex.49. Prove that the sum of two odd integers is even.

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$\therefore m + n = (2p + 1) + (2q + 1)$

$= 2p + 2q + 2$

$= 2(p + q + 1)$

$= 2$ (integer)

$\therefore$ Sum of two odd integers is even **Ans.**

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CS/CT/CE : W-06,10, S-08

IT : W-07, 11, 12, S-09, 12, 13

OR Prove that there is no rational number $\frac{p}{q}$ whose square is 2.

Soln. Suppose $\sqrt{2}$ is a rational number. Then there exists integers p and q such that $\sqrt{2} = p/q$, $q \neq 0$ and p and q can not have a common factor.

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$\Rightarrow p^2$ is an even number

$\Rightarrow$ p is an even number

$\therefore \exists$ an integer m such that $p = 2m$, then $p^2 = 4m^2$ ...(2)

From (1) and (2), we get

$2q^2 = 4m^2$

$\Rightarrow q^2 = 2m^2$

$\Rightarrow q^2$ is an even number

$\Rightarrow$ q is an even number

Thus p and q both are even numbers. Hence p and q both are multiples of 2. Thus p and q have a common factor 2, which leads to a contradiction.

$\therefore$ Our assumption that $\sqrt{2}$ is a rational number is wrong.

$\therefore \sqrt{2}$ is an irrational number.

On similar grounds, we can show that $\sqrt{3}$ is also an irrational number. **.....Ans.**

Ex.51. Show by contrapositive method :

x^2 is an even number if and only if x is an even number.

Soln. Let $p : x^2$ is an even number.

$q : x$ is an even number

To show : $p \leftrightarrow q$

We show that $(p \to q) \wedge (q \to p)$ $(p \to q)$

If part :

To show : $p \to q$

i.e. to prove: $\sim q \to \sim p$

Given : $\sim q$: x is an odd number

To show : $\sim p : x^2$ is an odd number

Now,

Since x is an odd number, then $\exists m \in Z$ such that

$x = 2m+1 \Rightarrow x^2 = (2m+1)^2$

$x^2 = 4m^2 + 4m + 1$

$x^2 = 2(2m^2 + 2m) + 1$

$= 2$ (integer) $+ 1$

$\Rightarrow x^2$ is an odd number

$\Rightarrow \sim p$

$\therefore \sim q \rightarrow \sim p = p \rightarrow q$

Only if part :

To show: $q \rightarrow p$

Given : x is an even number

To show : x^2 is an even number

Since x is an even number then $\exists n \in Z$ such that

$x = 2n \Rightarrow x^2 = 4n^2$

$\Rightarrow x^2 = 2(2n^2)$

$\Rightarrow x^2$ is an even number. ..... **Ans.**

Ex.52. Show that the product of two odd numbers is odd.

Soln. Let m and n are two odd numbers. Then there exists p and $q \in Z$ such that $m = 2p + 1$ and $n = 2q + 1$

Now $mn = (2p+1)(2q+1)$

$= 4pq + 2pq) + 1$

$= 2[2pq + (p+q)] + 1$

$= 2$ (integer) $+ 1$

= odd number.

Thus product of two odd numbers is odd. .... **Ans.**

Ex.53. Prove that sum of five consecutive numbers is divisible by 5. ***IT : S-08***

Soln. Let x, x + 1, x + 2, x + 3 and x + 4 be five consecutive number. Then its

Sum $= x + x + 1 + x + 2 + x + 3 + x + 4$

$= 5x + 10$

$= 5(x + 2)$

$\therefore$ Sum of 5 consecutive numbers is a multiple of 5 and, hence the sum is divisible by 5. .....**Ans.**

QUANTIFIELS

Predicate Calculus :

The part of declarative sentence describing the properties of an object relation among objects is called predicate. In logic, predicate can be obtained by removing any nouns from a statement. Capital letters are used to denote the predicate and small letters used to denote the name of the individuals or objects.

For example : Manish is an Engineer

Here x is a Manish and P(x) denotes that the he is Engineer.

Thus x P(x) denotes that Manish is an engineer. So x is known as predicate variable and P(x) is known as propositional function, because each choice of x produces a proposition P(x) that is either true or false.

Quantifier :

Quantifiers are words that refer the quantities such as 'some' or 'all' or there exists' and indicate how frequently a certain statement is true.

Following are the types of Quantifiers :

(i) Universal quantifier

(ii) Existential quantifier.

(i) Universal Quantifier :

The phrase 'for all' or 'for every' (denoted by $\forall$) is called universal quantifier.

For example : All teachers are scholars.

Let P(x) : x is a teacher

Then above sentence can written as $\forall \times P(x)$

(ii) Existential Quantifier :

The pharse "There exist" or 'some' (denoted by $\exists$) is called existential quantifiers.

For example : There exists a student such that he is clever.

Let p(x) : x is a student

Then above statement can be written as $\exists x : P(x)$.

Ex.54. Let P(x) : x is a rational number

Q(x) : x is an even number

S(x) : x is divisible by 2

Express the following using quantifiers.

(a) All rational number are real numbers.

(b) Every even number is divisible by 2.

Soln. The given sentences can be written as

(a) For all x, if x is rational number then it is real number, $\forall x [P(x) \rightarrow Q(x)]$.

(b) For all x, if x is an even number, then it is divisible by 2, $\forall x [R(x) \rightarrow S(x)]$. ..... **Ans.**

Ex.55. Symbolize the following:

(i) All birds can fly.

(ii) All babies are innocent.

(iii) Not all odd numbers are prime.

(iv) Not all men are happy.

Soln.

(i) Let $P(x)$: x is a bird

$Q(x)$: x can fly

Then given statement can be written as, for all x, if x is a bird then x can fly

Symbolically it is written as $\forall x\,[P(x) \to Q(x)]$

.... Ans.

(ii) $R(x)$: x is a baby, $S(x)$: x is innocent.

Then the given statement can be written as, for all x, if x is a baby then x is innocent.

Symbolically it is written as $\forall x\,[P(x) \to Q(x)]$

..... Ans.

(iii) Let $T(x)$: x is an odd number.

$U(x)$: x is prime

Then we have, there is x, x is an odd and x is prime

Symbolically it is written as $\exists x\,[T(x) \wedge U(x)]$

Hence the given statement symbolically can be written as $\sim \exists x\,[T(x) \wedge U(x)]$ **..... Ans.**

(iv) Let $V(x)$: x is man

$W(x)$: x is happy

Then we have, there is x, x is a man and x is happy

Symbolically it is written as $\forall x\,[V(x) \to W(x)]$

hence the given statement symbolically can be written as $\sim \forall x\,[V(x) \to W(x)]$. **..... Ans.**

Ex.56. Let A = {1, 2, 3, 4, 5, 6}. Determine the truth value of each of the following statements.

(a) $(\exists x \in A)(x + 3 = 10)$

(b) $(\forall x \in A)(x + 3) < 10$

(c) $(\forall x \in A)(x + 3 \le 7)$.

Soln.

(a) False, because no number in A is a solution to $x + 3 = 10$

(b) True, because every number in A satisfies $x + 3 < 10$

(c) False, because if $x = 5$, $x + 3 \le 7$ is not satisfied.

Negation of Quantified Statement :

Consider the statement, "all teachers are scholars". Its negation is " It negation is "It is not the case that all teachers are scholars" or equivalently.

"There exists atleast one teacher who is not a scholar"

Symbolically, using M to denote the set of teachers, the above statement can be written as

$\sim[(\forall x \in M)$ (x is a scholar)]

$= (\exists x \in M)$ (x is not a scholar)

or when $P(x)$ denotes "x is a scholar".

$\sim[(\forall x \in M)\,P(x)] = (\exists x \in M)[\sim P(x)]$.

Ex.57. Negate the following statements and hence write them symbolically.

(1) for all positive integers n, we have $n + 2 > 8$.

(2) Every four legged man is happy.

(3) For every man, there is a woman who loves him.

IT : S-06

Soln.

(1) There exists a positive integer n such that $n + 2 \le 8$

Let N denotes the set of positive integers, then

$\sim[(\forall n \in N)(n + 2) > 8] = (\exists x \in N)(n + 2 \le 8)$

(2) Let M denote the set of four legged an

Then $(\forall x \in M)$ (x is happy)

Its negation,

There exists atleast one four legged man who is unhappy"

i.e. $\sim[(\forall x \in M)$(x is happy)$] = (\exists x \in M)$

(x is unhappy)

(3) Let M denotes the set of man

Then $(\forall x \in M)$ (x is loved by a woman)

Its negations is

There exists a man such that no woman loves him

$\sim[(\forall x \in M)$ (x is loved by woman)]

$= (\exists x \in M)$ [x is not loved by woman]

Negation of above statements can also be written as:

(1) It is not the case that for every positive integer n, $n + 2 > 8$

(2) It is not the case that every four legged man is happy.

(3) It is not the case that for every man there is a woman who loves him. **..... Ans.**

Negation of Quantifiers:

Statement	Negation
$(\forall(x)A(x)$	$(\exists x(\sim A(x))$
$\exists[\sim A(x)]$	$(\forall x)A(x)$
$(\forall x)(\sim A(x))$	$(\exists x)A(x)$
$(\exists x)[A(x)]$	$(\forall x)(\sim A(x))$

Ex.58. Negate the following statement.

For all real numbers x, if $x \ge 5$ then $x^2 > 9$.

Soln. $P(x): x \ge 5$

$Q(x): x^2 > 9$

Given statement is $\forall x\,[P(x) \to Q(x)]$

Its negation is

$\sim\{(\forall x)[P(x) \rightarrow Q(x)]\}$

$= (\exists x)[P(x) \wedge \sim Q(x)]$

$\therefore$ There exists a real number x such that $x \geq 5$ and $x^2 > 9$. .... **Ans.**

Ex.59. Write the negation of the 'for every number x there is a number y such that y < x'. **CS/CT/CE : S-10**

Soln. Let M be any number

Then given statement is

$(\forall x \in M)$ (there is a number y such that $y < x$)

Its negation is

There is a number x for which no y exists such that $y < x$. .... **Ans.**

Ex.59.(a) Translate each of the following statement into symbolic, using quantifiers : **IT : S-09**

(i) There is a student who can speak Tamil and who knows C^{++}.

(ii) There is a student who can speak Tamil but does not know C^{++}.

(iii) Every student either can speak Tamil or knows C^{++}.

(iv) No student can speak Tamil or knows C^{++}.

Soln. Here p : There is a student who can speak Tamil.

q: There is a student who knows C^{++}.

Symbolically :

(i) $p \wedge q$

(ii) $p \wedge \sim q$

(iii) $p \vee q$

(iv) $\sim p \vee q$ .... **Ans.**

BASIC CONCEPTS OF SET THEORY, POWERSET

A set is usually defined as an unordered collection of well defined objects or things. The objects or things are called elements or members of the set.

For example : The collection of all states in India, the collection of all integers, the collection of fruits in the market, the collection of all the students in a college etc are all sets.

Notation :

The symbols $\in$ (read as belongs to) is used to denote the membership in a set. The symbol $\notin$ (read as does not belong to) denotes that a particular element is not a member of a set. Sets are usually denoted by capital letters.

Let A = {a, b, c, d, e}

then $c \in A, d \in A, x \notin A$

Specifying sets :

There are two different ways of specifying sets.

(i) Listing method or Roaster Method:

In this method all the elements of the set are separated by commas and the elements are unique.

For example: A = {1, 6, 4, 3}

(ii) Set Builder Method or Ruler Method :

In this method all the elements of the set are denoted by x where x satisfies certain property.

For example :

Let a = {x | x has the property}

Here the set A denotes all the elements x such that x has the property P.

For example :

Let $A = \{x \mid x \in R\}$

i.e. $A = \{\ldots, -3, -2, -1, 0, 1, 2, 3 \ldots\}$

where R is the set of real numbers

$D = \{y \mid 0 \leq y \leq 3, y \in Z\}$

where Z is the set of integers

i.e. D = {0, 1, 2, 3}.

TYPES OF SETS

(i) Finite Set :

A set A is said to be finite, if it contains finite number of elements. Finite sets are also called numerable sets.

For example :

A = { 0, 1, 2, 3), $B = \{x \mid 0 \leq x \leq 2, x \in Z\}$ are finite sets.

(ii) Infinte set :

A set A is said to be infinite, if it contains infinite number of elements.

For example :

A = {0, 1, 2, 3, ...}, B = {x | x is a real number} are infinite sets.

(iii) Singleton Set :

A set A is said to be singleton set, if it contains only one element.

For example :

A = {0}, D = {7} are singleton sets.

(iv) Null set or Empty Set :

A set A is said to be null set or empty set, if it contains no elements. It is denoted by ϕ where $\phi = \{\}$.

(v) Subset :

A set A is said to be a subset of the set B written as $A \subseteq B$, if and only if every element of A is also an element of B. We also say that A is contained in B or B contains A ($B \supseteq A$).

Symbolically, If $x \in A \Rightarrow x \in B$, then $A \subseteq B$.

For example :

Let $A = \{1, 2, 3, 4\}$,

$B = \{1, 2, 3, 4, 5, 6\}$

Here $A \subseteq B$ because every element of A is also an element of B.

If $A \subseteq B$, then it may be possible that $A = B$. This is the reason why we write $A \subseteq B$.

Note ;

(i) Every set is a subset of itself.

i.e. $A \subseteq A, B \subseteq B, C \subseteq C$ etc.

(ii) Empty set ϕ is the subset of all sets.

(vi) Proper subset :

A set A is called a proper subset of the set B if (i) A is the subset of B and (ii) B is not a subset of A. In other words, A is said to be a proper subset of B if every element of A belongs to the set B, but there is atleast one element of B which does not belong to A. If A is a proper subset of B, then it is denoted by $A \subset B$.

For example :

Let $A = \{x, y, z, w\}$

$B = \{x, y, z, w, r, s, t\}$

Here $A \subseteq B$ but $B \not\subseteq A$

$\therefore A \subset B$

(vii) Equal sets :

two non-empty sets A and B are said to be equal, if

(i) $A \subseteq B$

(ii) $B \subseteq A$

i.e. every element of A is an element of B and every element of B is an element of A.

For example :

If $A = \{1, 2, 3, 4\}$

and $B = \{3, 2, 1, 4\}$, then $A = B$

(viii) Universal sets :

All the sets are likely to be considered as a subset of a particular set. This set is called universal set and is denoted by U.

(ix) Power set :

For a given non-empty set A, the collection of all subsets of A is called the power set of A and is denoted by P(A).

For example :

Let $A = \{1, 2, 3\}$.

The subsets of A are $\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, (1, 2, 3\}$

$\therefore P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

Note :

Let a set A contains n elements. Then P(A) contains 2^n elements.

(x) Super set :

If A is a subset of B, then B is called superset of A.

For example :

Let $A = \{a, b, c\}$, $B = \{a, c, b, d\}$

Here $A \subseteq B$

$\therefore$ B is a superset of A

i.e. $B \supseteq A$

Note :

Universal set is the superset of all sets.

i.e. if A, B, C are any sets, then $U \supseteq A, U \supseteq B, U \supseteq C$ etc.

(xi) Disjoint sets :

Two non-empty sets A and B are said to be disjoint if they have no element in common.

For example :

Let $A = \{1, 2, 3\}, B] = \{-3, 4, 6\}$

Here A and B have no common element.

$\therefore$ A and B are disjoint sets.

(xii) Comparable and Non-comparable sets :

Let A and B be any two sets. Then A and B are said to be comparable if $A \subset B$ or $B \subset A$. Also A and B are said to be non-comparable if $A \not\subset B$ and $B \not\subset A$.

(xiii) Multiset and Multiplicity :

Multiset is a collection of objects, where the objects may not be distinct.

Let A be any set and $x \in A$. Then the multiplicity of x is defined to be the number of times the element x appears in the set A.

For example :

$A = \{1, 1, 2, 3, 3, 4, 5\}$ is a multiset.

Here multiplicity of 1 is 3,

multiplicity of 3 is 2 etc.

Also $B = \{x, x, x, y, z, z, w, w, w, w\}$ is a multiset.

Here multiplicity of x is 3

Multiplicity of y is 1

Multiplicity of z is 2

Multiplicity of w is 4

Note :

If A and B are multisets, then $A \cup B$ and $A \cap B$ are also multisets.

Let A = {1, 1, 2, 2, 3, 4, 4} and B = {1, 1, 2, 3, 3, 4, 4}

Then

$A \cup B = \{1, 1, 2, 2, 3, 3, 4, 4\}$ and

$A \cap B = \{1, 1, 2, 3, 4, 4\}$

(xiv) Partition of a set :

Let A be any non-empty set. Then A can be partitioned into several subsets such that each element of A is exactly in one of the subsets of A.

For example :

Let A = {1, 2, 3, 4, 5, 6, 7, 8}

then {1, 2, 3}, {4}, {5, 6}, {7, 8} are the different partitions of A.

Properties of sets :

(1) If A is any set, then $A \subseteq A$

(2) if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$ where A, B, C are sets.

OPERATIONS ON SETS

(i) Union of sets :

Let A and B be any two non-empty sets. Then the union of A and B denoted by $A \cup B$, is a set of all elements which belongs to A or B or both.

For example :

Let A = {1, 3, 5}

B = {1, 2, 4, 6},

Then $A \cup B$ = {1, 2, 3, 4, 5, 6}

(ii) Intersection of sets :

Let A and B be any two non-empty sets. Then the intersection of A and B denoted by $A \cap B$, is a set of all elements which belong to both A and B.

For example :

Let A = {a, b, d, g},

B = {b, d, f, e},

then $A \cap B$ = {b, d}

(iii) Complement of a set :

Let A be any non-empty set. Then $A \subseteq U$, where U is the universal set. Then the complement of A is denoted by A^C or A' or $\overline{A}$ and is defined as the collection of all elements which belong to U but does not belong to A.

i.e. $A'(A^C \text{ or } \overline{A}) = \{x \mid x \in U, x \notin A\}$

For example :

Let A = {1, 2, 3, 4}

U = {1, 2, 3, 4, 5, 6, ...}

Then

$A'(A^C \text{ or } \overline{A})$ = {5, 6, 7, 8, ...}

Note :

(1) The empty set ϕ is the complement of universal set U.

(2) The universal set U is the complement of empty set ϕ

(iv) Relative complement or difference of sets :

Let A and B be the subsets of the universal set U. Then the difference of A and B, denoted by A – B, is the set containing all the elements of A but not containing the elements of B.

i.e. $A - B = \{x \mid x \in A, x \in B\}$

For example :

Let A = {p, q, r, s}

B = {p, a, s, t, y}

Then

A – B = {q, r}, B – A – {a, t, y}

(v) Symmetric Difference : **IT: W-06**

The symmetric difference of two non-empty sets A and B is denoted by $A \Delta B$ or $A \oplus B$ and is defined as the union of A – B and B – A.

i.e. $A \Delta B (A \oplus B) = (A - B) \cup (B - A)$

$= (A \cap B') \cup (B \cap A')$

For example :

Let A = {a, b, c}, B = {b, c, d},

then A – B = {a},

B – A = {d}

$\therefore$ $A \Delta B (A \oplus B) = (A - B) \cup (B - A) = \{a, d\}$

PRINCIPLE OF DUALITY

The principle of duality states that any established result involving sets and complements and operations of union and intersection gives a corresponding dual result by replacing U by ϕ and $\cup$ by $\cap$ $\cup$ and vice-versa.

For example :

Consider $A \cup \overline{A} = U$

By principle of duality, $A \cap \overline{A} = \phi$

LAWS OF ALGEBRA ON SETS

(1) $A \cup A = A$

(2) $A \cap A = A$

(3) $A \cup \phi = A$

(4) $A \cup U = U$

(5) Commutative laws :

(i) $A \cup B = B \cup A$

(ii) $A \cap B = B \cap A$

(6) Associative laws :

(i) $(A \cup B) \cup C = A \cup (B \cup C)$

(ii) $(A \cap B) \cap C = A \cap (B \cap C)$

(7) Distributive Laws :

(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(8) Absorption Laws :

(i) $(A \cap B) \cup A = A$

(ii) $(A \cup B) \cap A = A$

Proof : (1) To prove : $A \cup A = A$

We prove that $A \cup A \subseteq A$ and $A \subseteq A \cup A$

Let $x \in A \cup A$

i.e. $x \in A$ or $x \in A$

i.e. $x \in A$

$\therefore\ x \in A \cup A \Rightarrow x \in A$

$A \cup A \subseteq A$ .....(a)

Let $x \in A$

i.e. $x \in A$ or $x \in A$

i.e. $x \in A \cup A$

$x \in A \Rightarrow x \in A \cup A$

$\therefore\ A \subseteq A \cup A$ .....(b)

From (a) and (b), we get

$\boxed{A \cup A = A}$

(2) To prove : $A \cap A = A$

We prove that $A \cap A \subseteq A$ and $A \subseteq A \cap A$

Let $x \in A \cap A$

i.e. $x \in A$ and $x \in A$

i.e. $x \in A$

$\therefore\ x \in A \cap A \Rightarrow x \in A$

$\therefore\ A \cap A \subseteq A$ ....(a)

Let $x \in A$

i.e. $x \in A$ and $x \in A$

i.e. $x \in A \cap A$

$\therefore\ x \in A \Rightarrow x \in A \cap A$

$\therefore\ A \subseteq A \cap A$ .....(b)

From (a) and (b), we get

$\boxed{A \cap A = A}$

(3) To prove : $A \cup \phi = A$

We prove that $A \cup \phi \subseteq A$ and $A \subseteq A \cup \phi$

Let $x \in A \cup \phi$

i.e. $x \in A$ or $x \in \phi$

i.e. $x \in A$

$\therefore\ x \in A \cup \phi \Rightarrow x \in A$

$\therefore\ A \cup \phi \subseteq A$ .....(a)

Clearly, $A \subseteq A \cup \phi$ .....(b)

From (a) and (b), we get

$\boxed{A \cup \phi = A}$

(4) To prove : $A \cup U = U$

We prove that $A \cup U \subseteq U$ and $U \subseteq A \cup U$

We know that every set is a subset of the universal set

i.e. $(A \cup U) \subseteq U$

Also $U \subseteq (A \cup U)$

$\therefore\ \boxed{A \cup U = U}$

(5) (i) To prove : $A \cup B = B \cup A$

We prove that $A \cup B \subseteq B \cup A$ and $B \cup A \subseteq A \cup B$

Let $x \in A \cup B$

i.e. $x \in A$ or $x \in B$

i.e. $x \in B$ or $x \in A$

i.e. $x \in B \cup A$

$\therefore\ x \in A \cup B \Rightarrow x \in B \cup A$

$\therefore\ A \cup B \subseteq B \cup A$ ....(a)

Let $x \in B \cup A$

i.e. $x \in B$ or $x \in A$

i.e. $x \in A$ or $x \in B$

i.e. $x \in A \cup B$

$\therefore\ x \in B \cup A \Rightarrow x \in A \cup B$

$\therefore\ B \cup A \subseteq A \cup B$ ....(b)

From (a) and (b), we get

$A \cup B = B \cup A$

(ii) To prove : $A \cap B = B \cap A$

We prove that $A \cap B \subseteq B \cap A$ and

$B \cap A \subseteq A \cap B$

Let $x \in A \cap B$

i.e. $x \in A$ and $x \in B$

i.e. $x \in B$ and $x \in A$

i.e. $x \in B \cap A$

$\therefore\ x \in A \cap B \Rightarrow x \in B \cap A$

$\therefore\ A \cap B \subseteq B \cap A$ ....(a)

Let $x \in B \cap A$

i.e. $x \in B$ and $x \in A$

i.e. $x \in A$ and $x \in B$

i.e. $x \in A \cap B$

$\therefore\ x \in B \cap A \Rightarrow x \in A \cap A$

$B \cap A \subseteq A \cap B$ ....(b)

From (a) and (b), we get

$\boxed{A \cap B = B \cap A}$

(6) (i) To prove : $(A \cup B) \cup C = A \cup (B \cup C)$

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We prove that $(A \cup B) \cup C \subseteq A \cup (B \cup C)$ and

$A \cup (B \cup C) \subseteq (A \cup B) \cup C$

Let $x \in (A \cup B) \cup C$

i.e. $x \in (A \cup B)$ or $x \in C$

i.e. $(x \in A$ or $x \in B)$ or $x \in C$

i.e. $x \in A$ or $(x \in B$ or $x \in C)$

i.e. $x \in A$ or $(x \in B \cup C)$

i.e. $x \in A \cup (B \cup C)$

$\therefore\ x \in (A \cup B) \cup C \Rightarrow x \in A \cup (B \cup C)$

$\therefore\ (A \cup B) \cup C \subseteq A \cup (B \cup C)$ ....(a)

Let $x \in A \cup (B \cup C)$

i.e. $x \in A$ or $x \in (B \cup C)$

i.e. $x \in A$ or $(x \in B$ or $x \in C)$

i.e. $(x \in A$ or $x \in B)$ or $x \in C$

i.e. $(x \in A \cup B)$ or $x \in C$

i.e. $x \in (A \cup B) \cup C$

$\therefore\ x \in A \cup (B \cup C) \Rightarrow x \in (A \cup B) \cup C$

$\therefore\ A \cup (B \cup C) \subseteq (A \cup B) \cup C$ .....(b)

From (a) and (b), we get

$\boxed{(A \cup B) \cup C = A \cup (B \cup C)}$

(ii) To prove : $(A \cap B) \cap C = A \cap (B \cap C)$

We prove that $(A \cap B) \cap C \subseteq A \cap (B \cap C)$

$A \cap (B \cap C) \subseteq (A \cap B) \cap C$

Let $x \in (A \cap B) \cap C$

i.e. $(x \in A$ and $x \in B)$ and $x \in C$

i.e. $x \in A$ and $(x \in B$ and $x \in C)$

i.e. $x \in A \cap (B \cap C)$

$\therefore\ x \in (A \cap B) \cap C \Rightarrow x \in A \cap (B \cap C)$

$\therefore\ (A \cap B) \cap C \subseteq A \cap (B \cap C)$ .....(a)

Let $x \in A \cap (B \cap C)$

i.e. $x \in A$ and $x \in (B \cap C)$

i.e. $(x \in A$ and $x \in B)$ and $x \in C$

i.e. $x \in (A \cap B)$ and $x \in C$

$\therefore\ x \in A \cap (B \cap C) \Rightarrow x \in (A \cap B) \cap C$

$\therefore\ A \cap (B \cap C) \subseteq (A \cap B) \cap C$ .....(b)

From (a) and (b), we get

$\boxed{(A \cap B) \cap C = A \cap (B \cap C)}$

(7) (i) To prove : $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

CS/CT/CE : S-12

We prove that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ and

$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

Let $x \in A \cap (B \cup C)$

i.e. $x \in A$ and $x \in (B \cup C)$

i.e. $x \in A$ and $(x \in B$ or $x \in C)$

i.e. $(x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$

i.e. $x \in (A \cap B)$ or $x \in (A \cap C)$

i.e. $x \in (A \cap B) \cup (A \cap C)$

$\therefore\ x \in A \cap (B \cup C) \Rightarrow x \in (A \cap B) \cup (A \cap C)$

$\therefore\ A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ ....(a)

Let $x \in (A \cap B) \cup (A \cap C)$

i.e. $x \in (A \cap B)$ or $x \in (A \cap C)$

i.e. $(x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$

i.e. $x \in A$ and $(x \in B$ or $x \in C)$

i.e. $x \in A$ and $x \in (B \cup C)$

i.e. $x \in A \cap (B \cup C)$

$\therefore\ x \in (A \cap B) \cup (A \cap C) \Rightarrow x \in A \cap (B \cup C)$

$\therefore\ (A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ ....(b)

From (a) and (b), we get

$\boxed{A \cap (B \cup C) = (A \cap B) \cup (A \cap C)}$

(ii) To prove : $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

CS/CT/CE : W-10, IT : W-10

We prove L.H.S. $\subseteq$ R.H.S. and R.H.S. $\subseteq$ L.H.S.

Let $x \in A \cup (B \cap C)$

$\Rightarrow x \in A$ or $x \in B \cap C$

$\Rightarrow x \in A$ or $x \in B$ and $x \in C$

$\Rightarrow (x \in A$ or $x \in B)$ and $(x \in A$ and $x \in C)$

$\Rightarrow x \in A \cup B$ and $x \in A \cup C$

$\Rightarrow x \in (A \cup B) \cap (A \cup C)$

$\therefore$ L.H.S. $\subseteq$ R.H.S.

Let $x \in (A \cup B) \cap (A \cup C)$

$\Rightarrow x \in A \cup B$ and $x \in A \cup C$

$\Rightarrow x \in A$ or $x \in B$ and $x \in A$ or $x \in C$

$\Rightarrow x \in A$ or $x \in B$ and $x \in C$

$\Rightarrow x \in A$ or $x \in B \cap C$

$\Rightarrow x \in A \cup (B \cap C)$

$\therefore$ R.H.S. $\subseteq$ L.H.S.

$\therefore$ $\boxed{A \cup (B \cap C) = (A \cup B) \cap (A \cup C)}$

(8) (i) To prove : $(A \cap B) \cup A = A$

L.H.S. $= (A \cap B) \cup A$

$= (A \cap B) \cup (A \cap U)$ *[$\because A \cup U = A$]*

$= A \cap (B \cup U)$ ..... (by distributive law)

$= A \cap U$ *[$\because B \cup U = U$]*

$= A$

= R.H.S.

$\therefore$ $\boxed{(A \cap B) \cup A = A}$

(ii) To prove : $(A \cup B) \cap A = A$

L.H.S. $= (A \cup B) \cap A$

$= A \cap (A \cup B)$ ..... (by commutative law)

$= A \cap (B \cup A)$

$= (A \cap B) \cup (A \cap A)$ ..... (by distribution law)

$= (A \cap B) \cup A$ *[$\because$ By (2)]*

$= A$ *[$\because$ By 8(i)]*

= R.H.S.

$\therefore$ $\boxed{(A \cup B) \cap A = A}$

DE MORGAN'S LAWS

(1) $(A \cup B)' = A' \cap B'$

(2) $(A \cap B)' = A' \cup B'$

CS/CT/CE: S-09,12, W-11, IT: S-07, W-11

Proof : (i) To prove : $(A \cup B)' = A' \cap B'$

We prove that

$(A \cup B)' \subseteq A' \cap B'$ and $A' \cap B' \subseteq (A \cup B)'$

Let $x \in (A \cup B)'$

i.e. $x \notin A \cup B$

i.e. $x \notin A$ or $x \notin B$

i.e. $x \in A'$ and $x \in B'$

i.e. $x \in A' \cap B'$

$\therefore x \in (A \cup B)' \Rightarrow x \in A' \cap B'$

$\therefore (A \cup B)' \subseteq A \cap B'$ ....(a)

Let $x \in A' \cap B'$

i.e. $x \in A'$ and $x \in B'$

i.e. $x \notin A$ or $x \notin B$

i.e. $x \notin (A \cup B)$

i.e. $x \in (A \cup B)'$

$\therefore x \in A' \cap B' \Rightarrow x \in (A \cup B)'$

$\therefore A' \cap B' \subseteq (A \cup B)'$ .....(b)

From (a) and (b), we get

$\boxed{(A \cup B)' = A' \cap B'}$

(2) To prove : $(A \cap B)' = A' \cup B'$

We prove that $(A \cap B)' \subseteq A' \cup B'$ and

$A' \cup B' \subseteq (A \cap B)'$

Let $x \in (A \cap B)'$

i.e. $x \notin A \cap B$

i.e. $x \notin A$ or $x \notin B$

i.e. $x \in A'$ or $x \in B'$

i.e. $x \in A' \cup B'$

$\therefore x \in (A \cap B)' \Rightarrow x \in A' \cup B'$

$\therefore (A \cap B)' \subseteq A' \cup B'$ .....(a)

Let $x \in A' \cup B'$

i.e. $x \in A'$ or $x \in B'$

i.e. $x \in A$ and $x \notin B$

i.e. $x \notin A \cap B$

i.e. $x \in (A \cap B)'$

$\therefore x \in A' \cup B' \Rightarrow x \in (A \cap B)'$

$\therefore A' \cup B' \subseteq (A \cap B)'$ ....(b)

From (a) and (b), we get

$\boxed{(A \cap B)' = A' \cup B'}$

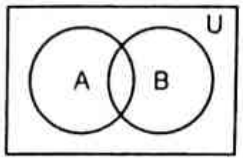
VENN DIAGRAMS

The operations on sets can be illustrated with the help of Venn Diagrams. A Venn Diagram is a pictorial representation of sets and is a useful device which expresses the relationship between sets. The universal set U is represented by the set of points inside and on the boundary of a single closed curve, which is usually a rectangle. The subsets of universal set U are usually represented by circles.

We can produce a geometrical interpretation involving sets and operations on sets using Venn diagrams. However, we should not that, Venn diagrams simply illustrate the possibility but, but do not prove the truth of a relationship between sets.

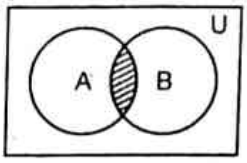
Some examples of Venn diagrams are given below.

(i)



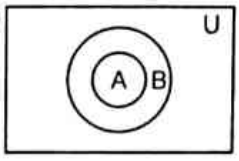
$A \cup B$

(ii)



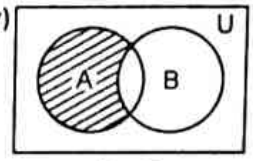
$A \cap B$

(iii)



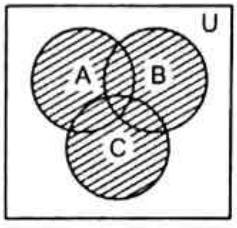
$A \subset B$

(iv)



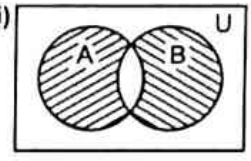
$A - B$

(v)



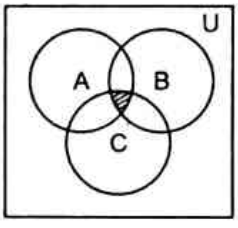
$A \cap (B \cup C) = (A \cup B) \cup C$

(vi)



$A \Delta B = (A - B) \cup (B - A)$

(vii)



$A \cap (B \cap C) = (A \cap B) \cap C$

Cardinality :

The number of elements in a finite set A is called the cardinality of A. It is denoted by n(A) or |A|.

For examples :

Let A = {a, b, c, d, e, x, y, z}

Then n (A) = 8

IMPORTANT FORMULAE :

(1) $n(A \cup B) = n(A) + n(B)$, if $A \cap B = \phi$

(2) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$,

if $A \cap B \neq \phi$

(3) $n(A - B) = n(A) - (A \cap B)$

(4) $n(A \cup B \cup C) = n(A) + n(B) + n(C)$

$- n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$.

Set theory :

(5) $A \cup A = A$

(6) $A \cap A = A$

(7) $A \cup \phi = A$

(8) $A \cup U = U$

(9) Commtative Laws :

(i) $A \cup B = B \cup A$

(ii) $A \cap B = B \cap A$

(10) Associative Laws :

(i) $(A \cup B) \cup C = A \cup (B \cup C)$

(ii) $(A \cap B) \cap C = A \cap (B \cap C)$

(11) Distributive laws :

(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(12) Absorption laws :

(i) $(A \cap B) \cup A = A$

(ii) $(A \cup B) \cap A = A$

(13) De-Morgan's Laws :

(i) $(A \cup B)' = A' \cap B'$

(ii) $(A \cap B)' = A' \cup B'$

Ex.60. Prove that: $A - (B \cup C) = (A - B) \cap (A - C)$

CS/CT/CE : S-07, 13, W-11

Soln. We prove that L.H.S. $\subseteq$ R.H.S. and R.H.S. $\subseteq$ L.H.S.

Let $x \in A - (B \cup C)$

i.e. $x \in A$ and $x \notin B \cup C$

i.e. $x \in A$ and $x \notin B$ or $x \notin C$

i.e. $x \in A - B$ and $x \in A - C$

i.e. $x \in (A - B) \cap (A - C)$

$\therefore$ $x \in A - (B \cup C) \Rightarrow x \in (A - B) \cap (A - C)$

$\therefore$ L.H.S. $\subseteq$ R.H.S. ....(a)

Let $x \in (A - B) \cap (A - C)$

i.e. $x \in A - B$ and $x \in A - C$

i.e. $x \in A$ and $x \notin B$ and $x \in A$ and $x \notin C$

i.e. $x \in A$ and $x \notin B$. $x \notin C$

i.e. $x \in A$ and $x \notin B \cup C$

i.e. $x \in A - (B \cup C)$

$\therefore$ $x \in (A - B) \cap (A - C) \Rightarrow x \in A - (B \cup C)$

$\therefore$ R.H.S. $\subseteq$ L.H.S. .....(b)

From (a) and (b), we get

$\boxed{A - (B \cup C) = (A - B) \cap (A - C)}$ **Ans.**

Ex.61. Prove that : $A - (B \cap C) = (A - B) \cup (A - C)$

CS/CT/CE :S-06,09,13, W-10

Soln. We prove that L.H.S. $\subseteq$ R.H.S. and R.H.S $\subseteq$ L.H.S.

Let $x \in A - (B \cap C)$, then

$x \in A$ and $x \notin B \cap C$

$\Rightarrow$ ($x \in A$ and $x \notin B$) or ($x \in A$ and $x \notin C$)

$\Rightarrow x \in (A - B)$ or $x \in (A - C)$

$\Rightarrow x \in (A - B) \cup (A - C)$

$\therefore$ $A - (B \cap C) \subseteq (A - B) \cup (A - C)$

$\therefore$ L.H.S. $\subseteq$ R.H.S. ...(a)

Let $x \in (A-B) \cup (A-C)$, then

$x \in (A-B)$ or $x \in (A-C)$

$\Rightarrow (x \in A$ and $x \notin B)$ or $(x \in A$ and $x \notin C)$

$\Rightarrow x \in A$ and $(x \notin B$ or $x \notin C)$

$\Rightarrow x \in A$ and $x \notin (B \cap C)$

$\Rightarrow x \in A-(B \cap C)$

$\therefore (A-B) \cup (A-C) \subseteq A-(B \cap C)$

$\therefore$ R. H. S. $\subseteq$ L. H. S.

$\therefore$ $\boxed{A-(B \cap C) = (A-B) \cup (A-C)}$ **Ans.**

Ex.62. Prove that : $(A \cap B) \cup (A \cap B') = A$

IT : S-07, W-12, CS/CT/CE: S-10

Soln. We prove this result by Laws of Algebra

Consider L.H.S. = $(A \cap B) \cup (A \cap B')$

$= A \cap (B \cup B')$ ..... (by distributive law)

$= A \cap U$

$= A =$ R.H.S.

$\therefore$ $\boxed{(A \cap B) \cup (A \cap B') = A}$ **Ans.**

Ex.63. Prove that : $A \cap (B-C) = (A \cup B) - (A \cup C)$

CS/CE/CT : S-10

Soln. We prove that L.H.S. $\subseteq$ R.H.S. and R.H.S. $\subseteq$ L.H.S.

Let $x \in A \cap (B-C)$

i.e. $x \in A$ and $x \in B-C$

i.e. $x \in A$ and $x \in B$ and $x \notin C$

i.e. $x \in A, x \in B$ and $x \in A, x \notin C$

i.e. $x \in (A \cup B)$ and $x \notin (A \cup C)$

i.e. $x \in (A \cup B) - (A \cup C)$

$\therefore x \in A \cap (B-C) \Rightarrow x \in (A \cup B) - (A \cup C)$

$\therefore$ L.H.S. $\subseteq$ R.H.S. ....(a)

Let $x \in (A \cup B) - (A \cup C)$

i.e. $x \in A \cup B$ and $x \notin A \cup C$

i.e. $x \in A$ or $x \in B$ and $x \notin A$ or $x \notin C$

But $x \in A$

i.e. $x \in A$ and $x \in B$, $x \notin C$

i.e. $x \in A$ and $x \in B-C$

i.e. $x \in A \cap (B-C)$

$\therefore x \in (A \cup B) - (A \cup C) \Rightarrow x \in A \cap (B-C)$

$\therefore$ R.H.S. $\subseteq$ L.H.S. ...(b)

From (a) and (b), we get

$\boxed{A \cap (B-C) = (A \cup B) - (A \cup C)}$ **Ans.**

Ex.64. Prove that :

$A \cup (B-C) = (A \cap B) - (A \cap C)$

Soln. We prove that L.H.S. $\subseteq$ R.H.S. and R.H.S. $\subseteq$ L.H.S.

Let $x \in A \cup (B-C)$

i.e. $x \in A$ or $x \in (B-C)$

i.e. $x \in A$ or $x \in B$ and $x \notin C$

i.e. $x \in A$ and $x \in B$ and $x \in A$ and $x \notin C$

i.e. $x \in (A \cap B)$ and $x \in A$ and $x \notin C$

i.e. $x \in (A \cap B)$ and $x \notin (A \cap C)$

i.e. $x \in (A \cap B) - (A \cap C)$

$\therefore x \in A \cup (B-C) \Rightarrow x \in (A \cap B) - (A \cap C)$

$\therefore A \cup (B-C) \subseteq (A \cap B) - (A \cap C)$ ....(a)

$\therefore$ L.H.S. $\subseteq$ R.H.S.

Now, let $x \in (A \cap B) - (A \cap C)$

i.e. $x \in (A \cap B)$ and $x \notin (A \cap C)$

i.e. $(x \in A$ and $x \in B)$ and $(x \notin A$ and $x \notin C)$

$\therefore x \in A$ or $x \in B$ and $x \notin C$

$\therefore x \in A$ or $x \in B-C$

$\therefore x \in A \cup (B-C)$

$\therefore x \in (A \cap B) - (A \cap C)$

$\Rightarrow x \in A \cup (B-C)$

$\therefore (A \cap B) - (A \cap C) \subseteq A \cup (B-C)$

$\therefore$ $\boxed{(A \cap B) - (A \cap C) = A \cup (B-C)}$ **Ans.**

Ex.65. Prove that : $(A \cap B) - C = (A-C) \cap (B \cup C)$.

Soln. We prove that L.H.S. $\subseteq$ R.H.S. and R.H.S. $\subseteq$ L.H.S.

Let $x \in (A \cap B) - C$

i.e. $x \in A \cap B$ and $x \notin C$

i.e. $x \in A$ and $x \in B$ and $x \notin C$

i.e. $x \in A$ and $x \notin C$ and $x \in B$ and $x \notin C$

i.e. $x \in A-C$ and $x \in B-C$

i.e. $x \in (A-C) \cap (B-C)$

$\therefore x \in (A \cap B) - C$

$\Rightarrow x \in (A-C) \cap (B-C)$

$\therefore$ L.H.S. $\subseteq$ R.H.S. ..... (a)

Let $x \in (A-C) \cap (B-C)$

i.e. $x \in A-C$ and $x \in B-C$

i.e. $x \in A$ and $x \notin C$ and $x \in B$ and $x \notin C$

i.e. $x \in A$ and $x \in B$ and $x \notin C$

i.e. $x \in (A \cap B)$ and $x \notin C$

i.e. $x \in (A \cap B) - C$

$\therefore\ x \in (A - C) \cap (B - C) \Rightarrow x \in (A \cap B) - C$ .... (b)

From (a) and (b), we get

$\boxed{(A \cap B) - C = (A - C) \cap (B - C)}$ **Ans.**

Ex.66. Prove that the following statements are equivalent.

(i) $A \subset B$

(ii) $A \cap B' = \phi$

(iii) $A \cap B = A$

(iv) $A \cup B = B$

(v) $B' \subset A'$ ***IT : W-07, S-12***

Soln. In order to prove that the given statements are equivalent, we prove that

(i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), (iii) $\Rightarrow$ (iv),

(iv) $\Rightarrow$ (v), (v) $\Rightarrow$ (i),

To prove : (i) $\Rightarrow$ (ii) :

i.e. **Given :** $A \subset B$

To prove : $A \cap B' = \phi$

If possible, let $A \cap B' \neq \phi$

Let $x \in A \cap B'$

i.e. $x \in A$ and $x \in B'$

i.e. $x \in B$ and $x \in B'$ $[\because A \subset B, x \in B \Rightarrow x \in B]$

But this is a contradiction,

$\therefore\ A \cap B' = \phi$

To prove : (ii) $\Rightarrow$ (iii) :

i.e. **Given :** $A \cap B' = \phi$

To prove : $A \cap B = A$

We prove that $A \cap B \subseteq A$ and $A \subseteq A \cap B$

$A \cap B \subseteq A$ is trivially true

We now prove that $A \subseteq A \cap B$

Let $x \in A$ that $x \notin B$

i.e. $x \in A$ and $x \cap B'$

i.e. $x \in A \cap B'$

But this is a contradiction as $A \cap B' = \phi$

$\therefore\ x \in A$ and $x \in B$

i.e. $x \in A \cap B$

$\therefore\ A \subseteq A \cap B$

$\therefore\ A \cap B = A$

To prove: (iii) $\Rightarrow$ (iv) :

i.e. **Given :** $A \cap B = A$

To prove : $A \cup B = B$

We prove that $A \cup B = B$

We prove that $A \cup B \subseteq B$ and $B \subseteq A \cup B$

$B \subseteq A \cup B$ is trivially true.

We now prove that $A \cup B \subseteq B$

Let $x \in A \cup B$

i.e. $x \in A$ or $x \in B$

i.e. $x \in B$

$\therefore\ A \cup B \subseteq B$

$\therefore\ A \cup B = B$

To prove (iv) $\Rightarrow$ (v) :

i.e. **Given :** $A \cup B = B$

To prove : $B' \subset A'$

Since $A \subset A \cup B$ is triviality true

$\therefore\ A \subset B$ $[\because A \cup B = B]$

i.e. $A' \supset B'$

$\therefore\ B' \subset A'$

To prove: (v) $\Rightarrow$ (i) :

i.e. **Given :** $B' \subset A'$

To prove : $A \subset B$

Since $B' \subset A'$

$\therefore\ (B')' \supset (A')'$

i.e. $B \supset A$

$\therefore\ A \subset B$. ..... **Ans.**

Ex. 67. A survey of 500 television watchers produced the following information. 285 watch football games; 115 watch basketball games, 45 watch football and basketball games, 70 watch football and hockey games, 50 watch hockey and basketball games and 50 do not watch any of the three kinds of games. Find:

(i) How many people in the survey watch all three kinds of games?

(i) How many people watch exactly one of the sports?

IT : S-06, S-13, W-06

Soln. Let F, B, H denote football, basketball, hockey games respectively

Given : $n(F) = 285$, $n(B) = 115$, $n(F \cap B) = 45$,

$n(F \cap H) = 70$, $n(B \cap H) = 50$

Since 50 do not watch any of the games

$\therefore\ n(F \cup B \cup H) = 500 - 50 = 450$

We know that

$n(F \cup B \cup H) = n(F) + n(B) + n(H) - n(F \cap B)$
$- n(B \cap H) - n(H \cap F) + n(F \cap B \cap H)$

$\therefore\ 450 = 285 + 115 + n(H) - 45$
$- 50 - 70 + n(F \cap B \cap H)$

$\therefore\ n(F \cap B \cap H) + n(H) = 215$ .....(1)

(i) Since $n(F \cap B) = 45$, $n(F \cap H) = 70$,

$n(B \cap H) = 50$

Hence there cannot be more than 45 watching all three games

$\therefore\ n(F \cap B \cap H) = 45$

(1) $\Rightarrow n(H) = 215 - 45 = 170$

(ii) The number of people watching exactly two of the games

$= n(F \cap B) + n(F \cap H) + n(B \cap H)$
$- 3 \times n(F \cap B \cap H)$

$= 45 + 70 + 50 - 3 \times 45$

$= 165 - 135 = 30$ ..... **Ans.**

Ex.68. A sample of 80 people have revealed that 24 like cinema and 62 like T.V. programmes. Find the number of people who like both. ***IT :S-07***

Soln. Let A be the set of people who like cinema

$\therefore\ n(A) = 24$

Let B be the set of people who like T.V. programmes

$\therefore\ n(B) = 62$

Also $n(A \cup B) = 80$

We know that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\therefore\ 80 = 24 + 62 - n(A \cap B)$

$\therefore\ n(A \cap B) = 6$

$\therefore$ The number of people who like both cinema and T.V. programmes is 6 **Ans.**

Ex.69.(a) If $A \cup B = A \cup C$, must $B = C$? Explain. ***IT: S-06***

Soln. Given: $A \cup B = A \cup C$

Let $x \in A \cup B$, then $x \in A \cup C$

i.e. $x \in A$ or $x \in B$, $x \in A$ or $x \in C$

i.e. $x \in B$, $x \in C$

Hence every element of B is also an element of C and vice-versa.

$\therefore$ $\boxed{B = C}$ **Ans.**

Ex.69.(b) Suppose that $A \oplus B = A \oplus C$.
Does this guarantee that $B = C$?
Justify your conclusion. ***IT: W-06***

Soln. If $A \oplus B = A \oplus C$, then it is not necessary that $B = C$

For example: Consider $A = \{a, b, c, d\}$
$B = \{c, d, e, f\}$
$C = \{d, e, f\}$

Hence, $A \oplus B = \{a, b, e, f\}$

and $A \oplus C = \{a, b, e, f\}$

Here, $A \oplus B = A \oplus C$, but $B \neq C$. .... **Ans.**

Ex.70. For any sets A and B, prove that
$A - B = A - (A \cap B)$. ***CS/CT/CE: S-07, W-09***

Soln. We prove that LHS $\subseteq$ RHS and RHS $\subseteq$ LHS

Let $x \in A - B$

i.e. $x \in A$ and $x \notin B$

i.e. $x \in A$ and $x \in A$, $x \notin B$

i.e. $x \in A$ and $x \notin A \cap B$

i.e. $x \in A - (A \cap B)$

$\therefore\ A - B \subseteq A - (A \cap B)$ ..... (i)

Let $x \in A - (A \cap B)$

i.e. $x \in A$ and $x \notin A$ and $x \notin B$

i.e. $x \in A$ and $x \notin B$

[$\because x \notin A$ contradicts $x \in A$]

i.e. $x \in A - B$

$\therefore\ A - (A \cap B) \subseteq A - B$ .... (ii)

From (i) & (ii), we get

$\boxed{A - B = A - (A \cap B)}$ **Ans.**

Ex.71. In a class of 25 students, 12 have taken mathematics, 8 have taken Mathematics but not Biology. Find the number of students who have taken Mathematics and Biology and those who have taken Biology but not Mathematics. ***IT: W-07***

Soln. Let A and B be sets of students who have taken Maths and Biology respectively.

Then A–B is the set of students who have taken Maths

but not Biology.

Here, $n(A) = 12$, $n(A \cup B) = 25$, $n(A - B) = 8$

We have $n(A - B) + n(A \cap B) = n(A)$

$\therefore\ 8 + n(A \cap B) = 12$

$\therefore\ n(A \cap B) = 4$

$\therefore$ 4 students have taken both Maths and Biology **Ans.**

Also, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\therefore\ 25 = 12 + n(B) - 4$

$\therefore\ n(B) = 17$

Now, $n(B - A) + n(A \cap B) = n(B)$

$\therefore\ n(B - A) = n(B) - n(A \cap B) = 17 - 4 = 13$

$\therefore$ 13 students have taken Biology but not Maths **Ans.**

Ex.72. Prove that:

If $A \cap (B - C) = (A \cap B) - (A \cap C)$, then

$A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$.

CS/CT/CE: W-07

Soln. Consider L.H.S. $= A \cap (B \Delta C)$

$= A \cap [(B - C) \cup (C - B)]$

$= [A \cap (B - C)] \cup [A \cap (C - B)]$

$= [(A \cap B) - (A \cap C)] \cup [(A \cap C) - (A \cap B)]$

$= (A \cap B) \Delta (A \cap C)$

$=$ R.H.S.

$\therefore\ A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$ **Ans.**

Ex.73. Let $U = \{a, b, c, d, e, f\}$

$A = \{a, b, c\}$ and $B = \{b, e, f\}$

Find $B' - A'$, $(A \cap B)'$, $A' \cup B'$.

CS/CT/CE: W-07

Soln. Given: $U = \{a, b, c, d, e, f\}$

$A = \{a, b, c\}$, $B = \{b, e, f\}$

$A' = \{d, e, f\}$, $B' = \{a, c, d\}$

$\therefore\ B' - A' = \{a, c\}$

$\therefore (A \cap B)' = A' \cup B' = \{a, c, d, e, f\}$ **.... Ans.**

Ex.74. Let A be a finite set containing n elements, then prove that the power set of A has exactly 2^n elements. ***IT: W-08***

Soln. We prove this by taking a counter example.

Let $A = \{a, b\}$ i.e. A contains 2 elements.

Here P(A) must contain $2^2 = 4$ elements.

The subsets of A are ϕ, $\{a\}$, $\{b\}$, A

$\therefore P(A) = \{\phi, \{a\}, \{b\}, A\}$

$\therefore$ P(A) contains 4 elements.

Let $A = \{1, 2, 3\}$ i.e. A contains 3 elements.

Hence P(A) must contain $2^3 = 8$ elements.

The subsets of A are ϕ, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, A

$\therefore P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, A\}$

$\therefore$ P(A) contains 8 elements.

Hence, if A contains n elements, then P(A) contains 2^n elements. **.... Ans.**

Ex.75. An investigator interviewed 100 students to determine their preferences for the three drinks- Milk (M), Coffee (C), Tea (T). He reported the following: 10 students had all the three drinks, 20 had 'M' and 'C', 30 had 'C' and 'T', 25 had 'M' and 'T', 12 had 'M' only, 5 had 'C' only and 8 had 'T' only.

(i) How many did not take any of the three drinks?

(ii) How many take milk but not coffee?

(iii) How many take tea and coffee but not milk?

IT: W-08

Soln. Given: $n(M) = 12$, $n(C) = 5$, $n(T) = 8$,

$n(M \cap C) = 20$, $n(C \cap T) = 30$, $n(M \cap T) = 25$,

$n(M \cap C \cap T) = 10$, $n(M \cup C \cup T) = 100$

(i) We have $n(\overline{M} \cap \overline{C} \cap \overline{T}) = n(M)$

$= 12$

(ii) We have $n(M - C) = n(M) - n(M \cap C) = 12 - 20 = 8$

(iii) We have $n((T \cap C) - M) = n(T \cap C) - n(T \cap C \cap M))$

$= 30 - 10 = 20$ **.... Ans.**

Ex.76. Out of 250 candidates who failed in an examination, it was revealed that 128 failed in Mathematics, 87 in Physics and 134 in aggregate. 31 failed in Mathematics and in Physics, 54 failed in aggregate and in Mathematics, 30 failed in aggregate and in Physics. Find how many candidates failed:

(i) in all the three subject.

(ii) in Physics but not in aggregate or in Mathematics.

(iii) in the aggregate or in Mathematics, but not in Physics. ***IT: S-09***

Soln. Let A, B, C denote the sets of candidates who failed in mathematics, physics and aggregate respectively.

Given: $n(A) = 128$, $n(B) = 87$, $n(C) = 134$,

$n(A \cap B) = 31$, $n(A \cap C) = 54$, $n(B \cap C) = 30$,
$n(A \cup B \cup C) = 250$

(i) The number of candidates who failed in all subjects

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

$$\Rightarrow 250 = 128 + 87 + 134 - 31 - 54 - 30 + n(A \cap B \cap C)$$

Hence, $n(A \cap B \cap C) = 16$.

(ii) The number of candidates failed in physics but not in aggregate or in mathematics is

$$n(A' \cap B \cap C') = n(B) - n(A \cap B) - n(B \cap C) + n(A \cap B \cap C)$$

$$= 87 - 31 - 30 + 16 = 42$$

(iii) The number of candidates who failed in aggregate or in mathematics, but not in physics is

$n[(A \cup C) \cap B']$

$$= n(A \cap B') + n(B' \cap C) - n(A \cap B' \cap C)$$

$$= [n(A) - n(A \cap B)] + [n(C) - n(B \cap C)] - [n(A \cap C) - n(A \cap B \cap C)]$$

$$= (128 - 31) + (134 - 30) - (54 - 16)$$

$= 97 + 104 - 38 = 163$. .... **Ans.**

Ex.77. Let $U = \{x : x \in N, 1 \le x \le 12\}$ be the universal set and $A = \{1, 9, 10\}$, $B = \{3, 4, 6, 11, 12\}$ and $C = \{2, 5, 6\}$ are subsets of U. Find the sets.

(i) $(A \cup B) \cap (A \cup C)$

(ii) $A \cup (B \cap C)$

(iii) $(A \cup B \cup C)^c$. **IT: W-09,11**

Soln.

(i) $(A \cup B) \cap (A \cup C)$

$= \{1, 9, 10, 3, 4, 6, 11, 12\} \cap \{1, 9, 10, 2, 5, 6\}$

$= \{1, 9, 10, 6\}$

(ii) $A \cup (B \cap C) = \{1, 9, 10\} \cup \{6\}$

$= \{1, 9, 10, 6\}$

(iii) Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

$A \cup B \cup C = \{1, 9, 10, 3, 4, 6, 11, 12, 2, 5\}$

$\therefore (A \cup B \cup C)^c = \{7, 8\}$ .... **Ans.**

Ex.78. A TV survey shows that 60% people see programme A, 50% see programme B, 50% see programme C, 30% see programme A and B, 20% see programme B and C, 30% see programme A and C, 10% do not see any programme. Find:

(i) What per cent see programme A, B, C?

(ii) What per cent see exactly two programmes?

(iii) What per cent see programme A only? **IT: W-09**

Soln. Let $n(A \cup B \cup C) = 100$, $n(A) = 60$, $n(B) = 50$, $n(C) = 50$, $n(A \cap B) = 30$, $n(B \cap C) = 20$, $n(A \cap C) = 30$, $n(\overline{A} \cap \overline{B} \cap \overline{C}) = 10$

(i) We have

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

$$\therefore 100 = 60 + 50 + 50 - 30 - 20 - 30 + n(A \cap B \cap C)$$

$$\therefore 100 = 80 + n(A \cap B \cap C)$$

$$\therefore n(A \cap B \cap C) = 20$$

(ii) No. of people watching exactly 2 programmes

$$= n(A \cap B) + n(B \cap C) + n(A \cap C) - 3 \times n(A \cap B \cap C)$$

$$= 30 + 20 + 30 - (3 \times 20) = 20$$

(iii) $n(\overline{A} \cap \overline{B} \cap \overline{C}) = n(A) = 60$ ..... **Ans.**

Ex.79. Let A and B be two arbitrary sets show that:

(i) $P(A \cap B) = P(A) \cap P(B)$

(ii) $P(A \cup B) = P(A) \cup P(B)$ **IT: W-09**

Soln. We prove L.H.S. $\subseteq$ R.H.S. and R.H.S. $\subseteq$ L.H.S.

(i) Let $x \in P(A \cap B)$

$\Rightarrow x \subseteq A \cap B$

$\Rightarrow x \subseteq A$ and $x \subseteq B$

$\Rightarrow x \in P(A)$ and $x \in P(B)$

$\Rightarrow x \in P(A) \cap P(B)$

$\therefore P(A \cap B) \subseteq P(A) \cap P(B)$

Let $x \in P(A) \cap P(B)$

$\Rightarrow x \in P(A)$ and $x \in P(B)$

$\Rightarrow x \subseteq A$ and $x \subseteq B$

$\Rightarrow x \subseteq A \cap B \Rightarrow x \in P(A \cap B)$

$\therefore P(A) \cap P(B) \subseteq P(A \cap B)$

$\therefore \boxed{P(A \cap B) = P(A) \cap P(B)}$ **Ans.**

(ii) Let $x \in P(A \cup B)$

$\Rightarrow x \subseteq A \cup B \Rightarrow x \subseteq A$ or $x \subseteq B$

$\Rightarrow x \in P(A)$ or $x \in P(B) \Rightarrow x \in P(A) \cup P(B)$

$\therefore P(A \cup B) \subseteq P(A) \cup P(B)$

Let $x \in P(A) \cup P(B)$

$\Rightarrow x \in P(A)$ or $x \in P(B) \Rightarrow x \subseteq A$ or $x \subseteq B$

$\Rightarrow x \subseteq A \cup B \Rightarrow x \in P(A \cup B)$

$\therefore P(A) \cup P(B) \subseteq P(A \cup B)$

$\therefore \boxed{P(A) \cup P(B) = P(A \cup B)}$ **Ans.**

Ex.80. Using laws of algebra of sets, prove that :

(i) $A' \subset B$, if $A \cup B = U$

(ii) $A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$. **IT: S-10**

Soln. Given :

(i) $A \cup B = U$

$\therefore\ U \cap A' = A'$

$\Rightarrow (A \cup B) \cap A' = A'$

$\Rightarrow (A \cap A') \cup (B \cap A') = A'$

$\Rightarrow \phi \cup (B \cap A') = A'$

$\Rightarrow B \cap A' = A'$

But $B \cap A' \subseteq B \Rightarrow A' \subseteq B$

$\therefore$ $\boxed{A' \subseteq B, \text{ if } A \cup B = U}$ **Ans.**

(ii) $A \cap (B \oplus C) = A \cap \{(B - C) \cup (C - B)\}$

$= \{A \cap (B - C)\} \cup \{A \cap (C - B)\}$

$= \{(A \cap B) - (A \cap C)\} \cup \{(A \cap C) - (A \cap B)\}$

$= (A \cap B) \oplus (A \cap C)$

$\therefore$ $\boxed{A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)}$ **Ans.**

Ex.81. Prove that : $(A - B) \cap B = \phi$. **IT: W-10**

Soln. To prove: $(A - B) \cap B = \phi$

We consider $(A - B) \cap B \neq \phi$

$\Rightarrow x \in (A - B) \cap B$

$\Rightarrow x \in A - B$ and $x \in B$

$\Rightarrow [x \in A$ and $x \notin B]$ and $x \in B$

$\Rightarrow [x \in A'$ and $x \in B]$ and $[x \notin B$ and $x \in B]$

$x \notin B$ and $x \in B$ which is not possible. Hence our assumption must be wrong.

$\therefore$ $\boxed{(A - B) \cap B = \phi}$ **Ans.**

Ex.82. A computer company receives 40 applications for a job of programmers. Among them 25 knew JAVA, 28 knew ORACLE and 7 did not know any of the Languages.

How many of them knew both the languages? **IT:S-11**

Soln. Let A and B denotes the set of application who knew Java and ORACLE.

$n(A \cup B) = 40 - 7 = 33$

Now, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\therefore 33 = 25 + 28 - n(A \cap B)$

$\therefore 33 = 53 - n(A \cap B)$

$\therefore$ $\boxed{n(A \cap B) = 20}$ **Ans.**

Ex.83. Prove that: $A \cap (\overline{A} \cup B) = A \cap B$. **IT: W-12**

Soln. Consider L.H.S. $= A \cap (\overline{A} \cup B)$

$= (A \cap \overline{A}) \cup (A \cap B)$

$= \phi \cup (A \cap B)$

$= A \cap B$

$=$ R.H.S.

$\therefore$ $\boxed{A \cap (\overline{A} \cup B) = A \cap B}$ **Ans.**

Ex.84. Prove that: $A \oplus B = (A \cup B) - (A \cap B)$.

IT: S-13

Soln. We prove this by taking a counter example

Let $A = \{1, 2, 3\}$, $B = \{2, 4, 5\}$

Now, $A - B = \{1, 3\}$, $B - A = \{4, 5\}$

$\therefore A \oplus B = (A - B) \cup (B - A) = \{1, 3, 4, 5\}$ ... (1)

Also, $A \cup B = \{1, 2, 3, 4, 5\}$, $A \cap B = \{2\}$

$\therefore (A \cup B) - (A \cap B) = \{1, 3, 4, 5)$ .... (2)

$\therefore$ (1) and (2) $\Rightarrow A \oplus B = (A \cup B) - (A \cap B)$

$\therefore$ $\boxed{A \oplus B = (A \cup B) - (A \cap B)}$ **Ans.**

Peano Axioms (Peano Postulates):

CS/CT/CE: W-10, S-11

(1) $O \in N$, where N is the set of natural numbers, $O = \phi$

(2) It $n \in N$, then $n^+ \in N$ where $n^+ = n \cup \{n\}$ (Here n^+ means $n + 1$).

(3) If a subset $S \subseteq N$ possess the properties

(i) $O \in S$ and

(ii) If $n \in S$, then $n^+ \in S$

Then $S = N$

PRINCIPLE OF MATHEMATICAL INDUCTION

CS/CT/CE: W-06

The principle of mathematical induction is proving a collection of statements which can be put in one-one correspondence with the set of natural numbers. The above peano axiom (3) is the basis of principle of mathematical induction.

The axiom (3) can be put in the following equivalent form.

If P(n) is any property (or predicate) defined over the set of natural numbers and

(i) P(0) is true

(ii) P(m) is true $\Rightarrow$ P(m + 1) is also true for $m \in N$.

Then P(n) holds true for all $n \in N$. ..... **Ans.**

Ex.85. Prove the following using principle of mathematical induction.

(i) $n < 2^n$

(ii) $2^n < n!,\ n \geq 4$

(iii) $n^3 + 2n$ **is divisible by 3**

(iv) $1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$

CS/CT/CE: W-11, IT: W-12

(v) $2^1 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2$

(vi) $\dfrac{1}{1.2} + \dfrac{1}{2.3} + \dfrac{1}{3.4} + \ldots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$

CS/CT/CE: W-08, S-13

(vii) $3^{4n+2} + 5^{2n+1}$ **is a multiple of 14,** $\forall\, n \geq 0$.

CS/CT/CE: S-11

Soln.

(i) Let $P(n) : n < 2^n$

$P(0) : 0 < 2^0$ i.e. $0 < 1$

$\therefore$ P(0) is true

Let P(n) be true for n = m

i.e. P(m) is true

We prove that P(m + 1) is also true

Now, $P(m) : m < 2^m$

i.e. $m + 1 < 2^m + 1$

i.e. $m + 1 < 2^m + 2^m$ $\quad [\because 1 < 2^m]$

i.e. $m + 1 < 2.2^m$

i.e. $m + 1 < 2^{m+1}$

$\therefore$ P(m + 1) is also true

$\therefore$ By principle of mathematical induction P(n) is true for all $n \in N$. **..... Ans.**

(ii) Let $P(n) : 2^n < n!,\ n \geq 4$

$P(4) : 2^4 < 4!$, i.e. $16 < 24$

$\therefore$ P(4) is true

Let P(n) be true for n = m

i.e. P(m) is true

We prove that P (m + 1) is also true

$P(m) : 2^m < m!$

$\Rightarrow 2.2^m < 2.m!$

$\Rightarrow 2^{m+1} < 2.m!$

$\Rightarrow 2^{m+1} < (m+1)m!$ $\quad [\because 2 < m + 1 \text{ for } m \geq 4]$

$\Rightarrow 2^{m+1} < (m+1)!$

$\therefore$ P(m + 1) is also true

$\therefore$ By principle of mathematical induction P(n) is true for all $n \in N$. **.... Ans.**

(iii) Let $P(n) : (n^3 + 2n)$ is divisible by 3

P(0): (0 + 0) is divisible by 3

i.e. 0 is divisible by 3

$\therefore$ P(0) is true

Let P(n) be true for n = m

i.e. P(m) is true

We prove that P(m + 1) is also true

$P(m) : (m^3 + 2m)$ is divisible by 3

Now $(m+1)^3 + 2(m+1)$

$= m^3 + 3m^2 + 3m + 1 + 2m + 2$

$= m^3 + 3m^2 + 5m + 3$

$= m^3 + 2m + 3m + 3m^2 + 3$

$= (m^3 + 2m) + 3(m^2 + m + 1)$

Both $(m^3 + 2m)$ (i.e. P(m)) and $3(m^2 + m + 1)$ are divisible by 3

$\therefore (m+1)^3 + 2(m+1)$ is divisible by 3

$\therefore$ P(m + 1) is also true

$\therefore$ By principle of mathematical induction P(n) true for all $n \in N$. **.... Ans.**

(iv) Let $P(n) : 1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$

$P(1) : 1 = \dfrac{1(1+1)}{2}$ i.e. $1 = 1$

$\therefore$ P(1) is true

Let P(n) be true for n = m

i.e. P(m) is true

We prove that P(m + 1) is also true

$P(m) : 1 + 2 + 3 + \ldots + m = \dfrac{m(m+1)}{2}$

Consider P(m + 1) :

$$1 + 2 + 3 + \ldots + m + (m+1) = \frac{m(m+1)}{2} + (m+1)$$

$$= \frac{m(m+1) + 2(m+1)}{2}$$

$$= \frac{(m+1)(m+1)}{2}$$

$$= \frac{(m+1)\,[(m+1)+1]}{2}$$

$\therefore$ P(m + 1) is also true

$\therefore$ By principle of mathematical induction P(n) is true for all $n \in N$. **... Ans.**

(v) Let P(n): $2^1 + 2^2 + 2^3 + ... + 2^n = 2^{n+1} - 2$

P(1) : $2^1 + 2^{1+1} - 2$ i.e. $2 = 4 - 2$ i.e. $2 = 2$

$\therefore$ P(1) is true

Let P(n) be true for n = m

i.e. P(m) is true

We prove that P(m + 1) is also true

P(m) : $2^1 + 2^2 + 2^3 + ... + 2^m = 2^{m+1} - 2$

Now, P(m + 1) : $2^1 + 2^2 + 2^3 + ... + 2^m + 2^{m+1}$

$$= 2^{m+1} - 2 + 2^{m+1}$$

$$= 2.2^{m+1} - 2$$

$$= 2.2^{m+2} - 2$$

$$= 2.2^{(m+1)+1} - 2$$

$\therefore$ P(m + 1) is also true

$\therefore$ By principle of mathematical induction P(n) is true for all $n \in N$ **.... Ans.**

(vi) Let P(n): $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}$

P(n) : $\frac{1}{1(1+1)} = \frac{1}{1+1}$ i.e. $\frac{1}{2} = \frac{1}{2}$

$\therefore$ P(1) is true

Let P(n) be true for n = m

i.e. P(m) : $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + ... + \frac{1}{m(m+1)} = \frac{m}{m+1}$

Now, P(m + 1) :

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + ... + \frac{1}{m(m+1)} + \frac{1}{(m+1)(m+2)}$$

$$= \frac{m}{m+1} + \frac{1}{(m+1)(m+2)}$$

$$= \frac{m(m+2)+1}{(m+1)(m+2)}$$

$$= \frac{m^2+2m+1}{(m+1)(m+2)}$$

$$= \frac{(m+1)\,(m+1)}{(m+1)(m+2)}$$

$$= \frac{m+1}{m+2}$$

$$= \frac{m+1}{(m+1)+1}$$

$\therefore$ P(m + 1) is also true

$\therefore$ By principle of mathematical induction P(n) is true for all $n \in N$. **... Ans.**

(vii) Let, P(n) : $3^{4n+2} + 5^{2n+1}$ is a multiple of 14

P(0) : $3^2 + 5^1 = 9 + 5 = 14$

$\therefore$ P(0) is true since 14 is a multiple of 14

Let P(n) be true for n = m

i.e. P(m) is true

We prove that P(m + 1) is also true

Now, P(m): $3^{4m+2} + 5^{2m+1}$ is a multiple of 14 ...(1)

Now, $P(m+1) = 3^{4(m+1)+2} + 5^{2(m+1)+1}$

$$= 3^{4m+6} + 5^{2m+3}$$

$$= 3^{4m+2}.3^4 + 5^{2m+1}.5^2$$

$$= 81.3^{4m+2} + 25.5^{2m+1}$$

$$= 81.3^{4m+2} + 81.5^{2m+1} - 81.5^{2m+1} + 25.5^{2m+1}$$

$$= 81(3^{4m+2} + 5^{2m+1}) - 56.5^{2m+1}$$

$$= 81(3^{4m+2} + 5^{2m+1}) - 14.4.5^{2m+1}$$

Here, $3^{4m+2} + 5^{2m+1}$ is a multiple of 14 *[$\because$ By (1)]*

$\therefore$ P(m + 1) is also true

$\therefore$ By principle of mathematical induction P(n) is true for all values of n. **... Ans.**

Ex.86. Show that:

$$1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)\,(2n+1)}{6},\ n \geq 1$$

by mathematical induction.

CS/CT/CE: S-06,09,12

Soln. Let P(n) : $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)\,(2n+1)}{6}$

P(1) : $1 = \frac{1(1+1)(2+1)}{6} = 1$

$\therefore$ P(1) is true

Let, P(n) be true for n = m i.e. P(m) is true.

We prove that P(m + 1) is also true.

P(m) : $1^2 + 2^2 + ... + m^2 = \frac{m(m+1)\,(2m+1)}{6}$

$\therefore$ P(m + 1) : $1^2 + 2^2 + ... + m^2 + (m+1)^2$

$$= \frac{m(m+1)\,(2m+1)}{6} + (m+1)^2$$

$$= \frac{m(m+1)(2m+1) + 6(m+1)^2}{6}$$

$$= \frac{(m+1)[2m^2 + m + 6m + 6]}{6}$$

$$= \frac{(m+1)[2m^2+7m+6]}{6}$$

$$= \frac{(m+1)(m+2)(2m+3)}{6}$$

$$= \frac{(m+1)((m+1)+1)(2(m+1)+1)}{6}$$

$\therefore$ P(m + 1) is also true

$\therefore$ By induction, P(n) is true for all n. .... **Ans.**

Ex.87. Prove or disprove that 3 divides $(n^3 - n)$ for every positive integer n. **IT: S-06**

Soln. Let $P(n) = n^3 - n$

Now, P(0) = 0 – 0 is divisible by 3

$\therefore$ P(0) is true.

Assume P(m) i.e. $m^3 - m$ is divisible by 3

Now, $P(m+1) = (m+1)^3 - (m+1)$

$$= m^3 + 3m^2 + 3m + 1 - m - 1$$

$$= m^3 - m + 3(m^2 + m)$$

Now by assumption $m^3 - m$ is divisible by 3 and $3(m^2 + m + 1)$ is divisible by 3.

$\therefore$ Their sum is divisible by 3.

i.e. P(m + 1) is true

$\therefore$ P(n) is true for all $n \in N$ .... **Ans.**

Ex.88. Prove by Mathematical induction that the sum of the cubes of three consecutive integers are divisible by Nine. **CS/CT/CE: W-07**

Soln. Let the three consecutive integers be n, n + 1, n + 2, then the statement

$P(n) = n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9

Here, P(1) is true

$\because$ $P(1) = 1^3 + (1+1)^3 + (1+2)^3$

$= 1 + 8 + 27 = 36$ is divisible by 9

We must now show that for $K \geq 0$, if P(K) is true, then P (K + 1) must also be true.

We assume that P(K) :

$K^3 + (K+1)^3 + (K+2)^3 = 9m$ for some integer m.

We now show the truth of P (K + 1) :

$$P(K+1): (K+1)^3 + (K+2)^3 + (K+3)^3 = K^3 + 3K^2 + K^3 + 3K^2 . 2 + 3K . 4 + 8 + 3K + 1 + K^3 + 3K^2 . 3 + 3K . 9 + 27$$

$$= 3K^3 + 18K^2 + 42K + 36$$

is also divisible by 9.

Thus, P(K+1) is true. By the principal of mathematical induction, it follows that P(n) is true for all n. ... **Ans.**

Ex.89. Prove or disprove:

$x^3 > x^2 \forall x$. **CS/CT/CE: W-08**

Soln. Let the given statement be denoted by S(n). Hence S(n) is false for n = 1.

(1) **Inductive base:** for n = 2

$2^3 > 2^2 \Rightarrow 8 > 4$ Hence S(2) is true.

(2) **Inductive hypothesis:** Assume S(k) is true

i.e. $k^3 > k^2$.

(3) **Inductive step:** To show the S(k + 1) is true.

i.e. $(k+1)^3 > (k+1)^2$

Now, $(k+1)^3 = k^3 + 3k^2 + 3k + 1 > k^2 + 3k^2 + 3k + 1$

$> k^2 + 2k + 1$

$\therefore$ $(k+1)^3 > (k+1)^2$

Thus, S(k+1) is true whenever S(k) is true. ... **Ans.**

Ex.90. Prove by mathematical Induction that $6^{n+2} + 7^{2n+1}$ is divisible by 43 for each positive integer n. **IT: S-09**

Soln. Let $P(n) = 6^{n+2} + 7^{2n+1}$

Basis of induction : For n = 1

$P(1) = 6^3 + 7^3 = 216 + 343 = 559 = 13 \times 43$ ...(1)

So, 43 divides P(1)

Induction step, For n = r

$P(r) = 6^{r+2} + 7^{2r+1} = 43 \times S$ ...(2)

Now for n = r + 1

$P(r+1) = 6^{r+2+1} + 7^{2r+3} = 6^{r+3} + 7^{2r+3}$

$= 6.6^{r+2} + 6.7^{2r+1} - 6.7^{2r+1} + 7^2 . 7^{2r+1}$

$= 6[6^{r+2} + 7^{2r+1}] + (43)\, 7^{2r+1}$

$= 6 . 43 . S + 43 . 7^{2r+1}$

$= 43[6S + 7^{2r+1}] = 43 \times t$ ....(3)

As (1), (2) and (3) all are true, hence P(n) is divisible by 43. ..... **Ans.**

Ex.91. Prove by mathematical induction

$$1^2 + 3^2 + 5^2 + ... + (2n-1)^2 = \frac{n(2n+1)(2n-1)}{3}.$$

CS/CT/CE : S-10, W-12

Soln. Let $P(n): 1^2 + 3^2 + 5^2 + ... + (2n-1)^2$

$$= \frac{n(2n+1)(2n-1)}{3}$$

For n = 1

$$P(1): 1 = \frac{1(2+1)(1)}{3}$$

So, 1 = 1

$\therefore$ P(1) is true

We assumed P (n) is true for n = m

$$\therefore P(m): 1^2 + 3^2 + 5^2 + ... + (2m-1)^2$$

$$= \frac{m(2m+1)(2m-1)}{3} \quad ... (1)$$

We prove that P (n) is true for n = m + 1

$$\therefore P(m+1): 1^2 + 3^2 + 5^2 + ... + [2(m+1)-1]^2$$

$$= 1^2 + 3^2 + 5^2 + ... + [2m+1]^2$$

$$= 1^2 + 3^2 + 5^2 + ... + (2m-1)^2 + (2m+1)^2$$

$$= \frac{m(2m+1)(2m-1)}{3} + (2m+1)^2$$

[$\because$ From (1)]

$$= (2m+1)\left[\frac{m(2m-1)+3(2m+1)}{3}\right]$$

$$= (2m+1)\left[\frac{2m^2 - m + 6m + 3}{3}\right]$$

$$= (2m+1)\left[\frac{2m^2 + 5m + 3}{3}\right]$$

$$= \frac{(2m+1)}{3}[2m^2 + 2m + 3m + 3]$$

$$= \frac{(2m+1)}{3}[(2m+3)(m+1)]$$

$$= \frac{(m+1)(2m+1)(2m+3)}{3}$$

$$= \frac{(m+1)(2(m+1)-1)(2(m+1)+1)}{6}$$

$\therefore$ P(m + 1) is true

Hence, P(n) is true for all mathematical induction.

..... Ans.

COMPLETE LIST OF FORMULAE:

(1) $p \leftrightarrow q = (p \rightarrow q) \wedge (q \rightarrow p)$

(2) $T \wedge F = F$ and $T \vee F = T$

(3) $p \wedge F = F$ and $p \vee F = p$

(4) $T \wedge p = p$ and $T \vee p = T$

(5) $\sim(p \wedge q) \equiv \sim p \vee \sim q$

(6) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

(7) $\sim(p \rightarrow q) \equiv p \wedge \sim q$

(8) $\sim(p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \vee \sim p)$

(9) Idempotent laws:

(a) $p \vee p \equiv p$

(b) $p \wedge p \equiv p$

(10) Associative laws:

(a) $p \wedge (q \wedge r) = (p \wedge q) \wedge r$

(b) $p \vee (q \vee r) = (p \vee q) \vee r$

(11) Commutative laws:

(a) $p \wedge q \equiv q \wedge p$

(b) $p \vee q = q \vee p$

(12) Distributive laws:

(a) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

(b) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(13) Identity laws:

(a) $p \vee F \equiv p$

(b) $p \wedge T \equiv p$

(c) $T \vee p \equiv T$

(d) $F \wedge p \equiv F$

(14) Complement laws:

(a) $p \vee \sim p \equiv T$

(b) $p \wedge \sim p = F$

(c) $\sim T = F$ and $\sim F = T$

(15) Absorption laws:

(a) $p \wedge (p \vee q) \equiv p$

(b) $p \vee (p \wedge q) = p$

(16) De-Morgan's laws:

(a) $\sim(p \wedge q) \equiv \sim p \vee \sim q$

(b) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

(17) $n(A \cup B) = n(A) + n(B)$, if $A \cap B = \phi$

(18) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$, if $A \cap B \neq \phi$

(19) $n(A - B) = n(A) - (A \cap B)$

(20) $n(A \cup B \cup C) = n(A) + n(B) + n(C)$
$- n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

Set theory :

(1) $A \cup A = A$

(2) $A \cap A = A$

(3) $A \cup \phi = A$

(4) $A \cup U = U$

(5) Commtative Laws :

(i) $A \cup B = B \cup A$

(ii) $A \cap B = B \cap A$

(6) Associative Laws :

(i) $(A \cup B) \cup C = A \cup (B \cup C)$

(ii) $(A \cap B) \cap C = A \cap (B \cap C)$

(7) Distributive laws :

(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(8) Absorption laws :

(i) $(A \cap B) \cup A = A$

(ii) $(A \cup B) \cap A = A$

(9) De-Morgan's Laws :

(i) $(A \cup B)' = A' \cap B'$

(ii) $(A \cap B)' = A' \cup B'$

PROBLEMS FOR PRACTICE :

Q.1. Show that : $p \Leftrightarrow q \equiv (p \vee q) \Rightarrow (p \wedge q)$

Q.2. If p and q are false, state whether $[(p \wedge q) \wedge \sim p] \vee (\sim q)$ is true or false.

Ans. True.

Q.3. Prove that : (i) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(ii) $p \Leftrightarrow q \equiv (p \rightarrow q) \Rightarrow (q \rightarrow p)$

Q.4. Negate the following :

If he works, then he will earn money.

Ans. He works and he will not earn money.

Q.5. Determine the validity of the following argument

If it rains, then the streets get wet

If it rains, then accidents happen

The streets get wet

$\therefore$ Accidents happen.

Ans. Valid.

UNIT - II
RELATIONS AND FUNCTIONS

SYLLABUS

Relations: Ordered pairs and n-tuples, Product Sets and Partitions, Relations and Digraphs, Matrix of Relation, Paths in Relations and Digraphs, Properties of Relations, Equivalence Relations & Partitions, Compatible Relation, Manipulation of Relations, Composition of Relations, Transitive Closure of a relation, Partial order relation, Partially ordered set, Hasse Diagrams.

Functions: Definition, Composition of functions, Types of Functions, Invertible Function, Permutation Function, Characteristics function of a set with Theorems.

UNIVERSITY PAPER SOLUTIONS
SINCE SUMMER - 2006

Note to the students :

- The questions given below are from previous Nagpur University examination papers.
- Though these questions are from old university papers, they have been included in the new syllabus.
- Questions which are out of new syllabus are not included here.

SUMMER - 06 (CS/CT/CE)

Q.1. Prove that for any three sets A, B, and C.
$A \times (B \cap C) = (A \times B) \cap (A \times C)$. **5M**

Ans. P.2-9, Refer property (4) of Cartesian Products.

Q.2. If $f : A \to B$ and $g : B \to C$ be one-to-one onto functions, then gof is also one-to-one on to and $(g\ of)^{-1} = f^{-1} o g^{-1}$. **7M**

Ans. P.2-33, Refer theorem (2).

Q.3. Consider the set Z of integers. Define a R b by $b = a^r$ for some positive integer 'r'. Show that R is a partial order on Z. **7M**

Ans. P.2-14, Ex-9.

SUMMER- 06 (IT)

Q.1. Define characteristics function. **4M**
Show that :
$f_{A \cup B}(x) = f_A(x) + f_B(x) - f_A(x)\, f_B(x)$.

Ans. P.2-39, Ex-68.

Q.2. Define a relation on the set X = {a, b, c, d} that is
(i) reflexive, symmetric, but not transitive,
(ii) reflexive and transitive, but not symmetric. **6M**

Ans. P.2-14, Ex-10.

Q.3. Let $A = R \times R$. Define the following relation R on A :
(a, b) R (c, d) if and only if $a^2 + b^2 = c^2 + d^2$. **7M**
(i) Show that R is an equivalence relation,
(ii) compute $\frac{A}{R}$.

Ans. P.2-16, Ex-13.

Q.4. Define a function from a set A into set B. **2M**

Ans. P.2-31, Refer definition of function.

Q.5. Let A = B = C = R and consider the functions.
$f : A \to B$ and $g : B \to C$ defined by f (a) = 2a + 1, g(b) = b/3. Verify theorem $(gof)^{-1} = f^{-1}\ o\ g^{-1}$. **7M**

Ans. P.2-36, Ex-58.

Q.6. Let A = {1, 2, 3, 4} and B = {a, b}. How many onto functions can be defined from a set A onto a set B? List functions which are not onto. **4M**

Ans. P.2-36, Ex-59.

WINTER - 06 (CS/CT/CE)

Q.1. Show the following identities for sets using characteristic function:
(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Ans. P.2-39, Ex-67.

(ii) A + (B+C) = (A + B) + C. **6M**

Ans. P.2-38, Ex- 66.

Q.2. Write examples of following relations on a set {1, 2, 3, 4} **3M**
(i) Equivalence Relation,
(ii) Compatibility Relation,
(iii) Partial ordered Relation.

Ans. P.2-24, Ex-34.

Q.3. Let the compatibility relation on a set $\{x_1, x_2, \ldots, x_6\}$ be given by the matrix: **4M**

x_2	1				
x_3	1	1			
x_4	0	0	1		
x_5	0	0	1	1	
x_6	1	0	1	0	1
	x_1	x_2	x_3	x_4	x_5

Draw the graph and find the maximal compatibility blocks of the relation.

Ans. **P.2-23, Ex-33.**

Q.4. Prove that:

If $g: X \to Y$ and $f: Y \to Z$ be 1–1 and onto, then $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$. **5M**

Ans. **P.2-33, Refer theorem (2).**

Q.5. Let A be a given finite set and P(A) its power set. Let $\subseteq$ be the inclusion relation on the elements of P(A). Draw Hasse diagrams of $< P(A), \subseteq >$ for: **6M**

(a) $A = \{a, b\}$

(b) $A = \{a, b, c\}$

(c) $A = \{a\}$

Ans. **P.2-30, Ex-50.**

WINTER - 06 (IT)

Q.1. Let S be the product set $A \times B$. If $|A| = m$ and $|B| = n$, then how many relations are there on S? **4M**

Ans. **P.2-24, Ex-35.**

Q.2. Determine whether the relation R on the set A is reflexive, irreflexive, symmetric, antisymmetric, or transitive. A is the set of all ordered pairs of real numbers : $(a, b) R (c, d) \Leftrightarrow a = c$. **5M**

Ans. **P.2-14, Ex-11.**

Q.3. Define partition of a set A. **4M**

Let $A = \{a, b, c, d, e, f, g, h\}$

Consider the following subsets of A:

$A_1 = \{a, b, c, d\}$, $A_2 = \{a, c, e, f, g, h\}$

$A_3 = \{a, c, e, g\}$, $A_4 = \{b, d\}$

$A_5 = \{f, h\}$

Write all possible partitions for the set A.

Ans. **P.2-15, Ex-11(a).**

Q.4. Let $A = B = C = IR$ (set of reals) and consider the functions $f: A \to B$ and $g: B \to C$ defined by $f(a) = 2a + 1$ and $g(b) = b/3$. Verify $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. **4M**

Ans. **P.2-36, Ex-58.**

Q.5. Define characteristic function on a set. Show that $f_{A \cup B} = f_A + f_B - f_{A \cap B}$, where f_A is characteristic function on A. **4M**

Ans. **P.2-39, Ex-69.**

Q.6. Determine the Hasse diagram of the relation on $A = \{1, 2, 3, 4, 5\}$ whose matrix is shown: **5M**

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Ans. **P.2-31, Ex-51.**

SUMMER - 07 (CS/CT/CE)

Q.1. Let $X = \{a, b, c\}$ and

$R = \{(a, a) (a, c) (b, a) (b, b) (c, a) (c, b) (c, c)\}$

Find R, R^2, R^3 and prove that $M_{ROR} = M_{R^2}$. Draw graph of R and R^2. **7M**

Ans. **P.2-24, Ex-36.**

Q.2. Consider the set $B = \{a, b, c\}$. Let P(B) be the power set of B, let $\subseteq$ be the inclusion relation on the element of P(B). Draw Hasse diagram of $(P(B), \subseteq)$ for the set B. **6M**

Ans. **P.2-30, Ex-50.**

SUMMER - 07 (IT)

Q.1. List all possible functions from the set $X = \{a, b, c\}$ to $Y = \{0, 1\}$. Indicate in each case whether the function is 1 – 1, onto or both. **6M**

Ans. **P.2-34, Ex-55.**

Q.2. Let $A = \{1, 2, 3, 4\}$

Let $R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (2, 4), (3, 4), (4, 1)\}$ and $R_2 = \{(1, 1), (1, 3), (2, 3), (3, 2), (3, 4)\}$. Write M_{R_1} and M_{R_2} and the digraphs of the relations $R_1 \cup R_2$. **7M**

Ans. **P.2-24, Ex-37.**

Q.3. Given the set $S = \{1, 2, 3, 4, 5\}$, find the relations R_A and R_B generating the partitions. **7M**

(i) $A = \{\{1, 2\}, \{3\}, \{4, 5\}\}$

(ii) $B = \{\{1, 2, 3\}, \{4, 5\}\}$.

Prove that R_A and R_B are equivalence relations. Draw the graphs of R_A, R_B and M_{R_A} and M_{R_B}.

Ans. **P.2-15, Ex-12.**

Q.4. Let F be the set of all 1 –1, on to mappings from X onto X where X = {1, 2, 3}. Find all elements of F and find inverse of each element. **6M**

Ans. P.2-34, Ex-56.

Q.5. Let A be a given finite set and $\rho(A)$ is power set. Let $\subseteq$ be the inclusion relation $\rho(A)$. Draw Hasse diagram of $(\rho(A), \subseteq)$ for A = {a, b, c}. **6M**

Ans. P.2-30, Ex-50.

WINTER - 07 (CS/CT/CE)

Q.1. Let X = {ball, bed, dog, let, egg} and R = {(x, y)/ x, y $\in$ X, x R y if x and y contain common letter}. Write M_R. Draw the graph. Prove that R is compatible. Is R an equivalence relation? **7M**

Ans. P.2-23, Ex-32.

Q.2. Define characteristics function and partially ordered set. Find all 1-1 onto mappings from X to X where X = {1, 2, 3}. Write inverse of each mapping. **7M**

Ans. P.2-38,34, Refer Definitions and Ex-56.

Q.3. Prove that : $A \times (B \cup C) = (A \times B) \cup (A \times C)$. **4M**

Ans. P.2-9, Refer property(3) of Cartesian product.

WINTER - 07 (IT)

Q.1. If relations R and S are equivalence relation, show that $R \cap S$ is also an equivalence relation. **6M**

Ans. P.2-13, Ex-7.

Q.2. Let A = R × R. Define the following relation R on A : (a, b) R (c, d) if and only if $a^2 + b^2 = c^2 + d^2$. **7M**

(i) Show that R is an equivalence relation.

(ii) Compute A/R.

Ans. P.2-16, Ex-13.

Q.3. Define :

(i) One-one function

(ii) On to function

List all possible functions from set X = {a, b, c} to Y = {0, 1}. Indicate in each case whether the function is 1 – 1, onto or 1 – 1 and onto. Represent pictorially in each case. **6M**

Ans. P.2-34, 32, Ex-55 & Definitions.

Q.4. Define Partial order relation on a set. Let A be a given finite set and e (A) its power set. Let $\leq$ be the inclusion relation on e(A). Draw Hasse diagram of (e(A), $\leq$) for

(i) A = {a}

(ii) A = {(a, b)}

(iii) A = {a, b, c}

Also give least member and greatest member if they exist. **7M**

Ans. P.2-30, Ex-50.

SUMMER - 08 (CS/CT/CE)

Q.1. Define characteristic function. Show the following identities for sets using characteristic function: **7M**

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Ans. P.2-39, Ex-67.

(ii) A + (B + C) = (A + B) + C.

Ans. P.2-38, Ex-66.

where + denotes union of two sets.

Q.2. Write the set of all possible function from A to B where A = {1, 2, 3} and B = {a, b}. Out of these which functions are onto? **6M**

Ans. P.2-34, Ex-55.

Q.3. Let X = {1, 2, 3, 6, 12, 24, 36}. Define the relation '$\leq$' x Ry if x divides y. Show that < X, $\leq$ > is a POSET. Draw its Hasse diagram. **7M**

Ans. P.2-29, Ex-46.

SUMMER - 08 (IT)

Q.1. Define characteristic function and hence prove that :

(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. **6M**

(ii) $(A')' = A$.

Ans. P.2-38, Ex-66.

Q.2. Draw tree diagram of A × B × C where A = {2, 3}, B = {1, 3, 5} and C = {3, 4} and hence find A × B × C. **4M**

Ans. P.2-10, Ex-1.

Q.3. If A = {a, b}, B = {2, 3} and C = {3, 4}, prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$. **4M**

Ans. P.2-11, Ex-2.

Q.4. If f: A→ B and g : B→ C are both one–one and onto functions then show that (g o f) is also one–one and onto. Also show that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. **6M**

Ans. P.2-33, Refer theorem 2.

Q.5. Let A = {1, 2, 3, 4} and R = {(1, 2), (2, 3), (3, 4), (2, 1)}. Find the transitive closure of R. **3M**

Ans. P.2-22, Ex-31.

WINTER - 08(CS/CT/CE)

Q.1. Prove that :

(ii) $A \times (B \cup C) = (A \times B) \cup (A \times C)$. **3M**

Ans. P.2-9, Refer property (3) of Cartesian product.

Q.2. Using properties of characteristic function, prove that :
(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
(ii) $(A')' = A$. **4M**

Ans. P.2-38, Ex-66.

Q.3. The relation in set defined by : **6M**
$R = \{(A, B) \mid A \subseteq B\}$
Show that R is partial order relation. Is this relation compatible?

Ans. P.2-16, Ex-17.

Q.4. Let Z be the set of integers and let R be the relation defined by
$R = \{(x, y) \mid x, y \in Z \text{ and } x - y \text{ is divisible by } 3\}$.
Determine the equivalence class generated by the element of Z. **7M**

Ans. P.2-25, Ex-38.

WINTER - 08 (IT)

Q.1. Let $X = \{1, 2, 3 \ldots. 9\}$ and
$R_1 = \{(x, y) / x - y \text{ is divisible by } 3\}$
$R_2 = \{(x, y) / x, y \in X \text{ and } x, y \text{ are in the same column of matrix } A\}$
where $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ **7M**
Find R_1 and R_2. Prove that R_1 and R_2 are equivalence relation. Prove that $R_1 = R_2$.

Ans. P.2-17, Ex-19.

Q.2. Let f: R→ R and g : R→ R where R is the set of real numbers. Find f o g and g o f where
$f(x) = x^2 - 2,\ g(x) = x + 4$. **6M**

Ans. P.2-36, Ex-60.

Q.3. Given f: X →Y and g: Y → Z, show that **6M**
(i) If g o f is one-one, then f is one-one
(ii) If g o f is onto, then g is onto
(iii) f^{-1} exists, iff f is bijective.

Ans. P.2-37, Ex-61.

Q.4. Define transitive closure.
Let A = {1, 2, 3, 4} and R = {(1, 2), (2, 3), (3, 4), (2,1)} be a relation on A. Find transitive closure of R and draw its digraph. **7M**

Ans. P.2-22, Ex-31.

SUMMER - 09 (CS/CT/CE)

Q.1. Let A = {1, 2, 3} and B = {a, b, c, d}, let R and S be the relations from A to B with the Boolean Matrices :

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \text{ and } M_S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Find :
(i) $M_{R^{-1}}$ and $M_{S^{-1}}$
(ii) Show that : $M_{(R \cap S)\ oR^{-1}} = M_{RoR^{-1}} \cap M_{SoR^{-1}}$
(iii) Find $M_{R^{-1} \cup S^{-1}}$ **7M**

Ans. P.2-22, Ex-30.

Q.2. List all possible functions from set X = {a, b, c} to Y = {0, 1}. Indicate in each case whether the function is 1–1, is on to, and is 1-1 on to. **5M**

Ans. P.2-34, Ex-55.

Q.3. Detine :
(i) Onto mapping
(ii) One to one mapping. **3M**

Ans. P.2-32, Refer Definitions.

SUMMER - 09 (IT)

Q.1. Prove that : $A \times (B \cap C) = (A \times B) \cap (A \times C)$. **4M**

Ans. P.2-9, Refer property (4) of Cartesian product.

Q.2. Let $X = \{1, 2, 3, \ldots. 9\}$
$R_1 = \{(x, y) / x - y \text{ is divisible by } 3\}$
$R_2 = \{(x, y) / x, y \in X \text{ and } x, y \text{ are in the same column of matrix } A\}$
where $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$
(i) Find R_1 and R_2.
(ii) Prove that R_1 and R_2 are equivalence relations.
(iii) Is $R_1 = R_2$?
(iv) Give partition of X generated by R_1. **7M**

Ans. P.2-17, Ex-19.

Q.3. Let X = { ball, bed, dog, egg, let} **6M**
and $R = \{(x, y) / x, y \in X, x R y$ if x and y contain some common letter}.
(i) Draw the graph of R.
(ii) Write M_R.
(iii) Prove that R is compatible.

Ans. P.2-23, Ex-32.

Q.4. If $f : X \to Y$ and $g : Y \to Z$ and both f and g are one-one, onto then show that **5M**
(i) g o f is also one-one, onto.
(ii) $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Ans. P.2-33, 38 Refer theorem 2 & Ex. 65.

Q.5. Let F be the set of all 1 – 1, onto mappings from X onto X where X = {1, 2, 3}. Find all elements of F and find inverse of each element. Write the composition table for elements of F. **6M**

Ans. P.2-34, Ex-56.

Q.6. Define partial order relation. **2M**

Ans. P.2-13, Refer types of relations.

WINTER - 09 (CS/CT/CE)

Q.1. Let X = {1, 2, 3, ... 9}

$R_1 = \{(x,y)/x - y \text{ is divisible by } 3\}$

$R_2 = \{(x,y)/x,y \in X$ and x, y are in the same column of matrix A}

where $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Find R_1 and R_2. Prove that R_1 and R_2 are equivalence relation. Is $R_1 = R_2$? Justify your answer. Give the partition of X generated by R_1. **7M**

Ans. P.2-17, Ex-19.

Q.2. If $f: X \to Y$ and A, B are two subsets of Y, then prove that:

(i) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

(ii) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$. **7M**

Ans. P.2-37, Ex-62.

Q.3. Let X be the set of factors of particular positive integer m and let '≤' be the relation 'divides' i.e. $\le = \{(x,y)/x,y \in X$ and x divides y}. Draw Hasse diagram for (X, ≤) and its dual for (i) m = 30, (ii) m = 45. **6M**

Ans. P.2-30, Ex-49.

WINTER - 09 (IT)

Q.1. Given a set S = {1, 2, 3, 4, 5}. Find the relation R_A and R_B generating the partitions:

A= {(1,2), (3), (4, 5)}, B = {(1, 2, 3), (4, 5)}.

Prove that R_A and R_B are equivalence relation. Draw graphs of R_A, R_B. Write M_{R_A} and M_{R_B}. **7M**

Ans. P.2-15, Ex-12.

Q.2. Let A = {1, 2, 3, 4} and let

$R_1 = \{(1,1),(1,2),(2,3),(2,4),(3,4),(4,1),(4,2)\}$ and

$R_2 = \{(3,1),(4,4),(2,3),(2,4),(1,1),(1,4)\}$.

Verify: **7M**

(i) $M_{R_1 \cdot R_2} = M_{R_1} \cdot M_{R_2}$

(ii) $M_{R_1^C}$ = transpose of M_{R_1}

(iii) $M_{(R_1 \cdot R_2)^o} = M_{R_2^C} \cdot M_{R_1^C}$.

Ans. P.2-25, Ex-39.

Q.3. Let R denotes a relation on the set of ordered pairs of positive integers defined by (x, y) R (u, v) if and only if xv = yu. Show that R is an equivalence relation. **6M**

Ans. P.2-17, Ex-18.

Q.4. Show that the function f defined by $f: Q \to Q$ such that f (x) = 3x + 4, for all $x \in Q$ is one-one, onto, where Q is the set of all rational numbers. Also find a formula that defines the inverse function f^{-1}. **7M**

Ans. P.2-35, Ex-57.

SUMMER - 10 (CS/CT/CE)

Q.1. Let R denotes a relation on the set of ordered pairs of positive integers defined by (x, y) R (u, v) if and only if x v = y u, show that R is an equivalence relation. **7M**

Ans. P.2-17, Ex-18.

Q.2. Let A = {1, 2} and B = {a, b, c}, find all functions g: $B \to A$ and for each such functions, determine whether it is one-to-one, onto both or neither. **6M**

Ans. P.2-34, Ex-55.

SUMMER - 10 (IT)

Q.1. Prove that : **6M**

(i) $f_{A \cup B} = f_A + f_B - f_{A \cap B}$.

(ii) $f_{A \cap B} = f_A f_B$.

where f_A denotes the characteristic function of A.

Ans. P.2-39, 40, Ex-69 & 70.

Q.2. Let X = {1, 2, 3, 4, 5}, R = {(1, 2), (3, 4), (2, 2)}. S = {(4, 2), (2, 5), (3, 1), (1, 3)}.

Find $R \circ S$, $S \circ R$, $R \circ R$, $S \circ S$, $(R \circ S) \circ R$ and prove that $M_{R \circ S} = M_R \circ M_S$. **7M**

Ans. P.2-26, Ex-40.

Q.3. Given the set S = {1, 2, 3, 4, 5}, find the relations R_A and R_B generating the partitions

(i) A = { {1, 2}, {3}, {(4, 5}}

(ii) B = {{1, 2, 3}, {4, 5}.

Prove that R_A and R_B are equivalence relations. Also draw the graph of R_A and R_B. **8M**

Ans. P.2-15, Ex-12.

Q.4. List all possible functions from the set X = {a, b, c} to Y = {0, 1}. Indicate in each case whether the function is 1 – 1, onto or bijective. **6M**

Ans. P.2-34, Ex-55.

WINTER - 10 (CS/CT/CE)

Q.1. Given

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \; M_S = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Find R, S, R o S, $\tilde{R}$, $\tilde{S}$, S o R and prove that :

$M_{\widetilde{ROS}} = M_{\tilde{S} o \tilde{R}}$ **6M**

Ans. P.2-27, Ex-41.

Q.2. Let F be the set of all 1 – 1 onto mappings from X onto X where X = {1, 2, 3}. Find all elements of F and find inverse of each element. Write the composition table for elements of F. **7M**

Ans. P.2-34, Ex-56.

WINTER - 10 (IT)

Q.1. Let I be the set of all integers. Consider the relation R = {(a, b)/ (a – b) is divisible by 5}.

Is R an equivalence relation? Write down its equivalence classes. **7M**

Ans. P.2-28, Ex-42.

Q.2. If f : X → Y and g : Y → Z are both 1 – 1 and onto functions show that f ∘ g is also 1 – 1 and onto. **6M**

Ans. P.2-38, Ex. 65

Q.3. If the relation set is :

R = {(1, 2), (3, 4), (2, 2)}

S = {(4, 2) (2, 5) (3, 1) (1, 3)}

find

(i) R ∘ S,

(ii) R ∘ (S ∘ R)

(iii) R ∘ (R ∘ R).

Write relational matrix for R and S. **7M**

Ans. P.2-25, Ex-40.

Q.4. Let A be a given finite given by A = {a, b, c}. let ⊆ be the inclusion on power set of A. Show that < P (A), ⊆ > is a POSET. Draw its Hasse diagram. **7M**

Ans. P.2-30, Ex-50.

SUMMER - 11 (CS/CT/CE)

Q.1. Define : **7M**

(i) Equivalence relation.

(ii) Partial ordered relation.

(iii) Compatibility relation.

(iv) Maximal compatibility blocks with examples.

Ans. P.2-13, Refer types of relations.

Q.2. Let R be a relation of N × N defined by (x, y) R (z, w) if xz = yw. Prove that R is an equivalence relation. **6M**

Ans. P.2-17, Ex-18.

Q.3. Define: **7M**

(i) Function

(ii) One to one

(iii) Onto

(iv) Characteristic function.

Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ using characteristic function.

Ans. P.2-31,32, 38, Refer Definitions and Ex-66.

SUMMER - 11 (IT)

Q.1. Define Cartesian product of two sets. For any non-empty sets A and B, prove that **7M**

$A \times B = B \times A \Leftrightarrow A = B$.

Ans. P.2-11, Ex-3.

Q.2. Find a non-empty set and a relation on the set that satisfy each of the following combinations of properties.

Also draw graph of each relation

(i) Reflexive and symmetric but not transitive.

(ii) Reflexive and transitive but not antisymmetric. **7M**

Ans. P.2-28, Ex-43.

Q.3. Let A = {1, 2, 3, 4} and

R = {(1, 2), (2, 3), (3, 4), (2, 1)}

Find the transitive closure of R. **6M**

Ans. P.2-22, Ex-31.

Q.4. If : R → R and g : R → R be defined by

$f(x) = x^3 - 4x, \; g(x) = \dfrac{1}{x^2+1}$ and

h : R → R defined by

$h(x) = x^4$, then

(i) Find composition function

(f ∘ g ∘ h) x

(ii) If f invertible? If yes, find f^{-1}. **7M**

Ans. P.2-38, Ex-63.

Q.5. Let U be a universal set and let X ⊆ U.

Define $C_X(x) = \begin{cases} 1, & \text{if } x \in X \\ 0, & \text{if } x \notin X \end{cases}$

where C_X the characteristic function of X show that

$C_{X \cup Y}(x) = C_X(x) + C_Y(x) - C_{X \cap Y}(x)$. **7M**

Ans. P.2-39, Ex-69.

WINTER - 11 (CS/CT/CE)

Q.1. Let A = {1, 2, 3}, B = {a, b, c, d }, let R and S be the relations from A to B with the Boolean matrices : **7M**

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \text{ and } M_S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Find :

(i) $M_{R^{-1}}$ and $M_{S^{-1}}$

(ii) Show that : $M_{(R \cap S) \circ R^{-1}} = M_{R \circ R^{-1}} \cap M_{S \circ R^{-1}}$

(iii) Find : $M_{R^{-1} \cup S^{-1}}$.

Ans. P.2-22, Ex-30.

Q.2. Define : **3M**

(i) Onto mapping,

(ii) One to one mapping.

Ans. P.2-32, Refer definitions.

Q.3. Let X = {2, 3, 6, 12, 24, 36}, define the relation '≤' x Ry , if x divides y. Show that < X, ≤> is a POSET. Draw its Hasse diagram. **7M**

Ans. P.2-29, Ex-46.

WINTER - 11 (IT)

Q.1. Let X = (ball, bed, dog, egg, let} and R = {(x, y)| x, y ∈ X, x R y if x and y contain some common letter}. **7M**

(i) Draw the graph of R.

(ii) Write M_R

(iii) Prove that R is compatible.

Ans. P.2-23, Ex-32.

Q.2. Let R denote a relation on the set of ordered pairs of positive integers defined by (x, y) R (u, v) if and only if xv = yu. Show that R is an equivalence relation. **6M**

Ans. P.2-17, Ex-18.

Q.3. List all possible functions from the set X = {a, b, c} to Y = {0, 1}. Indicate in each case whether the function is one-one, onto or bijective (one - one and onto). **6M**

Ans. P.2-34, Ex-55.

Q.4. Let A be a finite set given by A = {a, b, c}. let ⊆ be the inclusion on power set of A. Show that < P (A), ⊆ > is a POSET. Draw its Hasse diagram. **7M**

Ans. P.2-30, Ex-50.

SUMMER - 12 (CS/CT/CE)

Q.1. Show that the relation R defined on a set of ordered pairs is an equivalence relation :

(a, b) R (c, d) ⇔ a = c. **4M**

Ans. P.2-14, Ex-11.

Q.2. If f: A → B and g : B → C be one to one onto functions, then g o f is also one to one onto and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. **7M**

Ans. P.2-33, Refer theorem (2).

Q.3. Define characteristics function and partially ordered set. Find all 1-1 onto mappings from X to X where X = {1, 2, 3}. **7M**

Ans. P.2-34, 38, Ex-56 & Definitions.

SUMMER - 12 (IT)

Q.1. Let R be the relation in (N × N), where N is the set of natural numbers defined by (a, b) R (c, d) iff ad = bc. Prove that R is an equivalence relation. **6M**

Ans. P.2-17, Ex-18.

Q.2. Let f : A → B be a function. Then prove :

(i) f^{-1} is a function from B to A if and only if f is one to one.

(ii) the function f^{-1} is also one to one. **7M**

Ans. P.2-38, Ex-64.

Q.3. Given :

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } M_S = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Prove that $M_{\widetilde{(ROS)}} = M_{\widetilde{S} \circ \widetilde{R}}$. **6M**

Ans. P.2-27, Ex-41.

Q.4. Define partial order relation on a set. Let A = {a, b, c} and ρ (A) is a power set on A. Draw Hasse diagram of (ρ (A), ≤), where ≤ be the inclusion relation on ρ (A). **7M**

Ans. P.2-30, Ex-50.

WINTER - 12 (CS/CT/CE)

Q.1. Let R denote a relation on the set of ordered pairs of positive integers defined by (x, y) R (u, v) if and only if xv = xu. Show that R is an equivalence relation. **7M**

Ans. P.2-17, Ex-18.

Q.2. Let X be the set of factors a particular integer m and let ⊆ be the relation "divides" i.e.

$\subseteq = \{(x, y) / x, y \in X$ and (x divides y)$\}$.

Draw Hasse diagram for $(X, \subseteq)$ and its dual form = 6. **7M**

Ans. P.2-30, Ex-49.

WINTER - 12 (IT)

Q.1. If $A = \{1, 4\}$, $B = \{4, 5\}$, $C = \{5, 7\}$, find

(i) $A \times B$.

(ii) $A \times C$.

(iii) $(A \times B) \cap (A \times C)$. **6M**

Ans. P.2-11, Ex-4.

Q.2. If $f : X \to Y$, $g : Y \to Z$ and both f and g are onto, show that $g \circ f$ is also onto. Is $g \circ f$ one-to-one if both g and f are one-to-one. **7M**

Ans. P.2-38, Ex-65.

Q.3. Let $A = \{a, b, c, d\}$ and consider a relation $R = \{(a, a), (a, b), (a, c), (a, d), (b, b), (b, d), (c, c), (c, d), (d, d)\}$. Show that R is a partial ordering. **7M**

Ans. P.2-28, Ex-44.

Q.4. A relation R defined on $A = \{1, 2, 3, 4, 5\}$ as $R = \{(1, 2), (1, 6), (2, 3), (3, 3), (3, 4), (4, 1), (4,3), (4, 5), (6, 4)\}$. **6M**

Draw the graph of R and R^2 and hence find M_R^2.

Ans. P.2-28, Ex-45.

SUMMER - 13 (CS/CT/CE)

Q.1. Let 'R' be an relation on $N \times N$, N being the set of natural numbers.

Define

(i) (a,b) R (c, d), iff ad = bc.

(ii) (a,b) R (c, d), iff a + d = b + c.

Prove that R is an equivalence relation. **7M**

Ans. P.2-17, Ex.18.

Q.2. List all possible functions from the set $x = \{a, b, c\}$ to the set $Y = \{0, 1\}$ indicate in each case, whether the functions is one-one or onto or both. **6M**

Ans. P.2-34, Ex. 55.

SUMMER -13 (IT)

Q.1. Let R denote a relation on the set of ordered pairs of positive integers defined by (x, y) R (u, v) if and only if xv = yu. Show that R is an equivalence relation **7M**

Ans. P.2-17, Ex-18.

Q.2. Define characteristics function and hence prove that : $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. **6M**

Ans. P.2-38, Ex-66.

Q.3. If $f : X \to Y$ and $g : Y \to Z$ and both f and g are one-one, onto, then show that g of is also one-one and onto and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. **7M**

Ans. P.2-33, Refer theorem 2.

Q.4. Let A be given finite set P (A) its power set.

Let $\subseteq$ be the inclusion relation on the elements of P (A). Draw Hause diagram of $(P(A), \subseteq)$ for :

(i) $A = \{a, b\}$

(ii) $A = \{a, b, c\}$. **7M**

Ans. P.2-30, Ex-50.

SOLVED QUESTION BANK

[Sequence given as per syllabus]

INTRODUCTION

This chapter is concerned with relations and functions. The concept of relation is of primary importance in computer science especially in the study of data structure such as linked list, array, relational models etc. Relations are also important in the analysis of algorithms, information system etc.

Function or mapping is a special type of relation. It is an important concept in the field of mathematics and have numerous applications in computers science. Here we discuss some basic properties and types of functions.

ORDERD PAIR

An ordered pair consists of two objects (need not be distinct) in a given fixed order. An ordered pair is usually denoted by (a, b). The most common example of an ordered pair is the representation of a point in two dimension. The points (1, 3), (– 1, 4), (2, 0), (3, 4) and (3, 1) represent ordered pairs where the first element is referred as x - coordinate (abscissa) and the second element is referred as y-coordinate (ordinate). Here we should note that {1, 3} and {3, 1} represent the same sets, but (1, 3) and (3, 1) represent different ordered pairs.

An ordered triple is represented by (a, b, c) where a, b, c denote the first, second, third elements respectively of the triple. The ordered triple (a, b, c) can also be written in terms of ordered pair as ((a, b), c).

A ordered quadruple is an ordered pair (((a, b), c), d) where the first element ((a, b), c) is an ordered triple.

In general an ordered n-tuple is an ordered pair where the first element is and ordered (n – 1) tuple. For example, a set of 6 elements {l, m, n, o, p, q} can be represented as (((((l, m), n), o), p), q).

CARTESIAN PRODUCT

Let A and B be any two non-empty sets. Then the set of all ordered pairs (a, b) such that a is an element of A and b is an element of B is called the cartesian product of A and B and is denoted by $A \times B$.

i.e. $A \times B = \{ (a, b) \mid a \in A, b \in B\}$

For example : Let $A = \{1, 2, 3\}$, $B = \{a, b, c\}$

Then $A \times B = \{ (1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c)\}$

PROPERTIES OF CARTESIAN PRODUCT AND SETS

(1) $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$

(2) $(A - B) \times C = (A \times C) - (B \times C)$

(3) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

CS/CT/CE : W-07,08

(4) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

CS/CT/CE : S-06, IT : S-09

(5) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

Proof :

(1) **To prove :** $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$

We prove that L.H.S. $\subseteq$ R.H.S. and R.H.S. $\subseteq$ L.H.S.

Let $(x, y) \in (A \cap B) \times (C \cap D)$

i.e. $x \in A \cap B, y \in C \cap D$

i.e. $x \in A$ and $x \in B$, $y \in C$ and $y \in D$

i.e $x \in A, y \in C$ and $x \in B, y \in D$

i.e. $(x, y) \in (A \times C)$ and $(x, y) \in (B \times D)$

i.e. $(x, y) \in (A \times C) \times (B \times D)$

$\therefore$ L.H.S. $\subseteq$ R.H.S. .....(a)

Let $(x, y) \in (A \times C) \cap (B \times D)$

i.e. $(x, y) \in A \times C$ and $(x, y) \in B \times D$

i.e. $x \in A, y \in C$ and $x \in B, y \in D$

i.e. $x \in A$ and $x \in B$, $y \in C$ and $y \in D$

i.e. $x \in (A \cap B), y \in (C \cap D)$

i.e. $(x, y) \in (A \cap B), y \in (C \cap D)$

$\therefore (x, y) \in (A \times C) \cap (B \times D)$

$\Rightarrow (x, y) \in (A \cap B) \cap (C \cap D)$

$\therefore$ L.H.S. $\subseteq$ R.H.S. .....(b)

From (a) and (b), we get

$$\boxed{(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)}$$ **Ans.**

(2) **To Prove :** $(A - B) \times C = (A \times C) - (B \times C)$

We prove that L.H.S. $\subseteq$ R.H.S. and R.H.S. $\subseteq$ L.H.S.

Let $(x, y) \in (A - B) \times C$

i.e. $x \in A - B, y \in C$

i.e. $x \in A$ and $x \notin B$, $y \in C$

i.e. $x \in A, y \in C$ and $x \notin B, y \in C$

i.e. $(x, y) \in (A \times C)$ and $(x, y) \notin (B \times C)$

i.e. $(x, y) \in (A \times C) - (B \times C)$

$\therefore (x, y) \in (A - B) \times C \Rightarrow (x, y) \in (A \times C) - (B \times C)$

$\therefore$ L.H.S. $\subseteq$ L.H.S. .....(a)

Let $(x, y) \in (A \times C) - (B \times C)$

i.e. $(x, y) \in A \times C$ and $(x, y) \notin B \times C$

i.e. $x \in A, y \in C$ and $x \notin B, y \notin C$

But $y \in C$

i.e. $x \in A, y \in C$ and $x \notin B$

i.e $x \in A$ and $x \notin B, y \in C$

i.e. $x \in A - B, y \in C$

i.e. $(x, y) \in (A - B) \times C$

$\therefore$ R.H.S. $\subseteq$ L.H.S. .....(b)

From (a) and (b), we get

$\boxed{(A - B) \times C = (A \times C) - (B \times C)}$ **Ans.**

(3) **To prove :** $A \times (B \cup C) = (A \times B) \cup (A \times C)$

We prove that L.H.S. $\subseteq$ R.H.S. and R.H.S. $\subseteq$ L.H.S.

Let $(x, y) \in A \times (B \cup C)$

i.e. $x \in A, y \in B \cup C$

i.e. $x \in A, y \in B$ or $y \in C$

i.e. $x \in A, y \in B$ or $x \in A, y \in C$

i.e. $(x, y) \in A \times B$ or $(x, y) \in A \times C$

i.e. $(x, y) \in (A \times B) \cup (A \times C)$

$\therefore (x, y) \in A \times (B \cup C) \Rightarrow (x, y) \in (A \times B) \cup (A \times C)$

$\therefore$ L.H.S. $\subseteq$ R.H.S. .....(a)

Let $(x, y) \in (A \times B) \cup (A \times C)$

i.e. $(x, y) \in A \times B$ or $(x, y) \in A \times C$

i.e. $x \in A, y \in B$ or $x \in A, y \in C$

i.e. $x \in A, y \in B$ or $y \in C$

i.e. $x \in A, y \in B \cup C$

i.e. $(x, y) \in A \times (B \cup C)$

$\therefore (x, y) \in (A \times C) \cup (A \times C)$

$\Rightarrow (x, y) \in A \times (B \cup C)$

$\therefore$ L.H.S. $\subseteq$ R.H.S. .....(b)

From (a) and (b), we get

$\boxed{A \times (B \cup C) = (A \times B) \cup (A \times C)}$ **Ans.**

(4) **To prove :** $A \times (B \cap C) = (A \times B) \cap (A \times C)$ **S-06,09**

Let (x, y) be any element of $A \times (B \cap C)$, then

$(x, y) \in A \times (B \cap C) \Rightarrow x \in A$ and $y \in (B \cap C)$

$\Rightarrow x \in A$ and $(y \in B$ and $y \in C)$

$\Rightarrow (x \in A$ and $y \in B)$ and $(x \in A$ and $y \in C)$

$\Rightarrow (x, y) \in A \times B$ and

$(x, y) \in A \times C$

$\Rightarrow (x, y) \in (A \times B) \cap (A \times C)$

So $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$ .....(1)

Again let $(x, y) \in (A \times B) \cap (A \times C)$

$\Rightarrow (x, y) \in A \times B$ and $(x, y) \in A \times C$

$\Rightarrow (x \in A, y \in B)$ and $(x \in A, y \in C)$

$\Rightarrow x \in A$ and $(y \in B$ and $y \in C)$

$\Rightarrow x \in A$ and $y \in B \cap C$

$\Rightarrow (x, y) \in A \times (B \cap C)$

Hence

$(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$ .....(2)

From (1) and (2), we get

$\boxed{A \times (B \cap C) = (A \times B) \cap (A \times C)}$ **Ans.**

(5) **To prove :** $(A \cup B) \times C = (A \times C) \cup (B \times C)$

We prove that L.H.S. $\subseteq$ R.H.S. and R.H.S. $\subseteq$ L.H.S.

Let $(x, y) \in (A \cup B) \times C$

i.e. $x \in A \cup B, y \in C$

i.e. $x \in A$ or $x \in B, y \in C$

i.e. $x \in A, y \in C$ or $x \in B, y \in C$

i.e. $(x, y) \in A \times C$ or $(x, y) \in B \times C$

i.e. $(x, y) \in (A \times C) \cup (B \times C)$

$\therefore (x, y) \in (A \cup B) \times C$

$\Rightarrow (x, y) \in (A \times C) \cup (B \times C)$

$\therefore$ L.H.S. $\subseteq$ R.H.S. .....(a)

Let $(x, y) \in (A \times C) \cup (B \times C)$

i.e. $(x, y) \in A \times C$ or $(x, y) \in B \times C$

i.e. $x \in A, y \in C$ or $x \in B, y \in C$

i.e. $x \in A$ or $x \in B, y \in C$

i.e. $x \in A \cup B, y \in C$

i.e. $(x, y) \in (A \times C) \cup (B \times C)$

$\Rightarrow (x, y) \in (A \cup B) \times C$

$\therefore$ R.H.S. $\subseteq$ L.H.S. .....(b)

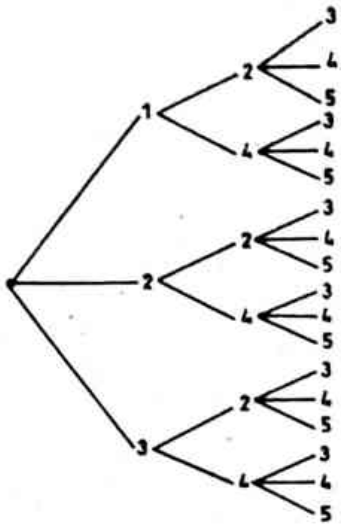
From (a) and (b), we get

$\boxed{(A \cup B) \times C = (A \times C) \cup (B \times C)}$ **Ans.**

Ex.1. Draw tree diagram of $A \times B \times C$ where $A = \{2, 3$ $B = \{1, 3, 5\}$ and $C = \{3, 4\}$ and hence find $A \times B \times C$. **IT : S-0**

Soln. We have

$A \times B \times C = \{ (213), (214), (233) (234) (253) (254) (313) (314) (333) (334) (353) (354)\}$



Ex.2. If $A=\{a, b\}$, $B=\{2, 3\}$ and $C=\{3, 4\}$, prove that $A\times (B\cap C) = (A\times B) \cap (A\times C)$. **IT : S-08**

Soln. Given : $A = \{a, b\}$, $B = \{2, 3\}$, $C = \{3, 4\}$

$B\cap C=\{3\}$

$A\times \{B\cap C\} = \{a, b\}\times\{3\} = \{(a, 3)(b, 3)\}$ .... (1)

$A \times B=\{(a, 2) (a,3) (b, 2) (b, 3)\}$

$A\times C= \{(a, 3)(b, 3)(a, 4)(b, 4)\}$

$(A\times B) \cap (A\times C)=\{(a, 3) (b,3)\}$ .....(2)

$\therefore$ From (1) and (2) we have

$\boxed{A\times (B\cap C)=(A\times B)\cap(A\times C)}$ **Ans.**

Ex.3. Define Cartesian product of two sets. For any non-empty sets A and B, prove that $A\times B=B\times A \Leftrightarrow A=B$. **IT : S-11**

Soln. For definition : Refer P.2-9

Now, **To Prove :** $A\times B=B\times A \Leftrightarrow A=B$

i.e. **To Prove :** (i) $A\times B=B\times A \Rightarrow A=B$

and (ii) $A=B \Rightarrow A\times B=B\times A$

First part is obviously trivially true, remaining To Prove that, $A=B \Rightarrow A\times B=B\times A$

i.e. $A\times B \subseteq B\times A$

and $B\times A \subseteq A\times B$

Let $(x, y) \in A\times B$

$\Rightarrow x \in A$ and $y \in B$

$\Rightarrow x \in B$ and $y \in A$ *[$\because$ By hypo]*

$\Rightarrow (x, y) \in B\times A$

$\therefore (x, y) \in A\times B \Rightarrow (x, y) \in B\times A$

$\therefore A\times B \subseteq B\times A$ .....(I)

and let $(x, y) \in B\times A$

$\Rightarrow x \in B$ and $y \in A$

$\Rightarrow x \in A$ and $y \in B$ *[$\because$ By hypo]*

$\Rightarrow (x, y) \in A\times B$

$\therefore (x, y) \in B\times A \Rightarrow (x, y) \in A\times B$

Thus, $B\times A \subseteq A\times B$ .....(II)

$\therefore$ From (I) and (II), we have

$A\times B=B\times A$,

$\therefore$ $\boxed{A\times B=B\times A \Leftrightarrow A=B}$ **Ans.**

Ex.4. If $A = \{1, 4\}$, $B = \{4, 5\}$, $C = \{5, 7\}$. Find

(i) $A\times B$

(ii) $A\times C$

(iii) $(A\times B)\cap(A\times C)$. **IT : W-12**

Soln. Given : $A = \{1, 4\}$, $B = \{4, 5\}$, $C = \{5, 7\}$

(i) $A\times B = \{1, 4\}\times\{4, 5\} = \{(1, 4), (1, 5), (4, 4), (4, 5)\}$

(ii) $A\times C = \{1, 4\}\times\{5, 7\} = \{(1, 5), (1, 7), (4, 5), (4, 7)\}$

(iii) $(A\times B)\cap(A\times C) = \{(1, 5), (4, 5)\}$..... **Ans.**

RELATIONS AND DIGRAPHS

The notation of a relation between any two sets of objects is quite common. If we suppose A to be the set of all male human beings and B to be the set of all female human beings, then the relation F (father) can be defined between the sets A and B. Hence if $x \in A$ and $y \in B$, then we say x is related to y by the relation F if x is the father of y and write x F y. In our daily life, we often use the relations of the type "is the son of", "is the brother of", "is the husband of" etc. In Mathematics we generally use the relations "less than", " greater than", "less than or equal to", "greater than or equal to" etc.

Let A and B be any non-empty sets. A relation R from A to B (usually called a binary relation) is a subset of $A\times B$. If $R \subseteq A\times B$ and $(a, b) \in R$, then we say that a is related to b by R and write a R b. If a is not related to b by R, then we write a **R** b.

If R is a relation form A to B, then R is a set of ordered pairs where the first element belongs to A and the second element belongs to B.

If R is a relation from a set A to itself i.e. R is a subset of $A^2 = A\times A$ ($R \subseteq A\times A$), then we say that R is a relation on A, instead of R is a relation form A to A.

Domain of a Relation :

The domain of a relation R is defined as the set of all first elements of the ordered pairs which belong to R.

Range of Relation :

The range of a relation R is defined as the set of all second elements of the ordered pairs which belong to R.

For example :

(a) Let A = {1, 2, 3}, B = {a, b, c}, R = {(1, b), (1, c), (3, a), (3, c)}.

Here $R \subseteq A \times B$ because all the ordered pairs of R have the first element from A and second element from B.

∴ R is a relation from A to B

The domain of R = {1, 3}

The range of R = {a, b, c}

(b) Let A = {0, 1, 2, 3, 4, 5}, B = {2, 3, 4, 5, 6}. Let us define a relation R (less than) as a R b iff a < b, a ∈ A, b ∈ B.

Then R = {(0, 2), (0, 3), (0, 4), (0, 5), (0, 6), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6), (4, 5), (4, 6), (5, 6)}.

The domain of R = {0, 1, 2, 3, 4, 5}

The range of R = {2, 3, 4, 5, 6}

(c) Let A = R, where R is the set of real numbers. We define the relation R on A as follows.

x R y if and only if x and y satisfy the equation $\frac{x^2}{16}+\frac{y^2}{25}=1$.

Then the set R consists of all the points on the ellipse as shown in the following figure.

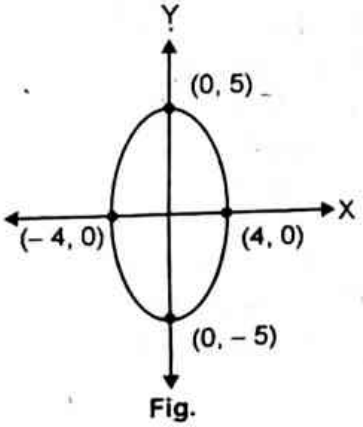


Fig.

Ex.5. Let P = {(1, 2), (2, 4), (3, 3)}, Q = {(1, 3), (2, 4), (4, 2)}. Show that

(i) D (P ∪ Q) = D (P) ∪ D (Q)

(ii) R (P ∩ Q) ⊆ R (P) ∩ R (Q)

[D-Domain, R-Range]

Soln. Given : P = {(1, 2), (2, 4), (3, 3)}

Q = {(1, 3), (2, 4), (4, 2)}

∴ D (P) = {1, 2, 3} and R (P) = {2, 3, 4}

D (Q) = {1, 2, 4} and R (Q) = {2, 3, 4}

P ∪ Q = {(1, 2), (1, 3), (2, 4), (3, 3), (4, 2)}

P ∩ Q = {(2, 4)}

∴ D (P ∪ Q) = {1, 2, 3, 4} and R (P ∩ Q) = {4}

(i) L.H.S. = D (P ∪ Q) = {1, 2, 3, 4}

R.H.S. = D (P) ∪ D (Q) = {1, 2, 3} ∪ {1, 2, 4}

= {1, 2, 3, 4}

∴ $\boxed{D(P \cup Q) = D(P) \cup D(Q)}$ **Ans.**

(ii) L.H.S. = R (P ∩ Q) = {4}

R.H.S. = R (P) ∩ R (Q) = {2, 3, 4} ∩ {2, 3, 4}

= {2, 3, 4}

∴ $\boxed{R(P \cap Q) \subseteq R(P) \cap R(Q).}$ **Ans.**

Ex.6. Find the range of the following relations.

$S = \{(x, x^2) \mid x \in N\}, T = \{(x, 2x) \mid x \in N\}$.

Find S ∪ T and S ∩ T.

Soln. Given : $S = \{(x, x^2) \mid x \in N\}$

i.e. S = {(1, 1), (2, 4), (3, 9), (4, 16), (5, 25), ...}

T = {(x, 2x) | x ∈ N}

i.e. T = {(1, 2), (2, 4), (3, 6), (4, 8), (5, 10), ...}

R (S) = {1, 4, 9, 16, 25, ...}

R (T) = {2, 4, 6, 8, 10, ...}

∴ S ∪ T = {(1, 1), (1, 2), (2, 4), (3, 9), (3, 6), (4, 16), (4, 8), (5, 25), (5, 10),.....} and S ∩ T = {(2, 4)}. **Ans.**

TYPE OF RELATIONS

(i) Reflexive Relation :

A relation R defined on a set A is said to be reflexive if a R a, ∀ a, b ∈ A.

i.e. (a, a) ∈ R, ∀ a ∈ A.

For example : Let A = {x, y, z} and R = {(x, x), (y, y), (z, z)}, then R is a reflexive relation on A.

(ii) Symmetric Relation :

A relation R defined on a set A is said to be symmetric, if a R b, then b R a, ∀ a, b ∈ A.

i.e. (a, b) ∈ R ⇒ (b, a) ∈ R, ∀ a, b ∈ A

For example : Let A = {1, 2, 3} and R = {(1, 2), (1, 3), (2, 3), (3, 3), (3, 2), (2, 1), (3, 1)}.

Here (1, 2) ∈ R ⇒ (2,1) ∈ R, (1, 3) ∈ R ⇒ (3, 1) ∈ R.

(2, 3) ∈ R ⇒ (3, 2) ∈ R

∴ R is a symmetric relation on A.

(iii) Transitive Relation :

A relation R defined on a set A is said to be transitive, if a R b, b R c, then a R c, $\forall$ a, b, c $\in$ A.

i.e. $(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R, \forall a, b, c \in A$

For example : Let A = {a, b, c} and R = {(a, a), (b. b), (b, c), (c, a), (c, c), (b, a)}, then R is a transitive relation on A.

(iv) Anti-Symmetric Relation :

A relation R defined on a set A is said to be anti-symmetric, if a R b, b R a, then a = b, $\forall$ a, b $\in$ A.

i.e. $(a, b) \in R, (b, a) \in R \Rightarrow a = b, \forall a, b \in A$.

For example : Let N be the set of natural numbers and R be a relation on N defined by " a is a divisor of b" i.e. a R b if a divides b, then R is anti-symmetric because a divides b and b divides a $\Rightarrow$ a = b.

(v) Asymmetric Relation :

A relation R defined on a set A is said to be asymmetric, if whenever a R b, then b $\not R$ a, $\forall$ a, b $\in$ A.

i.e. $(a, b) \in R$, but $(b, a) \notin R, \forall a, b \in A$

For example : Let A = {1, 2, 3} and R = {(1, 2), (1, 3), (3, 2)}, then R is an asymmetric relation on A.

(vi) Irreflexive Relation :

A relation R defined on a set A is said to be irreflexive if a $\not R$ a, $\forall$ a $\in$ A.

i.e. $(a, a) \notin R, \forall a \in A$

For example : Let A = {a, b. c, d} and R = {(a, b), (b, c), (b, d), (c, a)}, then R is an irreflexive relation on A.

(vii) Compatible Relation : ***CS/CT/CE : S-11***

A relation R defined on a set A is said to be compatible, if it is both reflexive and symmetric.

i.e. $(a, a) \in R, \forall a \in A$.

$(a, b) \in R \Rightarrow (b, a) \in R, \forall a, b \in A$

(viii) Equivalence Relation : ***CS/CT/CE : S-11***

A relation R defined on a set A is said to be equivalence relation if it is reflexive, symmetric and transitive,

i.e. $(a, a) \in R, \forall a, \in A$

$(a, b) \in R \Rightarrow (b, a) \in R, \forall a, b \in A$ and

$(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R, \forall a, b, c \in A$

(ix) Universal Relation :

A relation R defined on a set A is said to be universal relation if R = A × A

For example : Let A = {a, b, c}, then R = {(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)} is a universal relation on A.

(x) Partial order relation : ***IT : S-09, CS/CT/CE : S-11***

A relation R defined on a set A is called a partial order relation if it is reflexive, anti-symmetric and transitive.

i.e. $(a, a) \in R, \forall a \in A$

$(a, b) \in R, (b, a) \in R \Rightarrow a = b, \forall a, b \in A$

$(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R, \forall a, b, c \in A$

A set 'A' together with a partial order relation R is called partially ordered set or poset.

TRANSITIVE EXTENSION

Let R be a relation defined on a set A. Then a relation R_1 defined on the set A is called the transitive extension of R if $R \subseteq R_1$ and $(a, b) \in R \Rightarrow (a, c) \in R_1$

For example : Let A = {a, b, c, d},

R = { (a, b), (b, c), (c, b), (b, d) },

R_1 = { (a, b), (a, c), (b, b), (b, c), (b, d), (c, b), (c, c), (c, d) }

Here $R \subseteq R_1$ and clearly (a, b)

$\in R, (b, c) \in R \Rightarrow (a, c) \in R_1$

$\therefore$ R_1 is a transitive extension of R.

TRANSITIVE CLOSURE OF A RELATIONS

Let R be a relation defined on a set A. Let R_1 denote the transitive extension of R_1, R_2 denote the transitive extension of $R_1, \ldots\ldots\ldots, R_{n+1}$ denote then transitive extension of R_n, then the transitive closure of R is defined as the union of

$R, R_1, R_2, R_3, \ldots\ldots, R_{n+1}, \ldots\ldots$ It is denoted by R^+

$\therefore R^+ = R \cup R_1 \cup R_2 \cup \ldots\ldots \cup R_n \cup R_{n+1} \cup \ldots\ldots$

Ex.7. If both the relations A and B are reflexive, symmetric and transitive, then prove that $A \cap B$ is also reflexive, symmetric and transitive. ***IT : W-07***

Soln. (i) Since A and B are reflexive

$\therefore$ x A x, x B x

$\therefore$ x $(A \cap B)$ x = (x A x) $\cap$ (x B x)

$\therefore$ $A \cap B$ is reflexive.

(ii) Since A and B are symmetric

$\therefore$ x A y $\Rightarrow$ y A x, and

x B y $\Rightarrow$ y B x.

We now prove that x $(A \cap B)$ y $\Rightarrow$ y $(A \cap B)$ x

x $(A \cap B)$ y = (x A y) $\cap$ (x B y)

$\Rightarrow$ (y A x) $\cap$ (y B x)

$\Rightarrow y\,(A \cap B)\,x$

$\therefore$ $A \cap B$ is symmetric.

(iii) Since A and B are transitive

$\therefore x\,A\,y, y\,A\,z \Rightarrow x\,A\,z$ and

$x\,B\,y, y\,B\,z \Rightarrow x\,B\,z.$

We now prove that

$x\,(A \cap B)\,y, y\,(A \cap B)\,z \Rightarrow x\,(A \cap B)\,z$

$x\,(A \cap B)\,y, y\,(A \cap B)\,z$

$= (x\,A\,y) \cap (x\,B\,y), (y\,A\,z) \cap (y\,B\,z)$

$= (x\,A\,z), (y\,A\,z) \cap (x\,B\,y), (y\,B\,z)$

$\Rightarrow (x\,A\,z) \cap (x\,B\,z)$

$\Rightarrow x\,(A \cap B)\,z$

$\therefore$ $A \cap B$ is transitive. ..... **Ans.**

Ex.8. Prove that

$R = \{x\,R\,y \mid x^2 - 4xy + 3y^2 = 0, \forall x, y \in N\}$

is reflexive, but not symmetric.

Soln. (i) Reflexive :

We prove that $(x, x) \in R, \forall x \in N$

Now $x\,R\,y = x^2 - 4xy + 3y^2 = 0$

$x\,R\,x = x^2 - 4x^2 + 3x^2 = 0.$

This is true.

$\therefore$ R is reflexive

(ii) Symmetric :

We verify whether $x\,R\,y \Rightarrow y\,R\,x$

We have $x\,R\,y = x^2 - 4xy + 3y^2 = 0$

Then $y\,R\,x = y^2 - 4xy + 3x^2 = 0.$

But this is not true.

$\therefore$ R is not symmetric. ..... **Ans.**

Ex.9. If Z is a set of integers and a, b $\in$ Z, a R b if $b = a^r$ for some positive integer r, then show that R is a partial order on Z. ***CS/CT/CE: S-06***

Soln. Given : a R b, if $b = a^r$

where r is a positive integer and a, b $\in$ Z

To prove : R is a partial order on Z

We prove that R is reflexive, antisymmetric and transitive.

(i) Reflexive : Let $r = 1$, $a = a^r = a^1 = a$

$\therefore$ a R a, $\forall a \in Z$

$\therefore$ R is reflexive.

(ii) Antisymmetric : Let a R b and b R a, $\forall a, b \in Z$

Then $b = a^r$ and $a = b^r$

$\Rightarrow a = b$ (when $r = 1$)

i.e. a R b and b R a $\Rightarrow a = b\ \forall a, b \in Z$

$\therefore$ R is antisymmetric.

(iii) Transitive : Let a R b and b R c

Then $b = a^r$ and $c = b^r$

$\therefore c = (a^r)^r$ $[\because b = a^r]$

$\therefore c = a^{r^2}$

Since r is a positive integer, so is r^2

$\therefore$ a R c

i.e. a R b and b R c $\Rightarrow$ a R c

$\therefore$ R is transitive.

$\therefore$ $\boxed{\text{R is a partial order on Z.}}$ **Ans.**

Ex.10. Define a relation on the set S = {1, 2, 3, 4} which is

(i) reflexive symmetric but not transitive.

(ii) reflexive, transitive but not symmetric.

IT : S-06

Soln. Given : S = {1, 2, 3, 4}

(i) Let us define R as

R = { (1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (3, 4), (2, 4), (1, 4), (2, 1), (4, 1), (4, 3), (4, 2)}

Clearly R is reflexive as $(a, a) \in R, \forall a \in S$

Also R is symmetric as $(a, b) \in R$

$\Rightarrow (b, a) \in R, \forall a, b \in S$

But R is not transitive because $(3, 4) \in R, (4, 1) \in R$ but $(3, 1) \notin R$.

(ii) Let us define R as

R = { (1,1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 3), (1, 3), (3, 4), (2, 4), (2, 1), (3, 1), (1, 4) }

Clearly R is reflexive as $(a, a) \in R, \forall a \in S$

Also R is transitive as $(a, b) \in R, (b, c) \in R$

$\Rightarrow (a, c) \in R, \forall a, b, c, \in S$

But R is not symmetric because.

$(3, 4) \in R$ but $(4, 3) \notin R$. ..... **Ans.**

Ex.11. Show that the relation R defined on a set A is an equivalence relation. (a, b) R (c, d) $\Leftrightarrow$ a = c.

IT : W-06, CS/CT/CE: S-12

Soln. Given : (a, b) R (c, d) $\Leftrightarrow$ a = c

To Prove : R is an equivalence relation

We prove that R is reflexive, symmetric and transitive.

(i) Reflexive : We have (a, b) R (a, b) $\Leftrightarrow$ a = a

This holds true.

$\therefore$ R is reflexive.

(ii) **Symmetric :** We have (a, b) R (c, d)

$\Leftrightarrow a = c$

$\Leftrightarrow c = a$

$\Leftrightarrow$ (c, d) R (a, b)

$\therefore$ R is symmetric.

(iii) **Transitive :** We have (a, b) R (c, d) $\Leftrightarrow a = c$

And (c, d) R (e, f) $\Leftrightarrow c = e$

$\therefore$ (a, b) R (c, d), (c, d) R (e, f) $\Leftrightarrow a = c$ and $c = e$.

$\Leftrightarrow a = e$

$\Leftrightarrow$ (a, b) R (e, f)

$\therefore$ R is transitive.

$\therefore$ R is an equivalence relation. **Ans.**

Ex.11.(a) Let A = {a, b, c, d, f, g, h} consider the following subsets of A. A_1 = {a, b, c, d}, A_2 = {a, c, e, f, g, h}, A_3 = {a, c, e, g}, A_4 = {b, d}, A_5 = {f, h}. Write all possible partitions of A. ***IT : W-06***

Soln. The possible partitions of A are

$P_1 = \{A_2, A_4\}$ and $P_2 = \{A_3, A_4, A_5\}$

Ex.12. Given the set S = {1, 2, 3, 4, 5}. Find the relations R_A and R_B generating the partitions.

(i) A = { (1, 2), (3), (4, 5)} (ii) { (1, 2, 3), (4, 5)}

Prove that R_A and R_B are equivalence relations.

IT : W-09, S-07,10

Soln. Given : S = {1, 2, 3, 4, 5}

A = { (1, 2), (3), (4, 5) }

$\therefore$ R_A = { (1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4), (4, 5), (5, 4), (5, 5) }

B = { (1, 2, 3) , (4, 5) }

$\therefore$ R_B = { (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (4, 4), (4, 5), (5, 4), (5, 5) }

Now we prove that R_A and R_B are equivalence relations.

(i) **Reflexive :** We observe that (a, a) $\in R_A$ and $(a, a) \in R_B, \forall a \in S$.

$\therefore$ R_A and R_B are reflexive.

(ii) **Symmetric :** We observe that

$(a, b) \in R_A \Rightarrow (b, a) \in R_A, \forall a, b \in R_A$

Also $(a, b) \in R_B \Rightarrow (b, a) \in R_B, \forall a, b \in R_B$

$\therefore$ R_A and R_B are symmetric.

(iii) **Transitive :** We observe that

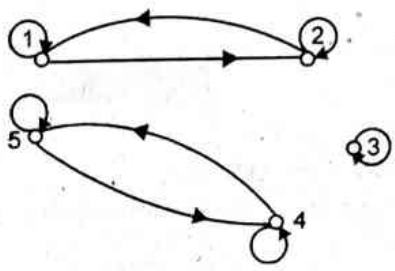
$(a, b), (b, c) \in R_A \Rightarrow (a, c) \in R_A, \forall a, b, c \in R_A$

Also $(a, b), (b, c) \in R_B \Rightarrow (a, c) \in R_B, \forall a, b, c \in R_B$

$\therefore$ R_A and R_B are transitive.

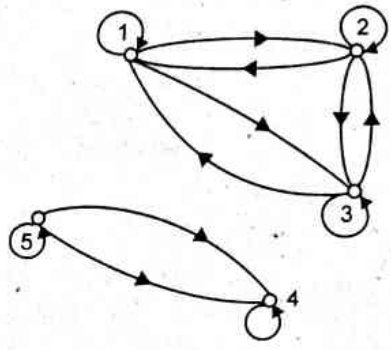
$\therefore$ R_A and R_B are equivalence relations. .....**Ans.**

Graph of R_A :



$$M_{R_A} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Graph of R_B :



$$M_{R_B} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

COVERING OF A SET

Lets S be a non-empty set and

$A = \{A_1, A_2, ..., A_n\}$ where each A_i,

$i = 1, 2, ..., n$ be a subset of S and $\bigcup_{i=1}^{n} A_i = S$

Then the set A is called a covering of S and the sets $A_1, A_2, ..., A_n$ are said to cover S.

Results :

(1) Let P be a partition of the set A. Let R be a relation defined on A by a R b $\Leftrightarrow$ a and b are members of the same block. Then R is an equivalence relation on A.

(2) Let R be an equivalence relation defined on a set A. Then R induces a partition on A. If P is a partition, then P = A / R. The elements of A / R are called quotient sets of A with respect to R.

(3) For any element $a \in A$, define [a] as

$[a] = \{x \in A \mid a R x\}$

Method of finding A/R :

(i) Choose $a \in A$ and find [a]

(ii) If $[a] \neq A$, then choose $b \in A$ and $b \notin [a]$ and find [b].

(iii) If $[a] \cup [b] \neq A$, then choose $c \in A$ and $c \notin [a] \cup [b]$ and find [c].

(iv) Repeat the above steps untill all elements of A are included in the computed equivalence classes.

Ex.13. Let A = R × R. A relation R on 'A' is defined as (a, b) R (c, d) iff $a^2 + b^2 = c^2 + d^2$.

(i) Show that R is an equivalence relation.

(ii) Compute A | R. ***IT : S-06, W-07***

Soln. (i) Given : (a, b) R (c, d), iff $a^2 + b^2 = c^2 + d^2$

To Prove : R is an equivalence relation.

We prove that R is reflexive, symmetric and transitive.

(a) **Reflexive :** We have

(a, b) R (a, b) $\Leftrightarrow a^2 + b^2 = a^2 + b^2$

This holds true

∴ R is reflexive.

(b) **Symmetric :** We have (a, b) R (c, d)

$\Leftrightarrow a^2 + b^2 = c^2 + d^2$

$\Leftrightarrow c^2 + d^2 = a^2 + b^2$

$\Leftrightarrow$ (c, d) R (a, b)

∴ R is symmetric.

(c) **Transitive :** We have

(a, b) R (c, d) $\Leftrightarrow a^2 + b^2 = c^2 + d^2$

And (c, d) R (e, f) $\Leftrightarrow c^2 + d^2 = e^2 + f^2$

∴ (a, b) R (c, d), (c, d) R (e, f) $\Leftrightarrow a^2 + b^2 = c^2 + d^2$

and $c^2 + d^2 = e^2 + f^2$

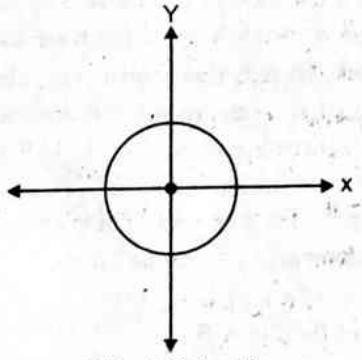
$\Leftrightarrow a^2 + b^2 = c^2 + f^2$

$\Rightarrow$ (a, b) R (e, f)

∴ R is transitive.

∴ R is an equivalence relation.

(ii) The collection of all equivalent classes of elements of A under the equivalence relation R is denoted by A/R and is given by



A/R = {[x]} |x∈ A}

Here the equivalent classes of A/R are circles with centre (0, 0) since for any point (a, b) the sum $a^2 + b^2$ is the square of its distance from the origin. ....**Ans.**

Ex.14. Let A = {a, b, c, d} and P = { {a, b, c}, {d} } be a partition of A. Find the equivalence relation determined by P?

Soln. Each element in the given blocks {a, b, c} and {d} is related to every other element in the same block and only to those elements of the block. Hence the required equivalence relation is

R = { (a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c), (d, d) } ..... **Ans.**

Ex.15. Let A = {a, b, c, d} and R = { (a, a), (a, b), (b, a), (b, b), (c, d), (c, c), (d, c), (d, d) } be an equivalence relation on R. Find A/R.

Soln. Here [a] = {a, b}

[b] = {b, a} = {a, b}

∴ [a] = [b]

[c] = {c, d}

[d] = {d, c} = {c, d}

∴ [c] = [d]

∴ A/R = { [a], [c]} or { [b], [d]} ....**Ans.**

Ex.16. Let Z be the set of integers and let R be the relation defined by, R = {(x, y) | x, y ∈ Z and x-y is divisible by 3}.

Determine the equivalence class generated by the elements of R.

Soln. The equivalence classes generated by the elements of R are

[0] = {....., – 6, – 3, 0, 3, 6, ...}

= Set of elements of Z related to '0'.

[1] = {....., – 5, – 2, 1, 4, 7, ...}

= Set of elements of Z related to '1'.

[2] = {....., – 4, – 1, 2, 5, 8, ...}

= Set of elements of Z related to '2'.

[3] = {....., – 6, – 3, 0, 3, 6, ...}

= Set of elements of Z related to '3'.

..... and so on. .....**Ans.**

Ex.17. A relation R is defined by R = { (A, B)/ A ⊆ B}. Show that R is a partial order relation. Is this releation compatible? ***CS/CT/CE : W-08***

Soln. Given : R = { (A, B) | A ⊆ B}

To Prove : R is a partial order relation.

We prove that R is reflexive, antisymmetric and transitive.

(i) **Reflexive :** We prove A R A

A ⊆ A is trivially true.

∴ A R A

∴ R is reflexive.

(ii) **Antisymmetric :** We prove A R B and B R A $\Rightarrow A = B$

If $A \subseteq B$ and $B \subseteq A$, then $A = B$

$\therefore$ R is antisymmetric.

(iii) **Transitive :** We prove A R B and B R C $\Rightarrow$ A R C

If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$

$\therefore$ R is transitive.

$\therefore$ R is a partial order relation.

(iv) **Symmetric :** We verify whether A R B $\Rightarrow$ B R A

If $A \subseteq B$, then $B \subseteq A$ is not true.

$\therefore$ R is not symmetric.

$\therefore$ R is not a compatible relation. ..... **Ans.**

Ex.18. Let R denote a relation on the set of ordered pairs of positive integers defined by (x, y) R (u, v) if and only if xv = yu, show that R is an equivalence relation.

IT : W-09,11, S-12,13, CS/CT/CE : S-10,11,13, W-12

Soln. Given : $(x, y) R (u, v) \Leftrightarrow xv = yu$

To Prove : R is an equivalence relation.

We prove that R is reflexive, symmetric transitive.

(i) **Reflexive :** We prove (x, y) R (x, y)

Since $(x, y) R (u, v) \Leftrightarrow xv = yu$

$\therefore (x, y) R (x, y) \Leftrightarrow xy = yx$

$\Leftrightarrow xy = xy$

This is trivially true.

$\therefore$ R is reflexive.

(ii) **Symmetric :** We prove (x, y) R (u, v)

$\Rightarrow (u, v) R (x, y)$

Since $(x, y) R (u, v) \Leftrightarrow xv = yu$

$(u, v) R (x, y) \Leftrightarrow uy = vx$

$\Leftrightarrow yu = xv \Leftrightarrow xv = yu$

$\Leftrightarrow (x, y) R (u, v)$

$\therefore (x, y) R (u, v) \Leftrightarrow (u, v) R (x, y)$

$\therefore$ R is symmetric.

(iii) **Transitive :** We prove (x, y) R (u, v) and (u, v) R (m, n) $\Rightarrow$ (x, y) R (m, n)

Since $(x, y) R (u, v) \Leftrightarrow xv = yu$

And $(u, v) R (m, n) \Leftrightarrow un = vm$

$\Leftrightarrow (xv)(un) = (yu)(vm)$

$\Leftrightarrow (xn)(vu) = (ym)(uv)$

$\Leftrightarrow (xn)(vu) = (ym)(vu)$

$\Leftrightarrow xn = ym$

$\Leftrightarrow (x, y) R (m, n)$

$\therefore (x, y) R (u, v)$ and $(u, v) R (m, n) \Rightarrow (x, y) R (m, n)$

$\therefore$ R is transitive.

$\therefore$ R is an equivalence relation. ..... **Ans.**

Ex.19. Let X = {1, 2, 3, 4, 5, 6, 7, 8, 9} and

$R_1 = \{(x,y) \mid x - y$ is divisible by 3}

$R_2 = \{(x,y) \mid x, y \in X$ and x, y are in the same column of matrix A }

where $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

IT : W-08, S-09, CS/CT/CE: W-09

Find R_1 and R_2. Prove that R_1 and R_2 are equivalence relations. Also prove that $R_1 = R_2$.

Soln. Given : X = {1, 2, 3, 4, 5, 6, 7, 8, 9}

$R_1 = \{(x, y) \mid x - y$ is divisible by 3}

$\therefore R_1 = \{(1, 1), (1, 4), (1, 7), (2, 2), (2, 5), (2, 8), (3, 3), (3, 6), (3, 9), (4, 1), (4, 4), (4, 7), (5, 2), (5, 5), (5, 8), (6, 3), (6, 6), (6, 9), (7, 1), (7, 4), (7, 7), (8, 2), (8, 5), (8, 8), (9, 3), (9, 6), (9, 9)\}$

Given : $R_2 = \{(x, y) \mid x, y \in X$ and x, y are in the same column of matrix A}

where $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$\therefore R_2 = \{(1, 1), (1, 4), (1, 7), (2, 2), (2, 5), (2, 8), (3, 3), (3, 6), (3, 9), (4, 1), (4, 4), (4, 7), (5, 2), (5, 5), (5, 8), (6, 3), (6, 6), (6, 9), (7, 1), (7, 4), (7, 7), (8, 2), (8, 5), (8, 8), (9, 3), (9, 6), (9, 9)\}$

Clearly $R_1 = R_2$.

To Prove : $R_1 = R_2$ are equivalence relations.

We prove that $R_1 = R_2$ are reflexive, symmetric and transitive.

(a) **Reflexive :** Consider R_1

Since $(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7), (8, 8), (9, 9) \in R_1$

Since $R_1 = R_2$

$\therefore R_2$ is also reflexive.

(b) **Symmetric :** Consider R_1

Clearly R_1 is symmetric because

$(a, b) \in R_1 \Rightarrow (b, a) \in R_1$ for all $a, b \in X$

Since $R_1 = R_2$

$\therefore R_2$ is also symmetric.

(c) **Transitive :** Consider R_1

Clearly R_1 is transitive because

$(a, b) \in R_1, (b, c) \in R_1 \Rightarrow (a, c) \in R_1$ for all $a, b, c \in X$

Since $R_1 = R_2$

$\therefore$ R_2 is also transitive

$\therefore$ R_1 and R_2 are equivalence relations. ..... **Ans.**

Ex.20. Draw the digraph corresponding to the matrix.

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

. Find the transitive closure of the digraph. ***W-06, S-10***

Soln. Given : $M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

The digraph corresponding to this matrix is drawn below.

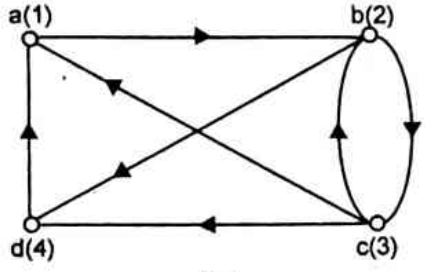


Fig.

Also, here R = { (1, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1)}

Now $R^2 = R \circ R$ = {(1, 3), (1, 4), (2, 1), (2, 2), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 2)}

$R^3 = R^2 \circ R$ = {(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 3), (4, 4)}

$R^4 = R^3 \circ R$ = {(1, 1), (1, 2), (1,3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 4)} = R^2.

$\therefore$ Transitive closure of $R = R \cup R^2 \cup R^3 \cup R^4$

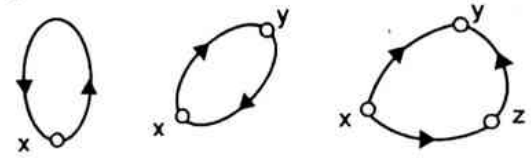
= {(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4)}. ..... **Ans.**

MATRIX OF RELATION

Let R be a relation from one set X to another set Y. Then R can be represented by a matrix called relation matrix M_R and the elements of the relation matrix are given by

$$M(R) = [m_{ij}] = \begin{cases} 1, & \text{if } x_i R y_j \\ 0, & \text{Otherwise} \end{cases}$$

Note : The graph of an equivalence relation can be represented as follows.



(i) Reflexive (ii) Symmetric (ii) Transitive

Ex.21. Let X = {1, 2, 3, 4} and R = { (x, y) | { x > y }. Draw the graph of R and find the relation matrix M_R

Soln. Given : X = {1, 2, 3, 4}

R = { (x, y) |{ x > y }

$\therefore$ R = {(2, 1), (3, 1), (3, 2), (4, 1), (4, 2) (4, 3)}

Graph of R :

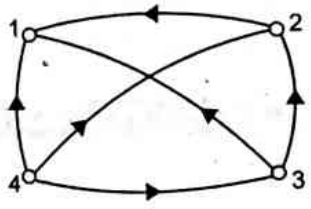


Fig.

Relation Matrix M_R :

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Ex.22. Let A = {a, b, c} and the subsets of A are denoted by

$B_0 = \phi$, $B_1 = \{c\}$, $B_2 = \{b\}$, $B_3 = \{b, c\}$, $B_4 = \{a\}$, $B_5 = \{a, c\}$, $B_6 = \{a, b\}$, $B_7 = \{a, b, c\}$. If R is a relation of proper inclusion on the subsets B_0 to B_7. find the relation matrix M_R and draw the graph.

Soln. Given : A = {a, b, c}.

$B_0 = \phi$, $B_1 = \{c\}$, $B_2 = \{b\}$,

$B_3 = \{b, c\}$, $B_4 = \{a\}$, $B_5 = \{a, c\}$,

$B_6 = \{a, b\}$, $B_7 = \{a, b, c\}$.

Relation R = {(B_0, B_1), (B_0, B_2), (B_0, B_3), (B_0, B_4), (B_0, B_5), (B_0, B_6), (B_0, B_7), (B_1, B_3), (B_1, B_5), (B_1, B_7), (B_2, B_3), (B_2, B_6), (B_2, B_7), (B_3, B_7), (B_4, B_5), (B_4, B_6), (B_4, B_7), (B_5, B_7), (B_6, B_7)}

Graph of R is drawn below.

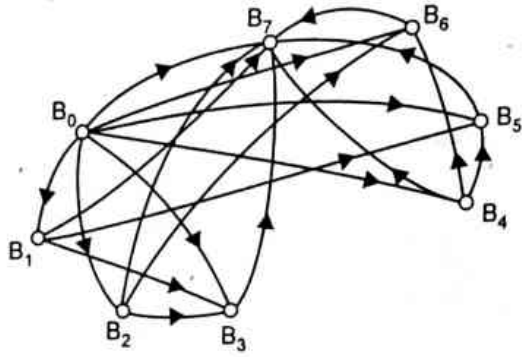


Fig.

Relation Matrix M_R :

$$M_R = \begin{array}{c} \\ B_0 \\ B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \\ B_7 \end{array} \begin{array}{c} \begin{array}{cccccccc} B_0 & B_1 & B_2 & B_3 & B_4 & B_5 & B_6 & B_7 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Ex.23. Let X = {1, 2, 3, 4, 5, 6, 7} and R = { (x, y) | x – y is divisible by 3}. Draw the graph of R and find the relation matrix M_R. Show that R is an equivalence relation.

Soln. Given : X = {1, 2, 3, 4, 5, 6, 7}

R = {(x, y) | x – y is divisible by 3}

∴ R = { (1, 1), (1, 4), (1, 7), (2, 2), (2, 5), (3, 3), (3, 6), (4, 1), (4, 4), (4, 7), (5, 2), (5, 5), (6, 3), (6, 6), (7, 1), (7, 4), (7, 7)}

Graph of R :

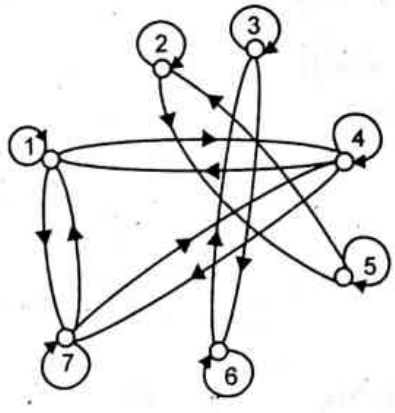


Fig.

Relation Matrix M_R :

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

To Prove : R is an equivalence relation.

We prove that R is reflexive, symmetric, transitive.

(a) **Reflexive :** We observe that

(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7) ∈ R

∴ R is reflexive

(b) **Symmetric :** We observe that

(a, b) ∈ R ⇒ (b, a) ∈ R, for all a, b ∈ R

∴ R is symmetric.

(c) **Transitive :** We observe that

(a, b) ∈ R, (b, c) ∈ R ⇒ (a, c) ∈ R, for all a, b, c ∈ R

∴ R is transitive

∴ R is equivalence relation. **...Ans.**

COMPOSITE RELATION (COMPOSITION OF RELATIONS)

Let R be a relation defined from x to y. Let S be another relation defined from y to z. Then the composition of these two relations R and S is denoted by R o S (read as R circle S) and is defined by

RoS = {(x, z) | ∃ y, (x, y) ∈ R, (y, z) ∈ S}

Note :

(1) Composition of relations obeys associative law, but does not obey commutative law, i.e. R o (S o T) = (R o S) o T, but R o S ≠ S o R

(2) $M_{RoS} = M_R . M_S$ or $M_{SoR} = M_S$ o M_R

(3) $M_{R^{-1}}$ = Transpose of M_R

(4) $M_{(RoS)^{-1}} = M_{S^{-1}}$ o $M_{R^{-1}}$

(5) $(M_R)'$ = Complement of M_R

Ex.24. Let R = {(1,2), (3, 4), (2, 2)} and S = {(2,5), (4, 2), (3, 1), (1, 3)}. Find (i) RoS (ii) SoR (iii) Ro(SoR) (iv) (RoS)oR (v) RoR (vi) SoS (vii) RoRoR

Soln. Given : R = {(1, 2), (3, 4), (2, 2)}

S = {(2, 5), (4, 2), (3, 1), (1, 3)}

(i) RoS = {(1, 5), (3, 2), (2, 5)}
(ii) SoR = {(4, 2), (3, 2), (1, 4)}
(iii) Ro (SoR) = {(3, 2)}
(iv) (RoS) oR = {(3, 2)}
(v) RoR = {(1, 2), (2, 2)}
(vi) SoS = {(4, 5), (3, 3), (1, 1)}
(vii) RoRoR = {(1, 2), (2, 2)}

Ex.25. Let R = {(1, 2), (3, 4), (2, 2) and S = {(4, 2), (2, 5), (3, 1), (1, 3)}. Find the relation matrix for R, S, RoS, SoR.

Soln. Given : R = {(1, 2), (3, 4), (2, 2)}
S = {(4, 2), (2, 5), (3, 1) , (1, 83)}
Let X = {1, 2, 3, 4, 5}
RoS = {{(1, 5), (3, 2), (2, 3)}
SoR = {(4, 1), (3, 2), (1, 4)}

Relation Matrix for R (M_R) :

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Relation Matrix for S (M_S) :

$$M_S = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Relation Matrix for RoS (M_{RoS}) :

$$M_{RoS} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Relation Matrix for SoR (M_{SoR}) :

$$M_{SoR} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

.....Ans.

CONVERSE OF A RELATION

Let R be a relation defined from X to Y. Then a relation defined from Y to X is called converse of relation R and is denoted by $\overline{R}$.

$\overline{R} = \{(x, y) | (y, x) \in R\} = R^{-1}$

If = $x \in X, y \in Y$, then $x R y \Leftrightarrow y \overline{R} x$

Note : $M_{\overline{R}} = M_{R^{-1}}$ = Transpose of M_R.

Ex.26. The relation matrices of two relations R and S are given by

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, M_S = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Find R, S, RoS and show that RoS is not an equivalence relation.

Soln. Given : $M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, M_S = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Let X = {1, 2, 3}
Here, R = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)}
S = {(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3)}
∴ RoS = {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3)}

$$M_{RoS} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Now we prove that RoS is not an equivalence relation.
Since (1, 1), (2, 2), (3, 3) ∈ RoS
∴ RoS is reflexive.
Also (1, 3) ∈ RoS, but (3, 1) ∉ RoS
∴ RoS is not symmetric.
∴ RoS is not an equivalence relation. **....Ans.**

Ex.27. Let A = {1, 2, 3}. The relation matrices M_R and M_S are given by.

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, M_S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find $M_{R^{-1}}, (M_R)', M_{S^{-1}}, (M_S)', M_{R \cup S}, M_{R \cap S}$

Soln. Given : $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, M_S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$M_{R^{-1}} = \text{Transpose of } M_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$(M_R)' = \text{Complement of } M_R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\text{Also } M_{S^{-1}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

$$(M_S)' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$M_{R \cup S} = M_R \vee M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore M_{R \cup S} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M_{R \cap S} = M_R \wedge M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \wedge \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore M_{R \cap S} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ ..... Ans.}$$

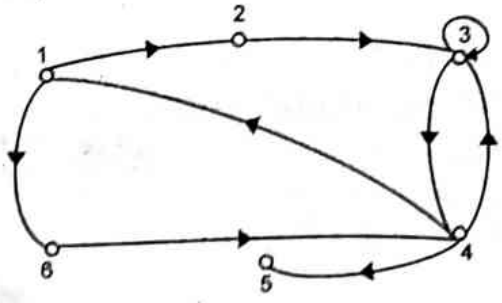
Ex.28. A relation R is defined on a set A where A = {1, 2, 3, 4, 5, 6} and R = {(1, 2), (1, 6), (2, 3), (3, 3), (3, 4), (4, 1), (4, 3), (4, 5), (6, 4)}. Draw the graph of R and R^2. Also find the matrix for R^2

Soln. Given : A = {1, 2, 3, 4, 5, 6}

R = {(1, 2), (1, 6), (2, 3), (3, 3), (3, 4), (4, 1), (4, 3), (4, 5), (6, 4)}

R^2 = R o R = {(1,3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 3), (3, 4), (3, 5), (4, 2), (4, 3), (4, 4), (4, 6), (6, 1), (6, 3), (6, 5)}

Graph of R



Graph of R^2

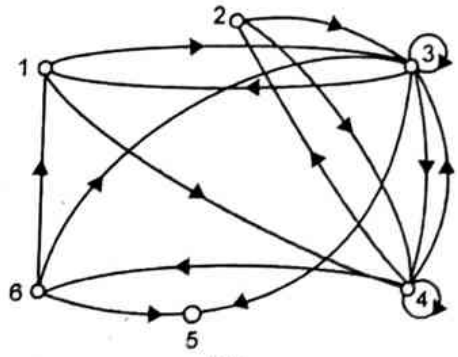


Fig.

Relation Matrix for R^2 (M_{RoR}):

$$M_{RoR} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

...Ans.

Ex.29. Let A = {0, 1, 2, 3} . Find the matrices for R = {(x, y) | x + y = 3}, S = { (x, y) | x + y ≤ 4}. Also find $(M_R . M_S)'$ and show that $(M_R . M_S)' \neq (M_S)' . (M_R)'$

Soln. Given : A = {0, 1, 2, 3}

R = {(x, y) | x + y = 3}

i.e. R = {(0, 3), (1, 2), (3, 0), (2, 1)}

S = {(x, y) | x + y ≤ 4}

i.e. S = {(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 1), (2, 2), (3, 0), (3, 1)}

Relation Matrix for R (M_R) and S (M_S) :

$$M_R = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad M_S = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$(M_R)' = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \quad (M_S)' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

$$M_R . M_S = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$(M_R \cdot M_S)' = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{.....(1)}$$

$$\text{Now } (M_S)'(M_R)' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 2 & 2 & 3 & 2 \end{bmatrix} \quad \text{.....(2)}$$

From (1) and (2), we observe that

$(M_R \cdot M_S)' \neq (M_S)' \cdot (M_R)'$ ..... **Ans.**

Ex.30. Let A = {1, 2, 3}, B = {a, b, c, d}. Let R and S be relations from A to B with relation matrices given by $M_R = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}, M_S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$. **Find**

(i) $M_{R^{-1}}$ **and** $M_{S^{-1}}$.

(ii) Show that : $M_{(R \cap S) \circ R^{-1}} = M_{RoR^{-1}} \cap M_{SoR^{-1}}$

(iii) Find : $M_{R^{-1} \cup S^{-1}}$ **CS/CT/CE : S-09, W-11**

Soln. Given : A = {1, 2, 3}, B = {a, b, c, d}

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \end{array}, \quad M_S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

i.e. R = {(1,a), (1, b), (1, c), (3, a), (3, b), (3, d)}

S = {(1, b), (2, a), (2, d), (3, b), (3, c)}

(i) $M_{R^{-1}} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, M_{S^{-1}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

(ii) $M_{(R \cap S)} = M_R \wedge M_S$

$$= \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \wedge \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$M_{(R \cap S)} \cdot R^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= M_{(R \cap S)} \cdot M_{R^{-1}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{.....(1)}$$

Now,

$M_{RoR^{-1}} = M_R \cdot M_{R^{-1}}$

$$= \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$M_{SoR^{-1}} = M_S \cdot M_{R^{-1}}$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\therefore M_{RoR^{-1}} \cap M_{SoR^{-1}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \wedge \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{.....(2)}$$

From (1) and (2), we observe that

$M_{(R \cap S) oR^{-1}} = M_{RoR^{-1}} \cap M_{SoR^{-1}}$

(iii) $M_{R^{-1} \cup S^{-1}} = M_{R^{-1}} \vee M_{S^{-1}}$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \vee \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \text{ ...Ans.}$$

Ex.31. For the relation R = {(1, 2), (2, 3), (3, 4), (2, 1)} Find M_R and draw the graph. Also find transitive closure of R. **IT : S-08,11, W-08**

Soln. Given : R = {(1, 2), (2, 3), (3, 4), (2, 1)}

Let X = {1, 2, 3, 4}

Relation Matrix M_R :

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Graph of R :

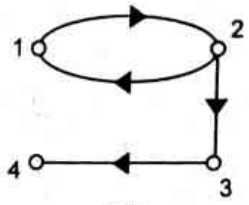


Fig.

To find : The transitive closure :

$$M_{R^2} = M_R \cdot M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{R^2} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

i.e. $R^2 = \{(1, 1), (1, 3), (2, 2), (2, 4)\}$

$$M_{R^3} = M_{R^2} \cdot M_R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_{R^3} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

i.e. $R^3 = \{(1, 2), (1, 4), (2, 1), (2, 3)\}$

$\therefore$ Transitive closure of $R = R \cup R^2 \cup R^3 \cup .....$

$R^+ = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), ...\}$ ..... **Ans.**

Ex.32. Let X = {ball, bed, dog, egg, let} and R = {(x, y) | x, y ∈ X, x R y if x and y contain same common letter}.

(i) Draw the graph of R

(ii) Write M_R.

(iii) Prove that R is compatible.

CS/CT/CE : W-07, IT : S-09, W-11

Soln. Given : X = {ball, bed, dog, egg, let}

Let us denote ball by a, bed by b, dog by c, egg by d and let by e

$\therefore X = \{a, b, c, d, e\}$

Given : $R = \{(x, y) \mid x, y \in X, x\ R\ y$ if x and y contain some letter$\}$

$\therefore R = \{(a, a), (a, b), (a, e), (b, a), (b, b), (b, c), (b, d), (b, e), (c, b), (c, c), (c, d), (d, b), (d, c), (d, d), (d, e), (e, a), (e, b), (e, d), (e, e)\}$

Relation Matrix M_R :

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Graph of R :

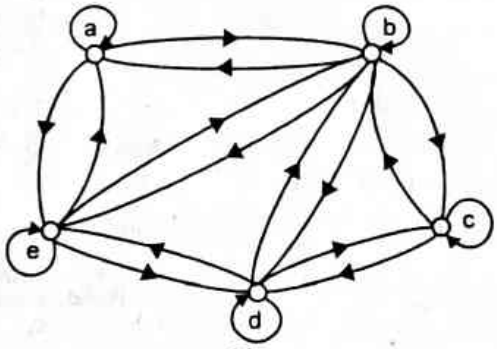


Fig.

From the graph it is clear R is reflexive and symmetric

$\therefore$ R is a compatible relation.

Also $(a, b) \in R$ and $(b, d) \in R$ but $(a, d) \notin R$

$\therefore$ R is not transitive. ..... **Ans.**

Ex.33. Let the compatibility relation on a set $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ be given as follows. Draw the graph. ***CS/CT/CE : W-06***

x_2	1	-	-	-	-
x_3	1	1	-	-	-
x_4	0	0	1	-	-
x_5	0	0	1	1	-
x_6	1	0	1	0	1
	x_1	x_2	x_3	x_4	x_5

Soln.

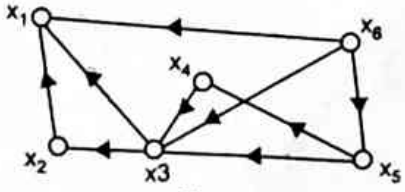


Fig.

Ex.34. Write examples of following relations on a set {1, 2, 3, 4}

(i) Equivalence Relation,

(ii) Compatibility Relation,

(iii) Partial ordered Relation. **CS/CT/CE : W-06**

Soln. (i) Equivalence relation:

$R_1 = \{(1,2), (1,1), (2,1), (2,2), (3,3), (4,4), (1,3), (2,3), (2,4), (1,4)\}$

(ii) Compatibility relation:

$R_3 = \{(1, 1), (1, 4), (2, 2), (2,3), (3,2), (3,1), (4,4), (4,3), (4,2)\}$

(iii) Partial ordered relation:

$R_2 = \{(1,1), (1, 2), (1, 3), (1, 4), (2, 2), (2,4), (3,3), (3,4), (4, 4)\}$ .....**Ans.**

Ex.35. Let S be the product set A × B.

If $|A| = m$ and $|B| = n$, then how many relations are there on S? **IT : W-06**

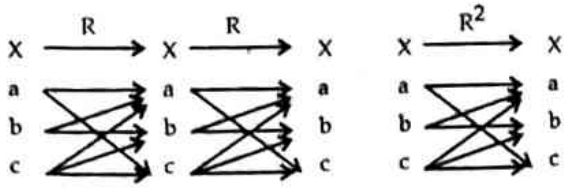
Soln. There will be mn relations on S.

Ex.36. Let X = {a, b, c} and R = {(a, a) (a, c) (b, a) (b, b) (c, a) (c, b) (c, c) }

Find R, R^2, R^3 and prove that $M_{ROR} = M_{R^2}$.

Draw graph of R and R^2. **CS/CT/CE : S-07**

Soln. Given : $R = \{(a, a)(a, c)(b, a)(b, b)(c, a)(c, b)(c, c)\}$



$R \cdot R = R^2 = \{(a, a)(a, b)(a, c), (b, a)(b, b)$
$(b, c)(c, a)(c, b)\ (c, c)\}$

Also $R^3 = R \cdot R \cdot R = \{(a, a)(a, b)(a, c)(b, a)(b, b)$
$(b, c)(c, a)(c, b)$

We know that

$M_{R \circ R} = M_R \circ M_R$

Since

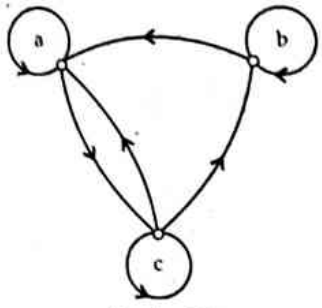
$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\text{L.H.S.} = M_{R \cdot R} = M_R \cdot M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

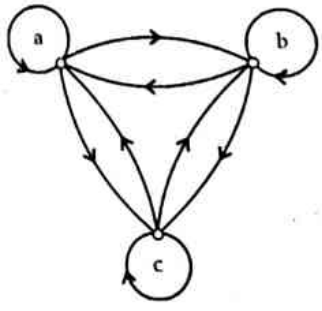
$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\text{And R.H.S.} = M_{R^2} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$\therefore\ M_{R \cdot R} = M_{R^2}$



Graph of R



Graph of R^2

Ex.37. Let A = {1, 2, 3, 4}

Let $R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (2, 4), (3, 4), (4, 1)\}$

$R_2 = \{(1, 1), (1, 3), (2, 3), (3, 2), (3, 4)\}$. **IT : S-07**

Write M_{R_1} and M_{R_2} and the di- graphs of the relations $R_1 \cup R_2$.

Soln. Let $A = \{1, 2, 3, 4\}$

$\therefore\ a_1 = 1,\ a_2 = 2,\ a_3 = 3,\ a_4 = 4$

and $b_1 = 1,\ b_2 = 2,\ b_3 = 3,\ b_4 = 4$

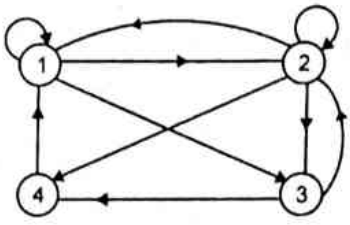
$$\therefore M_{R_1} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

and $M_{R_2} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Now $R_1 \cup R_2 = \{(1, 1)\ (1, 2), (1, 3), (2, 1)\ (2, 2)\ (2, 3), (3, 2), (3, 4)\ (4, 1)\ (2, 4)\}$

$\therefore$ Diagraph of the relation $R_1 \cup R_2$ is as follows



Graph of $R_1 \cup R_2$

Ex.38. Let Z be the set of integers and R be the relation defined by

$R = \{(x, y) : x \in Z, y \in Z \wedge (x - y)$ is divisible by $3\}$.

Determine the equivalence classes generated by R.

CS/CT/CE : W-08

Soln. (This relation R is also called "Congruence module 3")

The equivalence classes generated by R are

$[0]_R = \{ \ldots -6, -3, 0, 3, 6, \ldots \}$

= Set of elements of Z related to 0.

$[1]_R = \{ \ldots -5, -2, 1, 4, 7 \}$

= Set of elements of Z related to 1.

$[2]_R = \{ \ldots -4, -1, 2, 5, 8, \ldots \}$

The quotient set of Z by R is $\frac{Z}{R} = \{[0]_R, [1]_R, [2]_R\}$

Note that $[0]_R = [3]_R = [6]_R = \ldots\ldots$

$[1]_R = [4]_R = [7]_R = \ldots\ldots$

$\therefore Z = [0]_R \cup [1]_R \cup [2]_R$

We get a partition of Z by partitions

$[0]_R, [1]_R, [2]_R$

COMPATIBLE RELATION

Compatible relation : A relation R in X is said to be compatible if it is reflexive and symmetric.

Ex.39. Let A = {1, 2, 3, 4} and let

$R_1 = \{(1,1), (1,2), (2,3), (2,4), (3,4), (4,1), (4,2)\}$ and

$R_2 = \{(3,1), (4,4), (2,3), (2,4), (1,1), (1,4)\}$.

Verify:

(i) $M_{R_1 \cdot R_2} = M_{R_1} \cdot M_{R_2}$

(ii) $M_{R_1^c}$ = transpose of M_{R_1}

(iii) $M_{(R_1 \cdot R_2)^c} = M_{R_2^c} \cdot M_{R_1^c}$.

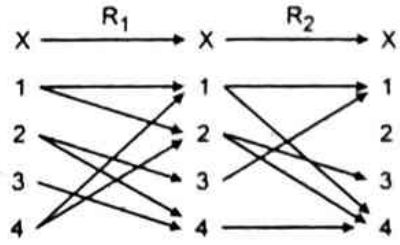
IT : W-09

Soln. Let A = {1, 2, 3, 4}

and let $R_1 = \{(1, 1), (1, 2), (2, 3), (2, 4), (3, 4), (4, 1), (4, 2)\}$ and

$R_2 = \{(3,1), (4,4), (2,3), (2,4)\ (1, 1), (1, 4)\}$

(i)



$R_1 \cdot R_2 = \{(1,1), (1,4), (1,3), (2,1), (2,4), (3,4), (4,1)\ (4,4), (4,3)\}$

$$M_{R_1 \cdot R_2} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \quad \ldots (1)$$

$$M_{R_1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \text{ and } M_{R_2} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_{R_1} \cdot M_{R_2} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \quad \ldots (2)$$

From (1) and (2),

$M_{R_1 \cdot R_2} = M_{R_1} \cdot M_{R_2}$

(ii) $$[M_{R_1}]^T = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad \ldots (3)$$

$\tilde{R}_1 = \{(1,1)(2,1), (3,2), (4,2), (4, 3), (1, 4), (2,4)\}$

$$M_{\tilde{R}_1} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad \ldots (4)$$

From (3) and (4)

$M_{R_1^c}$ = transpose of M_{R_1}

(iii) $\widetilde{(R_1 \cdot R_2)} = (R_1 \cdot R_2)^c = \{(1, 1), (4, 1), (3, 1), (1, 2), (4, 2), (4, 3), (1, 4), (4, 4), (3, 4)\}$

$(R_1)^c = \widetilde{R}_1 = \{(1, 1), (2, 1), (3, 2), (4, 2), (4, 3), (1, 4), (2, 4)\}$

$(R_2)^c = \widetilde{R}_2 = \{(1, 3), (4, 4), (3, 2), (4, 2), (1, 1), (4, 1)\}$

$$M_{(R\,R_2)^c} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{.... (5)}$$

$$M_{R_2^c} \cdot M_{R_1^c} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \quad \text{..... (6)}$$

From (5) and (6), we have

$M_{(R_1 \cdot R_2)^c} = M_{R_2^c} \cdot M_{R_1^c}$. .....**Ans.**

Ex.40. Let $X = \{1, 2, 3, 4, 5\}$

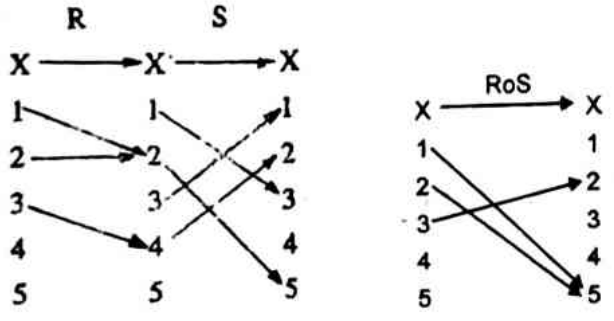
$R = \{(1, 2)\ (3, 4)\ (2, 2)\}$,

$S = \{(4, 2)\ (2, 5)\ (3, 1)\ (1, 3)\}$

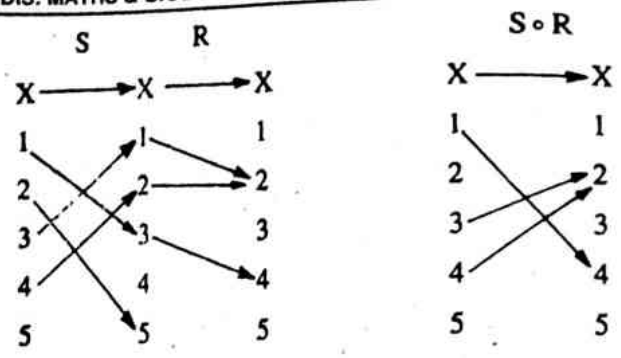
Find RoS, SoR, RoR, SoS, (RoS) oR, Ro (SoR) and prove that

$M_{RoS} = M_R \circ M_S$, $M_{SoR} = M_S \circ M_R$

$M_{RoR} = M_R \circ M_R$ *IT : S-10, W-10*



$\therefore$ RoS = $\{(1, 5), (2, 5), (3, 2)\}$



$S \circ R = \{(1, 4), (3, 2), (4, 2)\}$

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{5\times 5} \quad M_S = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{5\times 5}$$

$$M_{R\circ S} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{5\times 5}$$

$$M_R o M_S = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = M_{R\circ S}$$

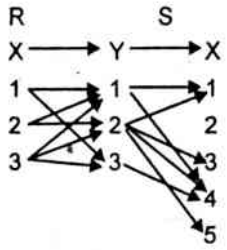
$\therefore$ $M_{R\circ S} = M_R \circ M_S$

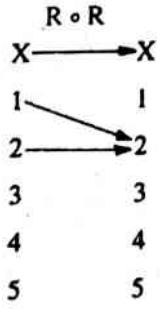
Also

$$M_S o M_R = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore M_{S \circ R} = M_S o M_R$





$\therefore \quad R \circ R = \{(1,2),(2,2)\}$

$$M_{R \circ R} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_R o M_R \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = M_{R \circ R}$$

Performing Boolean multiplication

+	0	1
0	0	1
1	1	1

$1 + |z|$

.	0	1
0	0	0
1	0	1

We get (RoS) oR = { (3, 2)} = Ro (SoR)

(SoS) = {(4,5), (3, 3), (1,1)}

(RoR) = {(1,2), (2,2)}

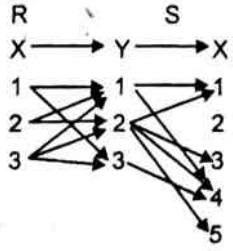
RoRoR = {(1, 2), (2, 2) } .....**Ans.**

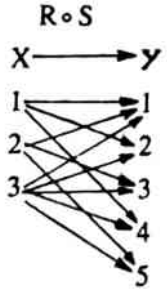
Ex.41. Given $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, $M_S = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$

Find R, S, $R \circ S$, $\tilde{R}$, $\tilde{S}$, $S \circ R$ and prove that $M_{\tilde{R} \circ \tilde{S}} = M_{\tilde{S} \circ \tilde{R}}$. ***CS/CT/CE : W-10, IT : S-12***

Soln. R = {(1, 1), (1, 3), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3)}

S = {(1, 1), (1, 4), (2, 1), (2, 3), (2, 5), (3, 2), (3, 4)}



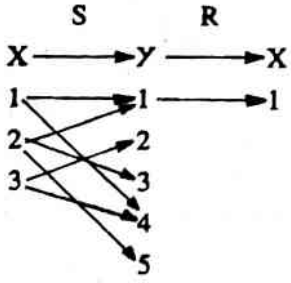


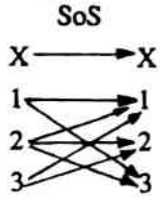
$$M_{\tilde{R}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad M_{\tilde{S}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

RoS = {(1, 1), (1, 2), (1, 4), (2, 1), (2, 3), (2, 4), (2, 5), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5)}

$\tilde{R}$ = {(1, 1), (3, 1), (1, 2), (2, 2), (1, 3), (2, 3), (3, 3)}

$\tilde{S}$ = {(1, 1), (4, 1), (1, 2), (3, 2), (5, 2), (2, 3), (4, 3)}





$$M_{R \circ S} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

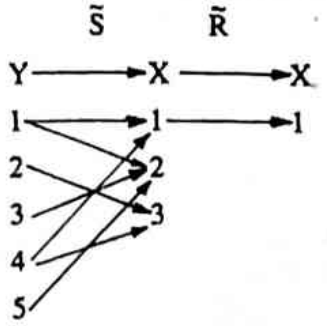
$S \circ R$ ={ (1, 1), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2)}

$\tilde{R} \circ \tilde{S}$ = {(1, 1), (2, 1), (4, 1), (1, 2), (3, 2), (4, 2), (5,2), (1, 3), (2, 3), (3, 3), (4, 3), (5,3)}

$$M_{\tilde{R} \circ \tilde{S}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$\tilde{S}$ = {(1, 1), (4, 1), (1, 2), (3, 2), (5, 2), (2, 3), (4, 3)}

$\tilde{R}$ = { (1, 1), (3, 1), (1, 2), (2, 2), (1, 3), (2, 3), (3, 3)}



$\tilde{S} \circ \tilde{R} = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 3), (3, 2), (3, 3), (4, 1), (4, 2), (4, 3), (5, 2), (5, 3)\}$

$$\therefore \quad M_{\tilde{S} \circ \tilde{R}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$\therefore \boxed{M_{\tilde{R} \circ \tilde{S}} = M_{\tilde{S} \circ \tilde{R}}}$

Ex.42. Let I be the set of all integers. Consider the relation

$R = \{(a, b) / (a - b)$ is divisible by 5$\}$.

Is R an equivalence relation? Write down its equivalence classes. ***IT : W-10***

Soln. The equivalence classes are as follows :

$[0] = \{ x \in I : x = 5k + 0$, for some integer k$)$
$= \{ x \in I : x = 5k$, for some integer k$\}$
$= \{\ldots, -10, -5, 0, 5, 10, \ldots\}$

$[1] = \{x \in I : x = 5k + 1$, for some integer k$\}$
$= \{\ldots, -9, -4, 1, 6, 11, \ldots\}$

$[2] = \{\ldots, -8, -3, 2, 7, 12, \ldots\}$

$[3] = \{\ldots, -7, -2, 3, 8, 13, \ldots\}$

$[4] = \{\ldots, -6, -1, 4, 9, 14, \ldots\}$

We observe that there are exactly five equivalence classes and they are disjoint also

$I = [0] \cup [1] \cup [2] \cup [3] \cup [4]$ .....**Ans.**

Ex.43. Find a nonempty set and a relation on the set that satisfy each of the following combinations of properties.

Also draw graph of each relation

(i) Reflexive and symmetric but not transitive.

(ii) Reflexive and transitive but not antisymmetric. ***IT : S-11***

Soln.

(i) $A = \{1, 2, 3\}$ then it is easy to verify that the relation $R = \{(1, 1)\ (2, 2)\ (3, 3)\ (1, 2)\ (2, 1)\ (2, 3)\ (3, 2)\}$ is reflexive and symmetric but not transitive since $(1, 2) \in R$ and $(2, 3) \in R$ but $(1, 3) \notin R$

(ii) Let Z^* be the set of all non-zero integers and R be the relation on Z^* given by $(a, b) \in R$ if a is a factor of b i.e. if a/b.

Since a/a for all $a \in Z^*$;

a/b and b/c $\Rightarrow$ a/c, hence R is reflective and transitive, 2/6 but 6/2 is not true; hence is not symmetric again 5/ – 5 and – 5 / 5 but $5 \neq -5$ hence R is not antisymmetric. .....**Ans.**

Ex. 44. Let A = {a, b, c, d} and consider a relation R = {(a, a), (a, b), (a, c), (a, d), (b, b), (b, d), (c, c), (c, d), (d, d)}. Show that R is a partial ordering. ***IT : W-12***

Soln. Given : A = {a, b, c, d}

R = ((a, a), (a, b), (a, c), (a, d), (b, b), (b, d), (c, c), (c, d), (d, d)}

To Prove : R is a partial order relation.

We prove that R is reflexive, antisymmetric and transitive.

(i) Reflexive : We observe that $(x, x) \in R, \forall x \in A$

$\therefore$ R is reflexive.

(ii) Antisymmetric : This holds trivially, because for every $(x, y) \in R$, there does not exist $(y, x) \in R$

(iii) Transitive : We observe that

$(x, y), (y, z) \in R \Rightarrow (x, z) \in R$

$\therefore$ R is transitive.

$\therefore$ $\boxed{\text{R is a partial order relation.}}$ **Ans.**

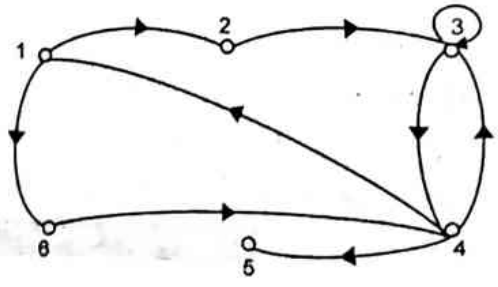
Ex.45. A relation R defined on A = {1, 2, 3, 4, 5, 6} as R = { (1, 2), (1, 6), (2, 3), (3, 3), (3, 4), (4, 1), (4, 3), (4, 5), (6, 4)}. Draw the graph of R and R^2 and hence find M_{R^2}. ***IT : W-12***

Soln. Given : A = {1, 2, 3, 4, 5, 6}

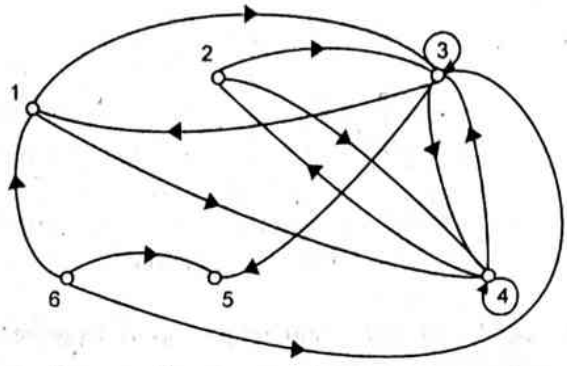
R = {(1, 2), (1, 6), (2, 3), (3, 3), (3, 4), (4, 1), (4, 3), (4, 5), (6, 4)}

$\therefore$ $R^2 = R \circ R$ = {(1, 2), (1, 6), (2, 3), (3, 3), (3, 4), (4,1), (4, 3), (4, 5), (6, 4)} o {(1, 2), (1, 6), (2, 3), (3, 3), (3, 4), (4, 1), (4, 3), (4, 5), (6, 4)}

$\therefore$ R^2 = {(1, 3), (1, 4), (2, 3), (2, 4), (3, 3), (3, 4), (3, 1), (3, 5), (4, 2), (4, 3), (4, 4), (6, 1), (6, 3), (6, 5)}



Graph of R



Graph of R^2

$$\therefore M_{R^2} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

PARTIALLY ORDERED SET

A relation R on a set A is called partial ordering, if it is reflexive, antisymmetric and transitive.

That is

(i) $a R a, \forall a \in A$ .....(reflexive)

(ii) $a R b$ and $b R a, \forall a, b \in A \Rightarrow a = b$ .....(antisymmetric)

(iii) $a R b$ and $b R c \Rightarrow a R c, \forall a, b, c \in A$ .....(transitive)

The set A together with partial order relation R is called a **partial order set** or **POSET.**

TOTALLY ORDERED SET OR CHAIN

Consider the partially ordered set $(A, \leq)$ and $a, b \in A$ such that $a < b$ or $b < a$. Then the relation $\leq$ is called totally ordered relation and $(A, \leq)$ is called **totally ordered set or chain.**

Cover :

If $(A, \leq)$ is a POSET and $a, b \in A$ such that $a < b$ and their does not exist any $c \in A$ in between a and b such that $a < c < b$, then we say that b covers a.

Power Set :

Power set is a set of all possible subsets of a given set.

If set A contains n elements, then the power set of A denoted by $\rho(A)$ contains 2^n elements.

Let $A = \{a, b, c\}$, then $\rho(A) = \{\{\phi\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$.

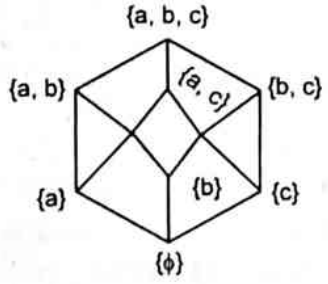
HASSE DIAGRAM

A partial order relation on a set A can be represented by a diagram known as **Hasse diagram.**

We represent the elements of A by points or small circles and if b covers a, then we take b at higher level than a and join a and b by a straight line so that we get well formed diagram known as Hasse diagram.

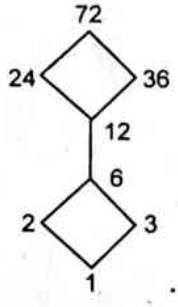
For example :

Let $A = \{a, b, c\}$, then $\rho(A) = \{\{\phi\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$. If $\subseteq$ is a partial order relation defined on A, then Hasse diagram of POSET $(\rho(A) \subseteq)$ is given by



Ex.46. Let A = {1, 2, 3, 6, 12, 24, 36}. Draw the Hasse diagram of POSET (A, /). ***CS/CT/CE : S-08, W-11***

Soln. The Hasse diagram of POSET (A, /) is



...Ans.

Ex.47. Describe the ordered pairs in the relations determined by Hasse diagram of POSET (A, ≤) on the set A = {1, 2, 3, 4}.

Fig.

Soln. Since the relation on A is a partial order relation, hence, (1, 1), (2, 2), (3, 3) and (4, 4) are all reflexive

pairs and must be in $\leq$. All edges when converted to upwards, towards their vertices give the ordered pairs (1, 2), (1, 3), (1, 4). The transitive relation gives the ordered pairs (1, 4), (2, 4), (3, 4). Therefore the {(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 3), (1, 4), (2, 3), (4, 4), (3, 4)}. ....**Ans.**

Ex.48. Determine the Hasse diagram of the relation R on set A = {a, b, c, d} and R = {(a, a), (a, b), (b, b), (b, d), (a, c), (c, c), (c, d), (a, d), (d,d)}

Soln. Let A = {a, b, c, d} and the relation R is
R = {(a, a), (a, b), (b, b), (b, d), (a, c), (c, c), (c, d), (a, d), (d,d)}.
It's Hasse diagram is given by

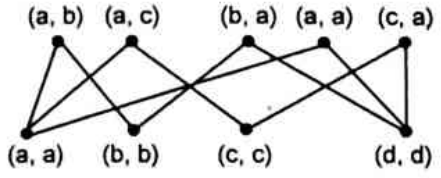


Fig. ...**Ans.**

Ex.49. For any integer n, S_n denotes the set of all divisors of n and D denotes the division operation. Find the Hasse diagram for n = 30, n = 36 and n = 45. ***CS/CT/CE : W-09,12***

Soln. For n = 30; S_n = {1, 2, 3, 5, 6, 10, 15, 30}
.. Hasse diagram is

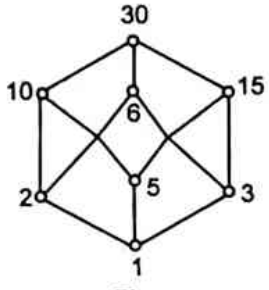


Fig.

For n = 36; S_n = {1, 2, 3, 4, 6, 9, 12, 18, 36}

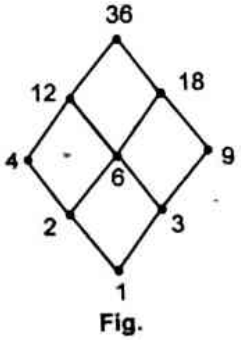


Fig.

For n = 45; S_n = {1, 3, 5, 9, 15, 45}

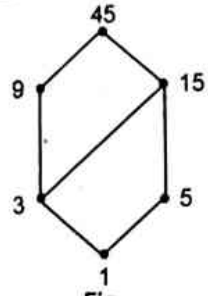


Fig. ....**Ans.**

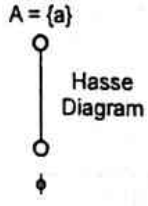
Ex.50. Let A be a given finite set and $\rho(A)$ its power set. Let $\leq$ be the inclusion relation on $\rho(A)$. Draw Hasse diagram of $(\rho(A), \leq)$ or $(\rho(A), \leq)$ for (1) A = {a}, (2) A = { a, b} (3) A = {a, b, c}. Also give least member and greatest member if they exist. ***CS/CT/CE : W-06, S-07***
IT : S-07,12,13 W-07,10,11

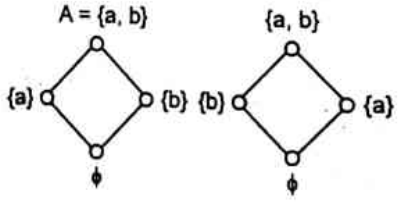
Soln.

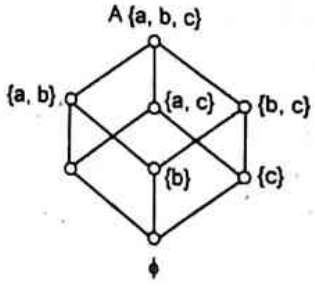
(1) A = {a} , $\rho(A) = \{\phi, \{a\}\}$
$l.m = 0 = \phi$, $g.m = 1 = A$

(2) A = { a, b}
$\rho(A) = \{\phi\ \{a\}\ \{b\}\ \{a\ b\}\}$
$l.m = 0 = \phi$, $g.m = 1 = A$

(3) A = {a, b, c}
$\rho(A) = \{\phi,\ \{a\}, \{b\}, \{c\},$
$\{ab\}, \{ac\}, \{bc\}\ \{a, b, c\}$
$l.m = 0 = \phi$, $g.m = 1 = A$.







For a given poset a Hasse diagram is not unique. $(P, \geq)$ is dual of $(P, \leq)$. Hasse diagram of $(P, \geq)$ can be 0 obtained from $(P, \leq)$ by rotating the diagram through 18^0. So that the points at the top becomes the point at the bottom. ....**Ans.**

Ex.51. Determine the Hasse diagram of the relation on $A = \{1, 2, 3, 4, 5\}$ whose matrix is shown:

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

IT : W-06

Soln. The required Hasse diagram is

...**Ans.**

FUNCTION (MAPPING OR TRANSFORMATION)

The mathematical concept of a function expresses the intuitive idea that one quantity (the argument of the function, also known as the input) completely determines another quantity (the value or the output). A function assigns exactly one value to each input of a specified type. Since functions are so widely used, many traditions have grown up around their use. The symbol for the input to a function is often called the letter x. The symbol for the output is called the dependent variable or value and is represented by the letter y. The function is usually represented by the letter f. Thus the notation $y = f(x)$ indicates that a function named f has an input named x and an output named y. The set of all input to a given function is called the domain of the function and the set of all resulting output is called the image or range of the function. The image is often a subset of some larger set, called the codomain of a function.

Function : ***CS/CT/CE : S-11***

Let A and B be any two non-empty sets. A relation f from the set A to the set B, written as $f : A \to B$ is called a function if every element of the set A is related to exactly one element of the set B.

Here the set A is called domain and the set B is called codomain. In other words, a function $f : A \to B$ is a set of ordered pairs with the property that for all $x \in A$, there exists a unique $y \in B$ such that $(x, y) \in f$.

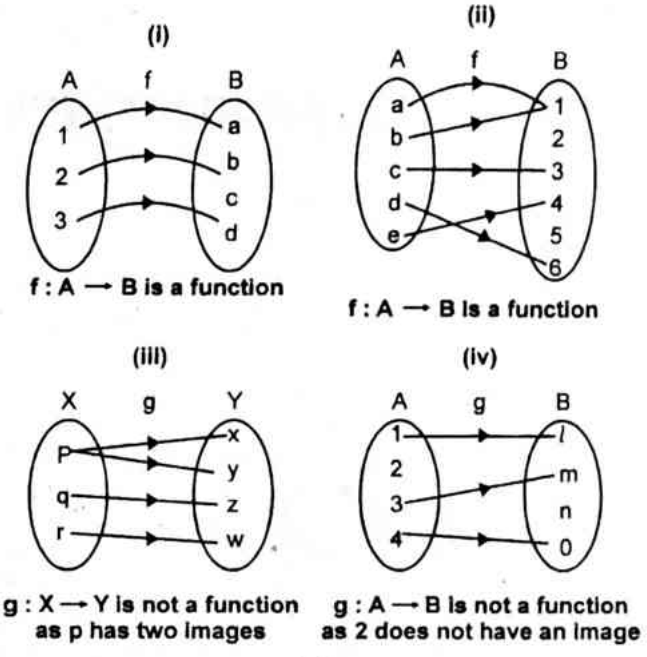


Fig.

Range : The set consisting of all the images of the element of A under the function f is called the range of f and is denoted by f (A).

For example :

(i)

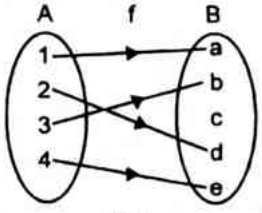


Fig.

$f : A \to B$ is a function

Domain = A = {1, 2, 3, 4}

Codomain = B = {a, b, c, d, e}

Range = {a, b, d, e}

(ii) Let A = {1, 2, 3, 4, 5}, B = {0, 1, 2, 3, 5, 7, 9, 12, 13},

$f : A \to B$

f = {(1, 1), (3, 7), (4, 3), (5, 2)}

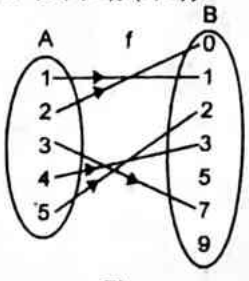


Fig.

Domain = A = {1, 2, 3, 4, 5}
Codomain = B = {0, 1, 2, 3, 5, 7, 9, 12, 13}
Range = {0, 1, 2, 3, 7}

TYPES OF FUNCTIONS

(i) One-One function :

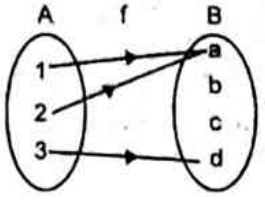
IT : W-07, CS/CT/CE : S-09,11,W-11

A function $f : A \to B$ is called one-one (injective or 1-1), if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

or $f(x_1) \neq f(x_2) \Rightarrow x_1 \neq x_2$

(ii) Many-One function : A function $f : A \to B$ is called many-one, if two or more elements of A have the same image in B.

For example :

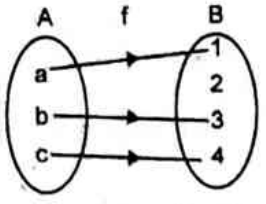


f : A → B is many one function

Fig.

(iii) Into function : A function $f : A \to B$ is called into, if there exists atleast one element in B which is not an image of some element in A.

For example :



f : A → B is into function

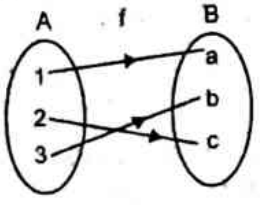
Fig.

(iv) Onto function :

IT : W-07, CS/CT/CE : S-09,11,W-11

A function $f : A \to B$ is called onto, (surjective) if every element of B is the image of some element of A.

For example :



f : A → B is onto function

Fig.

If $f : A \to B$ is onto function, then Range of f = Codomain.

(v) Bijective Function : A function $f : A \to B$ is called bijective if it is both one-one and onto.

(vi) Identity Function : A function $f : A \to A$ defined by $f(x) = x, \forall x \in A$ is called identity function and denoted by I_A.

For example :

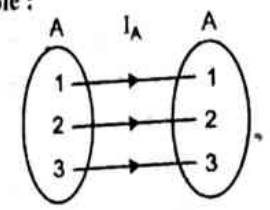


Fig.

COMPOSITION OF FUNCTIONS

Let f be a function from A to B ($f : A \to B$) and g be another function from B to C ($g : B \to C$) then the composition of f and g denoted by gof (read as g circle f) is a new function from A to C and is given by

$(gof)x = g[f(x)]$

For example :

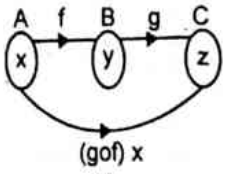


Fig.

Here $y = f(x)$ and $z = g(y)$

$(gof)x = g[f(x)] = g(y) = z$

Note : (i) The composition of functions is not commutative i.e. $fog \neq gof$ where f and g are functions.

(ii) gof is called the left composition of g with f.

Ex.52. Let $f : R \to R$, $f(x) = x + 2$ and $g : R \to R$, $g(x) = x^2$. Find fog, gof.

Soln. Given : $f(x) = x + 2$, $g(x) = x^2$

$(fog)_x = f[g(x)] = f(x^2) = x^2 + 2$

$(gof)_x = g[f(x)] = g(x + 2) = (x + 2)^2 = x^2 + 4x + 4$

Clearly $fog \neq gof$ .....Ans.

Theorem 1 : The composition of any function with identify function is the function itself .

OR $f \circ I_A = f$ or $I_B \circ f = f$

Proof : Consider

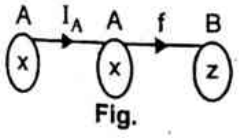


Fig.

Here $I_A(x) = x$ and $y = f(x)$

Now $(fo\, I_A)_x = f[I_A(x)]$

$= f(x)$

$\therefore$ $\boxed{fo\, I_A = f}$ **Ans.**

Consider

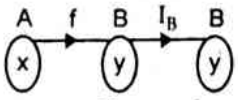


Fig.

Here $y = f(x)$ and $I_B(y) = y$

Now $(I_B\, o\, f)\, x = I_B[f(x)]$

$= I_B(y)$

$= y$

$= f(x)$

$\therefore$ $\boxed{I_B of = f}$ **Ans.**

INVERSE FUNCTIONS

Let $f : A \to B$ be one-one and onto. Then the inverse function of f is the function that assigns to an element $b \in B$, the unique element $a \in A$ such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence $f^{-1}(b) = a$ whenever $f(a) = b$.

A function $f : A \to B$ is said to be invertible if the inverse relation $f^{-1} : B \to A$ is also a function.

Note :

(i) In general if $f : A \to B$, then $f^{-1} : B \to A$ need not be a function, it may be a relation.

(ii) **If $f : A \to B$ is a bijection (one-one and onto) and** $f(a) = b$, then $a = f^{-1}(b)$ where $a \in A$, $b \in B$.

Theorem 2 : If $f : A \to B$ and $g : B \to C$ are one-one and onto functions, then $(gof)^{-1} = f^{-1}og^{-1}$.

CS/CT/CE : W-06, S-06,12, IT : S-08,09,13

Soln. Given : $f : A \to B$ and $g : B \to C$ are one-one and onto functions.

To Prove : $(gof)^{-1} = f^{-1}og^{-1}$

Since $f : A \to B$ and $g : B \to C$ are one-one and onto.

$\therefore$ $gof : A \to C$ is also one-one and onto

$\Rightarrow (gof)^{-1} : C \to A$ is also one-one.

Also $g^{-1} : C \to B$ and $f^{-1} : B \to A$ are one-one and onto

$\Rightarrow f^{-1}og^{-1} : C \to A$ is one-one and onto.

Let $c \in C$, then there exists an element $b \in B$ such that $g(b) = c$

$\Rightarrow b = g^{-1}(c)$

Also there exists an element $a \in A$ such that $f(a) = b$

$\Rightarrow a = f^{-1}(b)$

Now $(gof)_a = g[f(a)] = g(b) = c$

$\Rightarrow a = (gof)^{-1}c$ .....(1)

Also $(f^{-1}og^{-1})c = f^{-1}[g^{-1}(c)] = f^{-1}(b) = a$ ....(2)

From (1) and (2), we get

$(gof)^{-1}c = \left(f^{-1}og^{-1}\right)c$

$\therefore$ $\boxed{(gof)^{-1} = f^{-1}og^{-1}}$ **Ans.**

Ex.53. Let $f : R \to R$, $f(x) = x^3 + 1, \forall x \in R$. Then prove that f is a bijective function.

Soln. Given : $f : R \to R$, $f(x) = x^3 + 1, \forall x \in R$

To Prove : f is a bijective function

i.e. f is one-one and onto

One-One : Let $x_1, x_2 \in R$

if $f(x_1) = f(x_2)$

$\Rightarrow x_1^3 + 1 = x_2^3 + 1$

$\Rightarrow x_1^3 = x_2^3$

$\Rightarrow x_1 = x_2$

$\therefore$ f is one-one

Onto : Let $y \in R$ (codomain), then there exists $x \in R$ such that

$f(x) = y$

i.e $x^3 + 1 = y$

$\therefore$ f is onto

$\therefore$ $\boxed{\text{f is a bijective function.}}$ **Ans.**

Ex.54. A function is defined on the set of integers as follows.

$$f(x) = \begin{cases} x \;; & 0 \le x \le 1 \\ x + 2 \;; & 1 \le x \le 3 \\ 4x - 5 \;; & 3 \le x \le 5 \end{cases}$$

Find the domain and range and check whether it is one-one or many-one.

Soln. Given : $f(x) = \begin{cases} x \;; & x = 0 \\ x + 2 \;; & x = 1, 2 \\ 4x - 5 \;; & x = 3, 4 \end{cases}$

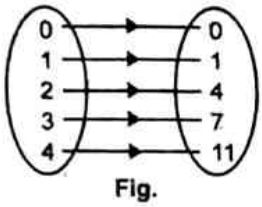


Fig.

Domain = {0, 1, 2, 3, 4}
Range = {0, 3, 4, 7, 11}
This function is one-one.

Ex.55. List all possible functions from the set X = {a, b, c} to the set Y = {0, 1}. Indicate in each case whether the function is one-one or onto or both.

IT : S-07,10, W-07,11, CS/CT/CE : S-08,09,10,13

Soln. Given : X = {a, b, c} and Y = {0, 1}

Let f : X→Y

Here there are $2^3 = 8$ possible distinct functions from X to Y.

(i) $f_0 = f_{000} = \{(a, 0), (b,0), (c, 0)\}$

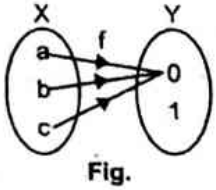


Fig.

From the figure it is clear that f : X→Y is many-one function and into function.

(ii) $f_1 = f_{001} = \{(a, 0), (b, 0), (c, 1)\}$

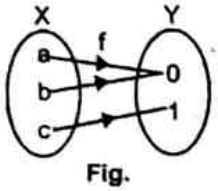


Fig.

From the figure it is clear that f : X→Y is many-one function and onto function.

(iii) $f_2 = f_{010} = \{(a, 0), (b, 1), (c, 0)\}$

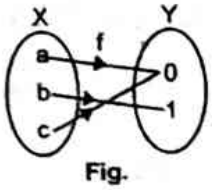


Fig.

From the figure it is clear that f : X→Y is many-one function and onto function.

(iv) $f_3 = f_{011} = \{(a, 0), (b, 1), (c, 1)\}$

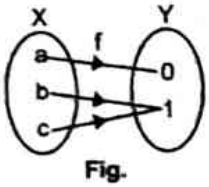


Fig.

From the figure it is clear that f : X→Y is many-one function and onto function.

(v) $f_4 = f_{100} = \{(a, 1), (b, 0), (c, 0)\}$

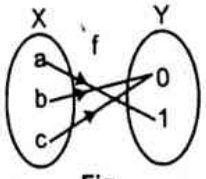


Fig.

From the figure it is clear that f : X→Y is many-one function and onto function.

(vi) $f_5 = f_{101} = \{(a, 1), (b, 0), (c, 1)\}$

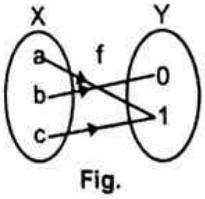


Fig.

From the figure it is clear that f : X→Y is many-one function and onto function.

(vii) $f_6 = f_{110} = \{(a, 1), (b, 1), (c, 0)\}$

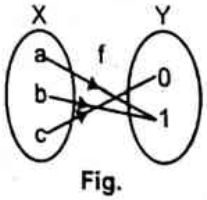


Fig.

From the figure it is clear that f : X→Y is many-one function and onto function.

(viii) $f_7 = f_{111} = \{(a, 1), (b, 1), (c, 1)\}$

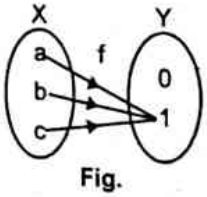


Fig.

From the figure it is clear that f : X→Y is many-one function and into function. ....**Ans.**

Ex.56. Let f be the set of all one-one and onto mappings from X to X, where X = {1, 2, 3}. Find all elements of f and also find inverse of each element.

IT : S-07,09 CS/CT/CE : W-07,10, S-12

Soln. Given : X = {1, 2, 3} and f : X→X

Since X has three elements.

∴ Number of one-one and onto mappings from X to X $= 3! = 3 \times 2 \times 1 = 6$

(i) $f_1 = \{(1, 1), (2, 2), (3, 3)\}$

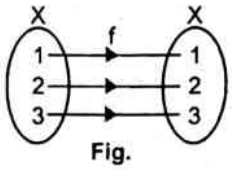


Fig.

(ii) $f_2 = \{(1, 1), (2, 3), (3, 2)\}$

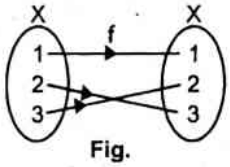


Fig.

(iii) $f_3 = \{(1, 3), (2, 2), (3, 1)\}$

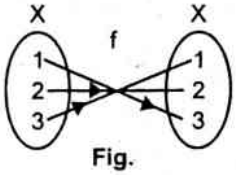


Fig.

(iv) $f_4 = \{(1, 2), (2, 1), (3, 3)\}$

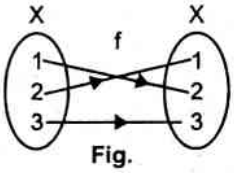


Fig.

(v) $f_5 = \{(1, 2), (2, 3), (3, 1)\}$

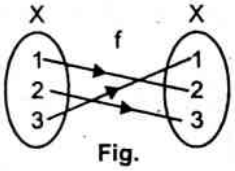


Fig.

(vi) $f_6 = \{(1, 3), (2, 1), (3, 2)\}$

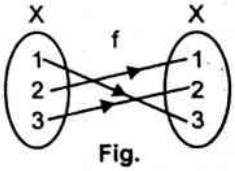


Fig.

$\therefore\ f = \{f_1, f_2, f_3, f_4, f_5, f_6\}$

Now we will find the inverse of each element.

(i) f_1^{-1}

$\therefore\ f_1^{-1} = f_1 = I$

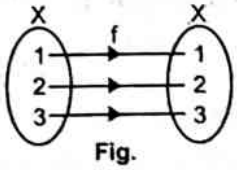


Fig.

(ii) f_2^{-1}

$\therefore\ f_2^{-1} = f_2$

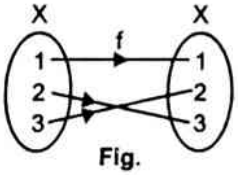


Fig.

(iii) f_3^{-1}

$\therefore\ f_3^{-1} = f_3$

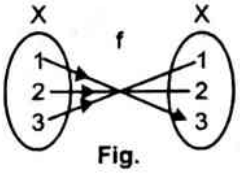
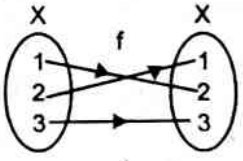


Fig.

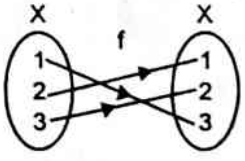
(iv) f_4^{-1}

$\therefore\ f_4^{-1} = f_4$



(v) f_5^{-1}

$\therefore\ f_5^{-1} = f_6$



(vi) f_6^{-1}

$\therefore\ f_6^{-1} = f_5$

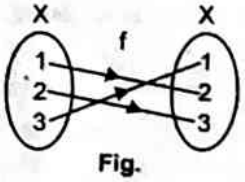


Fig. ...Ans.

Ex.57. Show that the function defined by $f : Q \to Q$ such that $f(x) = 3x + 4$ for all $x \in Q$ is one-one and onto where Q is the set of rational numbers. Also find a formula that defines the inverse function f^{-1}.

IT : W-09

Soln. Given : $f : Q \to Q$ such that $f(x) = 3x + 4$

VBD

To Prove : f is one-one and onto

(i) Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

i.e. $3x_1 + 4 = 3x_2 + 4$

$\Rightarrow 3x_1 = 3x_2$

$\Rightarrow x_1 = x_2$

$\therefore f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

$\therefore$ f is one-one

(ii) Let $y \in Q$ such that $y = f(x)$

i.e. $y = 3x + 4$

ie. $3x = y - 4$ i.e. $x = \frac{1}{3}(y-4)$

Hence for a given $y \in Q$, there exists an element $x = \frac{1}{3}(y-4) \in Q$

Such that $f(x) = y$

$\therefore$ f is onto.

Since f is one-one and onto

$\therefore$ f is invertible.

$\therefore f_y^{-1} = \frac{1}{3}(y-4)$ ....**Ans.**

Ex.58. Let A = B = C = R and consider the functions.

$f : A \to B$ and

$g : B \to C$ defined by

$f(a) = 2a + 1$, $g(b) = b/3$.

Verify theorem

$(gof)^{-1} = f^{-1} \circ g^{-1}$. ***IT : S-06, W-06***

Soln. We have $f : A \to B$ by $f(a) = 2a + 1$

and $g : B \to C$ by $g(b) = \frac{b}{3}$

$(g\,of)(a) = g\{f(a)\} = g(2a+1) = \frac{2a+1}{3}$

Here $(g\,of) : A \to C$ given by $(g\,of)\ a = \frac{2a+1}{3}$

Let 'b' be the image of 'a' under $f : A \to B$

$b = f(a) = 2a + 1$

Consequently 'a' will be the image of 'b' under the inverse function f^{-1}. Solve for 'a' in terms of 'b', we get $a = \frac{b-1}{2}$

Then $f^{-1}(b) = \frac{b-1}{2}$ or $f^{-1}(a) = \frac{a-1}{2}$

Let 'c' be the image of 'b' under $g : B \to C$

$c = g(b) = \frac{b}{3}$

Consequently 'b' will be the image of 'c' under the inverse function g^{-1}. Solve for b, we get

$b = 3c$

or $g^{-1}(c) = 3c$ or $g^{-1}(a) = 3a$

Again let 'c' be the image of 'a' under $g\,of : A \to C$

$c = \frac{2a+1}{3}$

Consequently, 'c' will be the image of 'a' under the inverse function $(g\,of)^{-1}$. Solve for 'a', we get

$3c = 2a + 1$ or $a = \frac{3c-1}{2}$

$(g\,of)^{-1}(c) = \frac{3c-1}{2}$ or $(g\,of)^{-1}(a) = \frac{3a-1}{2}$ .....(1)

$(f^{-1} \circ g^{-1})(a) = f^{-1}\{g^{-1}(a)\} = f^{-1}(3a)$

$= \frac{3a-1}{2}$ .....(2)

From (1) and (2), we get

$\boxed{(g\,of)^{-1} = f^{-1} \circ g^{-1}}$ **Ans.**

Ex.59. Let A = {1, 2, 3, 4} and B = {a, b}. How many onto functions can be defined from a set A onto a set B? List functions which are not onto. ***IT : S-06***

Soln. Given : A = {1, 2, 3, 4} and B = {a, b}

Let $f : A \to B$ then we have

$f_1 = \{(1, a), (2, a), (3, a), (4, a)\}$ many-one into function

$f_2 = \{(1, a), (2, a), (3, a), (4, b)\}$ many one onto function

$f_3 = \{(1, b), (2, a), (3, a), (4, a)\}$ many one onto function

$f_4 = \{(1, a), (2, b), (3, a), (4, a)\}$ many one onto function

$f_5 = \{(1, a), (2, a), (3, b), (4, a)\}$ many one onto function

$f_6 = \{(1, a), (2, b), (3, b), (4, b)\}$ many one onto function

$f_7 = \{(1, b), (2, a), (3, b), (4, b)\}$ many one onto function

$f_8 = \{(1, b), (2, b), (3, a), (4, b)\}$ many one onto function

$f_9 = \{(1, b), (2, b), (3, b), (4, a)\}$ many one onto function

$f_{10} = \{(1, b), (2, b), (3, b), (4, b)\}$ many one into function.

.....**Ans.**

Ex.60. Let $f : R \to R$ and $g : R \to R$, where R is the set of real numbers. Find $f \circ g$ and $g \circ f$, where $f(x) = x^2 - 2$ and $g(x) = x + 4$ whether these functions are injective, surjective and bijective. ***IT : W-08***

Soln. Let $x \in R$ $(f \circ g)(x) = f\{g(x)\} = f\{x+4\}$

$= (x+4)^2 - 2 = x^2 + 8x + 14$

$\therefore\ f \circ g = \{(x, y) \mid y = x^2 + 8x + 14,\ x \in R \Rightarrow y \in R\}$

$\Rightarrow f \circ g$ is 1 - 1 onto function i.e. bijective

Let $x \in R$, $(g \circ f)(x) = g\{f(x)\} = g\{x^2 - 2\}$

$= (x^2 - 2) + 4 = x^2 + 2$

$\therefore\ g \circ f = \{x, x^2 + 2) \mid x \in R\}$

$\therefore\ g \circ f$ is bijective. ...**Ans.**

Ex.61. Given $f : X \to Y$ and $g : Y \to Z$, show that

(i) if $g \circ f$ is 1–1 then f is 1–1

(ii) if $g \circ f$ is onto then g is onto.

(iii) f^{-1} exists, iff f is bijective. ***IT : W-08***

Soln. Proof : (i) Suppose f is not 1–1, then $\exists\, x, y \in X$ such that $x \neq y$ but $f(x) = f(y)$

$\Rightarrow gf(x) = gf(y)$

Thus we have $x \neq y$ but $gf(x) = gf(y)$

But as g o f is one-one, therefore we must have $x \neq y \Rightarrow g\ f(x) \neq g\ f(y)$

Hence we get a contradiction.

$\therefore$ f must be one one.

(ii) For any $x \in X$

$g \circ f(x) = g(f(x)) \in g(Y)$. This is true for any $x \in X$

$\Rightarrow (g \circ f)(X) \subseteq g(Y)$

Suppose g is not onto, then g (Y) is properly contained in Z

$\therefore (g \circ f)(X)$ is properly contained in Z

$\therefore$ g must be onto. .....**Ans.**

(iii) Suppose f^{-1} exist to prove f is bijective.

As f^{-1} exist and $f : X \to Y$, $f^{-1} : Y \to X$.

$\therefore\ f^{-1} \circ f = I_X,\ f \circ f^{-1} = I_Y$

Since, I_X is 1 – 1 therefore by **(i)** f is 1 – 1

Also since, I_Y is onto therefore by **(ii)** f is on to

$\Rightarrow$ f is bijective.

Conversely if f is bijective to prove f^{-1} exists f is 1–1 onto

$\therefore$ Each $y \in Y$ is image of unique $x \in X$ i.e. $f(x) = y$

Let g be a mapping from $Y \to X$ defined by $g(y) = x$

We have

$g \circ f(x) = g(y) = x, \forall x \in X \Rightarrow g \circ f = I_X$

$f \circ g(y) = f(x) = y, \forall\ y \in Y \Rightarrow f \circ g = I_Y$

$\therefore\ g = f^{-1}$ i.e. f^{-1} exists. ....**Ans.**

Ex.62. If $f : X \to Y$ and A, B are two subsets of Y, then prove that:

(i) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

(ii) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

CS/CT/CE : W-09

Soln. Let $f : X \to Y$, then we prove that

(i) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

(ii) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

(i) **To Prove :** $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

i.e. **To Prove :** $f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$

and $f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)$

We know that, $f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$ ... (1)

which is trivially true.

Now, we prove : $f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)$

Let $x \in f^{-1}(A) \cup f^{-1}(B)$

$\Rightarrow x \in f^{-1}(A)$ or $x \in f^{-1}(B) \Rightarrow x \subseteq A$ or $x \subseteq B$

$\Rightarrow x \subseteq A \cup B \Rightarrow x \in f^{-1}(A \cup B)$

$\Rightarrow f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)$ ... (2)

From (1) and (2), we have

$\boxed{f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)}$ **Ans.**

(ii) **To Prove :** $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

i.e. **To Prove :** $f^{-1}(A \cap B) \subseteq f^{-1}(A) \cap f^{-1}(B)$

and $f^{-1}(A) \cap f^{-1}(B) \subseteq f^{-1}(A \cap B)$

We have,

$f^{-1}(A \cap B) \subseteq f^{-1}(A) \cap f^{-1}(B)$ ... (3)

which is trivially true.

Now we Prove. $f^{-1}(A) \cap f^{-1}(B) \subseteq f^{-1}(A \cap B)$

Let $y \in f^{-1}(A) \cap f^{-1}(B)$

$\Rightarrow y \in f^{-1}(A)$ and $y \in f^{-1}(B)$

$\Rightarrow y \subseteq A\ \&\ y \subseteq B$

$\Rightarrow y \subseteq A \cap B \Rightarrow y \in f^{-1}(A \cap B)$

$\Rightarrow f^{-1}(A) \cap f^{-1}(B) \subseteq f^{-1}(A \cap B)$ .... (4)

From (3) and (4), we have

$\boxed{f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)}$ **Ans.**

Ex.63. If $f : R \to R$ and $g : R \to R$ be defined by

$f(x) = x^3 - 4x$, $g(x) = \dfrac{1}{x^2+1}$ and

$h : R \to R$ defined by

$h(x) = x^4$. Then

(i) Find composition function (f.g.h) x

(ii) Is f invertible? If yes, find f^{-1}. ***IT : S-11***

Soln.

(i) Let $f : R \to R$ and $g : R \to R$

be defined by $f(x) = x^3 - 4x$, $g(x) = \dfrac{1}{x^2+1}$

and $h : R \to R$ is defined by $h(x) = x^4$

then $f.g.h : R \to R$ is defined by

$f.g.h(x) = f.g\{h(x)\} = f.g\{x^4\} = f(g(x^4))$

$= f\left\{\dfrac{1}{x^8+1}\right\}$

$\therefore \boxed{(f.g.h)x = \left[\dfrac{1}{x^8+1}\right]^3 - 4\left[\dfrac{1}{x^8+1}\right]}$ **Ans.**

(ii) Yes, f is invertible, then

$f^{-1} : R \to R$ is defined by

$\boxed{f^{-1}(x^3 - 4x) = x}$ **Ans.**

Ex.64. Let $f : A \to B$ be a function. Then prove :

(i) f^{-1} is a function from B to A if and only if f is one to one.

(ii) the function f^{-1} is also one to one. ***IT : S-12***

Soln. Suppose that

$f = \{(x, y) : x \in A, y \in B\}$

then $f^{-1} = \{(y, x) : x \in A, y \in B, (x, y) \in f\}$

Let $(y_1, x), (y_2, x) \in f^{-1}$, then

$(x, y_1), (x, y_2) \in f$

But f is a mapping

$\therefore y_1 = y_2$

$(y_1, x), (y_2, x) \in f^{-1} \Rightarrow y_1 = y_2$

$\Rightarrow f^{-1}$ is one - one

Now, $R(f^{-1}) = \{x : (y, x) \in f^{-1}\}$

$= \{x : (x, y) \in f\} = D(f) = A$

which implies that f^{-1} is onto mapping. Hence f^{-1} is bijective. **...Ans.**

Ex.65. If $f : X \to Y$ and $g; Y \to Z$ and both f and g are 1 – 1 onto then show that $g \circ f$ is also 1 – 1 onto.

IT : S-09,W-10,12

Soln. Let z be an arbitrary element of Z then as g is onto $\exists y \in Y$ such that $g(y) = z$. Further as f is onto each $y \in Y \exists x \in X$ such that $f(x) = y$.

Now $g \circ f(x) = gf(x) = g(y) = z$

Thus to each $z \in Z \ \exists x \in X$ such that

$g \circ f(x) = z \quad \therefore g \circ f$ is onto

Now to prove $g \circ f$ is 1 – 1, we have

$g \circ f(x_1) = g \circ f(x_2)$

$\Rightarrow g(y_1) = g(y_2)$ where $f(x_1) = y_1$, $f(x_2) = y_2$

$\Rightarrow y_1 = y_2$ *[∵ g is 1 – 1]*

$\Rightarrow f(x_1) = f(x_2)$

$\Rightarrow x_1 = x_2$ *[∵ f is 1 – 1]*

$\therefore g \circ f$ is 1 – 1. **....Ans.**

CHARACTERISTIC FUNCTION

IT : W-06, S-13, CS/CT/CE : W-07, S-11,12

Let U be universal set and A be any subset of U, then the characteristic function f_A of A is defined by

$$f_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

Properties of Characteristic Function :

(i) $f_{A \cap B} = f_A . f_B$

(ii) $f_{A \cup B} = f_A + f_B - f_{A \cap B}$

(iii) $f_{A \cap (B \cap C)} = f_{(A \cap B) \cap (A \cap C)}$

(iv) $f_{A'} = 1 - f_A$

Ex.66. Using the properties of characteristic function, show that

(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

IT : S-08,13, CS/CT/CE : W-08, S-11

(ii) $A + (B + C) = (A + B) + C$

CS/CT/CE : W-06, S-08

(iii) $f_{(A \cup B)}(x) = f_A(x) + f_B(x) - f_A(x) . f_B(x)$

(iv) $(A')' = A$ ***IT : S-08, CS/CT/CE : W-08***

Soln.

(i) **To Prove :** $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

We know that $f_{A \cap B} = f_A . f_B$

$\therefore f_{A \cap (B \cup C)} = f_A . f_{B \cup C}$

$= f_A\,[f_B + f_C - f_{B\cap C}]$

$= f_A\, f_B + f_A\, f_C - f_A \cdot f_{B\cap C}$

$= f_{A\cap B} + f_{A\cap C} - f_{A\cap(B\cap C)}$

$= f_{A\cap B} + f_{A\cap C} - f_{(A\cap B)\cap(A\cap C)}$

$\therefore\ f_{A\cap(B\cup C)} = f_{(A\cap B)\cup(A\cap C)}$

$\therefore\ \boxed{A\cap(B\cup C) = (A\cap B)\cup(A\cap C)}$

(ii) To Prove : $A + (B + C) = (A + B) + C$

i.e. $A\cup(B\cup C) = (A\cup B)\cup C$

L.H.S. $= f_{A\cup(B\cup C)}$

$= f_A + f_{B\cup C} - f_{A\cap(B\cup C)}$

$= f_A + f_B + f_C - f_{B\cap C} - f_A \cdot f_{B\cup C}$

$= f_A + f_B + f_C - f_B \cdot f_C$
$\quad - f_A \cdot [f_B + f_C - f_{B\cap C}]$

$= f_A + f_B + f_C - f_B \cdot f_C$
$\quad - f_A \cdot [f_B + f_C - f_B \cdot f_C]$

$= f_A + f_B + f_C - f_B \cdot f_C - f_A \cdot f_B$
$\quad - f_A \cdot f_C + f_A \cdot f_B \cdot f_C$ .....(1)

R.H.S. $= f_{(A\cup B)\cup C}$

$= f_{A\cup B} + f_C - f_{(A\cup B)\cap C}$

$= f_A + f_B - f_{A\cap B} + f_C - f_{A\cup B} \cdot f_C$

$= f_A + f_B - f_A \cdot f_B + f_C$
$\quad - f_C \cdot [f_A + f_B - f_{A\cap B}]$

$= f_A + f_B - f_A \cdot f_B + f_C$
$\quad - f_C \cdot [f_A + f_B - f_A \cdot f_B]$

$= f_A + f_B + f_C - f_A \cdot f_B - f_A \cdot f_C$
$\quad - f_B \cdot f_C + f_A \cdot f_B \cdot f_C$ .....(2)

From (1) and (2), we get

$f_{A\cup(B\cup C)} = f_{(A\cup B)\cup C}$

$\therefore\ A\cup(B\cup C) = (A\cup B)\cup C$

$\therefore\ A + (B + C) = (A + B) + C$

(iii) To prove : $f_{A\cup B}(x) = f_A(x) + f_B(x) - f_{(A\cup B)}(x)$

Let $x \in A\cup B$, then $f_{A\cup B}(x) = 1$ .....(2) (by definition)

But if $x \in A\cup B$, then

$x \in A$ or $x \in B$

i.e $f_A(x) = 1, f_B(x) = 1$

If $x \in A\cap B$, then $f_{A\cap B}(x) = 1$

Now $f_A(x) + f_B(x) - f_{A\cap B}(x) = 1 + 1 - 1 = 1$ .....(2)

From (1) and (2), we get

$f_{A\cap B}(x) = f_A(x) + f_B(x) - f_{A\cap B}(x)$

(iv) To Prove : $(A')' = A$

We have

$f_{(A')'} = 1 - f_{A'}$

$= 1 - [1 - f_A] = 1 - 1 + f_A = f_A$

$\therefore\ f_{(A')'} = f_A$

$\therefore\ \boxed{(A')' = A}$ **Ans.**

Ex.67. Show the following identities for sets using characteristic function :

CS/CT/CE : W-06, S-08

$A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$

Soln. If $A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$

then $f_{A\cup(B\cap C)}(x) = f_{A\cup B}(x) f_{A\cup C}(x)$

Conversely for $f_{A\cup(B\cap C)}(x) = f_{A\cup B}(x) f_{A\cup C}(x)$

if $x \in A\cup(B\cap C)$, $f_{A\cup(B\cap C)}(x) = 1$

$\Rightarrow f_{A\cup B}(x) f_{A\cup C}(x) = 1$

$\Rightarrow x \in (A\cup B)\cap(A\cup C)$

Hence $A\cup(B\cap C) \subseteq (A\cup B)\cap(A\cup C)$ .....(1)

Also, if $x \in (A\cup B)\cap(A\cup C)$

$\Rightarrow f_{(A\cup B)\cap(A\cup C)}(x) = 1$

$\Rightarrow f_{A\cup(B\cap C)}(x) = 1$

$\Rightarrow x \in A\cup(B\cap C)$

$(A\cup B)\cap(A\cup C) \subseteq A\cup(B\cap C)$ .....(2)

From (1) and (2), we have

$\boxed{A\cup(B\cap C) = (A\cup B)\cap(A\cup C)}$ **Ans.**

Ex.68. Show that : **IT : S-06**

$f_{A\cup B}(x) = f_A(x) + f_B(x) - f_A(x)\, f_B(x)$

Soln. Let $x \in (A\cup B)$, then $f_{A\cup B}(x) = 1$ ....(1)

But $x \in A\cup B$

$\Rightarrow x \in A$ or $x \in B$ or $x \in A$ and $x \in B$

If $x \in A$, then $f_A(x) = 1$, If $x \in B$, then $f_{(B)}(x) = 1$

If $x \in A\cap B$, then $f_{A\cap B}(x) = 1$

$f_A(x) + f_B(x) - f_A(x)\, f_B(x) = 1 + 1 - 1 = 1$ .....(2)

From (1) and (2), we get

$\boxed{f_{A\cup B}(x) = f_A(x) + f_B(x) - f_A(x) f_B(x)}$ **Ans.**

Ex.69. Let U be a universal set and $X \subseteq U$.

Define $C_X(x) = \begin{cases} 1, & \text{if } x \in X \\ 0, & \text{if } x \notin X, \end{cases}$ **where C_X is the characteristic function of X.**

Show that $C_{X\cup Y}(x) = C_X(x) + C_Y(x) - C_{X\cap Y}(x)$. **IT : S-10,11**

Soln.

Case I : Let $x \in X$ and $x \notin Y$ so

$x \in X \cup Y$ and $x \notin X \cap Y$

Now, $C_{X\cup Y}(x) = 1$, $C_{X\cap Y}(x) = 0$, $C_X(x) = 1$,

$C_Y(x) = 0$

$\therefore 1 = 1 + 0 - 0$ i.e. $1 = 1$

i.e. $C_{X\cup Y}(x) = C_X(x) + C_Y(x) - C_{X\cap Y}(x)$

Case II : $x \notin X$ and $x \in Y$

so that $x \notin X \cap Y$ and $x \in X \cup Y$

$\therefore C_X(x) = 0$, $C_Y(x) = 1$, $C_{X\cap Y}(x) = 0$ and

$C_{X\cup Y}(x) = 1$

$\therefore 1 = 0 + 1 - 0 \Rightarrow 1 = 1$

i.e. $C_{X\cup Y}(x) = C_X(x) + C_Y(x) - C_{X\cap Y}(x)$

Case III : $x \in X$ and $x \in Y$

so that $x \in X \cup Y$ and $x \in X \cap Y$

$\therefore C_X(x) = 1$, $C_Y(x) = 1$, $C_{X\cup Y}(x) = 1$ and

$C_{X\cap Y}(x) = 1$

$\therefore 1 = 1 + 1 - 1 \Rightarrow 1 = 1$

i.e. $C_{X\cup Y}(x) = C_X(x) + C_Y(x) - C_{X\cap Y}(x)$

Case IV : $x \notin X$ and $x \notin Y$ so that $\notin x\ X \cup Y$ and $x \notin X \cap Y$

$\therefore C_X(x) = 0$, $C_Y(x) = 0$, $C_{X\cup Y}(x) = 0$

and $C_{X\cap Y}(x) = 0$

$\therefore 0 = 0 + 0 - 0 \Rightarrow 0 = 0$

i.e. $C_{X\cup Y}(x) = C_X(x) + C_X(y) - C_{X\cap Y}(x)$

Hence proved.

Ex.70. Prove that : $f_{A\cap B} = f_A \cdot f_B$. **IT : S-10**

Soln.

Case I : Let $x \in A$ and $x \notin B$ so $x \notin A \cap B$

Now $f_A(x) = 1$, $f_B(x) = 0$, $f_{A\cap B}(x) = 0$

$\therefore 0 = 1.0$

i.e. $f_{A\cap B} = f_A \cdot f_B$

Case II : Let $x \notin A$, $x \in B$ so that $x \notin A \cap B$

Now $f_A(x) = 0$, $f_B(x) = 1$, $f_{A\cap B}(x) = 0$

$0 = 0.1$

$\therefore f_{A\cap B} = f_A \cdot f_B$

Case III : Let $x \in A$, $x \in B$, $\therefore x \in A \cap B$

Now, $f_A(x) = 1$, $f_B(x) = 1$, $f_{A\cap B}(x) = 1$

$\therefore 1 = 1.1$

$\therefore f_{A\cap B} = f_A \cdot f_B$

Case IV : Let $x \notin A$, $x \notin B$ $\therefore x \notin A \cap B$

Now $f_A(x) = 0$, $f_B(x) = 0$ $f_{A\cap B}(x) = 0$

$\therefore 0 = 0.0$

$\therefore \boxed{f_{A\cap B} = f_A \cdot f_B}$ **Ans.**

PROBLEMS FOR PRACTICE :

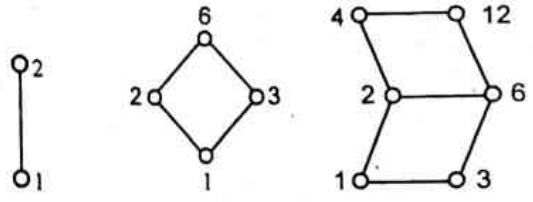
Q.1. Let $X = \{a, b, c\}$,

$R = \{(a, a), (a, c), (b, a), (b, b), (c, a), (c, b), (c, c)\}$

Prove that $M_{RoR} = M_R . M_R$, $M_{\tilde{R}} = [M_R]^T$

Q.2. Let X be the set of factors of a particular positive integer m and let $\le$ be the relation divides. Draw the Hasse diagram for $(X, \le)$ for (i) m = 2 (ii) m = 6 (iii) m = 12.

Ans. (i) (ii) (iii)



Q.3. Let $f : R \to R$, $f(x) = x^2 + 1$, $\forall x \in R$. Then prove that f is a bijective function.

Q.4. Using properties of characteristics function, prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Q.5. Let $X \{1, 2, 3, 4, 5\}$, $R = \{(1, 2), (3, 4), (2, 2)\}$, $S = \{(4, 2), (2, 5), (3, 1), (1, 3)\}$. Find SoR, RoS, Ro(SoR)

Ans. **SoR = {(4, 2), (3, 2), (1, 4)}**

RoS = {(1, 5), (3, 2), (2, 5)}

Ro (SoR) = {(3, 2)}

Q.6. If $A = B \cap C$, then

prove that $A \times A = (B \times B) \cap (C \times C)$

Q.7. **Given:** $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, $M_S = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$

Find RoS and show that $M_{\overline{RoS}} = M_{\overline{S}\,\overline{o}\,\overline{R}}$

Ans. **RoS = {(1, 1), (1, 2), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5)}**

Q.8. If $A = \{a, b\}$, $B = \{2, 3\}$, $C = \{3, 4\}$, then prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

UNIT - III
GROUP THEORY

SYLLABUS

Binary Operations, Properties, Semigroups, Monoids, Subsemigroup, Submonoid, Isomorphism & Homomorphism, Groups, (only definitions and examples) Subgroups and Homomorphism, Cosets and Lagrange's Theorem, Normal subgroups.

UNIVERSITY PAPER SOLUTIONS
SINCE SUMMER - 2006

Note to the students :
- The questions given below are from previous Nagpur University examination papers.
- Though these questions are from old university papers, they have been included in the new syllabus.
- Questions which are out of new syllabus are not included here.

SUMMER - 06 (CS/CT/CE)

Q.1. Prove that every homomorphic image of a group G is isomorphic to some quotient group of G. **7M**

Ans. P.3-36, Ex.64

Q.2. G is a group and there exist two relatively prime positive integers m and n such that $a^m b^m = b^m a^m$ and $a^n b^n = b^n a^n$ for all a, b $\in$ G. Prove that G is abelian. **6M**

Ans. P.3-21, Ex.26

Q.3. Prove that monoid homomorphism preserves associativity, identity, commutativity and invertibility. **7M**

Ans. P.3-9, Ex.2

Q.4. Prove that a finite semi group (S, *) is a group if and only if (S, *) satisfies the cancellation laws. **6M**

Ans. P.3-21, Ex.27

SUMMER - 06 (IT)

Q.1. Let G be the nonzero integers under the operation of multiplication and let $H = \{3n \mid n \in Z\}$. Is H a subgroup of G? **7M**

Ans. P.3-37, Ex.65

Q.2. Let G be the set of all nonzero real numbers and let $a * b = \left(\frac{ab}{2}\right)$, show that <G, *> is an abelian group. **6M**

Ans. P.3-12, Ex.7

Q.3. Define normal subgroup of a group.

Ans. P.3-30 Refer definitions

Q.4. Let <H, *> be a subgroup of a group <G, *>. If H is a normal subgroup of G, then show that $gh\,g^{-1} \in H$, for any $g \in G$ and $h \in H$. **6M**

Ans. P.3-31, Ex.50(a)

Q.5. Let <G, *> and <G′, *′> be two groups and let $f : G \to G'$ be a homomorphism from G to G′. Show that

(i) If e is identity in G and e′ is the identity from G′, then $f(e) = e'$.

(ii) If $a \in G$, then $f(a^{-1}) = \{f(a)\}^{-1}$ **6M**

Ans. P.3-35, Ex.61

Q.6. State Lagrange's theorem on group. **2M**

Ans. P.3-28, Ex.45

WINTER - 06 (CS/CT/CE)

Q.1. Let * denote a binary operation on the set of natural numbers given by x * y = x.
Show that * is not commutative, but associative. Which elements are idempotent and are there any left (or) right identities? **5M**

Ans. P.3-22, Ex.28

Q.2. For any commutative monoid (M, *), the set of idempotent elements of M forms a submonoid **4M**

Ans. P.3-9, Ex.3

Q.3. Let G be a finite group with identity e and let a be an arbitrary element of G, prove that there exists a non-negative integers n such that $a^n = e$. **5M**

Ans. P.3-22, Ex.29

WINTER - 06 (IT)

Q.1. Let R = <Q, *> be a mathematical structure, where $x * y = \frac{x+y}{2}$.

Determine which of the following properties hold for this structure:

(i) Closure

(ii) Commutative

(iii) Associative

(iv) An identity element

(v) An inverse for every element. **5M**

Ans. P.3-18, Ex.18

Q.2. If f is a homomorphism from a commutative semigroup (S, *) onto a semigroup (T, *′), then show that (T, *′) is also commutative. **4M**

Ans. P.3-9, Ex.2

Q.3. Consider the binary operation * defined on the set A = {a, b, c} by the following table:

*	a	b	c
a	b	c	b
b	a	b	c
c	c	a	b

(i) Is * a commutative operation?

(ii) Compute a * (b * c) and (a * b) * c. **4M**

Ans. P.3-22, Ex.30

Q.4. Define order of an element of a group.

Let G = {1, –1, i, –i} be a multiplicative group. Find the order of every element. **4M**

Ans. P.3-20, Ex.22

Q.5. Prove that intersection of any two subgroups of a group (G, *) is again a subgroup of (G, *).

Is union of two subgroups of a group and again a subgroup? Explain your answer. **4M**

Ans. P.3-27, Ex.42

Q.6. Let G be a group of real numbers under addition. Let G′ be the group of positive real numbers under multiplication.

Let $f: G \to G'$ defined by $f(a) = 2^a$

Prove that G and G′ are isomorphic. **6M**

Ans. P.3-33, Ex.55, 56

SUMMER - 07 (CS/CT/CE)

Q.1. Define semigroup and monoid. Prove that the set G of all rational numbers other than 1 with the operation defined by $a \cdot b = a + b - ab$ constitute an abelian group. **7M**

Ans. P.3-8, 22, Definitions and Ex.31

Q.2. Prove that if G is an abelian group, then for all a, b ∈ G and for all integer $(ab)^n = a^n b^n$. **6M**

Ans. P.3-17, Ex.16

Q.3. Prove that the intersection of two subgroups of a group G is a subgroup of G. Give an example to show that union of two subgroups is not necessarily a subgroup.

Ans. P.3-27, Ex.42

Q.4. Show that the group G = {{0, 1, 2, 3}, $\times_4$} is isomorphic to the group.

$G' = \{\{1,2,3,4\}. \times_5\}$. **4M**

Ans. P.3-37, Ex.66

SUMMER - 07 (IT)

Q.1. Show that the set of matrices

$$A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix},$$

α ∈ R forms a monoid.

Ans. P.3-8, Ex.1

Q.2. Prove that for any commutative monoid (M, *) the set of idempotent elements of M forms a sub monoid. **14M**

Ans. P.3-9, Ex.3

Q.3. Prove that the set of all n^{th} roots of unity forms a finite abelian group of order 'n' with respect to multiplication of complex numbers.

Ans. P.3-23, Ex.32

Q.4. Show that the multiplicative group G = {1, i, – 1, – i} is isomorphic to group G′ = (0, 1, 2, 3) with addition modulo 4 as composition. Also find the order of each element.

Ans. P.3-33, Ex.57

Q.5. State and prove Lagrange's theorem.

Ans. P.3-28, Ex.45

Q.6. Show that every subgroup of an abelian group is a normal subgroup.

Ans. P.3-32, Ex.53

WINTER - 07 (CS/CT/CE)

Q.1. Prove that if for every element a in a group G, $a^2 = e$ then G is an abelian group. **3M**

Ans. P.3-19, Ex.19

Q.2. Prove that the order of an element of a group is same as that of its inverse. **4M**

Ans. P.3-20, Ex.24

Q.3. Define Normal subgroup.
Prove that intersection of any two normal subgroups of a group G is a normal subgroup. **6M**

Ans. P.3-31, Ex.51 & Definition on P.3-30

WINTER - 07 (IT)

Q.1. Prove that monoid homomorphism preserves associativity, identity, commutativity. **7M**

Ans. P.3-9, Ex.2

Q.2. Prove that for any commutative monoid (M, *) the set of idempotent elements of M forms a submonoid. **6M**

Ans. P.3-9, Ex.3

Q.3. Prove that the set of all n, n^{th} root of unity forms a finite abelian group of order n with respect to multiplication of complex numbers. **6M**

Ans. P.3-23, Ex.32

Q.4. Prove that every homomorphic image of a group G is isomorphic to some quotient group of G. **7M**

Ans. P.3-36, Ex.64

SUMMER - 08 (CS/CT/CE)

Q.1. Describe an abelian group. Prove that the set $G = \{0, 1, 2, 3, 4, 5\}$ is a finite abelian group under addition modulo 6 as operation. **7M**

Ans. P.3-15, Ex.11(a)

Q.2. Prove that order of a subgroup of a finite group divides order of the group. **7M**

Ans. P.3-28, Ex.45

Q.3. Determine whether the set of integers Z, < Z, * > defined by
$a * b = a + b - ab$ is a monoid. **6M**

Ans. P.3-14, Ex.9

SUMMER - 08 (IT)

Q.1. Define order of an element of a group. Let $G = \{1, -1, i, -i\}$ be a multiplicative group. Find the order of every element. **3M**

Ans. P.3-20, Ex.22

Q.2. Prove that : $O(a) = O(x^{-1}ax)$. **5M**

Ans. P.3-21, Ex.25

Q.3. If in a group G, $a^5 = e$, $aba^{-1} = b^2$ for $a, b \in G$, find O (b). **5M**

Ans. P.3-23, Ex.33

Q.4. Prove that monoid homomorphism preserves associativity, identity, commutativity and invertibility. **6M**

Ans. P.3-9, Ex.2

Q.5. The order of each subgroup of a finite group is a divisor of the order of the group. **7M**

Ans. P.3-28, Ex.45

Q.6. Let N be a normal subgroup of G. Define the mapping $\phi : G \to G/N$ such that $\phi(x) = Nx, \forall x \in G$, then show that ϕ is a homomorphism of G onto G/N with kernel N. **7M**

Ans. P.3-37, Ex.67

Q.7. If ϕ is a homomorphism of a group G into a group G′ with kernel K then show that K is a normal subgroup of G. **6M**

Ans. P.3-35, Ex.62

WINTER - 08 (CS/CT/CE)

Q.1. Let G be the set of all non-zero real number and let $a * b = \frac{ab}{2}$. Show that (G,*) is an abelian group. **6M**

Ans. P.3-12, Ex.7

Q.2. Prove that monoid homomorphism preserves associativity, identity, commutativity and invertibility. **6M**

Ans. P.3-9, Ex.2

Q.3. Prove that an isomorphism preserves order property of element. **7M**

Ans. P.3-24, Ex.34

WINTER - 08 (IT)

Q.1. In a group G, prove that $(ab)^2 = a^2 \cdot b^2$ if and only if G is abelian. **7M**

Ans. P.3-18, Ex.17

Q.2. Show that the multiplicative group $G = \{1, i, -1, -i\}$ is isomorphic to group $G' = \{0,1,2,3\}$ with addition modules 4 as composition. **7M**

Ans. P.3-33, Ex.57

Q.3. Let G be an abelian group and 'n' a fixed integer. Prove that the function f: G → G defined by $f(a) = a^n$, for $a \in G$ is a homomorphism. **7M**

Ans. P.3-34, Ex.58

SUMMER - 09 (CS/CT/CE)

Q.1. Define algebra, semigroup and monoid. Prove that the set of matrices $A_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$, $\alpha \in R$, forms a monoid. **7M**

Ans. P.3-8, Ex.1

Q.2. Show that the multiplicative group $G = \{1, i, -1, -i\}$ is isomorphic to group $G' = \{0, 1, 2, 3\}$ with addition modulo 4 as composition. **7M**

Ans. P.3-33, Ex.57

SUMMER - 09 (IT)

Q.1. Show that set P(S) of all partitions of $S = \{a, b, c\}$ is a monoid with the operation of product of partitions. **6M**

Ans. P.3-24, Ex.35

Q.2. Define sub-semigroup, sub-monoid.

Ans. P.3-9, Refer Definitions.

Q.3. Prove that for any commutative monoid $(M, *)$ the set of idempotent elements of M forms a sub-monoid. **5M**

Ans. P.3-9, Ex.3

Q.4. G is a group and there exists two relatively prime positive integers m and n such that $a^m b^m = b^m a^m$ and $a^n b^n = b^n a^n$ for all $a, b \in G$. Prove that G is abelian. **6M**

Ans. P.3-21, Ex.26

Q.5. Let $(G, *)$ and $(G_1, *_1)$ be two groups and let $f : G \to G_1$ be a homomorphism from G to G_1, then show that :

(i) $f(e) = e'$

(ii) $f(a^{-1}) = (f(a))^{-1}$ for all $a \in G$.

(iii) If H is a subgroup of G, then $f(H) = \{f(h)/h \in H\}$ is a subgroup of G,

where e is the identity in G

and e' is the identity in G_1. **7M**

Ans. P.3-35, Ex.61

WINTER - 09 (CS/CT/CE)

Q.1. Let $G = \{(a, b) / a, b \in R, a \neq 0\}$. Define a binary operation * on G by $(a, b) * (c, d) = (ac, bc + d)$ for all $(a, b), (c, d) \in G$. Show that $(G, *)$ is a group. **7M**

Ans. P.3-19, Ex.20

Q.2. Show that if a, b are arbitrary elements of a group G, then $(ab)^2 = a^2b^2$ if G is abelian. **6M**

Ans. P.3-18, Ex.17

Q.3. Show that every homomorphic image of group G is isomorphic to some quotient group of G. **7M**

Ans. P.3-36, Ex.64

WINTER - 09 (IT)

Q.1. Determine whether the set of even integers with binary operation * defined by $a * b = \frac{ab}{2}$ is semigroup or monoid. Show whether it is commutative.

Ans. P.3-12, Ex.7

Q.2. Let $G = (Z_4, +_4)$ for each of the following subgroups H of G, determine all left cosets of H in G:

(i) $H = \{0\}$

(ii) $H = \{0, 2\}$

(iii) $H = \{0, 1, 2, 3\}$. **6M**

Ans. P.3-37, Ex.68

Q.3. Let $f : G \to G'$ be a homomorphism of group G into G'. Show that f is one-one if and only if $\ker(f) = \{e\}$ where e is the identity of G. **7M**

Ans. P.3-36, Ex.63

SUMMER - 10 (CS/CT/CE)

Q.1. Show that the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

form a multiplicative abelian group. **7M**

Ans. P.3-19, Ex.21

Q.2. Let $S = N \times N$. Let * be the operation on S defined by $(a, b) * (a', b') = (aa', bb')$

(i) Show that * is associative

(ii) Define $f : (S, *) \to (Z, +)$ by $f(a, b) = a/b$. Show that f is a homomorphism. **7M**

Ans. P.3-34, Ex.59

SUMMER - 10 (IT)

Q.1. Prove that every homomorphic image of a group G is isomorphic to some quotient group of G. **7M**

Ans. P.3-36, Ex.64

Q.2. Show that the set $\{1, w, w^2\}$ where w is imaginary cube root of unity forms a group under multiplication. **6M**

Ans. P.3-11, Ex.5

Q.3. Let N be a normal subgroup of G. Define the mapping $\phi : G \to \frac{G}{N}$ such that $\phi(x) = Nx, \forall x \in G$ then show that ϕ is a homomorphism of G onto $\frac{G}{N}$ with Kernel N. **6M**

Ans. P.3-37, Ex.67

Q.4. If ϕ is a homomorphism of a group G into a group G′ with Kernel K then show that K is a normal subgroup of G. **7M**

Ans. P.3-35, Ex.62

WINTER - 10 (CS/CT/CE)

Q.1. Show that the set of all positive rational numbers forms an abelian group under the composition defined by $a * b = (ab)/2$. **7M**

Ans. P.3-12, Ex.7

Q.2. G is a group and there exists two relatively prime positive integers m and n such that

$a^m b^m = b^m a^m$ and $a^n b^n = b^n a^n$

for all a, b $\in$ G. Prove that G is abelian. **6M**

Ans. P.3-21, Ex.26

Q.3. Let $(G,*)$ and $(G_1, *_1)$ be two groups and let $f : G \to G_1$ be a homomorphism from G to G_1 then show that :

(i) $f(e) = e_1$ where e is the identity in G and e_1 is the identity in G_1.

(ii) $f(a^{-1}) = [f(a)]^{-1}$ for all $a \in G$.

(iii) If H is a subgroup of G, then

$f(H) = \{f(h)/h \in H\}$

is a subgroup of G_1. **7M**

Ans. P.3-35, Ex.61

Q.4. Prove that a normal subgroup H of a group G is normal if and only if $g^{-1}hg \in H$ for every $h \in H$, $g \in G$. **6M**

Ans. P.3-31, Ex.50(a)

WINTER - 10 (IT)

Q.1. Determine whether the set of integers Z, < Z, * > is a monoid, where

$a * b = a + b - ab$. **6M**

Ans. P.3-14, Ex.9

Q.2. Show that the set of matrices :

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix};\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix};\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix},\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$$

forms a multiplicative group. **7M**

Ans. P.3-19, Ex.21

Q.3. Define an abelian group. Prove that G is abelian iff for all $a, b \in G$, $(ab)^2 = a^2b^2$. **6M**

Ans. P.3-18, Ex.17

Q.4. Show that the multiplicative group $G = \{1, i, -1, -i)$ is isomorphic to the group $G' = \{0, 1, 2, 3\}$ with addition modulo 4 as composition. **7M**

Ans. P.3-33, Ex.57

SUMMER - 11 (CS/CT/CE)

Q.1. Show that the fourth roots of unity form an abelian group w.r.to multiplication. **6M**

Ans. P.3-12, Ex.6

Q.2. Prove that the necessary and sufficient condition for a non empty subset H of a group G to be a subgroup of G is that $\forall\, a, b \in H \Rightarrow ab^{-1} \in H$ where b^{-1} is the inverse of b in G. **7M**

Ans. P.3-26, Ex.39.

Q.3. State and prove Lagrange's theorem.

Ans. P.3-28, Ex.45

Q.4. Show that the group $(G = \{0, 1, 2, 3\}, +_4)$ and the group $(G' = \{1, -1, i, -i\}, \times_4)$ are isomorphic **7M**

Ans. P.3-33, Ex.57

SUMMER - 11 (IT)

Q.1. If (G, *) is a group and if a and b be any elements of G, then

(i) The equation $a * x = b$ has a unique solution $x = a^{-1} * b$ in G

(ii) The equation $y * a = b$ has a unique solution $y = b * a^{-1}$ in G. **7M**

Ans. P.3-17, Ex.14

Q.2. Define left coset of H in G generated by a in G. Let H be a subgroup of G and a, b $\in$G, then show that $aH = bH$ if and only if $a^{-1}b \in H$. **6M**

Ans. P.3-38, Ex.69

Q.3. If G is a finite group and H is subgroup of G, then show that order of H divides the order of G. **7M**

Ans. P.3-28, Ex.45

Q.4. If for a group G, $f : G \to G$ is given by $f(x) = x^2$ $x \in G$ and f is a homomorphism then show that G is abelian group. **6M**

Ans. P.3-38, Ex.70

WINTER - 11 (CS/CT/CE)

Q.1. Let G be the set of all non-zero real numbers and let $a * b = \frac{ab}{2}$. Show that (G, *) is an abelian group. **7M**

Ans. P.3-12, Ex.7

Q.2. Let G be group and let a, b and c be elements of G, then show that :

(i) ab = ac implies that b = c (left cancellation property)

(ii) ba = ca implies that b = c (right cancellation property). **6M**

Ans. P.3-16, Ex.13

Q.3. Let G be the group of real numbers under addition and let G′ be the group of positive real numbers under multiplication. Let $f : G \to G'$ defined by $f(x) = e^x$, then prove that f is an isomorphism. **7M**

Ans. P.3-38, Ex.71

Q.4. Show that if G is an abelian group, then every subgroup of G is a normal subgroup. **6M**

Ans. P.3-32, Ex.53

WINTER - 11 (IT)

Q.1. Determine whether the set of even integers with binary operation * defined by $a * b = \frac{ab}{2}$ is semigroup or monoid. Show whether it is commutative. **6M**

Ans. P.3-12, Ex.7

Q.2. Prove that for any commutative monoid (M, *) the set of idempotent elements of M forms a submonoid. **7M**

Ans. P.3-9, Ex.3

Q.3. Prove that a subgroup H of a group G is normal if and only if $g^{-1}h\ g \in H$ for every $g \in G$ and $h \in H$. **7M**

Ans. P.3-31, Ex.50(a).

SUMMER - 12 (CS/CT/CE)

Q.1. Prove that for any element a in a group G if $a^2 = e$, then G must be an abelian group. **6M**

Ans. P.3-19, Ex.19

Q.2. G is a group and there exists two relatively prime positive integers m and n such that :

$a^m b^m = b^m a^m$ and $a^n b^n = b^n a^n$

for all a, b, $\in$ G. Prove that G is abelian. **7M**

Ans. P.3-21, Ex.26

Q.3. Show that the set of matrices

$$A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, \alpha \in R \text{ forms a monoid.}$$ **7M**

Ans. P.3-8, Ex.1

Q.4. If H and K are any two subgroups of a group G, then show that $H \cap K$ is also a subgroup of G. Is $H \cup K$ is a subgroup of G ? **6M**

Ans. P.3-27, Ex.42

SUMMER - 12 (IT)

Q.1. Let T be the set of all even integers. Show that semigroups (Z, +) and (T, +) are isomorphic. **6M**

Ans. P.3-34, Ex. 57(a)

Q.2. Prove that fourth roots of unity forms an abelian group under multiplication. **7M**

Ans. P.3-12, Ex.6

Q.3. If G is a group, then $\forall a \in G$, prove that $O(a) = O(a^{-1})$. **7M**

Ans. P.3-24, Ex.36

Q.4. Let G be an abelian group and n be a fixed integer. Prove that the function $f : G \to G$ defined by $f(a) = a^n$, $a \in G$ is a homomorphism. **4M**

Ans. P.3-34, Ex.58

WINTER - 12 (CS/CT/CE)

Q.1. In group G, show that $(ab)^{-1} = b^{-1}a^{-1}\ \forall a, b \in G$. **3M**

Ans. P.3-15, Ex.12

Q.2. Define normal subgroup of group. Prove that the intersection of any two normal sub groups of a group G is a normal subgroup. **4M**

Ans. P.3-31, Ex.51 & definition on P.3-30

Q.3. Show that the set $\{1, w, w^2\}$ where w is the imaginary cube root of unity forms a group under multiplication. **6M**

Ans. P.3-11, Ex.5

Q.4. Define semigroup and monoid. Prove that the set of matrices. $A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, $\alpha \in R$ forms a monoid. **7M**

Ans. P.3-8, Ex.1

WINTER - 12 (IT)

Q.1. Let M_2 be a set of 2×2 matrices over R.

i.e. $M_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} / a, b, c, d \in R, ad - bc \neq 0 \right\}$.

Show that $(M_2, +)$ is both associative and commutative and $(M_2, \bullet)$ is associative but not commutative. **7M**

Ans. P.3-25, Ex.37

Q.2. For any two subgroup H and K of any group G. Prove that $H \cap K$ is a subgroup of G. **6M**

Ans. P.3-27, Ex.42

Q.3. Let a be any real number and let $S = \{na : n \in Z\}$. Prove that (S, +) is a group. **7M**

Ans. P.3-25, Ex.38

SUMMER - 13 (CS/CT/CE)

Q.1. Prove that fourth roots of unity forms an abelian group under multiplication. **7M**

Ans. P.3-12, Ex.6

Q.2. Show that the set of matrices $A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, $\alpha \in R$ forms a monoid. **6M**

Ans. P.3-8, Ex.1

Q.3. Let G be a group of integers and H be the subset consisting of all the multiples of an integer n, then show that H is a subgroup of G. **6M**

Ans. P.3-27, Ex.41

Q.4. Show that the multiplicative group $G = \{1, i, -1, -i\}$ is isomorphic to the group $G' = \{0, 1, 2, 3\}$ with addition modulo 4 as composition. Write the composition tables for G and G'. **7M**

Ans. P.3-33, Ex.57

SUMMER - 13 (IT)

Q.1. Define group. Show that $G = \{1, -1, i, -i\}$ is abelian group. **6M**

Ans. P.3-12, Ex.6 & definition on P.3-10

Q.2. Prove that for any commutative monoid (M, *) the set of idempotent elements of M forms a sub monoid **7M**

Ans. P.3-9, Ex.3

Q.3. Let R = {Q, *} be an algebraic structure defined by $x * y = \frac{x+y}{2}$. Show that R is an abelian group. **7M**

Ans. P.3-18, Ex.18

Q.4. Define subgroup and coset. Prove that intersection of two subgroups is a subgroup. **6M**

Ans. P.3-27, Ex.42 & definition on P.3-25, 28

SOLVED QUESTION BANK

[Sequence given as per syllabus]

INTRODUCTION

In Mathematics and abstract algebra, group theory studies the algebraic structures known as group. The concept of a group is central to abstract algebra. Other well-known algebraic structures such as rings, fields and vector spaces can all be seen as groups with additional operations and axioms. Linear algebraic groups and Lie groups are two branches of graph theory that have experienced tremendous advances and have become subject are as in their own right.

Galois in 1830s, was the first to employ groups to determine the solvability of polynomial equation. Cayley and Cauchy pushed these investigations further by creating the theory of permutation group. The concept of Lie group is important in the study of differential equations and manifolds. They describe the symmetries of con tenuous geometric and analytical structures. Analysis on these and other groups is called harmonic analysis. Integral invariant under the translation in a Lie group are used for pattern recognition and other image processing techniques.

In combinatory, the notation of permutation group and the concept of group action are often used to simplify the counting of a set of objects. In physics groups are important because they describe the symmetries which the laws of physics obeys. Examples of the use of groups in physics include the standard model, Gauge theory, the Lorentz group and the Poincare group. In chemistry and material science, groups are used to classify crystal structures, regular polyhedral and the symmetries of molecules. The assigned point groups can then be used to determine physical properties (such as polarity and chirality), spectroscopic properties (useful for Raman spectroscopy and integrated spectroscopy) and to construct molecular orbital.

SEMIGROUPS, MONOIDS, SUBSEMIGROUP, SUBMONOID

Definitions :

Groupoid : Let G be a non-empty set and * be a binary operation on G. Then the algebraic structure (G, *) is called a groupoid if the operation * satisfies closure property.

G1 (Closure) : $a, b \in G$

$\Rightarrow a * b \in G$

Semi-Group : **CS/CT/CE : S-09**

Let G be a non-empty set and * be a binary operation on G. Then the algebraic structure (G, *) is called semi-group if the operation * satisfies closure and associative property.

(i) **G1 (Closure) :** $a, b \in G$
$\Rightarrow a * b \in G$

(ii) **G2 (Associative) :** $a*(b*c) = (a*b)*c, \forall a, b, c \in G$.

A semi-group (G, *) is called abelian, if
$a * b = b * a \ \forall a, b, \in G$

Sub-Algebra : Let (G, *) be an algebraic structure and $G' \subset G$ be closed under the operation *. Then (G', *) is called a sub-algebra of (G, *).

Monoid : **CS/CT/CE : S-09**

Let G be a non-empty set and * be a binary operation on G. Then the algebraic structure (G, *) is called a monoid, if the following properties are satisfied.

(i) **G1 (Closure) :** $a, b \in G$
$\Rightarrow a * b \in G$

(ii) **G2 (Associative) :** $a * (b * c) = (a * b) * c$, $\forall a, b, c \in G$.

(iii) **G3 (Identity) :** There exists an element $e \in G$ such that $a * e = a = e * a, \forall a \in G$.
e is called the identity element for * in G.

Semi-Group Homomorphism :

Let (G, *) and (G', •) be any two semi-groups where * and • are binary operations. Then a mapping $f : G \to G'$ such that $a, b, \in G$
$\Rightarrow f(a * b) = f(a) \bullet f(b)$ is called a semi-group homomorphism.

A homomorphism of a semi-group into itself is called a semi-group endomorphism.

Semi-Group Isomorphism :

Let (G, *) and (G', •) be any two semi-groups where * and • are binary operations. Then a homomorphism $f : G \to G'$ is called a semi-group isomorphism, if f is one-one and onto.

If $f : G \to G'$ is an isomorphism, then (G, *) and (G', •) area said to be isomorphic.

An isomorphism of a semi-group onto itself is called a semi-group automorphism.

Important Results :

(1) The set of all endomorphisms of a semi-group is a semi-group under the operation of composition.

(2) The set of all automorphisms of a semi-group is a semi-group under the operation of composition.

(3) If (G, *), (G', •) and (G'', o) are semi-groups and $f : G \to G'$ and $g : G \to G''$ are semi-group homomorphism, then $gof : G \to G''$ is a semi-group homomorphism from (G, *) to (G'', o).

Monoid Homomorphism :

Let (G, *) and (G', •) be any two monoids where * and • are binary operations and e_g and $e_{g'}$ denote the identity elements of (G, *) and (G', •) respectively. Then a mapping $f : G \to G'$ such that $a, b \in G \Rightarrow f(a * b) = f(a) \bullet f(b)$ and $f(e_g) = e_{g'}$ is called a **monoid homomorphism**.

Ex.1. Show that the set of matrices

$$A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, \alpha \in R \text{ forms a monoid.}$$

CS/CT/CE : S-09,12,13,W-12, IT : S-07

Soln. Given : $A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, \alpha \in R$

Let (G, *) be an algebraic structure,

where $G = \{A_\alpha \mid \alpha \in R\}$

We verify G1, G2, G3.

G1 (Closure) : Let $A_\alpha, A_\beta \in G$

where

$$A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, \quad A_\beta = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix},$$

$\alpha, \beta \in R$.

$$\text{Now } A_\alpha * A_\beta = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta & -\cos\alpha \cdot \sin\beta - \sin\alpha \cdot \cos\beta \\ \sin\alpha \cdot \cos\beta + \cos\alpha \sin\beta & -\sin\alpha \sin\beta + \cos\alpha \cdot \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

$\therefore A_\alpha, A_\beta \in G \Rightarrow A_\alpha * A_\beta \in G$ as $\alpha + \beta \in R$

$\therefore$ G1 is verified.

G2 (Associative) : Since associative law of multiplication holds in matrices.

$\therefore$ It holds in G.

i.e. $A_\alpha * (A_\beta * A_\gamma) = (A_\alpha * A_\beta) * A_\gamma$,

$\forall A_\alpha, A_\beta, A_\gamma \in G$

$\therefore$ G2 is verified

G3 (Identity) : We know that $0 \in R$

$\therefore\ A_0 = \begin{bmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in G$

$\therefore\ A_0 =$ The identity element such that

$A_0 * A_\alpha = A_\alpha = A_\alpha * A_0$

$\therefore$ G3 is verified.

$\therefore$ $\boxed{A_\alpha \text{ froms a monoid}}$ **Ans.**

Ex.2. Prove that monoid homomorphism preserves associativity, identity, commutativity and invertibility.

IT : W-06,07, S-08

CS/CT/CE : S-06, W-08

Soln. Let $(G, *)$ and $(G', \bullet)$ be any two monoids.

Let $f : G \to G'$ such that $f(a * b) = f(a).f(b)$ ...(1)

and $f(e_g) = e_{g'}$ ...(2)

be a monoid homomorphism where e_g and $e_{g'}$ are identity elements of $(G, *)$ and $(G', \bullet)$ respectively.

(i) Associatively : We prove that

$a*(b*c) = (a * b) * c, \forall a, b, c, \in G$

Consider $f[a * (b * c)]$

$= f(a) . f(b * c)$ *[∵ By(1)]*

$= f(a) . [f(b) . f(c)]$ *[∵ By(1)]*

$= [f(a) . f(b)] . f(c) = f(a * b) . f(c)$

$= f[(a * b) * c]$

$\therefore\ a * (b * c) = (a * b) * c, \forall a, b, c \in G$

$\therefore$ $\boxed{\text{Associatively is preserved.}}$

(ii) Identity : Let e_g be the identity element of $(G, *)$

To prove : $f(e_g)$ is the identity element of $(G', \bullet)$

From (2), we have $f(e_g) = e_{g'}$, where $e_{g'}$ is the identity element of $(G', \bullet)$

$\therefore$ $\boxed{\text{Identity is preserved.}}$

(iii) Commutativety : Let $a * b = b * a, \ \forall a, b \in N$

Then $f(a * b) = f(b * a)$

i.e. $f(a).(b) = f(b) \bullet f(a)$ *[∵ By(1)]*

$\therefore$ $\boxed{\text{Commutativity is preserved.}}$

(iv) Invertibility : Let $a \in G$ be invertible and $a^{-1} \in G$ be its inverse.

$\therefore\ a * a^{-1} = e_g = a^{-1} * a$, e_g is the identity element of $(G, *)$

i.e. $f(a * a^{-1}) = f(e_g) = f(a^{-1} * a)$

i.e. $f(a).f(a^{-1}) = f(a^{-1}).f(a) = f(e_g) = e_{g'}$

[∵ e'_g is the identity element of $(G', \bullet)$]

$\Rightarrow\ f(a^{-1}) = [f(a)]^{-1}$

$\therefore$ $\boxed{\text{Invertibility is preserved}}$

Sub Semi-group : ***IT : S-09***

A set $G' \subseteq G$ is said to be a sub semigroup of the semigroup $(G, *)$, if G' itself is a semigroup under the operation $*$.

Sub Monoid : ***IT : S-09***

A set $G' \subseteq G$ is said to be a sub monoid of the monoid $(G, *)$, if G' itself is a monoid under the operation $*$.

Ex.3. For any commutative monoid $(G, *)$, the set of idempotent elements of G forms a submonoid.

CS/CT/CE : W-06, IT : S-07,09,13, W-07,11

Soln. Let G' be the set of idempotent elements of G.

Let $G' = \{a,b,c,d,.....\}$

i.e. $a*a = a$, $b*b = b$ etc

To prove : G' is a submonoid.

We verify G1, G2, G3

G1 (Closure) : Let $a, b \in G'$

Now $(a * b) * (a * b) = (a * b) * (b * a)$

.... ($a * b = b * a$, as G is commutative)

$= a * (b * b) * a$

$= a * b * a$

$= (a * a) * b$

$= a * b$

$\therefore\ (a * b) * (a * b) = a * b$ *[∵ a * b is idempotent]*

$\therefore\ a, b \in G'$

$\Rightarrow\ a * b \in G'$

$\therefore$ G1 is verified.

G2 (Associative) : Associative property obviously holds in G'

$\therefore$ G2 is verified.

G3 (Identity) : Given $G' \subseteq G$

Let e_g be the identity element of $(G, *)$

$\therefore\ e_g * e_g = e_g$

$\therefore\ e_g \in G'$

$\therefore$ G3 is verified.

$\therefore$ $\boxed{G' \text{ is a submonoid}}$ **Ans.**

BINARY OPERATIONS

(i) Let a be any non-empty set. Then a symbol * is called a binary operation on A, if A is closed under *

i.e. $a, b \in A$

$\Rightarrow a * b \in A, \forall a\, b \in A$

For example : Consider the set of natural numbers $N = \{1, 2, 3, ...\}$.

Then usual addition and multiplication are binary operations on N because $a, b \in N \Rightarrow a + b \in N$ and $a, b \in N \Rightarrow a \times b \in N$,

But subtraction and division are not binary operations on N because $3,4 \in N$, but $3 - 4 \notin N$ and $2,3 \in N$ but $2/3 \notin N$

Binary operations are generally denoted by *, +, ..., o etc.

(ii) **Algebraic Structure (Algebraic System or Algebra) :** A non-empty set A together with a binary operation * is called an algebraic structure and it is written as (A, *).

(iii) **Group :** ***IT : S-13***

Let G be a non-empty set on which a binary operation * is defined. Then G is called a group if the following four properties are satisfied.

(a) **Closure Property (G1) :**

$a, b \in G$

$\Rightarrow a * b \in G$

(b) **Associative property (G2) :**

$a, b, c, \in G$

$\Rightarrow a * (b * c) = (a * b) * c$

(c) **Existence of Identity (G3) :**

There exists an element $e \in G$ such that

$a * e = a = e * a, \ \forall a \in G$, e is called identity element for * in G.

(d) **Existence of Inverse (G4) :**

For every $a \in G$, there exists an element $b \in G$ such that $a * b = e = b * a$.

Here b is called the inverse of a and is written as $b = a^{-1}$.

Note : (1) If * is usual addition, then $e = 0$ and $b = -a$.

(2) If * is usual multiplication, then $e = 1$ and $b = 1/a$.

(iv) **Abelian or Commutaive Group :** A group G is said to be abelian,

if $a, b \in G$

$\Rightarrow a * b = b * a$.

(v) **Finite Group :** A group G is said to be finite if it contains finite number of elements, otherwise it is called infinite group.

(vi) **Order of a group :** The number of elements in finite group G is called order of the group and is denoted by O(G).

Note : For the sake of convenience, we sometimes write $a * b = ab$ or $a.b$

Examples :

(i) Consider $N = \{1, 2. 3. 4....\}$...(Natural Numbers)

Here N is not a group under usual addition because G3 does not hold as $O \notin N$.

Also N is not a group under usual multiplication because G4 does not hold as $2 \in N$, but $\frac{1}{2} \notin N$.

(ii) $Z = \{0, \pm 1, \pm 2, \pm 3, ...\}$..... (Integers)

Here Z is an abelian group under usual addition because all the properties G1, G2, G3, G4 hold.

Also $a + b = b + a, \ \forall a, b \in Z$

Also Z is not a group under usual multiplication because G4 does not hold as $3 \in Z$, but $\frac{1}{3} \notin Z$.

(iii) $Q = \left\{\frac{a}{b} \middle| a, b \in Z, b \neq 0\right\}$ (Rational Numbers)

$R = \{0,1,2,3,.....\}$ (Real Numbers)

$C = \{a + ib, a, b \in R, i = \sqrt{-1}\}$ (Complex Numbers)

Here Q, R, C are abelian groups under usual addition, but they are not groups under usual multiplication.

If we consider the set of non-zero rational numbers, non-zero real numbers, non-zero complex numbers, then they are all abelian groups under usual multiplication, but they are not groups under usual addition.

(iv) $G = \{0, 1, -1\}$

G is not a group under usual addition because $1 \in G$, but $1 + 1 = 2 \notin G$.

G is not a group under usual multiplication because G4 does not hold as $0 \in G$, but $\frac{1}{0} \notin G$.

(v) $G = \{e\}$ where e is the identity element w.r.t. multiplication. Here G is an abelian group of order 1 under multiplication.

(vi) $G = \{1, -1\}$

Here G is a finite group of order 2 under usual multiplication.

Ex.4. Let G be the set of all 2×2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $ad - bc \neq 0$ and $a, b, c, d \in Q$. Show that G is a group under matrix multiplication.

Soln. Given : $G = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \begin{matrix} a, b, c, d \in Q \\ ad - bc \neq 0 \end{matrix}\right\}$ ...(1)

To prove : G is a group under matrix multiplication.
We verify G1, G2, G3, G4

G1 (Closure) : i.e. A, B $\in$ G

$\Rightarrow$ AB $\in$ G.

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, ad - bc \neq 0$$

$$B = \begin{bmatrix} \frac{1}{p} & \frac{1}{q} \\ \frac{1}{r} & \frac{1}{s} \end{bmatrix}, \frac{1}{ps} - \frac{1}{qr} \neq 0 \qquad \text{....(2)}$$

be two elements of G.

$$\text{Now } AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \frac{1}{p} & \frac{1}{q} \\ \frac{1}{r} & \frac{1}{s} \end{bmatrix} = \begin{bmatrix} \frac{a}{p}+\frac{b}{r} & \frac{a}{q}+\frac{b}{s} \\ \frac{c}{p}+\frac{d}{r} & \frac{c}{q}+\frac{d}{s} \end{bmatrix}$$

Clearly all entries are rational numbers

$$\text{Also } \left(\frac{a}{p}+\frac{b}{r}\right)\left(\frac{c}{q}+\frac{d}{s}\right) - \left(\frac{a}{q}+\frac{b}{s}\right)\left(\frac{c}{p}+\frac{d}{r}\right)$$

$$= (ad - bc)\left(\frac{1}{ps} - \frac{1}{qr}\right) \neq 0$$

$\therefore$ AB = G

$\therefore$ G1 is verified.

G2 (Associative) : Since associative law of multiplication holds in matrices.

$\therefore$ It holds in G.

$\therefore$ G2 is verified.

G3 (Identity) : Let $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $1.1 - 0.0 = 1 \neq 0$ $\therefore$

For A, I is the identity element.

$\therefore$ G3 is verified.

G4 (inverse) : i.e. $A \in G \Rightarrow A^{-1} \in G$.

Given : $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $ad - bc \neq 0$.

$\therefore$ A^{-1} exists and

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} \in G.$$

Clearly all entries are rational numbers.

$$\text{Also } \left(\frac{d}{ad-bc}\right)\left(\frac{a}{ad-bc}\right) - \left(\frac{-b}{ad-bc}\right)\left(\frac{-c}{ad-bc}\right)$$

$$= \frac{ad-bc}{(ad-bc)^2} = \frac{1}{ad-bc} \pm 0 \text{ as } ad - bc \neq 0$$

$\therefore$ $A^{-1} \in G$

$$\text{Now } AA^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$A^{-1}A = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore$ $A \in G$

$\Rightarrow$ $A^{-1} \in G$ and $AA^{-1} = I = A^{-1}A$

$\therefore$ G4 is verified

$\therefore$ $\boxed{\text{G is group under matrix multiplication}}$ **Ans.**

Note : Here after G1 means closure property, G2 means associative property, G3 means existence of identity, G4 means existence of inverse, G5 means commutative property.

Ex.5. Prove that cube roots of unity forms a group under multiplication. ***IT : S-10, CS/CT/CE : W-12***

Soln. Let G = {1, w, w^2}, $w^3 = 1$. Composition table for G is given below

x	1	w	w^2
1	1	w	w^2
w	w	w^2	$w^3 = 1$
w^2	w^2	1	w

$[\because w.w^2 = w^3 = 1, w^2.w^2 = w^4 = w.w^3 = w.1 = w]$

We prove that G is a group.

i.e. we verify G1, G2, G3, G4.

G1 : From the above table, we observe that

$ab \in G, \forall a, b \in G$

$\therefore$ G1 is verified.

G2 : We have $1 \times (w \times w^2) = (1 \times w) \times w^2$

$\Rightarrow$ $1 \times w^3 = w \times w^2$

$\Rightarrow$ $w^3 = w^3$

$\Rightarrow$ $1 = 1$

$\therefore$ $a(bc) = (ab)c, \forall a, b, c \in G$

$\therefore$ G2 is verified.

G3 : We have $1 \times 1 = 1\ w = w$, $1 \times w^2 = w^2$

$\therefore$ 1 is the identity element and $1 \in G$

$\therefore$ G3 is verified.

G4 : The multiplicative inverse of 1 is 1 and $1 \in G$.

The multiplicative inverse of w is $\frac{1}{w}$

Now $\frac{1}{w} = \frac{1}{w} \times \frac{w^2}{w^2}$

$= \frac{w^2}{w^3} = \frac{w^2}{1} = w^2 \in G$

i.e. $\frac{1}{w} \in G$

The multiplicative inverse of w^2 is $\frac{1}{w^2}$

$\frac{1}{w^2} = \frac{1}{w^2} \times \frac{w}{w}$

$= \frac{w}{w^3} = \frac{w}{1} = w \in G$

$\therefore \frac{1}{w^2} \in G$

$\therefore a \in G$

$\Rightarrow a^{-1} \in G, \forall a \in G$

$\therefore$ G4 is verified.

$\therefore$ $\boxed{\text{G is a group under multiplication}}$ **Ans.**

Ex.6. Prove that fourth roots of unity forms an abelian group under multiplication.

CS/CT/CE : S-11,13, IT : S-12,13

Soln. Let $G = \{1, -1, i, -i\}$

Composition table for G is given below.

×	1	−1	i	−i
1	1	−1	i	−i
−1	−1	1	−i	i
i	i	−i	−1	1
−i	−i	i	1	−1

We prove that G is an abelian group

i.e. we verify G1, G2, G3, G4, G5.

G1 : From the above table, we observe that

$ab \in G, \forall a, b \in G$

$\therefore$ G1 is verified.

G2 : We have $1 \times (-1 \times j) = (1 \times -1) \times i$

$\Rightarrow 1 \times -i = -1 \times i$

$\Rightarrow -i = -i$

$\therefore a(bc) = (ab)c, \forall a, b, c \in G.$

$\therefore$ G2 is verified

G3 : We have $1 \times 1 = 1, -1 \times 1 = -1,$

$i \times 1 = -i, -i \times 1 = -i$

$\therefore$ G3 is verified

G4 : The multiplicative inverse of 1 is 1 and − 1 is − 1 and $1, -1 \in G$.

The multiplicative inverse of i is $\frac{1}{i}$

Now $\frac{1}{i} = \frac{1}{i} \times \frac{i}{i} = \frac{i}{i^2}$

$= \frac{i}{-1} = -i \in G$

$\therefore \frac{1}{i} \in G$

The multiplicative inverse of − i is $\frac{-1}{i}$

Now $\frac{-1}{i} = \frac{-1}{i} \times \frac{i}{i} = \frac{-i}{i^2} = \frac{-i}{-1} = i \in G$

$\therefore \frac{-1}{i} \in G$

$\therefore a \in G$

$\Rightarrow a^{-1} \in G, \forall a \in G$

$\therefore$ G4 is verified.

G5 : From the table, we observe that

$ab = ba, \forall a, b \in G$

$\therefore$ G5 is verified.

$\therefore$ $\boxed{\text{G is an abelian group under multiplication}}$ Ans.

Ex.7. If * is a binary operation defined on Q^+ by $a * b = \frac{ab}{3}, \forall a, b \in Q^+$, then prove that $(Q^+, *)$ forms a group.

IT : S-06, W-09,11

CS/CT/CE : W-08,10,11

Soln. Given : $a * b = \frac{ab}{3}, \forall a, b \in Q^+$

To prove : $G = (Q^+, *)$ is a group.

We verify G1, G2, G3, G4.

G1 : Let $a, b \in Q^+$

Since multiplication of two positive rational numbers is again a positive rational number

$\therefore a * b = \frac{ab}{3} \in Q^+$

$\therefore$ G1 is verified.

G2 : Let $a, b, c \in Q^+$

$a * (b * c) = (a * b) * c$

$\Rightarrow a * \left(\frac{bc}{3}\right) = \left(\frac{ab}{3}\right) * c$

$$\Rightarrow \frac{a.\left(\frac{bc}{3}\right)}{3} = \frac{\left(\frac{ab}{3}\right).c}{3}$$

$$\Rightarrow \frac{abc}{9} = \frac{abc}{9}$$

$\therefore$ G2 is verified.

G3 : Let $a \in Q^+$

$a * e = \frac{ae}{3} = \frac{a.3}{3} = a$ for $e = 3$ where $3 \in Q^+$

Similarly $a * e = \frac{e.a}{3} = \frac{3.a}{3} = a$ for $e = 3$ where $3 \in Q^+$.

$\therefore$ $a * e = a = e * a$

$\therefore$ $e = 3$ is the identity element and $3 \in Q^+$.

$\therefore$ G3 is verified.

G4 : Let $a \in Q^+$.

Let b be the inverse of a such that $b = \frac{9}{a} \in Q^+$

$$\left[\because a \in Q^+ \Rightarrow \frac{9}{a} \in Q^+\right]$$

Now $a * b = a * \frac{9}{a} = \frac{a.\frac{9}{a}}{3} = 3$ and

$b * a = \frac{9}{a} * a$

$= \frac{\frac{9}{a}.a}{3} = 3$

$\therefore$ $a * b = 3 = b * a$

where 3 is the identity element and $3 \in Q^+$

$\therefore$ Every $a \in Q^+$ has an inverse $b \in Q^+$ such that $a * b = 3 = b * a$

$\therefore$ G4 is verified.

Commutative law :

let a, b, $\in Q^+$, then

$a * b = \frac{ab}{3} = \frac{ba}{3} = b * a$

$\therefore$ $\boxed{(Q^+, *) \text{ an abelian is group}}$ **Ans.**

Ex.8. Show that the set A = {1, 2, 3} under multiplication modulo 4 is not a group, but B = {1, 2, 3, 4} under multiplication modulo 5 is a group.

Soln. First part :

Given : A = {1, 2, 3}

To prove : $(A, \times_4)$ is not a group.

The composition table for $(A, \times_4)$ is given below.

$\times_4$	1	2	3
1	1	2	3
2	2	0	2
3	3	2	1

($2 \times 2 = 4$, when 4 is divided by 4 the remainder = 0

$2 \times 3 = 6$, when 6 is divided by 4 the remainder = 2)

Here we write ab instead of $a \times_4 b$.

From the above table we observe that

$2 \times 2 = 0$ and $0 \notin A$

$\therefore$ $ab \notin A$, $\forall a, b \in A$

$\therefore$ G1 is not satisfied.

$\therefore$ $\boxed{(A, \times_4) \text{ is not a group.}}$ **Ans.**

Second part :

Given : B = {1, 2, 3, 4}

To prove : $(B, \times_5)$ is a group.

The composition table for $(B, \times_5)$ is given below.

$\times_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

($2 \times 3 = 6$, when 6 is divided by 5 the remainder = 1

$3 \times 3 = 9$, when 9 is divided by 5 the remainder = 4)

Here we write a .b instead of $a \times_5 b$.

G1 : From the above table, we observe that

$\therefore$ $ab \in B$, $\forall a, b \in B$

$\therefore$ G1 is verified.

G2 : We observe that

$1.(2.3) = (1.2).3$

$\Rightarrow 1.1 = 2.3$

$\Rightarrow 1 = 1$

Hence $a.(b.c) = (a.b).c$, $\forall a, b \in B$

$\therefore$ G2 is verified.

G3 : We observe that $1 \times 1 = 1$, $2 \times 1 = 2$, $3 \times 1 = 3$, $4 \times 1 = 4$.

$\therefore$ 1 is the identity element and $1 \in B$.

$\therefore$ G3 is verified.

G4 : The multiplicative inverse of 1 is 1 and $1 \in B$.

Element	Inverse	Reason
2	3	$2 \times 3 = 1$ (from table)
3	2	$3 \times 2 = 1$ (from table)
4	4	$4 \times 4 = 1$ (from table)

Here 1 is the identity element.

Hence the inverse of every element of B belongs to B.

$\therefore$ G4 is verified.

$\therefore$ $\boxed{(B, \times_5) \text{ is a group}}$ **Ans.**

Ex.9. Check whether (Z, *) is a group where the operation * is defined by a * b = a + b − ab, ∀a, b ∈ Z. **CS/CT/CE : S-08, IT : W-10**

Soln. Given : $a * b = a + b - ab, \ \forall a, b \in Z$.

We verify G1,G2,G3,G4,G5.

G1 : Let a, b ∈ Z

Then a + b ∈ Z and ab ∈ Z

$\therefore$ $a + b - ab \in Z$ i.e. $a * b \in Z$

$\therefore$ G1 is verified.

G2 : Let a, b, c ∈ Z

Now $a * (b * c) = a * (b + c - bc)$

$= a + b + c - bc - [a(b + c - bc)]$

$= a + b + c - bc - ab - ac + abc$ ...(i)

$(a * b) * c = (a + b - ab) * c$

$= (a + b - ab + c) - [(a + b - ab)c]$

$= a + b + c - ab - ac - bc + abc$ ...(ii)

From (i) and (i), we get

$a * (b * c) = (a * b) * c, \ \forall a, b, c \in Z$

$\therefore$ G2 is verified.

G3 : We know that $0 \in Z$

Now $a * 0 = a + 0 - a.0 = a$

$0 * a = 0 + a - 0.a = a$

$\therefore$ $0 * a = 0 = a * 0, \forall a \in Z$

$\therefore$ 0 is the identity element

$\therefore$ G3 is verified.

G4 : Here we observe that for any given $a \in Z$, there does no exist an inverse b such that

$a * b = 0 = b * a$

where 0 is the identity element

$\therefore$ G4 does not hold

$\therefore$ $\boxed{(Z, *) \text{ is not a group}}$ **Ans.**

Ex.10. Prove that the set {0, 1, 2, 3, 4} is a finite a abelian group of order 5 under addition modulo 5 as composition.

Soln. Given : A = {0, 1, 2, 3, 4}

To Prove : $(A, +_5)$ is an abelian group.

We verify G1, G2, G3, G4, G5.

The composition table for $(A, +_5)$ is given below.

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

(1 + 4 = 5 and 5 − 5 = 0

2 + 4 = 6 and 6 − 5 = 1

4 + 4 = 8 and 8 − 5 = 3)

Here we write a + b instead of $a +_5 b$

G1 : From the above table, we observe that

$a + b \in A, \ \forall a, b, \in A$

$\therefore$ G1 is verified.

G2 : We observe that

$1 + (2 + 3) = (1 + 2) + 3$

$\Rightarrow 1 + 0 = 3 + 3$

$\Rightarrow 1 + 1$

$\therefore$ $a + (b + c) = (a + b) + c, \ \forall a, b, c \in A$

$\therefore$ G2 is verified.

G3 : Here $0 \in A$

and $a + 0 = a = 0 + a$

$\therefore$ 0 is the identity element

$\therefore$ G3 is verified

G4 : From the table, we observe that

1 + 4 = 0 = 4 + 1

2 + 3 = 0 = 3 + 2, where 0 is the identity element.

$\therefore$ $a + b = 0 = b + a, \ \forall a, b, \in A$, where b is the additive inverse of a.

$\therefore$ Additive inverse of each element of A exists

$\therefore$ G4 is verified.

G5 : From the table, we observe that

$a + b = b + a, \ \forall a, b, \in A$

$\therefore$ G5 is verified.

$\therefore$ $\boxed{(A, +_5) \text{ is an abelian group.}}$ **Ans.**

Ex.11. Prove that the set {1, 2, 3, 4, 5, 6} of order 6 is finite abelian group under multiplication modulo 7 as composition.

Soln. Given : {1, 2, 3, 4, 5, 6}

To prove : $(A, \times_7)$ is an abelian group.

We verify G1, G2, G3, G4, G5.

$\times_5$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

($2 \times 5 = 10$, when 10 is divided by 7, the remainder = 3.

$5 \times 4 = 20$, when 20 is divided by 7, the remainder = 6.)

Here we write a.b instead of $a \times_7 b$.

G1 : From the above table, we observe that $a.b \in A$, $\forall a, b \in A$

$\therefore$ G1 is verified.

G2 : We observe that

$1.(2.3) = (1.2).3$

$\Rightarrow 1.6 = 2.3$

$\Rightarrow 6 = 6$

$\therefore a.(b.c) = (a.b).c, \forall a, b, \in A$

$\therefore$ G2 is verified.

G3 : Here $1 \in A$ and $a.1 = a = 1.a$

$\therefore$ 1 is the identity element.

$\therefore$ G3 is verified.

G4 : From the table, we observe that

$2.4 = 1 = 4.2$

$3.5 = 1 = 5.3$, where 1 is the identity element.

$\therefore a.b = 1 = b.a, \forall a, b, \in A$, where b is the multiplicative inverse of a.

$\therefore$ Multiplicative inverse of each element of A exists

$\therefore$ G4 is verified.

G5 : From the table we observe that

$a.b = b.a, \forall a, b \in A$

$\therefore$ G5 is verified.

$\therefore$ $\boxed{(A, \times_5) \text{ is an abelian group}}$ **Ans.**

Ex.11.(a) Prove that the set G = {0, 1, 2, 3, 4, 5} is a finite abelian group under addition modulo 6 as operation. **CS/CT/CE : S-08**

Soln. Given : A = {0, 1, 2, 3, 4, 5 }

To prove : $(A, +_6)$ is an abelian group.

Composition table for $(A, +_6)$ is given below :

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Here we write a + b instead of $a +_6 b$

G1 : From the composition table, we observe that

$a + b \in A, \forall a, b \in A$

$\therefore$ G1 is verified.

G2 : We observe that

$1 + (3 + 4) = (1 + 3) + 4$

i.e. $1 + 1 = 4 + 4$

i.e. $2 = 2$

$\therefore a, b\ c \in A \Rightarrow a + (b + c) = (a + b) + c$

$\therefore$ G2 is verified.

G3 : Here $0 \in A$ and $a + 0 = a = 0 + a$

$\therefore$ 0 is the identity element

$\therefore$ G3 is verified.

G4 : From the table, we observe that

$1 + 5 = 0 = 5 + 1$

Also $2 + 4 = 0 = 4 + 2$,

where 0 is the identity element.

$\therefore a + b = 0 = b + a, \forall a, b \in A$,

where b is the additive inverse of a

$\therefore$ G4 is verified.

G5 : From the table, we observe that

$a + b = b + a, \forall a, b \in A$

$\therefore$ G5 is verified.

$\therefore$ $\boxed{(A, +_6) \text{ is an abelian group}}$ **Ans.**

Ex.12. If G is a group, then

(i) The identity element of G is unique

(ii) Every $a \in G$ has a unique inverse in G.

(iii) For every $a \in G$, $(a^{-1})^{-1} . a$.

(iv) For all $a, b \in G$, $(a.b)^{-1} = b^{-1}.a^{-1}$. (Reverse Law) **CS/CT/CE : W-12**

Soln. Given : G is a group.

(i) To prove :

The identity element of G is unique.

Let e and f be the identity element of G.

Then $e, f \in G$

If e is the identity element, then

$f.e = f = e.f$ ...(a)

If f is the identity element, then

$e . f = e = f . e$ ...(b)

From (a) and (b), we get

$e = f$

$\therefore$ The identity element of G is unique. **...Ans.**

(ii) To prove :

Every $a \in G$ has a unique inverse in G.

Let G be a group with identity e and let $a \in G$.

$\therefore$ Inverse of a exists in G (by G4)

Suppose x and y are the inverses of a in G.

$$\therefore \left.\begin{array}{l} a . x = e = x . a \\ a . y = e = y . a \end{array}\right| \quad ...(1)$$

Since $a, x, y \in G$ and G is a group

$\therefore x . (a . y) = (x . a) . y$ *[$\because$ By G2]*

$\Rightarrow x . e = e . y$ *[$\because$ By (1)]*

$\Rightarrow x = y$

$\therefore$ Every $a \in G$ has a unique inverse in G. **.....Ans.**

(iii) To prove :

For every $a \in G$, $(a^{-1})^{-1} = a$.

Let G be a group with identity e and let $a \in G$

$\Rightarrow a^{-1} \in G$ such that $a . a^{-1} = e = a^{-1} . a$ ...(2)

Consider $a . a^{-1} = e$

Multiply both sides by $(a^{-1})^{-1}$

$a . a^{-1} . (a^{-1})^{-1} = e . (a^{-1})^{-1}$

$\Rightarrow a . [a^{-1} . (a^{-1})^{-1}] = (a^{-1})^{-1}$

[$\because$ By G2 and G3]

$\Rightarrow a . e = (a^{-1})^{-1}$ *[$\because$ By (2)]*

$\Rightarrow a = (a^{-1})^{-1}$ *[$\because$ By G3]*

$\therefore \boxed{(a^{-1})^{-1} = a, \forall a \in G}$ **Ans.**

To prove :

(iv) For all $a, b \in G$, $(a . b)^{-1} = b^{-1} . a^{-1}$

Let G be a group with identity e and let $a, b \in G$

$\Rightarrow a^{-1} . b^{-1} \in G$ *[$\because$ By G4]*

$\Rightarrow b^{-1} . a^{-1} \in G$ *[$\because$ By G1]*

Consider $(a.b).(b^{-1} . a^{-1}) = a.[b.(b^{-1} . a^{-1})]$

[$\because$ By G2]

$= a . [(b . b^{-1}) . a^{-1}]$ *[$\because$ By G2]*

$= a . [e . a^{-1}]$ *[$\because$ By G4]*

$= a . a^{-1}$ *[$\because$ By G3]*

$\therefore (a.b).(b^{-1} . a^{-1}) = e$ – (3) *[$\because$ By G4]*

Consider $(b^{-1} . a^{-1}) . (a . b) = b^{-1} . [a^{-1} . (a . b)]$

[$\because$ By G2]

$= b^{-1} [(a^{-1} . a) . b]$ *[$\because$ By G2]*

$= b^{-1} . [e . b]$ *[$\because$ By G4]*

$= b^{-1} . b$ *[$\because$ By G3]*

$\therefore (b^{-1} . a^{-1}) . (a . b) = e$ .....(4)

[$\because$ By G4]

From (3) and (4), we get

$(a.b).(b^{-1} . a^{-1}) = (b^{-1} . a^{-1}) . (a . b) = e$

$\therefore \boxed{(a.b)^{-1} = b^{-1} . a^{-1}}$ **Ans.**

[$\because$ By G4, $x . y = e = y . x \Rightarrow y = x^{-1}$]

CANCELLATION LAWS

Ex.13. Let a, b, c be the elements of a group G, then

(i) $a . b = a . c \Rightarrow b = c$ (Left Cancellation Law LCL)

(ii) $b . a = c . a \Rightarrow b = c$ (Right Cancellation Law RCL). **CS/CT/CE : W-11**

Soln. Given : G is a group and $a, b, c \in G$.

To prove : (i) $a . b = a . c \Rightarrow b = c$

(ii) $b . a = c . a \Rightarrow b = c$

Let G be a group with identity e and let $a \in G$

$\Rightarrow a^{-1} \in G$ such that $a . a^{-1} = e = a^{-1} . a$ ...(1)

(i) Let $a . b = a . c$

Multiply by a^{-1} on both sides, we get

$a^{-1} . (a.b) = a^{-1} . (a.c)$

$\Rightarrow (a^{-1} . a) . b = (a^{-1} . a) . c$ *[$\because$ By G3]*

$\Rightarrow e . b = e . c$ *[$\because$ By (1)]*

$\Rightarrow b = c$ *[$\because$ By G3]*

$\therefore a . b = a . c \Rightarrow b = c$

(ii) Let $b . a = c . a$

Multiply by a^{-1} on both sides, we get

$(b.a) . a^{-1} = (c.a) . a^{-1}$

$\Rightarrow b . (a . a^{-1}) = c . (a . a^{-1})$ *[$\because$ By G2]*

$\Rightarrow b . e = c . e$ *[$\because$ By (1)]*

$\Rightarrow b = c$ *[$\because$ By G3]*

$\therefore \boxed{b . a = c . a \Rightarrow b = c}$ **Ans.**

Ex.14. Let a and b be any two elements of the group G. Then the equations $a.x = b$ and $y.a = b$ have unique solutions for x and y in G. **IT : S-11**

Soln. Given : G is a group and $a, b \in G$

To prove : (i) The equation $a.x = b$ and $y.a = b$ have solutions in G.

(ii) The solution are unique.

(i) Consider $a.x = b$

Let $a, b \in G$ and e be the identity element of G.

$\Rightarrow a^{-1}, b \in G$ *[∵ By G4]*

$\Rightarrow a^{-1}, b \in G$ *[∵ By G1]*

Put $x = a^{-1}.b$ in (1), we get

$a.(a^{-1}.b) = b$

$\Rightarrow (a.a^{-1}).b = b$ *[∵ By G2]*

$\Rightarrow e.b = b$ *[∵ By G4]*

$\Rightarrow b = b$ *[∵ By G3]*

$\therefore x = a^{-1}.b$ is a solution $a.x = b$ in G.

Consider $y.a = b$ ...(2)

Let $a, b \in G$ and e be the identity element of G.

$\Rightarrow a^{-1}, b \in G$ *[∵ By G4]*

$\Rightarrow b.a^{-1} \in G$ *[∵ By G1]*

Put $y = b.a^{-1}$ in (2), we get

$(b.a^{-1}).a = b$

$\Rightarrow b.(a^{-1}.a) = b$ *[∵ By G2]*

$\Rightarrow b.e = b$ *[∵ By G4]*

$\Rightarrow b = b$ *[∵ By G3]*

$\therefore y = b.a^{-1}$ is a solution of $y.a = b$ in G.

$\therefore$ $\boxed{\text{The equations } a.x = b \text{ and } y.a = b \text{ have solution in G.}}$ **Ans.**

(ii) Uniqueness :

Suppose $x = x_1$ and $x = x_2$ are the two solutions of $a.x = b$

$\therefore a.x_1 = b$ and $a.x_2 = b$

$\Rightarrow a.x_1 = a.x_2$

$\Rightarrow x_1 = x_2$ .... (by LCL, $a, x_1, x_2 \in G$)

$\therefore a.x = b$ has a unique solution in G.

Suppose $y = y_1$ and $y = y_2$ are two solutions of $y.a = b$

$\Rightarrow y_1.a = b$ and $y_2.a = b$

$\Rightarrow y_1.a = y_2.a$

$\Rightarrow y_1 = y_2$ .... (by RCL, $b, y_1, y_2 \in G$)

$\therefore$ $\boxed{y.a = b \text{ has a unique solution in G.}}$ **Ans.**

Note : If a is an element of a group G, then

(1) $a^0 = e$, where e is the identity element of G.

(2) $a^1 = a$

(3) $a^{n+1} = a^n.a = a.a^n, \forall n$

Ex.15. Show that y if every element of a group G is its own inverse, then G must be abelian.

Soln. Given : G is a group such that

$a \in G$

$\Rightarrow a^{-1} = a$ ...(1)

To prove : G is abelian.

Let $a, b \in G$

$\Rightarrow a^{-1} = a \;\&\; b^{-1} = b$ (2) *[∵ By (1)]*

Also $a.b \in G$ *[∵ By G1]*

$\Rightarrow a.b = (a.b)^{-1}$ *[∵ By (1)]*

$\Rightarrow a.b = b^{-1}.a^{-1}$ *[∵ By Ex.13]*

$\Rightarrow a.b = b.a$ *[∵ By G2]*

$\therefore$ $\boxed{\text{G is abelian}}$ **Ans.**

Ex.16. Show that if G is an abelian group then $(a.b)^n = a^n.b^n, \forall a, b \in G$ for every integer n. **CS/CT/CE : S-07**

Soln. Given : G is an abelian group.

To prove : $(a.b)^n = a^n.b^n, \forall a, b \in G, \forall n, n \in Z$.

Let $a, b \in G$

Then $a.b, a^n, b^n \in G$

We consider three cases.

Case (i) : Let n be a positive integer.

We prove $(a.b)^n = a^n.b^n$ be principle of mathematical induction.

Let $P(n) : (a.b)^n = a^n.b^n$

$P(1) : (a.b)^1 = a^1.b^1$ i.e. $a.b = a.b$

$\therefore P(1)$ is true

Let P(m) be true for some positive integer m.

We prove that $P(m+1)$ is also true.

$P(m): (a.b)^m = a^m . b^m$

Multiply both sides by a . b

$(a.b).(a.b)^m = (a^m . b^m).(a.b)$

$(a.b)^{m+1} = a^m.[b^m.(b.a)]$ *[∵ G is abelian]*

$= a^m.[(b^m.b).a]$

$= a^m.(b^{m+1}).a$

$= (a^m.b^{m+1}).a$

$= (b^{m+1}.a^m).a$

$= b^{m+1}.(a^m.a)$

$= b^{m+1}.(a^{m+1})$

$= a^{m+1}.b^{m+1}$

$\therefore (a.b)^{m+1} = a^{m+1}.b^{m+1}$

$\therefore$ P(m) is true $\Rightarrow$ P(m + 1) is also true.

$\therefore$ By the principle of mathematical induction, P(n) is true for all positive integers n.

Case (ii) : Let n = 0

We prove $(a.b)^n = a^n.b^n$

For n = 0, $(a.b)^0 = e$ as $a.b \in G$

Also, $a^0.b^0 = e.e = e$

$\therefore (a.b)^0 = a^0.b^0$

$\therefore (a.b)^n = a^n.b^n$ for n = 0

Case (iii) : Let n be a negative integer.

Put n = – m, where m is a positive integer

$\therefore (a.b)^n = (a.b)^{-m}$

$= b^{-m}.a^{-m}$ *[∵ By Ex.13]*

$= a^{-m}.b^{-m}$ *[∵ G is abelian]*

$= a^n.b^n$ *[∵ n = – m]*

$\therefore (a.b)^n = a^n.b^n$

Hence $(a.b)^n = a^n.b^n$ for every integer n. **...Ans.**

Ex.17. If G is a group such that $(a.b)^2 = a^2.b^2$, then show that G must be abelian.

IT : W-08,10, CS/CT/CE : W-09

Soln. Given : (i) G is a group

(ii) $(a.b)^2 = a^2.b^2$

To prove : G is abelian.

Let G be a group with identity e

Since $(a.b)^2 = a^2.b^2$,

$\Rightarrow (a.b).(a.b) = (a.a).(b.b)$

$\Rightarrow a.[b.(a.b)] = a.[a.(b.b)]$ *[∵ By G2]*

$\Rightarrow b.(a.b) = a.(b.b)$ *[∵ By LCL]*

$\Rightarrow (b.a).b = (a.b).b$ *[∵ By G2]*

$\Rightarrow b.a = a.b$ *[∵ By RCL]*

$\therefore a.b = b.a, \forall a, b \in G$

$\therefore$ $\boxed{\text{G is an abelian group}}$ **Ans.**

Ex.18. Let (Q, *) be an algebraic structure where $x * y = \frac{x+y}{2}$. Determine which of the following properties hold for this structure.

(i) Closure

(ii) Commutative

(iii) Associative

(iv) Identity

(v) Inverse.

IT : W-06, S-13

Soln. Given : $x * y = \frac{x+y}{2}, x, y \in Q$

G1 (Closure) : Let $x, y \in Q$

Then $\frac{x+y}{2} \in Q \Rightarrow x * y \in Q$

$\therefore$ G1 holds in (Q, *)

G2 (Associative) : Let $x, y, z, \in Q$

Now $(x * y) * z = \left(\frac{x+y}{2}\right) * z$

$$= \frac{\frac{x+y}{2} + z}{2} = \frac{x+y+2z}{4} \quad ...(i)$$

Also $x * (y * z) = x * \left(\frac{y+z}{2}\right) * z$

$$= \frac{x + \frac{y+z}{2}}{2} = \frac{2x+y+z}{4} \quad ...(ii)$$

From (i) and (ii), we get

$(x * y) * z \neq x * (y * z)$

$\therefore$ G2 does not hold in (Q, *)

G3 (Identity) : Let e be the identity element of Q such that $e * x = x = x * e$

Consider $e * x = x$

i.e. $\frac{e+x}{2} = x$

i.e. e = x and $x \in Q$

$\therefore$ G3 holds in (Q, *)

G4 (Inverse) : Let $x \in Q$

If y is to be the inverse of x, then we must have

$y * x = e = x * y$

Consider $y * x = e = x$ *[∵ e = x]*

i.e. $\frac{y+x}{2} = x$

i.e. $y + x = 2x$

i.e. $y = x$ and $x \in Q$.

∴ G4 holds in (Q, *)

G5 (Commutative) : Let $x, y \in Q$

Now $x * y = \frac{x+y}{2} = \frac{y+x}{2} = y * x$

∴ G5 holds in (Q, *). **...Ans.**

Ex.19. Prove that for any element a in a group G, if $a^2 = e$, then G must be an abelian group.

CS/CT/CE : W-07, S-12

Soln. Given : (i) G is a group and $a \in G$

(ii) $a^2 = e$

To prove : G is an abelian group.

i.e. we verify ab = ba

Let $a, b \in G$

Then $ab \in G$ *[∵ G is a group so by G1]*

∴ $(ab)^2 = e$ *[∵ $a \in G \Rightarrow a^2 = e$]*

i.e. (ab) (ab) = e

i.e. $ab = (ab)^{-1}$ *[∵ By G4]*

∴ $ab = b^{-1}a^{-1}$ ....(1)

But $a^2 = e$

i.e. a . a = e

i.e. $a = a^{-1}$ *[∵ By G4]*

Similarly, $b = b^{-1}$

∴ (1) ⇒ ab = ba

∴ $\boxed{\text{G is an abelian group}}$ **Ans.**

Ex.20. Let G = {(a, b) | a, b ∈ R, a ≠ 0}. Define a binary operation * on G by (a, b) * (c, d) = (ac, bc, + d) for all (a, b), (c, d) ∈ G. Show that (G, *) is a group.

CS/CT/CE : W-09

Soln. Given : (i) G = {(a, b) | a, b ∈ R, a ≠ 0}.

(ii) (a, b) * (c, d) = (ac, bc + d)

To prove : (G, *) is a group.

G1 : Let (a, b), (c, d) ∈ G, $a \neq 0$, $c \neq 0$

∴ $ac \neq 0$

∴ (a, b) * (c, d) = (ac, bc + d) ∈ G

∴ G1 is verified

G2 : Let (a, b), (c, d), (e, f) ∈ G, $a \neq 0, \neq 0, e \neq 0$

Now (a, b) * [(c, d) * (e, f)] = (a, b) * [ce, de + f]

= (ace, bce + de + f) ...(i)

Also [(a, b) * (c, d)] * (e, f) = (ac, bd + d) * (e, f)

= (ace, bce + de + f) ...(ii)

From (i) and (ii), we get

(a, b) * [(c, d) * (e, f)] = [(a, b) * (c, d)] * (e, f)

∴ G2 is verified

G3 : Let (x, y) ∈ G such that (x, y) * (a, b) = (a, b), ∀ (a, b) ∈ G

Then (xa, ya + b) = (a, b)

i.e. xa = a, ya + b = b

i.e. x = 1, y = 0 and (x, y) = (1, 0) ∈ G

∴ (1.0) is the identity element of G

∴ G3 is verified.

G4 : Let (a, b), (x, y) ∈ such that

(x, y) * (a, b) = (1, 0)

Then (xa, ya + b) = (1, 0)

i.e. xa = 1, ya + b = 0

i.e. $x = \frac{1}{a}, y = \frac{-b}{a}$

Since $a \neq 0$, hence x and y are real numbers.

∴ $(x, y) = \left(\frac{1}{a}, \frac{-b}{a}\right)$ is the inverse of (a, b)

∴ G4 is verified.

∴ $\boxed{\text{(G,*) is a group.}}$ **Ans.**

Ex.21. Show that the matrices $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}, \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$, forms a multiplicative abelian group.

IT: W-10, CS/CT/CE : S-10

Soln. G1 : Let $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix} \in G$

Then $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix} \in G$

∴ A, B ∈ G

⇒ AB ∈ G, ∀A, B

∴ G1 is verified.

G2 : Let $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} \in G$

Now $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\left\{\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\right\}$

$$= \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix} = \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix} \quad \text{...(i)}$$

Also $\left\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\right\}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$

$$= \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} = \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix} \quad \text{...(ii)}$$

From (i) and (ii), we get

$A . (B . C) = (A . B) . C, \forall A, B, C \in G$

$\therefore$ G2 is verified.

G3 : Here we observe that $I = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ is the identity matrix as $A . I = A = I . A, \forall A \in G$

$\therefore$ G3 is verified.

G4 : Here we observe that

$$\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix} \cdot \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

Hence each of the four given matrix is its own inverse

$\therefore$ G4 is verified.

G5 : Let $\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} \in G$

Now $\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix} \cdot \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} = \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$ ...(i)

Also $\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} \cdot \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$ ...(ii)

From (i) and (i), we get

$A . B = B . A, \forall A, B \in G$

$\therefore$ G5 is verified.

$\therefore$ $\boxed{\text{The given matrices form a multiplicative abelian group.}}$ **Ans.**

IT : W-06

Order of an element :

Let (G, *) be a group and $a \in G$, then the least positive integer n (if it exists) such that $a^n = e$ is called the order of a, $a \in G$. Here e is the identity element of G. The order of an element $a \in G$ is denoted by O(a).

Note : If $O(a) = n$, then $a^n = e$.

Ex.22. Find the order of each element of the groups $G = \{1, w, w^2\}$ and $G = \{1, -1, i, -i\}$ with respect to multiplication. **IT : W-06, S-08**

Soln.

(i) Let $G = \{1, w, w^2\}$ be a group w.r.t multiplication. Here 1 is the identity element.

Element	Order	Reason
1	$O(1) = 1$	$(1)^1 = 1$
w	$O(w) = 3$	$(w)^3 = 1$
w^2	$O(w^2) = 3$	$(w^2)^3 = w^6 = w^3 . w^3 = 1 . 1 = 1$

(ii) Let $G = \{1, -1, i, -i\}$ be a group w.r.t multiplication. Here 1 is identity element.

Element	Order	Reason
1	$O(1) = 1$	$(1)^1 = 1$
-1	$O(-1) = 2$	$(-1)^2 = 1$
i	$O(i) = 4$	$(i)^4 = (i^2)^2 = (-1)^2 = 1$
$-i$	$O(-i) = 4$	$(-i)^4 = i^4 = 1$

Ex.23. Find the order of each element of the group $G = \{0, 1, 2, 3\}$ with addition modulo 4.

Soln. Let $G = \{0, 1, 2, 3\}$ be a group with addition modulo 4.

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Here 0 is the identity element.

Element	Order	Reason
0	$O(0) = 1$	$(0)^1 = 0$
1	$O(1) = 4$	$(1)^4 = 0$ (From table $1 + 3 = 4 = 0$)
2	$O(2) = 2$	$(2)^2 = 4 = 0$
3	$O(3) = 4$	$(3)^4 = 12 = 0$

Ex.24. The order of an element of a group is same as that of its inverse. **IT : W-07**

Soln. Let $O(a) = m$ and $O(a^p) = +n$, where $p \in Z$

Since $O(a) = m \Rightarrow a^m = e$ *[∵ By definition]*

Also since $O(a^p) = n \Rightarrow (a^p)^n = e$

[∵ By definition]

Now $(a^p)^m = a^{pm} = a^{mp} = (a^m)^p = (e)^p = e$

$\Rightarrow O(a^p) \le m$

$\Rightarrow O(a^p) \le O(a)$

Hence

$O(a^{-1}) \le O(a)$ ...(1)

We know that $a = (a^{-1})^{-1}$

Hence $O(a) \le O(a^{-1})$ ...(2)

From (1) and (2), we get

$\boxed{O(a) = O(a^{-1})}$ **Ans.**

Ex.25. Prove that $O(a) = O(x^{-1}ax)$ where a, x, are any two elements of a group G. ***IT : S-08***

Soln. Let $O(a) = n$ and $O(x^{-1}ax) = m, a, x \in G, G$ is group.

Consider $(x^{-1}ax)^2 = (x^{-1}ax)(x^{-1}ax)$

$= x^{-1}a(xx^{-1})ax$ *[∵ By G2]*

$= x^{-1}a.e.ax$ *[∵ By G4]*

$= x^{-1}a.ax$ *[∵ By G3]*

$= x^{-1}a^2.x$

$\therefore (x^{-1}ax)^2 = x^{-1}a^2x$

In general, $(x^{-1}ax)^n = x^{-1}a^nx$

$= x^{-1}ex$ *[∵ O(a) = n ⇒ aⁿ = e]*

$= x^{-1}x$

$\therefore (x^{-1}ax)^n = e$ ...(1)

But $O(x^{-1}ax) = m \Rightarrow (x^{-1}ax)^m = e$ ...(2)

From (1) and (2), we get $m \le n$ ...(I)

Also since $O(x^{-1}ax) = m$

$\Rightarrow (x^{-1}ax)^m = e$

$\Rightarrow x^{-1}a^mx = e$ *[∵ By above results]*

$\Rightarrow xx^{-1}a^mxx^{-1} = xex^{-1}$

$\Rightarrow ea^me = xx^{-1}$

$\Rightarrow a^m = e$

i.e. $O(a) = m$

But $O(a) = n \Rightarrow n \le m$ ...(II)

$\therefore$ (I) and (II) $\Rightarrow m = n$

$\therefore \boxed{O(a) = O(x^{-1}ax)}$ **Ans.**

Ex.26. G is a group and there exist two relatively prime positive integers m and n such that $a^mb^m = b^ma^m$ and $a^nb^n = b^na^n$ for all a, b ∈ G. Prove that G is Abelian. ***CS/CT/CE : S-06,12, W-10 IT : S-09***

Soln. Since m and n are relatively prime, g c d (m, n) = 1, We get $mx + ny = 1$ for some x, y ∈ z

Now $(a^mb^n)^{mx} = a^m(b^na^m)^{mx-1}b^n$

$= a^m(b^na^m)^{mx}(b^na^m)^{-1}b^n$

$= (b^na^m)^{mx}a^ma^{-m}b^{-n}b^n$

$= (b^na^m)^{mx}$ ...(1)

Similarly it can be proved that $(a^mb^n)^{ny} = (b^na^m)^{ny}$ ...(2)

From (1) and (2), we get

$a^mb^n = (a^mb^n)^{mx+ny}$

$= (b^na^m)^{mx+ny} = b^na^m$ ...(3)

Finally $ab = a^{mx+ny}b^{mx+ny}$

$= a^{mx}(a^{ny}b^{mx})b^{ny}$

$= a^{mx}b^{mx}a^{ny}b^{ny}$ *[∵ By (3)]*

$= b^{mx}a^{mx}b^{ny}a^{ny}$

$= b^{mx+ny}a^{mx+ny}$

$= ba$

$\therefore$ $\boxed{\text{Hence G is abelian.}}$ **Ans.**

Ex.27. Prove that a finite semi group (S, *) is a group if and only if (S, *) satisfies the cancellation laws. ***CS/CT/CE : S-06***

Soln. Let (S, *) be a finite semi group satisfying cancellation law i.e. $a * b = a * c \Rightarrow b = c$

and $b * a = c * a \Rightarrow b = c$

Let $S = \{a_1, a_2, \ldots\ldots, a_n\}$ where a_i are all distinct element of S.

Consider now the elements $a_1 * a_1, a_1 * a_2, \ldots. a_1 * a_n$

These elements belongs to S and are distinct. If they are not distinct, let $a_1 * a_i = a_1 * a_j$

Then, by cancellation law, $a_i = a_j$, which contradicts the fact $a_i \neq a_j$.

Then composite elements $a_1 * a_1, a_1 * a_2, \ldots\ldots a_1 * a_n$, being all distinct, they are the n given elements of S in some order.

This shows that the equation $a * x = b$ for $a, b \in S$ has a solution in S.

Similarly, by forming the products $a_1 * a_1, a_2 * a_1, \ldots\ldots, a_n * a_1$ it can be shown that the equation $y * a = b$ for $a, b \in S$ has a solution in S.

Thus {S, *} is a semi group in which each of the equations $a * x = b$ and $y * a = b$ has a solution in S for all $a, b \in S$. Hence (S, *) is a group. ....**Ans.**

Ex.28. Let * denote a binary operation on the set of natural numbers given by x * y = x.
Show that * is not commutative, but associative. Which elements are idempotent and are there any left (or) right identities. ***CS/CT/CE : W-06***

Soln. $x * y = x$

$y * x = y$

Since $x * y \neq y * x$, the given operation * is not commutative

$(x * y) * z = x * z = x$ and $x * (y * z) = x * y = x$

the given operation is associative.

Since $x * x = x$ every element is idempotent.

Since $x * y = x$, y is the left identity of x or every element is its left identity also. ....**Ans.**

Ex.29. Let Gbe a finite group with identity e, and let a be an arbitrary element of G, prove that there exists a non-negative integers n such that $a^n = e$. ***CS/CT/CE : W-06***

Soln. If G be a finite group of order n and $n \in G$, then $a^n = e$.

Let $O(a) = m$ which implies $a^m = e$

Now, the subset H of G consisting of all the integral power of a is a sub group of G and the order of a is m.

Then by the above theorem, m is a divisor of n.

Let $n = m\,k$, then $a^n = a^{mk}$

$= (a^m)^k = e^k = e$

$\therefore$ $\boxed{a^n = e}$ **Ans.**

Ex.30. Consider the binary operation * defined on the set A = {a, b, c} by the following table:

*	a	b	c
a	b	c	b
b	a	b	c
c	c	a	b

(i) Is * a commutative operation?

(ii) Compute a * (b * c) and (a * b) * c. ***IT : W-06***

Soln. We are given

*	a	b	c
a	b	c	b
b	a	b	c
c	c	a	b

(i) Here * is not commutative since a * b = c where as b * a = a

(ii) a * (b * c) = a * c = b and
(a * b) * c = c * c = b ...**Ans.**

Ex.31. Define semigroup and monoid. Prove that the set of Z of all rational numbers other than 1 with the operation defined by $a \cdot b = a + b - ab$ constitute an abelian group. ***CS/CT/CE : S-07***

Soln. Semigroup : A semigroup $(S, \circ)$ is a non – empty set S together with operation $\circ$ satisfying following properties.

(i) **Closure property :**
If $x, y \in S$, then
$x \circ y \in S$

(ii) **Associative property :**
$x \circ (y \circ z) = (x \circ y) \circ z$
$\forall x, y, z \in S$

The semigroup $(S, \circ)$ is called commutative if $a \circ b = b \circ a \; \forall a, b \in S$

Monoid : Monoid $< M, \circ >$ is non empty set M with operation o satisfying.

(i) **Closure property :** If $x \in M$, $y \in M$, then $x \circ y \in M$

(ii) **Associative property :** $x(y \circ z) = (x \circ y) \circ z$
$\forall x, y, z \in M$

(iii) **Existence of identity :** For any $x \in M$ $\exists e \in M$
$x \circ e = e \circ x = x$

A monoid is commutative, if
$a \circ b = b \circ a \quad \forall a, b \in M$

Closure : Let a and $b \in Z$

$\Rightarrow a + b \in Z$, $ab \in Z$, then $a + b - ab \in Z$,

i.e. $a * b = a + b - ab \in Z$

Associativity : Let a, b, c $\in Z$, then

$(a * b) * c = (a + b - ab) * c$

$= a + b - ab + c - (ac + bc - abc)$

$= a + b + c - ab - ac - bc + abc$

Also, $a * (b * c) = a * (b + c - bc)$

$= (a + b + c - bc - \{ab + ac - abc\})$

$= a + b + c - bc - ab - ac + abc$

$\therefore\ a * (b * c) = (a * b) * c$

Identity : Since $0 \in Z$

We have $a * 0 = a + 0 - a(0) = a$

$0 * a = 0 + a - 0(a) = a$

$\therefore\ a * 0 = 0 * a = a$ for any $a \in z$

$(Z, *)$ defined in given equation is a monoid

Commutativity : $a * b = a + b - ab = b + a - ba = b * a$

$\therefore$ It is also commutative

$\therefore$ $\boxed{(Z, *) \text{ is an abelian group}}$ **Ans.**

Ex.32. Prove that the set of all n, n^{th} roots of unity forms a finite abelian group of order n with respect to multiplication of complex numbers.

IT : S-07, W-07

Soln. We have $(1)^{\frac{1}{n}} = (\cos 2r\pi + i\sin 2r\pi)^{\frac{1}{n}}$, r = 0, 1, 2..... n – 1.

$$= \cos\frac{2r\pi}{n} + i\sin\frac{2r\pi}{n}$$

$$= e^{i\left(\frac{2r\pi}{n}\right)}$$

Thus we get distinct n values for r = 0, 1, ..., n–1

$$\text{Let } G = \left\{1, e^{i\frac{2\pi}{n}}, e^{i\frac{4\pi}{n}}, \ldots\ldots\ldots, e^{i2\frac{(n-1)\pi}{n}}\right\}$$

$$= \left\{1, \omega, \omega^2, \ldots\ldots\ldots, \omega^{n-1}\right\}$$

where $\omega = e^{2i\pi/n}$

To prove G is a group.

(1) **Closure property :** Let a, b $\in$G then $a = \omega^p$, $b = \omega^q$

$\therefore\ ab = \omega^{p+q} = \omega^l$.

where l is an remainder after dividing p + q by n, clearly $l < n$.

$\therefore\ \omega^l \in G$

i.e. a b $\in$ G.

(2) **Associative property :** Since 1, ω, ω^2, ... are all complex numbers and product of complex number is associative so associative property is true for elements of G.

(3) **Existence of Identity :** 1 is the identity element because for any $\omega^p \in G$, $\omega^p . 1 = 1 . \omega^p = \omega^p$

(4) **Existence of inverse :** For any $\omega^p \in G$

we have $\omega^p . \omega^{n-p} = \omega^{n-p}\,\omega^p = \omega^n = 1$

$\therefore\ \omega^{n-p}$ is inverse of ω.

(5) Since product of complex numbers is commutative so elements of G are also commutative. Hence G is an abelian group.

Ex.33. If in a group G, $a^5 = e$, $aba^{-1} = b^2$ for a, b $\in$ G. Find O (b).

IT : S-08

Soln. We have $(aba^{-1})^2 = a\,b\,a^{-1}a\,b\,a^{-1}$

$= a\,b\,e\,b\,a^{-1}$

$= a\,b^2\,a^{-1}$

$= a\,(a\,b\,a^{-1})\,a^{-1}$ $\quad [\because b^2 = aba^{-1}]$

$\therefore\ (aba^{-1})^2 = a^2\,b\,a^{-2}$

Now $(aba^{-1})^4 = [(aba^{-1})^2]^2$

$= (a^2\,ba^{-2})^2$

$= (a^2\,ba^{-2})\,(a^2\,ba^{-2})$

$= a^2\,ba^{-2}\,a^2\,ba^{-2}$

$= a^2b^2a^{-2} = a^2\,aba^{-1}\,a^{-2}$

$\therefore\ (aba^{-1})^4 = a^3ba^{-3}$.

Similarly, $(aba^{-1})^8 = a^4\,ba^{-4}$

$(aba^{-1})^{16} = a^5\,ba^{-5} = ebe^{-1}$ $\quad [\because a^5 = e]$

$= b$

$\therefore\ (aba^{-1})^{16} = b$

$\therefore\ (b^2)^{16} = b$ $\quad [\because aba^{-1} = b^2]$

$\therefore\ b^{32} = b$

$\therefore\ b^{31} = e$ ...(1)

$\therefore$ O (b) is the least + ve integer n with $b^n = e$ ...(2)

From (1) and (2) n must divide 31 . but 31 is a prime number

$\therefore$ n = 1 or 31

if n = 1, O (b) = 1 i.e. b = e

if n = 31 then O(b) = 31 when $b \neq e$.

Ex.34. Prove that isomorphism preserves order property of elements. **CS/CT/CE : W-08**

Soln. Let $f : G \to G'$ be an isomorphism

For $a \in G$, $f(a) \in G'$

To prove : $O[f(a)] = O(a)$

Let e and e' be identity elements of G and G'

Let $O(a) = n$, $O[f(a)] = m$ to prove $m = n$

Now,

$O[f(a)] = m$

$\therefore [f(a)]^m = f(e)$

$\therefore f(a)\, f(a) \ldots m$ times

$= f(e)$ ...(1)

$\therefore f(a.a. \ldots m \text{ time}) = f(e)$

[∵ f is an isomorphism]

$\therefore f(a^m) = f(e)$

$\therefore a^m = e$ ...(2)

[∵ f is 1 – 1]

But $O(a) = n$ i.e. n is the least positive integer with

$a^n = e$ ...(3)

Also from (2) & (3), $n \le m$.

$a^n = e$

$\Rightarrow f(a^n) = f(e)$

$\Rightarrow f(a.a. \ldots n \text{ times}) = f(e)$

$\Rightarrow f(a)\, f(a) \ldots n \text{ times} = f(e)$

[∵ f is an isomorphism]

$\Rightarrow [f(a)]^n = f(e) = e'$ ...(4)

But $O[f(a)] = m$ i.e. m is the least positive integer

s.t. $[f(a)]^m = e'$ ...(5)

From (4) and (5) $m \le n$

Now $m \le n$, $n \le m$

$\Rightarrow m = n$

Thus isomorphism preserves order property of elements. **Ans.**

Ex.35. Show that set P(S) of all partitions of S = {a, b, c} is a monoid with the operation of product of partitions. **IT : S-09**

Soln. Product of partitions P, Q is defined as P*Q = {Set of intersection of every element of P with every element of Q leaving out empty set}

We have S = {a, b, c}

$\therefore P(S) = \{\{\bar{a}, \bar{b}, \bar{c}\}, \{\overline{ab}, \bar{c}\}, \{\bar{a}, \overline{bc}\}, \{\overline{ac}, \bar{b}\}, \{\overline{abc}\}\}$

$= \{P_1, P_2, P_3\, P_4, P_5\}$

We have $P_2 * P_3 = \{\overline{ab}, \bar{c}\} * \{\bar{a}, \overline{bc}\}$

$= \{\overline{ab} \cap \bar{a}, \overline{ab} \cap \overline{bc}, \bar{c} \cap \bar{a}, \bar{c} \cap \overline{bc}\}$

$= \{\bar{a}, \bar{b}, \bar{c}\} = P_1 \in P(S)$.

Similarly, it can seen that product of any two partition is again a member of P(S).

* is associative because operation of intersection on sets is associative.

The partition P_5 is the identity element because, e.g.

$P_5 * P_2 = \{\overline{abc}\} * \{\overline{ab}, \bar{c}\}$

$= \{\overline{abc} \cap \overline{ab}, \overline{abc} \cap \bar{c}\}$

$= \{\overline{ab}, \bar{c}\} = P_2$

$\therefore$ P(S) is a monoid. **Ans.**

Ex.36. The order of an element of a group is same as that of its inverse. **IT : S-12**

Soln. Let $O(a) = n$ and $O(a^{-1}) = m$

To prove : $n = m$

$O(a) = n$

$\Rightarrow a^n = e$

$\Rightarrow (a^n)^{-1} = e^{-1} = e$

$\Rightarrow (a^{-1})^n = e$ ...(1)

But $O(a^{-1})$ is m, m is the least positive integer with

$(a^{-1})^m = e$ ...(2)

From (1) & (2), $m \le n$ ...(A)

Also $O(a^{-1}) = m$

$\therefore (a^{-1})^m = e$

$\therefore (a^m)^{-1} = e$

$\therefore \left[(a^m)^{-1}\right]^{-1} = e^{-1} = e$

$\therefore a^m = e$ ....(3)

But order of a is n i.e. n is the least positive integer with $a^n = e$ ....(4)

From (3) & (4), $n \le m$ ...(B)

$\therefore$ From (A) & (B), we have

$m = n$

$\therefore$ The order of an element of a group is same as that of its inverse. **Ans.**

Ex.37. Let M_2 be a set of 2×2 matrices over R i.e.

$$M_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d, \in R, ad - bc \neq 0 \right\}.$$ **Show that $(M_2, +)$ is both associative and commutative and $(M_2, .)$ is associative but not commutative.**

IT : W-12

Soln. Given : $M_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d, \in R, ad - bc \neq 0 \right\}$

(i) To prove : $(M_2, +)$ is associative and commutative

(a) Associative : We prove that $A + (B + C)$

$= (A + B) + C$

Let $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$, $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$, $C = \begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix}$,

$\therefore\ A + B = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$,

$B + C = \begin{bmatrix} a_2 + a_3 & b_2 + b_3 \\ c_2 + c_3 & d_2 + d_3 \end{bmatrix}$

$\therefore$ L.H.S. $= A + (B + C)$

$= \begin{bmatrix} a_1 + a_2 + a_3 & b_1 + b_2 + b_3 \\ c_1 + c_2 + c_3 & d_1 + d_2 + d_3 \end{bmatrix}$

R.H.S. $= (A + B) + C$

$= \begin{bmatrix} a_1 + a_2 + a_3 & b_1 + b_2 + b_3 \\ c_1 + c_2 + c_3 & d_1 + d_2 + d_3 \end{bmatrix}$

$\therefore$ $(M_2, +)$ is associative. **...Ans.**

(b) Commutative : We prove $A + B = B + A$

$B + A = \begin{bmatrix} a_2 + a_1 & b_2 + b_1 \\ c_2 + c_1 & d_2 + d_1 \end{bmatrix}$

$= \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix} = A + B$

$\therefore$ $(M_2, +)$ is commutative. **...Ans.**

(ii) To prove : $(M_2, .)$ is associative but not commutative.

Associative law trivially holds in matrices

i.e. $A . (B . C) = (A . B) . C$

$\therefore$ $(M_2, .)$ is associative. **...Ans.**

Now $AB = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

$= \begin{bmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{bmatrix}$

Also $BA = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$

$= \begin{bmatrix} a_1 a_2 + b_2 c_1 & a_2 b_1 + b_2 d_1 \\ c_2 a_1 + d_2 c_1 & c_2 b_1 + d_2 d_1 \end{bmatrix}$

$\therefore\ AB \neq BA$

$\therefore$ $\boxed{(M_2, .) \text{ is not commutative.}}$ **Ans.**

Ex.38. Let a be any real no. and let $S = \{na \mid n \in Z\}$. Prove that (S, +) is a group. ***IT : W-12***

Soln. Given : $S = \{na \mid n \in Z\}$

To prove : (S, +) is a group

(i) G1 : Since $a \in R$, $n \in Z$

$\therefore\ na \in R$

$\therefore\ na + nb \in R,\ \forall a, b \in R, n \in Z$

$\therefore$ G1 is verified.

(ii) G2 : Let na, nb, nc $\in R$, where a, b, c $\in R$, $n \in Z$.

Now $na + (nb + nc) = na + nb + nc$

$= (na + nb) + nc$

$\therefore$ G2 is verified.

(iii) G3 : Here 0 is the identity element and

$na + 0 = na = 0 + na$

$\therefore$ G3 is verified.

(iv) G4 : For every $na \in R$, there exists $-na \in R$ such that

$na + (-na) = 0 = (-na) + na,\ \forall a \in R, n \in Z$

$\therefore$ G4 is verified.

$\therefore$ $\boxed{(S, +) \text{ is a group}}$ **Ans.**

SUBGROUP

IT : S-13

Let (G, *) be a group and H be a non-empty subset of G, then H is called a subgroup of G, if H is a group under the operation *.

For Example :

(i) Consider, $Z = \{0, \pm 1, \pm 2, ...\}$, $H = \{0, \pm 2, \pm 4, ...\}$

Here $H \subset Z$

Also both H and Z are groups under the operation addition.

$\therefore$ H is a subgroup of Z.

(ii) $G = \{1, -1, i, -i\}$, $H = \{i, -1\}$

Here $H \subset G$.

Also both H and G are groups under the operation multiplication.

$\therefore$ H is a subgroup of G.

(iii) $Z = \{0, \pm 1, \pm 2, ...\}$, $H = \{0, \pm 1\}$

Here $H \subset Z$.

Also Z is a group under usual addition, but H is not a group under usual addition as G_1 does not hold in H because $1, 1 \in H$, but

$1 + 1 = 2 \notin H$.

$\therefore$ H is not a subgroup of Z.

(iv) $G = \{1, -1, i, -i\}$ $H = \{i, -i\}$

Here $H \subset Z$.

Also G is a group under multiplication, but H is not a group under multiplication as G1 does not hold in H because $i, -i \in H$, but $i \times -i = 1 \notin H$.

$\therefore$ H is not a subgroup of G.

(v) The set of rational numbers is a subgroup of set of real numbers under the operation addition.

(vi) The set of integers is a subgroup of both the set of rational numbers and the set of real numbers under the operation addition.

Note : (1) If e is the identity element of a group G, then both the sets $\{e\}$ and G are subgroups of G. These subgroups $\{e\}$ and G are called **trivial** subgroups and the other subgroups of G are called **non-trivial** subgroups.

(2) If $H \subseteq G$, then associative property in H obviously holds because $a, b, c \in H$

$\Rightarrow a, b, c$

$\Rightarrow G$.

But G is group.

$\therefore a * (b * c) = (a * b) * c$.

Criteria for a subgroup :

Ex.39. A non-empty subset H of a group G is a subgroup of G, iff

(i) $a, b \in H$

$\Rightarrow a * b \in H$.

(ii) $a \in H$

$\Rightarrow a^{-1} \in H$.

CS/CT/CE : S-11

Soln. Given : (1) $H \subseteq G$

(2) G is a group.

If Part :

Given : H is a subgroup of G.

To Prove : (i) $a, b \in H$

$\Rightarrow a * b \in H$.

(ii) $a \in H$

$\Rightarrow a^{-1} \in H$.

Since H is a subgroup of G

$\therefore$ By definition, H is a group under the same operation as in G.

i.e. G1, G2, G3, G4 hold in H.

Now G1 : $a, b \in H$

$\Rightarrow a * b \in H$ holds in H

and G4 : $a \in H$

$\Rightarrow a^{-1} \in H$ holds in H

Since $a, b^{-1} \in H$

$\therefore a * b^{-1} \in H$

Only if part :

Given : (i) $a, b \in H$

$\Rightarrow a * b \in H$

(ii) $a \in H$

$\Rightarrow a^{-1} \in H$

To prove : H is subgroup of G.

i.e. we prove that H is a group under the same operation as in G.

G1 : $a, b \in H$

$\Rightarrow a * b \in H$.

This holds by hypothesis (i).

G2 : Let $a, b, c \in H$

$\Rightarrow a, b, c \in G$

But G is a group

$\therefore a * (b * c) = (a * b) * c$

$\therefore a, b, c, \in H$

$\Rightarrow a * (b * c) = (a * b) * c$

$\therefore$ G2 holds in H.

G3 : Let e be the identity element of G.

We prove that $e \in H$.

By (ii), $a \in H$

$\Rightarrow a^{-1} \in H$.

By (i), $a \in H$ and $a^{-1} \in H$

$\Rightarrow a * a^{-1} \in H$.

$\Rightarrow a * a^{-1} \in G$ as $H \subseteq G$.

But G is a group whose identity element is e.

$\therefore a * a^{-1} = e$

$\therefore e \in H$ *[$\because$ By (iii)]*

$\therefore$ G3 holds in H.

G4 : $a \in H \Rightarrow a^{-1} \in H$

This holds by hypothesis (ii).

$\therefore$ G4 holds in H.

$\therefore$ H is a group under the same operation as in G.

$\therefore$ $\boxed{\text{H is a subgroup of G.}}$ **Ans.**

Note : If H is a subgroup of G, then

(1) Identity of H = Identity of G.

(2) Inverse of $a \in H$ = Inverse of $a \in G$.

Ex.40. If H is a non-empty finite subset of a group G and H is closed under multiplication, then H is a subgroup of G.

Soln. Given : (i) H is a non-empty finite subset of a group G.

(ii) H is closed under multiplication.

To Prove : H is a subgroup of G.

i.e. We verify G1, G4.

G1 : Since H is closed under multiplication.

$\therefore$ $a, b \in H$

$\Rightarrow a * b \in H$.

G4 : Let $a \in H$

$\Rightarrow a^2 = a * a \in H$

$\Rightarrow a^3 = a^2 * a \in H$

$\vdots$

$a^m = a^{m-1} * a \in H$

$\vdots$

$a^n = a^{n-1} * a \in H$

i.e. H contains infinite collection of elements $a, a^2, a^3, \ldots, a^m, \ldots, a^n$ ...(iii)

But given that H is a finite set.

$\therefore$ The element in the infinite list of H must repeat.

$\therefore$ $a^p = a^q$ for some positive integers p and q such that $p > q$

i.e. $a^{p-q} * a^q = a^q$

i.e. $a^{p-q} * a^q = e * a^q$

i.e. $a^{p-q} = e$

Multiply by a^{-1}

$\Rightarrow a^{p-q} * a^{-1} = e * a^{-1}$

$\Rightarrow a^{p-q-1} = a^{-1}$ ...(iv)

But $p > q$

i.e. $p - q > 0$

i.e. $p - q - 1 \geq 0$

$\Rightarrow a^{p-q-1} \in H$ *[$\because$ By (III)]*

$\Rightarrow a^{-1} \in H$ ...(v)

$\therefore$ $a \in H \Rightarrow a^{-1} \in H$

$\therefore$ G_4 is verified.

$\therefore$ $\boxed{\text{H is a subgroup of G.}}$ **Ans.**

Ex.41. Let G be a group of integers and H be the subset consisting of all the multiples of an integer n, then show that H is a subgroup of G. ***CS/CT/CE : S-13***

Soln. Let $G = \{0, \pm 1, \pm 2, \ldots\}$ be a group under addition.

$H = \{0, \pm n, \pm 2n, \pm 3n \ldots\}$

i.e. $H = \left\{ \begin{matrix} nx, \\ ny \end{matrix} \middle| \begin{matrix} x \in G, \\ y \in G \end{matrix} \right\}$ be a subset of G. ...(1)

To prove : H is a subgroup of G.

i.e. we verify G1, G4.

We observe that 0 is the identity element of G and $0 \in H$.

$\therefore$ H is a non-empty subset of G.

G1 : Let $a, b \in H$

$\Rightarrow a = nx, b = ny$ for some $x, y \in G$ *[$\because$ By (1)]*

$\therefore$ $a + b = n(x + y) = nz$, where $z = x + y$

$\Rightarrow a, b \in H$

$\therefore$ $a, b \in H$

$\Rightarrow a + b \in H$

$\therefore$ G1 is verified.

G4 : Let $a \in H$

$\Rightarrow a = nx$ for some $x \in G$

$\Rightarrow a^{-1} = -a = -nx$

$\Rightarrow a^{-1} = n(-x) = ny$, where $y = -x$

$\Rightarrow a^{-1} \in H$

$\therefore$ $a \in H$

$\Rightarrow a^{-1} = -a \in H$

$\therefore$ G4 is verified.

$\therefore$ $\boxed{\text{H is a subgroup of G}}$ **Ans.**

Ex.42. If H and K are any two subgroups of a group G, then show that $H \cap K$ is also a subgroup of G. Is $H \cup K$ a subgroup of G ?

IT : W-06,12, S-13, CS/CT/CE : S-07,12

Soln.

(i) Let H and K be any two subgroups of G, where G is a group with identity element e.

$\Rightarrow e \in H$ and $e \in k$

$\Rightarrow e \in H \cap K$

$\Rightarrow H \cap K$ is a non-empty subset of G.

Now we verify G1, G4.

Let $a, b \in H \cap K$

$\Rightarrow a, b \in H$ and $a, b \in K$

But H and K are subgroups of G.

$\Rightarrow a * b \in H,\ a^{-1} \in H$ and $a * b \in K,\ a^{-1} \in K$

$\Rightarrow a * b \in H \cap K$ and $a^{-1} \in H \cap K, \forall a, b \in H \cap K$

$\therefore$ G1, G4 are verified.

$\therefore$ $\boxed{H \cap K \text{ is a subgroup of G.}}$ **Ans.**

(ii) By taking a counter example we check whether $H \cup K$ is a subgroup of G.

Let $G = \{0, \pm 1, \pm 2, \pm 3, \ldots 1\}$

$H = \{0, \pm 2, \pm 4, \pm 6, \ldots\} = \{2x / x \in G\}$

$K = \{0, \pm 3, \pm 6, \pm 9, \ldots\} = \{3x / x \in G\}$

We observe that $H \subseteq G, K \subseteq G$ are subgroups of G under the operation addition.

Now $H \cup K = \{0, \pm 2, \pm 3, \pm 4, \pm 6, \ldots\}$

We observe that $2, 3 \in H \cup K$,

but $2 + 3 = 5 \notin H \cup K$

$\therefore$ G1 does not hold in $H \cup K$.

$\therefore$ $\boxed{H \cup K \text{ is not a subgroup of G.}}$ **Ans.**

COSETS

IT : S-13

Cosets : Let H be a subgroup of a group G and $a \in G$, then

(i) The set $H * a = \{h * a / h \in H\}$ is called the right coset of H in G determined by a.

(ii) The set $a * H = \{a * h / h \in H\}$ is called the left coset of H in G determined by a.

Note :

(1) In addition notation, the above definition takes the form $H + a = [h + a / h \in H]$ and

$a + H = \{a + h / \in H\}$.

(2) If e is the identity element of G, then we have

$e * H = H = H * e$.

Hence H itself is a right as well as left coset determined by e.

(3) In general $a * H \neq H * a$

But if G is an abelian group, then

$a * H = H * a$.

Ex.43. Let $G = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$ and $H = \{\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\}$. Find right and left cosets.

Soln. Then $H + 0 = H$

$H + 1 = \{\ldots, -8, -5, -2, 1, 4, 7, 10, \ldots\}$

$H + 2 = \{\ldots, -7, -4, -1, 2, 5, 8, 11, \ldots\}$

$H + 3 = \{\ldots, -6, -3, 0, 3, 6, 9, \ldots\} = H$

$H + 4 = \{\ldots, -5, -2, 1, 4, 7, 10, \ldots\} = H + 1$

Similarly $H + 5 = H + 2$

$H + 6 = H + 3$ .....and so on.

Hence the right as well as left cosets are

$H, H + 1, H + 2$. ...Ans.

Ex.44. Let $G = \{1, -1, i, -i\}$ and $H = \{1, -1\}$. Find the right cosets of H in G.

Soln. Given : $G = \{1, -1, i, -i\}$ and $H = \{1, -1\}$

Here we observe that H is a subgroup of G under multiplication.

$H . 1 = \{1.1, -1.1\}$

$= \{1, -1\} = H$

$H . -1 = \{1.-1, -1.-1\}$

$= \{-1, 1\} = H$

$H . i = \{1.i, -1.i\}$

$= \{i, -i\}$

$H . -i = \{1.-i, -1.-i\}$

$= \{-i, i\}$

$\therefore$ $\{1, -1\}$ and $\{i, -i\}$ are the right cosets of H is G.

Since G is an abelian group, hence $\{1, -1\}$ and $\{i, -i\}$ are also the left cosets of H is G.

Important Results :

(1) If (H, *) is a subgroup of (G, *), then $a * H = H$, if and only if $a \in H$.

(2) If (H, *) is a subgroup of (G, *), then $a * H = b * H$, if and only if $a^{-1} * b \in H$.

(3) If (H, *) is a subgroup of (G, *), then any two left cosets of H are either identical or disjoint.

(4) There exists a one-one correspondence between the elements of subgroup H and those of any coset of H is G.

(5) If $a * H$ and $b * H$ are two distinct left cosets of H in G, then there exists a one-one correspondence between $a * H$ and $b * H$.

LAGRANGE'S THEOREM

Ex.45. The order of any subgroup of a finite group divides the order of the group.

IT : S-06,07,08,11, CS/CT/CE : S-08,11

Soln. Let (G, *) be a finite group of order h and (H, *) be a sub-group of G of order m.

We decompose the set G into a union of a finite number of distinct left cosets of H say K.

Let $a_1 H, a_2 H, \ldots a_k H$ denote the K distinct left cosets of H in G such that

$G = (a_1 * H) \cup (a_2 * H) \cup (a_3 * H) \cup .... \cup (a_k * H)$

where all the K left cosets appearing on the right are disjoint.

$\therefore\ O(G) = O(a_1 * H) + O(a_2 * H) + ... + O(a_k * H)$

i.e. n = m + m + ...K times'

i.e. n = Km

i.e. m/n

$\therefore$ $\boxed{O(H)/O(G).}$ **Ans.**

This proves the theorem.

Ex.46. The order of every element of a finite group is a divisor of the order of the group.

Soln. Let (G, *) be a finite group and $a \in G$.

Let O(a) = m

Consider the set $H = \{a, a^2, a^3, ..., a^m = e\}$

Obviously (H, +) is a subgroup of (G, *) and O(H) = m.

Hence by Lagrange's theorem, O(H)/O(G)

$\Rightarrow$ m/O(G)

$\Rightarrow$ O(a)/O(G)

This prove the theorem. .....**Ans.**

A counting principle : Let H and K be two subgroups of a group (G, *).

Define HK = {h*k/h∈H, k∈K}

i.e. $x \in HK \Leftrightarrow x = h * k$, for some $h \in H, k \in K$.

Ex.47. If (G, *) is a group and (H, *), (K, *) are any two subgroups of (G, *), then (HK, *) is a subgroup of (G, *) if and only if HK = KH.

Soln. If part :

Given : (HK, *) is a subgroup of (G, *)

To prove : HK = KH.

i.e. we prove (i) $HK \subseteq KH$

(ii) $KH \subseteq HK$

(i) Let $x \in HK$

$\Rightarrow x^{-1} \in HK$ *[∵ HK is subgroup of G]*

$\Rightarrow x^{-1} = h*k, h \in H, k \in K$

$\Rightarrow (x^{-1})^{-1} = (h*k)^{-1}$

$\Rightarrow x = k^{-1} * h^{-1} \in KH$ as $k^{-1} \in K, h^{-1} \in H$

$\therefore x \in HK \Rightarrow x \in KH$

$\therefore HK \subseteq KH$

(ii) Let $y \in KH$

$\Rightarrow y = k * h, k \in K, h \in H$

$\Rightarrow y^{-1} = (k*h)^{-1} = h^{-1} * k^{-1} \in HK$ as

$h^{-1} \in H, k^{-1} \in K$

$\Rightarrow (y^{-1})^{-1} \in HK$

$\Rightarrow y \in HK$

$\therefore y \in KH$

$\Rightarrow y \in HK$

$\therefore KH \subseteq HK$

$\therefore HK = KH$

Only if part :

Given : HK = KH

To prove : (HK, *) is a subgroup of (G, *)

i.e. we prove **(i)** HK is a non-empty subset of G

(ii) $x, y \in HK$

$\Rightarrow x * y \in HK$

(iii) $x \in HK$

$\Rightarrow x^{-1} \in HK.$

(i) Let e be the identity element of G.

$\therefore e \in H, e \in K$ as H and K are subgroups of G.

$\Rightarrow e = e * e \in HK$

$\Rightarrow e \in HK$

$\therefore$ HK is a non-empty subset of G.

(ii) Let $x, y \in HK$

$\Rightarrow x = h_1 * k_1, y = h_2 * k_2; h_1, h_2 \in H, k_1, k_2 \in K$

$\Rightarrow x * y = (h_1 * k_1) * (h_2 * k_2)$

$= h_1 * (k_1 * h_2) * k_2$ ...(a)

But $k_1 * h_2 \in KH$

Given that KH = HK

$\therefore k_1 * h_2 \in HK$

$\therefore k_1 * h_2 = h_3 * k_3, h_3 \in H, k_3 \in K$

$\therefore$ (a) $\Rightarrow x * y = h_1 * (h_3 * k_3) * k_2$

$= (h_1 * h_3) * (k_3 * k_2) \in HK$ as

$h_1 * h_3 \in H, k_1 * k_3 \in K$

$\therefore x, y \in HK \Rightarrow x * y \in HK$

(iii) Let $x \in HK$

$\Rightarrow x = h * k, h \in H, k \in K$

$\Rightarrow x^{-1} = (h * k)^{-1} = k^{-1} * h^{-1} \in KH$ as

$k^{-1} \in K, h^{-1} \in H$

But given that HK = KH

$\Rightarrow x^{-1} \in HK$

$\therefore x \in HK \Rightarrow x^{-1} \in HK$

$\therefore$ $\boxed{\text{(HK,*) is a subgroup of G if and only if HK = KH.}}$ **Ans.**

Index of a subgroup of a group :

Let G be a finite group and H is a subgroup of G. The number of a distinct right or left coset of H in G is called the index of H in G. It is denoted by [G : H] or $i_G(H)$.

Ex.48. If (H, *) and (K, *) are any two finite subgroups of a group (G, *) of order O(H) and O(K) respectively, then $O(H * K) = \dfrac{O(H) * O(K)}{O(H \cap K)}$

Soln. Let $D = H \cap K$

$\Rightarrow$ D is a subgroup of G as H and K are subgroups of G. Also D is a subgroup of K (and H)

$\therefore$ K can be decomposed into m number of distinct right cosets of D.

$$\therefore \quad K = \bigcup_{i=1}^{m} Dk_i \ldots\ldots -(1), k_1, k_2, \ldots k_m \in K$$

Hence index of D in K is $m = \dfrac{O(K)}{O(D)}$ ...(2)

as K is finite.

$$\text{Now } H * K = H * \bigcup_{i=1}^{m} Dk_i \qquad [\because \textit{By (1)}]$$

$$= \bigcup_{i=1}^{m} (H * Dk_i)$$

But $H * D = H$ as $D \subseteq H$

$$\therefore \quad H * K = \bigcup_{i=1}^{m} Hk_i \qquad \ldots(3)$$

Now we prove that $H * K_1, H * K_2, H * \ldots K_m$ are pairwise disjoint.

Suppose $H * k_i = H * k_j$

$\Rightarrow H * k_i * k_j^{-1} = H$

$\Rightarrow k_i * k_j^{-1} \in H \qquad [\because H * a = H \Leftrightarrow a \in H]$

But $k_i * k_j^{-1} \in K$

$\Rightarrow k_i * k_j^{-1} \in H \cap K$

But $H \cap K = D$

$\Rightarrow k_i * k_j^{-1} \in D$

$\Rightarrow D * (k_i * k_j^{-1}) = D \qquad [\because a \in H \Leftrightarrow H * a = H]$

$\Rightarrow D * k_i = D * k_j$

This is a contradiction to (1)

$\therefore$ $H * K_1, H * K_2 \ldots\ldots, H * K_m$ are distinct right cosets of H and each of them has O(H) number of elements.

$\therefore$ (3) $\Rightarrow O(H*K) = O(H) + O(H) + O(H) + \ldots$ m times $= m \cdot O(H)$

$$= \frac{O(K)}{O(D)} * O(H) \qquad [\because \textit{By (2)}]$$

$$\therefore \quad \boxed{O(H * K) = \frac{O(H) * O(K)}{O(H \cap K)}} \quad \textbf{Ans.}$$

Ex.49. If (H, *) and (K, *) are any two subgroups of a finite subgroup G and $O(H) > \sqrt{O(G)}$, $O(K) > \sqrt{O(G)}$.

Then $H \cap K \neq \{e\}$.

Soln. Since H and K are subgroups of G

$\therefore O(H * K) < O(G)$

i.e. $O(G) > O(H * K)$

But by previous theorem

$$O(H * K) = \frac{O(H) * O(K)}{O(H \cap K)}$$

$$\Rightarrow O(G) > \frac{O(H) * O(K)}{O(H \cap K)}$$

But given that $O(H) > \sqrt{O(G)}$ and $O(K) > \sqrt{O(G)}$

$$\Rightarrow O(G) > \frac{O(H) * O(K)}{O(H \cap K)} > \frac{\sqrt{O(G)} * \sqrt{O(G)}}{O(H \cap K)}$$

$$\Rightarrow O(G) > \frac{O(G)}{O(H \cap K)}$$

$$\Rightarrow O(H \cap K) > \frac{O(G)}{O(G)}$$

$\Rightarrow O(H \cap K) > 1$

$\Rightarrow O(H \cap K) \geq 2$

$\therefore O(H \cap K) \neq 1$

$\Rightarrow H \cap K \neq \{e\}$

Note : For the sake of convenience we hereafter represent any group or subgroup by simply the capital letters G, H etc instead of (G, *), (H, *) and use ab instead of a * b.

NORMAL SUBGROUP *IT : S-06, CS/CT/CE : W-07,12*

A subgroup N of a group G is said to be a normal subgroup iff, $gng^{-1} \in N, \forall n \in N, \forall g \in G$

Equivalently $gNg^{-1} \subseteq N, \forall g \in G$

where $gNg^{-1} = \{gng^{-1} | n \in N\}$

i.e. $x \in gNg^{-1} \Leftrightarrow x = gng^{-1}$

Ex.50. N is a normal subgroup of G iff $gNg^{-1} = N$, $\forall g \in G$. **IT : S-06, W-11, CS/CT/CE : W-10**

Soln. If part

Given : Let N be a normal subgroup of G

To prove : $gNg^{-1} = N, \forall g \in G$

Since N is a normal subgroup of G

$\therefore$ By definition $gng^{-1} \forall \in N, \ \forall g \in G, \ \forall n \in N$.

$\Rightarrow gNg^{-1} \subseteq N, \ \forall g \in G$ ...(i)

Also $g^{-1} N(g^{-1})^{-1} \subseteq N$ $[\because g^{-1} \in G]$

i.e. $g^{-1} Ng \subseteq N$

$\Rightarrow g(g^{-1} Ng) g^{-1} \subseteq gNg^{-1}$

$\Rightarrow (gg^{-1}) N(gg^{-1}) \subseteq gNg^{-1}$

$\Rightarrow eNe \subseteq gNg^{-1}$

$\Rightarrow N \subseteq gNg^{-1}$ ...(ii)

From (i) and (ii), we get

$gNg^{-1} = N$

Only if part :

Given : $gNg^{-1} = N$

To prove : N is a normal subgroup of G

Given : $gNg^{-1} = N, \forall g \in G$

$\Rightarrow gNg^{-1} \subseteq N, \forall g \in G$

$\Rightarrow gNg^{-1} \in N, \ \forall g \in G, \forall n \in N$

$\Rightarrow$ N is a normal subgroup of G. **...Ans.**

Ex.50.(a) A subgroup H of a group G is normal iff $g^{-1} hg \in H$ for every $h \in H, g \in G$. **IT : S-06, W-11** **CS/CT/CE : W-10**

Soln. Let H be a normal subgroup of G

Let $h \in H, g \in G$, then

$Hg = gH$ *($\because$ Definition of normal subgroup)*

Now $hg \in Hg = gH$

So $hg = gh_1$ for some $h_1 \in H$

i.e. $g^{-1} hg = h_1 \in H$

Conversely, let H be such that

$g^{-1} hg \in H \ \forall h \in H, g \in G$

Consider $a \in G$

for any $h \in H$, $a^{-1} ha \in H$

Therefore, $ha = a(a^{-1} ha) \in aH$

Consequently, $Ha \subseteq aH$

let $b = a^{-1}$, then $b^{-1} hb \in H$

But $b^{-1} hb = (a^{-1})^{-1} ha^{-1} = aha^{-1}$

This gives $aha^{-1} \in H$ so that

$ah = (aha^{-1}) a \in Ha$

which proves that $aH \subseteq Ha$

Hence $aH = Ha$

This theorem shows that, equivalently a subgroup H of group G can be defined to be a normal subgroup if $g^{-1} hg \in H \ \forall h \in H, g \in G$. **...Hence proved.**

Ex.51. Show that the intersection of any two normal subgroups of a group G is a normal subgroup of G. **CS/CT/CE : W-07,12**

Soln. Let N and M be any two normal subgroups of G

$\Rightarrow$ N and M are subgroups of G

$\Rightarrow N \cap M$ is also a subgroup of G

We now prove that $N \cap M$ is normal in G

i.e. $gxg^{-1} \in N \cap M, \forall x \in N \cap M$

Let,

$x \in N \cap M$

$\Rightarrow x \in N$ and $x \in M$

But N and M are normal subgroups of G

$\Rightarrow gxg^{-1} \in N$ and $gxg^{-1} \in M, \forall g \in G$

$\Rightarrow gxg^{-1} \in N \cap M, \forall g \in G, \forall x \in N \cap M$.

$\therefore N \cap M$ is normal in G

$\therefore$ $\boxed{N \cap M \text{ is a normal subgroup of } G}$ **Ans.**

Ex.52. Show that if H is a subgroup of a group G and N is a normal subgroup of G, then $H \cap N$ is a normal subgroup of G.

Soln. Since H is a subgroup of G and N is a normal subgroup of G.

$\therefore H \cap N$ is a subgroup of G.

Also $H \cap N \subseteq H$

$\Rightarrow H \cap N$ is a subgroup of H.

We now prove that $H \cap N$ is normal in H

i.e. $gxg^{-1} \in H \cap N, \forall x \in H \cap N, \forall g \in H$

Let $x \in H \cap N, g \in H$

$\Rightarrow x \in H$ and $x \in N, g \in H$

$\Rightarrow g x g^{-1} \in H$ and $gxg^{-1} \in N$

[∵ N is a normal subgroup of G]

$\Rightarrow gxg^{-1} \in H \cap N, \forall g \in H, \forall x \in H \cap N$

$\therefore$ $H \cap N$ is normal in H

$\therefore$ $\boxed{H \cap N \text{ is a normal subgroup of } H}$ **Ans.**

Ex.53. Show that every subgroup of an abelian group is normal. **IT: S-07, CS/CT/CE : W-11**

Soln. Let N be a subgroup of an abelian group G.

We have to prove that N is normal in G.

i.e $gng^{-1} \in N, \forall n \in N, \forall g \in G$

Let $n \in N, g \in G$

Then $gng^{-1} = (gn) g^{-1}$

$= (ng)g^{-1}$ *[∵ G is abelian]*

$= n (gg^{-1})$

$= ne$

$= n$

$\therefore gng^{-1} = n \in N$

$\therefore gng^{-1} \in N, \forall n \in N, \forall g \in G$

$\therefore$ $\boxed{N \text{ is a normal subgroup of } G.}$ **Ans.**

QUOTIENT GROUP OR FACTOR GROUP

Let N be a normal subgroup of a group G. Then G/N denotes the set of all right cosets or left cosets of N in G.

i.e. $G/N = \{Na \mid a \in G\}$ or $\{aN \mid a \in G\}$

i.e. $x \in G|N \Leftrightarrow x = Na$ OR $x = aN$, for some $a \in G$.

Here G/N is called quotient group or factor group.

Ex.54. Let G be a group and N be a normal subgroup of G. Then G/N is also a group.

Soln. Given : (i) N is a normal subgroup of G.

(ii) $G/N = \{Na / a \in G\}$

To Prove : G/N is a group

We verify G1, G2, G3, G4 in G/N

G1 : Let $x, y \in G/N$

$\therefore x = Na, y = Nb$, for some $a, b \in G$

$\Rightarrow x, y = (Na)(Nb) = N(ab) \in G/N$ as $ab \in G$

$\therefore x, y \in G/N \Rightarrow xy \in G/N$

$\therefore$ G1 is verified.

G2 : Let $x, y, z \in G/N$

$\therefore x = Na, y = Nb, z = Nc$, for some $a, b, c, \in G$.

Now $x(yz) = Na[(Nb)(Nc)]$

$= Na[N(bc)]$

$= Na(bc)$

$= N(ab)c$

[∵ G is a group, hence G2 holds in G]

$= [N(ab)] Nc$

$= [N(a) N(b)] Nc$

$= (xy)z$

$\therefore x(yz) = (xy)z, x, y, z \in G$

$\therefore$ G2 is verified.

G3 : Let e be the identity element of G.

Then $N = Ne \in G/N$

Let $x \in G/N$

$\Rightarrow x = Na$, for some $a \in G$

Then $Nx = (Ne)(Na)$

$= N(ea)$

$= Na$

$= x$

$\therefore Nx = x = xN, \forall x \in G/N$

$\therefore$ N or Ne is the identity element of G/N.

$\therefore$ G3 is verified.

G4 : Let $x \in G/N$

$\therefore x = Na$, for some $a \in G$

Since $a \in G$, $a^{-1} \in G$ *[∵ By G4 in G]*

Hence $y = Na^{-1} \in G/N$

Now $xy = (Na)(Na^{-1}) = N(aa^{-1})$

$= Ne = N =$ identity of G/N.

Also $yx = (Na^{-1})(Na)$

$= N(a^{-1}a) = Ne = N$

$\therefore xy = N = yx$

$\therefore$ Inverse of every $x \in G/N$ exists in G/N

$\therefore$ G4 is verified

$\therefore$ $\boxed{G/N \text{ is a group}}$ **Ans.**

ISOMORPHISM & HOMOMORPHISM OF GROUPS

Homomorphism of Groups :

Let G and G′ be any two groups. A function $f : G \to G'$ is said to be a group homomorphism if $f(ab) = f(a) . f(b), \forall a, b \in G$.

Isomorphism of Groups :

A homomorphism $f : G \to G'$ is said to be an isomorphism if f is one-one.

i.e. $f(a) = f(b)$

$\Rightarrow a = b$

Isomorphic Groups :

Two groups G and G' are said to be isomorphic if there exists a homomorphism $f : G \to G'$ which is one-one and onto and is written as $G \approx G'$.

$f : G \to G'$ is onto, then for every $y \in G'$, there exists some $x \in G$ such that $f(x) = y$.

KERNEL OF HOMOMORPHISM

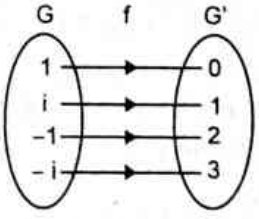
Let $f : G \to G'$ be a group homomorphism.

Then a set $\{x \in G / f(x) = e'\}$ where e' is the identity of G' is called kernel of homomorphism and is denoted by Kerf or Kf.

$\therefore$ Kerf = Kf

$= \{x \in G / f(x) = e'\}$

i.e. $x \in \text{Kerf} \Leftrightarrow f(x) = e'$



Here Kerf = {1, 2, 5}

Ex.55. Let G be a group of real numbers under addition and G' be a group of non-zero real numbers under multiplication. Define $f : G \to G'$ by $f(a) = 2^a$. Show f is a homomorphism and find Kerf. **IT : W-06**

Soln. Given : $f : G \to G'$ defined by $f(a) = 2^a$. ...(1)

Homomorphism : Let $a, b \in G$

$\Rightarrow a + b \in G$

$\therefore$ By (1), $f(a) = 2^a$

$f(b) = 2^b$ ....(2)

$f(a + b) = 2^{a+b}$

Now $f(a + b) = 2^{a+b} = 2^a . 2^b$

$= f(a) . f(b)$ *[$\because$ By (2)]*

$\therefore f(a + b) = f(a) . f(b), \forall a, b \in G$

$\therefore$ f is a homomorphism

To find Kerf : Since G' is a group under multiplication

$\therefore$ The identity element of $G' = e' = 1$

Let $x \in$ Kerf

$\Rightarrow f(x) = 1$ *[$\because$ By definition of kernel]*

$\Rightarrow 2^x = 1$

$\Rightarrow 2^x = 2^0$

$\Rightarrow x = 0$

$\therefore x \in$ Ker f

$\Rightarrow x = 0$

$\therefore$ $\boxed{\text{Ker } f = \{0\}}$ **Ans.**

Ex.56. In the previus example 55, Prove that G and G' are isomorphic. **IT : W-06**

Soln. We have already proved in previous example that $f : G \to G'$ is a homomorphism. In order to prove that G and G' are isomorphic, we prove that $f : G \to G'$ is one-one and onto.

one-one : Let $f(a) = f(b)$

$\Rightarrow 2^a = 2^b$ *[$\because$ By (2)]*

$\Rightarrow a = b$

$\therefore f(a) = f(b)$

$\Rightarrow a = b, \forall a, b \in G$

$\therefore f : G \to G'$ is one-one

Onto : Since $f : G \to G'$ defined by $f(a) = 2^a$ is one-one thus every element of G' (Set of non-zero reals) is the image of some element of G (set of reals).

$\therefore f : G \to G'$ is onto.

$\therefore$ $\boxed{\text{G and } G' \text{ are isomorphic}}$ **Ans.**

Ex.57. Show that the multiplicative group $G = \{1, i, -1, -i\}$ is isomorphic to the group $G' = \{0, 1, 2, 3\}$ with addition modulo 4 as composition. Write the composition tables for G and G'.

IT : S-07, W-08,10, CS/CT/CE : S-09,11,13

Soln. We define a mapping $f : G \to G'$

Composition tables for G and G' are given below.

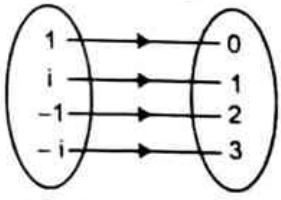
×	1	i	−1	−i
1	1	i	−1	−i
i	i	−1	−i	1
−1	−1	−i	1	i
−i	−i	1	i	−1

(i) G

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

(ii) G′

The above composition tables are illustrated in following figure.



i.e. $f(1) = 0$, $f(i) = 1$, $f(-1) = 2$, $f(-i) = 3$.

Clearly $f : G \to G'$ is one-one and onto.

We now prove that $f : G \to G'$ is homomorphism.

Let $a = i$, $b = -1$

$\therefore\ f(ab) = f(i \cdot -1) = f(-i)$

$= 3$

$= 1 + 2$

$= f(i) + f(-1)$

$= f(a) + f(b)$

$\therefore\ f(ab) = f(a) + f(b)$

$\therefore\ f : G \to G'$ is homomorphism

$\therefore$ $\boxed{\text{G and G}' \text{ are isomorphic}}$ **Ans.**

Ex.57.(a) Let T be the set of all even integers. Show that semigroups (Z, +) and (T, +) are isomorphic. ***IT : S-12***

Soln. We know that the set of positive integers (Z, +) is a group.

$\therefore$ The set of even integers (T, +) is also a group.

To prove : $f : Z \to T$ is an isomorphism.

We prove that f is a homomorphism, one-to-one and onto.

Homomorphism : Let $a, b \in Z$

Let us define f by $f(a) = a$

Then $f(b) = b$, $f(ab) = ab$

$\therefore\ f(ab) = ab = f(a) \cdot f(b)$

$\therefore$ f is a homomorphism.

One-to-one : Let $f(a) = f(b)$

$\Rightarrow a = b$

$\therefore$ f is one-to-one.

Clearly f is onto

$\therefore$ $\boxed{f : Z \to T \text{ is an isomorphism.}}$ **Ans.**

Ex.58. Let G be an abelian group and n be a fixed integer. Prove that the function $f : G \to G$ defined by $f(a) = a^n$, $a \in G$ is a homomorphism. ***IT : W-08, S-12***

Soln. Given : (i) G is an abelian group

(ii) $f : G \to G$ defined by $f(a) = a^n$, $a \in G, n \in Z$

To prove : $f : G \to G$ is a homomorphism

Since $f : G \to G$ is defined by $f(a) = a^n$

$\therefore\ f(b) = b^n, b \in G$

$\therefore\ f(ab) = (ab)^n$

$= a^n b^n$ *[∵ G is abelian]*

$= f(a) \cdot f(b)$

$\therefore\ f(ab) = f(a) \cdot f(b), \forall a, b \in G$

$\therefore$ $\boxed{f : G \to G \text{ is a homomorphism}}$ **Ans.**

Ex.59. Let $S = N \times N$. Let * be the operation on S defined by $(a, b) * (a', b') = (aa', bb')$

(i) Show that * is associative

(ii) Define $f : (S, *) \to (Q, *)$ by $f(a, b) = a/b$. Show that f is a homomorphism. ***CS/CT/CE : S-10***

Soln.

(i) Let $(a, b), (a', b'), (\bar{a}, \bar{b}) \in N \times N = S$

We prove that * is associative

Consider $[(a, b) * (a', b')] * (\bar{a}, \bar{b})$

$= (aa', bb') * (\bar{a}, \bar{b})$

$= (aa'\bar{a}, bb'\bar{b})$ ...(i)

Also $(a, b) * [(a', b') * (\bar{a}, \bar{b})]$

$= (a, b) * (a'\bar{a}, b'\bar{b}) = (aa'\bar{a}, bb'\bar{b})$ ....(ii)

From (i) and (ii), we get

$[(a, b) * (a', b')] * (\bar{a}, \bar{b}) = (a, b) * [(a', b') * (\bar{a}, \bar{b})]$

$\therefore$ $\boxed{* \text{ is associative}}$ **Ans.**

(ii) $f : (S, *) \to (Q, *)$ is defined by $f(a, b) = a/b$

We prove that f is a homomorphism.

Consider $f[(a, b) * (a', b')] = f(aa', bb')$

$= \dfrac{aa'}{bb'}$

$= \dfrac{a}{b} \cdot \dfrac{a'}{b'}$

$= f(a, b) \cdot f(a', b')$

$\therefore$ $f[(a, b) * (a', b')] = f(a, b) f(a', b')$

$\therefore$ $\boxed{\text{f is a homomorphism.}}$ **Ans.**

Ex.60. Let G be a group and N be a normal subgroup of G, Define $f : G \to G/N$ by $f(x) = Nx, \forall x \in G$. Then f is a homomorphism of G onto G/N.

Soln. Given : $f : G \to G/N$ defined by

$f(x) = N(x), \forall x \in G$ ...(1)

To Prove : f is a homomorphism of G onto G/N.

Homomorphism :

Let $x, y \in G \Rightarrow xy \in G$ *[$\because$ G1 in G]*

$$\text{(i)} \Rightarrow \left.\begin{aligned} f(x) &= Nx \\ \therefore \quad f(y) &= Ny \\ f(xy) &= N(xy) \end{aligned}\right| \quad ...(2)$$

Now $f(xy) = N(xy)$

$= (Nx)(Ny)$

$= f(x) . f(y)$ *[$\because$ By (2)]*

$\therefore$ $f(xy) = f(x) . f(y)$

$\therefore$ f is a homomorphism

Onto :

Let $y = Nx \in G/N$

Then by definition of G/N, $x \in G$ such that $f(x) = Nx$

$= y$

$\therefore$ f is onto

$\therefore$ $\boxed{\text{f is a homomorphism of G onto G/N}}$ **Ans.**

Ex.61. If f is a homomorphism of G into G′, then

(i) $f(e) = e'$ where e' is the identity element of G′

(ii) $f(x^{-1}) = [f(x)]^{-1}, \forall x \in G$.

(iii) If H is a subgroup of G, then

$f(H) = \{f(h) \mid h \in H\}$ is a subgroup of G.

IT : S-06,09, CS/CT/CE : W-10

Soln. Given : $f : G \to G'$ is a homomorphism

To prove : (i) $f(e) = e'$

(ii) $f(x^{-1}) = [f(x)]^{-1}$

(i) Let $x \in G$, then $f(x) \in G'$ because e' is the identity element of G′

$\therefore$ $f(x) . e' = f(x)$

$= f(xe)$, where e is the identity element of G

$= f(x). f(e)$ *[$\because$ f is a homomophism]*

$\therefore$ $f(x) . e' = f(x) . f(e), \forall f(x) \in G'$

$\Rightarrow$ $e' = f(e)$ *[$\because$ By LCL]*

$\therefore$ $\boxed{f(e) = e'}$ **Ans.**

(ii) By part (i), we have

$f(e) = e'$

$\Rightarrow$ $f(xx^{-1}) = e', \forall x \in G$

$\Rightarrow$ $f(x) . f(x^{-1}) = e'$ *[$\because$ f is a homomorphism]*

$\therefore$ $\boxed{f(x^{-1}) = [f(x)]^{-1}}$ **Ans.**

[$\because ab = e \Rightarrow b = a^{-1}$]

(iii) Since H is a subgroup of G

$\therefore$ G1, G4 holds in H

Let $a, b \in H$

$\Rightarrow a * b \in H$ and $a^{-1} \in H$

Since $f(H) = \{f(h) / h \in H\}$

$\therefore$ G1, G4 holds in f(H)

$\therefore$ f(H) is a subgroup of G .... **Ans.**

Ex.62. If f is a homomorphism of G into G′ with kernel K, then K is a normal subgroup of G. **IT : S-08,10**

Soln. Given : (i) $f : G \to G'$ is homomorphism

(ii) K is the kernel of f

To prove : (i) K is a subgroup of G

(ii) K is normal in G.

Since K is the kernel of f

$\therefore$ $x \in K \Leftrightarrow f(x) = e'$ ...(1)

(i) Let e be the identity element of G.

Then we have, $f(e) = e'$

[$\because$ By theorem definition of kernel]

$\Rightarrow$ $e \in K$ *[$\because$ By theorem definition of kernel]*

$\therefore$ K is a non-empty subset of G.

Let $x, y \in K$

$\Rightarrow$ $f(x) = e', f(y) = e'$ *[$\because$ By definition of kernel]*

$\Rightarrow$ $f(x) . f(y) = e' . e$

$\Rightarrow$ $f(xy) = e'$ *[$\because$ f is a homomorphism]*

$\Rightarrow$ $xy \in K$ *[$\because$ By definition of kernel]*

Also $f(x^{-1}) = [f(x)]^{-1}$

[$\because$ By theorem definition of kernel]

$= (e')^{-1}$

$\therefore$ $f(x^{-1}) = e'$

$\Rightarrow$ $x^{-1} \in K$ *[$\because$ By definition of kernel]*

$\therefore$ $x, y \in K$

$\Rightarrow$ $xy \in K$ and $x^{-1} \in K$

$\therefore$ K is a subgroup of G.

(ii) We now prove that K is normal in G.

i.e. $gxg^{-1} \in K, \forall x \in K, \forall g \in K$

Let $x \in K, g \in G$

$\therefore f(x) = e'$ *[$\because$ By definition of kernel]*

Now $f(gxg^{-1}) = f(g).f(x).f(g^{-1})$

[$\because$ f is a homomorphism]

$= f(g).e'.f(g^{-1})$

$= f(g).f(g^{-1})$

$= f(g).[f(g)]^{-1}$ *[$\because$ By theorem]*

$\therefore f(gxg^{-1}) = e'$

$\therefore gxg^{-1} \in K$ *[$\because$ By definition of kernel]*

$\forall x \in K, \forall g \in G$

$\therefore$ K is normal in G

$\therefore$ $\boxed{\text{K is a normal subgroup of G}}$ **Ans.**

(iii) Since H is a subgroup of G

$\therefore$ G1, G4 holds in H

Let $a, b \in H$

$\Rightarrow a * b \in H$ and

since $f(H)\ \{f(n) / h \in H\}$

$\therefore$ G1, G4 holds in f(H)

$\therefore$ f(H) is a subgroup of G .... **Ans.**

Ex.63. A homomorphism f from G into G′ with Kernel K is an isomorphism of G into G, if K = {e}.

IT : W-09

Soln. Given : $f : G \to G'$ is a homomorphism with kernel K.

If Part :

Given : f is an isomorphism i.e. f is one-one.

To Prove : K = {e}

Let $x \in K$

$\Rightarrow f(x) = e'$, where e′ is the identity element of G′.

But $f(e) = e'$ *[$\because$ By theorem]*

$\Rightarrow f(x) = f(e)$

$\Rightarrow x = e$ *[$\because$ f is one-one]*

$\therefore x \in K \Rightarrow x = e$

$\therefore K = \{e\}$

Only if part :

Given : K = {e}

To prove : f is an isomorphism.

We prove that f is one-one

Let $x, y \in G$ and suppose $f(x) = f(y)$

$\Rightarrow f(x).[f(y)]^{-1} = f(y).[f(y)]^{-1}$

$\Rightarrow f(x).f(y^{-1}) = e'$ *[$\because$ By theorem]*

$\Rightarrow f(xy^{-1}) = e'$ *[$\because$ f is a homomorphism]*

$\Rightarrow xy^{-1} \in K$

But K = {e}

$\Rightarrow xy^{-1} = e$

$\Rightarrow x = y$

$\therefore f(x) = f(y) \Rightarrow x = y$

$\therefore$ f is one-one

$\therefore$ f is an isomorphism. ...**Ans.**

Note : we conclude that $f : G \to G'$ is one-one iff K = {e}.

FUNDAMENTAL THEOREM OF GROUP HOMOMORPHISM

Ex.64. Let f be homomorphism of G onto G′ with kernel K, then $G|K \simeq G'$.

(i.e. $G|K$ is isomorphic to G′)

CS/CT/CE : S-06, W-09, IT : W-07, S-10

OR Every homomorphic image of a group G is isomorphic to some quotient group of G.

Soln. Given : f is a homomorphism with kernel K.

To prove : $G/K \approx G'$

Since $f : G \to G'$ is onto homomorphism with kernel K

$\therefore$ K is a normal subgroup of G.

Also if $a \in G$, then $Ka \in G/K$ and $f(a) \in G'$

We define a mapping $\phi : G/K \to G'$ such that $\phi(Ka) = f(a), \forall a \in G$. ...(1)

First we show the mapping ϕ is well defined, i.e. if a, $b \in G$ and Ka = kb, then

$\phi(Ka) = \phi(Kb)$

We have $Ka = Kb \Rightarrow ab^{-1} \in K$

$\Rightarrow f(ab^{-1}) = e'$ *[e′ is identity of G′]*

$\Rightarrow f(a).f(b^{-1}) = e'$

$\Rightarrow f(a).(f(b))^{-1} = e'$

$\Rightarrow f(a)(f(b))^{-1} f(b) = e' f(b)$

$\Rightarrow f(a).e' = f(b)$

$\Rightarrow f(a) = f(b)$

$\Rightarrow \phi(Ka) = \phi(Kb)$ *[∵ By (1)]*

$\Rightarrow \phi$ is well defined

Now we show ϕ is one-one

We have $\phi(Ka) = \phi(Kb)$

$\Rightarrow f(a) = f(b)$

$\Rightarrow f(a)(f(b))^{-1} = f(b)(f(b))^{-1}$

$\Rightarrow f(a)f(b^{-1}) = e'$

$\Rightarrow f(ab^{-1}) = e'$

$\Rightarrow ab^{-1} \in K$ *[∵ K is kernel]*

$\Rightarrow Ka = Kb$

$\therefore \phi$ is one-one

Now we show ϕ is onto G'.

Let $y \in G'$ then $y = f(a)$ for some $a \in G$ because f is onto G'.

Now $Ka \in G/K$ and we have

$\phi = (Ka) = f(a) = y$

ϕ is onto G'

Finally we have $\phi\{(Ka)(Kb)\} = \phi(Ka\,b)$

$= f(ab)$

$= f(a)f(b)$

$= \phi(Ka).\phi(Kb)$

$\therefore \phi$ is an isomorphism of G/K into G'.

Hence $G/K \cong G'$. **...Ans.**

Ex.65. Let G be the nonzero integers under the operation of multiplication and let $H = \{3n \mid n \in Z\}$. Is H a subgroup of G? ***IT : S-06***

Soln. It is a subgroup of G since

(i) H is non empty

(ii) Let $x, y \in H$. Then there exist $p, q \in G$ such that $x = 3p, y = 3q$.

Now $xy^{-1} = 3p - 3q = 3(p-q)$ where $p - q \in G$

Thus $xy^{-1} \in H$. Hence H is a subgroup of G. **...Ans.**

Ex.66. Show that the group $G = \{\{0, 1, 2, 3\}, \times_4\}$ is isomorphic to the group. $G' = \{\{1, 2, 3, 4\}, \times_5\}$. ***CS/CT/CE : S-07***

Soln. Let the $F: G \to G'$ by $F(0) = 1, F(1) = 2, F(2) = 3, F(3) = 4$

Suppose $F(a_1) = F(a_2)$, then $a_1 + 1 = a_2 + 1$

$\Rightarrow a_1 = a_2$

$\therefore$ F is a one-one function

Also suppose that $a_2 \in G'$, then

$F(a_1) = a_1 + 1 = a_2 + 1 = F(a_2)$

So, F is onto function

For $2 \in G$, $0 \in G$, we have

$F(0+2) = F(2) = 3$

and $F(0) \times F(2) = 1 \times 3 = 3$

$\therefore F(0) \times F(2) = F(0+2)$

$\therefore$ The function is homomorphism

$\therefore$ $\boxed{\text{The group G is isomorphic to the group } G'}$ **Ans.**

Ex.67. Suppose G is a group and N is a normal sub group of G. Let f be a mapping of G to G/N defined by f(x) = Nx. Then f is a homomorphism of G on to G/N and Kernel (f) = N. ***IT : S-08,10***

Soln. Let $G = [x, y, \ldots, a, b, \ldots]$

$G/N = [Nx, Ny, Na, Nb \ldots]$

To prove : f is on to.

It is clear from the fact that to each Nx of G/N $\exists\, x \in G$ such that $f(x) = Nx$. Thus each element of G/N has pre image. Hence f is onto.

Also $f(ab) = N\,ab$ *[by definition of f]*

$= N\,Nab$ *[∵ Hh = H, hH = H]*

$= Na\,Nb$ *[∵ N is normal, ∴ Na = aN]*

$= f(a)f(b)$

$\therefore$ f preserves the composition

$\therefore$ f is a homomorphism

Now to prove kernel (f) = N

The identity of group G/N is N

By definition, $K = \{x : x \in G / f(x) = N\}$

Let $k \in K$, then $f(k) = N$

$\Rightarrow Nk = N$

$\Rightarrow k \in N$ *[∵ Hx = H ⇒ x ∈ H]*

$\Rightarrow K \subseteq N$

Conversely, $n \in N$

$\Rightarrow Nn = N$

$\Rightarrow f(n) = N$

$\Rightarrow n \in K$

$\therefore N \subseteq K$

$\therefore$ $\boxed{\ker(f) = N}$ **Ans.**

Ex.68. Let $G = (Z_4, +_4)$ for each of the following subgroups H of G, determine all left cosets of H in G:

(i) $H = \{0\}$

(ii) $H = \{0, 2\}$

(iii) $H = \{0, 1, 2, 3\}$. **IT : W-09**

Soln. H is a subgroup of $< Z_4, +_4 >$ as

$+_4$	[0]	[1]	[2]	[3]
[0]	[0]	[1]	[2]	[3]
[1]	[1]	[2]	[3]	[0]
[2]	[2]	[3]	[0]	[1]
[3]	[3]	[0]	[1]	[2]

The left coset of H in G determined by the elements of G are

(i) $\{0\}$ and $\{1, 2, 3\}$ which is a partition of Z_4.

(ii) $\{0, 2\}$ and $\{1, 3\}$ which is a partition of Z_4.

(iii) $\{\ \}$ and $\{0, 1, 2, 3\}$ which is a partition of Z_4 ....**Ans.**

Ex.69. If a and b are any two elements of a group G and H is a subgroup of G then $H a = H b \Leftrightarrow a b^{-1} \in H$ and $a H = b H \Leftrightarrow a^{-1} b \in H$. **IT : S-11**

Soln. Given : $H a = H b$

$\Rightarrow a \in H b$

$\Rightarrow a b^{-1} \in H b b^{-1}$

$\Rightarrow a b^{-1} \in H$ *[$\because H h = H$ if $h \in H$]*

Conversely, if $a b^{-1} \in H$

$\Rightarrow H a b^{-1} = H$

$\Rightarrow H a b^{-1} b = H b$

$\Rightarrow H a e = H b$ *[$\because b b^{-1} = e$]*

$\Rightarrow Ha = Hb$

Similarly we can prove $a H = b H \Leftrightarrow a^{-1} b \in H$.

...**Ans.**

Ex.70. If for a group G, $f : G \to G$ is given by $f(x) = x^2$, $x \in G$ and f is a homomorphism then show that G is abelian group. **IT : S-11**

Soln. Let $f : G \to G$ is given by $f(x) = x^2$, $\forall x \in G$

To Prove : G is abelian group.

Let $a, b \in G$

then by closure property.

$a . b \in G$

Now, By definition,

$f(ab) = (ab)^2$

But f is homomorphism.

$\therefore f(a) . f(b) = ab . ab$

$\Rightarrow a^2 . b^2 = ab . ab$ *[$\because$ By definition]*

$\Rightarrow a.a. b.b = ab.ab$

$\Rightarrow ab = ba$

[$\because$ by left and right cancellation law]

$\therefore$ $\boxed{\text{G is an abelian group.}}$ **Ans.**

Ex.71. Let G be the group of real numbers under addition and let G′ be the group of positive real numbers under multiplication. Let $f : G \to G'$ defined by $f(x) = e^x$, then prove that f is an isomorphism. **CS/CE/CT : W-11**

Soln. We have $f : G \to G'$ defined by $f(x) = e^x$

If $f(x_1) = f(x_2)$ so that $e^{x_1} = e^{x_2}$,

then $\log e^{x_1} = \log e^{x_2}$

$\Rightarrow x_1 \log e = x_2 \log e$

$\Rightarrow x_1 = x_2$

Thus f is 1 – 1

If $x = \log y \in R^+$, then

$f(x) = e^x = e^{\log y} = y$

Thus each element of R^+ is the image of some elements of R (i.e. G)& hence f is onto

Again $f(x_1 + x_2) = e^{x_1 + x_2} = e^{x_1} . e^{x_2}$

$= f(x_1) . f(x_2)$

Hence f is an isomorphism ...**Ans.**

PROBLEMS FOR PRACTICE :

Q.1. Show that the following are groups.

(a) The set C of all complex numbers w.r.t. addition of complex number.

(b) The set of all rational number w.r.t addition.

Q.2. Show that the set $G = \{x + y\sqrt{2} / x, y \in Q\}$ is a group w.r.t. addition.

Q.3. Show by means of examples that union of two subgroups may or may not be subgroup.

Q.4. Show that {0, 1, 2, 3} forms a group with respect to addition modulus 4.

Q.5. Show that if $f : G \to G'$ is an isomorphism, then $f^{-1} : G' \to G$ is also an isomorphism.

UNIT - IV
RINGS, LATTICES AND BOOLEAN ALGEBRA

SYLLABUS

Rings, Fields, Integral Domain, Ring Homomorphism (definitions & examples), Lattices: Properties, Types of Lattices, Sub lattices, Isomorphic Lattices, Complemented & Modular Lattices (definitions & examples), Boolean, Algebra: Definition, Properties, Simplification of Switching Circuits.

UNIVERSITY PAPER SOLUTIONS
SINCE SUMMER - 2006

Note to the students :
- The questions given below are from previous Nagpur University examination papers.
- Though these questions are from old university papers, they have been included in the new syllabus.
- Questions which are out of new syllabus are not included here.

SUMMER - 06 (CS/CT/CE)

Q.1. If R is a ring such that $a^2 = a$, $\forall\ a \in R$, prove that :

(i) $a + a = 0\ \forall\ a \in R$,

(ii) $a + b = 0\ \forall\ a, b \in R$,

(iii) R is a commutative ring. **7M**

Ans. P.4-11, Ex.5.

Q.2. Show that $S = \{a + b\sqrt{2}, a, b \in Z\}$ for the operation +, × is an integral domain but not a field. **7M**

Ans. P.4-15, Ex.11.

Q.3. Show that there are 5 partitions of a set of three elements. Draw the diagram of the corresponding lattice. **6M**

Ans. P.4-29, Ex.41.

Q.4. Construct switching circuit for the Boolean polynomial $A \cdot B + A \cdot B' + A' \cdot B'$. Simplify this and construct an equivalent circuit. Verify equivalence by truth table. **7M**

Ans. P.4-34, Ex.51.

SUMMER - 06 (IT)

Q.1. Show that $< 2 \times 2$ matrices, + . * > is a ring with zero divisors; where * denotes matrix multiplication and + denotes addition of matrices. **7M**

Ans. P.4-12, Ex.7.

Q.2. How does a field differ from a ring? **2M**

Ans. P.4-13, Definitions.

Q.3. Let $<R, +, \cdot>$ be a ring. Then show that $x \cdot 0 = 0 \cdot x = 0$ for all $x \in R$. **4M**

Ans. P.4-11, Ex.6.

Q.4. Define Polynomial ring. **3M**

Ans. P.4-18, Definitions.

WINTER - 06 (CS/CT/CE)

Q.1. Show that $< Z_7, +_7, \times_7 >$ is a commutative ring with identity. **6M**

Ans. P.4-9, Ex.4.

Q.2. Show that in a Boolean Algebra for any a, b and c, if $a \le b$, then :

i) $a \vee (b \wedge c) = b \wedge (a \vee c)$

ii) $\{(a \vee c) \wedge (b' \vee c)\}' = (a' \vee b) \wedge c'$ **3M**

Ans. P.4-36, Ex.55.

Q.3. If R is a ring, then for all $a, b, c \in R$

i) $a.0 = 0.a = 0$

ii) $a(-b) = -(ab) = (-a)b$

iii) $(-a)(-b) = ab$

iv) $a(b - c) = ab - ac$

v) $(b - c)a = ba - ca$ **7M**

Ans. P.4-11, Ex.6.

Q.4. Define distributive lattice. **1M**

If $(L_1, \le_1)$ and $(L_2, \le_2)$ are lattices, then show that $(L_1 \times L_2, \le)$ is also a lattice. **5M**

Ans. P.4-30, Ex.45(b).

WINTER - 06 (IT)

Q.1. Show that $(Z_6, +_6, \times_6)$ is a ring but not an integral domain. **7M**

Ans. **P.4-10, Ex.4(a).**

Q.2. Let (R, +, .) be a ring in which 0 is the additive identity and –a denotes the additive inverse of $a \in R$. Prove that :

i) $a \cdot 0 = 0, \forall a \in R$

ii) $a(b - c) = ab - ac$. **6M**

Ans. **P.4-11, Ex.6.**

Q.3. If R is a ring such that $a^2 = a, \forall a \in R$, then prove that:

i) $a + a = 0, \forall a \in R$

ii) R is a commutative ring

iii) $a + b = 0 \Rightarrow a = b$. **6M**

Ans. **P.4-11, Ex.5.**

SUMMER - 07 (CS/CT/CE)

Q.1. For the set $I_4 = \{0, 1, 2, 3\}$, show that the modulo 4 system is a ring. **6M**

Ans. **P.4-11, Ex.4(b).**

Q.2. Show that the set of number $\{a + b\sqrt{3}\ ;\ a, b \in R\}$ is an integral domain with respect to addition and multiplication. **7M**

Ans. **P.4-15, Ex.11.**

Q.3. Construct a switching circuit for Boolean polynomial $A \cdot B + C + A'C'$. Simplify this and draw equivalence switching circuit. Verify result by truth table. **7M**

Ans. **P.4-34, Ex.52.**

Q.4. Is the poset A = {2, 3, 6, 12, 36, 72} under the relation of divisibility lattice? **6M**

Ans. **P.4-29, Ex.42.**

SUMMER - 07 (IT)

Q.1. Define zero divisor of a ring. Suppose M is a set of all 2×2 matrices with their elements as integers with the addition and multiplication of matrices being the two binary operations defined on the set M. Prove that M is a ring with zero divisors. **7M**

Ans. **P.4-12, Ex.7 & Definition.**

Q.2. Prove that a ring R is without zero divisors if and only if the cancellation laws hold in R. **6M**

Ans. **P.4-13, Ex.8.**

Q.3. Define an integral domain. Prove that the set of all real numbers of the form $a + b\sqrt{2}$, with a and b as integers is an integral domain with respect to ordinary addition and multiplication. **7M**

Ans. **P.4-15, Ex.11 & Definition on P.4-13.**

WINTER - 07 (CS/CT/CE)

Q.1. Test whether (M, +, .) is a ring where M is matrices with real entries and '+', '•' denote operation of matrix addition and multiplication. **7M**

Ans. **P.4-12, Ex.7.**

Q.2. If R is a ring such that $a^2 = a, \forall a \in R$

Prove that

(i) $a + a = 0 \forall a \in R$

(ii) $a + b = 0 \Rightarrow a = b$

(iii) R is a commutative ring. **6M**

Ans. **P.4-11, Ex.5.**

Q.3. Construct switching circuit for the Boolean polynomial $A.B + C + A'.C'$. Simplify this and draw equivalent switching circuit. **7M**

Ans. **P.4-34, Ex.52.**

Q.4. Show that the lattice < P (S), ⊆ > is isomorphic to $< L^3, \leq >$ where S = {a, b, c}, L = {0, 1}. **6M**

Ans. **P.4-30, Ex.43.**

WINTER - 07 (IT)

Q.1. Define :

(i) Ring

(ii) Integral Domain

(iii) Field. **3M**

Ans. **P.4-7 & 13, Definitions.**

Q.2. Show that the set of numbers of the form $a + b\sqrt{2}$ with a and b as rational numbers is a field. **7M**

Ans. **P.4-13, Ex.10(a).**

Q.3. Define Polynomial Ring. **3M**

Ans. **P.4-18, Definitions.**

Q.4. Define ring homomorphism

Let $R_1 = \left\{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in R\right\}$ where R is a ring

$f : R_1 \to R$ by $f\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = a$ for all $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in R_1$.

Show that f is isomorphism. **7M**

Ans. **P.4-20, Ex.19.**

SUMMER - 08 (CS/CT/CE)

Q.1. Prove that

$S = \{a + b\sqrt{2},\ a, b \in Z\}$

for the operation ' + ' and ' · ' is an integral domain but not a field where Z is set of integers. **7M**

Ans. P.4-15, Ex.11.

Q.2. Define a Lattice. Draw the Hasse diagram of Lattice D_{30}. Is it an complemented Lattice? Write complement of each element of D_{30}. **7M**

Ans. P.4-25, Ex.29 & Definition on P.4-24.

Q.3. Construct a circuit for Boolean expression : **7M**

$(p+q)\cdot(p+t)+t\cdot(p+q\cdot t)$

Simplify and construct its equivalent circuit.

Ans. P.4-35, Ex.53.

SUMMER - 08 (IT)

Q.1. Show that the set of complex numbers a + ib where 'a' and 'b' are integers forms a ring under ordinary addition and multiplication of complex number. Is it an integral domain or is it a field? **7M**

Ans. P.4-16, Ex.12.

Q.2. Show that $<Z_7, +_7, \times_7>$ is a commutative ring with identity. **7M**

Ans. P.4-9, Ex.4.

WINTER - 08 (CS/CT/CE)

Q.1. Define :

(i) Ring with unit element.

(ii) Zero divisor.

(iii) Integral domain.

(iv) Division ring. **4M**

Ans. P.4-8,12 & 13, Definitions.

Q.2. Show that the set {0, 1, 2, 3, 4, 5, 6} is commutative ring w.r.to. addition modulo 7 and multiplication modulo 7 as the two composition. **7M**

Ans. P.4-9, Ex.4.

Q.3. Show that in a Boolean Algebra for any a, b and c if $a \le b$ then,

$\{(a \vee c) \wedge (b' \vee c)\}' = (a' \vee b) \wedge c'$. **3M**

Ans. P.4-36, Ex.55.

Q.4. Show that $S = \{a + b\sqrt{2} \mid a, b \in Z\}$ for the operation +, × is an integral domain but not a field. **7M**

Ans. P.4-15, Ex.11.

Q.5. Construct switching circuit for the Boolean polynomial :

$(A\cdot B)+(A\cdot B')+(A'\cdot B')$

Simplify this and construct equivalent simplified circuit. **6M**

Ans. P.4-34, Ex.51.

WINTER - 08 (IT)

Q.1. Show that the set of residue class of mod 7 i.e. $(Z_7, +_7, \times_7)$ is a commutative ring with identity. **7M**

Ans. P.4-9, Ex.4.

Q.2. Prove the following :

i) A non-empty subset S of a ring (R, +, •) is a ring if and only if $a - b \in S$, $ab \in S$ for all a, $b \in S$. **3M**

Ans. P.4-18, Ex.14.

ii) Intersection of two subrings is also a subring. **3M**

Ans. P.4-19, Ex.15.

Q.3. Let S be a set and P(S) is its power set. On P(S) we define operations + and • as follows :

$A + B = \{x \in S / x \in A \text{ or } x \in B \text{ and } x \notin A \cap B\}$

$A \cdot B = A \cup B \ \forall\ A, B \in P(S)$.

Show that (P(S), +, •) is a ring called the ring of subsets of S. **7M**

Ans. P.4-20, Ex.20.

SUMMER - 09 (CS/CT/CE)

Q.1. Prove that the set of all real numbers of the form $a + b\sqrt{2}$, with a, b as integers is an integral domain with respect to addition and multiplication. Is it a field? **7M**

Ans. P.4-15, Ex.11.

Q.2. Draw the following : **6M**

(i) The lattice (S_n, D) for n = 45 and write the complement of every element.

Ans. P.4-26, Ex.30.

(ii) The lattice $(L^3, \le)$ where L = {0, 1}.

Ans. P.4-26, Ex.32.

Q.3. Construct switching circuit for $A.B + C + A'.C'$.

Simplify this and draw equivalent circuit. Verify result by truth table. **6M**

Ans. P.4-34, Ex.52.

SUMMER - 09 (IT)

Q.1. Define commutative ring and ring with unity. If R is a ring such that $a^2 = a$ for all $a \in R$, prove the following :

(i) $a + a = 0$ for all $a \in R$

(ii) $a + b = 0 \Rightarrow a = b$

(iii) R is a commutative ring. **6M**

Ans. P.4-11, Ex.5 & Definition on P.4-8.

Q.2. Show that $S = \{a + b\sqrt{3} / a, b \in Z\}$ for the operations $+, \times$ is an integral domain but not a field. **7M**

Ans. P.4-15, Ex.11.

WINTER - 09 (CS/CT/CE)

Q.1. If R is a ring such that $a^2 = a, \forall a \in R$, prove that :

(i) $a + a = 0, \forall a \in R$

(ii) $a + b = 0 \Rightarrow a = b$

(iii) R is a commutative ring. **6M**

Ans. P.4-11, Ex.5.

Q.2. Show that the set S of all matrices of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, where a, b $\in$ R is a field with respect to matrix addition and matrix multiplication. **7M**

Ans. P.4-17, Ex.13.

Q.3. If S = {a, b, c} and L = {0, 1} then draw Hasse diagram of the Lattices < e (s), ≤ > and < L^3, ≤ > and show that they are isomorphic, **7M**

Ans. P.4-30, Ex.43.

Q.4. Construct the switching circuit for $abc + ab'c + abc'$. Simplify and draw equivalent simplified circuit. Prepare the truth table and state when circuit is on. **7M**

Ans. P.4-36, Ex.54.

WINTER - 09 (IT)

Q.1. Show that the set of number given by $a + b\sqrt{2}$ where a and b are integers is a ring with addition and multiplication as compositions. **7M**

Ans. P.4-13, Ex.10(a).

Q.2. Prove that if a, b $\in$ R where (R, +, .) is a ring, then $(a + b)^2 = a^2 + a \cdot b + b \cdot a + b^2$ where $a^2 = a \cdot a$. **6M**

Ans. P.4-8, Ex.2.

Q.3. Let (A, +, .) be any ring with zero then for every a, b, $\in$ A, show that:

(i) $0 \cdot a = a \cdot 0 = 0$

(ii) $a \cdot (-b) = (-a) \cdot b = -(a \cdot b)$

(iii) $(-a) \cdot (-b) = a \cdot b$

(iv) $a \cdot (b - c) = a \cdot b - a \cdot c$. **7M**

Ans. P.4-11, Ex.6.

Q.4. Prove that the set R = {0, 2, 4, 6, 8) is a commutative ring under addition and multiplication modulo 10. Verify that it is an integral domain. **6M**

Ans. P.4-15, Ex.10(b).

SUMMER - 10 (CS/CT/CE)

Q.1. Show that the set S of all matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, where a, b $\in$ R is a field with respect to matrix addition and matrix multiplication. **7M**

Ans. P.4-17, Ex.13.

Q.2. Simplify the following expression using Boolean algebra:

(i) $(AB'C' + AB'C + ABC + ABC')(A + B)$

(ii) $P + P'QR' + (Q + R)'$. **6M**

Ans. P.4-36, Ex.56.

Q.3. Define distributive lattice. If $(L_1, \leq_1)$ and $(L_2, \leq_2)$ are lattices, then show that $(L_1 \times L_2, \leq)$ is also lattice **6M**

Ans. P.4-30, Ex.45(b).

SUMMER - 10 (IT)

Q.1. Define zero divisor of a ring. Suppose M is a set of all 2 × 2 matrices with their elements as integers w. r. to addition and multiplication of matrices being the two binary operations defined on the set M. Prove that M is a ring with zero divisors. **7M**

Ans. P.4-12, Ex.7 & Definition.

Q.2. If (L, $\wedge$, $\vee$) is a complemented distributive lattice, then prove that

(i) $(a \vee b)' = a' \wedge b'$

(ii) $(a \wedge b)' = a' \vee b'$. **6M**

Ans. P.4-30, Ex.44.

Q.3. If R is a ring such that $a^2 = a, \forall a \in R$, prove that

(i) $a + a = 0, \forall a \in R$

(ii) $a + b = 0 \Rightarrow a = b$

(iii) R is a commutative ring. **7M**

Ans. P.4-11, Ex.5.

WINTER - 10 (CS/CT/CE)

Q.1. Let R be a ring and a, b, c $\in$ R, then prove that :

(i) $a \cdot 0 = 0 \cdot a = 0$

(ii) $a \cdot (-b) = (-a) \cdot b = -(a \cdot b)$

(iii) $(-a) \cdot (-b) = a \cdot b$

(iv) $a \cdot (b - c) = a \cdot b - a \cdot c$. **6M**

Ans. P.4-11, Ex.6.

Q.2. Show that $S = \{a + b\sqrt{2} / a, b \in Z\}$ for the operation $+, \times$ is an integral domain but not a field. **7M**

Ans. P.4-15, Ex.11.

Q.3. In any lattice $(L, \wedge, \vee)$ the following statements are equivalent : **6M**

(i) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge v) \forall a, b, c \in L$

(ii) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \forall a, b, c, \in L$.

Ans. P.4-30, Ex.45(a).

Q.4. Define distributive lattice.

If $(L_1, \leq_1)$ and $(L_2, \leq_2)$ are lattices, then show that $(L_1 \times L_2, \leq)$ is also Lattice. **6M**

Ans. P.4-30, Ex.45(b) & Definition on P.4-25.

Q.5. Define partial order relation and POSET. **2M**

Ans. P.4-23.

WINTER - 10 (IT)

Q.1. Define a Ring. Show that set of integers with operations '+' and '×" forms a ring. **7M**

Ans. P.4-8 & 7, Ex.1 & Definition.

Q.2. Show that $S = \{a + b\sqrt{2}, a, b \in z\}$ for the operation $+, \cdot$ is an integral domain but not a field. **7M**

Ans. P.4-15, Ex.11.

SUMMER - 11 (CS/CT/CE)

Q.1. If R is a Boolean Ring, prove that : **6M**

(i) $a + a = 0, \forall a \in R$

(ii) $a + b = 0 \Rightarrow a = b$

(iii) $ab = ba, \forall a, b \in R$.

Ans. P.4-11, Ex.5.

Q.2. Prove that the set R = {0, 1, 2, 3, 4, 5, 6} is a field w.r.to addition and multiplication modulo 7. **7M**

Ans. P.4-9, Ex.4.

Q.3. If $(L, \wedge, \vee)$ is a complemented distributive lattice, then prove that $(a \wedge b)' = a' \vee b'$ and $(a \vee b)' = a' \wedge b' \forall a, b \in L$. **6M**

Ans. P.4-30, Ex.44.

Q.4. Construct switching circuit for the Boolean polynomial $(A + B) . (B' + C) + (C' + A) . (C + B)$. Simplify this and construct an equivalent simplified circuit. **7M**

Ans. P.4-37, Ex.57.

SUMMER - 11 (IT)

Q.1. Show that every finite integral domain is a field. **6M**

Ans. P.4-21, Ex.22.

Q.2. Show that intersection of two subrings of a ring R is a subring of R. Is union of two subrings of a ring R a subring of R? **6M**

Ans. P.4-19, Ex.15.

Q.3. Consider the ring

$< Z_4, +_4, \times_4 >$

Let $f(x) = 2x^2 + 4x + 4$ be a polynomial defined over z_4 find the zeros of f(x) over Z_4. Also find additive inverse of f(x). **7M**

Ans. P.4-21, Ex.21.

WINTER - 11 (CS/CT/CE)

Q.1. Prove that the set S = {0, 1, 2, 3, 4, 5} is a ring with respect to the operation of addition and multiplication modulo 6. **6M**

Ans. P.4-10, Ex.4(a).

Q.2. Let R be set of ordered pairs (a, b) of real numbers. Let addition and multiplication composition be defined as :

$(a, b) + (c, d) = (a + c, b + d)$

$(a, b)(c, d) = (ac - bd, bc + ad)$.

Show that R is a field. **7M**

Ans. P.4-22, Ex.23.

Q.3. Draw the following : **7M**

(i) The lattice (S_n, D) for n = 45 and write the complement of every element.

Ans. P.4-26, Ex.30.

(ii) The Lattice $[L^3, \leq]$ where L = {0, 1}.

Ans. P.4-26, Ex.32.

Q.4. Construct a circuit for Boolean expression :

$(p + q) \cdot (p + t) + t \cdot (p + q \cdot t)$.

Simplify and construct its equivalent circuit. **7M**

Ans. P.4-35, Ex.53.

WINTER - 11 (IT)

Q.1. Prove that if a, b ∈ R where (R, +, .) is a ring, then $(a + b)^2 = a^2 + a . b + b . a + b^2$ where $a^2 = a . a$ **7M**

Ans. P.4-8, Ex.2.

Q.2. Let (A, +, .) be any ring with zero then for every a, b ∈ A, show that **7M**

(i) $0 \cdot a = a \cdot 0 = 0$

(ii) $a \cdot (-b) = (-a) \cdot b = -(a \cdot b)$

(iii) $(-a)\cdot(-b) = a\cdot b$

(iv) $a\cdot(b-c) = a\cdot b - a\cdot c$

Ans. **P.4-11, Ex.6.**

Q.3. Prove that set $R = \{0, 2, 4, 6, 8\}$ is a commutative ring under addition and multiplication modulo 10. Verify that it is an integral domain. **7M**

Ans. **P.4-15, Ex.10(b).**

Q.4. Define ring homomorphism and Kernel of homomorphism. If f is a homomorphism mapping of ring R to ring R' then prove that $f(0) = (0)'$ and $f(-a) = -f(a)$. **6M**

Ans. **P.4-19, Ex.17.**

SUMMER - 12 (CS/CT/CE)

Q.1. Show that the set $\{0, 1, 2, 3, 4, 5, 6\}$ is commutative ring with respect to addition modulo 7 and multiplication modulo 7 as the composition. **8M**

Ans. **P.4-9, Ex.4.**

Q.2. Draw the following : **6M**

(i) The Lattice (S_n, D) for $n = 45$ and write the complement of every element.

Ans. **P.4-26, Ex.30.**

(ii) The Lattice $(L^3, \leq)$ where $L = \{0, 1\}$.

Ans. **P.4-26, Ex.32.**

Q.3. Construct the logic diagram for the Boolean expression :

$A.B + A'[(A. + B + B')]$. Simplify this to obtain the equivalent circuit. Also draw the equivalent switching circuit. **7M**

Ans. **P.4-33, Ex.50.**

SUMMER - 12 (IT)

Q.1. Prove that a non-empty subset S of a ring (R, +, .) is a ring if and only if $a - b \in S$ and $a . b \in S$, $\forall$ a, b $\in$ S. **5M**

Ans. **P.4-18, Ex.14.**

Q.2. Prove that the intersection of two subrings is also a subring. **4M**

Ans. **P.4-19, Ex.15.**

Q.3. Prove that a ring R is without zero divisors if and only if the cancellation laws hold in R. **4M**

Ans. **P.4-13, Ex.8.**

Q.4. Show that $S = \{a + b\sqrt{2} \mid a, b \in Z)$ with operations '+' and '.' is an integral domain but not a field. **7M**

Ans. **P.4-15, Ex.11.**

WINTER - 12 (CS/CT/CE)

Q.1. Show that $(Z_7, +_7, \times_7)$ is a commutative ring with identify. **6M**

Ans. **P.4-9, Ex.4.**

Q.2. Define :-

i) Zero Divisors.

(ii) Integral Domain

(iii) Field. **3M**

Ans. **P.4-12 & 13, Definition.**

Q.3. Prove that, every field is an integral domain. **4M**

Ans. **P.4-13, Ex.9.**

Q.4. Show that in a Boolean algebra for any a, b and c if $a \leq b$, then

(i) $a \vee (b \wedge c) = b \wedge (a \vee c)$

(ii) $\{(a \vee c) \wedge (b' \vee c)' = (a' \vee b) \wedge c'$ **4M**

Ans. **P.4-36, Ex.55.**

Q.5. Define distributive lattice

If $(L_1, \leq_2)$ and $(L_2, \leq_2)$ are the lattices, then show that $(L_1 \times L_2, \leq)$ is also a lattice. **5M**

Ans. **P.4-25 & 30, Ex.45(b).**

Q.6. Construct a switching circuit for Boolean polynomial $A\cdot B + C + A'C'$. Simplify this and draw equivalence switching circuits. **5M**

Ans. **P.4-34, Ex.52.**

WINTER - 12 (IT)

Q.1. Let R^+ be the group of non-zero real number under multiplication, and r be the (+) ve integer. Show that $f(x) = x^r$ is a homomorphism from R^+ to R^+. **6M**

Ans. **P.4-23, Ex.24.**

Q.2. Show that the set of integers with the composition o, * by $a \circ b = a + b + 1$ and $a * b = ab + a + b$ is a ring. **7M**

Ans. **P.4-23, Ex.25.**

SUMMER - 13 (CS/CT/CE)

Q.1. Show that the set $\{0, 1, 2, 3, 4, 5, 6\}$ is a commutative ring w.r. to addition modulo 7 and multiplication modulo 7 as the composition. **7M**

Ans. **P.4-9, Ex.4.**

Q.2. Construct a circuit for the Boolean expression $(A.B) + [A'.(A + B + B')]$ simplify and construct the equivalent circuit. **6M**

Ans. **P.4-33, Ex.50.**

Q.3. If R is a ring such that $a^2 = a, \forall a$, then show that

(i) $a + a = 0, \forall \ a \in R$

(ii) $a + b = 0 \Rightarrow a = b, \forall a, b \in R$

(iii) R is commulative ring **7M**

Ans. P.4-11, Ex.5.

Q.4. Draw the lattice.

$(L, \leq 1), (L^2 \leq 2)$, if $L = \{0, 1\}$ **6M**

Ans. P.4-26, Ex.32.

SUMMER - 13 (IT)

Q.1. Show that $(Z_6, +_6, \times_6)$ is a ring but not an integral domain. **7M**

Ans. P.4-10, Ex.4(a).

Q.2. Let (R, +, .) be a ring in which 0 is the additive identity and – a denotes the additive inverse of $a \in R$. Prove that

(i) $a \,.\, 0 = 0, \forall a \in R$

(ii) $a(b - c) = ab \,.\, ac$ **6M**

Ans. P.4-11, Ex.6.

Q.3. Prove that the intersection of two subring is also a subring. **6M**

Ans. P.4-19, Ex.15.

SOLVED QUESTION BANK

[Sequence given as per syllabus]

INTRODUCTION

In Group Theory, we have studied the binary operations of addition and multiplication among many objects viz. Complex numbers, Square matrices with entries as rationals, real functions defined on a set, integers with modular arithmetic etc. Ring Theory deals with such objects. Groups are sets of objects with one binary operation, while rings are set of objects with two binary operations addition and multiplication, interrelated to each other by distributive laws. The theorems obtained as a result of abstract study of rings apply to the diverse objects mentioned above which can then be used to solve problems arising in number theory, geometry and many other fields. Coding theory is one area which has developed techniques for introducing information in transmitted data which help in detecting and correcting errors. Some of these techniques make use of abstract algebra. A wide variety of algebraic concepts are used to describe codes and their properties.

Rings are distinguished in two different theories namely commutative and non-commutative rings. Non-commutative rings have been an object of systematic study only quite recently, during the 20th century. Commutative rings on the contrary have appeared in a hidden way much before in Fermat's Last Theorem, but Liouville noticed that the proof was depended on a unique decomposition into primes, which he thought was unlikely to be true. Kummer was the one who finally published an example to show that the uniqueness of prime decomposition failed. Two years later he restored the uniqueness by introducing what he called "ideal complex numbers" (referred as ideals now-a-days) and used it to prove Fermat's Last Theorem for all $n<100$ except $n = 37, 59, 67$ and 74.

RINGS, FIELDS, INTEGRAL DOMAIN

Ring : ***IT : W-07,10***

Let R be a non-empty set. An algebraic system or structure (R, +, •) is called an associative ring if the following axioms (properties) are satisfied.

R_1 : (R, +) is an abelian group.

G_1 : $a, b \in R \Rightarrow a + b \in R$

G_2 : $a, b, c \in R \Rightarrow a + (b + c) = (a + b) + c$

G_3 : There exists an element 0 in R, such that

$a + 0 = a = 0 + a, \forall a \in R$, 0 is called zero element of R.

G_4 : For every $a \in R$, there exists $-a \in R$, such that

$a + (-a) = 0 = (-a) + a$, $-a$ is called negative (or additive inverse) of a.

G_5 : $a, b, \in R \Rightarrow a + b = b + a$ (commutative property)

R_2 : The operation $*$ is binary and associative.

G_1 : $a, b \in R \Rightarrow a * b \in R$

G_2 : $a, b, c \in R \Rightarrow a * (b * c) = (a * b) * c$

R_3 : The operation $*$ is distributive over +

$$\text{i.e. } a, b, c \in R \Rightarrow \begin{cases} a*(b+c) = a*b + a*c \\ \qquad \text{.....(Left distributive – LD)} \\ (b+c)*a = b*a + c*a \\ \qquad \text{.....(Right distributive – RD)} \end{cases}$$

Note : For the sake of convenience, hereafter we call an associative ring simply by a ring and denote (R, +, *) by simply R.

Ring With Unity : ***CS/CT/CE : W-08, IT : S-09***

Let R be a ring. If there exists and element $1 \in R$ such that $a * 1 = a = 1 * a, \forall a \in R$, then R is called a ring with unity. Here 1 is the multiplicative identity, it need not be integer 1.

Commutative Ring : ***IT : S-09***

A ring R is said to be commutative, if $a * b = b * a$, $\forall a, b \in R$. Otherwise it is called non-commutiative ring.

Note : Hereafter by a . b or ab, we mean $a * b$.

Examples on Ring :

(i) Consider $Z = \{0, \pm 1, \pm 2, \pm 3, ...\}$.

Here Z is a commutative ring with unit. Here unit element is 1 and zero element is 0.

(ii) The set of real numbers, rational numbers, complex numbers are all commutative rings with unity under usual addition and multiplication.

(iii) Consider $A = \{0, \pm 2, \pm 4, \pm 6, ...\}$ i.e. the set of all even integers. Here A is a commutative ring without unity $(1 \notin R)$.

(iv) Consider $B = \{\pm 1, \pm 3, \pm 5, ...\}$ i.e. the set of all odd integers. Here $(0 \notin B)$, hence G3 does not hold. Hence B is not a ring.

(v) The set of all square matrices of order 2×2 is a non-commutative ring with unity w.r.t addition and multiplication of matrices.

Ex.1. Show that the set I of all integers is a ring with respect to addition and multiplication of integers as the two ring compositions. ***IT : W-10***

Soln. We should first prove that I is an abelian group with respect to addition of integers.

Closure : We know that the sum of two integers is also an integer i.e. $a + b \in I, \forall a, b \in I$,

Associativity : We know that addition of integers is an associative composition.

$\therefore a + (b + c) = (a + b) + c, \forall a, b, c \in I$

Existence of Identity : The number $0 \in I$

Also we have $0 + a = a = a + 0, \forall a \in I$. Therefore, the integer 0 is the identity.

Existence of inverse : If $a \in I$, then $-a \in I$, such that $(-a) + a = 0 = a + (-a)$. Thus, every integer possesses additive inverse.

Since addition of integers is a commutative composition, therefore (I, +) is an abelian group.

Further we observe that

(i) The product of two integers is also an integer. Therefore, I is closed with respect to multiplication of integers.

(ii) Multiplication of an integers is an associative composition.

(iii) Multiplication of integers is distributive with respect to addition of integers. i.e. if a,b,c are any elements of I, then

$a(b + c) = ab + ac$ and $(b + c)a = ba + ca$.

Therefore, I is a ring with respect to addition and multiplication of integers. The integer 0 is the zero element of this ring.

Also the multiplicative identity exists and is the integer 1.

We have $1a = a = a1, \forall a \in I$. Thus the ring of integers is a ring with unity. The integer 1 is the unit element of this ring. The multiplication of integers is a commutative composition. Therefore, it is also a commutative ring. **..... Ans.**

Ex.2. If R is a ring and $a, b \in R$, then show that $(a + b)^2 = a^2 + ab + ba + b^2 \neq a^2 + 2ab + b^2$. ***IT : W-09,11***

Soln. Let R be a ring and $a, b \in R$

$\therefore (a + b)^2 = (a + b)(a + b)$

$= (a + b)\,x$ where $x = a + b \in R$

$= ax + bx$ *[∵ By RD]*

$= a(a + b) + b(a + b)$

$= a^2 + ab + ba + b^2$ *[∵ By LD]*

$\neq a^2 + 2ab + b^2$ unless R is commutative ring.

$\therefore$ $\boxed{(a + b)^2 = a^2 + ab + ba + b^2}$ **Ans.**

Ex.3. Let R be a ring $x^2 = x, \forall x \in R$. Show that R is a commutative ring.

Soln. Given : (i) R is a ring

(ii) $x^2 = x, \forall x \in R$ ....(1)

To find : R is a commutative ring.

i.e. $xy = yx, \forall x, y \in R$

First we show that **(a)** $x + x = 0$,

(b) $xy + yx = 0, \forall x, y \in R$

(a) Let $x \in R$

$\Rightarrow x + x \in R$ *[∵ By G1]*

$\Rightarrow (x + x)^2 = (x + x)$ *[∵ By (1)]*

$\Rightarrow (x + x)(x + x) = (x + x)$

$\Rightarrow x^2 + x^2 + x^2 + x^2 = x + x$

$\Rightarrow x + x + x + x = x + x$ *[∵ By (1)]*

$\Rightarrow (x + x) + (x + x) = (x + x) + 0$

$\Rightarrow x + x = 0$ *[∵ By LCL in (R, +)]*

(b) Let $x, y \in R$

$\Rightarrow x + y \in R$ *[∵ By G1]*

$\Rightarrow (x + y)^2 = (x + y)$ *[∵ By (1)]*

$\Rightarrow (x + y)(x + y) = (x + y)$

$\Rightarrow x^2 + xy + yx + y^2 = x + y$

$\Rightarrow x + xy + yx + y = x + y$ *[∵ By (1)]*

$\Rightarrow (x + y) + (xy + yx) = (x + y) + 0$

$\Rightarrow xy + yx = 0$ *[∵ By LCL in (R, +)]*

We now prove that $xy = yx, \forall x, y \in R$

Let $x, y \in R$

$\Rightarrow xy \in R$

$\Rightarrow xy + xy = 0$ *[∵ By (a)]*

$\Rightarrow xy + xy = xy + yx$ *[∵ By (b)]*

$\Rightarrow xy = yx$ *[∵ By LCL in (R, +)]*

$\therefore xy = yx, \forall x, y \in R$

$\therefore$ $\boxed{\text{R is commutative ring}}$ **Ans.**

Ex.4. Show that the set {0, 1, 2, 3, 4, 5, 6} is a commutative ring w.r.t. addition modulo 7 and multiplication modulo 7.

CS/CT/CE : W-06,08,12, S-11,12,13, IT : S-08, W-08

OR Show that $< Z_7, +_7, \times_7 >$ is a commutative ring with identity.

Soln. Composition table for $+_7$

$+_7$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

R_1 : We prove that $(R, +_7)$ i.e. (R, +) is an abelian group.

G_1 : From the composition table, we observe that $a, b \in R \Rightarrow a + b \in R$

$\therefore G_1$ is verified

G_2 : Consider $1 + (4 + 5) = (1 + 4) + 5$

$\Rightarrow 1 + 2 = 5 + 5$

$\Rightarrow 3 = 3$

$\therefore a + (b + c) = (a + b) + c, \forall a, b\, c \in R$

$\therefore G_2$ is verified

G_3 : We observe that $0 \in R$ and $a + 0 = a = 0 + a$, $\forall a \in R$

$\therefore$ 0 is the zero element of R.

$\therefore G_3$ is verified

G_4 : From the table we observe that additive inverses of 0, 1, 2, 3, 4, 5, 6 are 0, 6, 5, 4, 3, 2, 1 respectively.

$\therefore a + (-a) = 0 = (-a) + a, \forall a \in R$

$\therefore G_4$ is verified.

G_5 : Consider $2 + 5 = 5 + 2$

$\Rightarrow 0 = 0$

Similary consider $3 + 5 = 5 + 3$

$\Rightarrow 1 = 1$

$\therefore a + b = b + a, \forall a, b \in R$

$\therefore G_5$ is verified.

$\therefore (R, +_7)$ is an abelian group.

$\therefore R_1$ is verified.

Composition table for $(R, \times_7)$

$\times_7$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

R_2 : From the table, we observe that $a, b \in R \Rightarrow ab \in R$

Consider $2.(3.4) = (2.3).4$

$\Rightarrow 2.(5) = 6.4$

$\Rightarrow 3 = 3$

$a.(b.c) = (a.b).c, \forall a, b, c \in R$

R_2 is verified

R_3 : Consider $2.(3 + 4) = 2.3 + 2.4$

$\Rightarrow 2.0 = 6 + 1$

$\Rightarrow 0 = 0$

Consider $(4 + 5).2 = 4.2 + 5.2$

$\Rightarrow 2.2 = 1 + 3$

$\Rightarrow 4 = 4$

$a.(b + c) = a.b + a.c$ and

$(b + c).a = b.a + c.a, \forall a, b, c \in R$

R_3 is verified

Also from the composition table for $\times_7$ we observe that $a.b = b.a, \forall a, b \in R$

$\boxed{(R, +_7, \times_7) \text{ is a commutative ring}}$ **Ans.**

From the composition table for $\times_7$ we observe that every non-zero element has a multiplicative inverse as given below

Element	Multiplicative Inverse
1	1
2	4
3	5
4	2
5	3
6	6

$\boxed{R \text{ is a field}}$ **Ans.**

Ex.4.(a) The set R = {0, 1, 2, 3, 4, 5} is a commutative ring with respect to '$+_6$' and '$\times_6$' as the two ring compositions. **CS/CT/CE : W-11** **IT : W-06**

Soln. We should first prove that R is an abelian group with respect to '$+_6$'

Let us form the composition table $+_6$

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

We see that all the entries in the composition table are elements of the set R. Therefore R is closed with respect to addition modulo 6.

The composition '$+_6$' is associative. If a, b, c are any three elements of R, then

$a +_6 (b +_6 c) = a +_6 (b + c)$ $[\because b +_6 c = b + c \pmod 6]$

= least non-negative remainder when $a + (b + c)$ is divided by 6.

= least non-negative remainder when $(a + b) + c$ is divided by 6 $[\because a + (b + c) = (a + b) + c]$

$= (a + b) +_6 c = (a +_6 b) +_6 c$

$[\because a + b = a +_6 b \pmod 6]$

Existence of identity. We have $0 \in R$ such that

$0 +_6 a = a = a +_6 0, a \in R$

Therefore, 0 is the identity element.

Existence of inverse : From the table we see that the additive inverse of 0, 1, 2, 3, 4, 5 are 0, 5, 4, 3, 2, 1 respectively.

Commutative : Since $a +_6 b = b +_6 a$ the composition $+_6$ is commutative.

$\therefore$ R is an abelian group with respect to the operation addition modulo 6.

Now we form the composition table for R for the composition $\times_6$

$\times_6$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

From the composition table we see that R is closed with respect to the composition '$\times_6$'

Also we know that '$\times_6$' is an associative composition in R i.e

$a \times_6 (b \times_6 c) = (a \times_6 b) \times_6, \forall a, b, c \in R$

Further '$\times_6$' is distributive in R with respect to '$+_6$'

If a, b,c are any elements of R, then

$a \times_6 (b +_6 c) = a \times_6 (b + c)$

$[\because b + c = b +_6 c \ (mod\ 6)]$

= least non-negative remainder when $a(b + c)$ is divided by 6

= least non-negative remainder when $ab + ac$ is divided by 6

$= (ab) +_6 (ac)$

$= (a \times_6 b) +_6 (ac)$ $[\because a \times_6 b = ab \ (mod\ 6)]$

$= (a \times_6 b) +_6 (a \times_6 c)$

$[\because a \times_6 c = ac \ (mod\ 6)]$

Similarly we can prove that

$(b +_6 c) \times_6 a = (b \times_6 a) +_6 (c \times_6 a)$

$\therefore$ R is a ring with respect to the given compositions. Since '$\times_6$' is a commutative composition in R as is clear from the composition table also, therefore R is a commutative ring. Also 1 is the identity element for the composition '$\times_6$'. Therefore R is a ring with unity. The integer 0 is the zero element of this ring. .....**Ans.**

From the table of $\times_6$, we observe that $2 \times 3 = 0$

i.e. $ab = 0 \Rightarrow a = 0$ or $b = 0$ failed.

$\therefore$ $\boxed{\text{R is not an integral domain.}}$ **Ans.**

Ex.4.(b) For the set $I_4 = \{0, 1, 2, 3\}$, show that the modulo 4 system is a ring. **CS/CT/CE : S-07**

Soln. The composition table for $(Z_4, +_4)$ is given as

$+_4$	[0]	[1]	[2]	[3]
[0]	[0]	[1]	[2]	[3]
[1]	[1]	[2]	[3]	[0]
[2]	[2]	[3]	[0]	[1]
[3]	[3]	[0]	[1]	[2]

Closure : Let $[1], [2] \in Z_4 = [1] +_4 [2] = [3] \in Z_4$

Associative : Let $[2], [3], [1] \in Z_4$,

then $[2] +_4 ([3] +_4 [1]) = [2] +_4 [0] = [2]$

Also $([2] +_4 [3]) +_4 [1] = [1] +_4 [1] = [2]$

Identity : $[0] \in Z_4$ is identity. Since

$[0] +_4 [1] = [1]$

$[0] +_4 [2] = [2]$

$[0] +_4 [3] = [3]$

Inverse : Inverse of each element exists

Since $[0] +_4 [0] = [0]$

$[1] +_4 [3] = [0]$

$[2] +_4 [2] = [0]$

Commutative : Let $[2], [3] \in Z_4$

Then $[2] +_4 [3] = [1] = [3] +_4 [2]$

Also the distributive law holds in I_4

$\therefore$ $\boxed{(Z_4, +_4) \text{ is algebraic system with commutativity \& is a ring}}$ **Ans.**

Ex.5. If R is a ring such that $a^2 = a, \forall a \in R$, then show that (i) $a + a = 0, \forall a \in R$

(ii) $a + b = 0 \Rightarrow a = b, \forall a, b \in R$

(iii) R is a commutative ring (i.e. $a.b = b.a$)

CS/CT/CE : S-06,11,13, M-07,09.

IT : M-06, S-09,10

Soln. Given : R is a ring such that $a^2 = a, \forall a \in R$

To prove : (i) $a + a = 0, \forall a \in R$

(ii) $a + b = 0 \Rightarrow a = b, \forall a, b \in R$

(iii) $a.b = b.a$

(i) Let $a \in R$

$\Rightarrow a + a \in R$

$\Rightarrow (a + a)^2 = a + a$

$\Rightarrow (a + a)(a + a) = a + a$

$\Rightarrow a^2 + a^2 + a^2 + a^2 = a + a$

$\Rightarrow a + a + a + a = a + a$ $[\because a^2 = a]$

$\Rightarrow (a + a) + (a + a) = (a + a) + 0$

$\Rightarrow a + a = 0$ $[\because \text{By LCL}]$

(ii) We know that

$a + a = 0$.....(1) $[\because \text{By (i)}]$

Let $a, b \in R$, Now $a + b = 0$

$\Rightarrow a + b = a + a$ $[\because \text{By (1)}]$

$\Rightarrow b = a$ $[\because \text{By LCL}]$

$\therefore a + b = 0 \Rightarrow a = b$

(iii) Let $a, b \in R$, then $a^2 = a, b^2 = b$

$\Rightarrow a + b \in R$

$\Rightarrow (a + b)^2 = a + b$

$\Rightarrow (a + b) \times (a + b) = a + b$

$\Rightarrow a^2 + ab + ba + b^2 = a + b$

$\Rightarrow a + ab + ba + b = a + b$ $[\because a^2 = a, b^2 = b]$

$\Rightarrow (a + b) + (ab + ba) = (a + b) + 0$

$\Rightarrow ab + ba = 0$ $[\because \text{By LCL}]$

$\Rightarrow ab = ba$ $[\because \text{By (ii)}, a + b = 0 \Rightarrow a = b]$

$\therefore$ $\boxed{\text{R is a commutative ring}}$ **Ans.**

Ex.6. If R is a ring, then $\forall a, b \in R$.

(i) $a.0 = 0 = 0.a$

(ii) $a.(-b) = (-a).b = -(a.b)$

(iii) $(-a).(-b) = a.b$.

If R has a unit element 1, then

(iv) $(-1)a = -a$ (v) $(-1).(-1) = 1$

(vi) $a . (b - c) = a. b - a.c$

(vii) $(b - c) a = b a - c. a.$

IT : S-06,13,W-06,09,11.
CS/CT/CE : W-06,10

Soln. Given : R is a ring.

(i) **To prove :** $a . 0 = 0 = 0.a$

Let $a \in R$

Now $a . 0 = a . (0 + 0)$ *[∵ 0 is zero element of R]*

$= a . 0 + a. 0$ *[∵ By LD]*

$\Rightarrow a. 0 + 0 = a . 0 + a. 0$

$\Rightarrow a. 0 = 0$ *[∵ By LCL]*

Now $0.a = (0 + 0). a$

$= 0.a + 0.a$

$\Rightarrow 0.a + 0 = 0.a + 0.a$ *[∵ By LD]*

$\Rightarrow 0.a = 0$ *[∵ By LCL]*

$\therefore a.0 = 0 = 0.a$

(i) **To prove :** $a (- b) = (- a) . b = - (a . b)$

Let $a, b \in R$

Consider $a . b + a. (- b) = a. (b + (- b))$ *[∵ By LD]*

$= a . 0$

$= 0$ *[∵ By (i)]*

$\therefore a . b + a. (- b) = 0$

$\therefore a . (- b) = - (ab)$

Consider $a .b + (- a) . b = (a + (- a)).b$ *[∵ By RD]*

$= 0 . b$

$= 0$ *[∵ By (i)]*

$\therefore a . b + (- a) . b = 0$

$\therefore (- a) . b = - (a . b)$

$\therefore a . (- b) = (- a) . b = - (a . b)$

(iii) **To prove :** $(- a) . (- b) = a . b$

Consider $(- a) . (- b) = - (a . (- b))$ *[∵ By (ii)]*

$= - (- a . b)$ *[∵ By (ii)]*

$= a .b$ *[∵ $(a^{-1}) = a, \forall a \in R$]*

$\therefore (- a) . (- b) = a . b$ ..... **Ans.**

(iv) **To prove :** $(- 1) . a = - a$

Let R be a ring with unit element 1.

Consider $a + (- 1) . a = 1 . a + (-1) .a$

[∵ $a = 1.a, 1 \in R$]

$= (1 + (- 1)) . a$ *[∵ By RD]*

$= 0.a$

$= 0$ *[∵ By (i)]*

$\therefore a + (- 1) . a = 0$

$\therefore (- 1) . a = - a$

(v) **To prove :** $(- 1) . (-1) = 1$

By (iv), we have $(- 1) . a = - a, \forall a \in R$ ....(1)

Since $1 \in R$, hence $- 1 \in R$

Put $a = - 1$ in (1)

(i) $\Rightarrow (- 1) . (-1)) = - (-1)$

$= 1$ *[∵ $(a^{-1})^{-1} = a, \forall a \in R$]*

$\therefore (- 1) . (- 1) = 1$ ..... **Ans.**

(vi) **To prove :** $a . (b - c) = a. b - a.c$

Consider L.H.S. $= a. (b - c)$

$= a. (b + (- c))$

$= a . b + a. (- c)$ *[∵ By LD]*

$= a .b - a . c$ *[∵ By (ii)]*

$=$ R.H.S.

$\therefore a . (b - c) = a . b - a . c$

(vii) **To prove :** $(b - c) . a = b . a - c . a$

Consider L.H.S. $= (b - c) .a$

$= (b + (- c)) .a$ *[∵ By RD]*

$= b. a - c . a$

$=$ R.H.S.

$\therefore (b - c) a = b . a - c . a$ ..... **Ans.**

SPECIAL CLASSES OF RINGS

Zero Divisor :

IT : S-07,10
CS/CT/CE : W-08,12

Let R be a commutative ring. An element $a (\neq 0)$ in R is called zero divisor, if there exists an element $b (\neq 0)$ in R such that $ab = 0$.

i.e. If $ab = 0 \Rightarrow a \neq 0$ and $b \neq 0$, then a and b are called zero divisors.

If $ab = 0 \Rightarrow a = 0$ or $b = 0$, then a and b are called non zero divisors.

Ex.7. Let R be a ring of all 2×2 matrices with entries as integers under addition and multiplication of matrices. Show that R is a ring with zero divisors.

IT : S-06,07,10.
CS/CT/CE : W-07

Soln. Let R be a ring of 2×2 matrices under matrix addition and multiplication.

Let $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in R$

Then $ab = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

But $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq 0$ and $b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \neq 0$

$\therefore$ a and b are zero divisors.

Here we say that R is a ring with zero divisors a and b.

Note : Zero divisors generally do not arise in the rings of numbers, because in numbers $ab = 0 \Rightarrow a = 0$ or $b = 0$. Hence R is a ring without zero divisors because if $ab = 0 \Rightarrow a = 0$ or $b = 0$.

Integral Domain : **IT : S-07, W-07** **CS/CT/CE : W-08,12**

A commutative ring R is called an integral domain if it has no zero divisors. In other words a commutative ring R is an integral domain, if

$ab = 0 \Rightarrow a = 0$ or $b = 0, \forall a, b \in R$

Division Ring : **CS/CT/CE : W-08**

A ring whose non-zero elements form a group under multiplication is called a division ring. In other words a ring with unit element in which every non-zero element has multiplicative inverse is called a division ring.

Field : **IT : S-06, W-07** **CS/CT/CE : W-12**

A commutative division ring is called a field. In other words a commutative ring with unit element in which every non-zero element has multiplicative inverse is called a field.

Ex.8. Show that commutative ring D is an integral domain, iff for a, b, c $\in$ D with a = 0, the relation $ab = ac \Rightarrow b = c$

OR Prove that a ring D is without zero divisors, iff the cancellation laws hold in D. **IT : S-07,12**

Soln. If Part :

Given : D is an integral domain.

To prove : $ab = ac, a \neq 0 \Rightarrow b = c$

Since D is an integral domain, hence D has no zero divisors.

Consider $ab = ac, a \neq 0$

$\Rightarrow ab + (-ac) = 0, a \neq 0$

$\Rightarrow ab - ac = 0, a \neq 0$

$\Rightarrow a(b - c) = 0, a \neq 0$

$\Rightarrow b - c = 0$ $[\because a \neq 0]$

$\Rightarrow b = c$

Only If part :

Given : (i) D is a commutative ring

(ii) $ab = ac, a \neq 0 \Rightarrow b = c$

To prove : D is an integral domain.

We prove that D has no zero divisors.

Since $ab = ac, a \neq 0 \Rightarrow b = c$

$\therefore ab + (-ac) = 0, a \neq 0 \Rightarrow b = c$

$\therefore ab - ac = 0, a \neq 0 \Rightarrow b = c$

$\therefore a(b - c) = 0, a \neq 0 \Rightarrow b + (-c) = 0$

Thus for $a(b - c) = 0, a \neq 0$ then $b - c = 0$

$\therefore$ D has no zero divisors.

$\therefore$ $\boxed{\text{D is an integral domain}}$ **Ans.**

Ex.9. Prove that every field is an integral domain. Does its converse hold?

OR Prove that a field has no zero divisors. **CS/CT/CE : W-12**

Soln. Let F be a field.

$\therefore$ By definition, F is a commutative ring with unity 1 in which every non zero element has multiplicative inverse in F.

In order to prove F is an integral domain, we prove that F has no zero divisors.

i.e. $ab = 0 \Rightarrow a = 0$ or $b = 0, \forall a, b \in F$

Suppose $ab = 0$ .....(1) and $a \neq 0$.

Since $a \neq 0$, hence a^{-1} exists in F

$\therefore (1) \Rightarrow a^{-1}(ab) = a^{-1}(0)$

$\Rightarrow (a^{-1}a)b = 0$

$\Rightarrow 1.b = 0$

$\Rightarrow b = 0$

$\therefore ab = 0, a \neq 0 \Rightarrow b = 0$

i.e. $ab = 0 \Rightarrow a = 0$ or $b = 0, \forall a, b \in F$

$\therefore$ F has no zero divisors

$\therefore$ $\boxed{\text{F is an integral domain}}$ **Ans.**

Now we check whether every integral domain is a field.

Consider $Z = \{0, \pm 1, \pm 2, \pm 3, ...\}$

Here Z is an integral domain under usual addition and multiplication.

But Z is not a field because $2 \in Z$, but $\frac{1}{2} \notin Z$

$\therefore$ $\boxed{\text{Every integral domain is not a field}}$ **Ans.**

Ex.10.(a) Show that the set of numbers of the form $a + b\sqrt{2}$ where a and b are rational numbers is a field, **IT : W-07,09**

Soln. Let $R = \{a + b\sqrt{2} \mid a, b \in Q\}$

To prove : R is a field.

We prove that R is a commutative ring with unit element in which every non zero element has multiplicative inverse.

R_1 : Let $a_1 + b_1\sqrt{2} \in R, a_2 + b_2\sqrt{2} \in R,$

$\forall a_1, a_2, b_1, b_2 \in Q$

Then $(a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2})$

$= (a_1 + a_2) + (b_1 + b_2)\sqrt{2} \in R$

$[\because a_1 + a_2 \in Q, b_1 + b_2 \in Q]$

$\therefore$ G_1 is verified.

G_2 : Let $a_1 + b_1\sqrt{2}, a_2 + b_2\sqrt{2}, a_3 + b_3\sqrt{2} \in R,$

$\forall a_1, a_2, a_3, b_1, b_2, b_3 \in Q$

Now $[(a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2})] + (a_3 + b_3\sqrt{2})$

$= (a_1 + b_1\sqrt{2}) + [(a_2 + b_2\sqrt{2})$

$+ (a_3 + b_3\sqrt{2})]$

$\therefore$ G_2 is verified.

G_3 : Since $0 \in Q, 0 + 0\sqrt{2} \in R$

Further if $a + b\sqrt{2} \in R$, then

$(a + b\sqrt{2}) + (0 + 0\sqrt{2}) = a + b\sqrt{2} = (0 + 0\sqrt{2})$

$+ (a + b\sqrt{2})$

$\therefore$ $0 + 0\sqrt{2}$ is the zero element.

$\therefore$ G_3 is verified.

G_4 : If $a + b\sqrt{2} \in R$, then $(-a) + (-b)\sqrt{2} \in R$

as $-a, b \in Q$ if $a, b \in Q$

$\therefore (a + b\sqrt{2}) + ((-a) + (-b)\sqrt{2})$

$= 0 + 0\sqrt{2}$

$= ((-a) + (-b)\sqrt{2}) + (a + b\sqrt{2})$

$\therefore$ G_4 is verified.

G_5 : We know that $a + b\sqrt{2} = b\sqrt{2} + a, \forall a, b \in Q$

$\therefore$ G_5 is verified.

$\therefore$ R_1 is verified.

R_2 : Let $a_1 + b_1\sqrt{2}, a_2 + b_2\sqrt{2} \in R,$

$\forall a_1, a_2, b_1, b_2 \in Q$

Then $(a_1 + b_1\sqrt{2})(a_2 + b_2\sqrt{2})$

$= a_1 a_2 + a_1 b_2\sqrt{2} + a_2 b_1\sqrt{2} + 2b_1 b_2$

$= (a_1 a_2 + 2b_1 b_2) + (a_1 b_2 + a_2 b_1)\sqrt{2} \in R$

$[\because a_1 a_2 + 2b_1 b_2, a_1 b_2 + a_2 b_1 \in Q]$

Let $a_1 + b_1\sqrt{2}, a_2 + b_2\sqrt{2}, a_3 + b_3\sqrt{2} \in R,$

$\forall a_1, a_2, a_3, b_1, b_2, b_3 \in Q$

We know that

$(a_1 + b_1\sqrt{2})[(a_2 + b_2\sqrt{2})(a_3 + b_3\sqrt{2})]$

$= [(a_1 + b_1\sqrt{2}).(a_2 + b_2\sqrt{2})](a_3 + b_3\sqrt{2})$

[$\because$ Q is set of rational numbers]

$\therefore$ R_2 is verified.

R_3 : We know that

$(a_1 + b_1\sqrt{2}).[(a_2 + b_2\sqrt{2}) + (a_3 + b_3\sqrt{2})$

$= (a_1 + b_1\sqrt{2}).(a_2 + b_2\sqrt{2})$

$+ (a_1 + b_1\sqrt{2})(a_3 + b_3\sqrt{2})$

and

$[(a_2 + b_2\sqrt{2}) + (a_3 + b_3\sqrt{2})].(a_1 + b_1\sqrt{2})$

$= (a_2 + b_2\sqrt{2}).(a_1 + b_1\sqrt{2})$

$+ (a_3 + b_3\sqrt{2})(a_1 + b_1\sqrt{2})$

$\forall a_1 + b_1\sqrt{2}, a_2 + b_2\sqrt{2}, a_3 + b_3\sqrt{2} \in R$

$\therefore$ R_3 is verified.

Also $(a_1 + b_1\sqrt{2}).(a_2 + b_2\sqrt{2})$

$= (a_2 + b_2\sqrt{2}).(a_1 + b_1\sqrt{2})$

$\therefore$ R is a commutative ring.

Let $a + b\sqrt{2}, 1 + 0\sqrt{2} \in R$ where $a, b, 1, 0 \in Q$

Then $(a + b\sqrt{2})(1 + 0\sqrt{2}) = a + b\sqrt{2}$

$= (1 + 0\sqrt{2}) \times (a + b\sqrt{2})$.

$\therefore$ $1 + 0\sqrt{2}$ is the unit element.

Now we prove that every non zero element of R has a multiplicative inverse.

Let $a + b\sqrt{2} \in R$ where atleast one of a and b is not zero.

$$\therefore \frac{1}{a + b\sqrt{2}} = \frac{1}{a + b\sqrt{2}} \frac{a - b\sqrt{2}}{a - b\sqrt{2}}$$

$$= \left(\frac{a}{a^2 - 2b^2}\right) + \left(\frac{-b}{a^2 - 2b^2}\right)\sqrt{2}$$

Now $a^2 - 2b^2 = 0$ if atleast one of a and b is zero.

But atleast one of a and b is not zero.

$\therefore a^2 - 2b^2 \neq 0$

$\therefore$ $\left(\frac{a}{a^2 - 2b^2}\right)$ and $\left(\frac{-b}{a^2 - 2b^2}\right)$ are both rational numbers and atleast one of them is not zero.

$\therefore$ $a + b\sqrt{2} \in R$ has a multiplicative inverse.

$\therefore$ Every non zero element of R has a multiplicative inverse.

$\therefore$ $\boxed{\text{R is a field.}}$ **Ans.**

Ex.10.(b) Prove that the set R = {0, 2, 4, 6, 8} is a commutative ring under addition and multiplication modulo 10. Verify that it is an integral domain. ***IT : W-09,11***

Soln. Composition table for $+_{10}$ is given by

$+_{10}$	0	2	4	6	8
0	0	2	4	6	8
2	2	4	6	8	0
4	4	6	8	0	2
6	6	8	0	2	4
8	8	0	2	4	6

To prove (R, +) is abelian group.

G_1 : From the composition table, we observe that

$a + b \in R, \forall a, b \in R$

$\therefore$ G_1 is verified.

G_2 : Consider $2 + (4 + 6) = (2 + 4) + 6$

i.e. $2 + 0 = 6 + 6$

i.e. $2 = 2$

$\therefore$ $a + (b + c) = (a + b) + c, \forall a, b, c \in R$

$\therefore$ G_2 is verified.

G_3 : Here 0 is the identity element such that

$a + 0 = a = 0 + a, \forall a \in R$

$\therefore$ G_3 is verified.

G_4 :

Element	Inverse
0	0
2	8
4	6
6	4
8	2

$\therefore$ Inverse of every $a \in R$ belong to R.

$\therefore$ G_4 is verified.

G_5 : Consider $2 + 6 = 6 + 2$

$\Rightarrow$ $8 = 8$

Also $6 + 8 = 8 + 6$

$\Rightarrow$ $4 = 4$

$\therefore$ $a + b = b + a, \forall a, b \in R$

$\therefore$ G_5 is verified.

$\therefore$ (R, +) is an abelian group.

$\times_{10}$	0	2	4	6	8
0	0	0	0	0	0
2	0	4	8	2	6
4	0	8	6	4	2
6	0	2	4	6	8
8	0	6	2	8	4

R_2 : From the table we observe that

$a, b \in R \Rightarrow ab \in R$

Consider $2.(4.6) = (2.4).6$

i.e. $2.4 = 8.6$

i.e. $8 = 8$

$\therefore$ $a . (b . c) = (a . b) . c, \forall a, b, c \in R$

$\therefore$ R_2 is verified.

R_3 : Consider $2 . (4 + 8) = (2 . 4) + (2 . 8)$

$\Rightarrow$ $2 . (2) = 8 + 6$

$\Rightarrow$ $4 = 4$

$\therefore$ $a . (b + c) = a . b + a . c$ and similarly

$(b + c) . a = b . a + c . a, \forall a, b, c \in R$

$\therefore$ R_3 is verified.

Also $a . b = b . a, \forall a, b \in R$

$\therefore$ R is a commutative ring.

We observe that $ab = 0 \Rightarrow a = 0$ or $b = 0, \forall a, b \in R$

$\therefore$ $\boxed{\text{R is an integral domain.}}$ **Ans.**

Ex.11. Show that the set of real numbers of the form $a + b\sqrt{2}$; where a and b are integers is an integral domain but not a field.

CS/CT/CE : S-06,07,08,09, W-08,10

IT : S-07,09,12, W-10

Soln. Using previous example (10) we can easily prove that $a + b\sqrt{2}$ is a ring where a and b are integers. Now we prove that $a + b\sqrt{2}$ is an integral domain but not a field.

Let $0 + 0\sqrt{2}$ is the zero element of R.

Now to prove R is an integral domain, we prove that R is a ring without zero divisions.

Let

$a_1 + b_1\sqrt{2}, a_2 + b_2\sqrt{2} \in R, \forall a_1, a_2, b_1, b_2 \in Z$

Consider $(a_1 + b_1\sqrt{2})(a_2 + b_2\sqrt{2}) = 0 = 0 + 0\sqrt{2}$

$\Rightarrow (a_1 a_2 + 2b_1 b_2)(a_1 b_2 + a_2 b_1)\sqrt{2} = 0 + 0\sqrt{2}$

i.e. $a_1 a_2 + 2b_1 b_2 = 0$ or $a_1 b_2 + a_2 b_1 = 0$

This is possible only when $a_1 = 0$ and $b_1 = 0$ or $a_2 = 0$ and $b_2 = 0$

$\therefore (a_1 + b_1\sqrt{2})(a_2 + b_2\sqrt{2}) = 0 + 0\sqrt{2} = 0$

or $a_2 + b_2\sqrt{2} = 0$

$\therefore$ R is a ring without zero divisors.

Now we prove that R is not a field.

Let $a + b\sqrt{2} \in R, \forall a, b \in Z$

$$\therefore \frac{1}{a + b\sqrt{2}} = \left(\frac{a}{a^2 - 2b^2}\right) + \left(\frac{-b}{a^2 - 2b^2}\right)\sqrt{2}$$

But $\frac{a}{a^2 - 2b^2}, \frac{-b}{a^2 - 2b^2} \notin Z$

$\therefore$ Every non-zero element of R does not have a multiplicative inverse.

$\therefore$ $\boxed{\text{R is not a field}}$ **Ans.**

Gaussian Integer : A Gaussian integer is a complex number of the form a + ib where a and b are integers.

Ex.12. Show that Gaussian integer forms a ring under usual addition and multiplication of complex numbers. It an integral domain or a field?

IT : S-08

Soln. Let $R = \{a + ib \mid a, b \in Z\}$

To prove : R is a ring

R_1 : (R, +) is an abelian group

G_1 : Let $a_1 + ib_1, a_2 + ib_2 \in R, \forall a_1, a_2, b_1, b_2 \in Z$

$\therefore (a_1 + ib_1) + (a_2 + ib_2)$

$= (a_1 + a_2) + (b_1 + b_2)i \in R$

as $a_1 + a_2, b_1 + b_2 \in Z$

$\therefore G_1$ is verified.

G_2 : Let $a_1 + ib_1, a_2 + ib_2, a_3 + ib_3 \in R$

Consider $(a_1 + ib_1) + [(a_2 + ib_2) + (a_3 + ib_3)]$

$= [(a_1 + ib_1) + (a_2 + ib_2)] + (a_3 + ib_3)$

$\Rightarrow (a_1 + a_2 + a_3) + (b_1 + b_2 + b_3)i$

$= (a_1 + a_2 + a_3) + (b_1 + b_2 + b_3)i$

$\therefore G_2$ is verified.

G_3 : Since $0 \in Z, 0 + i0 \in R$

Further if $a + ib \in R$, then

$(a + ib) + (0 + i0) = a + ib = (0 + i0) + (a + ib)$

$\therefore 0 + i0$ is the zero element.

G_3 is verified.

G_4 : If $a + ib \in R$, then $(-a) + i(-b) \in R$

as $-a, -b \in Z$ if $a, b \in Z$

$\therefore (a + ib) + ((-a) + i(-b)) = 0 + i0$

$= ((-a) + i(-b)) + (a + ib)$

$\therefore G_4$ is verified.

G_5 : We know that $a + ib = ib + a, \forall a, b \in Z$

G_5 is verified.

$\therefore R_1$ is verified.

R_2 : Let $a_1 + ib_2, a_2 + ib_2 \in R$

$\therefore (a_1 + ib_1)(a_2 + ib_2) = (a_1a_2 - b_1b_2)$

$+ i(a_1b_2 + b_1a_2) \in R$

as $a_1 a_2 - b_1 b_2, a_1 b_2 + b_1 a_2 \in Z$

Also $(a_1 + ib_1)[(a_2 + ib_2)(a_3 + ib_3)]$

$= [(a_1 + ib_1)(a_2 + ib_2)](a_3 + ib_3)$

$\therefore R_2$ is verified.

R_3 : We know that distributive property of multiplication over addition is preserved in complex numbers.

$\therefore (a_1 + ib_1)[(a_2 + ib_2) + (a_3 + ib_3)]$

$= (a_1 + ib_1)(a_2 + ib_2) + (a_1 + ib_1)(a_3 + ib_3)$

and

$[(a_2 + ib_2) + (a_3 + ib_3)](a_1 + ib_1)$

$= (a_2 + ib_2)(a_1 + ib_1) + (a_3 + ib_3)(a_1 + ib_1)$

$\forall a_1 + ib_1, a_2 + ib_2, a_3 + ib_3 \in R$ and

$\forall a_1, a_2, a_3, b_1, b_2, b_3 \in Z$

$\therefore R_3$ is verified.

$\therefore$ R is ring.

We know that complex multiplication is commutative

i.e. $(a_1 + ib_1)(a_2 + ib_2) = (a_2 + ib_2)(a1 + ib_1)$

$\therefore$R is a commutative ring.

Now we check whether R is an integral domain.

R will be an integral domain if it is a ring without zero divisors.

Let $a_1 + ib_1, a_2 + ib_2 \in R, \forall a_1, a_2, b_1, b_2 \in Z$

Consider $(a_1 + ib_1)(a_2 + ib_2) = 0 = 0 + i0$

$(a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2) = 0 + i0$

i.e. $(a_1 a_2 - b_1 b_2) = 0$ or $(a_1 b_2 + b_1 a_2) = 0$

This is possible only when $a_1 = 0$ and $b_1 = 0$ or $a_2 = 0$ $b_2 = 0$

$\therefore (a_1 + ib_1)(a_2 + ib_2) = 0 + i0 \Rightarrow a_1 a_2 - b_1 b_2 = 0$

or $a_1 b_2 + b_1 a_2 = 0$

$\therefore$ R is a ring without zero divisors

$\therefore$ R is an integral domain.

Now we check whether R is a field.

R will be a field if every non-zero element of R has a multiplicative inverse.

Let $a + ib \in R, \forall a, b \in Z$

$$\therefore \frac{1}{a+ib} = \frac{1}{a+ib} \cdot \frac{a-ib}{a-ib} = \left(\frac{a}{a^2+b^2}\right) + \left(\frac{-b}{a^2+b^2}\right)$$

But $\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2} \notin Z$

$\therefore$ Every non-zero element of R does not have a multiplicative inverse.

$\therefore$ R is not a field. .....Ans.

Ex.13. Show that the set of all matrices of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ where a, b are real numbers is a field w.r.t. addition and multiplication of matrices.

CS/CT/CE : W-09, S-10

Soln. Let $F = \left\{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \middle| a, b \in R\right\}$

To prove : F is a field.

$\mathbf{R_1}$: (F, +) is an abelian group.

$\mathbf{G_1}$: Let $F_1 = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$,

$F_2 = \begin{pmatrix} c & d \\ -d & c \end{pmatrix} \in F, a, b, c, d \in R$

$\therefore F_1 + F_2 = \begin{pmatrix} a+c & b+d \\ -b-d & a+c \end{pmatrix} \in F$ as $a + c, b + d \in R$

$\therefore G_1$ is verified.

$\mathbf{G_2}$: We know that associative property over addition holds in matrices.

$\therefore$ It holds in F

$\therefore G_2$ is verified.

$\mathbf{G_3}$: Since $0 \in R, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in F$ such that

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

where $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in F, a, b \in R$

$\therefore \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is the zero element of F.

$\therefore G_3$ is verified.

$\mathbf{G_4}$: Let $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in F, a, b \in R$

Then its additive inverse $\begin{pmatrix} -a & -b \\ b & -a \end{pmatrix} \in F$

G_4 is verified.

$\mathbf{G_5}$: Let $F_1 = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$,

$F_2 = \begin{pmatrix} c & d \\ -d & c \end{pmatrix} \in F, a, b, c, d \in R$

Then $F_1 + F_2 = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} + \begin{pmatrix} c & d \\ -d & c \end{pmatrix}$

$= \begin{pmatrix} a+c & b+d \\ -b-d & a+c \end{pmatrix}$

Also $F_2 + F_1 = \begin{pmatrix} c & d \\ -d & c \end{pmatrix} + \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$

$= \begin{pmatrix} c+a & d+b \\ -d-b & c+a \end{pmatrix}$

$\therefore F_1 + F_2 = F_2 + F_1$

$\therefore G_5$ is verified.

$\therefore R_1$ is verified.

$\mathbf{R_2}$: We know that if $F_1 = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$,

$F_2 = \begin{pmatrix} c & d \\ -d & c \end{pmatrix} \in F, a, b, c, d \in R$,

then $F_1 F_2 = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \cdot \begin{pmatrix} c & d \\ -d & c \end{pmatrix}$

$= \begin{pmatrix} ac-bd & ad+bc \\ -bc-ad & -bd+ac \end{pmatrix} \in F$

Also associative property under multiplication holds in matrices.

$\therefore$ It holds in F.

$\therefore R_2$ is verified.

$\mathbf{R_3}$: We know that if $F_1 = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$,

$F_2 = \begin{pmatrix} c & d \\ -d & c \end{pmatrix}, F_3 = \begin{pmatrix} e & f \\ -f & e \end{pmatrix} \in F,$

a, b, c, d, e, f ∈ R, then left distribution, i.e.

$F_1 (F_2 + F_3) = F_1 . F_2 + F_1 . F_3$ and

right distribution $(F_2 + F_3) . F_1 = F_2 . F_1 + F_3 . F_1$

holds in matrices.

∴ It holds in F.

∴ R_3 is verified.

Now, $F_1 F_2 = \begin{pmatrix} ac - bd & ad + bc \\ -bc - ad & -bd + ac \end{pmatrix}$ and

$F_2 F_1 = \begin{pmatrix} ac - bd & bc + ad \\ -ad - bc & -bd + ac \end{pmatrix}$

$\therefore F_1 F_2 = F_2 F_1$

∴ F is a commutative ring.

Now in order to prove that F is a field, we prove that every non zero element of F has a multiplicative inverse.

Let $F_1 = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in F, a, b \in R$

$\therefore F_1^{-1} = \frac{1}{a^2 + b^2} \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

$\begin{pmatrix} \frac{a}{a^2 + b^2} & \frac{-b}{a^2 + b^2} \\ \frac{b}{a^2 + b^2} & \frac{a}{a^2 + b^2} \end{pmatrix} \in$ F because all entries are reals

Every non zero element of R has a multiplicative inverse

∴ $\boxed{\text{F is a field}}$ **Ans.**

POLYNOMIAL RING

IT : S-06, W-07

Let R be a ring. A polynomial ring over R is defined by an expression of the form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

where $a_n \in R$ and $a_n = 0$ for all,

but a finite number of values of n.

If for some $n > 0$, $a_n \neq 0$, then such a largest value of n is called the degree of f(x). If no such $a_n \neq 0$, for $n > 0$ exists then the degree of f(x) is zero.

A polynomial consisting of only a single term a_0 is called constant polynomial.

Subring : Let (R, +, *) be a ring. Then a subset $S \subseteq R$ is called a subring if (S, +, *) is itself a ring with the same binary operations + and *.

Ex.14. A non-empty subset S of a ring (R, +, *) is a subring iff $a - b \in S$, $a * b \in S$ for all a, b ∈ S.

IT : W-08, S-12

Soln. If part :

Given : (S, +, *) is a subring of (R, +, *) where $S \subseteq R$.

To prove : $a - b \in S$, $a * b \in S$ for all a, b ∈ S.

Since S is a subring of R, hence by definition S itself is a ring.

Hence (S, +) is an abelian group

Let a, b, ∈S, then a, – b ∈S

∴ a + (– b) ∈S *[∵ By G_1 in S]*

∴ a – b ∈S

Also a * b ∈S, ∀a, b∈S *[∵ S is a ring]*

∴a – b ∈S, a * b ∈S, ∀a, b ∈S

Only If part :

Given : a – b ∈S, a * b ∈S, ∀a, b ∈S

To prove : S is a subring of R.

We prove that S is a ring with binary operations + and *

S_1 : (S, +) is an abelian group.

G_1 : Let a, b ∈S, then a – b ∈S ....(1)

and a * b ∈S ....(2)

Put b = a in (1), we get

a – a ∈S i.e. 0 ∈S ....(3)

Also since 0, a ∈S, then 0 – a ∈S *[∵ By (1)]*

i.e. – a ∈S

a ∈S ⇒ – a ∈S ....(4)

Similarly b ∈S ⇒ – b ∈S ....(5)

Now a, b ∈S ⇒ a – b ∈S *[∵ By (1)]*

⇒ a + (– b) ∈S *[∵ By (5)]*

∴ a, b ∈S ⇒ a + b ∈S

∴ G_1 is verified.

G_2 : Let a, b, c, ∈S, then a + b, b + c ∈S *[∵ By G_1]*

Consider a + (b + c) = (a + b) + c

This is obviously true

∴ G_2 is verified.

G_3 : By (3), 0 ∈S such that a + 0 = a = 0 + a, ∀a ∈S

∴ 0 is the zero elements of S

G_3 is verified.

G_4 : By (4), $a \in S \Rightarrow -a \in S$

$\therefore$ G_4 is verified.

G_5 : Let a, b $\in$S, then a + b $\in$S *[$\because$ By G_1]*

Also b + a $\in$S

$\therefore$ a + b = b + a is obviously true.

$\therefore$ G_5 is verified.

$\therefore$ (S, +) is an abelian group.

S_2 : Let a, b, $\in$S, then a * b $\in$S *[$\because$ By (2)]*

Let a, b, c $\in$S, then a * b, b * c $\in$S

$\therefore$ a * (b * c) = (a * b) * c is obviously true.

$\therefore$ S_2 is verified.

S_3 : Since a, b, c $\in$S $\Rightarrow$ a – b, a + b, a * b, b * c $\in$S

$\therefore$ LD and RD hold is $S \subseteq R$

$\therefore$ S_3 is verified.

$\therefore$ (S, +, *) is a ring.

$\therefore$ S is a subring of R .... **Ans.**

Ex.15. Show that the intersection of any two subrings is also a subring. **IT : W-08, S-11,12,13**

Soln. Let R be a ring and R_1, R_2 be any two subrings of R.

Let 0 be the zero element of R.

$\therefore$ 0 $\in$ R_1 and 0 $\in$ R_2 as R_1, R_2 are subrings of R.

$\therefore$ R_1 and R_2 are non-empty subsets of R.

$\therefore$ $R_1 \cap R_2$ is also a non-empty subset of R.

Let a, b $\in$ $R_1 \cap R_2$

$\Rightarrow$ a, b $\in$ R_1 and $\Rightarrow$ a, b $\in$ R_2

$\Rightarrow$ a – b $\in$ R_1, a * b $\in$ R_1 and a – b $\in$ R_2 *[$\because$ Ex – 14]*

$\Rightarrow$ a – b $\in$ $R_1 \cap R_2$ and a * b $\in$ $R_1 \cap R_2$

$\therefore$ a, b $\in$ $R_1 \cap R_2 \Rightarrow$ a – b $\in$ $R_1 \cap R_2$, a * b $\in$ $R_1 \cap R_2$

$\therefore$ $\boxed{R_1 \cap R_2 \text{ is also a subring of R}}$ *[$\because$ Ex – 14]*

RING HOMOMORPHISM

Homomorphism of Rings : **IT : W-11**

(1) A mapping f from a ring, R into a ring R' is said to be a homomorphism, if f (a + b) = f (a) + f (b) and f (ab) = f (a). f (b), $\forall$a, b $\in$R.

(2) A homomorphism f : R $\rightarrow$ R' is said to be an isomorphism, if is one-one, i.e. f (a) = f (b) $\Rightarrow$ a = b

(3) Two rings R and R' are said to be isomorphic, if there is an isomorphism of R onto R'.

Kernel of Homorphism in a Ring : If is a homomorphism of rings R into R' then the set {a $\in$ R | f (a) = 0}, where 0 is the zero element of R' is called kernel of f and is denoted by I (f).

Thus I (f) = {a $\in$ R | f (a) = 0}

i.e. a $\in$ I (f) $\Rightarrow$ f (a) = 0

Ex.16. Let R be a ring R' = R and define f (x) = x, $\forall$x $\in$ R. Show that f is an isomorphism. Find I (f).

Soln. Given : f (x) = x, $\forall$x $\in$R. ..... (1)

To prove : f is an isomorphism

We prove that f is a homomorphism and one-one.

Let a, b $\in$ R $\Rightarrow$ a + b $\in$ R, ab $\in$ R

Then by (1),

$\left.\begin{array}{l} f(a) = a \\ f(b) = b \end{array}\right|$ ..... (2)

f (a + b) = a + b ..... (3)

f (ab) = ab ..... (4)

Now (3) $\Rightarrow$ f (a + b) = a + b

= f (a) + f (b) *[$\because$ By (2)]*

(4) $\Rightarrow$ f (ab) = ab

= f (a) . f (b)

$\therefore$ f is a homomorphism.

Let f (a) = f (b)

$\Rightarrow$ a = b *[$\because$ By (2)]*

$\therefore$ f is one – one

$\therefore$ $\boxed{\text{f is an isomorphism}}$ **Ans.**

To find I (f) :

Let 0 be the zero element of ring R'

Then I (f) = {a $\in$R | f (a) = 0}

But f (a) = a

$\therefore$ I (f) = {a$\in$R | a = 0} = {0}

$\therefore$ $\boxed{I(f) = \{0\}}$ **Ans.**

Ex.17. If f : R $\rightarrow$ R' is a ring homomorphism, then (i) f (0) = 0 (ii) f (– a) = – f (a), $\forall$a $\in$R. **IT : W-11**

Soln. Given : f : R $\rightarrow$ R' is a ring homomorphism

To prove : (i) f (0) = 0

(ii) f (– a) = – f (a), $\forall$a $\in$R

Let a $\in$ R $\Rightarrow$ f (a) $\in$R'

Let 0 be the zero element of R, also R'.

Then $f(a) + 0 = f(a)$

$= f(a + 0)$

$= f(a) + f(0)$ *[∵ f is a homomorphism]*

$\therefore f(a) + 0 = f(a) + f(0)$

$\therefore f(0) = 0$ *[∵ By LCL]*

∴ (i) is proved

Let $a \in R$

Now $f(a) + f(-a) = f(a + (-a))$ *[∵ f is a homomorphsim]*

$= f(0)$

$= 0$ *[∵ By (i)]*

$\therefore f(a) + f(-a) = 0$

$\therefore f(-a) = -f(a), \forall a \in R$

∴ (ii) is proved. ....**Ans.**

Ex.18. The homomorphism $f : R \to R'$ is an isomorphism, iff $I(f) = \{0\}$

Soln. If Part :

Given : Homomorphism $f : R \to R'$ is an isomorphism

To prove : $I(f) = (0)$

Let $a \in I(f)$

$\Rightarrow f(a) = 0$, the zero element of R'

$\Rightarrow f(a) = f(0)$ *[∵ Ex-17]*

But f is one-one

$\Rightarrow a = 0$

$\therefore a \in I(f) \Rightarrow a = 0$

$\therefore I(f) = \{0\}$

Only If Part :

Given : (i) $I(f) = \{0\}$ (ii) $f : R \to R'$ is a homomorphism

To prove : Homomophism $f : R \to R'$ is an isomorphism.

Let $a, b \in R$ and suppose $f(a) = f(b)$

$\therefore f(a) + f(-b) = f(b) + f(-b)$

$f(a + (-b)) = f(b + (-b))$ *[∵ f is a homomorphism]*

$f(a + (-b)) = f(0)$

But $f(0) = 0$

$\therefore f(a + (-b)) = 0$

$\Rightarrow a + (-b) \in I(f)$ *[∵ By definition of kernel]*

But $I(f) = \{0\}$

$\therefore a + (-b) = 0$

$\therefore a = b$

$\therefore f(a) = f(b) \Rightarrow a = b$

f is one - one

∴ f is an isomorphism ..... **Ans.**

Ex.19. Let $R_1 = \left\{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \middle| a \in R\right\}$ where R is a ring. Define $f : R_1 \to R$ by $f\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = a, \forall \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in R_1$. Show that f is an isomorphism. **IT : W-07**

Soln. Let $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \in R_1, a, b \in R$

$$\therefore f\left[\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}\right] = f\begin{pmatrix} a+b & 0 \\ 0 & 0 \end{pmatrix} = a + b$$

$$= f\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + f\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}$$

$$\text{Also } f\left[\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}\right] = f\begin{pmatrix} ab & 0 \\ 0 & 0 \end{pmatrix} = a.b$$

$$= f\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} . f\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}$$

∴ f is a homomorphism

Now let $a = b$

$$\Rightarrow f\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = f\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}$$

∴ f is one-one

∴ $\boxed{\text{f is an isomorphism}}$ **Ans.**

Ex.20. Let $S = \{A, B\}$ and P(S) its power set. On P(S) we define operations + & . as follows :

$A + B = \{x \in S / (x \in A \vee x \in B) \wedge (x \notin A \cap B)\}$

$A . B = A \cup B$ for all $A, B \in P(S)$

Show that (P(S), +, .) is a ring. **IT : W-08**

Soln. We prove the following properties :

R_1 : Let $A, B \in P(S)$, then

$A + B = \{x \in S / (x \in A \vee x \in B) \wedge (x \notin A \cap B)\}$

$\therefore A + B \leq S \Rightarrow A + B \in P(S)$

R_2 : Let $A, B, C \in P(S)$, then

$(A + B) + C = \{x \in S \setminus (x \in A \vee x \in B)$

$\wedge (x \notin A \cap B)\} + C$

$= \{y \in S / (y \in A \vee y \in B \in y \vee C)$

$\wedge (y \notin A \cap B \cap C)\}$

Also, $A + (B + C) = A + \{z \in S / (z \in B \vee z \in C)$

$\wedge (z \notin B \cap C)\}$

$= \{y \in S / (y \in B \vee y \in C \vee y \in A)$

$\wedge (y \notin A \cap B \cap C)$

$\therefore (A + B) + C = A + (B + C)$

R_3 : For $A \in P(S), \exists \phi \in P(S)$ such that $A + \phi = A$

Consider, $A + \phi = \{x \in S / (x \in A \vee x \in \phi)$

$\wedge (x \notin A \cap \phi)\}$

$= \{x \in S / (x \in A \vee x \in \phi) \wedge x \notin \phi)\}$

[$\because A \cap \phi = \phi$]

$= \{x \in S / x \in A\}$

$= A$

$\therefore \phi \in P(S)$ is identity in $P(S)$

R_4 : For any $A \in P(S)$, $\exists B \in P(S)$ such that $A + B = \phi$

Consider, $A + B = \{x \in S / (x \in A \vee x \in B)$

$\wedge (x \notin A \cap B)\}$

$= \{x \in S / (x \in A \vee x \in B) \wedge (x \in (A \cap B)^C\}$

$= x \in S / (x \in A \vee x \in B) \wedge (x \in A^C \cup B^C)\}$

$= \{x \in S / (x \in A \vee x \in B) \wedge (x \in A^C \vee x \in B^C)\}$

[$\because$ De Morgan's Law]

$= \{x \in S / (x \in A \wedge x \in A^C) \vee (x \in B \wedge x \in B^C)\}$

$= \{x \in S / (x \in A \wedge A^C) \vee (x \in B \wedge x \in B^C)\}$

$= \{x \in S / (x \in \phi) \vee (x \in \phi)\}$

$= \{x \in S / x \in \phi\} = \phi$

$\exists$ an inverse set for any $A \in P(S)$ which belongs to set

R_5 : Let $A, B \in P(S)$, then

$A + B = \{x \in S / (x \in A \vee x \in B) \wedge (x \notin A \cap B)\}$

$= \{x \in S / x \in B \vee x \in A) \wedge (x \notin B \cap A)\}$

$= B + A$

$\therefore R_1$ to $R_5 \Rightarrow (P(S), +)$ is an abelian group

R_6 : Let $A, B \in P(S)$ then $A . B = A \cup B \in P(S)$

R_7 : Let $A, B, C \in P(S)$

then $A \cdot (B \cdot C) = A \cdot (B \cup C) = A \cdot (B \cup C)$

$= A \cup B \cup C$

and $(A \cdot B) \cdot C = (A \cup B) \cdot C = A \cup B \cup C$ .....

.....(By definition)

$\therefore R_6$ and R_7 shows $(P(S), .)$ is semigroup

R_8 : Let $A, B, C \in P(S)$ and $x \in A$, $y \in B$, $z \in C$

Consider, $A . (B + C) = A \cup (B + C)$

$= \{x \wedge (y \vee z)\ y, z \notin B \cap C\}$

Also, $A \cdot B + A \cdot C = (A \cup B) + (A \cup C)$

$= \{(x \wedge y) \vee (x \wedge z) / x, y, z \notin (A \cup B) \cap (A \cup C)\}$

$= (x \wedge y) \vee (x \wedge z) / x, y, z \notin A \cup (B \cap C)\}$

$= \{x \wedge (y \vee z) / y \cdot z \notin B \cap C\}$

$\therefore$ From R_1 to R_8, $(P(S), +, \cdot)$ is a ring....**Ans.**

Ex.21. Consider the ring $< Z_4, +_4, \times_4 >$

Let $f(x) = 2x^2 + 4x + 4$ be a polynomial defined over Z_4 find the zeros of $f(x)$ over Z_4. Also find additive inverse of $f(x)$. ***IT : S-11***

Soln. To find : a root or zero of $f(x)$ substitute each of the four elements of Z_4 into $f(x)$ to see which elements yield 0

we have

$z_4 = \{0, 1, 2, 3\}$

$f(0) = 4 = 0$

$f(1) = 10 = 2$

$f(2) = 44 = 0$

$f(3) = 502 = 2$

Thus the roots are 0 and 2.

Additive inverse of $f(x)$: By $-a$ in a ring R we mean that element such that

$a +_4 (-a) = (-a) +_4 a = 0$

Hence $-1 = 3$ since $1 +_4 3 = 3 +_4 1 = 0$ in z_4.

Similarly $-2 = 2$

By a^{-1} in a ring R we mean that element such that $a . a^{-1} = a^{-1} . a = 1$.

Here multiplicative inverse does not exist. .....**Ans.**

Ex.22. A finite commutative ring without zero divisors is a field OR Every finite integral domain is a field. ***IT : S-11***

Soln. Let D be a finite commutative ring without zero divisors having n elements $a_1, a_2, \ldots\ldots, a_n$.

In order to prove that D is a field, we must produce an element $1 \in D$ such that $1\,a = a\ \forall a \in D$. Also we should show that for every element $a \neq 0 \in D$ there exists an element $b \in D$ such that $ba = 1$.

Let $a \neq 0 \in D$. Consider the n products $aa_1, aa_2, \ldots\ldots, aa_n$, these are elements of D. Also they are all distinct i for suppose that $a a_i = a a_j$ for $i \neq j$.

Then $a(a_i - a_j) = 0$ .....(1)

Since D is without zero divisors and $a \neq 0$, therefore (1) implies $a_i - a_j = 0 \Rightarrow a_i = a_j$, contradicting $i \neq j$.

$\therefore$ $aa_1, aa_2 .....aa_n$ are all the n distinct elements of D placed in some order. So one of these elements will be equal to a. Thus there exists an element, say $c \in D$ such that $ac = a = ca$ *[$\because$ D is commutative]*

We shall show that this element c is the multiplicative identity of D. Let y be any element of D. Then from the above discussion for some $x \in D$, we shal have $ax = y = xa$.

Now $cy = c(ax)$ *[$\because ax = y$]*

$= (ca)x$

$= ax$ *[$\because ca = a$]*

$= y$ *[$\because ax = y$]*

$= yc$ *[$\because$ D is commutative]*

Thus $cy = y = yc$, $\forall$ $y \in D$. Therefore c is the unit element of the ring D and let us denote it by 1.

Now $1 \in D$. Therefore from the above discussion one of the n products $aa_1, aa_2 .....aa_n$ will be equal to 1. Thus, there exists an element, say $b \in D$ such that $ab = 1 = ba$.

$\therefore$ b is the multiplicative inverse of the non-zero element $a \in D$. Thus every non-zero element of D is inversible.

Hence D is a field. .....**Ans.**

Ex.23. Let C be the set of the ordered pairs (a, b) of real numbers.

Define addition and multiplication in CT by the equations.

$(a, b) + (c, d) = (a + c, b + d)$

$(a, b)(c, d) = (ac - bd, bc + ad)$

Prove that CT is a field. **CS/CT/CE : W-11**

Soln. We see that C is closed with respect to the two composition since a+c, b+d, ac – bd, bc + ad are all real numbers. Now let (a, b), (c, d) (e, f) be any elements of C. Then we make the following observations.

Associativity of addition :

We have, $[(a, b) + (c, d)] + (e, f)$

$= (a + c, b + d) + (e, f)$

$= ([a + c] + e, [b + d] + f)$

$= (a + [c + e], b + [d + f])$

$= (a, b) + (c + e, d + f)$

$= (a, b) + [(c, d) + (e, f)]$

Commutativity of addition :

We have $(a, b) + (c, d)$

$= (a + c, b + d)$

$= (c + a, d + b) = (c, d) + (a, b)$

Existence of additive identity:

We have $(0, 0) \in C$

Also $(0,0) + (a, b) = (0 + a, 0 + b) = (a, b)$

$\therefore$ (0, 0) is the additive identity.

Existence of additive inverse :

We have if $(a, b) \in C$, then $(-a, -b) \in C$.

We have $(-a, -b) + (a, b) = (-a + a, -b + b) = (0, 0)$

$\therefore$ $(-a, -b)$ is the additive inverse of (a, b).

Associativity of multiplication :

We have $[(a,b)(c,d)](e,f) = (ac - bd, bc + ad)(e, f)$

$= ([ac - bd]e - [bc + ad]f, [bc + ad]e + [ac - bd]f)$

$= (a[ce - df] - b[de + cf], b[ce - df] + a[de + cf])$

$= (a, b)(ce - df, de + cf) = (a, b)[(c,d)(e,f)]$

Distributive laws :

We have

$(a,b)[(c,d) + (e, f)] = (a, b)(c + e, d + f)$

$= (a[c + e] - b[d + f], b[c + e] + a[d + f])$

$= ([ac - bd] + [ae - bf], [bc + ad] + [be + af])$

$= (ac - bd, bc + ad) + (ae - bf, be + af)$

$= (a,b)(c, d) + (a, b)(e, f)$

Similarly we can show that other distributive law also holds good. Therefore C is a ring with respect to the two composition. The ordered pair (0, 0) is the zero **element of the ring.**

Commutativity of multiplication :

We have $(a, b)(c,d) = (ac - bd, bc + ad)$

$= (ca - db, da + cb)$

$= (c,d)(a,b)$

Existence of multiplicative identity :

We have $(1,0) \in C$.

If $(a,b) \in C$, then $(a, b)(1, 0) = (a1 - b0, b1 + a0)$

$= (a + b) = (1, 0)(a, b)$

Therefore (1,0) is the unity element of the ring.

Existence of multiplicative inverse of each non-zero element of C :

Let (a,b) be any non-zero element of C. Then a and b are not both simultaneously zero. If (c, d) is the multiplicative inverse of (a, b) then we should have $(a, b)(c, d) = (1, 0)$ or $(ac - bd, bc + ad) = (1, 0)$ or by the definition of the equality of two ordered pairs, we have $ac - bd = 1$ and $bc + ad = 0$

Solving these equations for c, d we get

$$c = \frac{a}{a^2+b^2}, \quad d = \frac{-b}{a^2+b^2}$$

Now $a \neq 0$ or $b \neq 0 \Rightarrow a^2+b^2 \neq 0$.

Therefore either c and d or both non-zero real numbers.

Thus, $\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}$ is the multiplicative inverse of (a, b). Hence C is a field. ..... **Ans.**

Ex.24. R^+ be the group of non-zero real number under multiplication and r be a positive integer. Show that $f(x) = x^r$ is a homomorphsim from R^+ to R^+. **IT : W-12**

Soln. We know that the set of non-zero real numbers under multiplication forms a ring.

$\therefore R^+$ is a ring.

Given : $f(x) = x^r$

To prove : $f: R^+ \to R^+$ is a homomorphism

We prove that $f(x+y) = f(x) + f(y)$ and

$f(xy) = f(x).f(y), \forall x, y \in R^+$.

Let $x, y \in R^+$

$\Rightarrow x^r, y^r \in R^+$

Also $x+y, x^r+y^r, xy, x^r y^r \in R^+$

Since $f(x) = x^r \Rightarrow f(y) = y^r$

Now $f(x+y) = x^r + y^r = f(x) + f(y)$

Also $f(xy) = x^r y^r = f(x).f(y)$

$\therefore$ $\boxed{f: R^+ \to R^+ \text{ is a homomorphsim}}$ **Ans.**

Ex.25. Show that the set of integers with the composition $\circ$, * defined by $a \circ b = a + b + 1$ and $a * b = ab + a + b$ is a ring. **IT : W-12**

Soln. Let Z be the set of integers

To prove : $(Z, \circ, *)$ is a ring.

R_1 : $(Z, \circ)$ is an abelian group

G_1 : Let $a, b \in Z$, then $a \circ b = a + b + 1 \in Z$

$\therefore G_1$ is verified.

G_2 : Let $a, b, c \in Z$

Consider $a \circ (b \circ c) = (a \circ b) \circ c$

$\Rightarrow a \circ (b + c + 1) = (a + b + 1) \circ c$

$\Rightarrow a + b + c + 1 + 1 = a + b + 1 + c + 1$

$\Rightarrow a + b + c + 2 = a + b + c + 2$

$\therefore G_2$ is verified

G_3 : There exists $0 \in Z$, such that $a \circ 0 = 0 = 0 \circ a$, $\forall a \in Z$

$\therefore$ 0 is the zero element.

G_4 : For every $a \in Z$, there always exists $-a \in Z$, s.t. $a \circ (-a) = 0 = 0 (-a) \circ a$

$\therefore G_4$ is verified.

G_5 : Let $a, b \in Z$

Consider $a \circ b = b \circ a$

i.e. $a + b + 1 = b + a + 1$

i.e. $a + b + 1 = a + b + 1$

$\therefore G_5$ is verified

$\therefore R_1$ is verified.

R_2 : The operation * is binary and associative.

G_1 : Let $a, b \in Z \Rightarrow ab, a + b \in Z$

Then $a * b = ab + a + b \in Z$

G_2 : Let $a, b, c \in Z \Rightarrow ab, bc, a + b, b + c \in Z$

Consider $a * (b * c) = (a * b) * c$

$a * (bc + b + c) = (ab + a + b) * c$

$abc + ab + ac + a + bc + b + c$

$= abc + ac + bc + ab + a + b + c$

$\therefore G_2$ is verified.

$\therefore R_2$ is verified.

R_3 : The operation * is distributive over +.

We know that if $a, b, c \in Z$, then

$a * (b + c) = (a * b) + (a * c)$ and

$(b + c) * a = (b * a) + (c * a)$

$\therefore R_3$ is verified.

$\therefore$ $\boxed{(Z, \circ, *) \text{ is a ring}}$ **Ans.**

LATTICES

Posets and Lattices : Lattice is a special kind of partially ordered set. Lattices has a wide applications in studying the relations of two operations. Partial order relation has a great application in programme evaluation and review techniques and in critical path method. These programmes are important in scheduling the activities of the individuals needed to complete the large projects.

Basic Concepts :

(1) Partially Ordered Set : **CS/CT/CE : W-10**

A relations R on a set A is called partial ordering if it is reflexive, antisymmetirc and transitive. That is

(i) $A R a. \forall a \in A$ (reflexive)

(ii) $a R b$ and $b R a, \forall a, b \in A \Rightarrow a = b$ (antisymmetic)

(iii) $a R b$ and $b R c \Rightarrow a R c, \forall a, b, c \in A$ (transitive)

The set A together with partial order relation R is called a partial order set or POSET.

(2) **Totally Ordered Set or Chain :** Consider the partially ordered set $(A, \leq)$ and $a, b \in A$ such that $a < b$ or $b < a$. Then the relation $\leq$ is called totally ordered relation and $(A, \leq)$ is called totally ordered set or chain.

(3) **Upper Bound and Lower Bound :** Let B be a subset of poset $(A, \leq)$. An element $c \in A$ is called an upper bound of B if $a \leq c, \forall a \in B$.

An element $d \in A$ is called a lower bound of B if $d \leq a, \forall a \in B$.

(4) **Least Upper Bound (lub) and Greatest Lower Bound (glb) :** An element $a \in A$ is called least upper bound of B if a is an upper bound of B and $a < a'$, whenever a' is an upper bound of B. Thus a = lub (B).

An element $a \in A$ is called greatest lower bound of B if a is a lower bound of B and $a' < a$, whenever a' is a lower bound of B. Thus a = glb (B).

Ex.26. Consider $A = \{a \in R / 1 < a < 2\}$ with $\leq$ as the partial order. Find glb and lub of A.

Soln. 1 is the glb and 2 is the lub of A. ..... **Ans.**

Definition of Lattice : ***CS/CT/CE : S-08***

Let $(L, \leq)$ be a POSET. If every pair of elements of L has greatest lower bound and least upper bound then the partially ordered set $(L, \leq)$ is called a lattice. i.e. for every $a, b \in L$ if lub (a, b) and glb (a, b) exists, then $(L, \leq)$ is a lattice.

$a \wedge b = a * b = \text{glb}(a, b)$ read as meet of a and b.

and $a \vee b = a \oplus b = \text{lub}(a, b)$ read as join of a and b.

It is to be noted that every chain is a lattice since any two elements a, b of chain are comparable.

$a \vee b = \text{lub}(a, b) = b$

$a \wedge b = \text{glb}(a, b) = a$

Thus lattice is an algebraic structure with two operations meet and join.

Law of Algebra in Lattice : Let $(L, \leq, *, \oplus)$ be a lattice. Then it satisfies the following laws.

(i) **Idempotent Law :** $a \oplus a = a = a * a$

(ii) **Commutative Laws :** $a \oplus b = b \oplus a$ and $a * b = b * a$

(iii) **Associative Laws :** $a \oplus (b \oplus c) = (a \oplus b) \oplus c$

$a * (b * c) = (a * b) * c$

(iv) **Absorption Law :** $a \oplus (a * b) = a = a * (a \oplus b)$

Principle of Duality : In a lattice the operations * and $\oplus$ and the relations $\leq$ and $\geq$ remains true if * is replaced by $\oplus$ and relation $\leq$ by $\geq$.

For example : Consider $a * (b \oplus C) \leq (a * b) \oplus (a * c)$

Then its dual is $a \oplus (b * c) \geq (a \oplus b) * (a \oplus c)$

Note :

(1) Let L be a lattice. Then $a \wedge b = a$ if and only if $a \vee b = b, \forall a, b \in L$.

(2) Let $(L, \leq)$ be a lattice with binary operations * and $\oplus$, then for any $a, b, c \in L$

$b < c \Rightarrow a * b \leq a * c$

$a \oplus b \leq a \oplus c$.

(3) Let $(L, \leq)$ be a lattice. Then for $a, b\ c \in L$, we have

(i) $a \leq b$ and $a \leq c \Rightarrow a \leq b \oplus c$

(ii) $a \leq b$ and $a \leq c \Rightarrow a \leq b * c$

(iii) $a \geq b$ and $a \geq c \Rightarrow a \geq b * c$

(iv) $a \geq b$ and $a \geq c \Rightarrow a \geq b \oplus c$

Type of Lattices :

(1) **Complete Lattice :** A lattice is said to be complete, if each of its nonempty subset has a least upper bound and greatest lower bound.

Thus every finite lattice is complete because every subset here is finite.

(2) **Bounded Lattice :** A lattice is said to be bounded, if it is complete and contains 0 and 1 as least & greatest element resp.

In a bounded lattice, the bounds 0 and 1 clearly satisfy

$0 \wedge a = 0 = a \wedge 0$ for all $a \in L$

$0 \vee a = a = a \vee 0$ for all $a \in L$

$1 \wedge a = a = a \wedge 1$ for all $a \in L$

$1 \vee a = 1 = a \vee 1$ for all $a \in L$

(3) **Complemented Lattice :** In a bounded lattice $(L, \vee, \wedge, 0, 1)$; an element $b \in L$ is called a complement of $a \in L$, if

$a \wedge b = 0$ and $a \vee b = 1$

where 0 and 1 are lower and upper bound of L respectively.

A lattice L is called complemented Lattice if it is bounded and every element in L has a complement.

Note :

(i) Any element $a \in L$ may or may not have a complement.

(ii) Any element $a \in L$ may have more than one complement in L.

(iii) In any bounded lattice, the bounds 0 and 1 are unique complements of each other i.e. $0 \vee 1 = 1$ and $0 \wedge 1 = 0$.

(4) Distributive Lattice : **CS/CT/CE : W-10,12**

A lattice $(L, \vee, \wedge)$ is called a distributive lattice, if for any $a, b, c \in L$.

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

For example : $L = (P(S), \cup, \cap)$ is a distributive lattice since the laws of $\cup$ over $\cap$ and $\cup$ over $\cup$ hold.

(5) Modular Lattice : For all $a, b, c \in L$, if $a \vee (b \wedge c) = (a \vee b) \wedge c$ with $a < b$ and $a < c$, then the lattice $(L, \leq)$ is called a modular lattice.

Every distributive lattice is modular.

IMPORTATN FORMULAE :

Laws of Algebra in lattice :

(1) Idempotent law :

$a \oplus a = a = a * a$

(2) Commutative laws :

$a \oplus b = b \oplus a$ and $a * b = b * a$

(3) Associative laws :

$a \oplus (b \oplus c) = (a \oplus b) \oplus c$

$a * (b * c) = (a * b) * c$

(4) Absorption law :

$a \oplus (a * b) = a = a * (a \oplus b)$

(5) (i) $a \leq b$ and $a \leq c \Rightarrow a \leq b \oplus c$

(ii) $a \leq b$ and $a \leq c \Rightarrow a \leq b * c$

(iii) $a \geq b$ and $a \geq c \Rightarrow a \geq b * c$

(iv) $a \geq b$ and $a \geq c \Rightarrow a \geq b \oplus c$

(6) In a bounded Lattice :

$0 \wedge a = 0 = a \wedge 0$ for all $a \in L$

$0 \vee a = a = a \vee 0$ for all $a \in L$

$1 \wedge a = a = a \wedge 1$ for all $a \in L$

$1 \vee a = 1 = a \vee 1$ for all $a \in L$

(7) In Distributive lattice :

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

Ex.27. In a distributive lattice $(L, \leq)$, $a \wedge b = a \wedge c$ and $a \vee b = a \vee c \Rightarrow b = c$

Soln. We have

$b = b \wedge (a \vee b)$ .....(by absorption law)

$= b \wedge (a \vee c)$ $[\because a \vee b = a \vee c]$

$= (b \wedge a) \vee (b \wedge c)$ .....(by distributive law)

$= (a \wedge c) \vee (b \wedge c)$ $[\because a \wedge b = a \wedge c]$

$= c \wedge (a \vee c)$ .....(distributive law)

$= c$ ..... (absorption law)

$\therefore$ $\boxed{b = c}$ **Ans.**

Ex.28. In a distributive lattice if the complement exists then it is unique.

Soln. Let b and c be two complements of a such that $a \vee b = 1$ and $a \wedge b = 0$

$a \vee c = 1$ and $a \wedge c = 0$

We have

$b = b \wedge 1$

$= b \wedge (a \vee c)$

$= (b \wedge a) \vee (b \wedge c)$

$= 0 \vee (b \wedge c)$

$= (a \wedge c) \vee (b \wedge c)$

$= (a \vee b) \wedge c$

$= 1 \wedge c$

$= c$

$\therefore$ $\boxed{\text{Element 'a' has a unique complement}}$ **Ans.**

Ex.29. Show that the Lattice (S_n, D), $n = 30$ is a complemented lattice. **CS/CT/CE : S-08**

Soln. For $n = 30$, $S_n = \{1, 2, 3, 5, 6, 10, 15, 30\}$

Its lattice is

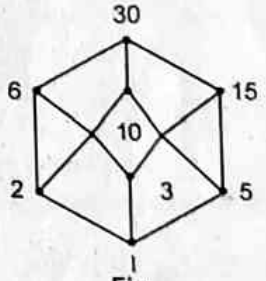


Fig.

Here 1 is the lower bound, 30 is an upper bound. Also

$1 * 30 = 1, 1 \oplus 30 = 30$

$\Rightarrow$ 1 and 30 are complements of each other.

For 2 and 15;

$2 * 15$ glb $(2, 15) = \gcd(2, 15) = \text{lcm}(2, 15) = 30$

$\therefore$ 2 and 15 are complements of each other for 3 and 10,

$2 * 10 = \text{lub}(3, 10) = \text{lcm}(3, 10) = 30$

$\Rightarrow$ 3 and 10 are complements of each other

For 5 and 6

$5 * 6 = \text{glb}(5, 6) = \gcd(5, 6) = 1$

$5 \oplus 6 = \text{lub}(5, 6) = \text{lcm}(5, 6) = 30$

$\Rightarrow$ 5 and 6 are complements of each other.

Thus every element in the lattice (S_{30}, D) has a complement. So (S_{30}, D) is a complemented lattice.

Since every complemented lattice is distributive so is (S_{30}, D). ....**Ans.**

Ex.30. Is the lattice (S_{45}, D) complemented.

CS/CT/CE : S-09,12, W-11

Soln. $S_{45} = \{1, 3, 5, 9, 15, 45\}$

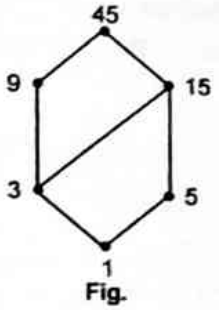


Fig.

Here 1 is the lower bound and 45 is the upper bound.

Also $1 * 45$ 1 and $1 \oplus 45 = 30$.

$\Rightarrow$ 1 and 45 are the complements of each other.

For 3 and 15; $3 * 15 = \text{gcd}(3, 15) = \text{glb}(3, 15) = 3 \neq 1$

and $3 \oplus 15 = \text{lcm}(3, 15) = \text{lub}(3, 15) = 15 \neq 45$

$\Rightarrow$ 3 and 15 are not complements of each other

Thus we can say that every element of the lattice (S_{45}, D) does not have complement. Therefore it is not a complemented lattice and hence it is not a distributive lattice.

SUBLATTICE

Let $(L, *, \oplus)$ be a lattice. If $S \subseteq L$ and S follows the closed property w.r.t. given binary operations $*$ and $\oplus$, then $(S, *, \oplus)$ is called sublattice of the given lattice $(L, *, \oplus)$.

For Example :

(1) Consider the set $\{x \in L / a \leq x \leq b\}$ .....(1)

Let $(L, *, \oplus)$ be a lattice and $a, b \in L$ with $a \leq b$. Then (1) is a sublattice of L.

(2) The lattice (S_n, D) is a sublattice of lattice (I_+, D).

Ex.31. Show that (S_6, D) is a sublattice of (S_{24}, D).

Soln. $S_6 = \{1, 2, 3, 6\}$ and $S_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$, hence $S_6 \subseteq S_{24}$

Moreover (S_6, D) and (S_{24}, D) are Lattices represented by

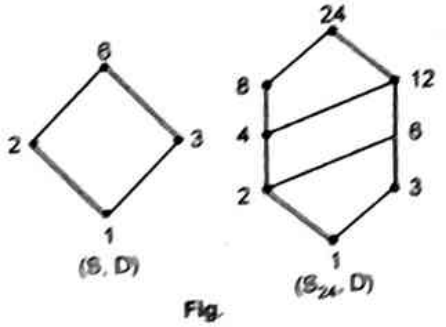


Fig.

$\therefore$ (S_6, D) is a sublattice of (S_{24}, D).

Product of Lattices :

Let L and M be lattices. The set of ordered pairs $\{(x, y) / x \in L, y \in M\}$ with operations $*$ and $\oplus$ defined by

$(x_1, y_1) \oplus (x_2, y_2) = (x_1 \oplus x_2, y_1 \oplus y_2)$

$(x_1, y_1) * (x_2, y_2) = (x_1 * x_2, y_1 * y_2)$,

is known as direct product of L and M, denoted by $L \times M$ w.r.t. binary operations $\oplus$ and $*$.

Note : (1) Product of two lattices is a lattice.

(2) $L^2 = L \times L$, $L^3 = L \times L \times L$, $L^4 = L \times L \times L \times L$.

Ex.32. Draw the lattice $(L, \leq_1)$, $(L^2, \leq_2)$, $(L^3, \leq_3)$ if $L = \{0, 1\}$

CS/CT/CE : S-09,12,13, W-11

Soln. Given : $L = \{0, 1\}$

Fig.

$L^2 = L \times L = \{0, 1\} = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$

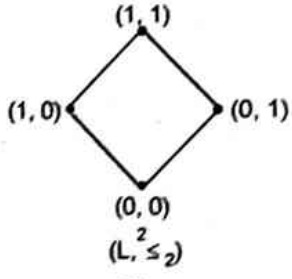


Fig.

$L^3 = L \times L \times L = \{(0, 0), (0, 1), (1, 0), (1, 1)\} \times \{0, 1\}$

$= \{(0, 0, 0), (0, 0, 1), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}$

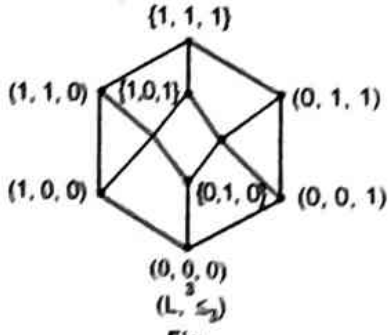


Fig.

Homomorphism of Lattice : Let L and M be a Lattices with the binary operations $*$ and $\oplus$, Then the mapping $f : L \to M$ is called a homomorphism if

$f(x * y) = f(x) * f(y)$ and

$f(x \oplus y) = f(x) \oplus f(y)$

Note :

1) If the homomorphism f is one - one and onto, then f is an isomorphism.

2) If f is an isomorphism from L to L, then f is called an automorphism.

Ex.33. Let (L_1, D_6) and $L_2 = (P(S), S)$ be two lattices, where $D_6 = \{1, 2, 3, 6\}$ and $S = \{a, b\}$. Then show that L_1 and L_2 are isomorphic.

Soln. The lattices L_1 and L_2 with Hasse diagram representation are given below.

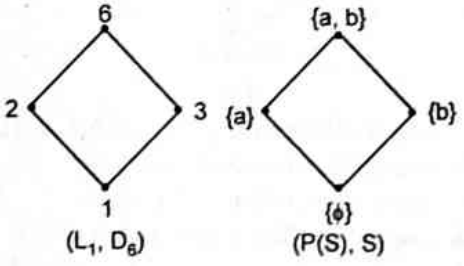


Fig.

Let the mapping $f : L_1 \to L_2$ be defined by

$f(1) = \phi$, $f(2) = \{a\}$, $f(3) = \{b\}$, $f(6) = \{a, b\}$

Since $f(2 \wedge 3) = 1 = \{a\} \wedge \{b\} = \phi$

$f(2 \vee 3) = 6 = \{a\} \vee \{b\} = \{a, b\}$

$\Rightarrow$ f is a homomorphism and also f is one-one

Also $\forall\, y \in L_2$, $\exists\, x \in L_1$ such that $f(x) = y$

$\therefore$ f is onto

$\Rightarrow$ f is one-one, onto and homomorphism.

$\therefore$ f is an isomorphism

$\therefore$ L_1 and L_2 are isomorphic lattices

Note : (1) Isomorphic lattices are identical in look. Each lattice is a mirror image of the other with the elements simply relabelled.

Ex.34. Let S and T be two finite sets such that $S = \{a, b, c\}$ and $T = \{1, 2, 3\}$. Then show that $(P(S), \subseteq)$ and $(P(T), \subseteq)$ are isomorphic.

Soln. Let $S = \{a, b, c\}$ and $T = \{1, 2, 3\}$

$P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$

$P(T) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$

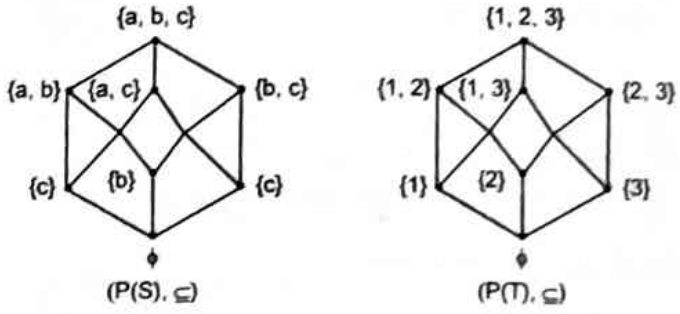


Fig.

The mapping $f : S \to T$ is give by

$f\{1\} = \{a\}$, $f\{2\} = \{b\}$, $f\{3\} = \{c\}$

$f\{1, 2\} = \{a, b\}$, $f\{1, 3\} = \{a, c\}$, $f\{2, 3\} = \{b, c\}$,

$f\{1, 2, 3\} = \{a, b, c\}$

Hence it is clear that f is homomorphism, one - one and onto and hence f is an isomorphism.

$\Rightarrow$ The lattices $(P(S), \subseteq)$ and $(P(T), \subseteq)$ are isomorphic. **.....Ans.**

Ex.35. Prove that every chain is a distributive lattice.

Soln. Let $(L, \wedge, \vee)$ be a chain, and $a, b, c \in L$

Since L is a chain, either $a \le b$ or $b \le a$. If $a \le b$, then $a \vee b = b$ and $a \wedge b = a$

$\therefore$ $a \wedge b$ and $a \vee b$ exists in L for any two elements a, $b \in L$. Suppose $a \le b$

Case (i) : $b \le c \Rightarrow b \vee c = c$ and $b \wedge c = b$

$\therefore$ $a \wedge (b \vee c) = a \wedge c = a$ $\quad [\because a \le b \le c \Rightarrow a \le c,$

$\therefore a \wedge c = a$ and $a \vee c = c]$ ..... (1)

and $(a \wedge b) = (a \wedge c) = a \vee a = a$ ..... (2)

$\therefore$ Form (1) and (2), we get

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

Case (ii) : $c \le b \Rightarrow c \vee b = b$ and $c \wedge b = c$

Let $a \le c$, then $a \le c \le b$

Now $a \wedge (b \vee c) = a \wedge b = a$ and ..... (3)

$(a \wedge b) \vee (a \wedge c) = a \vee a = a$ ..... (4)

$\therefore$ From (3) and (4), we get

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

Similarly, if $b \le a$, then $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

$\therefore$ Distributive property holds in L

$\therefore$ L is a distributive lattice. **.....Ans.**

Ex.36. Prove that Every Distributive lattice is modular.

Soln. Let $(L, \le)$ be a distributive lattice and

$a, b, c \in L$ such that $a \le c$

$\therefore$ $a \le c \Rightarrow a \vee c = c$

Now as L is distributive lattice

$\therefore \forall a, b, c \in L$

$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

$= (a \vee b) \wedge c$

$\therefore$ L is modular lattice.

Note :

(i) Every chain is modular.

(ii) Converse of above theorem is not true.

Ex.37. Consider the lattice

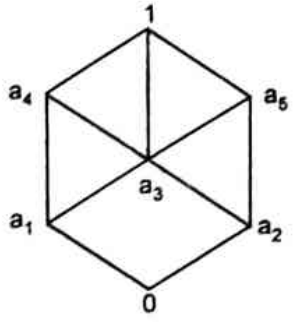


Fig.

(a) Which of the following are the sublattices of $L_1 = \{0, a_1, a_2, 1\}$, $L_2 = \{0, a_1, a_5, 1\}$, $L_3 = \{a_1, a_2, a_4, 1\}$,

(b) Find the complements of a_1, a_2, a_3 if they exist.

(c) Is L a complemented lattice.

(d) Is L a distributive lattice.

Soln.

(a) By the definition of sublattice, a lattice is said to be a sublattice, if it is closed under $\wedge$ and $\vee$

(1) L_1 is not sublattice as

$a_1 \vee a_2 = a_3 \notin L_1$

(2) L_2 is a sublattice as

$a_1 \vee a_5 = 1$ and $a_1 \wedge a_5 = 0$ and $1, 0 \in L_2$

(3) L_3 is a sublattice as

$a_1 \vee a_3 = 1 \in L_3$ and $a_1 \wedge a_3 = a_4 \in L_3$

$a_1 \vee a_4 = 1 \in L_3$ and $a_1 \wedge a_4 = a_3 \in L$

$a_3 \vee a_4 = 1 \in L_3$ and $a_1 \wedge a_4 = a_3 \in L$

L_3 is closed w.r.t. $\wedge$ and $\vee$

(b) Here a_1, a_2, a_5, a_4 all have a complements i.e.

$a_1 \wedge a_5 = 0$ and $a_1 \vee a_5 = 1 \Rightarrow a_1$ and a_5 are complements of each other

$a_2 \wedge a_4 = 0$ and $a_2 \vee a_4 = 1 \Rightarrow a_2$ and a_4 are complements of each other??

But a_3 has no complement

(c) Since a_1, a_2, a_4, a_5 all have the complements except a_3 and hence L is not a complemented lattice.

(d) As L is not complemented lattice therefore it is not a distributive lattice. .....**Ans.**

Ex.38. Show that the lattice $(L^3, \leq_3)$ of tuples of 0 and 1 is complemented.

Soln. Since $L = \{0, 1\}$

$L \times L \times L = L^3 = \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 1, 0), (1, 0, 1), (1, 1, 1)\}$

Hasse diagram for L^3 is

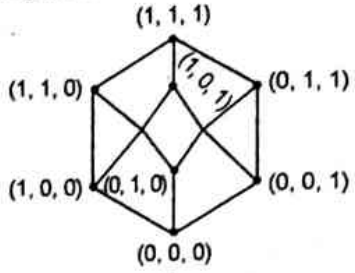


Fig.

The bounds of lattice are (0, 0, 0) and (1, 1, 1)

The complement of (1, 0, 0) is (0, 1, 1)

The complement of (0, 1, 0) is (1, 0, 1)

The complement of (1, 1, 0) is (0, 0, 1).

$\therefore (L^3, \leq)$ is a complemented lattice. .....**Ans.**

Note : In general the lattice $(L^n, \leq_n)$ is itself a complemented lattice, in which complement can be obtained by interchanging 0 and 1.

Ex.39. Show that :

(i) (S_{30}, D) is a complemented lattice

(ii) In (S_{20}, D), 2 and 10 have no complements.

(iii) Draw the Hasse diagram of the lattice (S_{20}, D) and (S_{30}, D).

Soln. (i) $S_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$

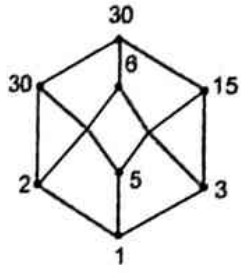


Fig.

We now find the complement of each element

Here lower and upper bounds in the latteice (S_{30}, D) are 1 and 30 repectively.

Moreover 1 and 30 are complements of each other.

For 2 and 15 : lub (2, 15) = lcm (2, 15) = 30
glb (2, 15) = gcd (2, 15) = 1
∴ 2 and 15 are complements of each other.
For 3 and 10 : lub(3, 10) = lcm (3, 10) = 30
glb (3, 10) = gcd (3, 10) = 30
∴ 3 and 10 are complements of each other.
For 5 and 6 : lub (5, 6) = lcm (5, 6) = 30
glb (5, 6) = gcd (5, 6) = 1
∴ 5 and 6 are complements of each other.
∴ Each element of lattice (S_{30}, D) has a complement
∴ (S_{30}, D) is a complemented lattice.

(ii) and (iii) $S_{20} = \{1, 2, 4, 5, 10, 20\}$

The Hasse diagram for the lattice (S_{20}, D) is

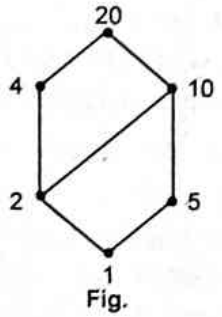


Fig.

Here lower and upper bounds are 1 and 20 respectively.
We now find the complement of each element.
For 1 and 20 : lub (1, 20) = lcm (1, 20) = 20
glb (1, 20) = gcd(1, 20) = 1
∴ 4 and 5 are complements of each other.
For 2 and 10 : lub (2, 10) = lcm (2, 10) = 20
glb(2, 10) = gcd(2, 10) = 2 ≠ 1
∴ 2 and 10 are not complements of each toher.
Since two elements 2 and 10 do not have complements, hence (S_{20}, D) is not a complemented lattice. .....Ans.

Ex.40. Consider the POSETS L_1 and L_2 shown in the following figure (a) and (b) respectively. Draw the Hasse diagram for the lattice $(L_1 \times L_2, \leq)$.

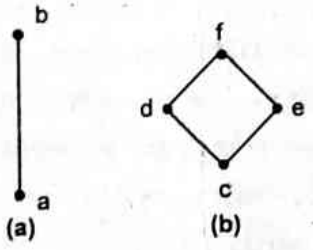
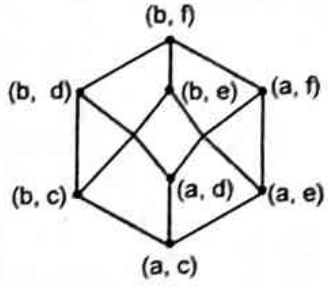


Fig.

Soln. Given : $L_1 = \{a, b\}$ and $L_2 = (c, d, e, f)$

Then $L_1 \times L_2$ = {(a, c), (a, d), (a, e), (a, f), (b, c), (b, d), (b, e), (b, f)}



$(L_1 \times L_2, \leq)$
Fig.

Ex.41. Show that there are 5 partitions of a set of three elements. Draw the diagram of the corresponding lattice. **CS/CT/CE : S-06**

Soln. Let S = {a, b, c} then let π(S) be the set of all partitions of S.

$\pi(S) = \{\pi_1, \pi_2, \pi_3, \pi_4, \pi_5\}$ where

$\pi_1 = \{\overline{a, b, c}\}$

$\pi_2 = \{\overline{a, b}, c\}$

$\pi_3 = \{\overline{a, c}, \overline{b}\}$

$\pi_4 = \{\overline{a}, \overline{b, c}\}$

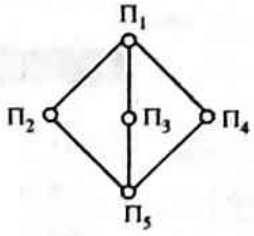
$\pi_5 = \{\overline{a}, \overline{b}, \overline{c}\}$

We define partial ordering relation ≤ on π(S) by $\pi i \leq \pi j$ iff every block of πi is a subset of some block πj.

We define product of partitions

πi, πj = G.L.B. {πi, πj}

= Set of intersections of every element of πi with every element of πj.



$\pi_1.\pi_2 = \{\overline{a, b}, \overline{c}\}$

$\pi_1.\pi_3 = \{\overline{ac}, \overline{b}\}$

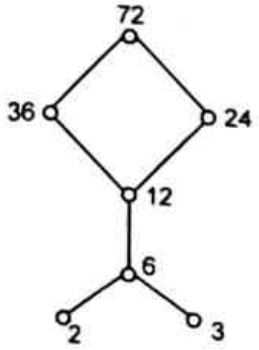
Ex.42. Is the poset A = (2, 3, 6, 12, 36, 72} under the relation of divisibility lattice? **CS/CT/CE : S-07**

Soln. In the poset A, for any 2, 3 ∈ A

G. L. B $\{2, 3\} = 2.3 = 2 \wedge 3 = 1$

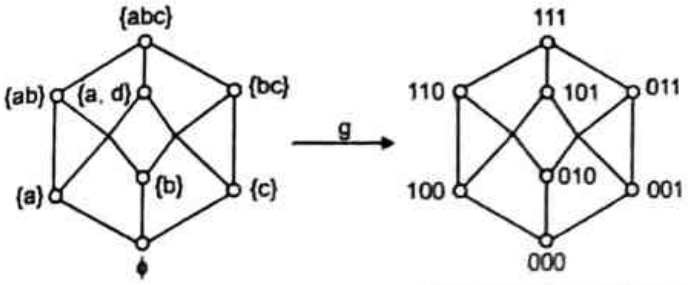
And L.C.M. $\{2, 3\} = 6$

g.c.d. is not given in set A



$\therefore$ $\boxed{\text{Poset A is not a lattice of divisibility.}}$ **Ans.**

Ex.43. Show that the lattice $\langle P(s), \leq \rangle$ is isomorphic to $\langle L^3, \leq \rangle$ where $S = \{a, b, c\}$, $L = \{0, 1\}$



CS/CT/CE : W-07,09

Soln. The mapping g defined as in above figure

e.g. $g(\phi) = 000$ is clearly

$g(a) = 100$ one-one onto

$g(b) = 010$

$\therefore$ $\boxed{\text{The two lattices are isomorphic.}}$ **Ans.**

Ex.44. If $(L, \wedge, V)$ is complemented distributive lattice, then prove that

$(a \vee b)' = a' \wedge b'$ and

$(a \wedge b)' = a' \vee b'$. **IT : S-10, CS/CT/CE : S-11**

Soln. Consider

$(a \vee b).(a' \wedge b') = (a' \wedge b').(a \vee b)$ *[$\because$ commutative]*

$= [(a' \wedge b').a] \vee [(a' \wedge b').b]$ *[$\because$ Distributive]*

$= [(b' \wedge a').a] \vee [a' \wedge (b'.b')]$

[$\because$ commutative]

$= [b' \wedge (a'.a)] \vee [a' \wedge (b'.b)]$

[$\because$ Associative]

$= (b' \wedge 0)\ (a' \wedge 0)$

$= 0 \vee 0$

$= 0$ .....(Indempotent)

$\Rightarrow (a \vee b) \cdot (a' \wedge b') = 0$

$\Rightarrow (a \vee b)$ is complement of $(a' \wedge b')$

$\therefore$ $\boxed{(a \vee b)' = a' \wedge b'}$ **Ans.**

The other part is obvious from principle of duality (i.e. replacing $\vee$ by $\wedge$ and $\wedge$ by $\vee$)

i.e. we get $(a \wedge b)' = a' \vee b'$ (by duality)

Ex.45.(a) In any lattice $(L, \wedge, \vee)$ the following statements are equivalent :

(i) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge v), \forall\ a, b, c \in L$

(ii) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c), \forall\ a, b, c, \in L$.

CS/CT/CE : W-10

Soln. (i) Let a, b, c, $\in$ L,

then $a \wedge (b \vee c) = \min(a, b \vee c) = \min(a, \max(b, c))$

$= \max(\min(a, b), \min(a, c))$

$\therefore$ $\boxed{a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)}$

(ii) Let a, b, c $\in$ L

then $a \vee (b \wedge c) = \max(a, b \wedge c)$

$= \max(a, \min(b, c))$

$= \min(\max(a, b), \max(a, c))$

$\therefore$ $\boxed{a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)}$

Ex.45.(b) If $(L_1, \leq_1)$ and $(L_2, \leq_2)$ are lattices, then show that $(L_1 \times L_2 \leq)$ is also a lattice.

CS/CT/CE : W-06,10,12, S-10

Soln. Let $(L_1, \leq_1)$ and $(L_2, \leq_2)$ be any two lattices

Then $L_1 \times L_2 = \{(x, y) \mid x \in L_1, y \in L_2\}$ is a poset under the relation $\leq$ defined by

$(x_1, y_1) \leq (x_2, y_2)$ if $x_1 \leq x_2$ in L_1 and $y_1 \leq y_2$ in L_2.

Now we prove that

$(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \vee y_2)$ and

$(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2, y_1 \wedge y_2)$

Let $(x_1, y_1), (x_2, y_2) \in L_1 \times L_2$

Then $x_1, x_2 \in L_1$ and $y_1, y_2 \in L_2$

Since L_1 and L_2 are lattices $\{x_1, x_2\}$ and $\{y_1, y_2\}$ have sup and inf in L_1 and L_2 respectively.

Let $x_1 \vee x_2 = \sup\{x_1, x_2\}$

$y_1 \vee y_2 = \sup\{y_1, y_2\}$

Then $x_1 \vee x_2 \geq x_1$ and x_2 ; $y_1 \vee y_2 \geq y_1$ and y_2

$\Rightarrow$ $(x_1 \vee x_2,\ y_1 \vee y_2) \geq (x_1, y_1)$ and

$(x_1 \vee x_2,\ y_1 \vee y_2) \geq (x_2, y_2)$

Hence $(x_1 \vee x_2,\ y_1 \vee y_2)$ is an upper bound of $\{(x_1, y_1), (x_2, y_2)\}$.

$\therefore$ $(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \vee y_2)$ and

$(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2, y_1 \wedge y_2)$.

$\therefore$ $\boxed{(L_1 \times L_2, \leq) \text{ is a lattice}}$ **Ans.**

BOOLEAN ALGEBRA

Boolean Algebra : Binary algebra is a part of binary logic which deals with variable 0 and 1 which shows the states off and on respectively. Many digital logical circuits of various families for different operations can be built using basic electronics circuits viz flip-flop, inverter and gates. We know that all digital circuits and digital devices perform operations using only binary number system. So the only mathematical tool for the analysis and design of digital circuit is Boolean Algebra. Thus only Boolean Algebra is supposed to be a greatest tool for the Engineers and computer scientist for the operations of the logic circuits in the modern digital systems. Boolean algebra has wide applications in every field of core engineering to analyse and design digital computers.

Finite Boolean Algebra : From the definition of lattice, it is clear that a lattice L is an algebraic system with two operations meet and join denoted by $\wedge$ and $\vee$ respectively or $\bullet$ and $\oplus$ both of which satisfy the properties like idempotent, commutative, associative and absorption.

A Boolean Algebra is a complemented distributive lattice in which bounds are denoted by 1 and 0.

Thus the Boolean Algebra with the operations $\vee$ and $\wedge$ and bounds 1 and 0 satisfy the following properties.

(1) $a \vee a = a;\ a \wedge a = a$ ....(idempotent property)

(2) $a \vee b = b \vee a;\ a \wedge b = b \wedge a$ ....(commutative property)

(3) $a \vee (b \vee c) = (a \vee b) \vee c;\ a \wedge (b \wedge c) = (a \wedge b) \wedge c$ ....(associative property)

(4) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c);$
$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ ....(distributive property)

(5) $a \vee (a \wedge c) = a;\ a \wedge (a \vee c) = a$ ....(absorption property)

(6) $0 \leq a, \forall a \in L;\ a \leq 1, \forall a \in L$ ....(properties of bounds)

(7) $a \vee 0 = a;\ a \wedge 1 = a$

(8) $a \vee 1 = 1$ and $a \wedge 0 = 0$

(9) $a \vee a' = 1;\ a \wedge a' = 0$ [where a' is the complement of a]

(10) $1' = 0;\ 0' = 1$

(11) $(a \vee b)' = a' \wedge b'$ and $(a \wedge b)' = a' \vee b'$ (De-Morgan's laws)

IMPORTANT FORMULAE :

(1) $a \vee a = a;\ a \wedge a = a$ ....(idempotent property)

(2) $a \vee b = b \vee a;\ a \wedge b = b \wedge a$ ....(commutative property)

(3) $a \vee (b \vee c) = (a \vee b) \vee c;\ a \wedge (b \wedge c) = (a \wedge b) \wedge c$ ....(associative property)

(4) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c);$
$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ ....(distributive property)

(5) $a \vee (a \wedge c) = a;\ a \wedge (a \vee c) = a$ ....(absorption property)

(6) $0 \leq a, \forall a \in L;\ a \leq 1, \forall a \in L$ ....(properties of bounds)

(7) $a \vee 0 = a;\ a \wedge 1 = a$

(8) $a \vee 1 = 1$ and $a \wedge 0 = 0$

(9) $a \vee a' = 1;\ a \wedge a' = 0$ [where a' is the complement of a]

(10) $1' = 0;\ 0' = 1$

(11) $(a \vee b)' = a' \wedge b'$ and $(a \wedge b)' = a' \vee b'$ (De-Morgan's laws)

Ex.46. Prove that :

(i) $a \vee (a' \wedge b) = a \vee b$

(ii) $a \wedge (a' \vee b) = a \wedge b$

(iii) $(a \wedge b) \vee (a \wedge b') = a$

Proof : (i) L.H.S. $= a \vee (a' \wedge b)$

$= (a \vee a') \wedge (a \vee b)$ .....(By distributive law)

$= 1 \wedge (a \vee b)$ $[\because a \vee a' = 1]$

$= a \vee b$ $[\because a \wedge 1 = a]$

$=$ R.H.S.

(ii) Consider L.H.S. $= a \vee (a' \vee b)$

$= (a \wedge a') \vee (a \wedge b)$ ....(By distributive law)

$= 0 \vee (a \wedge b)$ $[\because a \wedge a' = 0]$

$= a \wedge b$ $[\because a \vee 0 = a]$

$=$ R.H.S.

(iii) Consider L.H.S. $= (a \wedge b) \vee (a \wedge b')$

$= a \wedge (b \vee b')$

....(corollary of distributive property)

$= a \wedge 1$ $[\because a \vee a' = 1]$

$= a =$ R.H.S.

Ex.47. Which of the following lattices D_{20} and D_{30} under partial order divisibility is a Boolean algebra.

Soln. Consider

$D_{20} = \{1, 2, 4, 5, 10, 20\}$ has 6 ($\neq 2^n$ for any positive integer n) elements and hence D_{20} is not a Boolean algebra where as $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$ has $8 = 2^3$ elements and hence is a Boolean algebra.

Note :

1) Let $B = \{0, 1\}$. The operation $\vee$ and $\wedge$ are given in the following tables.

$\vee$	0	1
0	0	1
1	1	1

$\wedge$	0	1
0	0	0
1	0	1

Thus the complement of 0 is 1 and vice-versa. Then $(B, \vee, \wedge)$ is a Boolean algebra.

2) Consider the set of n tuples of 0's and 1's for a, b given by

$a = \{a_1, a_2, ..., a_n\}$ and

$b = \{b_1, b_2, ..., b_n\}$, defined by

$a \vee b = \{a_1 \vee b_1, a_2 \vee b_2, ...a_n \vee b_n\}$

$a \wedge b = \{a_1 \wedge b_1, a_2 \wedge b_2, ...a_n \wedge b_n\}$

Now $a_j = 0$ or 1 and $b_j = 0$ or 1

And also

$a' = \{a'_1, a'_2, a'_3 ..... a'_n\}$ and $b' = \{b'_1, b'_2, b'_3 ..... b'_n\}$

If we denote this set of n-tuples by

B_n, then $(B_n, \wedge, \vee)$ is a Boolean algebra with bounds 0_n and 1_n where

$0_n = (0, 0, 0.....)$ and $1_n = (1, 1, 1, ...1)$

It is to be noted that B_n is known as switching algebra and represents a switching network.

Note : Any finite Boolean algebra has exactly 2^n elements for some positive integer n.

Ex.48. Simplify the following boolean expressions.

(i) $(1 \wedge a) \vee (0 \wedge a')$

(ii) $(a \wedge b)' \wedge (a \vee b)'$

(iii) $(a \wedge c) \vee c \vee [(b \vee b') \wedge c]$

(iv) $(a' \wedge b' \wedge c) \vee (a \wedge b' \wedge c) \vee (a \wedge b' \wedge c')$

Soln. (i) $(1 \wedge a) \vee (0 \wedge a') = a \vee 0 = a$

(ii) $(a \wedge b)' \wedge (a \vee b)' = (a' \vee b') \wedge (a' \wedge b')$

$= [(a' \vee b') \wedge a'] \wedge [(a' \vee b') \wedge b']$

$= (a' \vee b') \wedge (a' \vee b')$

$= a' \vee b'$

(iii) $(a \wedge c) \vee c \vee [(b \vee b') \wedge c]$

$= [(a \wedge c) \vee c] \vee [(b \vee b') \wedge c]$

$= c \vee [(b \vee b') \wedge c]$

$= c \vee (1 \wedge c)$

$= c \vee c$

$= c$

(iv) $(a' \wedge b' \wedge c) \vee (a \wedge b' \wedge c) \vee (a \wedge b' \wedge c')$

$= (a' \wedge c \wedge b') \vee (a \wedge c \wedge b') \vee (a \wedge c' \wedge b')$

$= (a' \wedge c) \vee (a \wedge c) \vee (a \wedge c') \wedge b'$

$= [(a' \vee a) \wedge c] \vee (a \wedge c') \wedge b'$

$= (1 \vee c) \vee (a \wedge c') \wedge b'$

$= [c \vee (a \wedge c')] \wedge b'$

$= [(c \vee a) \wedge (c \vee c')] \wedge b'$

$= [(c \vee a) \wedge 1] \wedge b'$

$= (c \vee a) \wedge b'$

$= (a \vee c) \wedge b'$ **.....Ans.**

Ex.49. Let [B, ∧, ∨, ′, 0, 1] be a Boolean algebra. Define the operations + and . on the elements of B by

$a + b = (a \wedge b') \vee (a' \wedge b)$

$a.b = a \wedge b$ for all $a, b \in L$

Show that

(i) $a + b + b = a$ (ii) $a + a = 0$

(iii) $a + 0 = a$ (iv) $a + 1 = a'$

(v) $a . (b + c) = a . b + a . c$

Soln.

(i) Consider L.H.S. $= (a + b) + b = [(a + b) \wedge b'] \vee [(a + b)' \wedge b]$

$= \{[(a \wedge b') \vee (a' \wedge b)] \wedge b'\} \vee \{[(a \wedge b') \vee (a' \wedge b) \wedge b]\}$

$= \{[(a \wedge b' \wedge b') \vee (a' \wedge b \wedge b')] \vee \{[(a \wedge b')' \wedge [(a' \wedge b)'] \wedge b\}$

$= \{(a \wedge b') \vee (a' \wedge 0)\} \vee \{[(a' \vee b) \wedge [(a \vee b')] \wedge b\}$

$= (a \wedge b') \vee \{b \wedge [(a \wedge b) \vee (b \wedge b')]\}$

$= (a \wedge b') \vee \{b \wedge (a \wedge b) \vee 0\}$

$= (a \wedge b') \vee (a \wedge b)$

$= a \wedge (b' \vee b)$

$= a \wedge 1$

$= a =$ R.H.S.

(ii) Consider L.H.S. $= a + a = (a \wedge a') \vee (a' \wedge a)$ *[∵ By (i)]*

$= 0 \vee 0$

$= 0 =$ R.H.S.

(iii) Consider L.H.S. $= a + 0 = (a \wedge 0') \vee (0 \wedge a')$

$= (a \wedge 1) \vee 0$

$= a =$ R.H.S.

(iv) Consider L.H.S. $= a + 1 = (a \wedge 1') \vee (a' \wedge 1)$

$= (a \wedge 0) \vee a'$

$= 0 \vee a' = a' =$ R.H.S.

(v) Consider L.H.S. $= a.(b + c) = a \wedge (b + c)$

$= a \wedge \{(b \wedge c') \vee (b' \wedge c)\}$

$= [a \wedge (b \wedge c')] \vee [a \wedge (b' \wedge c)]$ ..... (i)

Also

R.H.S. $= a.b + a.c = [(a . b) \wedge (a . c)'] \vee [(a . b)' \wedge (a . c)]$

$= [(a \wedge b) \wedge (a \wedge c)'] \vee [(a \wedge b)' \wedge (a \wedge c)]$

$= [(a \wedge b) \wedge (a' \wedge c')] \vee [(a' \vee b') \wedge (a \wedge c)]$

$= \{[(a \wedge b) \wedge a'] \vee [(a \wedge b) \wedge c')]\} \vee \{[a' \wedge (a \wedge c)] \vee [b' \wedge (a \wedge c)]\}$

$= 0 \vee [(a \wedge b) \wedge c'] \vee \{[0] \vee [b' \wedge (a \wedge c)]$

$= [(a \wedge b) \wedge c'] \vee [b' \wedge (a \wedge c)\}$

$= [a \wedge b \wedge c'] \vee [a \wedge (b' \wedge c)]$ ..... (2)

From (1) and (2) it is clear that

$a . (b + c) = (a . b) + (a . c)$. **.....Ans.**

GRAPHICAL REPRESENTATION OF BOOLEAN EXPRESSIONS :

Boolean expression can also be written by graphical way. If A and B are variables, then $A \wedge B$, $A \vee B$, A' can be represented graphically as

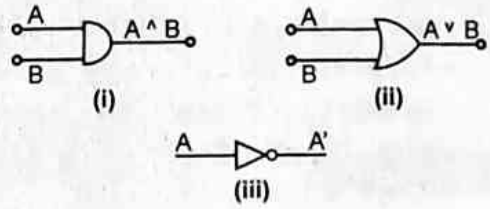
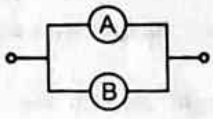


Fig. (i), (ii), (iii) are known as AND, OR and NOT gates respectively. Moreover complete graphical representation of Boolean expression is known as logic diagram and AND, OR and NOT gates are known as logic gates.

It is to be noted that in order to represent the switching circuit design, we denote the series combinations by

A . B and parallel combination



by A + B.

Note : (i) A and B denote the electric switches

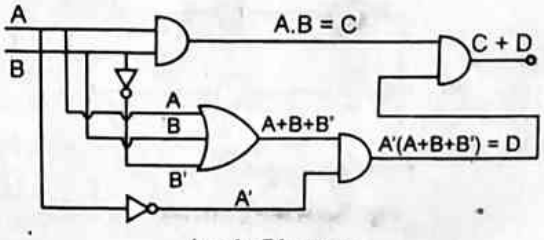
(ii) If A is on then A′ is off

(iii) $A \wedge B$ can also be written as A . B

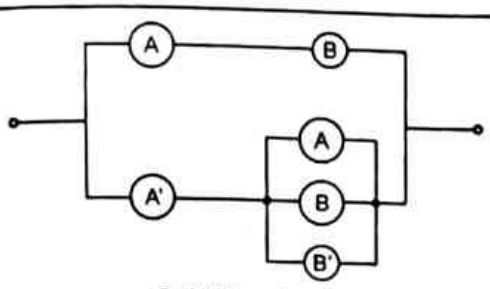
(iv) $A \vee B$ can also be written as A + B

Ex.50. Construct the logic diagram for the Boolean expression (A . B) + [A′ . (A + B + B′)]. Simplify this to obtain the equivalent circuit. Also draw the equivalent switching circuit. ***CS/CT/CE : S-12,13***

Soln.



Logic Diagram



Switching circuit

Logic Diagram

Simplification

$(A \cdot B) + A' \cdot (A + B + B')$

$= (A \cdot B) + [(A \cdot A') + A' \cdot B + A' \cdot B']$

$= (A \cdot B) + [0 + (A' \cdot B) + (A' \cdot B')]$

$= (A \cdot B) + (A' \cdot B) + (A' \cdot B')$

$= A \cdot B + A' \cdot (B + B')$

$= A \cdot B + A' \cdot 1$

$= A \cdot B + A'$

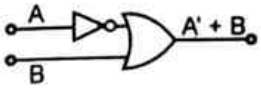
$= A' + (A \cdot B)$

$= (A' + A) \cdot (A' + B)$

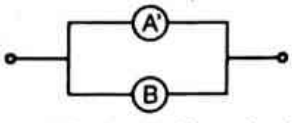
$= 1 \cdot (A' + B)$

$= A' + B$

$A' + B$ is a required equivalent simplified form.



Simplified Logic Diagram



Simplified switching circuit

Ex.51. Construct the logic circuit for the following Boolean expression. $A \cdot B + A \cdot B' + A \cdot B'$ Verify the equivalent by truth table. **CS/CT/CE : S-06, W-08**

Soln.

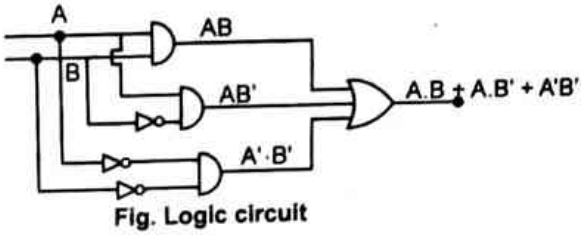


Fig. Logic circuit

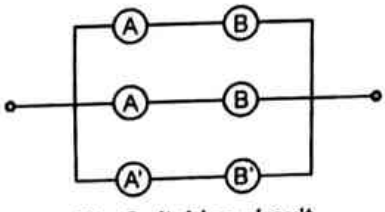


Fig. Switching circuit

Simplification :

$(A \cdot B) + (A \cdot B') + (A' \cdot B')$

$= A(B + B') + (A' \cdot B')$

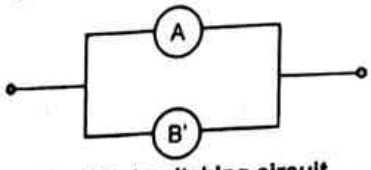
$= A \cdot 1 + (A' \cdot B')$

$= A + (A' \cdot B')$

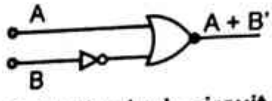
$= (A + A') \cdot (A + B')$

$= 1 \cdot (A + B')$

$= (A + B')$



Simplified switching circuit



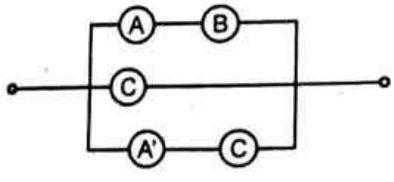
Switching logic circuit

Verification :

A	B	A'	B'	A.B	A.B'	A'.B'	(A.B) + (A.B') + (A'.B')	A+B'
0	0	1	1	0	0	1	1	1
0	1	1	0	0	0	0	0	0
1	0	0	1	0	1	0	1	1
1	1	0	0	1	0	0	1	1

From the last two coloumns it is clear that obtained circuit is equivalent to the given circuit. ....Ans.

Ex.52. Construct a switching circuit for Boolean polynomial $A \cdot B + C + A' C'$. Simplify this and draw equivalence switching circuit. Verify result by truth table. **CS/CT/CE : S-07,09, W-07,12**



Simplified circuit

Circuit for $A \cdot B + C + A' \cdot C'$

Soln. Simplification : $A \cdot B + C + A' \cdot C'$

$= A \cdot B + [(C + A') \cdot (C + C')]$

....(Distributive law)

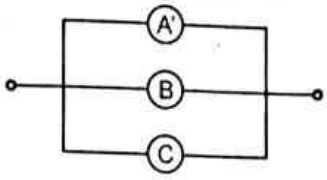
$= A \cdot B + [(C + A') \cdot 1]$

$= A \cdot B + (C + A')$

$= C + A' + (A \cdot B)$

$= C + (A' + A) \cdot (A' + B) = C + 1 \cdot (A' + B)$

$= C + A' + B$



Simplified circuit

Simplified circuit $C + A' + B$

Truth table

A	B	C	A'	C'	C + A' + B	A . B + C + A'. C'
0	0	0	1	1	1	1
0	0	1	1	0	1	1
0	1	0	1	1	1	1
0	1	1	1	0	1	1
1	0	0	0	1	0	0
1	0	1	0	0	1	1
1	1	0	0	1	1	1
1	1	1	0	0	1	1

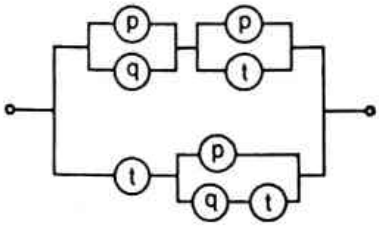
From truth table $A \cdot B + C + A' \cdot C' = C + A' + B$.

Ex.53. Construct a circuit for Boolean Expression :

$(p + q) \cdot (p + t) + t \cdot (p + q \cdot t)$

Simplify and construct its equivalent circuit. **CS/CT/CE : S-08, W-11**

Soln. Switching circuit :

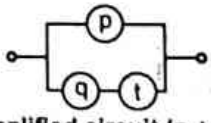


Circuit $(p + q) \cdot (p + t) + t \cdot (p + q \cdot t)$

Simplification : $(p + q) \cdot (p + t) + t \cdot (p + q \cdot t)$

$= [p + (q.t)] + t \cdot (p + q.t) = (p + q \cdot t)\ (1 + t)$

$= (p + q.t)$ *[∵ 1 + t = 1]*



Simplified circuit (p + q.t)

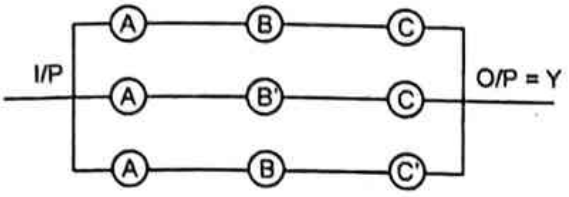
p	q	t	(p+q)	(p+t)	q.t	P+q.t	t.(p+q.t)	(p+q).(p+t)	(p+q).(p+t) + t.(p + q . t)	(p + q . t)
0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	1	0	0	0	0	0	0
0	1	0	1	0	0	0	0	0	0	0
0	1	1	1	1	1	1	1	1	1	1
1	0	0	1	1	0	1	0	1	1	1
1	0	1	1	1	0	1	1	1	1	1
1	1	0	1	1	0	1	0	1	1	1
1	1	1	1	1	1	1	1	1	1	1

From the truth table

$(p + q) . (p + t) + t . (p + q . t) = (p + q . t)$

Ex.54. Construct the switching circuit for $ABC + AB'C + ABC'$. Simplify and draw equivalent simplified circuit. Prepare the truth table and state when circuit is on. **CS/CT/CE : W-09**

Soln. The switching circuit for the given expression is



Let F (A, B, C) be the Boolean function of the given circuit, then

$F(A, B, C) = (A.B.C) + (A.B'C) + (ABC')$

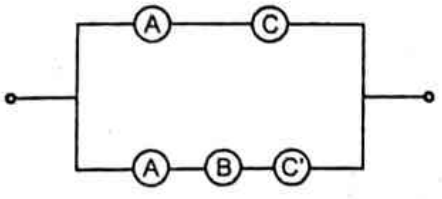
$= ABC + AB'C + ABC'$

$= AC(B + B') + ABC'$ $\quad$ $[\because B + B' = 1]$

$= AC(1) + ABC'$

$= AC + ABC'$

Simplified circuit :



Verification by truth tables :

A	B	C	ABC	B'	C'	AB'C	ABC'	ABC+ AB'C + ABC'
0	0	0	0	1	1	0	0	0
0	0	1	0	1	0	0	0	0
0	1	0	0	0	1	0	0	0
1	0	0	0	1	1	0	0	0
0	1	1	0	0	0	0	0	0
1	0	1	0	1	0	1	0	1
1	1	0	0	0	1	0	1	1
1	1	1	1	0	0	0	0	1

A	B	C	AC	AB C'	AC + AB C'
0	0	0	0	0	0
0	0	1	0	0	0
0	1	0	0	0	0
1	0	0	0	0	0
0	1	1	0	0	0
1	0	1	1	0	1
1	1	0	0	1	1
1	1	1	1	0	1

Circuit is on when output of truth table is 1. ..... **Ans.**

Ex.55. Show that in a Boolean Algebra for any a, b and c if $a \le b$, then :

i) $a \vee (b \wedge c) = b \wedge (a \vee c)$

ii) $\{(a \vee c) \wedge (b' \vee c)\}' = (a' \vee b) \wedge c'$

CS/CT/CE : W-06,08,12

Soln.

(i) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

$= b \wedge (a \vee c)$ $\quad$ $[\because a \le b, a \vee b = b]$

(ii) $\{(a \vee c) \wedge (b' \vee c)\}' = (a \vee c)' \vee (b' \vee c)'$

$= (a' \wedge c') \vee (b')' \wedge (c')$

$= a' \wedge c' \vee b \wedge c'$

$= (a' \vee b) \wedge c'$ ....**Ans.**

Ex.56. Simplify the following expression using Boolean algebra :

(i) $(AB'C' + AB'C + ABC + ABC')(A + B)$

(ii) $P + P'QR' + (Q + R)'$. **CS/CT/CE : S-10**

Soln.

(i) Consider $(AB'C' + AB'C + ABC + ABC')(A + B)$

Simplification :

$(AB'C' + AB'C + ABC + ABC')(A + B)$

$= (AB'(C' + C) + AB(C + C'))(A + B)$

$= (AB'.1 + AB.1)(A + B)$ .....(by Complement law)

$= (AB' + AB)(A + B)$ .....(by Identity law)

$= A.(B + B')(A + B)$

$= A.1(A + B)$ .....(by complement law)

$= A.(A + B)$ .....(by Identity law)

$= A$ ....(by Law of absorption)

(ii) Consider $P + P'QR' + (Q + R)'$

Simplification :

$P + P'QR' + (Q + R)'$

$= P + P'QR' + Q'.R'$ .....(by DeMorgan's law)

$= (P + P') \cdot (P + QR') + Q'R'$ .....(by Distributive law)

$= 1.(P + QR') + Q'R'$ .....(by Complement law)

$= P + QR' + Q'R'$ .....(by Identity law)

$= P + (Q + Q').R'$

$= P + 1.R'$ .....(by Complement law)

$= P + R'$ .....(by Identity law)

Ex.57. Construct switching circuit for Boolean polynomial $(A+B)\cdot(B'+C)+(C'+A)(C+B)$

Simplify this and construct equivalent switching circuit. **CS/CT/CE : S-11**

Soln. Simplification :

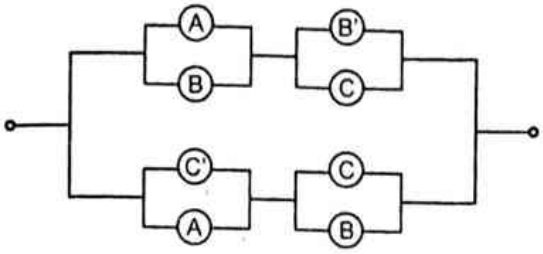


Fig. Circuit for $(A+B)\cdot(B'+C)+(C'+A)(C+B)$

Simplification : $(A+B)\cdot(B'+C)+(C'+A)(C+B)$

$= AB' + AC + BB' + BC + CC'$

$+C'B + AC + AB$ ....(Distributive law)

$= AB' + AC + 0 + BC + 0 + C'B + AC + AB$

$= AB' + AC + BC + C'B + AC + AB,$

$= AB' + AB + AC + BC + BC',$

$[\because AC + AC = AC]$

$= A(B+B') + AC + B(C+C')$

$= A\cdot 1 + AC + B\cdot 1 = A + AC + B$

$= A + B$ ....(Absorption law)

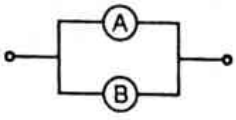


Fig. Simplified circuit for A + B.

COMPLETE LIST OF FORMULAE :

Laws of Algebra in lattice :

(1) Idempotent law :

$a \oplus a = a = a * a$

(2) Commutative laws :

$a \oplus b = b \oplus a$ and $a * b = b * a$

(3) Associative laws :

$a \oplus (b \oplus c) = (a \oplus b) \oplus c$

$a * (b * c) = (a * b) * c$

(4) Absorption law :

$a \oplus (a * b) = a = a * (a \oplus b)$

(5) (i) $a \le b$ and $a \le c \Rightarrow a \le b \oplus c$

(ii) $a \le b$ and $a \le c \Rightarrow a \le b * c$

(iii) $a \ge b$ and $a \ge c \Rightarrow a \ge b * c$

(iv) $a \ge b$ and $a \ge c \Rightarrow a \ge b \oplus c$

(6) In a bounded Lattice :

$0 \wedge a = 0 = a \wedge 0$ for all $a \in L$

$0 \vee a = a = a \vee 0$ for all $a \in L$

$1 \wedge a = a = a \wedge 1$ for all $a \in L$

$1 \vee a = 1 = a \vee 1$ for all $a \in L$

(7) In Distributive lattice :

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(8) $a \vee a = a;\ a \wedge a = a$ ....(idempotent property)

(9) $a \vee b = b \vee a;\ a \wedge b = b \wedge a$ ....(commutative property)

(10) $a \vee (b \vee c) = (a \vee b) \vee c;\ a \wedge (b \wedge c) = (a \wedge b) \wedge c$ ....(associative property)

(11) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c);$

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ ....(distributive property)

(12) $a \vee (a \wedge c) = a;\ a \wedge (a \vee c) = a$ ....(absorption property)

(13) $0 \le a, \forall a \in L;\ a \le 1, \forall a \in L$ ....(properties of bounds)

(14) $a \vee 0 = a;\ a \wedge 1 = a$

(15) $a \vee 1 = 1$ and $a \wedge 0 = 0$

(16) $a \vee a' = 1;\ a \wedge a' = 0$

[where a' is the complement of a]

(17) $1' = 0;\ 0' = 1$

(18) $(a \vee b)' = a' \wedge b'$ and $(a \wedge b)' = a' \vee b'$ (De-Morgan's laws)

NOTES

UNIT - V
GRAPH THEORY

SYLLABUS

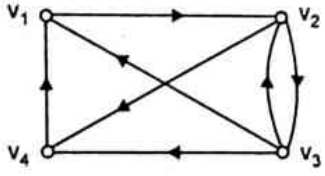
Basic concepts of Graph Theory, Digraphs, Basic definitions, Paths and Circuits, Reachability and Connectedness, Matrix representation of graphs, Subgraphs and Quotient Graphs, Isomorphic digraphs and Transitive Closure digraph, Euler's Path and Circuit (only definitions and examples). Trees, Binary Tree, Labelled Trees, Undirected Trees, Spanning Trees of Connected Relations, Prim's Algorithm to construct Spanning Trees, Weighted Graphs, Minimal Spanning Trees by Prim's Algorithm and Kruskal's Algorithm.

UNIVERSITY PAPER SOLUTIONS
SINCE SUMMER - 2006

Note to the students :
- The questions given below are from previous Nagpur University examination papers.
- Though these questions are from old university papers, they have been included in the new syllabus.
- Questions which are out of new syllabus are not included here.

SUMMER - 06 (CS/CT/CE)

Q.1. Find transitive closure of the following digraph : **5M**



Ans. P. 5-31, Ex.16

Q.2. Draw binary tree for the relation given by

R = {(1, 2), (1, 3), (1, 4), (2, 5), (4, 6), (4, 7)} on set A = {1, 2, 3, 4, 5, 6, 7}. **5M**

Ans. P. 5-41, Ex.34

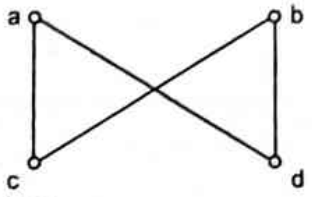
Q.3. Define :

(i) Path of graph

(ii) Tree.

Ans. Refer Definitions on P.5-22, 36.

Q.4. Apply Prim's algorithm to find the spanning tree for the following graph : **7M**



Use a as the root.

Ans. P. 5-40, Ex.32

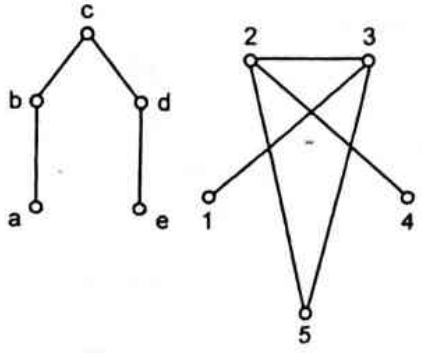
Q.5. Draw digraphs corresponding to adjacency matrices and show that they are isomorphic,

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$ **6M**

Ans. P. 5-30, Ex.13(a)

SUMMER - 06 (IT)

Q.1. Prove or disprove that graphs shown in the following figures are isomorphic. **6M**



Ans. P. 5-31, Ex.15

Q.2. Let $A = \{V_1, V_2, V_3, V_4, V_5, V_6, V_7, V_8, V_9, V_{10}\}$ and let $T = \{(V_2, V_3), (V_2, V_1), (V_4, V_5), (V_4, V_6), (V_5, V_8), (V_6, V_7), (V_4, V_2), (V_7, V_9), (V_7, V_{10})\}$. Show that T is a rooted tree and identify the root. **6M**

Ans. P. 5-41, Ex.35

Q.3. Define complete binary tree. **2M**

Ans. P. 5-41, Refer Definitions.

Q.4. Construct the tree of the following algebraic expression. $((2 \times x) + (3 - (4 \times x))) + (x - (3 \times 11))$. Draw Venn Diagram. **5M**

Ans. P. 5-44, Ex.40

Q.5. Define the following terms: **4M**

(i) Path,

(ii) Simple path,

(iii) Euler path,

(iv) Connected graph.

Ans. P. 5-22, 30, 28, Refer Definitions.

Q.6. Draw the complete graphs on 5 vertices. **3M**

Ans. P. 5-31, Ex.14

WINTER - 06 (CS/CT/CE)

Q.1. Define: **3M**

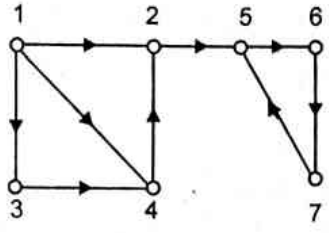
(i) Directed graph

(ii) Path

(iii) Reachability of graph.

Ans. P. 5-13, 22, 23, Refer Definitions.

Q.2. Find node base for the graph given below:



Ans. P. 5-26, Ex.11

Q.3. Construct binary tree for the following expression: $\{(3 - (2x)) + [(x-2) - (3 + x)]\} + xy$. **7M**

Ans. P. 5-44, Ex.40

Q.4. Draw the digraph corresponding to the matrix : **7M**

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Find the transitive closure of the digraph.

Ans. P. 5-31, Ex.16

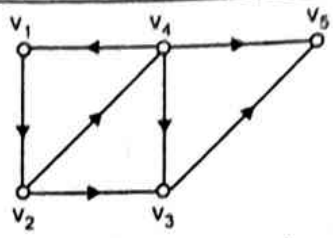
WINTER - 06 (IT)

Q.1. Define :

Node base of the digraph. **2M**

Ans. P. 5-24, Refer Definitions.

Q.2. Find a node base of the digraphs given in the figure.



Ans. P. 5-26, Ex.11

Q.3. Given a digraph whose adjacency matrix is:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Find A^T, AA^T, and A^2. Give also the interpretation of elements AA^T and A^2. **7M**

Ans. P. 5-23, Ex.8(a)

Q.4. Define the terms : **2M**

(i) Multigraph

(ii) Tree.

Ans. P. 5-14, 36, Refer Definitions.

Q.5. Consider the algebraic expression : **6M**

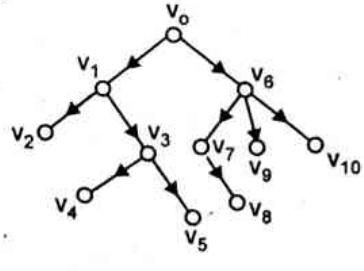
$E = (2x - y)^3 \{a + (b \div c)\}$

(i) Draw the tree which corresponds to the expression E.

(ii) Find the subtree at the exponential operator.

Ans. P. 5-44, Ex.41

Q.6. Obtain the binary tree corresponding to the tree given in the figure. **5M**



Ans. P. 5-42, Ex.36

SUMMER - 07 (CS/CT/CE)

Q.1. Draw digraph corresponding to $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ and interpret, AA^T, A^TA and A^2. **7M**

Ans. P. 5-23, Ex.8(a)

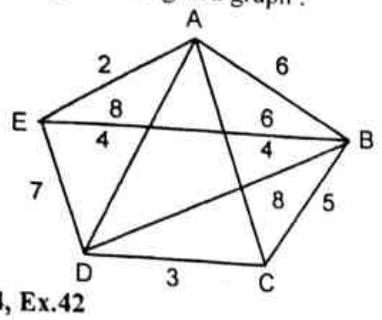
Q.2. Draw binary tree for the tree given by :

$R = \{(1, 2), (1, 3), (1, 4), (2, 5), (4, 6), (4, 7)\}$

on set $A = \{1, 2, 3, 4, 5, 6, 7\}$. **6M**

Ans. P. 5-41, Ex.34

Q.3. Use Prim's algorithm to find the minimum spanning tree for the given weighted graph : **7M**



Ans. P. 5-44, Ex.42

Q.4. Construct the tree of the algebraic expression :

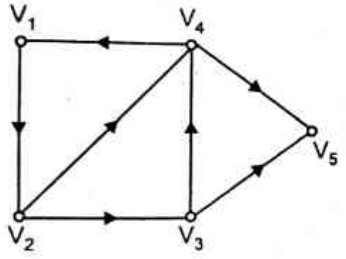
(i) $(x + (y - (x + y))) \times ((3 \div (2 \times 7)) \times 4)$

(ii) $3 - (x + (6 \times (4 \div (2 - 3))))$. **7M**

Ans. P. 5-45, Ex.43

SUMMER - 07 (IT)

Q.1. Find all the indegrees and outdegrees of the node of the graph. Give all the elementary cycle of this graph. Obtain an acyclic digraph by deleting one edge of the graph. List all the nodes which are reachable from another node of the digraph.



Ans. P. 5-25, Ex.10(a)

Q.2. Construct the tree of the following algebraic expression $(a - b) * [c + (d + e)]$

Ans. P. 5-45, Ex.44

Q.3. Draw digraphs corresponding to the following adjacent matrices and show that they are isomorphic.

$$M_1 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

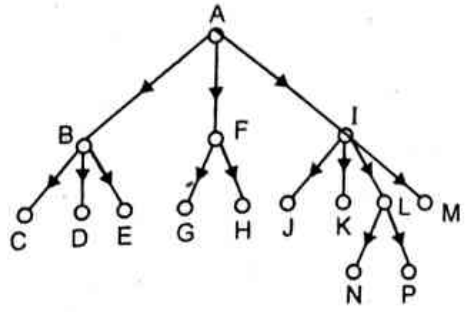
Ans. P.5-30, Ex.13(a).

Q.4. Draw the digraph corresponding to the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$ Interpret the results AA^T, A^TA, A^2, A^3 and A^4.

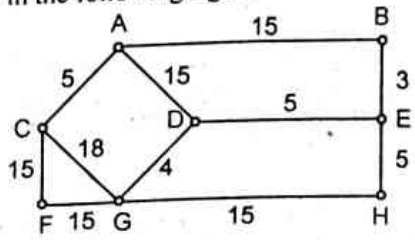
Ans. P. 5-23, Ex.8(a)

Q.5. Give a binary tree representation to the following tree



Ans. P.5-45, Ex.45

Q.6. Determine a railway network of minimal cost for the cities in the following figure



Ans. P. 5-39, Ex.31

WINTER - 07 (CS/CT/CE)

Q.1. Draw the digraphs corresponding to adjacency matrices :

$$M_1 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Are these two graphs isomorphic to each other ?

Ans. P. 5-30, Ex.13(a)

Q.2. Define spanning tree, binary tree. Draw binary tree for the relation given by

$R = \{(1, 2), (1, 3), (1, 4), (2, 5), (4, 6), (4, 7)\}$

on set $A = \{1, 2, 3, 4, 5, 6, 7\}$. **6M**

Ans. P. 5-41, Ex.34 and refer definition on P. 5-36, 40.

Q.3. Construct tree for the following mathematical expression

$[x + \{y - (x + y)\}] \ . \ [\{3 \div (2x)\} \ . \ y]$ **2M**

Ans. P. 5-46, Ex.46

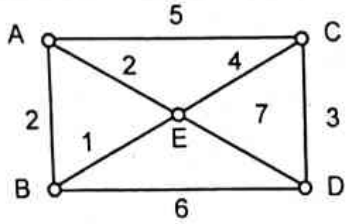
Q.4. If relation set

$R = \{(1, 2), (2, 3), (3, 4), (2, 1)\}$

Draw the digraph, find the matrix of relation M_R. Find the transitive closure of R. **7M**

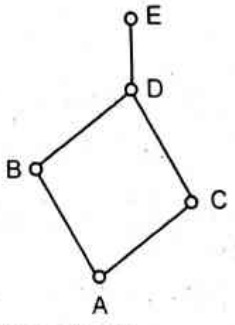
Ans. P. 5-32, Ex.17

Q.5. Using Prim's Algorithm determine a minimal spanning tree for the graph. **3M**



Ans. P. 5-38, Ex.30

Q.6. Define Eulerian path and Eulerian circuit. Write the Eulerian path for the following figure. **3M**



Ans. P. 5-30, See Example (ii)

WINTER - 07 (IT)

Q.1. Draw the digraphs corresponding to adjacency matrices :

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$ and prove that digraphs of A and B are isomorphic. **6M**

Ans. P. 5-32, Ex.18

Q.2. Define : **4M**

(i) Path (ii) Simple path

(iii) Euler path (iv) Connected graph.

Ans. P. 5-22, 30, 28, Refer Definitions.

Q.3. Give tree diagram and Venn diagram representation of

(i) $(V_1 V_2 + V_3) + \left(V_4 + \frac{V_5}{V_6}\right) V_7$

(ii) $[x + \{y - (x + y)\}] \times [\{3 \div (2x)\} y]$. **7M**

Ans. P. 5-46, Ex.46

Q.4. Draw tree representation for the tree given by $A = \{1, 2, 3, 4, 5, 6, 7\}$; $R = \{(1, 2), (1, 3), (1, 4), (2, 5), (4, 6), (4, 7)\}$ and draw corresponding binary tree. **6M**

Ans. P. 5-41, Ex.34

SUMMER - 08 (CS/CT/CE)

Q.1. Draw a graph corresponding to the adjacency matrix

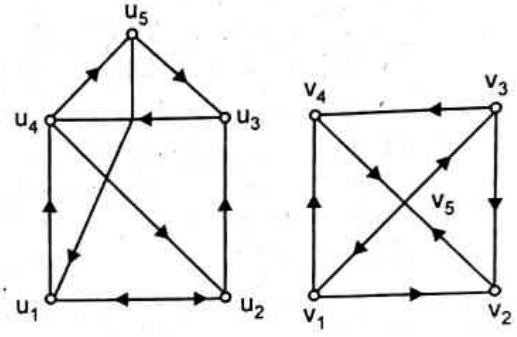
$$A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 2 & 1 & 3 \\ 2 & 1 & 0 & 1 \\ 0 & 3 & 1 & 0 \end{bmatrix}.$$ **6M**

Ans. P. 5-32, Ex.19

Q.2. Construct a tree corresponding to algebraic expression: $[3 - \{2 - (11 - (9 - x))\}] \div [2 + \{3 + (4 + y)\}]$. **7M**

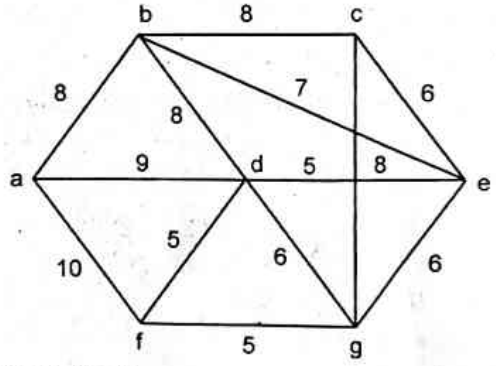
Ans. P. 5-46, Ex.46

Q.3. Define isomorphism of two graphs. Are the following graphs isomorphic? **6M**



Ans. P. 5-19, Ex.6(b) & Definition on P.5-18.

Q.4. Apply Prim's algorithm to construct a minimal spanning tree for the weighted graph given below : **7M**



Ans. P. 5-46, Ex.47

SUMMER - 08 (IT)

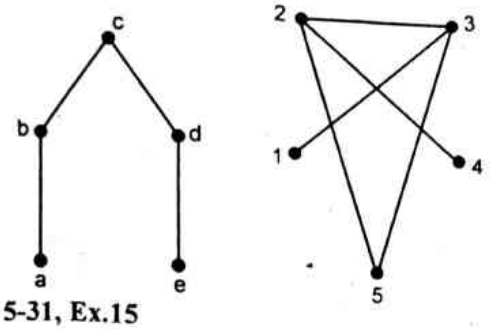
Q.1. Draw the diagram corresponding to adjacency matrices A and B

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Are these two digraphs isomorphic ? **7M**

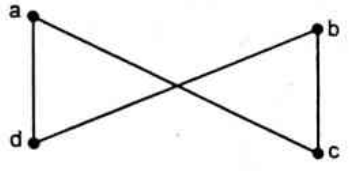
Ans. P. 5-19, Ex.7

Q.2. Prove or disprove that graphs shown in the following figures are isomorphic. **6M**



Ans. P. 5-31, Ex.15

Q.3. Apply Prim's algorithm to find the spanning tree for the following graph. **7M**



Ans. P. 5-40, Ex.32

Q.4. Construct the tree of the following algebraic expression : $(3-(2-(11-(9-4))))\div(2+(3+(4+7)))$. **6M**

Ans. P. 5-46, Ex.46

WINTER - 08 (CS/CT/CE)

Q.1. Define : **3M**

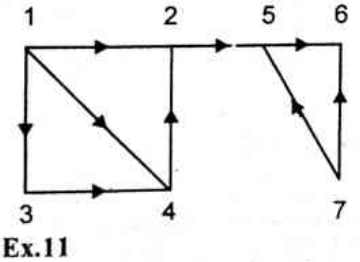
(i) Directed graph

(ii) Path

(iii) Reachability of graph.

Ans. P. 5-13, 22, 23, Refer Definitions.

Q.2. Find the node base for the graph given below : **3M**



Ans. P. 5-26, Ex.11

Q.3. Draw binary tree for the tree given by : **7M**

$R = \{(1,2),(1,3),(1,4),(2,5),(4,6),(4,7)\}$

on set $A = \{1,2,3,4,5,6,7\}$.

Ans. P. 5-41, Ex.34

Q.4. Draw digraph corresponding to $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ and interpret AA^T, A^TA and A^2. **7M**

Ans. P. 5-23, Ex.8(a)

Q.5. Construct the tree of the following algebraic expression : **6M**

(i) $[5(1-x)]\div[\{5-(y+3)\}\times\{7+(x\div y)\}]$

(ii) $\{(3-(2x))+[(x-2)-(3+x)]\}+xy$.

Ans. P. 5-46, Ex.48

WINTER - 08 (IT)

Q.1. Draw digraphs corresponding to the adjacency matrices A and B:

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Are these two digraphs isomorphic? **6M**

Ans. P. 5-32, Ex.20

Q.2. Draw the digraph corresponding to the matrix:

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Find the transitive closure of the digraph. Draw its digraph. **7M**

Ans. P. 5-25, Ex.9

Q.3. Construct tree diagram and its Venn diagram for following expressions :

(i) $[3-(2-(11-(9-x)))]\div[2+(3+(4+y))]$

(ii) $[x+(y-(x+y))]\times[(3\div(2x))-y]$

(iii) $[7+(6-x)]-[x-(y-4)]$. **6M**

Ans. P. 5-46, Ex.46

Q.4. Draw a directed tree with four nodes at level 1, six nodes at level 2. Obtain the corresponding binary tree. **7M**

Ans. P. 5-47, Ex.49

SUMMER - 09 (CS/CT/CE)

Q.1. Define : **3M**

(i) Euler Path

(ii) Spanning Tree

(iii) Strongly connected graph.

Ans. P. 5-30, 36,29, Refer Definitions.

Q.2. Draw binary tree for the relation given by : **5M**

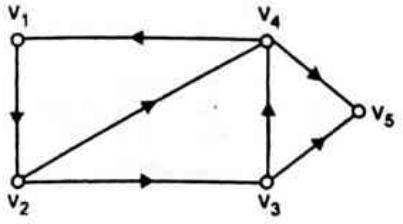
$R = \{(1,2),(1,3),(1,4),(2,5),(4,6),(4,7)\}$

on set $A = \{1,2,3,4,5,6,7\}$

Ans. P. 5-41, Ex.34

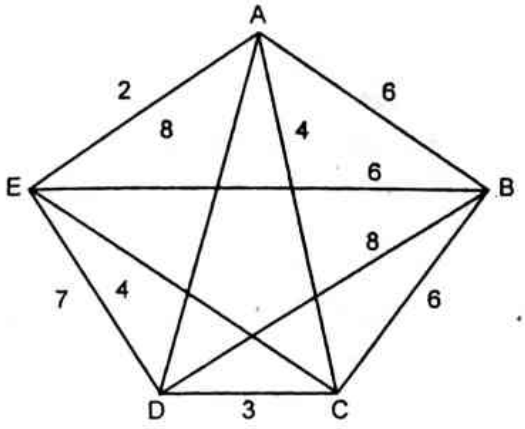
Q.3. Find all the indegree's and outdegrees of the node of the graph. Give all elementary cycles of this graph.

Obtain an acyclic diagraph by deleting one edge of the given graph. List all the nodes which are reachable from another node of the graph. **6M**



Ans. P. 5-25, Ex.10(a)

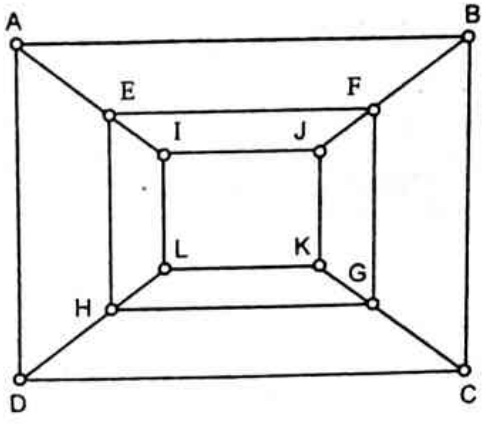
Q.4. Use Prim's Algorithm to find the minimum spanning tree for the given graph. **6M**



Ans. P. 5-44, Ex.42

SUMMER - 09 (IT)

Q.1. Let G be the graph and R be equivalence relation of V defined by partition {{I, J, K, L}, {A, E}, {F, B, C}, {D}, {H}} then find quotient graph G^R. **6M**



Ans. P. 5-33, Ex.21

Q.2. Define :

(i) Directed graph.

(ii) Simple path and elementary path.

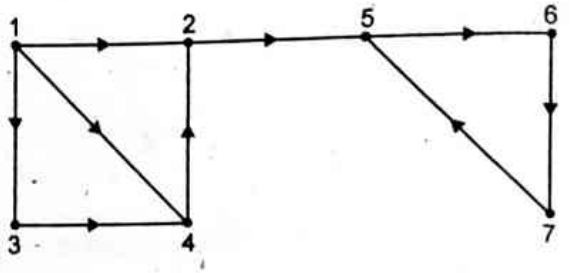
(iii) Reachability of graph.

Ans. P. 5-13, 22, 23, Refer Definitions.

Q.3. Construct the tree of following expression. Draw Venn diagram $(x + (y - (x + y))) \times (3 \div (2 \times 7) \times 4)$. **5M**

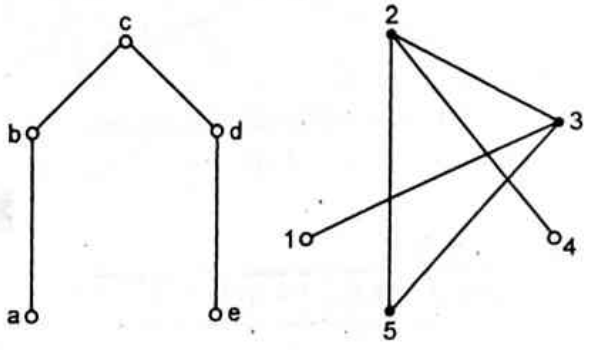
Ans. P. 5-45, Ex.43

Q.4. Find node base for the graph given below : **3M**



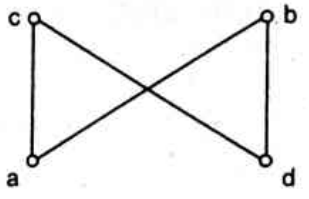
Ans. P. 5-26, Ex.11

Q.5. Prove or disprove that graphs shown in the following figures are isomorphic. **5M**



Ans. P. 5-31, Ex.15

Q.6. Apply Prim's algorithm to find the spanning tree for the following graph : **5M**



Use 'a' as the root.

Ans. P. 5-40, Ex.32

WINTER - 09 (CS/CT/CE)

Q.1. Draw the digraphs corresponding to adjacency matrix A, B, A^T and B^T where

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Show that the digraphs corresponding to A^T and B^T are isomorphic. **7M**

Ans. P. 5-33, Ex.22

Q.2. Define : **6M**

(i) Out degree of vertex.

(ii) Regular graph.

(iii) Path.

(iv) Reachability of graph.

(v) Strongly connected graph.

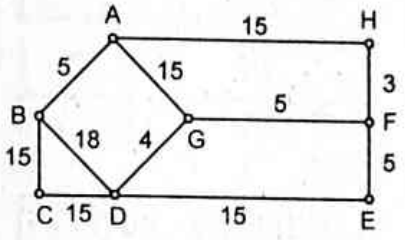
(vi) Complete binary tree.

Ans. P. 5-15, 20, 22, 23, 29, 41, Refer Definitions.

Q.3. Define a tree. Draw a directed tree with 2 nodes at level 1, 5 nodes at level 2 and 3 nodes at level 3. Obtain corresponding binary tree. **6M**

Ans. P. 5-42, Ex.36 and Definitions on P. 5-36

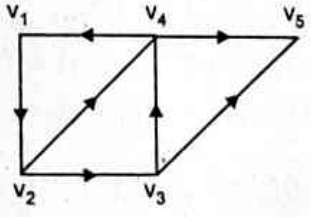
Q.4. Determine a railway network of minimal cost for the cities in the following figure : **7M**



Ans. P. 5-39, Ex.31

WINTER - 09 (IT)

Q.1. Find all the indegrees and outdegrees of the nodes of the graph given below. Give all elementary cycles of this graph. Obtain an acyclic digraph by deleting one edge of the given digraph. List all the nodes which are reachable from another node of the digraph. Find the diameter of the given digraph. **7M**



Ans. P. 5-25, Ex.10(a)

Q.2. Determine the number of loops and multiple edges in a multigraph G from its adjacency matrix :

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 2 & 1 & 3 \\ 2 & 1 & 0 & 1 \\ 0 & 3 & 1 & 0 \end{bmatrix}.$$ **7M**

Ans. P. 5-32, Ex.19

Q.3. Define the following terms of graph theory:

(i) Indegree and outdegree of a node.

(ii) Simple path and elementary path.

(iii) Unilaterally connected and strongly connected graph.

(iv) Root of a tree and leaf of a tree. **6M**

Ans. P. 5-15, 22, 29, 36, Refer Definitions.

Q.4. Construct tree for the algebraic expression : **7M**

(i) $\{[3-(2x)]+[(x-2)-(3+x)]\}+xy$

Ans. P. 5-44, Ex.40

(ii) $[3(1-x)]+[\{4+(7-(y+2))\}\times\{7+(x+y)\}]$

Ans. P. 5-47, Ex.50(a)

SUMMER - 10 (CS/CT/CE)

Q.1. Draw the undirected graph represented by adjacency matrix A given by $A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$ **3M**

Ans. P. 5-16, Ex.2

Q.2. Draw the digraph G corresponding to adjacency matrix : $A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ **3M**

Ans. P. 5-17, Ex.4

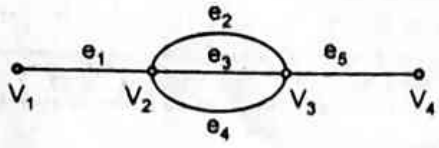
Q.3. Construct tree for the following formula. Also give its Venn diagram :

(i) $(P\vee(P\wedge Q))\wedge((P\vee Q)\wedge R)$

(ii) $[3]-[x+\{6(4+(2-x)\}]$.

Ans. P. 5-48, Ex.51

Q.4. Consider the following graph : **6M**



(i) How many simple paths are there from V_1 to V_4 ?

(ii) How many trails are there from V_1 to V_4 ?

(iii) How many paths are there from V_1 to V_4 ?

Ans. P. 5-27, Ex.12

Q.5. Define a tree and a binary tree. Find the product $\{1, 2, 3\} \times \{2, 4\} \times \{2, 3, 4\}$ by constructing the appropriate tree diagram. **7M**

Ans. P. 5-42, Ex.37 and refer definition on P. 5-36, 40.

SUMMER - 10 (IT)

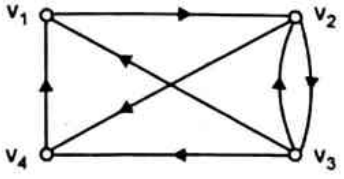
Q.1. Define:
(i) Graph
(ii) Path of graph
(iii) Adjacency matrix
(iv) Weakly Connected Graph.

Ans. **P. 5-13, 22, 34, 29, Refer Definitions.**

Q.2. Draw all possible simple digraphs having three nodes. Show that there is only one digraph with no edge, one with one edge, four with two edges. **4M**

Ans. **P. 5-34, Ex.23**

Q.3. Find transitive closure of the following digraph : **5M**



Ans. **P. 5-31, Ex.16**

Q.4. Construct a tree of the following algebraic expression : **6M**

(i) $[3(1-x)]\div[\{4+(7-(y+2))\}\times\{7\div(x\div y)\}]$

Ans. **P. 5-47, Ex.50(a)**

(ii) $[3-\{2-\{11-(9-x)\}\}]\div[2+\{3\div(4+y)\}]$.

Ans. **P. 5-46, Ex.46**

Q.5. Draw the digraphs corresponding to adjacency matrices A, B, A^T, B^T where

$$A=\begin{bmatrix}0&0&1&1\\1&0&1&0\\0&1&0&0\\0&1&1&0\end{bmatrix},\ B=\begin{bmatrix}0&1&1&0\\1&0&0&0\\0&1&0&1\\1&1&0&0\end{bmatrix}$$

Show that the digraphs corresponding to A^T and B^T are isomorphic. **7M**

Ans. **P. 5-33, Ex.22**

WINTER - 10 (CS/CT/CE)

Q.1. Define : **6M**
(i) Indegree of the node
(ii) Complete graph
(iii) Path
(iv) Reachability of graph
(v) Weakly connected graph
(vi) Binary tree.

Ans. **P. 5-15, 20, 22, 23, 29, 40, Refer Definitions.**

Q.2. From the given matrices A and B, draw the digraphs corresponding to the adjacency matrices A, B, A^T and B^T. Prove that A^T and B^T are isomorphic. **7M**
where :

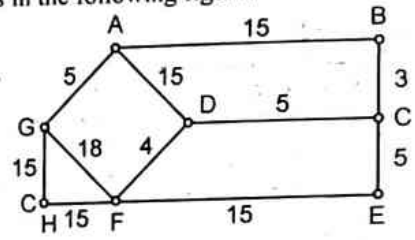
$$A=\begin{bmatrix}0&1&0&0\\0&0&1&1\\1&1&0&1\\1&0&0&0\end{bmatrix},\ B=\begin{bmatrix}0&1&0&1\\1&0&1&1\\1&0&0&0\\0&0&1&0\end{bmatrix}$$

Ans. **P. 5-19, Ex.7**

Q.3. Draw a directed tree with 4 nodes at level 1, 6 nodes at level 2. Obtain the corresponding binary tree. **6M**

Ans. **P. 5-47, Ex.49**

Q.4. Determine a railway network of minimal cost for the cities in the following figure. **7M**



Ans. **P. 5-39, Ex.31**

WINTER - 10 (IT)

Q.1. Define adjacency matrix of a digraph. Draw a digraph corresponding to the adjacency matrix $\begin{bmatrix}0&1&0&1\\1&0&1&1\\1&1&0&1\\1&0&1&0\end{bmatrix}$

Give all possible paths of length 2. Are there any cycles of length 2 ? **7M**

Ans. **P. 5-34, Ex.24**

Q.2. Construct a tree diagram of the following algebraic expression : **6M**

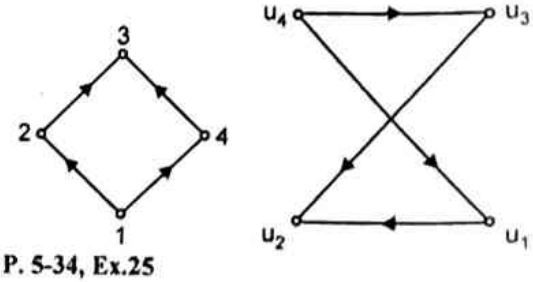
(i) $[3(1-x)]\div[\{(4\div(7-(y\div z))\}\times\{7+(x\div y)\}]$

Ans. **P. 5-47, Ex.50(b)**

(ii) $v_1 v_2+\left(v_4+\dfrac{v_5}{v_6}\right)v_3$.

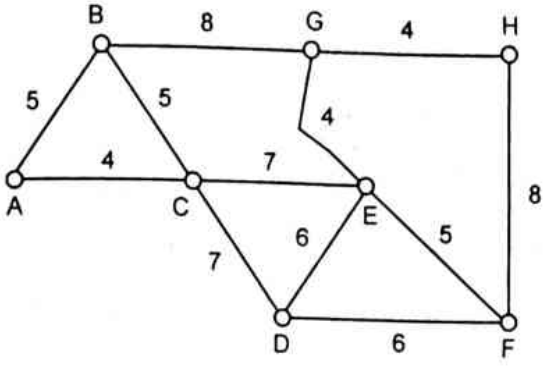
Ans. **P. 5-46, Ex.46**

Q.3. Define isomorphism of graphs. Show that the two graphs G_1 and G_2 are isomorphic. **7M**



Ans. **P. 5-34, Ex.25**

Q.4. Find a minimal spanning tree for connected weighted graph shown below. The weights of each edge is given in terms of kilometers. **6M**



Ans. P. 5-48, Ex.52

SUMMER - 11 (CS/CT/CE)

Q.1. Define : **7M**

(i) Directed graph
(ii) Regular graph
(iii) Path
(iv) Strongly connected graph
(v) Tree
(vi) Minimal spanning tree
(vii) Subgraph.

Ans. P. 5-13,20,22, 29, 36, 37, 21, Refer Definitions.

Q.2. Define: Isomorphism of graphs.
Draw digraphs corresponding to adjacency matrices and show that they are isomorphic. **7M**

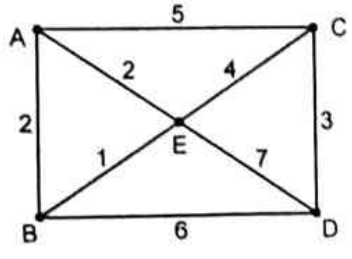
$$M_1 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}, M_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Ans. P. 5-30, Ex.13(a) and Definition on P. 5-18

Q.3. Draw binary tree for the tree given by
R = {(a, b), (a, c), (a, d), (b, e), (d, f), (d, g)}
on set A = {a, b, c, d, e, f, g}. **7M**

Ans. P. 5-41, Ex.34

Q.4. Find minimal spanning tree for the graph. **3M**



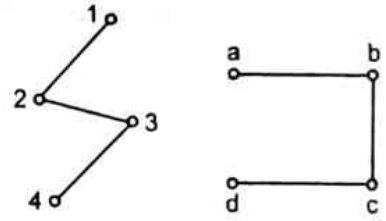
Ans. P. 5-38, Ex.30

Q.5. Construct tree for the following mathematical expression. Also give its Venn diagram representation.
[x + {y – (x + y)}] . [{3 + (2x)} . y] **3M**

Ans. P. 5-46, Ex.46

SUMMER - 11 (IT)

Q.1. Show that the two graphs as shown in figure are isomorphic. **6M**



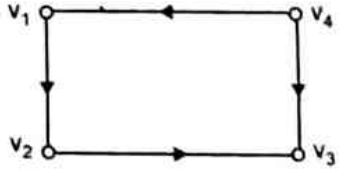
Graph G Graph G'

Ans. P. 5-18, Ex.6(a)

Q.2. Draw binary tree for the relation given by
R = {(1, 2), (1, 3), (1, 4), (2, 5), (4, 6), (4, 7)}
on set A = {1, 2, 3, 4, 5, 6, 7}. **7M**

Ans. P. 5-41, Ex.34

Q.3. Find all the indegrees and outdegrees of the node of the graph. Give all elementary cycles of this graph obtain an acyclic digraph by deleting one edge of the given graph.
List all the nodes which are reachable from the node v_1. **7M**



Ans. P. 5-26, Ex.10(b)

Q.4. Construct a tree corresponding to algebraic expression
$(3-\{2-(11-(9-x))\}) \div (2+\{3+(4+y)\})$
Also draw Venn diagram. **7M**

Ans. P. 5-46, Ex.46

WINTER - 11 (CS/CT/CE)

Q.1. Define : **6M**

(i) Euler path
(ii) Complete graph
(iii) Binary tree
(iv) Degree of a vertex
(v) Circuit
(vi) Hamiltonian path.

Ans. P. 5-30, 20, 40, 15, 22, 30, Refer Definitions.

Q.2. Construct a tree corresponding to algebraic expression
$(3 \times (1-x)) \div ((4+(7-(y+2))) \times (7+(x+y)))$.

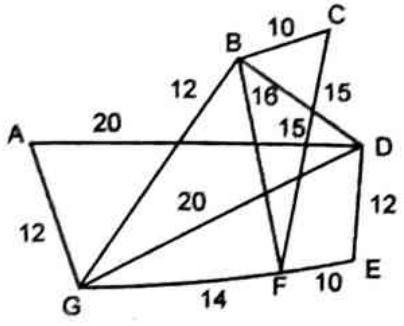
Ans. P. 5-47, Ex.50(a)

Q.3. Draw the digraph corresponding to adjacency matrix :

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$ **6M**

Ans. P. 5-35, Ex.26

Q.4. Apply Prim's algorithm to construct a minimal spanning tree for the weighted graph given below. **7M**



Ans. P. 5-48, Ex.53

WINTER - 11 (IT)

Q.1. Draw the digraphs corresponding to adjacency matrices A, B, A^T, B^T, where

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Show that the digraphs corresponding to A^T and B^T are isomorphic. **7M**

Ans. P. 5-33, Ex.22

Q.2. Define : **3M**

(i) Graph.

(ii) Simple Path

(iii) Weakly Connected Graph.

Ans. P. 5-13, 22, 29, Refer Definitions.

Q.3. Draw the digraph G corresponding to adjacency matrix :

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$ **3M**

Ans. P. 5-35, Ex.26

Q.4. Construct a tree diagram of the following algebraic expression : **6M**

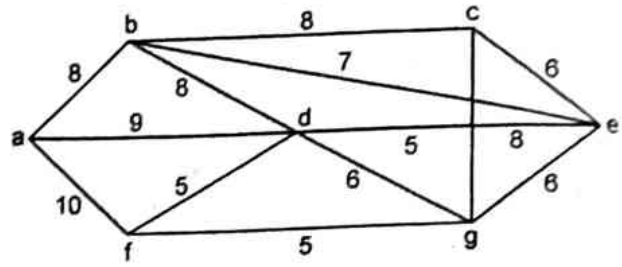
(i) $[3(1-x)] \div [\{4+(7-(y+z))\} \times \{7+(x+y)\}]$.

Ans. P. 5-47, Ex.50(b)

(ii) $(V_1V_2 + V_3) + \left(V_4 + \frac{V_5}{V_6}\right) V_7$.

Ans. P. 5-46, Ex.46

Q.5. Apply Prim's algorithm to construct a minimal spanning tree for weighted graph given below : **7M**



Ans. P. 5-46, Ex.47

SUMMER - 12 (CS/CT/CE)

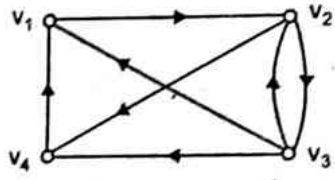
Q.1. Draw digraph corresponding to

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \text{ and}$$

interpret AA^T, A^TA, A^2. **6M**

Ans. P. 5-23, Ex.8(a)

Q.2. Find transitive closure of the following digraph. **5M**



Ans. P. 5-31, Ex.16

Q.3. Draw binary tree corresponding to the following expressions : **4M**

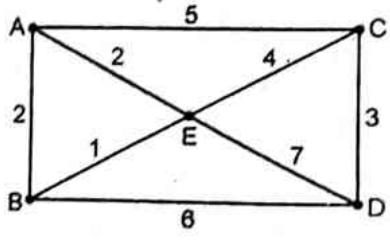
(i) $(x + (y - (x + y))) . ([3 \div (12 \times 7)] \times 4)$

Ans. P. 5-45, Ex.43

(ii) $[x + \{y - (x + y)\}] . [\{3 \div (2x)\} . y]$.

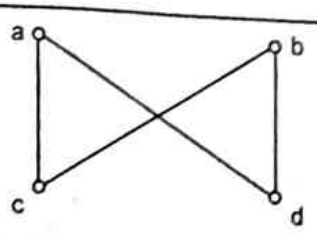
Ans. P. 5-46, Ex.46

Q.4. Find minimal spanning tree for the graph : **4M**



Ans. P. 5-38, Ex.30

Q.5. Using Prim's algorithm find the minimal spanning tree for the following graph using 'a' as the root : **7M**



Ans. P. 5-40, Ex.32

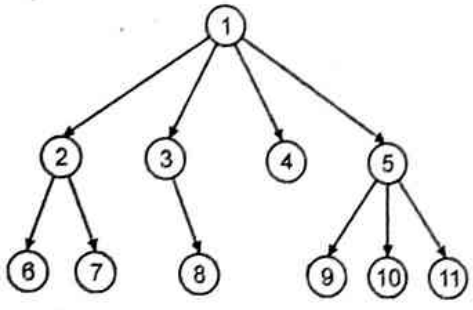
Q.6. Draw the digraphs corresponding to the following adjacency matrices and determine whether they are isomorphic : **6M**

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Ans. P. 5-33, Ex.22

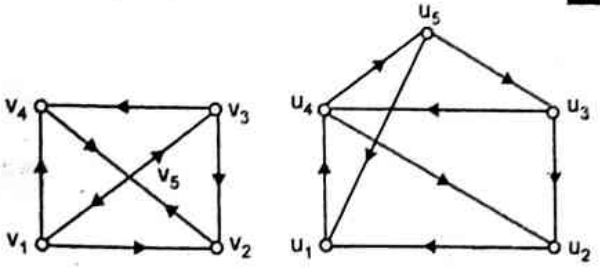
SUMMER - 12 (IT)

Q.1. Draw binary representation of the following tree : **7M**



Ans. P. 5-47, Ex.49

Q.2. Check whether the digraphs given below are isomorphic : **6M**



Ans. P. 5-19, Ex.6(c)

Q.3. Construct tree diagram and Venn diagram representation of : **6M**

(i) $[7 + (6 - x)] \times [x - (y - 4)]$

(ii) $[x + (y - (x + y))] - [(3 + (2x)) - y]$

Ans. P. 5-46, Ex.46

Q.4. Draw transitive closure digraph of the digraph represented by matrix $M_R = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. **8M**

Ans. P. 5-31, Ex.16

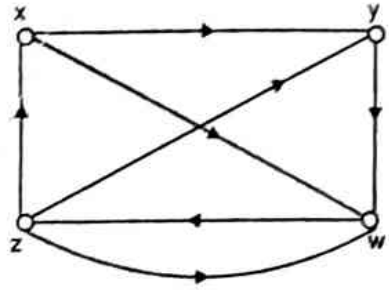
WINTER - 12 (CS/CT/CE)

Q.1. Construct the tree of following expression.

$(x + (y - (x + y))) \times (3 \div (2 \times 7)) \times 4)$. **4M**

Ans. P. 5-45, Ex.43

Q.2. In the following graph find simple path from x and z and all cycles. **3M**



Ans. P. 5-35, Ex.27

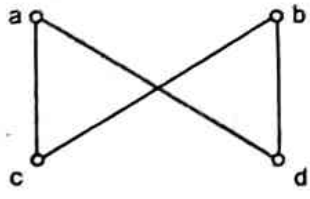
Q.3. Draw digraph corresponding to

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and interpret AA^T and A^2. **6M**

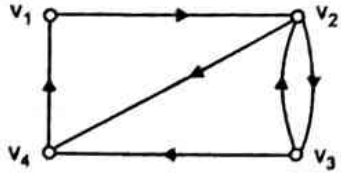
Ans. P. 5-25, Ex.8(c)

Q.4. Apply Prim's algorithm to find the spanning tree for the following graph. **7M**



Ans. P. 5-40, Ex.32

Q.5. Find transitive closure of the following digraph. **6M**

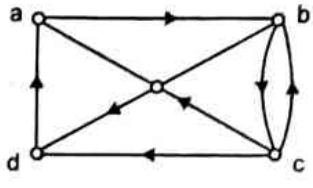


Ans. P. 5-31, Ex.16

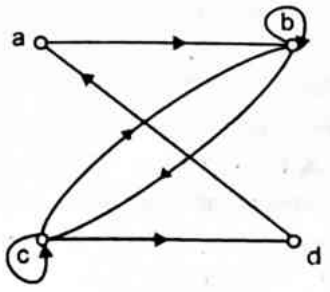
WINTER - 12 (IT)

Q.1. Find the adjacency matrix of the following graph :

(i) **7M**

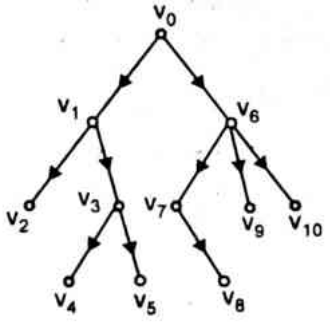


(ii)



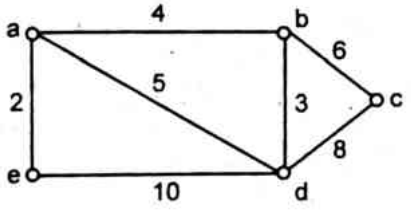
Ans. P. 5-35, Ex.28

Q.2. Obtain the binary tree corresponding to the tree given in the following figure.Describle all the steps. **6M**



Ans. P. 5-42, Ex.36

Q.3. Find the minimal spanning tree for the given weighted graph : **7M**



Ans. P. 5-38, Ex.30

SUMMER - 13 (CS/CT/CE)

Q.1. Define : **4M**

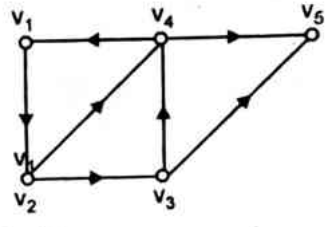
(i) Weighted graph

(ii) Indegree and outdegree of graph.

(iii) Cycle.

Ans. P. 5-14, 15, 22, Refer Definitions.

Q.2. Find the node base for the graph : **4M**



Ans. P. 5-26, Ex.11

Q.3. Construct binary tree for the following expression : **6M**

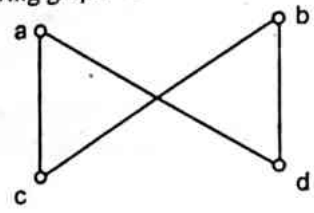
(i) $(3 - 2(-(11 - (9-4)))) \div (2 + (3 + (4 + 7)))$.

Ans. P. 5-43, Ex.39

(ii) $(2x + (3 - 4x)) + (x - (3 \times 11))$.

Ans. P. 5-44, Ex.40

Q.4. Apply Prim's algorithm to find the spanning tree for the following graph. **7M**



Ans. P. 5-40, Ex.32

Q.5. Draw the digraph represented by the given adjacency matrix and interprete AA^T, A^TA, A^2

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$ **6M**

Ans. P. 5-24, Ex.8(b)

SUMMER - 13 (IT)

Q.1. Draw the graphs corresponding to adjacency matrices :

$$M_1 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} \quad M_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Find whether they are issomorphic. **7M**

Ans. P. 5-31, Ex.13(b).

Q.2. Draw binary tree for the relation given by $R = \{(1, 2), (1, 3), (1, 4), (2, 5), (4, 6), (4, 7)\}$ on a set $A = \{1, 2, 3, 4, 5, 6, 7\}$. **7M**

Ans. P. 5-41, Ex.34

Q.3. Define : **6M**

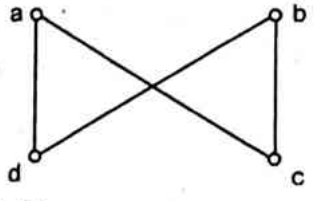
(i) Graph

(ii) Simple path

(iii) Weakly connected graph.

Ans. P. 5-13, 22, 29, Refer Definitions.

Q.4. Apply Prim's algorithm to find the spanning tree for the following graphs : **7M**



Ans. P. 5-40, Ex.32

SOLVED QUESTION BANK

[Sequence given as per syllabus]

INTRODUCTION

Graph Therory was developed in 1736 with Euler's papers in which he solved the Konigsberg Bridge Problem. In 1847, G.R Kirchoff developed the theory of trees for their applications in electrical network. In 1857, A. Cayley discovered trees while he was trying to enumerate the isomers of saturated hydrocarbons C_nH_{2n+2}. At this time, the four color conjecture was laid, which states that four colors are sufficient for coloring any atlas (a map on a plane) such that the countries with common boundaries have different colors.

The concepts of graph theory are widely used to study and model various applications in different areas. They include study of molecules, construction of bands in chemistry and the study of atoms. Grapyh theory is used in biology and conservation efforts where a vertex Represents regions where certain species exist and the edges represernt migration path or movement between the regions. This information is important when looking at breeding patterns or tracking the spread of disease, parasites and to study the impact of migration that affect other species. In operations Research, graph theory is used in the traveling salesman problem. Using the minimal spanning tree of a weighted graph one can obtain an optimal match of jobs and men and can locate the shortest path between two vertices in a graph. It is also used in modeling transport networks, activity networks.

BASIC CONCEPT OF GRAPH THEOTY

Graph : ***IT: S-10, 13, W-11***

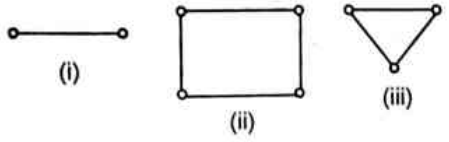
A graph $G = (V, E)$ consists of a non-empty set V, whose elements are called nodes or vertices or points and E is the set of edges whose elements are pairs of elements of V.

$\therefore$ $G = (V, E)$ is a graph where,

V = set of vertices or nodes or points

E = set of edges.

For example :



(i) (ii) (iii)

Note : If (u, v) is an edge in a graph G = (V, E), then u and v are said to be adjacent nodes.

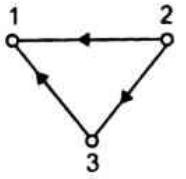
Definitions :

(i) **Undirected Graph :**

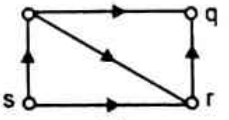
Let G = (V, E) be a graph. If the elements of E are represented by unordered pairs of vertices of G, then G is called an undirected graph. The above two graphs (ii), (iii) are undirected graphs.

(ii) **Directed Graph :** ***CS/CT/CE : W-06,08, S-11*** ***IT : S-09***

Let G = (V, E) be a graph. If the elements of E are represented by ordered pairs of vertices of G, then G is called a directed graph. Following are the examples of directed graphs (or digraphs).



(i)



(ii)

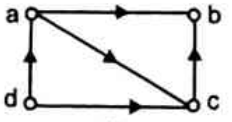
The edge (u, v) of a digraph G = (V, E) is an ordered pair where u is called the initial node (or vertex) and v is called the terminal node (or vertex).

(iii) **Mixed Graph :**

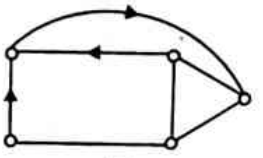
A graph G = (V, E) is said to be a mixed graph if it is neither a digraph nor an undirected graph.

For example :

Consider the following graph



(i)



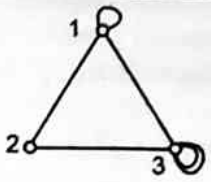
(ii)

(iv) **Loop :**

Let G = (V, E) be a graph. An edge of G, which joins a node v to itself is called a loop.

For example :

Consider the following graph.



Here node 1 has one loop, node 3 has 2 loops and node 2 does not have a loop.

(v) Null or Void Graph :

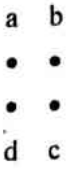
Let G = (V, E) be a graph. If the set of edges E is empty, then the graph G is said to be null graph or void graph.

For example :

Consider the following graph

(a)

(b)



(vi) Simple Graph :

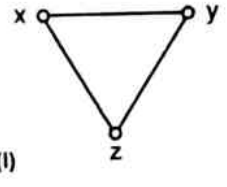
Let G = (V, E) be a graph. Then G is said to be a simple graph if

(1) G has no loops

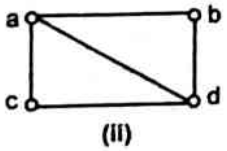
(2) There is exactly one edge between any given pair of vertices.

For example :

Consider the following graph



(I)

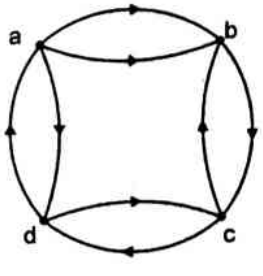


(II)

(vii) Parallel Edges :

Let G = (V, E) be a graph. Two edges of G are said to be parallel if they have same beginging and ending point. **For example :**

Consider the following graph



Here (a, b) is a parallel edge. Also (a, d) is a parallel edge. But (d, c) and (b, c) are not parallel edges.

IT : W-06

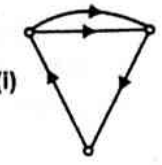
(viii) Multigraph :

Let G = (V, E) be a graph. Tlhen G is said to be a multigraph if it has some multiple edges.

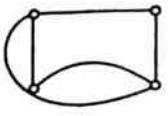
For example :

Consider the following graph

(I)



(ii)



CS/CT/CE : S-13

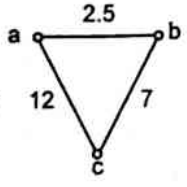
(ix) Weighted Graph :

Let G = (V, E) be a graph. If every edge in G is assigned some weight, then G is said to be a weighted graph.

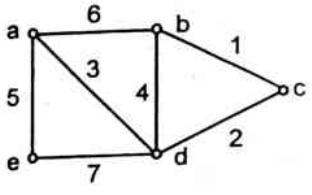
For example :

Consider the following graph

(I)



(ii)

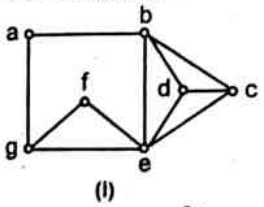


(x) Finite Graph :

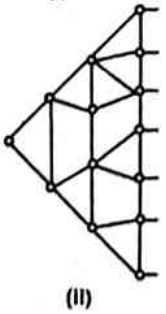
Let G = (V, E) be a graph. Then G is said to be finite if bot the sets V and E are finite. Otherwise it is said to be an infinite graph.

For example :

Consider the following graph



(I)



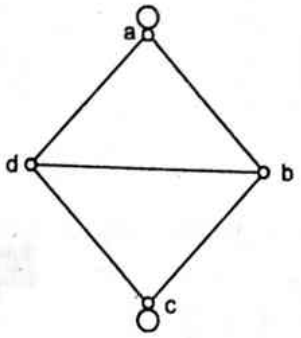
(II)

(xi) Pseudo Graph :

Let G = (V. E) be a graph. Then G is said to be a pseudograph if G has loops but does not have multiple edges.

For example :

Consider the following graph

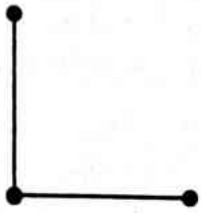


(xii) Isolated node :

In a graph G = (V, E) a node which is not adjacent to any other node is called an isolated node

For example :

Consider the following graph in which node x is an isolated node.



(xiii) Incidence and Degree :

(1) Incidence for undirected graph :

Let G = (V, E) be an undirected graph. Then the ordered pair (u,v) is an edge incident on u and v.

For example :

(2) Incidence for digraph :

Let G = (V, E) be a digraph. Then the ordered pair (u, v) is an edge and is said to be incident from u and to be incident to v.

For example :

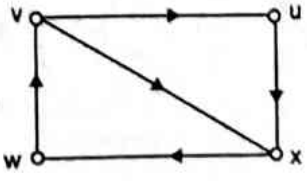
(3) Indegree and outdegree of a Digraph :

CS/CT/CE : W-09,10, S-13, IT : W-09

Let G = (V, E) be a digraph. Then the number of edges ending at a vertex v is called the indegree of v. Similarly the number of edges begining at a vertex v is called the outdegree of v.

For example :

Consider the following graph.



Vertex	Indegree	Outdegree	Total Degree
u	1	1	2
v	1	2	3
w	1	1	2
x	2	1	3

A vertex whose indegree is zero is called a source and a vertex whose outdegree is zero is called a sink.

Note : In a digraph, a loop contributes 1 to indegree and 1 to outdegree of the vertex.

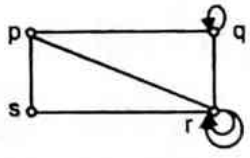
(4) Degree of a vertex in an undirected Graph :

CS/CT/CE : W-11

Let G = (V, E) is an undirected graph. Then the number of edges incident with a vertex v with self loops counted twice is called the degree of the vertex v.

For example :

Consider the following graph.



Vertex	Degree
p	3
q	4
r	7
s	2

A vertex whose degree is zero is called an isolated vertex. A vertex whose degree is one is called a pendant vertex.

MATRIX REPRESENTATION OF GRAPHS

Representing graphs by means of matrices, we can study the structural properties of graph from algebraic point of view. There are various ways of representing graphs by matrices.

(A) Adjacency Matrix :

(i) Representation of Undirected Graph :

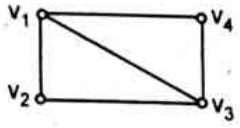
Let G = (V, E) be a simple graph with n vertices. Then an n × n matrix A = (a_{ij}) defined by

$$a_{ij} = \begin{cases} 1; & \text{if } (v_i, v_j) \text{ is an edge} \\ 0; & \text{otherwise} \end{cases}$$

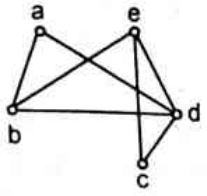
is called the adjacency matrix. Hence adjacency matrix is a bit matrix or a Boolean matrix as the elements are either 0 or 1.

Ex.1. Represent the following graphs using adjacency matrix.

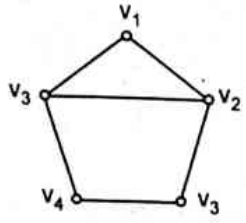
(i)



(ii)



(iii)



Soln.

(i) Here we order the vertices as v_1, v_2, v_3, v_4. Since there are four vertices, the adjacency matrix representing the graph will be a square matrix of order 4. (4 × 4 matrix).

The required adjacency matrix is

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

(ii) Here we order the vertices as a, b, c, d, e. Since there are five vertices, the adjacency matrix representing the graph will be a square matrix of order 5. (5 × 5 matrix).

The required adjacency matrix is.

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

(iii) Here we order the vertices as v_1, v_2, v_3, v_4, v_5. Since there are five vertices, the adjacency matrix representing the graph will be a square matrix of order 5. (5 × 5 matrix).

The required adjacency matrix

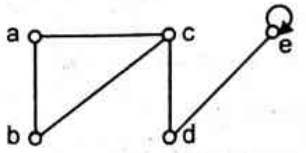
$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Ex.2. Draw an undirected graph representing the following adjacency matrices. **CS/CT/CE : S-10**

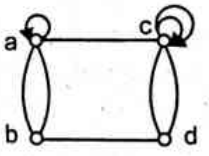
$$\textbf{(A)}\ A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \textbf{(B)}\ A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

Soln.

(A) The given adjacency matrix is a square matrix of order 5 × 5. Hence the graph will have 5 vertices say a, b, c, d, e. The required graph for the given adjacency matrix is.



(B) The given adjacency matrix is a square matrix of order 4 × 4. Hence the graph will have 4 vertices say a, b, c, d. The required graph for the given adjacency matrix is.



(ii) Representation of Digraph :

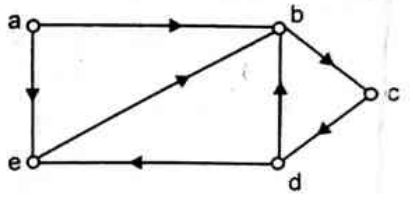
Let G = (V, E) be a simple graph with n vertices. Then an n × n matrix $A = (a_{ij})$ defined by

$$a_{ij} = \begin{cases} 1; & \text{if there is an edge from } v_i \text{ to } v_j \\ 0; & \text{Otherwise} \end{cases}$$

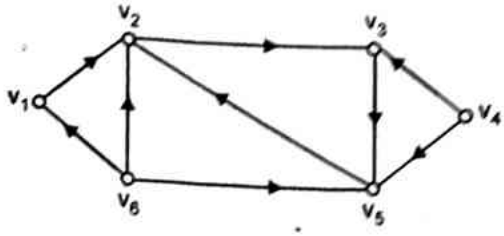
is called the adjacency matrix.

Ex.3. Represent the following digraphs by adjacency matrix.

(i)



(ii)



Soln.

(i) Here we order the vertices as a, b, c, d, e. Since there are five vertices, the adjacency matrix representing the graph will be a square matrix of order 5. (5 × 5 matrix).

The required matrix is.

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

(ii) Here we order the verticdes as v_1, v_2, v_3, v_4, v_5. Since there are six vertices, the adjacency matrix representing the graph will be a square matrix of order 6. (6 × 6 matrix).

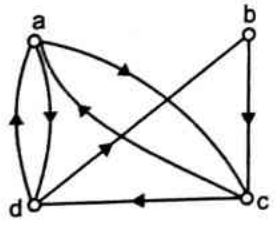
The required matrix is.

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Ex.4. Draw a digraph corresponding to the following adjacency matrix A. **CS/CT/CE : S-10**

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Soln. The given matrix is a square matrix of order 4 × 4. Hence the graph will have 4 vertices say a, b, c, d. the required digraph for the given adjacency matrix is



(B) Incidence Matrix :

(i) Representaiton of Undirected Graph :

Let G = (V, E) be an undirected graph with n vertices and m edges. The incidence matrix $B = (b_{ij})$, is an $n \times m$ matrix whose elements are given by

$$b_{ij} = \begin{cases} 1; & \text{if edge } e_j \text{ is incident with vertex } v_i \\ 0; & \text{otherwise} \end{cases}$$

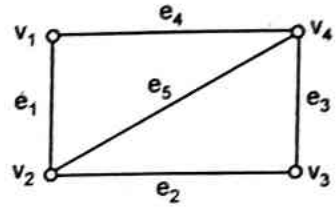
(ii) Representaiton of Digraph :

Let G = (V, E) be an digraph with n vertices and m edges. The incidence matrix $B = (b_{ij})$, is an $n \times m$ matrix whose elements are given by

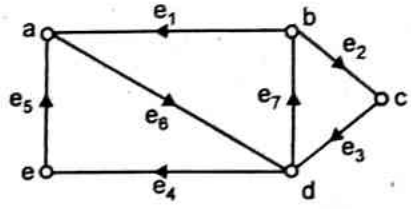
$$b_{ij} = \begin{cases} -1; & \text{if arc } j \text{ is directed towards vertex } v_i \\ 1; & \text{if arc } j \text{ is directed away from vertex } v_i \\ 0; & \text{otherwise} \end{cases}$$

Ex.5. Represent the following graphs by means of incidence matrix.

(i)



(ii)



Soln.

(i) Here we order the vertices as v_1, v_2, v_3, v_4 and edges as e_1, e_2, e_3, e_4, e_5. These are 4 vertices and 5 edges. Hence the incidence matrix will be have order 4 × 5. The required incidence matrix is

$$B = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

(ii) Here we order the vertices as a, b, c, d, e and edges as $e_1, e_2, e_3, e_4, e_5, e_6, e_7$. These are 5 vertices and 7 edges. Hence the incidence matrix will be have order 5 × 7. The required incidence matrix is

$$B = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \end{bmatrix} \end{array}.$$

ISOMORPHIC GRAPHS

CS/CT/CE : S-08, IT : W-10

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be any two graphs. Then G_1 and G_2 are said to be isomorphic if there exists a function $f: V_1 \to V_2$ such that

(i) f is one-one and onto

(ii) $(a, b) \in E_1 \Rightarrow (f(a), f(b)) \in E_2, \forall a, b \in V_1$

If G_1 is isomorphic to G_2 then it is denoted by $G_1 \approx G_2$ or $G_1 \cong G_2$.

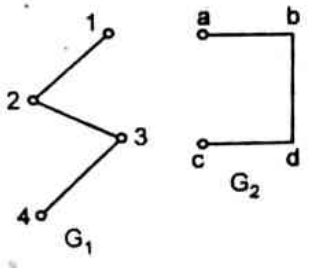
Note :

(i) If the number of vertices in G_1 and G_2 are unequal, then G_1 and G_2 cannot be isomorphic.

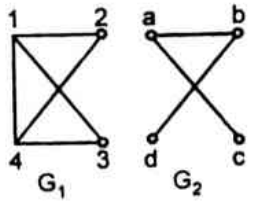
(ii) If the number of edges in G_1 and G_2 are unequal, then G_1 and G_2 cannot be isomorphic.

Ex.6.(a) Find which of the following graphs are isomorphic. **IT : S-11**

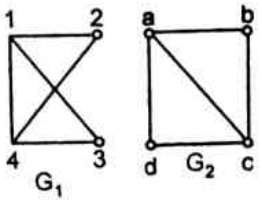
(i)



(ii)



(iii)



Soln.

(i) Here $V_1 = V(G_1) = \{1, 2, 3, 4\}$,

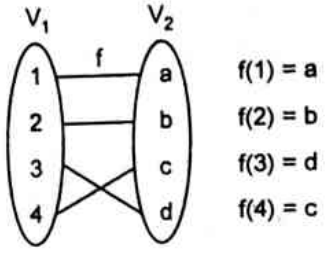
$V_2 = V(G_2) = \{a, b, c, d\}$

$E_1 = E(G_1) = \{(1, 2), (2, 3), (3, 4)\}$

$E_2 = E(G_2) = \{(a, b), (b, d), (d, c)\}$

Let $f: V_1 \to V_2$

From the figure it is clear that $f: V_2 \to V_2$ is both one-one and onto.



Also $(1, 2) \in E_1 \Rightarrow (f(1), f(2)) = (a, b) \in E_2$

Similarly $(2, 3) \in E_1 \Rightarrow (f(2), f(3)) = (b, d) \in E_2$

and $(3, 4) \in E_1 \Rightarrow (f(3), f(4)) = (d, c) \in E_2$

$\therefore\ (a, b) \in E_1 \Rightarrow (f(a), f(b)) \in E_2, \forall a, b \in V_1$

$\therefore\ G_1 \cong G_2$

(ii) Here $V_1 = V(G_1) = \{1, 2, 3, 4\}$,

$V_2 = V(G_2) = \{a, b, c, d\}$

$E_1 = (G_1) = \{(1, 2), (1, 3), (1, 4), (2, 4), (3, 4)\}$

$E_2 = (G_2) = \{(a, b), (a, c)\ (b, d)\}$

$n(E_1) = 5, n(E_2) = 3$

$\therefore\ n(E_1) \neq n(E_2)$

$\therefore\ G_1 \ncong G_2$

(iii) Here $V_1 = (G_1) = \{1, 2, 3, 4\}$,

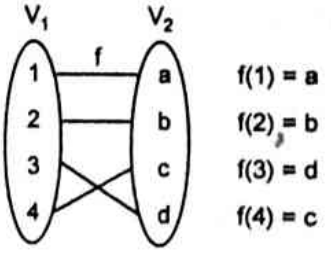
$V_2 = V(G_2) = \{a, b, c, d\}$

$E_1 = (G_1) = \{(1, 2), (1, 3), (1, 4), (2, 4), (3, 4)\}$

$E_2 = (G_2) = \{(a, b), (a, c), (a, d), (b, c), (d, c)\}$

Let $f: V_1 \to V_2$

From the figure it is clear that $f: V_1 \to V_2$ is both one-one and onto.



Also $(1, 2) \in E_1 \Rightarrow (f(1), f(2)) = (a, b) \in E_2$

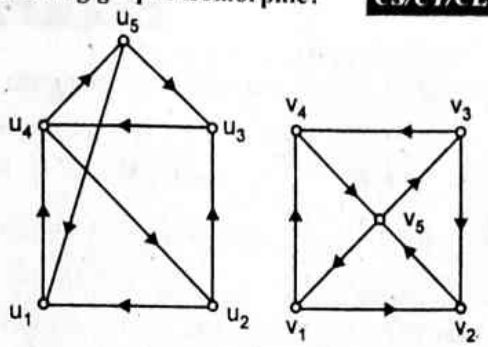
$(1, 4) \in E_1 \Rightarrow (f(1), f(4)) = (d, c) \in E_2$

$(3, 4) \in E_1 \Rightarrow (f(3), f(4)) = (d, c) \in E_2$

$\therefore\ (a, b) \in E_1 \Rightarrow (f(a), f(b)) \in E_2, \forall a, b \in V_1$

$\therefore\ G_1 \cong G_2$.

Ex.6.(b) Define isomorphism of two graphs. Are the following graphs isomorphic? **CS/CT/CE : S-08**



Soln. We define the correspondence as :

$v_1 \to u_2$,

$v_2 \to u_1$,

$v_3 \to u_5$,

$v_4 \to u_3$,

$v_5 \to u_4$.

Also

$(v_3, v_2) \to (u_5, u_1)$,

$(v_4, v_5) \to (u_3, u_4)$,

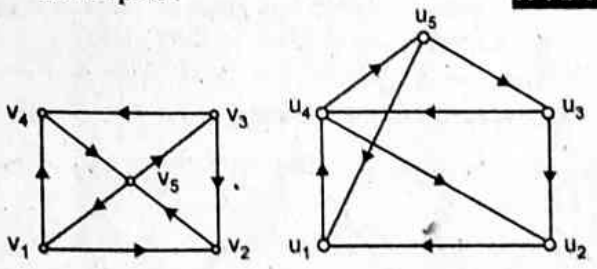
$(v_3, v_4) \to (u_5, u_3)$,

$(v_5, v_1) \to (u_4, u_2)$,

$(v_5, v_3) \to (u_4, u_5)$,

By the above correspondence, above two digraphs are isomorphic.

Ex.6.(c) Check whether the digraphs given below are isomorphic : **IT : S-12**



Soln. We define the correspondence as :

$v_1 \to u_5$,

$v_2 \to u_1$,

$v_5 \to u_4$,

$v_3 \to u_3$,

$v_4 \to u_2$,

Also

$(v_1, v_2) \to (u_5, u_1)$,

$(v_5, v_1) \to (u_4, u_5)$,

$(v_5, v_3) \to (u_4, u_3)$,

$(v_3, v_4) \to (u_3, u_2)$,

$(v_1, v_2) \to (u_5, u_1)$,

By the above correspondence, above two digraphs are isomorphic.

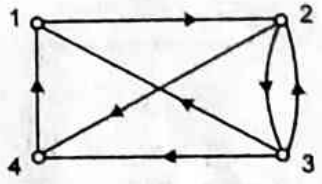
Ex. 7. Draw the digraphs of the following adjacency matrices A and B. Are these graphs isomorphic? Also draw the digraph corresponding to A^T and B^T. Are these graphs isomorphic?

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

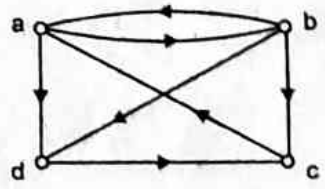
CS/CT/CE : W-10, IT : S-08

Soln.

(i) Since the order of matrices A and B is 4 × 4 hence their digraphs will have 4 vertices say 1, 2, 3, 4 and a, b, c, d resp. the required digraphs are



Graph of A (G_A)



Graph of B (G_B)

Here $V_A = V(G_A) = \{1, 2, 3, 4\}$,

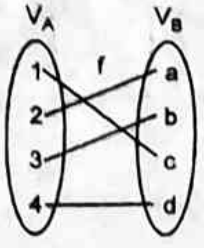
$V_B = V(G_B) = \{a, b, c, d\}$

$E_A = E(G_A) = \{(1, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1)\}$

$E_B = E(G_B) = \{(a, b), (a, d), (b, a), (b, c), (b, d), (c, a), (d, c)\}$.

Let $f : V_A \to V_B$

From the figure it is clear that $f : V_A \to V_B$ is both one-one and onto.



f(1) = c

f(2) = a

f(3) = b

f(4) = d

Also $(2, 3) \in E_A \Rightarrow (f(2), f(3)) = (a, b) \in E_B$

$(3, 1) \in E_A \Rightarrow (f(3), f(1)) = (b, c) \in E_B$

$(3, 4) \in E_A \Rightarrow (f(3), f(4)) = (b, d) \in E_B$

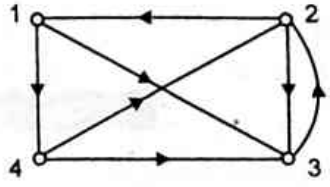
$\therefore\ (a, b) \in E_A \Rightarrow (f(a), f(b)) \in E_B, \forall a, b \in E_A$

$\therefore\ G_A \cong G_B$

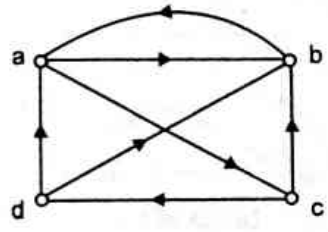
Now

$$A^T = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad B^T = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

(ii) The digraphs of A^T and B^T are drawn below.



Graph of A^T (G_{A^T})



Graph of B^T (G_{B^T})

Here,

$V_{A^T} = V(G_{A^T}) = \{1, 2, 3, 4\}$,

$V_{B^T} = V(G_{B^T}) = \{a, b, c, d\}$

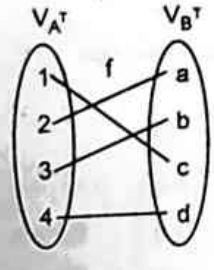
$E_{A^T} = E(G_{A^T}) = \{(1, 3), (1, 4), (2, 1), (2, 3), (3, 2), (4, 2), (4, 3)\}$

$E_{B^T} = E(G_{B^T}) = \{(a, b), (a, c), (b, a), (c, b), (c, d), (d, a), (d, b)\}$.

Let $f : V_{A^T} \to V_{B^T}$

This figure is similar to that of the previous figure

$\therefore\ G_{A^T} \cong G_{B^T}$



$f(1) = c$

$f(2) = a$

$f(3) = b$

$f(4) = d$

TYPES OF GRAPHS

CS/CT/CE : W-10,11

(1) Complete Graph :

A graph $G = (V, E)$ is said to be complete if every vertex of G is adjacent to every other vertex. A complete graph with n vertices is denoted by K_n.

There are $\frac{n(n-1)}{2}$ edges in K_n.

For example :

Consider the following graph.

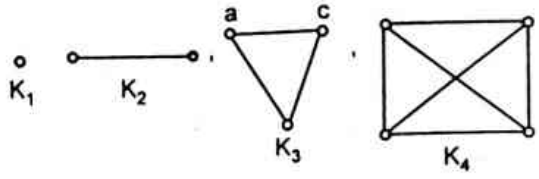


Fig.Complete Graph

(2) Linear Graph :

A graph $G = (V, E)$ is said to be linear if all the vertices of G appear along a straight line. A linear graph with n vertices is denoted by L_n.

For example :

Consider the following graph.

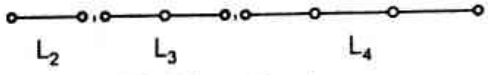


Fig. Linear Graph

(3) Regular Graph : **CS/CT/CE : W-09, S-11**

A graph G = (V, E) is said to be regular if all the vertices of G have the same degree. If the degree of each vertex is r, then the graph is called a regular graph of degree r.

For example :

Consider the following graph.

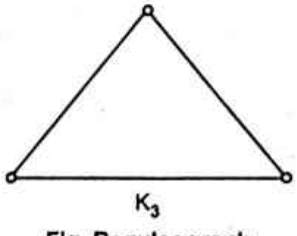


Fig. Regular graph

In K_3, each vertex has degree 2. Hence K_3 is a regular graph.

(4) Cycle :

The cycle C_n $(n \geq 3)$ consists of n vertices namely $v_1, v_2, \ldots, v_n$ and edges (v_1, v_2), (v_2, v_3), ... (v_{n-1}, v_n), (v_n, v_1)

For example :

Consider the following graph.

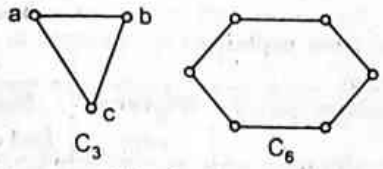


Fig. Cycles

(5) Wheel :

The wheel W_n is obtained when an additional vertex to the cycle C_n is connected with all other vertices.

For example :

Consider the following graph.

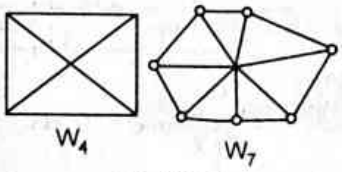


Fig. Wheel

(6) Bi-Partite Graph :

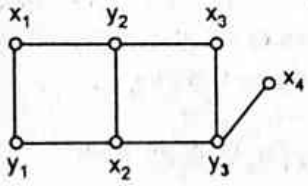
A graph G = (V, E) is said to be bi-pratite if the vertex set V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge in E connects a vertex in V_1 and a vertex in V_2.

For example :

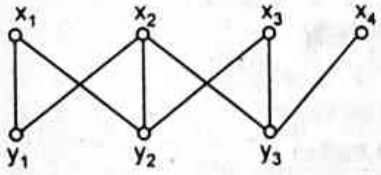
(i) Consider the graph.

Let $V_1 \{x_1, x_2, x_3, x_4\}$

$V_2 = \{y_1, y_2, y_3\}$



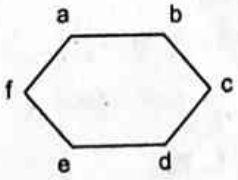
Then



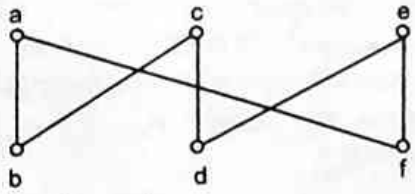
is a bi-partite graph.

(ii) Consider the graph.

Let $V_1 = \{a, b, c\}$, $V_2 = \{d, e, f\}$.



Then



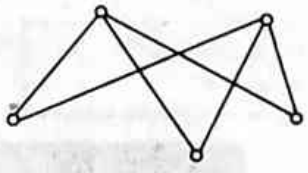
is a bi-partite graph.

(7) Complete Bi-Partite Graph :

A complete bi-partite graph is a bi-partite graph in which each vertex in V_1 is adjacent to each vertex in V_2. It is denoted by $K_{m,n}$ where V_1 contains m vertices and V_2 contains n vertices.

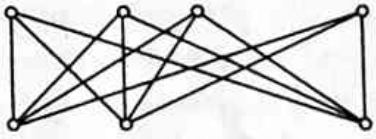
For example :

Consider the following graph.



$K_{2,3}$

Fig. Spanning graphics



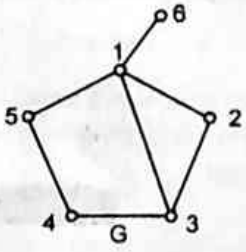
$K_{4,3}$

(8) Subgraph : **CS/CT/CE : S-11**

If G and H are any two graphs with vertex sets V(G) and V(H) and edge sets E(G) and E(H) respectively such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is called the subgraph of G and G is called the supergraph of H.

For example :

Let G be the graph.



Then

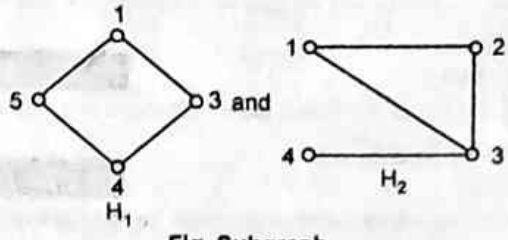


Fig. Subgraph

If $V(H) \subset V(G)$ and $E(H) \subset (E(G)$, then H is called a proper subgraph of G. If $V(H) = V(G)$, then H is called a spanning subgraph of G. A spanning subgraph need not contain all the edges of G.

For example :

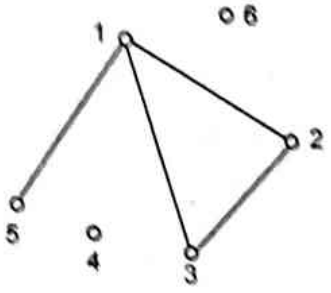


Fig. Spanning graphics

PATHS, REACHABILITY AND CONNECTEDNESS

(i) Path : **CS/CT/CE : W-06,08,10, S-06,09,11** **IT : S-06,10, W-07**

A path in a graph G = (V, E) is a sequency of the form $v_1, v_2, v_3 \ldots, v_k$ where each vertex is adjacent to the next. In other words, starting with a vertex v_1 and traveling along the vertices (v_1, v_2), (v_2, v_3), ..., One can reach a vertex v_k.

(ii) Length :

The number of edges appearing in the sequence of a path is called the length of the path. If the length is zero, then the path has no edges. A path which contains only a single vertex is called Trivial path and such a vertex is called Trivial vertex.

(iii) Closed Path :

A closed path is a path whose beginning and ending points are same, otherwise the path is called an open path.

(iv) Trail :

A trail from a vertex v to a vertex u is a path that does not contain any repeated edge. Repeated vertices are allowed in a trail.

(v) Simple path : **IT : S-06,09,13, W-07,09,11**

A simple path from a vertex v to a vertex u is a path that does not contain any repeated vertex and repeated edge.

(vi) Circuit : **CS/CT/CE : W-11**

A circuit is a closed path that does not contain any repeated edge.

(vii) Cycle : **CS/CT/CE : S-13**

A cycle is a circuit that does not have a repeated vertex except the first and the last.

(viii) Elementary path : **IT : S-09, W-09**

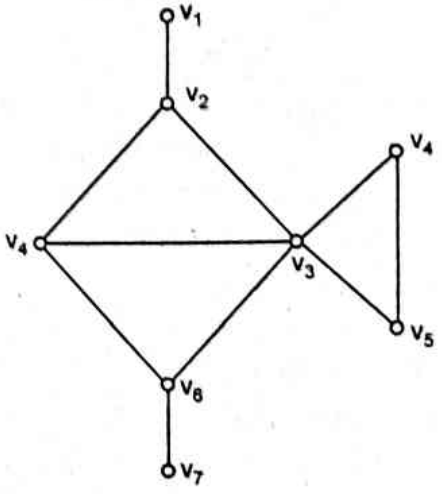
A path in graph G = (V, E) is called elementary if no vertex occurs more than once.

All the above definitions are arranged in tabular form as follows :

	Repeated vertex	Repeated edge	Same starting and ending point
Path	allowed	allowed	allowed
Closed Path	allowed	allowed	Yes
Trail	allowed	no	allowed
Simple Path	no	no	no
Circuit	allowed	no	yes
Cycle	First and last only	no	yes

For example :

Consider the graph.



(1) Path :

(i) $v_1\ v_2\ v_3\ v_6\ v_8$

(ii) $v_8\ v_3\ v_4\ v_5\ v_3\ v_6\ v_7$

(2) Trail :

(i) $v_8\ v_2\ v_3\ v_4\ v_5\ v_3 v_6$

(ii) $v_7\ v_6\ v_3\ v_2\ v_8\ v_6$

(3) Simple Path :

(i) $v_8\ v_2\ v_3\ v_4\ v_5$

(ii) $v_2\ v_3\ v_6\ v_7$

(4) Closed Paths :

(i) $v_8\ v_2\ v_3\ v_6\ v_8$

(ii) $v_4\ v_5\ v_3\ v_8\ v_2\ v_3\ v_4$

(5) Circuit :

(i) $v_6\ v_8\ v_2\ v_3\ v_6$

(ii) $v_5\ v_4\ v_3\ v_2\ v_8\ v_3\ v_5$

(6) Cycle :

(i) $v_3\ v_4\ v_5\ v_3$

(ii) $v_2\ v_3\ v_6\ v_8\ v_2$

Reachability : ***CS/CT/CE : W-06,08,09,10, IT : S-09***

A node v in a simple gralph G is said to be reachable from the node u of G if there exists a path from u to v. The set of nodes which are reachable from a given node v is called the reachable set of v and is denoted by R(v).

For any subset S of the vertex set V, the reachable set of S is the set of all nodes which are reachable from any node of S and this set is denoted by R(S).

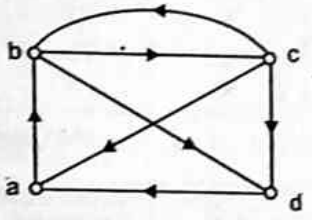
Ex.8.(a) Draw a digraph corresponding to the following adjacency matrix and interprete the results AA^T, A^TA, A^2, A^3, A^4.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

CS/CT/CE :S-07,12, W-08,

IT : W-06, S-07

Soln. The given adjacency matrix is a square matrix of order 4×4. Hence the graph will have 4 vertices say a, b, c, d. The required digraph for the given adjacency matrix is



$$\text{Now } A^T = \text{Transpose of } A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 1 & 3 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad A^TA = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

Interpretation of AA^T : The diagonal entries of AA^T shows the outdegree of the nodes.

Node	Outdegree
a	1
b	2
c	3
d	1

Interpretation of A^TA : The diagonal entries of A^TA shows the indegree of the nodes.

Node	Outdegree
a	2
b	2
c	1
d	2

$$\text{Now } A^2 = A \,.\, A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Interpretation of A^2 : The elements of first row of A^2 show the number of paths of length 2 from vertex a to the vertices a, b, c, d respectively.

Edges	No. of paths of length 2
(a, a)	0
(a, b)	0
(a, c)	1
(a, d)	1

$$\text{Now } A^3 = A^2 \,.\, A = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ 2 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Interpretation of A^3 : The elements of first row of A^3 show the number of paths of length 3 from vertex a to the vertices a, b, c, d respectively.

Edges	No. of paths of length 3
(a, a)	2
(a, b)	1
(a, c)	0
(a, d)	1

$$\text{Now } A^4 = A^3 \,.\, A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 2 & 2 & 3 \\ 3 & 3 & 2 & 3 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

Interpretation of A^4 : The elements of first row of A^4 show the number of paths of length 4 from vertex a to the vertices a, b, c, d respectively.

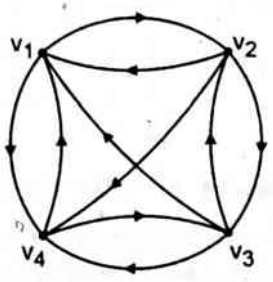
Edges	No. of paths of length 4
(a, a)	1
(a, b)	2
(a, c)	1
(a, d)	1

Node Base : **IT : W-06**

- In a digraph G = (V, E), a subser $A \subseteq V$ is called a node base if its reachable set is V and if no proper subset of A has this property.
- Every isolated point of a digraph must be present in a node base. Any node whose indegree is zero must be present in a node base.
- No node in the node base is reachable from another node in the node base. From the nodes lying on an elementary cycle any one node could be chosen as an element of a node base.
- Any node that does not have indegree zero and does not lie on a cycle connot be present in anode base. In an acyclic graph, a node base contains only those nodes whose indegree is zero.
- For a given digaph, we may have more than one node base, however every node base has the same number of nodes.

For example :

Consider the digraph.



Here $R(v_1) = \{v_2, v_3, v_4\}$

$R(v_2) = \{v_1, v_3, v_4\}$

$R(v_3) = \{v_1, v_2\ v_4\}$, $R(v_4) = \{v_1, v_2, v_3\}$

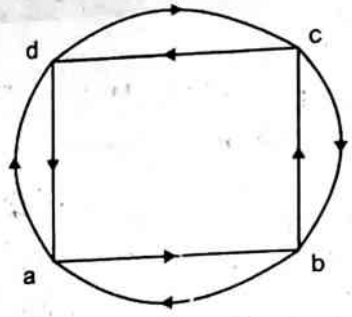
$R(\{v_1, v_2\}) = \{v_3, v_4\}$ ... and so on

Ex.8.(b) Draw the digraph represented by the given adjacency matrix and interpret AA^T, A^TA, A^2.

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

CS/CT/CE : S-13

Soln. The given adjacency matrix is a square matrix of order 4×4. Hence the graph will have four vertices say a, b, c, d. The required digraph for the given adjacency matrix is



Now A^T = Transpose of $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

Now

$$AA^T = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}, \quad A^TA = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}$$

Interpretation of AA^T : The diagonal of AA^T shows the outdegree of nodes.

Node	Outdegree
a	2
b	2
c	2
d	2

Interpretation of A^TA : The diagonal of A^TA shows the indegree of node.

Node	Outdegree
a	2
b	2
c	2
d	2

Now $A^2 = A \circ A = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}$.

Interpretation of A^2 : The elements of first row of A^2 shows the number of paths of length 2 from vertex a to the vertices a, b, c, d respectively.

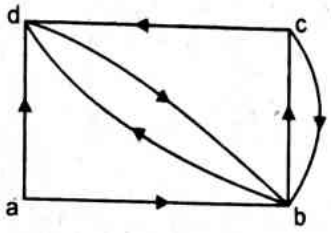
Edges	No. of paths of length 2
a	2
b	0
c	2
d	0

Ex.8.(c) Draw digraph corresponding to

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$ **and interpret AA^T, A^TA, A^2.**

CS/CT/CE : W-12

Soln. The given adjacency matrix is a square matrix of order 4×4. Hence the graph will have four vertices say a, b, c, d. The required digraph for the given adjacency matrix is



Now A^T = Transpose of $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

Now

$$AA^T = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \quad A^TA = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 3 \end{bmatrix}$$

Interpretation of AA^T: The diagonal of AA^T shows the outdegree of nodes.

Node	Outdegree
a	2
b	2
c	2
d	1

Interpretation of A^TA: The diagonal of A^TA shows the indegree of node.

Node	Indegree
a	0
b	3
c	1
d	3

Now $A^2 = A \circ A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

Interpretation of A^2: The elements of first row of A^2 shows the number of paths of length 2 from vertex a to the vertices a, b, c, d respectively.

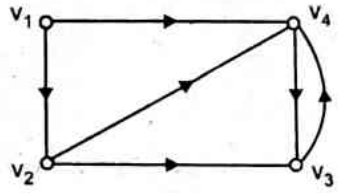
Edges	No. of paths of length 2
a	0
b	1
c	1
d	1

Ex.9. Draw the digraph correcponding to the adjacency matrix.

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Find three different elementary paths from v_1 to v_3. What is the shortest distance between v_1 and v_3? Is there a cycle in this graph? **IT : W-08**

Soln. Since the given adjacency matrix is of order 4×4, the digraph will have 4 vertices say v_1, v_2, v_3, v_4. The required digraph is



Three different elementary paths from v_1 to v_3 are

$P_1 = \{v_1\ v_4\ v_3\}$,

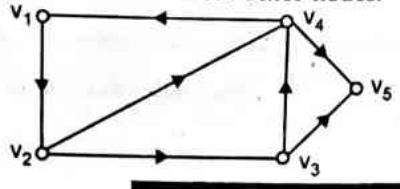
$P_2 = \{v_1\ v_2\ v_3\}$,

$P_3 = \{v_1\ v_2\ v_4\ v_3\}$.

From P_1, P_2, P_3 it is clear that the shortest distance between v_1 and v_3 is 2.

In this digraph the paths $v_3\ v_4\ v_3$ and $v_4\ v_3\ v_4$ are cycles.

Ex.10(a) Find all the indegrees and outdegrees of the nodes of the following digraph. Give all the elementary cycles of this digraph. Obtain an acyclic digraph by deleting one edge. List all the nodes which are reachable from other nodes.



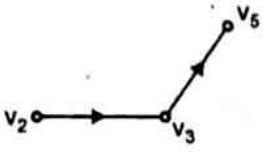
CS/CT/CE : S-09, IT : S-07, W-09

Soln.

Node	Indegree	Outdegree	Total degree
v_1	1	1	2
v_2	1	2	3
v_3	1	2	3
v_4	2	2	4
v_5	2	0	2

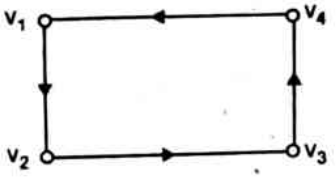
The elementay cycles of this digraph are :

$v_1\ v_2\ v_4\ v_1$ and $v_1\ v_2\ v_3\ v_4\ v_1$. By deleting the edge (v_4, v_1) we can obtain an acyclic digraph as below.



Node	Reachable from nodes
v_1	v_2 , v_3, v_4
v_2	v_1 , v_3, v_4
v_3	v_1, v_2 , v_4
v_4	v_1, v_2 , v_3

Ex.10.(b) Find all the indegrees and outdegrees of the nodes of the following diagram. Give all the elementary cycles of this digraph. Obtain an acyaic diagrph by deleting one edge. List all the nodes which are reachable from other nodes. ***IT : S-11***



Soln.

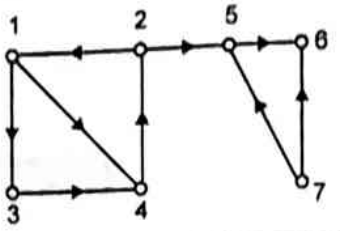
Node	Indegree	Outdegree	Total degree
v_1	1	1	2
v_2	1	1	2
v_3	1	1	2
v_4	1	1	2

The elementay cycles of this digraph are $v_1\ v_2\ v_3\ v_4$.

Node	Reachable from nodes
v_1	v_2 , v_3, v_4
v_2	v_1 , v_3, v_4
v_3	v_1, v_2 , v_4

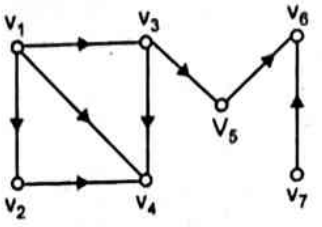
Ex.11. Find the node base of the following digraphs.

(i)

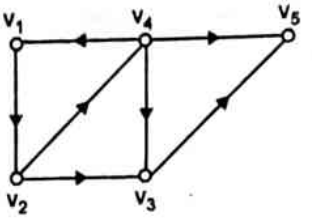


CS/CT/CE : W-06,08, IT : S-09

(ii)

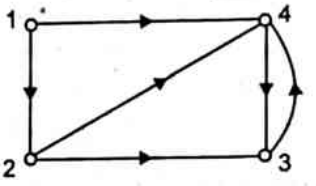


(iii)

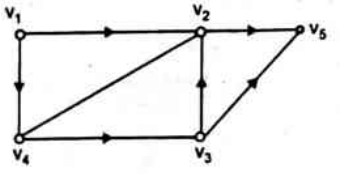


CS/CT/CE : S-13, IT : W-06

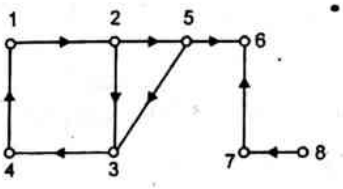
(iv)



(v)

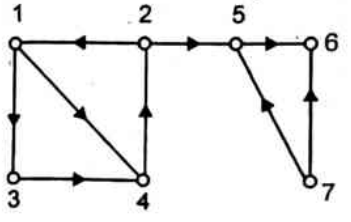


(vi)



Soln.

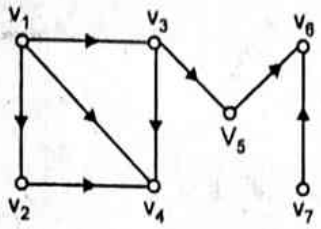
(i)



Here R (1) = {1, 2, 3, 4, 5, 6, 7},
R (2) = {2, 5, 6, 7},
R (3) = {2, 3, 4, 5, 6, 7},

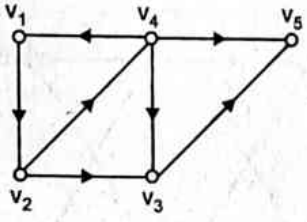
R (4) = {2, 4, 5, 6, 7},
R (5) = {5, 6, 7},
R (6) = {5, 6, 7}
R (7) = {5, 6, 7}
Vertex set = V = {1, 2, 3, 4, 5, 6, 7}
We observe that R (1) = V
∴ Node Base = {1}

(ii)



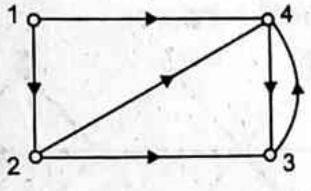
Here $R(v_1) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$,
$R(v_2) = \{v_2, v_4\}$,
$R(v_3) = \{v_3, v_4, v_5, v_6\}$,
$R(v_4) = \{v_4\}$, $R(v_5) = \{v_5, v_6\}$,
$R(v_6) = \{v_6\}$, $R(v_7) = \{v_6, v_7\}$,
Vertex set = V = $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$
We observe that none of the reachable sets is equal to the set of vertices V.
∴ This digraph does not contain node base.

(iii)



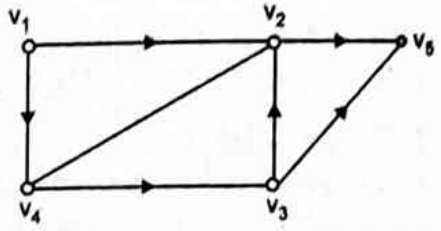
Here $R(v_1) = \{v_1, v_2, v_3, v_4, v_5\}$,
$R(v_2) = \{v_1, v_2, v_3, v_4, v_5\}$,
$R(v_3) = \{v_1, v_2, v_3, v_4, v_5\}$,
$R(v_4) = \{v_1, v_2, v_3, v_4, v_5\}$,
$R(v_5) = \{v_5\}$,
Vertex set = V = $\{v_1, v_2, v_3, v_4, v_5\}$
We observe that
$R(v_1) = R(v_2) = R(v_3) = R(v_4) = V$
∴ All the sets $\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}$ are node base.

(iv)



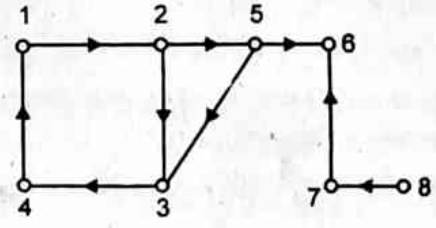
Here R(1) = {1, 2, 3, 4},
R (2) = {2, 3, 4}
R (3) = {3, 4},
R (4) = {3, 4}
Vertex set = V = {1, 2, 3, 4}
We observe that R(1) = V
∴ Node base = {1}

(v)



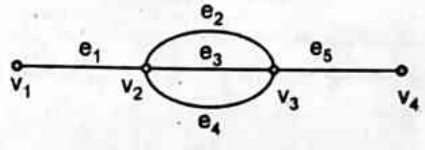
Here $R(v_1) = \{v_1, v_2, v_3, v_4, v_5\}$,
$R(v_2) = \{v_1, v_2, v_3, v_4, v_5\}$,
$R(v_3) = \{v_1, v_2, v_3, v_4, v_5\}$,
$R(v_4) = \{v_1, v_2, v_3, v_4, v_5\}$,
$R(v_5) = \{v_5\}$
Vertex set = V = $\{v_1, v_2, v_3, v_4, v_5\}$
We observe that
$R(v_1) = R(v_2) = R(v_3) = R(v_4) = V$
All the sets $\{V_1\}, \{V_2\}, \{V_3\}, \{V_4\}, \{V_5\}$ are node base.

(vi)



Here R(1) = {1, 2, 3, 4, 5, 6},
R (2) = {1, 2, 3, 4, 5, 6},
R (3) = {1, 2, 3, 4, 5, 6},
R (4) = {1, 2, 3, 4, 5, 6},
R(5) = {5, 6}, R {6}, R (7) = {6, 7},
R (8) = {7, 8}
Vertex set = V = {1, 2, 3, 4, 5, 6, 7, 8}
We observe that none of the reachable sets is equal to the set of vertices V. This digraph does not contain a node base.

Ex.12. Consider the following graph. CS/CT/CE : S-10



(i) How many simple paths are there from v_1 to v_4?

(ii) How many trails are there from v_1 to v_4?

(iii) How many paths are there from v_1 to v_4?

Soln.

(i) Simple paths from v_1 to v_4 are

$(v_1\ e_1\ v_2\ e_2\ v_3\ e_5\ v_4)$,

$(v_1\ e_1\ v_2\ e_3\ v_3\ e_5\ v_4)$,

$(v_1\ e_1\ v_2\ e_4\ v_3\ e_5\ v_4)$. Hence there are three simple paths from v_1 to v_4.

(ii) Trails from v_1 to v_4 are $(v_1\ e_1\ v_2\ e_2\ v_3\ e_5\ v_4)$,

$(v_1\ e_1\ v_2\ e_3\ v_3\ e_5\ v_4)$,

$(v_1\ e_1\ v_2\ e_4\ v_3\ e_5\ v_4)$,

$(v_1\ e_1\ v_2\ e_2\ v_3\ e_3\ v_2\ e_4\ v_3\ e_5\ v_4)$,

$(v_1\ e_1\ v_2\ e_2\ v_3\ e_4\ v_2\ e_3\ v_3\ e_5\ v_4)$,

$(v_1\ e_1\ v_2\ e_3\ v_3\ e_2\ v_2\ e_4\ v_3\ e_5\ v_4)$,

$(v_1\ e_1\ v_2\ e_3\ v_3\ e_4\ v_2\ e_2\ v_3\ e_5\ v_4)$,

$(v_1\ e_1\ v_2\ e_4\ v_3\ e_3\ v_2\ e_2\ v_3\ e_5\ v_4)$,

$(v_1\ e_1\ v_2\ e_4\ v_3\ e_2\ v_2\ e_3\ v_3\ e_5\ v_4)$. Hence there are nine trails from v_1 to v_4.

(iii) Paths from v_1 to v_4 will be infinite, because in a path a vertex as well as an edge may repeat. Some of the paths are $(v_1\ e_1\ v_2\ e_2\ v_3\ e_5\ v_4)$,

$(v_1\ e_1\ v_2\ e_4\ v_3\ e_3\ v_2\ e_1\ v_1\ v_2\ e_2\ v_3\ e_5\ v_4)$ etc.

Connectedness in Undirected graphs :

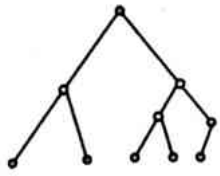
IT : S-06, W-07

- An undirected graph G = (V, E) is said to be connected if any pair of nodes are reachable from one another. That is, if u and v are any two nodes then there exists a path between u and v. if a graph is not connected, then it is called disconnected graph.
- A disconnected graph is the union of two or more connected subgraphs each pair of which has no common node.
- These disjoint connected subgrapys are called connected components. A graph A is a connected component of a graph G if and only if

(i) A is a subgraph of G

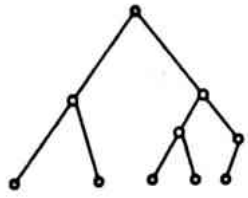
(ii) A is connected

(a)

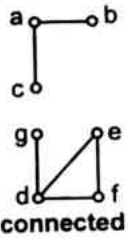


Connected Graph

(b)



Connected Graph



Disconnected Graph

(c)

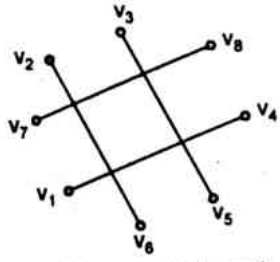
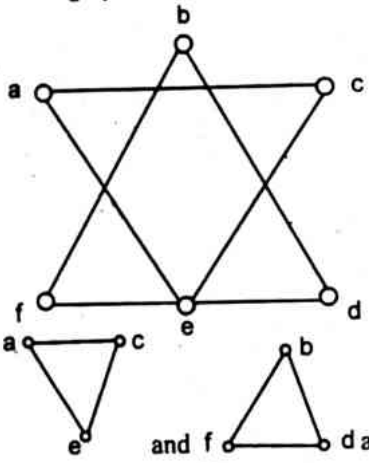


Fig. Disconnected graph

Consider the graph G



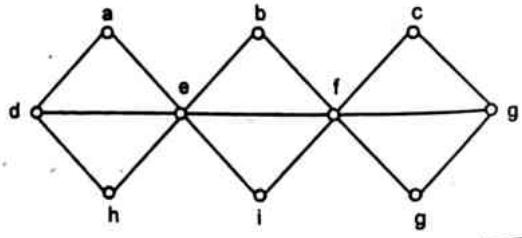
Then and are the connected components of G.

Distance :

Let G = (V, E) be a connected graph. Then the distance between the vertices a and b of G is denoted by d(a, b) and is defined as the length of shortest path between a and b.

For example :

Consider the following graph.



Then d (d, g) = 3, d (a, c) = 3, d (h, i) = 2 etc.

The distance function of a connected graph G = (V, E) with any vertices say a, b, c, has the following properties.

(i) d (a, b) ≥ 0 and d (a, b) = 0 iff a = b.

(ii) d (a, b) = d(b, a).

(iii) d (a, c) ≥ d (a, b) + d (b, c).

Cut Vertex :

Let G (V, E) be a connected graph. A vertex V of G is said to be a cut vertex, if the removal of V and all the edges incident with V produces more number of components. If v is a cut vertex of G, then G-V is a disconnected graph. Cut vertex is also called cut point or cut node.

Cut Edge or Bridge :

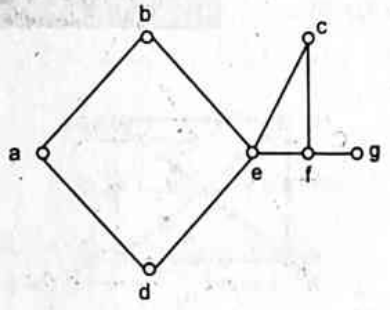
Let G = (V, E) be a connected graph. An edge e of G is said to be a cut edge or bridge if its removal produces a subgraph with more connected components. If e is a cut edge of G, then G-e is a disconnected graph.

Cut set :

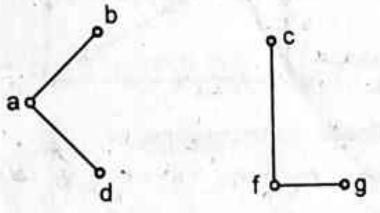
Let (G = (V, E) be a connected graph. Then the set of all minimum number of edges of G whose removal disconnects G is called cut set of G.

For example :

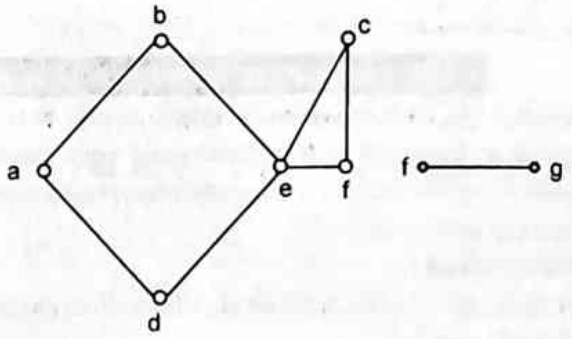
Consider the following connected graph.



Here e is the cut vertex. After removing vertex e and the edges incident with e viz (e, d), (e, b), (e, c) (e, f) the graph becomes.



The edge (f, g) is the bridge and the set {(f, g)} is a cut set.



Connectedness in Digraphs :

We know that in a digraph, if there is an edge from vertex v to vertex u, then there need not be an edge from vertex u to vertex v.

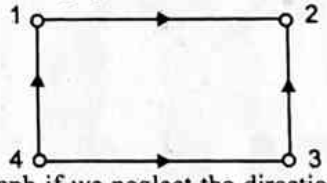
(i) **Weakly connected :**

CS/CT/CE : W-10, IT : S-10,13,W-11

A digraph is said to be weakly connected if it is connected as an undirected graph in which the direction of edge is neglected.

For example :

Consider the digraph



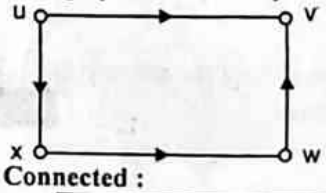
In this graph if we neglect the direction of edges, then it becomes an undirected graph and hence a weakly connected graph.

(ii) **Unilaterally Connected :** **IT : W-09**

A digraph is said to be unilaterally connected if for any pair of vertices u and v either ther is an edge from u to v or an edge from v to u.

For example :

Consider the digraph is unilaterally connected



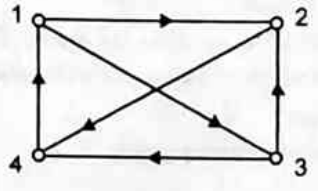
(iii) **Strongly Connected :**

CS/CT/CE : S-09,11, W-09, IT : W-09

A digraph is said to be strongly connected if for any pair of vertices u and v, there is a path from u to v and a path from v to u.

For example :

The following digraph is strongly connected.



EULER PATH

CS/CT/CE : W-07,11, S-09, IT : S-06, W-07

Let G = (V, E) be a connected graph. A path in G is called an Euler path if it includes every edge exactly once. Since the path includes every edge exactly once, it is also called Euler trail.

Euler Circuit :

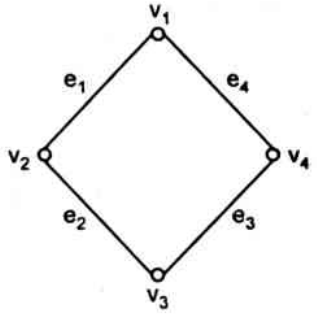
A Euler path which is a circuit is called a Euler circuit.

Euler Graph :

Let G = (V, E) be a connected graph. Then G is called a Euler graph if it contains a Euler circuit.

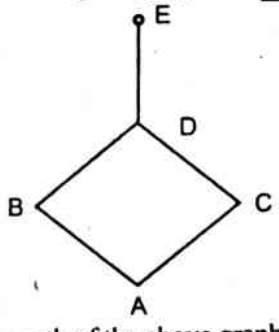
For example :

(i) Consider the following graph,



Here $v_1\, e_1\, v_2\, e_2\, v_3\, e_3\, v_4\, e_4\, v_1$ is a Euler path as well as a Euler circuit. Hence this is a Euler graph.

(ii) Now, consider the given graph. **CS/CT/CE : W-07**



The Eulerian path of the above graph is E-D-B-A-C.

Hamilton path : **CS/CT/CE : W-11**

Let G = (V, E) be a connected graph. A path in G is called a Hamilton path if it contains every vertex exactly once.

Hamilton circuit :

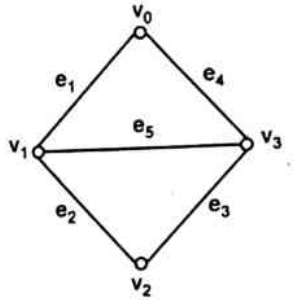
A Hamilton circuit is a circuit which contains every vertex exactly once except the first which appears twice.

Hamilton Graph :

Let G = (V, E) be a connected graph. Then G is called a Hamilton graph if it contains a Hamilton circuit.

For example :

Consider the following graph,



Hence $v_0\, e_1\, v_1\, e_2\, v_2\, e_3\, v_3\, e_4\, v_0$ is a Hamilton circuit. Hence this is a Hamilton graph.

Also $v_0\, e_1\, v_1\, e_2\, v_2\, e_3\, v_3$ is a Hamilton path.

Note :

(1) Let G be a graph with n vertices, then G has a Hamilton circuit if for any two non-adjacent vertices u and v, deg (u) + deg (v) $\geq$ n

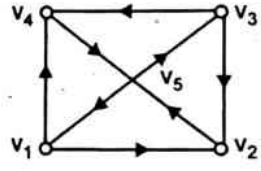
(2) G has Hamilttonian circuit if deg (u) $\geq \frac{n}{2}$ for each vertex u.

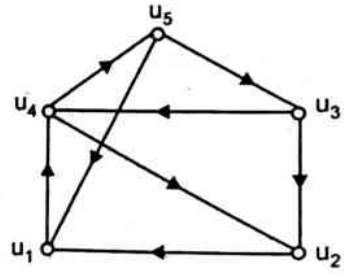
Ex.13.(a) Draw graphs corresponding to adjacency matrix and show that they are isomorphic.

$$M_1 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} \quad M_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Graph for M_1. **CS/CT/CE : S-06, W-07, S-11** **IT : S-07**

Soln.





We define the correspondence as :

$v_1 \to u_2$, $v_2 \to u_1$, $v_3 \to u_5$, $v_4 \to u_3$ & $v_5 \to u_4$

Also

$(v_3, v_2) \to (u_5, u_1)$, $(v_3, v_4) \to (u_5, u_3)$

$(v_4, v_5) \to (u_3, u_4)$, $(v_5, v_1) \to (u_4, u_2)$,
$(v_5, v_3) \to (u_4, u_5)$

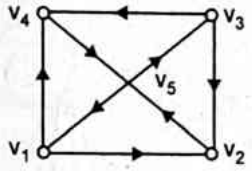
By the above correspondence two digraphs are isomorphic if two digraphs are such that the adjacency matrix of one can be obtained from the adjacency matrix of the other by interchanging some of the rows and the corresponding columns, then the digraphs are isomorphic. If G is graph corresponding to adjacency matrix A then $\tilde{G}$ (converse of G) is graph corresponding to matrix A' or A^T.

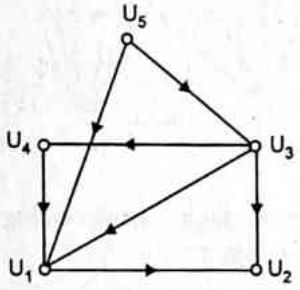
Ex.13.(b) Draw the graphs corresponding to adjacency matrices. ***IT : S-13***

$$M_1 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Find whether they are isomorphic.

Soln.





We define the correspondence as :

$v_2 \to u_3$,, $v_4 \to u_4$, $v_5 \to u_5$

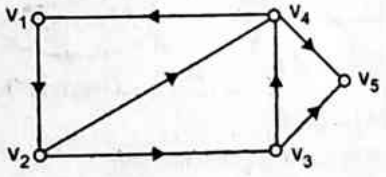
Also $(v_2, v_5) \to (u_3, u_5)$

$(v_4, v_5) \to (u_4, u_5)$

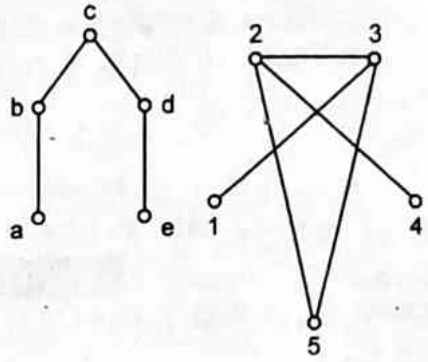
By the above correspondence two digraphs are not isomorphic.

Ex.14. Draw the complete graphs on 5 vertices. ***IT : S-06***

Soln.



Ex.15. Prove or disprove that graphs shown in the following figures are isomorphic. ***IT : S-06,08,09***



Soln. They are not isomorphic since Ist graph has 3 vertices of degree two and two vertices has degree 1 where as in the 2nd graph 2 vertices has degree 3, one vertices has degree 2 and 2 vertices has degree 1.

Ex.16. Draw the digraph corresponding to the matrix

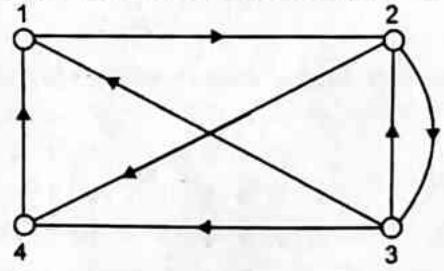
$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Find the transitive closure of the digraph.

CS/CT/CE : S-06,12, W-06,12, IT : S-10

Soln. Given : $M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

The digraph corresponding to this matrix is



Here R = {(1, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1)}

Now $R^2 = R \circ R$ = {(1, 3), (1, 4), (2, 1), (2, 2), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 2)}

$R^3 = R^2 \circ R$ = {(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 3), (4, 4)}

$R^4 = R^3 \circ R$ = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 4)}

$\therefore$ Transitive closure of

$R = R \cup R^2 \cup R^3 \cup R^4 \cup \ldots\ldots.$

$= \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 4)\}$

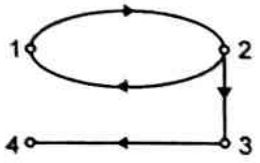
Ex.17. If R = {(1, 2), (2, 3), (3, 4), (2, 1)}, then draw the digraph and find M_R. Also find the transitive closure of R. **CS/CT/CE : W-07**

Soln. Given : $R = \{(1, 2), (2, 3), (2, 1), (3, 4)\}$

Let $X = \{1, 2, 3, 4\}$

$$\therefore\ M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Digraph of R :



Now $R^2 = R \circ R = \{(1, 1), (1, 3), (2, 2), (2, 4)\}$

$R^3 = R^2 \circ R = \{(1, 2), (1, 4), (2, 1), (2, 3)\}$

$R^4 = R^3 \circ R = \{(1, 1), (1, 3), (2, 2), (2, 4)\}$

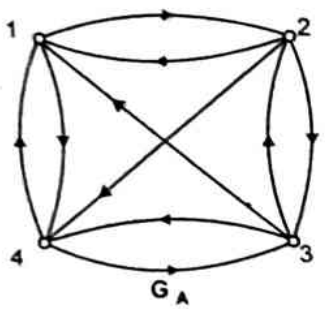
$\therefore$ Transitive closure of

$R = R \cup R^2 \cup R^3 \cup R^4 \cup \ldots\ldots$

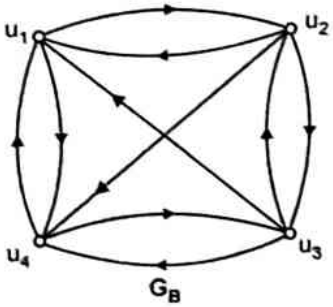
$\therefore\ R^+ = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 4)\}$

Ex.18. Draw digraphs corresponding to adjacency matrices $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ **and prove that digraphs of A and B are isomorphic.** **IT : W-07**

Soln.



G_A



G_B

We define the node correspondence as :

$1 \to u_2,\ 2 \to u_3,\ 3 \to u_4,\ 4 \to u_1$ so that

$(1, 2) \to (u_2, u_3),\ (2, 1) \to (u_3, u_2),$

$(3, 1) \to (u_4, u_2),$

$(1, 4) \to (u_2, u_1),\ (2, 3) \to (u_3, u_4),$

$(3, 2) \to (u_4, u_3),$

$(2, 4) \to (u_3, u_1),\ (3, 4) \to (u_4, u_1),$

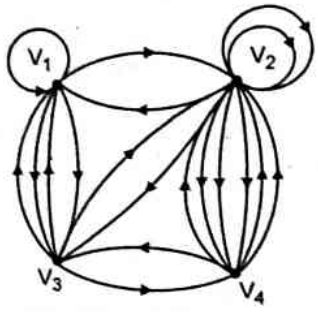
$(4, 1) \to (u_1, u_2),\ (4, 3) \to (u_1, u_4)$

$\therefore\ G_A \cong G_B$.

Ex.19. Draw a graph corresponding to the adjacency matrix

$$A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 2 & 1 & 3 \\ 2 & 1 & 0 & 1 \\ 0 & 3 & 1 & 0 \end{bmatrix}.$$ **CS/CT/CE : S-08, IT : W-09**

Soln. The given matrix is a square matrix of 4×4. hence the graph will have 4 vertices say V_1, V_2, V_3, V_4. The required graph for given adjacency matrix is

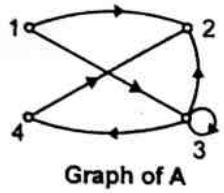


Ex.20. Draw the digraph corresponding to adjacency matrices A and B

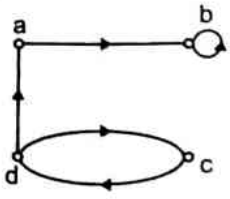
$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Are these two digraphs isomorphic? **IT : W-08**

Soln. Di-graphs corresponding to A and B are



Graph of A



Graph of B

From the di-graphs,

Graph of A has 5 edges and 1 loop.

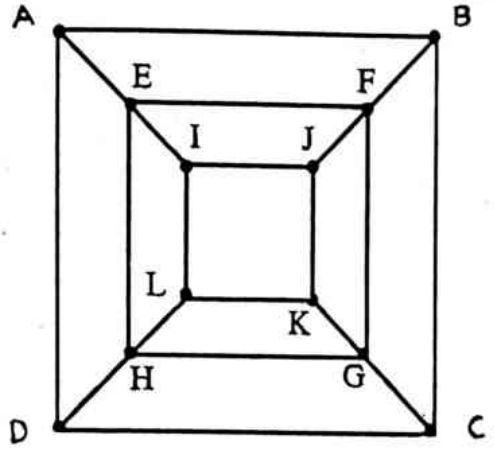
Graph of B has 4 edges and 1 loop.

Since there is one-one correspondence between vertices, but no one-one correspondence between edges.

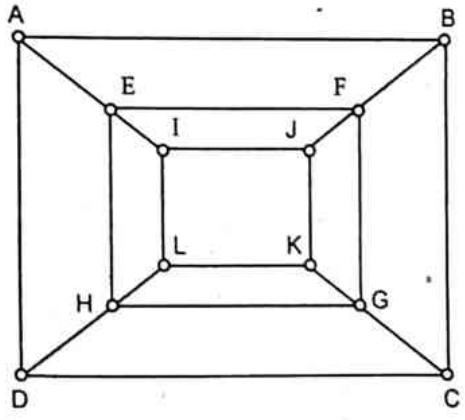
∴ These two digraphs are not isomorphic.

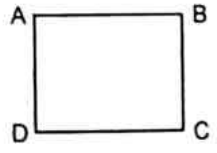
Ex.21. Let G be the graph and R be equivalence relation on V defined by partition.

{{I, J, K, L}, {A,E}, {F, B, C}, {D}, {H}} then find quotient graph G^R. **IT : S-09**



Soln. Quotient Graphs : Let G = (V, E, ϕ) be a graph and R be an equivalence relation on the set V. The quotient graph, G^R is a graph whose vertices are equivalence classes of V produced by R i.e. We obtain G^R by merging all the vertices in each equivalence class into a single vertex and combining the edge that are superimposed by such a process.



Then G^R = 

G^R is a quotient graph of graph G.

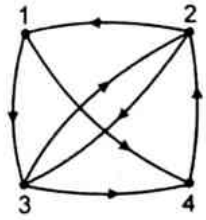
Ex.22. Draw the digraphs corresponding to adjacency matrix A, B, A^T and B^T where

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

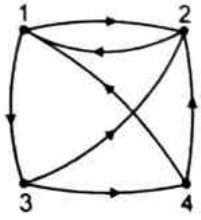
Show that the digraphs corresponding to A^T and B^T are isomorphic. **CS/CT/CE : W-09, S-12** **IT : S-10, W-11**

Soln. A = {(1, 3), (1, 4), (2, 1), (2, 3), (3, 2), (4, 2), (4, 3)}

B = {(1, 2), (1, 3), (2, 1), (3, 2), (3, 4), (4, 1), (4, 2)}



Diagraph of A

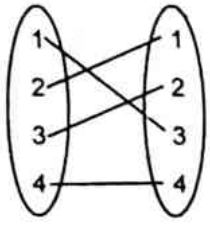


Diagraph of B

Here, $V_A = \{1, 2, 3, 4\}$

$V_B = \{1, 2, 3, 4\}$

Let $F : V_A \to V_B$



F(1) = 3

F(2) = 1

F(3) = 2

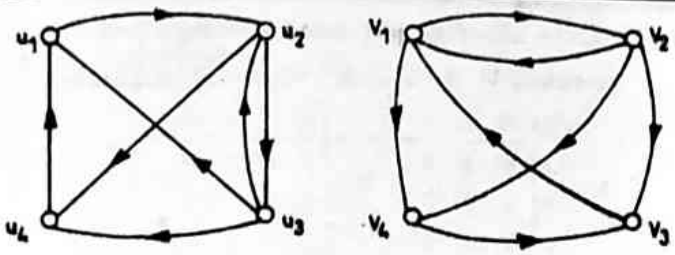
F(4) = 4

From the Fig. it is clear that $f : V_A \to V_B$ is one to one and onto.

∴ $G_A \cong G_B$

$$A^T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$



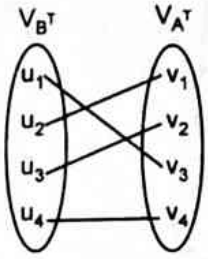
GRAPH OF A GRAPH OF B

Here $V_{A^T} = V(G_{A^T}) = \{u_1, u_2, u_3, u_4\}$

$V_{B^T} = V(G_{B^T})$

$= \{v_1, v_2, v_3, v_4\}$

Let $f : V_{A^T} \to V_{B^T}$



$f(u_1) = v_3$

$f(u_2) = v_1$

$f(u_3) = v_2$

$f(u_4) = v_4$

From the above Fig, it is clear that $f : V_{A^T} \to V_{B^T}$ is one to one and onto.

$\therefore\ G_{A^T} \cong G_{B^T}$

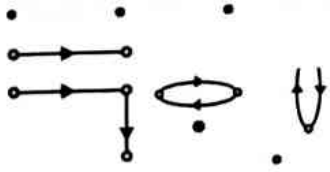
Ex.23. Draw all possible simple digraphs having 3 nodes. Show that there is only one digraph with no edge, one with one edge, four with two edges. ***IT : S-10***

Soln. The following are the digraphs with 3 nodes.

Graph with no edges :

Graphs with one edge :

Graphs with two edges :



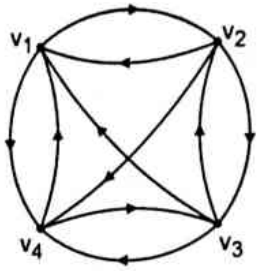
Ex.24. Define adjacency matrix of a diagraph. Draw a digraph corresponding to the adjacency matrix

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Give all possible paths of length 2. Are there any cycles of length 2? ***IT : S-10, W-10***

Soln. Adjacency matrix of a diagraph : A adjacency matrix A is defined as

$$a_{ij} = \begin{cases} 1, & \text{there is an edge between } V_i \text{ and } V_j \\ 0, & \text{otherwise} \end{cases}$$



There are many paths of length 2

$p_1 = \{< v_1, v_2 > < v_2, v_1 >\}$

$p_2 = \{< v_2, v_3 > < v_3, v_2 >\}$

$p_3 = \{< v_4, v_3 > < v_3, v_4 >\}$

$p_4 = \{< v_4, v_1 > < v_1, v_4 >\}$

and so on.

Cycles are as follows,

$C_1 = \{< v_1\ v_2 > < v_2\ v_1 >\}$

$C_2 = \{< v_2\ v_1 >, < v_1, v_2 >\}$

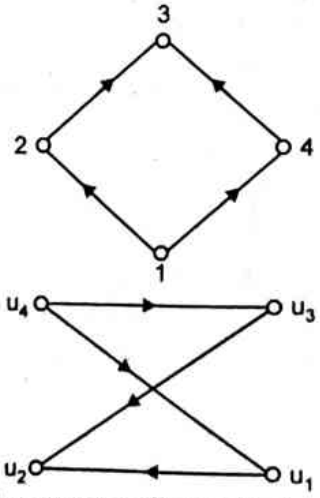
$C_3 = \{< v_2\ v_3 > < v_3\ v_2 >\}$

$C_4 = \{< v_3\ v_2 >, < v_2, v_3 >\}$

$C_5 = \{< v_4\ v_1 > < v_1\ v_4 >\}$ and

$C_6 = \{< v_1\ v_4 >, < v_4, v_1 >\}$

Ex.25. Show that the following graphs are isomorphic.



IT : W-10

Soln. The matrices corresponding to the given digraphs are

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Also $V_1 = V(G_1) = \{1, 2, 3, 4\}$,

$V_2 = V(G_2) = \{a, b, c, d\}$,

$E_1 = E(G_1) = \{(1, 2), (1, 4), (2, 3), (4, 3)\}$,

$E_2 = E(G_2) = \{(u_1, u_2), (u_3, u_2), (u_4, u_1), (u_4, u_3)\}$

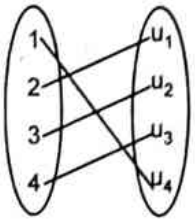
Let $f : V_1 \to V_2$

$f(1) = u_4$

$f(2) = u_1$

$f(3) = u_2$

$f(4) = u_3$



Clearly, $f : V_1 \to V_2$ is one-one and onto.

Also $(a, b) \in E_1 \Rightarrow (f(a), f(b)) \in E_2, \forall a, b \in V_1$

$\therefore\ G_1 \cong G_2$

Ex.26. Draw the digraph corresponding to the adjacency matrix.

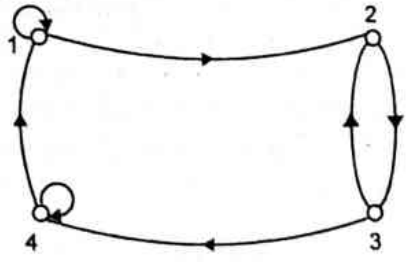
(i) $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ **CS/CT/CE : W-11**

(ii) $A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ **IT : W-11**

Soln.

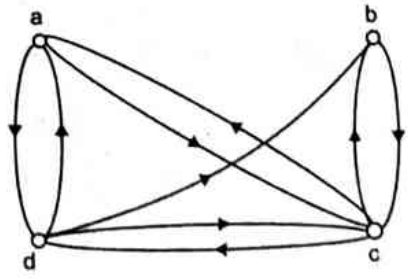
(i) Let $X = \{1, 2, 3, 4\}$

The required digraph is drawn below

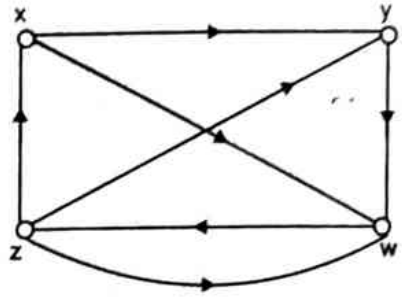


(ii) Let $X = \{a, b, c, d\}$

The required digraph is drawn below



Ex.27. In the following graph find the simple path from x to z and all cycles. **CS/CT/CE : W-12**



Soln. Simple paths from x to z :

(i) $x - y - w - z$

(ii) $x - w - z$

Cycles in the Graph :

(i) $x - y - w - z - x$

(ii) $x - w - z - x$

(iii) $y - w - z - y$

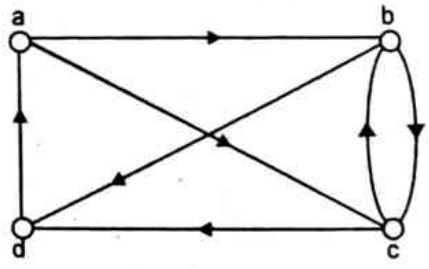
(iv) $w - z - w$

(v) $z - x - y - w - z$

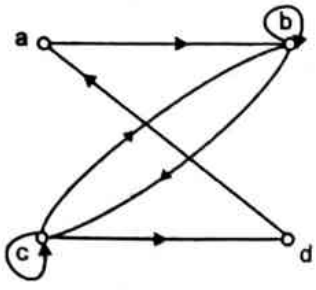
(vi) $z - y - w - z$

Ex.28. Find the adjacency matrix of the following graph.

(i)



(ii)



IT : W-12

Soln.

(i) The adjacency matrix is drawn below

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

(ii) The adjacency matrix is drawn below

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

TREE

Tree : **CS/CE/CE : S-06,10,11,W-09, IT : W-06**

A directed tree (or simply a tree) is a connected acyclic graph i.e. a connected graph having no cycle. The edges of a tree are called branches.

Examples :

(i)

1 Vertex

(ii)

2 Vertices

(iii)

3 Vertices

(iv)

4 Vertices

(v)

(vi)

The following graphs are not trees, because they contain a cycle.

(i)

(ii)

Properties :

(1) There is one and only one path between every pair of vertices in a tree T.

(2) If in a graph G, there is one and only one path between every pair of vertices, then G is a tree.

(3) A tree T with n vertices has n-1 edges. **IT : W-09**

Rooted tree :

A rooted tree is a tree in which a particular vertex is distinguished from the other vertices and this vertex is called the root. Rooted trees in graph theory are drawn with their roots at the top.

(1) The level of any vertex in a rooted tree is the length of the path from that vertex to the root. The level of root is 0.

(2) The height of a rooted tree is the maximum level of any vertex of the tree.

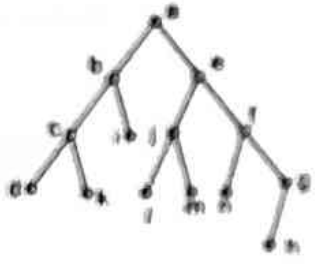
(3) Given any vertex v of a rooted tree, the children of v are the vertices that are immediate adjacent to v. **IT : W-09**

Leaf :

A vertex v having no children is called a leaf.

For example :

Consider the following rooted tree



Here, a is the root b and e are the children of a, c and d are the children of f.

Vertex : a b c d e f g h i j k l m n

Level : 0 1 2 3 1 2 3 4 2 2 3 3 3 3

Here, height = maximum level = 4

Here d, k, l, m, n, h, i are leaves.

Spanning tree : **CS/CE/CE : W-07, S-09**

A subgraph T of a connected graph G = (V, E) is called a spanning tree, if T is a tree and T includes every vertex of G i.e. V(T) = V(G).

Note : If |V| = n and |E| = m, then the spanning tree of must have n vertices and n-1 edges.

For example :

Here the graph has four vertices and hence the spanning tree must have four vertices and 4 - 1 edges. All the spanning trees are drawn below.

Note : A simple graph G has a spanning tree iff G is connected.

CAYLAY'S THEOREM

The complete graph K_n has n^{n-2} different spanning trees.

For example :

We have K_3.

Here n = 3.

Number of spanning trees

$$= n^{n-2} = 3^{3-2} = 3$$

The three different spanning trees are

MINIMAL SPANNING TREE

CS/IT/EE : 5-11

Let G = (V, E) be a connected weighted graph. The weight of a spanning tree of G is the sum of the weights of the edges. A minimal spanning tree of G is a spanning tree of G with minimum weight.

Prim's Algorithm to find minimal spanning tree :

Input : A connected weighted graph G.

Output : A minimal spanning tree T

Step (1) : Select an edge in G with minimum weight and select any vertex of this edge. Among all the edges incident with the selected vertex, select an edge with minimum weight. Include it in T.

Step (2) : At each stage, select an edge with minimum weight joining a vertex already included in T and a vertex which is yet to be included. (Make sure that a circuit is not formed). Include it in T.

Step (3) : Repeat this process until all the vertices of G are included in T.

Kruskal's Algorithm :

This algorithm provides an acyclic subgraph T of a connected weighted graph G which is a minimal spanning tree of G. The algorithm involves the following steps :

Input : A connected weighted graph G.

Output : A minimal spanning tree T

Step 1 : List all the edges (which do not form a loop) of G in non-decreasing order of their weights.

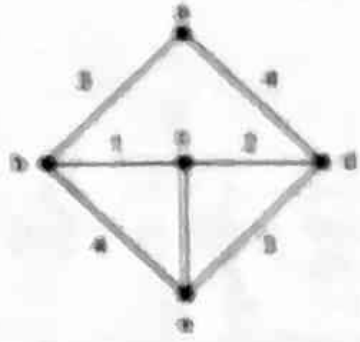
Step 2 : Select an edge of minimum weight (If more than one edge of minimum weight, arbitrarily choose one of them). This the first edge of T.

Step 3 : At each stage, select an edge of minimum weight from all the remaining edges of G if it does not form a circuit with the previously selected edges in T, add the edge to T.

Step 4 : Repeat step 3 until n – 1 edges have been selected, where n is the number of vertices in G. The following examples illustrate these steps.

For example :

Show how Kruskal's algorithm find a minimal spanning tree for the graph of Fig. given below.



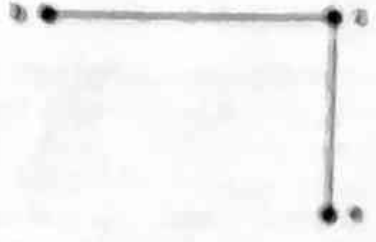
Soln.

Step 1 : List the edges in non-decreasing order of their weights.

Edge :	(b, c)	(a, c)	(c, d)	(a, b)	(c, d)	(a, d)	(b, e)
Weight :	1	2	2	3	3	4	4

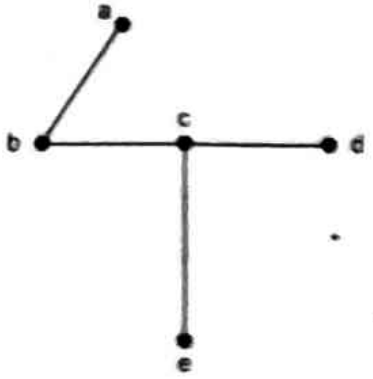
Step 2 : Select the edge (b, c) since it has the smallest weight, include it in T.

Step 3 : Select an edge with the next smallest weight (c, e) since it does not form circuit with the existing edges in T, so include it in T.



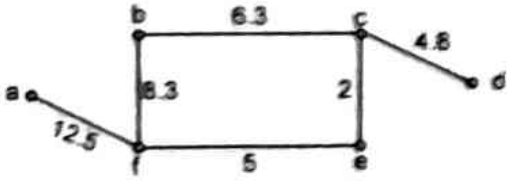
Step 4 : Select an edge with the next smallest weight (c, d) since it does not form circuit with the existing edges in T, so include it in T.

Step 5 : Select an edge with the next smallest weight (a, b) since it does not form circuit with the existing edges in T, so include it in T.



Since G contains 5 vertices and we have chosen 4 edges, we stop the algorithm and the minimal spanning tree is produced.

Ex.29. Find the minimal spanning tree of

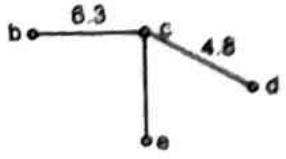


Using Prim's algorithm.

Soln.

(1) Let us select vertex e as the edge (e, c) has minimum weight.

(2) Now there are two edges incident with c i.e. (c, d), (c, b) and w (c, d) = 4.8, w (c, b) = 6.3. Hence minimum weight is 4.8.



(3) Now there is one vertex incident with e i.e. (e, f) and one vertex incident with c i.e. (c, b).
W(e, f) = 5, W (c, b) = 6.3.
Hence minimum weight is 5.

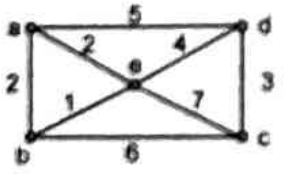
(4) Now there is one vertex incident with c i.e. (c, b) and two vertices incident with f i.e. (f, a) (f, b).
W (c, b) = 6.3, w (F, b) = 8.3, w (f, a) = 12.5. Hence minimum weight is 6.3.

(5) Now only one vertex a incident with f is left.

This is the required minimal spanning tree.
Its weight = 2 + 4.8 + 5 + 6.3 + 12.5 = 30.6.

Ex.30. Using Prim's algorithm find the minimal spanning tree of

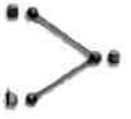
CS/CT/CE : W-07, S-11,12
IT : W-12



Soln.

(1) Let us select vertex b as edge (b, e) has minimum weight.

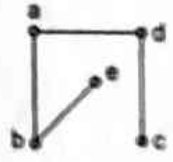
(2) Now there are two edges incident with b i.e. (b, c), (b, e) and three edges incident with e i.e. (e, a), (e, d), (e, c) and w (b, c) = 6, w (b, e) = 2, w (e, a) = 2, w (e, d) = 4, w (e, c) = 7.
Hence minimum weight is 2.



(3) Now there is one edge incident b i.e. (b, c) and one edge incident with a i.e. (a, d) and two edges incident with e i.e. (e, c), (e, d).
w (b, c) = 6, w (a, d) = 5, w (e, c) = 7, w (e, d) = 4.
Hence minimum weight is 4.

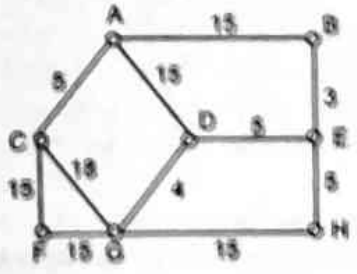
(4) Now only one vertex c is left which is incident with vertices b, e, d w (b, c) = 6, w (e, c) = 7, w (d, c) = 3.
Hence minimum weight is 3.

(5) This is the required minimal spanning tree.
(Its weight = 10)
If we would have gone for edge (b, a) in step (2) instead of edge (e, a) then the spanning tree would have been



This would not have been a minimal spanning tree because ist weight is 11.

Ex.31. Determine a railway network of minimal cost for the following cities. CS/CT/CE : 10-09,10, IT : 8-07

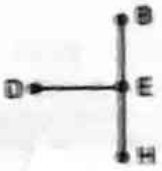


Soln.

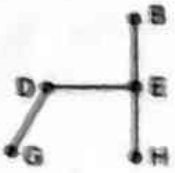
(1) Let us select vertex B as edge (B, E) has minimum weight.

(2) Now there is one edge incident with B i.e. (B, A) and two edges incident with E i.e. (E, D), (E, H)
w (B, A) = 15, w (E, D) = 5,
w (E, H) = 5
Hence minimum weight is 5.

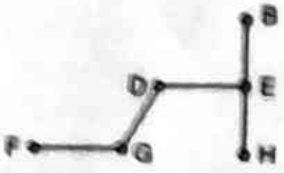
(3) Now there is one edge incident with B i.e. (B, A) and one edge incident with E
i.e. (E, D) and one edge incident with H i.e. (H, G).
w (B, A) = 15, w (E, D) = 5, w (H, G) = 15.
Hence minimum weight is 5.



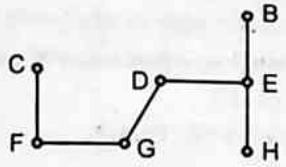
(4) Now there is one edge incidpnt with B i.e. (B, A) and one edge incident with D i.e. (D, G) and one edge incident with H
i.e. (H, G) and w (D, G) = 4,
w (B, A) = 15, w (H, G) = 15.
Hence minimum weight is 4.



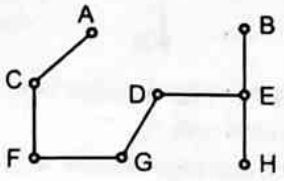
(5) Now there is one edge incident with B i.e (B, A) and two edges incident with G i.e. (G, F), (G, C)
w (B, A) = 15, w (G, F) = 15, w (G, C) = 18.
Hence minimum weight is 15.



(6) Now there is one edge incident with F i.e. (F, C) and one edge incident with G i.e. (G, C) and one edge incident with B i.e. (B, A) and
w (F, C) = 15,
w (G, C) = 18, w (B, A) = 15.
Hence minimum weight is 15.



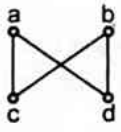
(7) Now only one vertex A is left which is incident with B and C.
w (B, A) = 15, w (C, A) = 5.
Hence minimum weight is 5.



This is the required minimal spanning tree.
Minimal cost = 3 + 5 + 5 + 4 + 15 + 15 + 5 = 52

Ex.32. Using Prim's algorithm find the minimal spanning tree for the following graph using 'a' as the root.

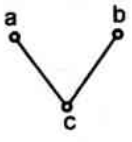
CS/CT/CE : S-06,12,13, W-12, IT : S-08,09,13



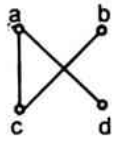
Soln.

(1) Let us select vertex a. There are two edges incident with a i.e. (a, d), (a, c). since the weight are not given, let us select edge (a, c).

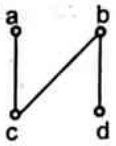
(2) Now there is only one edge incident with a i.e. (a, d) and one edge incident with c i.e. (c, b). Since the weight are not given let us select edge (c, b).



(3) Now only one vertex d is left which is incident with a and b ie (a, d), (b, d). Since the weights are not given let us select edge (a, d).



This is the required minimal spanning tree.
In the above step (3), if we would have selected edge (b, d), then the minimal spanning tree would have been.



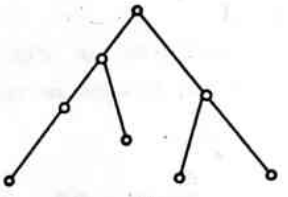
BINARY TREE

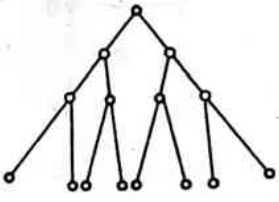
CS/CT/CE : W-07,11, S-10, W-10

Binary tree : A binary tree is a rooted tree in which each vertex has almost two children. That is a binary tree contains 0, 1 or 2 children. Each child is referred as either a left child or a right child (not both).

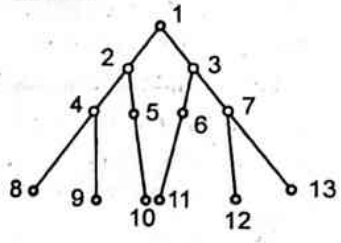
A full binary tree is a binary tree in which each vertex has exactly zero or two children.

For example :

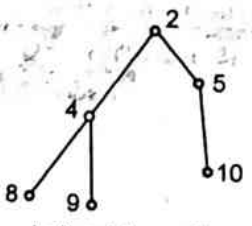




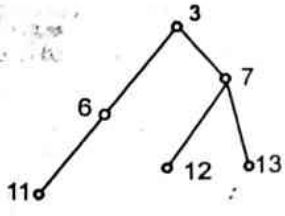
Ex.33. Find the left and right children of 2 in the following figure. What are the left and right subtrees of 1 ?



Soln. The left and right children of 2 are 4 and 5 respectively. The following trees represent the left and right subtrees of 1.



Left subtree of I.



Right subtree of I.

Complete Binary Tree : **CS/CT/CE : W-09, IT : S-06**

If all the leaves of a full binary tree are at level m, then such a tree is called complete binary tree of depth m.

For example :

Consider the following tree.

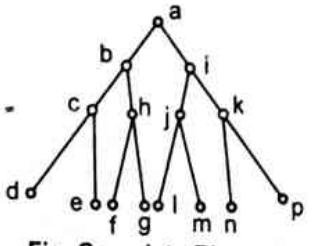


Fig. Complete Binary tree

Here d, e, f, g, l, m, n, p are the leaves and they all are att level 3 from the root a.

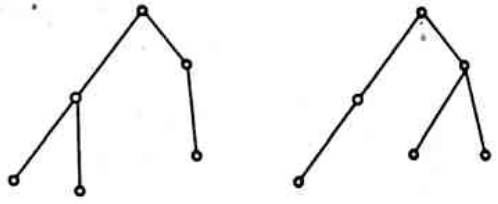
Hence this is a complete binary tree.

Forest :

A set of disjoint trees is called a forest.

For example :

The following figure is a forest.



Algorithm :

The following algorithm gives every tree or a forest a unique binary tree representation.

Step 1 : Delete all the edges originating in every vertex except the left most edge.

Step 2 : Draw edges from a vertex to the vertex on the right if any of them is at the same level.

Step 3 : Choose the left child and right child off any vertex as follows. The left child is the vertex immediately below the given vertex and the right child is to the immediate right of the given vertex on the same horizontal line.

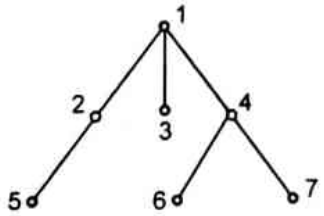
Ex.34. Draw a tree for the relation R = {(1, 2), (1, 3), (1, 4), (2, 5), (4, 6), (4, 7)} on a set A = {1, 2, 3, 4, 5, 6, 7}. Also give the corresponding binary tree. **CS/CT/CE : S-06,07,09,11 W-07,08, IT : W-07, S-11,13**

Soln. Given : R = {(1, 2), (1, 3), (1, 4), (2, 5), (4, 6), (4, 7)}

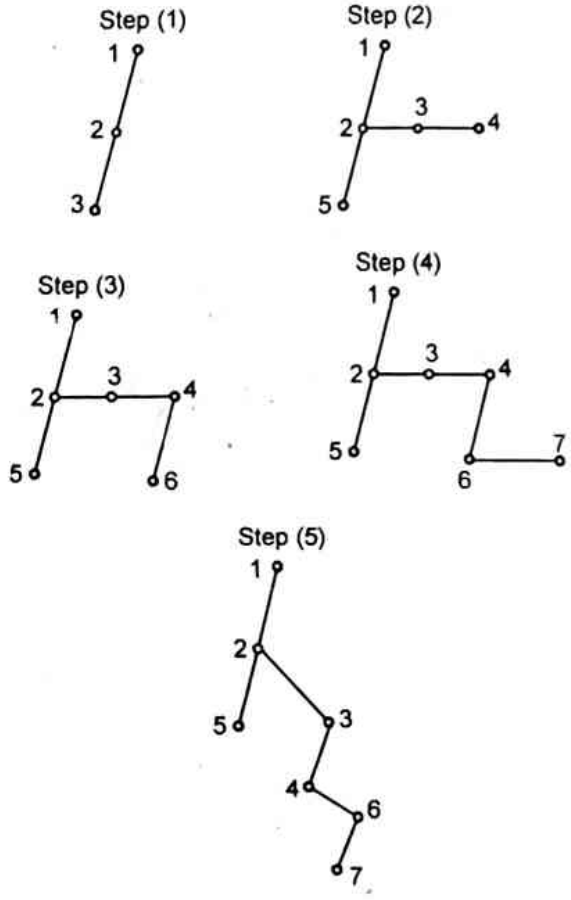
A = {1, 2, 3, 4, 5, 6, 7}.

We observe that the vertex with highest number of edges is 1.

Hence we select 1 as the root. The required tree for the given relation R is drawn below.



The corresponding binary tree is drawn below :



This is the required binary tree.

Ex.35. Let A = {1, 2, 3, 4, 5, , 6, 7, 8, 9, 10},

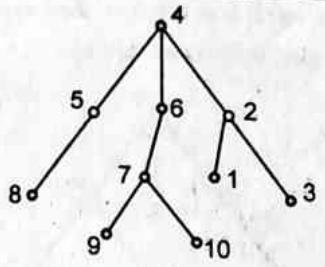
T = {(2, 3), (2, 1), (4, 5), (4, 6), (5, 8), (6, 7), (4, 2), (7, 9), (7, 10)}

Indentify the root and show that T is a rooted tree. Also give the coresrsponding binary tree. **IT : S-06**

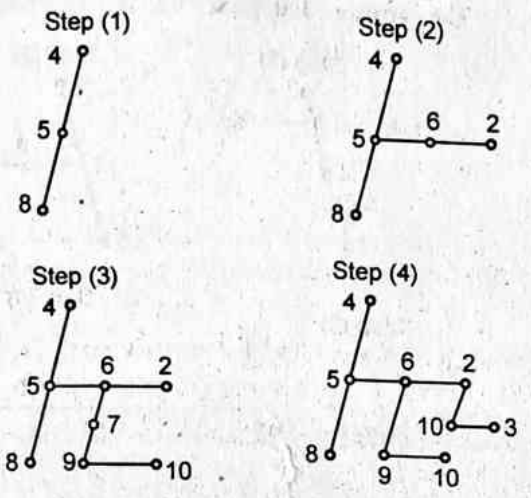
Soln. Given : A = {1, 2, 3, 4, 5, , 6, 7, 8, 9, 10} and

T = {(2, 3), (2, 1), (4, 5), (4, 6), (5, 8), (6, 7), (4, 2), (7, 9), (7, 10)}

We observe that the vertex with highest number of edges is 4. Hence we select 4 as the root. The required rooted tree is drawn below.



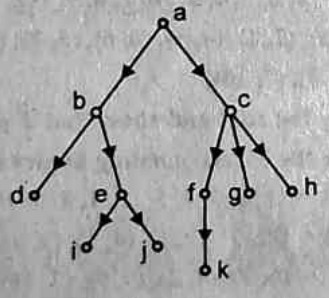
The corresponding binary tree is drawn below :



This is the required binary tree.

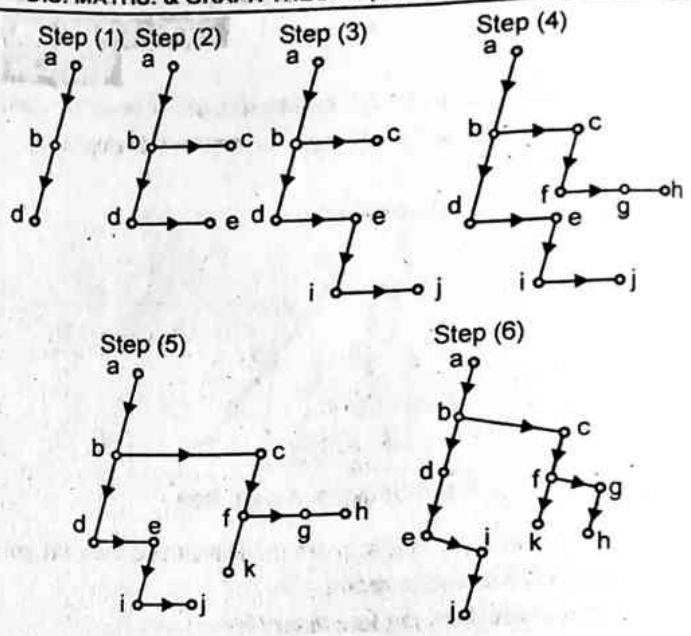
Ex.36. Draw a directed tree with 2 nodes at level 1, 5 nodes at level 2, 3 nodes at level 3. obtain the corresponding binary tree. ***CS/CT/CE : W-09, IT : W-06,12***

Soln.



The above tree is a directed tree with root a, 2 nodes b and c at level 1, 5 nodes b, c, f, g and h at level 2, 3 nodes i, j and k at level 3.

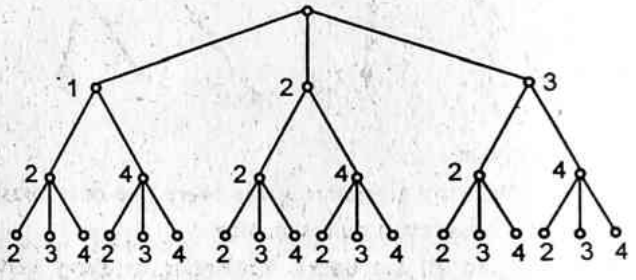
The corresponding binary tree is drawn below :



This is the required binary tree.

Ex.37. Find the product {1, 2, 3} × {2, 4} × {2, 3, 4} by constructing appropriate tree diagram. ***CS/CT/CE : S-10***

Soln. The required tree diagram is drawn below :

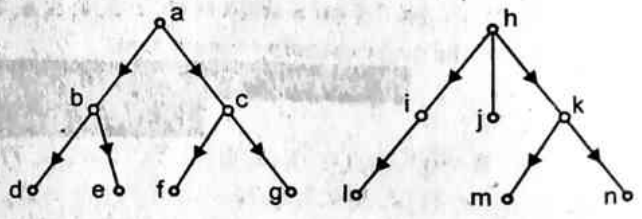


$\therefore \{1, 2, 3\} \times \{2, 4\} \times \{2, 3, 4\}$

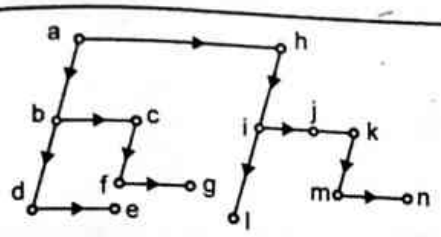
$= \{(1, 2, 2), (1, 2, 3), (1, 2, 4), (1, 4, 2), (1, 4, 3), (1, 4, 4), (2, 2, 2), (2, 2, 3), (2, 2, 4), (2, 4, 2), (2, 4, 3), (2, 4, 4), (3, 2, 2), (3, 2, 3), (3, 2, 4), (3, 4, 2), (3, 4, 3), (3, 4, 4)\}$

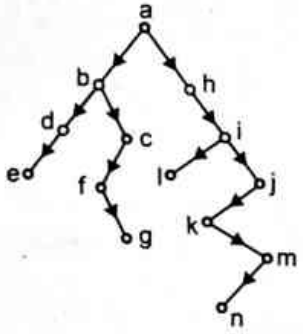
The element of the above set are all ordered triples.

Ex.38. Give the binary tree representation for the following forest.



Soln. The two trees in the given forest can be combined to form a tree as follows.





This is the required binary tree.

Representation of Algebraic Expressions by Binary trees :

- In an algebraic expression, we assume that no operations such +, –, ×, ÷ etc. can be performed until both of its arguments have been evaluated.
- Every expression contains a central operator corresponding to the last computation that can be performed. The Central operator corresponds to the root of the binary tree. the left and right arguments of the central operator are the left and right child of the root respectively. This process continues until the expression is exhausted.

Ex.39. Represent the following algebraic expression using binary tree. Also draw the venn diagrams.

(i) x * y

(ii) (a * b) | (c + d)

(iii) ((a + b) | c) + (x + y)

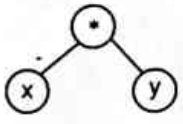
(iv) (x + (y + z)) * (a × (b + c))

(v) (3 – 2 (– (11 – (9 – 4)))) ÷ (2 + (3 + (4 + 7))).

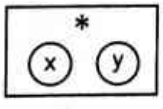
CS/CT/CE : S-13

Soln.

(i) Given algebraic expression is x * y. Here * is the central operator and hence it is the root. The required binary tree and Venn diagram are drawn below.

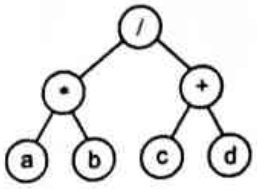


Binary Tree

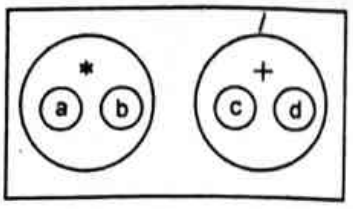


Venn Diagram

(ii) Given algebraic expression is (a * b) | (c + d)(. Here | is the central operator and hence it is the root. The required binary tree and Venn diagram are drawn below.

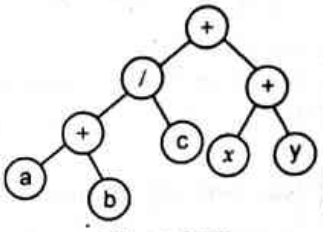


Binary Tree

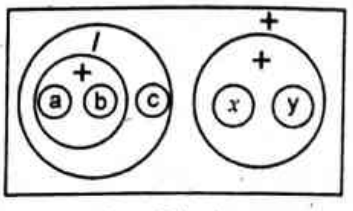


Venn Diagram

(iii) Given algebraic expression is ((a + b) | c) + (x + y). Here + is the central operator and hence it is the root. The required binary tree and Venn diagram are drawn below.

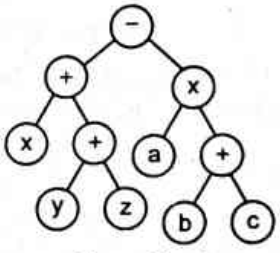


Binary Tree

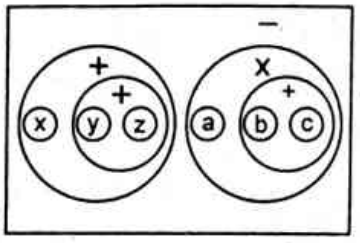


Venn Diagram

(iv) Given algebraic expression is (x + (y + z)) – (a × (b + c)). Here – is the central operator and hence it is the root. The required binary tree and Venn diagram are drawn below.

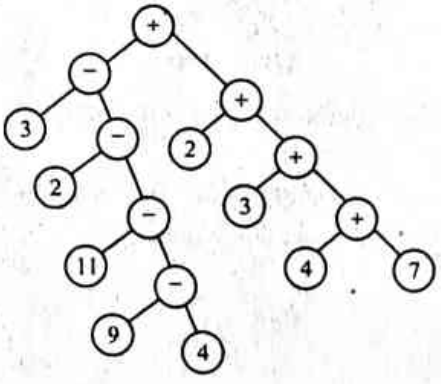


Binary Tree



Venn Diagram

(v) Given algebraic expression is (3 – 2 (– (11 – (9 – 4)))) ÷ (2 + (3 + (4 + 7))). Here ÷ is the central operator and hence it is the root. The required binary tree and Venn diagram are drawn below.



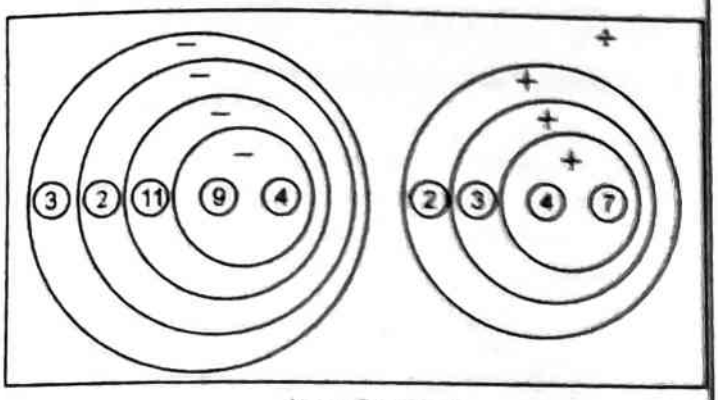
Binary Tree

$\therefore (3 - 2(-(11 - (9 - 4)))) \div (2 + (3 + (4 + 7)))$

$= (3 - 2(-11 + 5))) \div (2 + (3 + 11))$

$= (3 - 2 - 6) \div (2 + 14)$

$= \frac{-5}{16}$



Venn Diagram

Ex.40. Represent the following algebraic expressions by binary tree.

(i) $\{(3 - (2x)) + \{(x - 2) - (3 + x)\}\} + xy$.

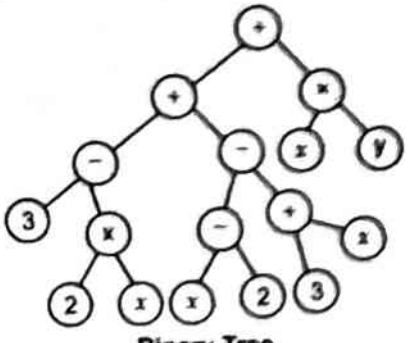
CS/CT/CE : W-06, IT : W-09

(ii) $(2x + (3 - 4x)) + (x - (3 \times 11))$

CS/CT/CE : S-13, IT : S-06

Soln.

(i) The given algebraic expression is $\{3 - (2x) + \{(x - 2) - (3 + x)\} + xy$. Hence + is the central operator and hence it is the root. The required binary tree is drawn below.

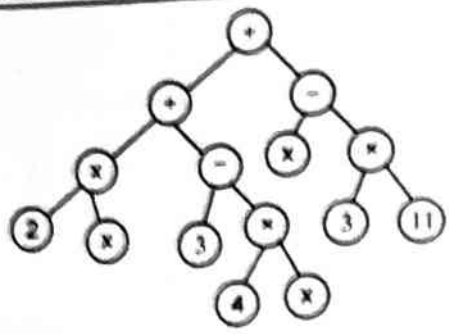


Binary Tree

(ii) The given algebraic expression is $(2x + (3 - 4x)) + (x - (3 \times 11)$.

Here + is the central operator and hence it is the root.

The required binary tree is drawn below.



Binary Tree

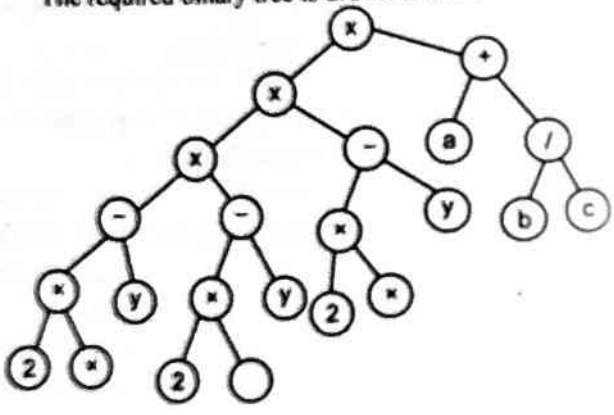
Ex.41. Draw the binary tree for $(2x - y)^3$ $[a + (b/c)]$ and find the subtree at the exponential operator.

IT : W-06

Soln. The given algebraic expression is $(2x - y)^3$ $(a + (b / c)]$

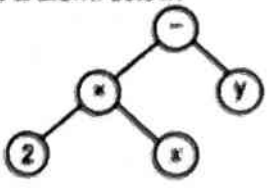
Here × is the central operator and hence it is the root

The required binary tree is drawn below.

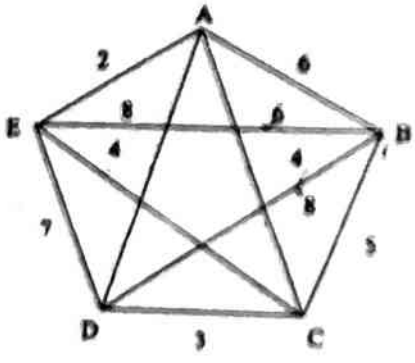


Binary tree

Here $2x - y$ is the exponential operator with exponent 3. its subtree is drawn below.



Ex.42. Use Prim's algorithm to find the minimum spanning tree for the given weighted graph :



CS/CT/CE : S-07,09

Soln. Select the vertex A. Among all edges incident on A with smallest weight is (A, E) = 2

Among all edges incident on vertex E with smallest weight is (E, C) = 4

Among all edges incident on vertex C with smallest weight is (C, D) = 3

Also among all edges incident on vertex D with smallest weight is (D, B) = 4

Thus a minimal spanning tree of weight is 2 + 4 + 3 + 4 = 13 for G.

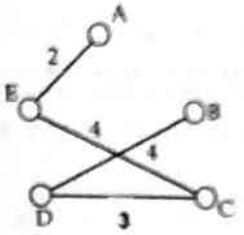


Fig. Minimal spanning tree

Ex.43. Construct the tree of the algebraic expression :

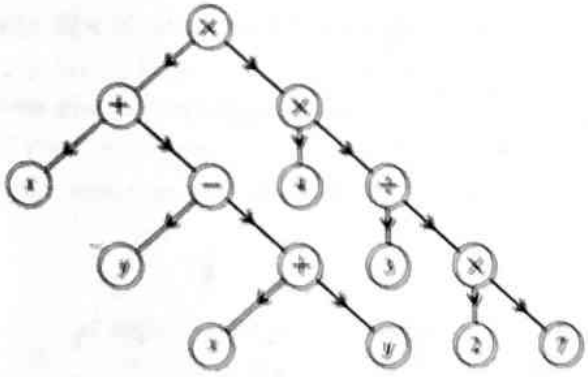
(i) $(x + (y - (x + y))) \times ((3 \div (2 \times 7)) \times 4)$

(ii) $3 - (x + (6 \times (4 \div (2 - 3))))$.

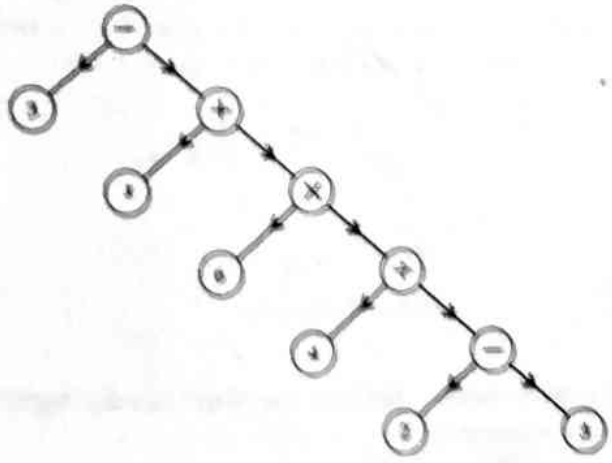
CS/CT/CE :S-07,12, W-12, IT : S-09

Soln.

(i)



(ii)

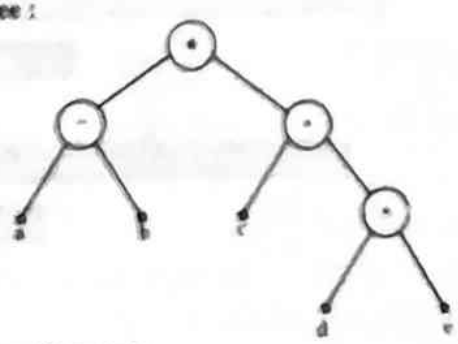


Ex.44. Construct the tree of the following algebraic expression (a − b) * [c + (d ÷ e)]

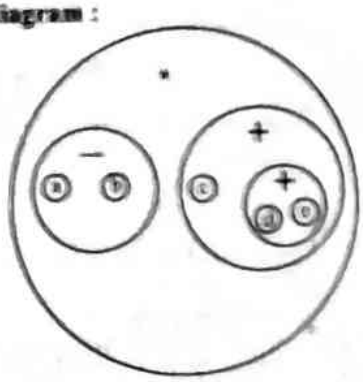
Also give the Venn Diagram. **IT : S-07**

Soln. Given expression is (a − b) * [c + (d ÷ e)].

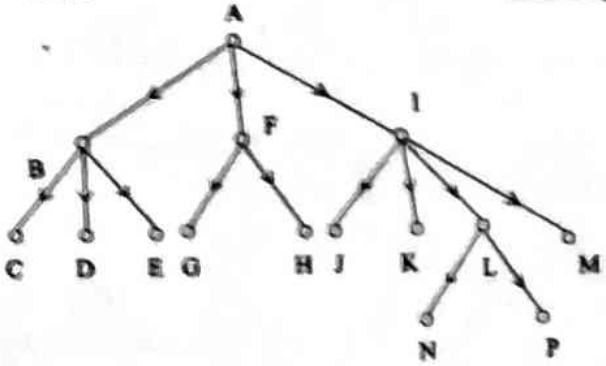
Tree :



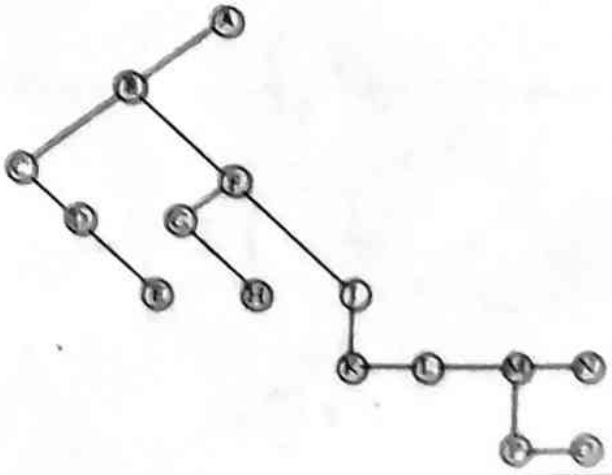
Venn diagram :



Ex.45. Give a binary tree representation to the following tree : **IT : S-07**



Soln. The binary tree representation of the given tree is :



Ex.46. Give a directed tree representation and venn diagram for the following formula.

(i) $[7 + (6 - x)] - [x - (y - 4)]$ **IT : W-08, S-12**

(ii) $[x + (y - (x + y)] \times [\{3 + (2x)\}.y]$

CS/CT/CE : W-07, S-11,12, IT : W-07,08

(iii) $(V_1 V_2 + V_3) + \left(V_4 + \frac{V_5}{V_6}\right) V_7$. **IT : W-07,11**

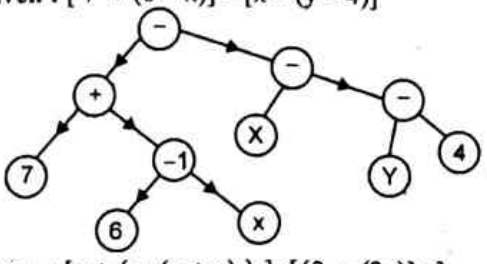
(iv) $[3 - \{2 - (11 - (9 - x))\}] \div [2 + \{3 + (4 + y)\}]$.

CS/CT/CE : S-08, IT : W-08, S-08,11

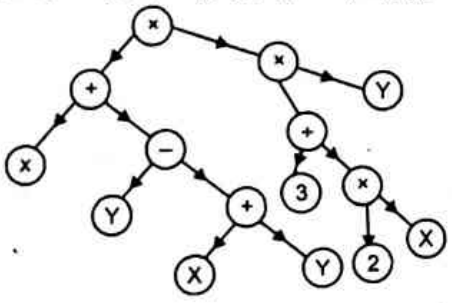
(v) $v_1 v_2 + \left[v_4 + \frac{v_5}{v_6}\right] v_3$. **IT : W-10**

Soln.

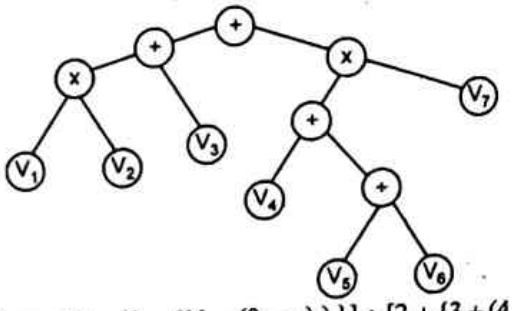
(i) Given : $[7 + (6 - x)] - [x - (y - 4)]$



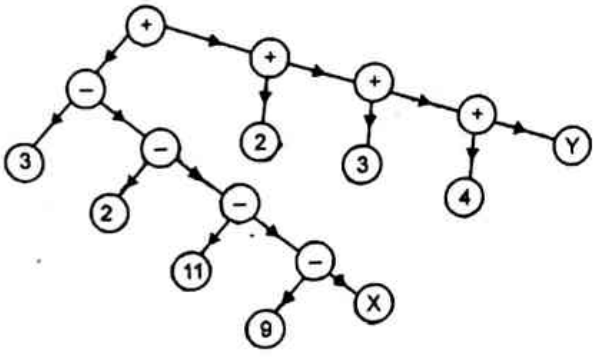
(ii) Given : $[x + \{y-(x + y)\}] \times [\{3 \div (2x)\}.y]$



(iii) Given : $(V_1 V_2 + V_3) + \left(V_4 + \frac{V_5}{V_6}\right) V_7$



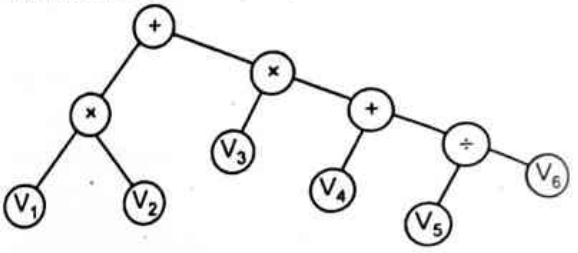
(iv) Given : $[3 - \{2 - (11 - (9 - x))\}] \div [2 + \{3 + (4 + y)\}]$



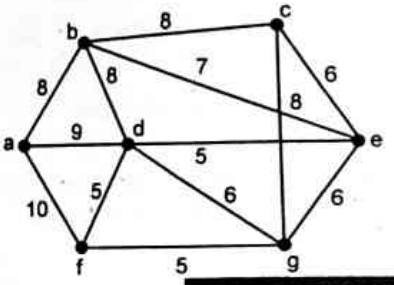
(v) Given algebraic expression is :

$$v_1 v_2 + \left[v_4 + \frac{v_5}{v_6}\right] v_3$$

The tree from the above expression is



Ex.47. Apply Prim's algorithm to construct a minimal spanning tree for the weighted graph given below:



CS/CT/CE : S-08, IT : W-11

Soln. Select the vertex 'a'. Among all edges incident on 'a' with smallest weight is (a, b) = 8

Among all edges incident on vertex 'b' with smallest weight is (b, e) = 7.

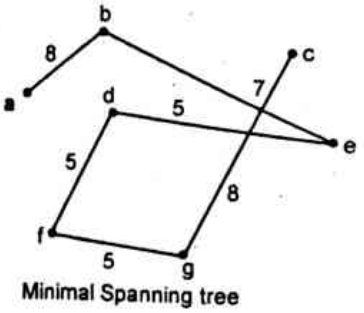
Among all edges incident on vertex 'e' with smallest weight is (e, d) = 5

Also among all edges incident on vertex 'd' with smallest weight is (d, f) = 5

Similarly (f, g) = 5 & (g, c) = 8 .

Thus, a minimal spanning tree of weight is

8 + 7 + 5 + 5 + 5 + 8 = 38 for G.



Minimal Spanning tree

Ex.48. Construct the tree for the following algebraic expression.

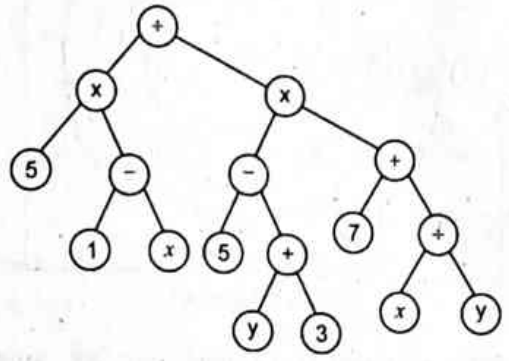
(i) $[5(1 - x)] \div [\{5- (y + 3)\} \times \{7 + (x \div y)\}]$

(ii) $\{(3-(2x))+[(x-2)-(3+x)]\}+xy$

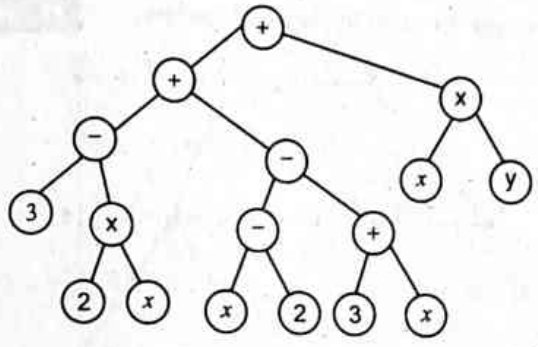
CS/CT/CE : W-08

Soln.

(i) **Given :** $[5(1-x)] \div [\{5-(y+3)\} \times \{7+(x \div y)\}]$

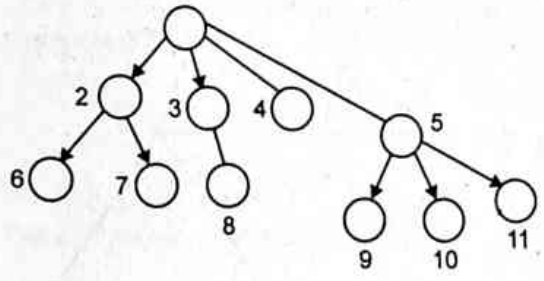


(ii) **Given :** $\{(3-(2x))+[(x-2)-(3+x)]\}+xy$

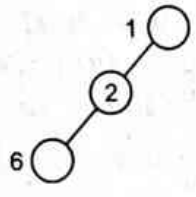


Ex.49. Draw a directed tree with 4 nodes at level '1', 6 nodes at level 2. Obtain the corresponding binary tree. **CS/CT/CE : W-10, IT : W-08, S-12**

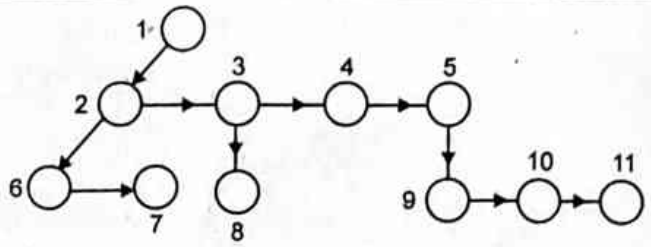
Soln. The binary tree is



Step I: We delete all the branches originating in every node except the left most branch. Thus we get

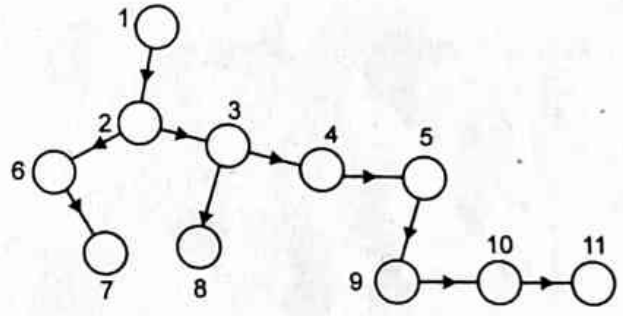


Step II: We draw edges from a node to the node on the right if any, which is situated at the same level. Thus we get



Step III :

- For a particular node we choose its left and right sons in the following manner. The left son is the node which is immediately below the given node and the right son is the node to the immediate right of the given node on the same horizontal line.
- Left son of node 2 is node 6 while right son of node 2 is node 3. Thus we get our binary representation as



Ex.50. Construct tree for the following algebraic expression.

(a) $[3(1-x)] \div [\{4+(7-(y+2))\} \times \{7+(x \div y)\}]$

CS/CT/CE : W-11, IT : W-09, S-10

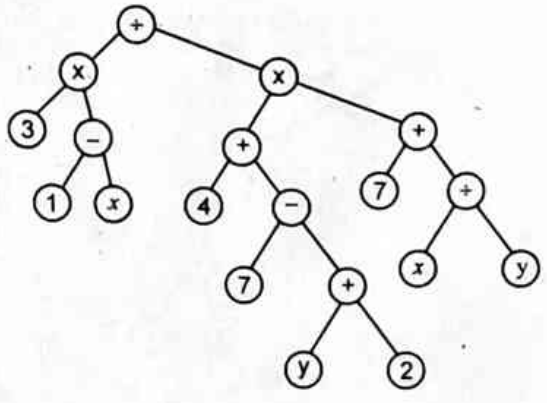
(b) $[3(1-x)] \div [\{4 \div 7-(y \div z)\} \times \{7+(x \div y)\}]$

IT : W-10,11

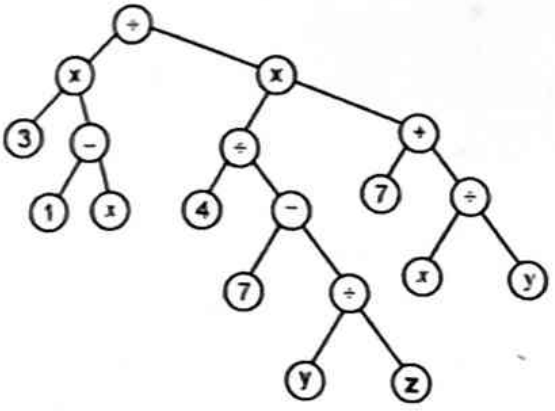
Soln.

(a) Given expression is :

$[3(1-x)] \div [\{4+(7-(y+2)\} \times \{7+(x \div y)\}]$



(b) $[3(1-x)] \div [\{4 \div 7 - (y \div z)\} \times \{7 \div (x \div y)\}$



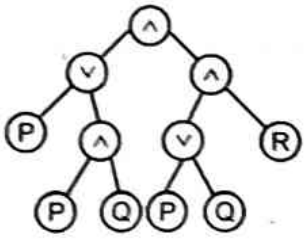
Ex.51. Construct tree and draw the Venn Diagram of

(i) $(P \vee (P \wedge Q)) \wedge ((P \vee Q) \wedge R)$

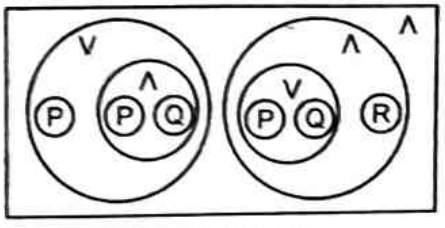
(ii) $[3] - [x + \{6(4 \div (2 - x))\}]$ **CS/CT/CE : S-10**

Soln.

(i) **Given :** $(P \vee (P \wedge Q)) \wedge ((P \vee Q) \wedge R)$

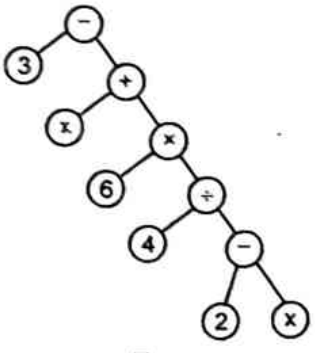


Tree

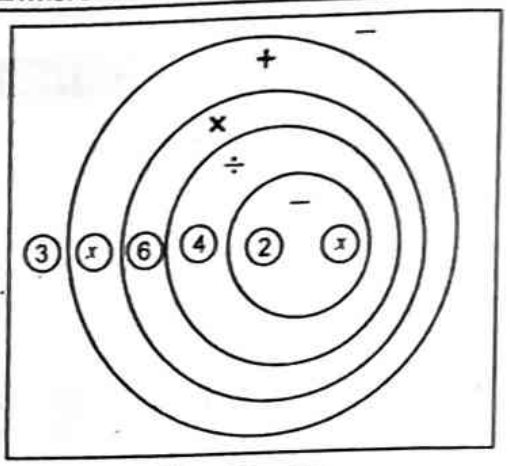


Venn Diagram

(ii) **Given :** $[3] - [x + \{6(4 \div (2 - x))\}]$.

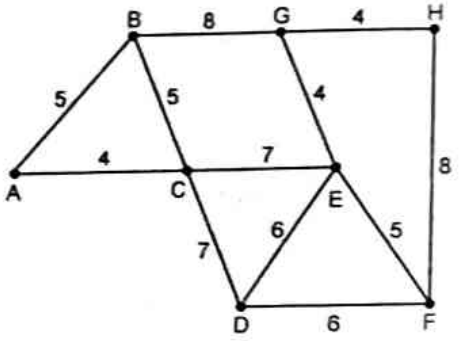


Tree

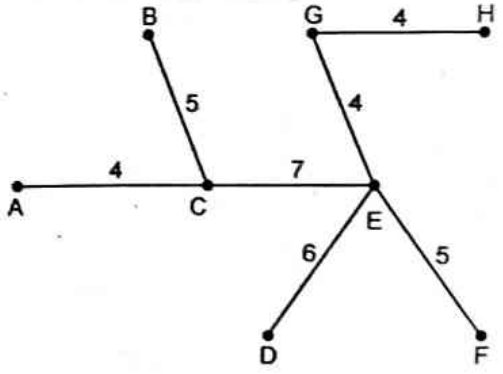


Venn Diagram

Ex.52. Find a minimal spanning tree for connected weighted graph shown below. The weights of each edge is given in terms of kilometers. **IT : W-10**



Soln. The minimal spanning tree for connected weighted graph is :



Minimal spanning tree

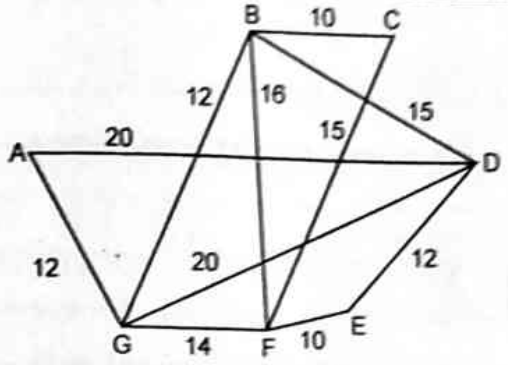
$= l(AC) + l(BC) + l(CE)$

$+ l(EG) + l(GH) + l(EF) + l(ED)$

$= 4 + 5 + 7 + 4 + 6 + 5 + 4$

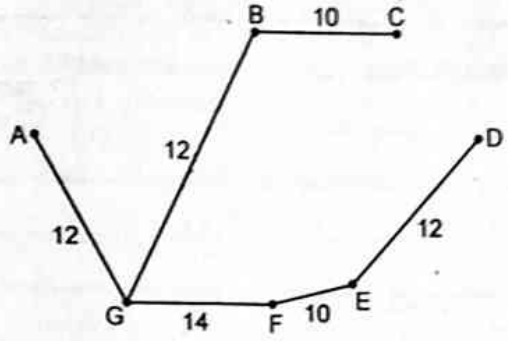
$= 35$ km.

Ex.53. Apply Prim's algorithm to construct a minimal spanning tree for the weighted graph given below.



CS/CT/CE :W-11

Soln. The minimal spanning tree for connected weighted graph is :



Minimal spanning tree = $l(AG) + l(GB) + l(BC) + l(GF) + l(EF) + l(ED)$

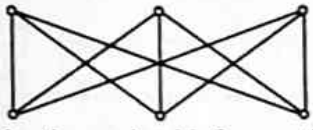
$= 12 + 14 + 12 + 10 + 10 + 12$

$= 70$

PROBLEMS FOR PRACTICE :

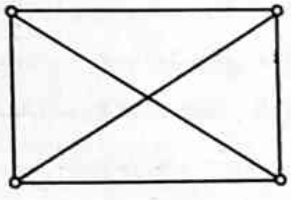
Q.1. Draw a graph representing the problem of 3 houses and 3 utilities.

Ans.

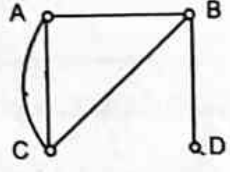


Q.2. Draw a simple graph with four vertices

Ans.

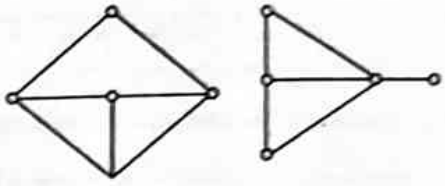


Q.3. Find adjacency matrix of the graph

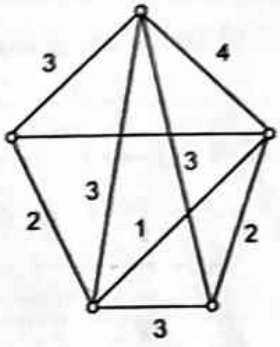


Ans. $\begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

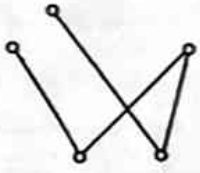
Q.4. Show that the following graphs are isomorphic.



Q.5. Using Prim's algorihm find the minimal spanning tree for the following tree.



Ans.



Notes :-

UNIT - VI
COMBINATORICS

SYLLABUS

Generating functions, Recurrence relations, Counting: Permutations and Combinations, Pigeonhole Principle with simple applications.

UNIVERSITY PAPER SOLUTIONS
SINCE SUMMER - 2006

Note to the students :
- The questions given below are from previous Nagpur University examination papers.
- Though these questions are from old university papers, they have been included in the new syllabus.
- Questions which are out of new syllabus are not included here.

SUMMER - 06 (CS/CT/CE)

Q.1. Solve the recurrence relation, $a_n = a_{n-1} + 2, n \geq 2$ subject to initial condition $a_0 = 3$. **7M**

Ans. P.6-23, Ex.41

Q.2. Find the generating function for the sequence 1, a, a^2, ...where a is a fixed constant. **3M**

Ans. P.6-21, Ex.34

WINTER - 06 (CS/CT/CE)

Q.1. Define recurrence relation and solve:
$F_n = F_{n-1} + F_{n-2}$, $F_1 = F_2 = 1$, $N \geq 3$. **6M**

Ans. P.6-12, Ex.28

Q.2. State Pigeonhole Principle and show that there must be at least 90 ways to choose six numbers from 1 to 15 so that all the choices have same sum. **6M**

Ans. P.6-8, 23, Definition and Ex.42

SUMMER - 07 (CS/CT/CE)

Q.1. Find the generating function for the sequence :
$1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n+1}, \ldots$ **4M**

Ans. P.6-21, Ex.35

Q.2. Find the particular solution of
$a_r + a_{r-1} = 3r\,2^r$. **5M**

Ans. P.6-21, Ex.36

WINTER - 07 (CS/CT/CE)

Q.1. Solve, using generating functions
$a_n = a_{n-1} + a_{n-2}$
given $a_0 = a_1 = 1$. **7M**

Ans. P.6-12, Ex.28

SUMMER - 08 (CS/CT/CE)

Q.1. Solve the following recurrence relation:
$a_n = 3a_{n-1} - 2a_{n-2}$ with $a_1 = 5$, $a_2 = 3$ **6M**

Ans. P.6-21, Ex.37

Q.2. State Pigeonhole principle and show that there must be at least 90 ways to choose six numbers from 1 to 15 so that all the choices have same sum. **7M**

Ans. P.6-23, 8, Ex.42

WINTER - 08 (CS/CT/CE)

Q.1. Solve: $a_n + 5a_{n-1} + 6a_{n-2} = 3n^2$. **6M**

Ans. P.6-22, Ex.38

SUMMER - 09 (CS/CT/CE)

Q.1. Show that if any 30 people are selected from a group, then one may choose a subset of five so that all five were born on the same day of week. **4M**

Ans. P.6-9, Ex.23

Q.2. Define recurrence relation and solve :
$F_n = F_{n-1} + F_{n-2}$, $F_1 = F_2 = 1$, $N \geq 3$. **6M**

Ans. P.6-12, Ex.28

WINTER - 09 (CS/CT/CE)

Q.1. Define recurrence relation and solve:
$F_n = F_{n-1} + F_{n-2}$, $F_1 = F_2 = 1$, $n \geq 3$. **6M**

Ans. P.6-12, Ex.28

Q.2. Find the particular solution of:

$a_r + a_{r-1} = 3r2^r$. **5M**

Ans. P.6-21, Ex.36

Q.3. Define generating function. **2M**

Ans. P.6-14, Refer Definition

SUMMER - 10 (CS/CT/CE)

Q.1. Solve the recurrence relation

$a_n = a_{n-1} + 2,\ n \geq 2$

subject to initial condition $a_0 = 3$. **7M**

Ans. P.6-23, Ex.41

Q.2. Find a closed form for the generating function for the following:

(i) 0, 0, 1, 1, 1, ...

(ii) 1, 0, –1, 0, 1, 0 –1, 0, 1, ...

(iii) 3, –3, 3, – 3, 3, – 3, ... **6M**

Ans. P.6-15, Ex.30

WINTER - 10 (CS/CT/CE)

Q.1. Evaluate the sum

$1^2 + 2^2 + 3^2 + ... + r^2$ by determining its generating function **6M**

Ans. P.6-22, Ex.39

SUMMER - 11 (CS/CT/CE)

Q.1. Solve :

$a_n - 5a_{n-1} - 6a_{n-2} = 2^n + n : n \geq 2\ ;\ a_0 = a_1 = 1$

Using generating function. **7M**

Ans. P.6-23, Ex.40

Q.2. Show that if any five numbers from 1 to 8 are chosen then two of them add up to 9 using Pigeonhole principle. **3M**

Ans. P.6-10, Ex.25

WINTER - 11 (CS/CT/CE)

Q.1. Solve the following recurrence relation :

$a_n = 3a_{n-1} - 2a_{n-2}$ with $a_1 = 5, a_2 = 3$. **6M**

Ans. P.6-21, Ex.37

SUMMER - 12 (CS/CT/CE)

Q.1. Find the generating function of the sequence $\{a_k\}$, if

$a_k = 2 + 3k$. **7M**

Ans. P.6-16, Ex.31

Q.2. Find the particular solution of :

$a_r + a_{r-1} = 3r2^r$. **6M**

Ans. P.6-21, Ex.36

WINTER - 12 (CS/CT/CE)

Q.1. State pigeonhole principle and show that there must be at least 90 ways to choose six numbers from 1 to 15 so that all the choices have same sum. **6M**

Ans. P.6-8, 23, Ex.42

Q.2. Find explicit formula for the recurrence relation.

$a_n = a_{n-1} + 3, a_0 = 3$. **7M**

Ans. P.6-23, Ex.43

SUMMER - 13 (CS/CT/CE)

Q.1. Solve the following recurrence relation.

$a_{n+2} - 6a_{n+1} + 8a_n = 3$ **6M**

Ans. P.6-23, Ex.44

Q.2. State pigeonhole principle and find the minimum number of students in a class to be sure that four out of them are born in the same month. **7M**

Ans. P.6-9, 8, Ex.24

SOLVED QUESTION BANK

[Sequence given as per syllabus]

INTRODUCTION

Combinatorics or combinatorial analysis, called as the science of counting, is the branch of Mathematics concerned with the selection, arrangement and operation of elements within sets. Combinatorial Theory deals with the existence (does a particular arrangement exist?), enumeration (how many such arrangements are there?) and structure (what are the properties of each arrangement?). It has a wide range of applications in managing computer and telecommunication networks, predicting poker hands, dividing tasks among workers etc. Combinatorial problems also arise in many areas of Pure Mathematics viz algebra, probability theory, topology, geometry. Combinatorics also plays a vital role in optimization, statistical physics.

Many combinatorial questions have historically been considered in isolation, giving an adhoc solution to a problem arising in some mathematical contex. In the later 20th century, however, powerful and general theoretical methods were developed, making combinatorics an independent branch of Mathematics. One of the oldest and most accessible parts of combinatorics is Graph theory, which has numerous natural connections to other areas. Combinatorics is frequently used in Computer Science to obtain formula and estimates in the analysis of algorithms.

COUNTING TECHINQUES

How many passwords can be constructed using 4 letters of the alphabets, 1 special character and 1 digit from 0 to 9 ? How long the computer will take to construct all such passwords as a way of compromising computer security? In how many ways can 10 persons seat in a round table conference? How many even numbers between 500 and 2000 have no digits repeated? Counting Techniques provide tools for answering questions like these.

Counting Techniques are used in the study of probability to determine the number of events in a sample space and the number of successful outcomes of an experiment. In computer science, elementary counting methods provide useful tools and techniques for dealing with problems that a program is correct. More generally, counting methods are used at several stages in the development of correct and efficient program. For example, a first decision about which algorithm should be used in a particular application is often based on knowing how the various alternative algorithms compare with respect to running time and storage. A step in this process includes one or more counting arguments.

In this chapter, we discuss the fundamental counting principles, permutation and combinations, Pigeon-hole principle and proofs of various combinatorial identities.

THE FUNDAMENTAL COUNTING PRINCIPLES

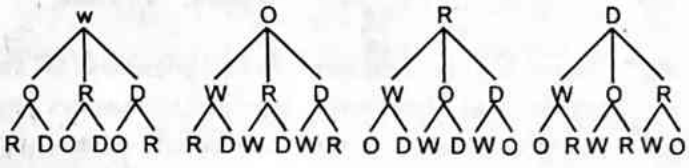
The Multiplication Principle or Product Rule :

If several operations are carried out in a certain order, then the number of ways of performing all the operations is the product of the number of ways of performing each operation, More precisely, if an event can be performed in T successive steps and step 1 can be done N_1 ways, step 2 can be done in N_2 ways, ..., step T can be done in N_T ways, then the number of different possible events is $N_1 . N_2 . N_3 ..... N_T$.

If $A \times B$ is the Cartesian product of sets A and B, then $|A \times B| = |A| . |B|$.

Ex.1. How many 3 letter words can be made from "WORD" if no letter is repeated?

Soln.



From the above diagram, the different possibilities are

WOR, WOD, WRO, WRD, WDO, WDR

OWR, OWD, ORW, ORD, ODW, ODR

RWO, RWD, ROW, ROD, RDW, RDO

DWO, DWR, DOW, DOR, DRW, DRO

$\therefore$ Required number of 3 letter words

$= 6 + 6 + 6 + 6 = 24$

We can solve the above problem by product rule as follows. In the required 3 letter word, the first letter can be selected in 4 ways, the second letter can be selected in 3 ways, the third letter can be selected in 3 ways.

$\therefore$ $\boxed{\text{Required number of 3 letter words} = 4 \times 3 \times 2 = 24}$ **Ans.**

Ex.2. A house has 4 doors and 8 windows. In how many ways can be burglar rob the house by entering through a window and exiting through a door?

Soln. Here there are 8 ways of entering through a window and 4 ways of exiting through a door.

Hence by product rule,

$\therefore$ $\boxed{\text{The total number of ways} = 8 \times 4 = 32}$ **Ans.**

Ex.3. How many strings of length 4 can be formed using the letters ABCDE if repetitions are not allowed?

Soln. Here we have to form a string of length 4. The first letter can be filled in 5 ways, the second letter can be filled in 4 ways, similarly the third and fourth letters can be filled in 3 and 2 ways respectively. Hence by product rule,

$\therefore$ $\boxed{\text{The total number of ways} = 5 \times 4 \times 3 \times 2 = 120}$ **Ans.**

Ex.4. Using multiplication principle, slow that a set $\{x_1, x_2, \ldots\ldots\ldots, x_n\}$ with n elements has 2^n subsets.

Soln. Here a subset can be formed in n successive steps. Select or do not select x_1, select or do not select $x_2, \ldots\ldots\ldots$, select or do not select x_n. This shows that each step can be done in 2 ways.

Hence by multiplication principle,

The total number of possible subsets

$= 2 \times 2 \times 2 \times \ldots\ldots \times 2$ (n times $= 2^n$) **Ans.**

Ex.5. There are 10 multiple choice questions in an examination. How many sequence of answers are possible, if the first 5 questions have 4 choices and the next 5 have 5 choices each?

Soln. Here each one of the first 5 questions can be answered in 4 ways and each one of the last 5 questions can be answered in 5 ways.

Hence by product rule,

The total number of possible sequences

$= 4 \times 4 \times 4 \times 4 \times 4 \times 5 \times 5 \times 5 \times 5 \times 5$

$= 3200000$. **....Ans.**

Ex.6. How many numbers are there between 100 and 1000 in which all the digits are distinct?

Soln. We know that any number between 100 and 1000 has 3 digits. Here we have to find all the possible 3-digit number with distinct digits.

Here hundread's place cannot be filled by 0 (otherwise the number will be2 digit number). So, the hundread's place can be filled by any one of the digits 1, , 2, 3, ..., 9. Hence hundread's place can be filled in 9 ways.

Hence by product rule,

Total number of ways $= 9 \times 9 \times 8 = 648$. **....Ans.**

Ex.7. In how many can 5 letters be posted in 4 letter boxes?

Soln. Here any letter (of 5 letters) can be posted in any one of the 4 letter boxes. The first letter can be posted in 4 ways. The second letter again can be posted in 4 ways. Similarly the third, fourth and fifth letters can be posted in 4, 4 & 4 ways.

Hence by product rule,

The total number of ways $= 4 \times 4 \times 4 \times 4 \times 4$

$= 1024$. **....Ans.**

The Addition Principle or Sum Rule :

If two operations are mutually exclusive (i.e. they do not overlap), then the number of ways of performing one operation or the other operation is the sum of the number of ways of performing each operation.

More precisely, if an even x_1 can occur in N_1 ways even x_2 can occur in N_2 ways, ..., event x_T can occur in N_T ways, then one of the events viz $x_1, x_2 \ldots\ldots, x_T$ can occur in $N_1 + N_2 + \ldots\ldots + N_T$ ways.

If A and B are disjoint sets, then $|A + B|$

$= |A| + |B|$.

Ex.8. How many 3 letters words starting or ending with 'O' can be made from the letters of "WORD"?

Soln. No words starting or ending with 'O' are mutually exclusive i.e. they do not overlap. Hence we will find the number of words starting with 'O' and the number of words ending with 'O' and add them.

Starting with 'O' : Here the first place is fixed which is 'O' i.e. the first place can be filled in 1 way.

The second place can be filled in 3 ways by using any one of the letters W, R, D.

The third place can be filled in 2 ways.

Hence by product rule,

The number of words starting with 'O'

$= 1 \times 3 \times 2 = 6$.

Ending with 'O' : Here the last place is fixed which is 'O' i.e. the last place can be filled in 1 way.

The first place can be filled in 3 ways by using any one of the letters W, R, R.

The second place can be filled in 2 ways.

Hence by product rule,

The number of words ending with 'O'

$= 3 \times 2 \times 1 = 6.$

$\therefore$ By sum rule

The total of words starting or ending with 'O' = 6 + 6 = 12	**Ans.**

Ex.9. A computer password consists of a letter followed by 3 or 4 digits. Find (i) The total number of passwords that can be formed (ii) The number of passwords in which no digit repeats.

Soln.

(i) There are 26 alphabets and 10 digits (0 to 9) and the digits can be repeated (given). Here the required password consists of one letter (alphabet) and 3 or 4 digits. Hence passwords of length 4 characters and 5 characters can be formed.

4 character Passwords : The first place is a letter and can be filled in 26 ways. The second place is a digit and can be filled in 10 ways. The third and fourth places are also digits and can be filled in 10 and 10 ways respectively (since the digits can repeat).

Hence by product rule,

The number of 4 character passwords

$= 26 \times 10 \times 10 \times 10 = 26000$

5 character Passwords : The first place is a letter and can be filled in 26 ways. The second, third, fourth and fifth places are digits and can be filled in 10, 10, 10 and 10 ways respectively (since the digits can repeat).

The number of 5 character passwords

$= 26 \times 10 \times 10 \times 10 \times 10 = 260000.$

$\therefore$ By sum rule,

The total number of passwords that can be formed.

$= 26000 + 26000$

$= 286000$ ....**Ans.**

(ii) There are 26 alphabets and 10 digits (0 to 9) and the digits cannot be repeated (given).

4 character Passwords : The first place is a letter and can be filled in 26 ways. The second third and fourth places are digits and can be filled in 10, 9 and 8 ways respectively (since the digits cannot be repeated).

Hence by product rule,

The number of 4 character passwords

$= 26 \times 10 \times 9 \times 8 = 18720$

5 character Passwords : The first place is a letter and can be filled in 26 ways. The second, third, fourth and fifth places are digits and can be filled in 10, 9, 8 and 70 ways respectively (Since the digits cannot be repeated).

Hence by product rule,

The number of 5 character passwords

$= 26 \times 10 \times 9 \times 8 \times 7 = 131040.$

$\therefore$ By sum rule,

The total number of passwords that can be formed.

$= 18720 + 131040$

$= 149760$ ....**Ans.**

Factorial Notation : The product of first n natural number viz 1, 2, 3 ....., n is denoted by n! and is read as factorial n or n factorial.

Hence $n! = 1 . 2 . 3 . 4 \ldots\ldots (n-1)\, n$

$= n\,(n-1)!$

$= n\,(n-1)\,(n-2)\,!$

Note : (i) If $n = 0$, then $n! = 0! = 1$

(ii) If $n > 0$, then $n! = n\,(n-1)\,!$

PERMUTATIONS AND COMBINATIONS

Permutation : Each of the arrangements which can be made by taking some or all, of a number of things is called a permutation. More precisely any arrangement of a set of n objects in a given order taken r at a time is called a permutation.

(i) BDCA, ACDB, DABC are permutations of the four letters taken all at a time.

(ii) BAD, DAC, ABD, CBA are permutations of the four letters taken 3 at a time.

(iii) AB, BC, DA, CD, DC are permutations of the four letter taken 2 at a time.

The number of permutations of n objects taken r at a time is denoted by P(n, r) and is defined by P (n, r)

$$= \frac{n!}{(n-r)!}.$$

In simple language, permutation is an ordered selection or arrangement of r objects from a set of n objects in which the order matters.

Permutation with Repetition :

Let $P\ (n\,;n_1,\ n_2,\ldots,n_r)$ denote the number of permutations of n objects of which n_1 are alike, n_2 are alike, ..., n_r are alike.

$$\text{Then } P(n\,;n_1,\ n_2,\ldots,n_r) = \frac{n!}{(n_1!\,n_2!\ldots\ldots n_r!)}$$

When there is no restriction on the number of times a particular object may occur in the permutation of n objects taken r at a time, then there n ways to select an object of the set for each of the r positions..

Hence by multiplication principle, there are

$n \times n \times n \times \ldots\ldots$ (r times) = 'n' possibilities.

Permutations Under Certain Conditions :

(i) The total number of permutations of n different objects taken r at a time, when k particular objects are always to be included is $P(n-k, r-k)$.

(ii) The total number of permutations of n different objects taken r at a time, when k particular objects are never taken is $P(n-k, r)$.

IMPORTANT FORMULAE :

(1) $P(n,r) = \frac{n!}{(n-r)!}$

(2) **Permutation with Repetition :**

$$P(n : n_1, n_2, ..., n_r) = \frac{n!}{(n!, n_2! ... n_r!)}$$

(3) **Permutation with condition :**

(i) $P(n-k, r-k)$, when k objects to be included

(ii) $P(n-k, r)$, when k objects are never taken.

Ex.10. In how many ways can be the letters of the word "SUNDAY" be arranged. How many of them begin with S and end with Y? How many of them begin with S and do not end with Y?

Soln. The word 'SUNDAY' contains 6 alphabets.

i.e. $n = 6$

Since we have to arrange all 6 alphabets

$\therefore r = 6$

This can be done in P (6, 6) ways

Using $P(n, r) = \frac{n!}{(n-r)!}$, we get

$P(6, 6) = \frac{6!}{(6-6)!} = 720$ ways ....**Ans.**

If S occupies the first place and Y occupies the last place, then there are four alphabets U, N, D, A. These 4 alphabets can be arraged in P (4, 4) ways.

i.e. in $P(4, 4) = \frac{4!}{(4-4)!} = 24$ ways ....**Ans.**

Let S occupy the first place. Since Y cannot occupy the last place, the last place can be filled in 4 ways using any one of the alphabets U, N, D, A.

Now the second place can be filled in 4 ways (including Y). Similarly the third, fourth and fifth places can be filled in 3, 2, and 1 ways respectively.

Hence by product rule,

$\therefore$ The total number of ways $= 4 \times 3 \times 2 \times 1 \times 4 = 96$ **Ans.**

Ex.11. A bit is either 0 or 1. A byte is a sequence of 8 bits. Find (a) The number of bytes that can be formed from 8 bits. (b) The number of bytes that begin with 11 and end with 11. (c) The number of bytes that begin will 11 and do not end with 11 (d) The number of bytes that begin with 11 or end with 11.

Soln. Here the digits 0 and 1 can repeat.

(a) This is problem of permutation with repetition. Hence the 8 positions of the byte can be filled in $2^8 = 256$ ways. Hence the number of bytes that can be formed is 256.

(b) Here out of the 8 positions of the byte, the first two and the last two (i.e. total four) positions are fixed. Hence the remaining 4 positions can be filled in $2^4 = 16$ ways.

(c) Here out of the 8 positions of the byte, the first two are fixed. Hence the remaining six positions can be filled in $2^6 = 64$ ways. But the last two positions can not be 11

Hence the required number of bytes.

$= 64 -$ Numbers of bytes that end with 11

$= 64 - 16 = 48$

(d) From (c) part, we known that 64 bytes begin with 11. Similarly 64 bytes end with 11. Here we have to count each byte that begins and ends with 11, twice i.e. $64 + 64 = 128$.

Hence the required number of bytes

$= 128 -$ Number of bytes that begin and end with 11

$= 128 - 16 = 112$. ....**Ans.**

Ex.12. How many 7 letter words can be formed using the word 'BENZENE'.

Soln. Here $n = 7$

Let n_1 = Number of E's = 3

n_2 = Number of N's = 2

Here we have to find the number of permutations of 7 objects of which 3 are alike, 2 are alike.

$\therefore P(n ; n_1, n_2) = \frac{n!}{(n_1! n_2!)}$

$\therefore$ The number of 7 letter words that can be formed

$= P(7, 3, 2) = \frac{7!}{3!\,2!} = 420$ ....**Ans.**

Ex.13. How many different signals, each consisting of eight flags hung in a vertical line, can be formed from a set of 4 red flags, 3 white flags, 1 blue flag?

Soln. Here $n = 4 + 3 + 1 = 8$

Let n_1 = Number of red flags = 4

n_2 = Number of white flags = 3

n_3 = Number of blue flags = 1

Here we have to find the number of permutation s of which 4 are alike, 3 are a like, 1 is alike.

$$\therefore P(n; n_1, n_2, n_3) = \frac{n!}{(n_1!\, n_2!\, n_3!)}$$

$\therefore$ The number of different signals that can be formed

$$= P(8; 4, 3, 1) = \frac{8!}{2!\,3!\,1!} = 280 \quad \text{....Ans.}$$

Combination : Each of the different selections made by taking some or all, of a number of things, irrespective of their order is called a combinations. More precisely any selections of a set of n objects taken 'r' at a time in any order is called a combination.

For example : Consider the combination of letters A, B, C taken 3 at a time given by ABC, ACD, DAC, BAC, DCA. Here since the order is not important the combinations ABC, BAC are same i.e. they represent the objects of the set {A, B, C} taken in any order.

The number of a combinations of n objects taken r at a time is denoted by C (n, r) or $\binom{n}{r}$ and is defined by

$$C(n, r) = \frac{n!}{r!\,(n-r)!}$$

Difference Between a Permutation and a Combination :

(1) In a combination only selection is made, whereas in a permutation not only a selection is made, but also an arrangement in a definite order is made.

(2) In a combination, the ordering of the selected objects is immaterial, whereas in a permutation, the order is essential.

For example: AB and BA are same as combinations, but different as permutations.

(3) A particular combination corresponds to many permutations.

For example: The combination of 3 letters ABC corresponds to the permutations ABC, ACB, BCA, CAB, CBA, etc.

(4) Usually the number of permutations exceeds the number of combinations.

(5) We generally use the word "Arrangement" for permutation and the word "Selection" for combination.

Combination with repetition : When repetition is allowed, the number of combinations of n objects taken r at a time = C (n + r – 1, r).

Combinations Under Certain Conditions :

(1) The number of combinations of n different object taken r at a time, when k particular objects are always to be included is C (n – k, r – k).

(2) The number of combinations of n different objects taken r at a time, when k particular objects are never taken is C (n – k, r).

IMPORTANT FORMULAE :

(1) $C(n, r) = \dfrac{n!}{r!\,(n-r)!}$

(2) **Combination with repetition :**
C (n + r – 1, r).

(3) **Combination with condition :**
(i) C (n – k, r – k),
when k objects to be included
(ii) C (n – k, r), when k objects are never taken.

Ex.14. Consider the set {A, B, C, D}. In how many ways can be select two of these letters when repetition is allowed?

Soln. Here n = 4 = number of objects

r = 2 = Number of objects to be selected

Since repetition is allowed

$\therefore$ The required number of ways = C (n + r – 1, r)

= C (4 + 2 – 1, 2)

= C (5, 2)

= 10

$\therefore$ $\boxed{\text{Required no. of ways} = 10}$ **Ans.**

Ex.15. A question paper has two sections, section A and section B each containing 5 questions. If the student has to choose 4 questions from section A and 3 questions from section B, in how many ways can he choose the questions?

Soln. Total number of questions = 5 + 5 = 10

4 questions out of 5 from section A can be selected in C (5, 4) ways.

Similarly 3 questions out of 5 from section B can be selected in C(5, 3) ways.

Hence by produce rule,

The total number of ways = C (5, 4) × C (5, 3)

$$= \frac{5!}{4!\,1!} \times \frac{5!}{3!\,2!} = 5 \times 10 = 50$$

$\therefore$ $\boxed{\text{Required no. of ways} = 50}$ **Ans.**

Ex.16. In how many ways can a cricket team be selected from a group of 25 players containing 10 batsmen, 8 bowlers, 5 all - rounders and 2 wicket keepers? Assume that the team of 11 players requires 5 batsmen, 3 all rounders, 2 bowlers and 1 wicket keeper.

Soln. 5 batsmen out of 10 batsmen can be selected in C (10, 5) ways 3 all rounders out of 5 all rounders can be selected in C (5, 3) ways. 2 bowlers out of 8 bowlers can be selected in C (8, 2) ways. 1 wicket keeper out of 2 can be selected in C (2, 1) ways.

Hence by product rule,

The total number of ways

$$= C(10, 5) \times C(5, 3) \times C(8, 2) \times C(2,1)$$

$$= \frac{10!}{5!\,5!} \times \frac{5!}{3!\,2!} \times \frac{8!}{2!\,6!} \times \frac{2!}{1!\,1!}$$

$$= 141120.$$

$\therefore$ $\boxed{\text{Total number of ways} = 141120}$ **Ans.**

Ex.17. A box contains 5 red and 6 white marbles. In how many ways can 6 marbles be selected so that there are atleast two balls of each colour?

Soln. Here we can select 6 marbles from 5 red marbles and 6 white marbles such that atleast two marbles of each colour are selected in the following ways.

(i) Select 2 red marbles out of 5 and 4 white marbles out of 6. This can be done in C (5, 2) × C (6, 4) ways.

(ii) Select 3 marbles out of 5 and 3 white marbles out of 6. This can be done in C (5, 3) × C (6, 3) ways.

(iii) Select 4 red marbles out of 5 and 2 white marbles out of 6. This can be done in C (5, 4) × C (6, 2) ways.

Since the selection of 6 marbles can be done in any one of the above ways, hence by sum rule,

Total Number of ways

$$= C(5, 2) \times C(6, 4) + C(5, 3) \times C(6,3) + C(5, 4) \times C(6, 2)$$

$$= \frac{5!}{2!\,3!} \times \frac{6!}{4!\,2!} + \frac{5!}{3!\,2!} \times \frac{6!}{3!\,3!} + \frac{5!}{4!\,1!} \times \frac{6!}{2!\,4!}$$

$$= (10 \times 15) + (10 \times 20) + (5 \times 15) = 425.$$

$\therefore$ $\boxed{\text{Required no. of ways} = 425}$ **Ans.**

Ex.18. There are 10 professors and 20 students, out of whom a committee of 2 professors and 3 students is to be formed. Find the number of ways in which this can be done. Further, find in how many of these committees

(i) A particular professor is included.

(ii) A particular student is included.

(iii) A particular student is excluded.

Soln. Two professors out of 10 professors can be selected in C (10, 2) ways. Similarly 3 students out of 20 students can be selected in C (20, 3) ways. Hence by product rule,

The total number of ways = C (10, 2) × C (20, 3)

$$= \frac{10!}{2!\,8!} \times \frac{20!}{3!\,17!} = 45 \times 1140 = 51300$$

(i) Here we have to include a particular professor

i.e. n = 10, r = 2, k = 1

$\therefore$ The number of ways = C (n – k, r – k)

= C (10 – 1, 2 – 1)

= C (9, 1) = 9

(ii) Here we have to include a particular student

i.e. n = 20, r = 3, k = 1

$\therefore$ The number of ways = C (n – k, r – k)

= C (20 – 1, 3 – 1)

= C (19, 2) = 171

(iii) Here we have to exclude a particular student

i.e. n = 20, r = 3, k = 1

$\therefore$ The number of ways = C (n – k, r)

= C (20 – 1, 3)

= C (19, 3)

= 969..... **Ans.**

Ex.19. How many integer solutions are there to $x_1 + x_2 + x_3 = 0$ with $x_i > -5$; i = 1, 2, 3 ?

Soln. Given equation is $x_1 + x_2 + x_3 = 0$ ..... (1)

Also $x_i \geq -5$; i = 1, 2, 3

i.e. $x_1 \geq -5$, $x_2 \geq -5$, $x_3 \geq -5$

Let $x_1 = -5 + p$, $x_2 = -5 + q$, $x_3 = -5 + s$

$\therefore$ (1) $\Rightarrow (-5 + p) + (-5 + q) + (-5 + s) = 0$

$p + q + s = 15$

Here n = Number of unknowns = 3, r = 15

Since the values of x, y, z are not unique, this is a problem of combination with repetition.

$\therefore$ The required number of solutions = C (n + r – 1, r)

= C (3 + 15 – 1, 15)

= C (17, 15)

$\therefore$ $\boxed{\text{Required number of solution} = 136}$ **Ans.**

PIGEON-HOLE PRINCIPLE

CS/CT/CE : W-06,12, S-08,13

The Pigeon-Hole Principle, also known as Dirichlet Drawer Principle is applicable in many problems where we

want to show that a given situation can occur. Many results in combinatorial theory come from this principle. The description of this principle is often given interms of pigeons and nesting holes (pigeon holes).

Suppose that m pigeons are placed into n nesting holes, where m > n. Then atleast one nesting hole contains two or more pigeons.

Generalized Pigeon-Hole Principle : If m pigeons are placed into n nesting holes, then one of the nesting holes must contain atleast $\left[\frac{m-1}{n}\right]+1$ pigeons.

Note : If m and n are positive integers, then $\left[\frac{m}{n}\right]$ stands for the largest integer less than or equal to the rational number $\frac{m}{n}$.

Hence $\left[\frac{3}{2}\right]=[1.5]=1, \left[\frac{9}{4}\right]=[2.25]=2,$

$\left[\frac{19}{3}\right]=[6.33]=6, \left[\frac{6}{2}\right]=[3]=3$ etc.

Ex.20. Show that if 8 persons are chosen from a group, then atleast 2 of them are born on the same day of the week.

Soln. By pigeon hole principle we know that if m pigeons are placed into n nesting holes (m > n), then atleast one nesting hole contains two or more pigeons.

Here we treat the number of persons as pigeons (m) and 7 days of a week as nesting holes or pigeon holes (n).

i.e. m = 8, n = 7 clearly m > n.

Hence by pigeon-hole principle, atleast 2 persons were born on the same day of the week. ...**Ans.**

Ex.21. Show that if an institute contains 13 professors, then two of them are born in the same month.

Soln. By pigeon hole principle we know that if m pigeons are placed into n nesting holes (m > n), then atleast one nesting hole contains two or more pigeons.

Here we treat the number of professors as pigeons (m) and 12 months of a year as nesting holes (n).

Hence by pigeon hole principle, ateast 2 professors were born in the same month. ....**Ans.**

Ex.22. If 9 books are to kept 4 shelves, the show that there must be atleast one shelf which contains ateast 3 books.

Soln. By generalized pigeon hole principle we known that if m pigaons are placed into n nesting holes, then one of the nesting holes must contain $\left[\frac{m-1}{n}\right]+1$ pigeons.

Hence we treat the 9 books as pigeons (m) and 4 shelves as nesting holes (n).

i.e. m = 9, n = 4

Hence by generalized pigeon hole principle.

$$\left[\frac{m-1}{n}\right]+1=\left[\frac{9-1}{4}\right]+1=\left[\frac{8}{4}\right]+1=[2]+1=2+1=3$$

i.e. atleast one shelf contains 3 books atleast. ....**Ans.**

Ex.23. Show that if any 30 people are selected from a group, then we may select a subset of 5 so that all 5 were born on the same day of the week.

CS/CT/CE : S-09

Soln. By generalized pigeon hole principle we know that if m pigeons are placed into n nesting holes, then one of the nesting holes must contain $\left[\frac{m-1}{n}\right]+1$

Hence we treat that 30 people as pigeons (m) and 7 days of a week as nesting holes (n).

i.e. m = 30, n = 7

Hence by generalized pigeon hole principle,

$$\left[\frac{m-1}{n}\right]+1=\left[\frac{30-1}{7}\right]+1=\left[\frac{29}{7}\right]+1$$
$$=[4.14]+1=4+1=5$$

i.e. atleast 5 people have been born on the same day of the week. ....**Ans.**

Ex.24. Find the minimum number of students in a class to be sure that four out of them are born in the same month?

CS/CT/CE : S-13

Soln. By generalized pigeon hole principle we know that if m pigeons are placed into n nesting holes, then one of the nesting holes must contain $\left[\frac{m-1}{n}\right]$ pigeons.

Here we treat the required number of students as pigeons (m) and 12 months of an year a nesting holes (n). Here we have to find the minimum number of students such that 4 out of them are born in the same month.

Hence by generalized pigeon hole principle.

$$\left[\frac{m-1}{n}\right]=4$$

$$\Rightarrow \left[\frac{m-1}{12}\right]=3$$

$\Rightarrow m-1=3\times 12=36$

$\therefore m=37$

$\therefore$ $\boxed{\text{The required number of students is 37.}}$ **Ans.**

Ex.25. Show that if 5 integers from 1 to 8 are chosen, then two of them will have a sum 9. **CS/CT/CE : S-11**

Soln. Here we construct 4 different sets, each containing 2 integres that will give a sum of 9 as follows.

$A_1 = \{1, 8\}, A_2 = \{2, 7\}, A_3 = \{3, 6\}, A_4 = \{4, 5\}$

Since we have to select any 5 integers, each of them must belong to one of these sets.

Hence we treat the 5 integers as pigeons (m) and the 4 sets as nesting holes (n).

i.e. $m = 5, n = 4 (m > n)$

Hence by pigeon hole principle two of the integers chosen must belong to the same set. ....**Ans.**

RECURRENCE RELATION

CS/CT/CE : W-06,09,S-09

A recurrence relation (sometimes called a difference equation) for a sequence $\{a_n\}$ is an equation that expresses a_n interms of one or more of the previous terms in the sequence, namely $a_0, a_1 \ldots\ldots, a_{n-1}$ for all integers $n \geq n_0$, where n_0 is a non-negative integer. A sequence is said to be a solution for a recurrence relation if its terms satisfy the recurrence relation for all $n \geq n_0$.

Explicit Formula : A formula which defines the positions of nth term only is called explicit formula.

For example : The sequence representing the explicit formula $a_n = (-4)^n$, $n \geq 1$ is $a_n = \{-4, 16, -64, 256, \ldots\ldots\}$

(i) Consider the sequence $a_n = \{3, 8, 13, 18, \ldots..\}$. The explicit formula for the sequence is given by the recurrence relation $a_n = a_{n-1} + 6$; $n \geq 2$ with the initial conditions $a_1 = 3$.

(ii) Consider the famous fibonacci sequence given by $a_n = \{1, 1, 2, 3, 5, 8, 13, \ldots.\}$. The recurrence relation for this sequence is given by $a_n = a_{n-1} + a_{n-2}, n \geq 3$ with the initial conditions $a_1 = a_2 = 1$.

(iii) Consider the recurrence relations

$a_n = a_{n-1} - a_{n-2}, n \geq 2$

If $a_0 = 3, a_1 = 6$, then $a_2 = a_1 - a_0 = 6 - 3 = 3$

$a_3 = a_2 - a_1 = 3 - 6 = -3$

$a_4 = a_3 - a_2 = -3 - 3 = -6$

In this way we can generate all other values of $\{a_n\}$:

Here we observe that example (i) requires only one initial condition because a_n is expressed interms of previous one term only whereas examples (ii) and (iii) require two initial conditions because a_n is expressed interms of previous two terms.

Ex.26. A person deposits Rs. 1,00,000/- in his savings bank A/C The annual interest of the bank is 8%. Find a recursive (explicit) formula to compute the amount he will get after n years?

Soln. Let a_n denote the amount after n years.

Since deposit $a_0 = 1,00,000$

Since the amount in the A/C after n years equals the amount in the A/C after (n – 1) years plus interest for the nth year, we have

a_n = (Compound amount at the end of $(n-1)^{th}$ year)

+ (Interest earned during n^{th} year)

$\therefore a_n = a_{n-1} + (a_{n-1})(0.08)$

$\therefore a_n = (1.08)\, a_{n-1}$ with the condition $a_0 = 1,00,000$.

....**Ans.**

Solving Recurrence Relations : In general, in order to solve a recurrence relation, we prefer to have an explicit formula to compute the value of a_n rather than conducting n iterations. For one class of recurrence relations we can obtain such formulas in systematic way. These are the recurrence are relations that express the terms of a sequence as linear combinations of previous terms.

Linear Homogenerous Recurrence Relation With Constant Coefficients : A linear homogeneous recurrence relation of degree k with contant coefficients is a recurrence relation of the from

$$a_n = C_1 a_{n-1} + C_2 a_{n-2} + \ldots.. + C_k a_{n-k} \quad \ldots.(1)$$

where $C_1, C_2, \ldots\ldots, C_k$ are real numbers and $C_k \neq 0$

Any sequence which satisfies such a recurrence relation is called its solution. This solution is uniquely determined by the recurrence relation and the k initial conditions $a_0 = C_0, a_1 = C_1, a_2 = C_2, \ldots\ldots, a_{k-1} = C_{k-1}$. The recurrence relation $a_n = (1.08)\, a_{n-1}$ is a linear homogeneous recurrence relation of degree 1. The recurrence relation $a_n = a_{n-1} + a_{n-2}$ is a linear homogeneous recurrence relation of degree 2. The recurrence relation $a_n = a_{n-5}$ is a linear homogeneous recurrence relation of degree 5.

Method of solving recurrence relation : Basically there are two methods of solving recurrence relations.

(i) Characterisitc Roots (ii) Generating functions.

(i) **Characterisitc Roots :** While solving recurrence relations we try to find solutions of the form $a_n = r^n$, where r

is a constant $a_n = r^n$ is a solution of the linear homogeneous recurrence relation (1), iff

$$r^n = C_1 r^{n-1} + C_2 r^{n-2} + \ldots\ldots + C_k r^{n-k} \quad \ldots..(2)$$

Multiply by r^{k-n}, we get

$$r^k = C_1 r^{k-1} + C_2 r^{k-2} + \ldots\ldots + C_k$$

i.e. $r^k - C_1 r^{k-1} - C_2 r^{k-2} \ldots\ldots - C_k = 0 \quad \ldots.(3)$

Equation (3) is called characteristic equation of the linear homogeneous recurrence relation given by equation (1). The solutions of this equation (3) are called characteristic roots of equation (1).

Nature of Roots :

(a) **Real and Distinct Roots :** If the roots $r_1, r_2, \ldots\ldots, r_k$ of the characteristic equation are real and distinct, then the general solution is

$$a_n = b_1 (r_1)^n + b_2 (r_2)^n + \ldots\ldots + b_k (r_k)^n$$

where $b_1, b_2, \ldots.., b_k$ are constants which can be determined from the given initial conditions.

(b) **Real and Equal Roots :** Let two roots r_1, r_2, of the characteristic equation are real and equal, then the general solution is

$$a_n = [b_1 + b_2 n](r)^n, r_1 = r_2 = r \text{ (say)}$$

If three roots r_1, r_2, r_3 of the characteristic equation are real and equal, then the general solution is

$$a_n = [b_1 + b_2 n + b_3 n^2](r)^n \quad , r_1 = r_2 = r_3 = r \text{ (say)}$$

(c) **Imaginary and Distinct Roots :** Let $\alpha \pm i\beta$ be the imaginary roots of the characteristic equation where α and β are real. Then the general solution is

$$a_n = b_1 (\alpha + i\beta)^n + b_2 (\alpha - i\beta)^n$$

(d) **Imaginary and equal Roots :** Let $\alpha \pm i\beta$ be repeated twice, then the general solution is

$$a_n = (b_1 + b_2 n)(\alpha + i\beta)^n + (b_3 + b_4 n)(\alpha - i\beta)^n$$

Let $a \pm ib$ be repeated twice. Then the general solution is

$$a_n = (b_1 + b_2 n + b_3 n^2)(\alpha + i\beta)^n + (b_3 + b_4 n + b_5 n^2)(\alpha - i\beta)^n$$

Ex.27. Solve the following recurrence relations :

(i) $a_n - 8a_{n-1} + 16a_{n-2} = 0$ with $a_2 = 16, a_3 = 80$

(ii) $a_n = 3a_{n-1} + 4a_{n-2} = 0$ with $a_0 = 0, a_1 = 5$

(iii) $a_{n+2} - 4a_{n+1} + 4a_n = 0$ with $a_0 = 1, a_1 = 3$

(iv) $a_n - 8a_{n-1} + 21a_{n-2} 18\, a_{n-3} = 0$

Soln.

(i) Given recurrence relation is

$$a_n - 8a_{n-1} + 16a_{n-2} = 0 \quad \ldots.(1)$$

This is a linear homogeneous recurrence relation of order 2.

Let $a_n = r^n$ be a solution of equation (1)

$\therefore (1) \Rightarrow r^n - 8r^{n-1} + 16r^{n-2} = 0$

Multiply by r^{2-n} $(r^{k-n}, k = 2)$

$\Rightarrow r^2 - 8r + 16 = 0$

$\Rightarrow (r-4)^2 = 0$ i.e. $r = 4, 4$

The roots are real and equal

$\therefore$ The general solution is $a_n = (b_1 + b_2 n)(4)^n \ldots.(2)$

Given $a_2 = 16$ i.e. put $n = 2$ in equation (2)

$\therefore (2) \Rightarrow 16 = (b_1 + 2b_2)(4)^2 \Rightarrow b_1 + 2b_2 = 1 \quad \ldots..(a)$

Also $a_3 = 80$ i.e. put $n = 3$ in equation (2)

$\therefore (2) \Rightarrow 80 = (b_1 + 3b_2)(4)^3$ i.e. $b_1 + 3b_2 = \frac{5}{4} \quad \ldots..(b)$

Solving equations (a) and (b), we get

$b_1 = \frac{1}{2}, b_2 = \frac{1}{4}$

$\therefore (2) \Rightarrow a_n = \left(\frac{1}{2} + \frac{1}{4}n\right)(4)^n$

This is the required solution. **....Ans.**

(ii) Given recurrence relation is

$$a_n - 3a_{n-1} + 4a_{n-2} = 0 \quad \ldots.(1)$$

This is a linear homogeneous recurrence relation of order 2.

Let $a_n = r^n$ be a solution of equation (1)

$\therefore (1) \Rightarrow r^n - 3r^{n-1} - 4r^{n-2} = 0$

Multiply by r^{2-n}

$\Rightarrow r^2 - 3r - 4 = 0$

$\Rightarrow r = -1, 4$

$\therefore$ The roots are real and distinct

$\therefore$ The general solution is $a_n = b_1(-1)^n + b_2(4)^n \ldots.(2)$

Given $a_0 = 0$ i.e. put $n = 2$ in equation (2)

$\therefore (2) \Rightarrow 0 = b_1 + 2b_2 \Rightarrow b_1 + b_2 = 0 \quad \ldots..(a)$

Also $a_1 = 5$ i.e. put $n = 1$ in equation (2)

$\therefore (2) \Rightarrow 5 = -b_1 + 4b_2$ i.e. $-b_1 + 4b_2 = 5 \quad \ldots..(b)$

Solving equations (a) and (b), we get

$b_1 = -1, b_2 = 1$

$\therefore (2) \Rightarrow a_n = (4)^n - (-1)^n$

This is the required solution. **...Ans.**

(iii) Given recurrence relation is

$a_{n+2} - 4a_{n+1} + 4a_n = 0$ ....(1)

Let $a_n = r^n$ be a solution of equation (1)

$\therefore (1) \Rightarrow r^{n+2} - 4r^{n+1} + 4r^n = 0$

Multiply by r^{-n}

$\Rightarrow r^2 - 4r + 4 = 0$

$\Rightarrow r = 2, 4$

The roots are real and equal

$\therefore$ The general solution is $a_n = (b_1 + b_2 n)(2)^n$ ....(2)

Given $a_0 = 1$ i.e. put $n = 0$ in equation (2)

$\therefore (2) \Rightarrow 1 = b_1$ i.e. $b_1 = 1$

Also $a_1 = 3$ i.e. put $n = 1$ in equation (2)

$(2) \Rightarrow 3 = (b_1 + b_2)\,2$ i.e. $b_1 + b_2 = \frac{3}{2} \Rightarrow b_2 = \frac{1}{2}$

$[\because b_1 = 1]$

$\therefore (2) \Rightarrow a_n = \left(1 - \frac{1}{2}n\right)(2)^n$ is the required solution.

...Ans.

(iv) Given recurrence relation is

$a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} = 0$ ....(1)

This is a linear homogeneous recurrence relation of order 3.

Let $a_n = r^n$ be a solution of equation (1)

$\therefore (1) \Rightarrow r^n - 8r^{n-1} + 21r^{n-2} - 18r^{n-3} = 0$

Multiply by r^{3-n}

$\therefore r^3 - 8r^2 + 21r - 18 = 0$

$\therefore r = 2, 3, 3$

$\therefore$ $\boxed{\text{The general solution is } a_n = b_1(2)^n + (b_2 + b_3 n)(3)^n}$ **Ans.**

Ex.28. The famous fibonacci sequence is defined by $f(n) = f(n-1) + f(n-2)$ with $f(0) = 1 = f(1)$. Find $f(n)$ in the explicit form.

CS/CT/CE : W-06,07,09, S-09

OR Find the solution of the fibonacci sequence define d by $f(n) = f(n-1) + f(n-2)$ with $f(0) = 1 = f(1)$.

Soln. The fibonacci sequence is given by

$f(n) = f(n-1) + f(n-2)$ with $f(0) = 1 = f(1)$

Let us write this as $a_n = a_{n-1} + a_{n-2}$ with $a_0 = 1 = a_1$

i.e. $a_n - a_{n-1} - a_{n-2} = 0$ ....(1)

This is a linear homogenous recurrence of order 2

Let $a_n = r^n$ be a solution of equation (1)

$\therefore (1) \Rightarrow r^n - r^{n-1} - r^{n-2} = 0$

Multiply by r^{2-n}, we get

$r^2 - r - 1 = 0$

Using quadratic formula $r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$;

$a = 1, b = -1, c = -1$, we get

$r = \frac{1 \pm \sqrt{5}}{2}$ i.e. $r = \frac{1+\sqrt{5}}{2}, r = \frac{1-\sqrt{5}}{2}$

$\therefore$ The roots are real and distinct

$\therefore$ The general solution is

$$a_n = b_1\left(\frac{1+\sqrt{5}}{2}\right)^n + b_2\left(\frac{1-\sqrt{5}}{2}\right)^n \quad ....(2)$$

Given $a_0 = 1$ i.e. put $n = 0$ in equation (2)

$\therefore (2) \Rightarrow 1 = b_1 + b_2$ i.e. $b_1 + b_2 = 1$ .....(a)

Also $a_1 = 1$ i.e. put $n = 1$ in equation (2)

$$\therefore (2) \Rightarrow 1 = b_1\left(\frac{1+\sqrt{5}}{2}\right) + b_2\left(\frac{1-\sqrt{5}}{2}\right)$$

$$\text{i.e. } b_1\left(\frac{1+\sqrt{5}}{2}\right) + b_2\left(\frac{1-\sqrt{5}}{2}\right) = 1 \quad ....(b)$$

$$(a) \times \left(\frac{1+\sqrt{5}}{2}\right) \Rightarrow \left(\frac{1+\sqrt{5}}{2}\right) = b_1\left(\frac{1+\sqrt{5}}{2}\right) + \left(\frac{1+\sqrt{5}}{2}\right)b_2$$

$(c) - (b) \Rightarrow$

$$b_2\left(\frac{1+\sqrt{5}}{2} - \frac{(1-\sqrt{5})}{2}\right) = \left(\frac{1+\sqrt{5}}{2}\right) = 1$$

$$\Rightarrow b_2(\sqrt{5}) = \left(\frac{1+\sqrt{5}-2}{2}\right)$$

$$\therefore b_2 = -\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)$$

$$\text{Now (a)} \Rightarrow b_1 = 1 - b_2 = 1 - \frac{(\sqrt{5}-1)}{2\sqrt{5}} = \frac{\sqrt{5}+1}{2\sqrt{5}}$$

$$\therefore b_1 = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)$$

$$\therefore (2) \Rightarrow a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

i.e. $f(n) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$

This is the required solution.

Note : If we solve the above fibonacci sequence for $n > 3$ and $f(1) = 1 = f(2)$, then we get

$b_1 = \frac{1}{\sqrt{5}}, b_2 = \frac{-1}{\sqrt{5}}$.

Ex.29. Find the explicit formula of the recurrence relation $d_n = 2d_{n-1} - d_{n-2}$ with initial conditions $d_1 = 1.5$ and $d_2 = 3$. ***CS/CT/CE : S-10,W-10***

Soln. Given recurrence relation is

$d_n = 2d_{n-1} - d_{n-2}$

i.e. $d_n - 2d_{n-1} + d_{n-2} = 0$ ....(1)

This is a linear homogeneous recurrence relation of order 2.

Let $d_n = r^n$ be a solution of equation (1)

$\therefore (1) \Rightarrow r^n - 2r^{n-1} + r^{n-2} = 0$

Multiply by r^{2-n}

$\therefore r^2 - 2r + 1 = 0$

$\therefore r = 1, 1$

The roots are real and equal

$\therefore d_n = (b_1 + nb_2)(1)^n\ b_1 + nb_2$ ....(2)

Given $d_1 = 1.5$ i.e. put $n = 1$ in equation (2)

$\therefore (2) \Rightarrow 1.5 = b_1 + b_2$ i.e. $b_1 + b_2 = 1.5$ .....(a)

Also $d_2 = 3$ i.e. put $n = 3$ in equation (2)

$\therefore (2) \Rightarrow 3 = b_1 + 2b_2$ i.e. $b_1 + 2b_2 = 3$ .....(b)

Solving equations (a) and (b), we get

$b_1 = 0, b_2 = 1.5$

$\therefore (2) \Rightarrow d_n = 1.5\,n$

This is the required solution. ...**Ans.**

Operators E and Δ :

These methods are based on the calculus of finite differences that deals with the changes in the value of the function (dependent variable) $y = f(x)$ due to changes in the value of the independent variable x.

The operator Δ :

Let $y = f(x)$ be the given function. The values of y are called entries and the values of x are called arguments. Let x takes the values a, a + h, a + 2h, ---, a + nh with an increment h.

i.e. $x : a, a + h, a + 2h, \ldots\ldots, a + nh$

$y = f(x) : f(x)\ f(a + h), f(a + 2h), \ldots, f(a + nh)$

The first order forward difference of f (a) is denoted by $\Delta f(a)$ and is given by

$\Delta f(a) = f(a + h) - f(a)$

The first order forward difference of f (a + h) is denoted by $\Delta f(a + h)$ and is given by

$\Delta f(a + h) = f(a + 2h) - f(a + h)$

Here Δ is called first order forward difference operator.

The operator E :

The operator E (also known as shift operator) increases the value of the argument by one interval.

i.e. $E\,f(x) = f(x + h)$, where h is the interval

For example :

$E \sin x = \sin(x + h)$, $E \log(x + h) = \log(x + 2h)$ etc.

Relation between E and D :

$E = 1 + \Delta$

or $\Delta = E - 1$

Factorial polynomial :

The continued product of n factors in Arithmetic progression whose first factor is x and the successive factors decrease by a constant (usually unity) is called factor polynomial of nth degree and is denoted by $x^{(n)}$ (read as x to the power n factorial).

Thus

$x^{(n)} = x(x-1)(x-2)(x-3)\ldots(x-(n-1))$

For example : $x^{(0)} = 1, x^{(1)} = x, x^{(2)} = x(x-1)$,

$x^{(3)} = x(x-1)(x-2), x^{(4)} = x(x-1)(x-2)(x-3)$ etc.

Use of operators E and Δ :

Consider the recurrence relation

$a_{n+k} + C_1 a_{n+k-1} + C_2 a_{n+k-2} + C_k a_n = f(n)$ ....(1)

with order k, Equation (1) interms of operator E can be written as

$E^k a_n + C_1 E^{k-1} - 1a_n + C_2 E^{k-2} a_n + \ldots\ldots + C_k a_n = f(n)$

i.e. $(E^k + C_1 E^{k-1} + C_2 E^{k-2} + \ldots\ldots + C_k)\,a_n = f(n)$

i.e. $f(E)\,a_n = f(n)$

where $f(E) = E^k + C_1 E^{k-1} + C_2 E^{k-2} + \ldots\ldots + C_k$

The particular solution (P.S.) is

$P.S. = \frac{1}{f(E)} . f(n)$

The particular solution PS of the given recurrence relation will depend on the different forms of f(n). Here we will study the following forms :

(a) When $f(n) = A^n$, where A is a constant, then Then

$$P.S. = \frac{1}{f(E)} f(n) = \frac{1}{f(E)} A^n = \frac{1}{f(A)} A^n \text{ provided}$$

$f(A) \neq 0$

If $f(A) = 0$, then $E - A$ must be a factor of $f(E)$. Then

(i) $\frac{1}{E-A} A^n = nA^{n-1}$

(ii) $\frac{1}{(E-A)^2} A^n = \frac{n(n-1)}{2!} A^{n-2}$

(iii) $\frac{1}{(E-A)^3} A^n = \frac{n(n-1)(n-2)}{3!} A^{n-3}$ and so on

(b) When $f(n) = n^p$. where n^p is polynomical of degree p, then

$$P.S. = \frac{1}{f(E)} f(n) = \frac{1}{f(E)} n^p$$

$$= \frac{1}{f(1+\Delta)} n^p = [f(1+\Delta)]^{-1} n^p$$

Here we express n^p interms of factorial polynomial and expand $[f(1+\Delta)]^{-1}$ by taking lowest degree terms common from the denominator and use

$(1+\Delta)^{-1} = 1 - \Delta + \Delta^2 - \Delta^3 + \ldots..$ or

$(1-\Delta)^{-1} = 1 + \Delta + \Delta^2 + \Delta^3 + \ldots..$

(c) When $f(n) = A^n . F(n)$ where F(n) is a polynomial in n of degree p.

$$P.S. = \frac{1}{f(E)} f(n) = \frac{1}{f(E)} A^n F(n)$$

$$= A^n \frac{1}{f(AE)} F(n)$$

Since F(n) is a polynomial, this can be solved by using above case (b).

Generating Function : **CS/CT/CE : W-09**

A generating function is a normal power series in one indeterminate, whose coefficients encode information about a sequence of numbers a_n, where n is a non-negative integer. One can generalize the formal power series in one more than one indeterminate to encode information about arrays of numbers.

There are various types of generating functions viz ordinary generating functions, exponential generating functions, Lambert series, Bell series and Dirichlet series.

The ordinary generating functions (OGF) for infinite sequence $\{a_n\}$ of real numbers a_0, a_1, a_2 $a_n, \ldots.$ is defined and denoted by

$$G(a_n; x) = \sum_{n=0}^{\infty} a_n x^n \text{ or } G(x) = \sum_{n=0}^{\infty} a_n x^n$$

i.e. $G(x) = a_0 + a_1 + a_2 x^2 + a_3 x^3 + \ldots. + a_n x^n + \ldots$

The exponential generating function (EGF) for infinite sequence $\{a_n\}$ of real numbers $a_0, a_1, a_2, \ldots, a_n$ is defined and denoted by

$$E(a_n; x) = \sum_{n=0}^{\infty} \frac{1}{n!} a_n x^n \text{ or } E(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$$

i.e. $E(x) = a_0 + a_1 \frac{x}{1!} + a_2 \frac{x^2}{2!} + \ldots. + a_n \frac{x^2}{n!} + \ldots$

For Example :

The OGF of the sequence 1, 1, 1, 1, ... is

$$G(x) = 1 + x + x^2 + x^3 + \ldots. = \sum_{n=0}^{\infty} x^n$$

i.e. $G(x) = \frac{1}{1-x}$

Similarly the EGF of the same sequence 1, 1, 1, 1,

$$E(x) = 1 + \frac{1}{1!} x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 \ldots.. = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

i.e. $E(x) = e^x$

Most of the times x is treated as a real variable unless or otherwise stated. Actually EGF is not an independent concept, because the BGF of a sequence $\{a_n\}$ is precisely the OGF of the sequence $\left\{\frac{1}{n} a_n\right\}$ and vice-versa. The reason for using the word "exponential" is that the EGF of the sequence 1, 1, 1, 1, ... is $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$, which is precisely the power series expansion of the exponential function e^x.

Addition and Multiplication of Two Generating Functions :

Let us define two generating functions as follows :

$$F(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots.$$

$$+ a_n x^n + \ldots.. \text{ and}$$

$$G(x) = \sum_{n=0}^{\infty} b_n x^n = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \ldots.$$

$$+ \ldots.. + b_n x^n + \ldots..$$

Addition of these two generating functions simply means combining like powers of x.

$$\therefore F(x)+G(x)=(a_0+b_0)+(a_1+b_1)x$$
$$+(a_2+b_2)x^2+.....+(a_n+b_n)x^n+....$$
$$\therefore F(x)+G(x)=\sum_{n=0}^{\infty}(a_n+b_n)x^n$$

We can multiply the above two generating functions F (x) and G(x) as follows.

$$F(x).G(x)=(a_0+a_1x+a_2x^2+a_3x^3+.....)$$
$$(b_0+b_1x+b_2x^2+b^3x^3+.....)$$
$$=a_0(b_0+b_1x+b_2x^2+b_3x^3+....)$$
$$+a_1x(b_0+b_1x+b_2x^2+b_3x^3+....)$$
$$+a_2x^2(b_0+b_1x+b_2x^2+b_3x^3+....)$$
$$+a_3x^3(b_0+b_1x+b_2x^2+b_3x^3+.....)$$
$$=(a_0b_0)+(a_0b_1+a_1b_0)x$$
$$+(a_0b_2+a_1b_1+a_2b_0)x^2$$
$$+(a_0b_3+a_1b_2+a_2b_1+a_3b_0)x^3$$
$$+.....+\sum_{k=0}^{\infty}(a_kb_{n-k})x^n+.....$$
$$\therefore F(x)G(x)=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{\infty}(a_ka_{n-k})x^n\right]$$

Note :

(1) $(1+x)^{-1}=\frac{1}{1+x}=1-x+x^2-x^3+....,|x|<1$

(2) $(1-x)^{-1}=\frac{1}{1-x}=1+x+x^2+x^3+....,|x|<1$

Some Examples of Generating Functions :

(1) The generating functions for the sequences whose n^{th} terms are $a_n=2, a_n=3+n, a_n=2+n^2$, $a_n=n+1$ are

$$G(x)=\sum_{n=0}^{\infty}2x^n, \qquad G(x)=\sum_{n=0}^{\infty}(3+n)x^n,$$
$$G(x)=\sum_{n=0}^{\infty}(2+n^2)x^n, \quad G(x)=\sum_{n=0}^{\infty}(n+1)x^n$$

respectively.

(2) The generating function for the sequence 1, 0, 1, 0, 1, 0 ... is

$$G(x)=1+0x+1x^2+0x^3+1x^4+....=1+x^2+x^4+..$$
$$\text{i.e. } G(x)=1+x^2+(x^2)^2+.....=\frac{1}{1-x^2}$$

(3) The generating function for the sequence 1, 2, 3, 4, 5, ... is

$$G(x)=1+2x+3x^2+4x^3+.....=\frac{1}{(1-x)^2}$$

It is often convenient to express generating functions by means of an explicit formula (closed form) rather than a power series. Hence we formulate the above results in the table given below.

n^{th} term of the sequence $\{a_n\}$	Corresponding generating function
1	$\frac{1}{1-x}$
n	$\frac{x}{(1-x)^2}$
n+1	$\frac{1}{(1-x)^2}$
n (n + 1)	$\frac{2x}{(1-x)^3}$
a^n	$\frac{1}{1-ax}$

IMPORTANT FORMULAE :

(1) $\Delta f(a)=f(a+h)-f(a)$

(2) $Ef(x)=f(x+h)$

(3) $E=1+\Delta$

(4) $(x)^{(n)}=x(x-1)(x-2)(x-3)...(x-(n-1))$

(5) $G(x)=\sum_{n=0}^{\infty}a_nx^n$

(6) $E(x)=\sum_{n=0}^{\infty}a_n\frac{x^n}{n!}$

Ex.30. Find the closed form of generating function for each of the following sequences.

(a) 0, 0, 1, 1, 1, ...
(b) 1, 1, 0, 1, 1, 1, 1, ...
(c) 1, 0, –1, 0, 1, 0, –1, 0...
(d) 3, – 3, 3, – 3, 3, –3, ...
(e) 0, 0, 0, 1, 1, 1, ...
(f) 0, 0, 1, 2, 3, 4, ...
(g) 3, 9, 27, 81. .....
(h) 1, –2, 3, –4

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Soln. (a) The given sequence is 0, 0, 1, 1, 1, ...

$$\therefore G(x)=0+0x+1x^2+1x^3+1x^4+....$$
$$\therefore G(x)=x^2+x^3+x^4+..... \qquad (1)$$

$$= x^2(1 + x + x^2 + ...) = x^2(1-x)^{-1} = \frac{x^2}{1-x}$$

$$\therefore \boxed{G(x) = \frac{x^2}{1-x}} \text{ Ans.}$$

(b) The given sequence is 1, 1, 0, 1, 1, 1,.....

$$\therefore G(x) = 1 + 1x + 0x^2 + 1x^3 + 1x^4 + ...$$

$$\therefore G(x) = 1 + x + x^3 + x^4 + ... \quad ....(1)$$

$$= (1 + x + x^2 + x^3 + x^4 + ...) - x^2$$

$$= (1-x)^{-1} - x^2$$

$$= \frac{1}{1-x} - x^2$$

$$= \frac{x^3 - x^2 + 1}{1-x}$$

$$\therefore \boxed{G(x) = \frac{x^3 - x^2 + 1}{1-x}} \text{ Ans.}$$

(c) The given sequence is 1, 0, – 1, 0, –1, 0.....

$$\therefore G(x) = 1 + 0x - 1x^2 + 0x^2 + 0x^3 + 1x^4 + 0x^5 - 1x^6 + ...$$

$$\therefore G(x) = 1 - x^2 + x^4 - x^6 + ... \quad ....(1)$$

We know that

$$\frac{1}{1+x^2} = (1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + ... \quad ....(2)$$

$$\therefore (1) \text{ and } (2) \Rightarrow \boxed{G(x) = \frac{1}{1+x^2}} \text{ Ans.}$$

(d) The given sequence is 3, –3, 3, –3, 3, –3, ...

$$\therefore G(x) = 3 - 3x + 3x^2 - 3x^2 - 3x^3 + 3x^4 + ...$$

$$\therefore G(x) = 3(1 - x + x^2 - x^3 + ...) \quad .....(1)$$

$$= (1+x)^{-1} = \frac{3}{1+x}$$

$$\therefore \boxed{G(x) = \frac{3}{1+x}} \text{ Ans.}$$

(e) The given sequence is 0, 0, 0, 1, 1, 1, 1, ...

$$\therefore G(x) = 0 + 0x + 0x^2 + 1x^3 + 1x^4 + 1x^5 + 1x^6 + ...$$

$$\therefore G(x) = x^3 + x^4 + x^5 + x^6 + ...$$

$$= x^3(1 + x + x^2 + x^3 + ...)$$

$$= x^3(1-x)^{-1} = \frac{x^3}{1-x}$$

$$\therefore \boxed{G(x) = \frac{x^3}{1-x}} \text{ Ans.}$$

(f) The given sequence is 0, 0, 1, 2, 3, 4, ...

$$\therefore G(x) = 0 + 0x + 1x^2 + 2x^3 + 3x^4 + 4x^5 + ...$$

$$\therefore G(x) = x^2 + 2x^3 + 3x^4 + 4x^5 + ...$$

$$= x^2(1 + 2x + 3x^2 + 4x^3 + ...) \quad ....(1)$$

We know that

$$\frac{1}{(1-x)^2} = (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + ... \quad ...(2)$$

$$\therefore (1) \text{ and } (2) \Rightarrow \boxed{G(x) = \frac{x^2}{(1-x)^2}} \text{ Ans.}$$

(g) The given sequence is 3, 9, 27, 81,.....

$$\therefore G(x) = 3 + 9x + 27x^2 + 81x^3 + ...$$

$$= 3(1 + 3x + 9x^2 + 27x^3 + ...)$$

$$G(x) = 3[1 + 3x + (3x)^2 + (3x)^3 + ...] \quad ....(1)$$

We know that

$$\frac{1}{1-3x} = (1-3x)^{-1} = 1 + 3x + (3x)^2 + (3x)^3 + ... \quad ....(2)$$

$$\therefore (1) \text{ and } (2) \Rightarrow \boxed{G(x) = \frac{3}{1-3x}} \text{ Ans.}$$

(h) The given sequence is 1, –2, 3, – 4, ...

$$\therefore G(x) = 1 - 2x + 3x^2 - 4x^3 + ... \quad ....(1)$$

We know that

$$\frac{1}{(1+x)^2} = (1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + ... \quad ...(2)$$

$$\therefore (1) \text{ and } (2) \Rightarrow \boxed{G(x) = \frac{1}{(1+x)^2}} \text{ Ans.}$$

Ex.31. Find the closed form of generating function for each of the following sequences $\{a_n\}$-

(a) $a_n = 3$

(b) $a_n = 2 + 3n$ **CS/CT/CE : S-12**

(c) $a_n = 3^n$

(d) $a_n = n(3 + 5n)$

Soln.

(a) **Given : $a_n = 3$**

We know that

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$\therefore G(x) = \sum_{n=0}^{\infty} 3x^n$$

$$= 3\sum_{n=0}^{\infty} x^n = 3(x^0 + x^1 + x^2 + x^3 + ...)$$

$$= 3(1 + x + x^2 + x^3 + ...) = 3\left(\frac{1}{1-x}\right)$$

$$\therefore \boxed{G(x) = \frac{3}{1-x}} \text{ Ans.}$$

(b) **Given :** $a_n = 2 + 3n$

We know that

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} (2+3n)\, x^n$$

$$= \sum_{n=0}^{\infty} (2x^n + 3nx^n)$$

$$= \sum_{n=0}^{\infty} 2x^n + 3\sum_{n=0}^{\infty} nx^n$$

$$= 2(x^0 + x^1 + x^2 + x^3 + ...)$$

$$+ 3(0x^0 + 1x^1 + 2x^2 + 3x^3 + ...)$$

$$= 2(1 + x + x^2 + x^3 + ...)$$

$$+ 3(x + 2x^2 + 3x^3 + ...)$$

$$= 2(1 + x + x^2 + ...) + 3x(1 + 2x + 3x^2 + ...)$$

$$\therefore \boxed{G(x) = \frac{3}{1-x} + \frac{3x}{(1-x)^2}} \text{ Ans.}$$

(c) **Given :** $a_n = 3^n$

We know that

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} 3^n x^n$$

$$= 3^0 x^0 + 3^1 x^1 + 3^2 x^2 + 3^3 x^3 + ...$$

$$= 1 + 3x + (3x)^2 + (3x)^3 + ...$$

$$\therefore \boxed{G(x) = \frac{1}{1-3x}} \text{ Ans.}$$

(d) **Given :** $a_n = n(3+5n) = 3n + 5n^2$

We know that

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} (3n + 5n^2)\, x^n$$

$$= \sum_{n=0}^{\infty} 3nx^2 + \sum_{n=0}^{\infty} 5n^2 x^n$$

$$= 3\sum_{n=0}^{\infty} nx^n + 5\sum_{n=0}^{\infty} n^2 x^n$$

$$= 3(0x^0 + 1x^1 + 2x^2 + 3x^3 + ...)$$

$$+ 5(0x^0 + 1^2x^1 + 2^2x^2 + 3^2x^3$$

$$+ 4^2x^4 + ...)$$

$$= 3(x + 2x^2 + 3x^3 + ...)$$

$$+ 5(x + 4x^2 + 9x^3 + 16x^4 + ...)$$

$$= 3x(1 + 2x + 3x^2 + ...)$$

$$+ 5x(1 + 4x + 9x^2 + 16x^3 + ...)$$

$$\therefore \boxed{G(x) = \frac{3x}{(1-x)^2} = \frac{5x(1+x)}{(1-x)^3}} \text{ Ans.}$$

Ex.32. Find the sequence $\{a_n\}$ for the following generating functions.

(a) $\dfrac{x}{1-2x}$ **(b)** $\dfrac{2x^2}{1-x}$ **(c)** $\dfrac{x}{(1-x)^2} + \dfrac{x}{1-x}$

Soln.

(a) **Given :** $G(x) = \dfrac{x}{1-2x}$

We know that $\dfrac{1}{1-2x} = (1-2x)^{-1}$

$$= 1 + 2x + (2x)^2 + (2x)^3 + ...(2x)^n + ...$$

$$\therefore \frac{1}{1-2x} = 1 + 2x + 4x^2 + 8x^3 + ... + 2^n x^n + ...$$

$$\therefore \frac{x}{1-2x} = x + 2x^2 + 4x^3 + 8x^4 + ...$$

$$+ 2^n x^n + 1 + ...$$

i.e. $\dfrac{x}{1-2x} = x + 2x^2 + 4x^3 + ...$

$$+ 2^{n-1}x^n + 2^n x^{n+1} + ...$$

$\therefore$ $\boxed{\text{The general term is } a_n = 2^{n-1}, a_0 = 0}$ **Ans.**

(b) **Given :** $G(x) = \dfrac{2x^2}{1-x}$

We know that $\dfrac{1}{1-x} = (1-x)^{-1}$

$$= 1 + x + x^2 + x^3 + ... + x^n + ...$$

$$\therefore \frac{2x^2}{1-x} = 2x^2 + 2x^3 + 2x^4 + \ldots + 2x^{n+2} + \ldots$$

$$= 2x^2 + 2x^3 + 2x^4 \ldots + 2x^n + 2x^{n+1} + 2x^{n+2} + \ldots$$

$\therefore$ $\boxed{\text{The general term is } a_n = 2, a_0 = 0, a_1 = 0}$ **Ans.**

(c) **Given :** $G(x) = \frac{x}{(1-x)^2} + \frac{x}{1-x}$

We know that $\frac{1}{(1-x)^2} = (1-x)^{-2}$

$$= 1 + 2x + 3x^2 + 4x^3 + \ldots + (n+1)x^n + \ldots$$

$$\therefore \frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + \ldots + nx^n + (n+1)x^{n+1} + \ldots \quad \ldots(1)$$

Also $\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots$

$$\Rightarrow \frac{x}{1-x} = x + x^2 + x^3 + \ldots + x^{n+1} + \ldots$$

i.e. $\frac{x}{1-x} = x + x^2 + x^3 + \ldots + x^n + x^{n+1} + \ldots \quad \ldots(2)$

$$\therefore (1) + (2) \Rightarrow \frac{x}{(1-x)^2} + \frac{x}{(1-x)}$$

$$= 2x + 3x^2 + 4x^3 + \ldots + (n+1)x^n + \ldots$$

$\therefore$ $\boxed{\text{The general term is } a_n = n+1,\ a_0 = 0}$ **Ans.**

Ex.33. Solve the following recurrence relations using generating function

(a) $a_n - 9a_{n-1} + 20a_{n-2} = 0, a_0 = -3, a_1 = -10$

(b) $a_n = a_{n-2} + 4n, a_0 = 3, a_1 = 2$

(c) $a_n = 3a_{n-1} + 2,\ a_0 = 1$

(d) $a_n = 6a_{n-1} - 9a_{n-2}, a_0 = 2, a_1 = 3$

Soln.

(a) We know that

$$G(x) = \sum_{n=0}^{\infty} a_n x^n \quad \ldots(1)$$

i.e. $G(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots \quad \ldots(2)$

The given recurrence relation is

$a_n - 9a_{n-1} + 20a_{n-2} = 0$

This is a linear homogeneous recurrence relation (LHRR) of order k = 2

Multiply both sides by x^n and take $\sum_{n=2}^{\infty}$, we get

$$\sum_{n=2}^{\infty} a_n x^n - 9\sum_{n=2}^{\infty} a_{n-1}x^n + 20\sum_{n=2}^{\infty} a_{n-2}x^n = 0$$

$$\Rightarrow (a^2x^2 + a_3x^3 + a_4x^4 + \ldots) - 9(a_1x^2 + a_2x^3 + a_3x^4 + \ldots) + 20(a_0x^2 + a_1x^3 + a_2x^4 + \ldots) = 0$$

$$\Rightarrow G(x) - a_0 - a_1x - 9x(a_1x + a_2x^2 + a_3x^3 + \ldots) + 20x^2(a_0 + a_1x + a_2x^2 + \ldots) = 0$$

[$\because$ By(2)]

$$\Rightarrow G(x) - a_0 - a_1x - 9x[G(x) - a_0] + 20x^2G(x) = 0$$

[$\because$ By(2)]

Put $a_0 = -3,\ a_1 = -10$

$$\therefore G(x) + 3 + 10x - 9x[G(x) + 3] + 20x^2G(x) = 0$$

$$\Rightarrow G(x) + 3 + 10x - 9x\,G(x) - 27x + 20x^2G(x) = 0$$

$$\Rightarrow (20x^2 - 9x + 1)\,G(x) = 17x - 3$$

$$\Rightarrow G(x) = \frac{17x-3}{20x^2 - 9x + 1} = \frac{17x-3}{(1-5x)(1-4x)}$$

$$\therefore G(x) - \frac{17x-3}{(1-5x)(1-4x)} = \frac{A}{1-5x} + \frac{B}{1-4x} \quad \ldots(3)$$

[$\because$ By partial fractions]

$$\Rightarrow 17x - 3 = A(1-4x) + B(1-5x)$$

$$\therefore A = 2, B = -5$$

$$(3) \Rightarrow G(x) = \frac{2}{1-5x} - \frac{5}{1-4x} \quad \ldots(4)$$

Now $\frac{2}{1-5x} = 2(1-5x)^{-1}$

$$= 2(1 + 5x + 5^2x^2 + 5^3x^3 + \ldots + 5^nx^n + \ldots)$$

$$= 2 + 2.5x + 2.5^2x^2 + 2.5^3x^3 + \ldots + 2.5^nx^n + \ldots$$

Also $\frac{5}{1-4x} = 5(1-4x)^{-1}$

$$= 5(1 + 4x + 4^2x^2 + 4^3x^3 + \ldots + 5^nx^n + \ldots)$$

$$= 5 + 5.4x + 5.4^2x^2 + 5.4^3x^3 + \ldots + 5.4^nx^n + \ldots$$

$$\therefore (4) \Rightarrow a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots$$

$$= (2 + 2.5^2 + 2.5^2x^2 + \ldots + 2.5^nx^n + \ldots) - (5 + 5.4 + 5.4^2x^2 + \ldots + 5.4^nx^n + \ldots)$$

$\boxed{a_n = 2.5^n - 5.4^n \text{ This is the required solution}}$ **Ans.**

This is the required solution

(b) We know that

$$G(x) = \sum_{n=0}^{\infty} a_n x^n \quad \ldots(1)$$

i.e. $G(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots \quad \ldots(2)$

The given recurrence relation is

$a_n = a_{n-2} + 4n$

This is LHRR of order $k = 2$

Multiply both sides by x^n and take $\sum_{n=2}^{\infty}$

$$\therefore \sum_{n=2}^{\infty} a_n x^n = \sum_{n=2}^{\infty} a_{n-2} x^n + 4\sum_{n=2}^{\infty} n x^n$$

$$\Rightarrow (a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots)$$
$$= (a_0 x^2 + a_1 x^3 + a_2 x^4 + \ldots)$$
$$+ 4(2x^2 + 3x^3 + 4x^4 + \ldots)$$

$$\Rightarrow G(x) - a_0 - a_1 x = x^2(a_0 + a_1 x + a_2 x^2 + \ldots)$$
$$+ 4x(2x + 3x^2 + 4x^3 + \ldots) \quad [\because By(2)]$$
$$= x^2 G(x) + 4x[(1 + 2x + 3x^2 + 4x^3 + \ldots) - 1]$$
$$[\because By(2)]$$

$$\Rightarrow G(x) - a_0 - a_1 x = x^2 G(x) + 4x\left[\frac{1}{(1-x)^2} - 1\right]$$

Put $a_0 = 3, a_1 = 2$

$$\Rightarrow G(x) - 3 - 2x = x^2 G(x) + \frac{4x}{(1-x)^2} - 4x$$

$$\Rightarrow G(x) - x^2 G(x) = \frac{4x}{(1-x)^2} - 4x + 2x + 3$$

$$\Rightarrow (1 - x^2) G(x) = \frac{4x}{(1-x)^2} - 2x + 3$$

$$\therefore G(x) = \frac{4x}{(1-x)^2(1-x)^2} - \frac{2x}{(1-x^2)} + \frac{3}{(1-x^2)}$$
$$= \frac{4x}{(1+x)(1-x)(1-x)^2} - \frac{2x}{(1-x)(1+x)}$$
$$+ \frac{3}{(1-x)(1+x)} \quad \ldots(3)$$

Now

$$\frac{4x}{(1-x)^3(1+x)} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{(1-x)^3} + \frac{D}{1+x}$$

$$\therefore 4x = A(1-x)^2(1+x) + B(1-x)(1+x)$$
$$+ C(1+x) + D(1-x)^3$$

$$\therefore A = \frac{-7}{2},\ B = 5, C = 2,\ D = \frac{1}{2}$$

Also $\dfrac{2x}{(1-x)(1+x)} = \dfrac{E}{(1-x)} + \dfrac{F}{1+x}$

$\therefore 2x = E(1+x) + F(1-x)$

$\therefore E = 1, F = -1$

Also $\dfrac{3}{(1-x)(1+x)} = \dfrac{G}{1-x} + \dfrac{H}{1+x}$

$G = \dfrac{3}{2},\ H = \dfrac{-3}{2}$

$$\therefore (3) \Rightarrow G(x) = \frac{-7}{2}\frac{1}{1-x} + 5\frac{1}{(1-x)^2}$$
$$+ 2\frac{1}{(1-x)^3} + \frac{1}{2}\frac{1}{1+x}$$
$$- \frac{1}{1-x} + \frac{1}{1+x} + \frac{3}{2}\frac{1}{1-x} - \frac{3}{2}\frac{1}{1+x}$$

$$\therefore G(x) = 2\frac{1}{(1-x)^3} + 5\frac{1}{(1-x)^2} - 3\frac{1}{1-x} \quad \ldots(4)$$

Now $2\dfrac{1}{(1-x)^3} = 2(1-x)^3$

$$= 2[2 + 2.3x + 3.4x^2 + 4.5x^3 + \ldots\ldots$$
$$+ (n+1)(n+2)x^n + \ldots\ldots]$$
$$= 2.2 + 2.2.3x + 2.3.4x^2 + 2.4.5x^3 + \ldots$$
$$+ 2(n+1)(n+2)x^n + \ldots\ldots$$

$5\dfrac{1}{(1-x)^2} = 5(1-x)^{-2}$

$$= 5[1 + 2x + 3x^2 + 4x^3 + \ldots\ldots$$
$$+ (n+1)x^n + \ldots\ldots]$$
$$= 5 + 5.2x + 5.3x^2 + 5.4x^3 + \ldots\ldots$$
$$+ 5(n+1)x^n + \ldots$$

$3\dfrac{1}{1-x} = 3(1-x)^{-1}$

$$= 3(1 + x + x^2 + x^3 + \ldots + x^n + \ldots\ldots)$$
$$= 3 + 3x + 3x^2 + 3x^3 \ldots + 3x^n + \ldots$$

$$\therefore (4) \Rightarrow a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n + \ldots$$
$$= [2.2 + 2.2.3x + 2.3.4x^2 + \ldots$$
$$+ 2(n+1)(n+2)x^n + \ldots]$$
$$+ [5 + 5.2x + 5.3x^2 + \ldots + 5(n+1)x^n + \ldots]$$
$$- [3 + 3x + 3x^2 + \ldots\ldots + 3x^n + \ldots]$$

$\therefore\ a_n = 2(n+1)(n+2) + 5(n+1) - 3$

This is the required solution. ....**Ans.**

(c) We know that

$$G(x) = \sum_{n=0}^{\infty} a_n x^n \qquad ....(1)$$

i.e. $G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$ ....(2)

The given recurrence relation is

$a_n = 3a_{n-1} + 2$

This is a LHRR of order $k = 1$

Multiply both sides by x^n and take $\sum_{n=1}^{\infty}$

$$\therefore \sum_{n=1}^{\infty} a_n x^n = 3\sum_{n=1}^{\infty} a_{n-1} x^n + 2\sum_{n=1}^{\infty} x^n$$

$$\Rightarrow a_1 x + a_2 x^2 + a_3 x^3 + ...$$
$$= 3(a_0 x + a_1 x^2 + a_2 x^3 + ...)$$
$$+ 2(x + x^2 + x^3 + x^4 + ...)$$
$$\Rightarrow G(x) - a_0 = 3x(a_0 + a_1 x + a_2 x^2 + ...)$$
$$+ 2x(1 + x + x^2 + x^3 + ...) \qquad [\because By(2)]$$
$$= 3x\,G(x) + 2x\frac{1}{1-x}$$

Put $a_0 = 1$

$$\therefore G(x) - 1 = 3x\,G(x) + \frac{2x}{1-x}$$
$$\therefore G(x) - 3x\,G(x) = \frac{2x}{1-x} + 1$$
$$\therefore G(x)(1-3x) = \frac{2x+1-x}{1-x} = \frac{1+x}{1-x}$$
$$\therefore G(x) = \frac{1+x}{(1-x)(1-3x)} = \frac{A}{1-x} + \frac{B}{1-3x} \qquad .....(3)$$

[∵ By partial fraction]

$\therefore\ 1 + x = A(1-3x) + B(1-x)$

$\therefore\ A = -1, B = 2.$

$$\therefore (3) \Rightarrow G(x) = \frac{-1}{1-x} + \frac{2}{1-3x} = \frac{2}{1-3x} - \frac{1}{1-x} \qquad ....(4)$$

Now $\frac{2}{1-3x} = 2(1-3x)^{-1}$

$$= 2(1 + 3x + 3^2x^2 + 3^3x^3 + ... + 3^n x^n + ...)$$
$$= 2 + 2.3x + 2.3^2x^2 + 2.3^3x^3 + ...$$
$$+ 2.3^n x^n + ...$$

$$\& \ \frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + x^3 + ... + x^n + ...$$

$$\therefore (4) \Rightarrow a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + ...$$
$$= (2 + 2.3x + 2.3^2x^2 + ... + 2.3^n x^n + ...)$$
$$- (1 + x + x^2 + ... + x^n + ...)$$

$\therefore\ a_n = 2.3^n - 1$

This is the required solution ....**Ans.**

(d) We know that

$$G(x) = \sum_{n=0}^{\infty} a_n x^n \qquad ....(1)$$

i.e. $G(x) = a_0 + a_1 x + a_2 x^2 + ...a_n x^n + ...$ ...(2)

The given recurrence relation is

$a^n = 6a_{n-1} - 9a_{n-2}$

This is a LHRR of order $k = 2$

Multiply both sides by x^n and take $\sum_{n=2}^{\infty}$

$$\therefore \sum_{n=2}^{\infty} a_n x^n = 6\sum_{n=2}^{\infty} a_{n-1} x^n - 9\sum_{n=2}^{\infty} a_{n-2} x^n$$
$$\Rightarrow a_2 x^2 + a_3 x^3 + a_4 x^4 + ...$$
$$= 6(a_1 x^2 + a_2 x^3 + a_3 x^4 + ...$$
$$- 9(a_0 x^2 + a_1 x^3 + a_2 x^4 + ...)$$
$$\Rightarrow G(x) - a_0 - a_1 x = 6x(a_1 x + a_2 x^2 + a_3 x^3 + ...)$$
$$- 9x^2(a_0 + a_1 x + a_2 x^2 + ...)$$

[∵ By(2)]

$$\Rightarrow G(x) - a_0 - a_1 x = 6x[G(x) - a_0] - 9x^2 G(x)$$

Put $a_0 = 2, a_1 = 3$ in above equation

$$\therefore G(x) - 2 - 3x = 6x[G(x) - 2] - 9x^2 G(x)$$
$$\Rightarrow G(x) - 2 - 3x = 6x\,G(x) - 12x - 9x^2 G(x)$$
$$\Rightarrow 9x^2 G(x) - 6x\,G(x) + G(x) = -12x + 2 + 3x$$
$$\Rightarrow (9x^2 - 6x + 1)G(x) = 2 - 9x$$
$$G(x) = \frac{2-9x}{9x^2 - 6x + 1} = \frac{2}{(1-3x)^2} - \frac{9x}{(1-3x)^2} \qquad ...(3)$$

Now $\frac{9x}{(1-3x)^2} = \frac{A}{1-3(x)} + \frac{B}{(1-3x)^2}$

$\therefore\ 9x = A(1-3x) + B$ *[∵ By partial fraction]*

$\therefore\ A = -3, B = 3$

i.e. $\frac{9x}{(1-3x)^2} = \frac{-3}{1-3x} + \frac{-3}{(1-3x)^2}$

$$\therefore (3) \Rightarrow G(x) = \frac{2}{(1-3x)^2} + \frac{3}{1-3x} - \frac{3}{(1-3x)^2}$$

$$= \frac{3}{1-3x} - \frac{1}{(1-3x)^2} \quad \ldots(4)$$

Now $\frac{3}{1-3x} = 3(1-3x)^{-1}$

$$= 3(1 + 3x + 3^2x^2 + \ldots + 3^n x^n + \ldots)$$

$$= 3 + 3.3x + 3.3^2x^2 + \ldots + 3.3^n x^n + \ldots$$

Now $\frac{1}{(1-3x)^2} = (1-3x)^{-2}$

$$= 1 + 2(3x) + 3(3x)^2 + 4(3x)^3 + \ldots + (n+1)(3x)^n + \ldots$$

$$\therefore (4) \Rightarrow a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots$$

$$= 3 + 3.3x + 3.3^2x^2 + \ldots + 3.3^n x^n + \ldots)$$

$$- (1 + 2.3x + 3.3^2x^2 + 4.3^3x^3 + \ldots + (n+1)3^n x^n + \ldots)$$

$\therefore$ $\boxed{a_n = 3.3^n - (n+1)3^n}$ **Ans.**

Ex.34. Find the generating function for the sequence 1, a, a^2, ... where a is a fixed constant.

CS/CT/CE : S-06

Soln. Let $G(x) = 1 + ax + a^2x^2 + a^3x^3 + \ldots$ ...(1)

$$\Rightarrow G(x) - 1 = ax + a^2x^2 + a^3x^3 + \ldots$$

$$\Rightarrow G(x) - 1 = ax(1 + ax + a^2x^2 + \ldots)$$

$$\Rightarrow \frac{G(x)-1}{ax} = 1 + ax + a^2x^2 + \ldots$$

$$\therefore \frac{G(x)-1}{ax} = G(x)$$

$$\Rightarrow G(x) = \frac{1}{1-ax} \quad [\because By(1)]$$

$\therefore$ $\boxed{\text{The required generating function is } \frac{1}{1-ax}}$ **Ans.**

Ex.35. Find the generating function for the sequence :

$1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n+1}, \ldots$

CS/CT/CE : S-07

Soln. Given sequence is

$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n+1}, \ldots$$

$$\therefore G(x) = 1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + \ldots \frac{1}{n+1}x^n + \ldots$$

$$\therefore G(x) = \sum_{n=0}^{\infty} \frac{1}{n+1}x^n$$

i.e. $a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots = \sum_{n=0}^{\infty} \frac{1}{n+1}x^n$

$\therefore$ $\boxed{a_n = \frac{1}{n+1}}$ **Ans.**

Ex.36. Find the particular solution of

$a_r + a_{r-1} = 3r\,2^r$.

CS/CT/CE : S-07,12, W-09

Soln. We have

$$a_r + a_{r-1} = 3r2^r \quad \ldots(1)$$

The general form of the particular solution is

$$(P_1r + P_2)2^r \quad \ldots(2)$$

Substituting (2) into (1), we obtain

$$(P_1r + P_2)2^r + [P_1(r-1) + P_2]2^{r-1} = 3r2^r$$

which simplifies to

$$\frac{3}{2}P_1 r 2^r + \left(\frac{-1}{2}P_1 + \frac{3}{2}P_2\right)2^r = 3r2^r \quad \ldots(3)$$

Comparing the two sides of (3), we obtain the equation

$$\frac{3}{2}P_1 = 3$$

$$\frac{-1}{2}P_1 + \frac{3}{2}P_2 = 0$$

Thus, $P_1 = 2$, $P_2 = \frac{2}{3}$

$\therefore$ $\boxed{\text{The particular solution is } a_r(P) = \left(2r + \frac{2}{3}\right)2^r}$

Ans.

Ex.37. Solve the following recurrence relation:

$a_n = 3a_{n-1} - 2a_{n-2}$ **with** $a_1 = 5, a_2 = 3$.

CS/CT/CE : S-08, W-11

Soln. Recurrence relation is

$$a_n - 3a_{n-1} + 2a_{n-2} = 0 \quad \ldots(1)$$

The equation (1) can also be written as,

$$(S^2 - 3S + 2)a_n = 0 \Rightarrow (S-2)(S-1) = 0 \Rightarrow S = 1, 2$$

$\therefore$ The solution is $a_n = C_1 + C_2\,2^n$ ....(2)

Put n = 1 in equation (2), we have

$$a_1 = C_1 + C_2 \cdot 2 = 5 \quad \ldots(3)$$

Put n = 2 in equation (2), we have

$$a_2 = C_1 + C_2.4 = 3 \quad \ldots(4)$$

On solving equation (3) & (4), we get

$C_1 = 7$, $C_2 = -1$

$\therefore$ The solution satisfying the initial conditions is $a_n = 7 - 2^n$ **Ans.**

Ex.38. Solve: $a_n + 5a_{n-1} + 6a_{n-2} = 3n^2$.

CS/CT/CE : W-08

Soln. Given equation is $a_n + 5a_{n-1} + 6a_{n-2} = 3n^2$

This can be written as

$(E^2 + 5E + 6)\, a_n = 3n^2$

The A.E. is $m^2 + 5m + 6 = 0$

$\therefore m = -2, -3$

$\therefore$ C.F. $= b_1(-2)^n + b_2(-3)^n$

$$\text{Now } a_n = \frac{1}{(E^2 + 5E + 6)} 3n^2$$

$$= \frac{1}{((\Delta+1)^2 + 5(\Delta+1) + 6)} 3n^2$$

$$\therefore a_n = \frac{1}{\Delta^2 + 2\Delta + 1 + 5\Delta + 5 + 6} 3n^2$$

$$= \frac{1}{\Delta^2 + 7\Delta + 12} 3n^2$$

$$= \frac{3}{12} \cdot \frac{1}{\left(1 + \frac{\Delta^2 + 7\Delta}{12}\right)} \cdot n^2$$

$$= \frac{1}{4}\left(1 + \frac{\Delta^2 + 7\Delta}{12}\right)^{-1} [(n)^2 + (n)]$$

$$= \frac{1}{4}\left(1 - \left(\frac{\Delta^2 + 7\Delta}{12}\right) + \left(\frac{\Delta^2 + 7\Delta}{12}\right)^2\right)([n]^2 + [n])$$

$$= \frac{1}{4}\left[1 - \frac{1}{12}(\Delta^2 + 7\Delta) + \frac{\Delta^4 + 14\Delta^3 + 49\Delta^2}{144}\right][n^{(2)} + n^{(1)}]$$

$$= \frac{1}{4}\left[1 - \frac{1}{12}\Delta^2 - \frac{7}{12}\Delta + \frac{49\Delta^2}{144}\right][n^{(2)} + n^{(1)}]$$

....(ignoring higher powers of Δ)

$$= \frac{1}{4}\left[n^{(2)} + n^{(1)} - \frac{1}{12}\Delta^2(n^{(2)} + n^{(1)}) - \frac{7}{12}\Delta(n^{(2)} + n^{(1)}) + \frac{49}{144}\Delta^2(n^{(2)} + n^{(1)})\right]$$

$$= \frac{1}{4}\left[n(n-1) + n - \frac{1}{12}[2] - \frac{7}{12}[2n+1] + \frac{49}{144}(2)\right]$$

$$= \frac{1}{4}\left[n^2 - \frac{1}{6} - \frac{7}{6}n - \frac{7}{12} + \frac{98}{144}\right]$$

$$\therefore a_n = \frac{1}{4}\left[n^2 - \frac{7}{6}n - \frac{5}{72}\right] \text{ ...Ans.}$$

Ex.39. Evaluate the sum $1^2 + 2^2 + 3^2 + ... + r^2$.

CS/CT/CE : W-10

Soln. Let us first determine the generating function of the numeric function $(0^2, 1^2, 2^2, 3^2, ...r^2, ...)$.

Differentiating both sides to the identity .

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + z^4 + ... + z^r + ...$$

$$\frac{1}{(1-z)^2} = 1 + 2z + 3z^2 + 4z^3 + ... + rz^{r-1} + ...$$

It follows that

$$\frac{d}{dz}\frac{z}{(1-z)^2} = 1^2 + 2^2 z + 3^2 z^2 + 4^2 z^3 + ... + r^2 z^{r-1} + ...$$

And that

$$z\frac{d}{dz}\frac{z}{(1-z)^2} = 0^2 + 1^2 z + 2^2 z^2 + 3^2 z^3 + 4^2 z^4 + ... + r^2 z^r + ...$$

Thus $z\dfrac{d}{dz}\dfrac{z}{(1-z)^2}$, which is equal to $[z\,(1+z)\,] / (1-z)^3$, is the generating function of the numeric function $(0^2, 1^2, 2^2, 3^2, ...r^2, ...)$ and it follows that to $[\,z\,(1+z)\,] / (1-z)^4$, is the generating function of the numeric function $(0^2, 0^2 + 1^2,\ 0^2 + 1^2 + 2^2,\ 0^2 + 1^2 + 2^2 + 3^2, ...,0^2 + 1^2 + 2^2 + 3^2 + ... + r^2, ...)$.

According to the Binomial theorem, the coefficient of $z^r . \dfrac{1}{(1-z)^4}$ is

$$\frac{(-4)(-4-1)(-4-2).....(-4-r+1)}{r!}(-1)^r$$

$$= \frac{4 \times 5 \times 6 \times ..... \times (r+3)}{r!}$$

$$= \frac{(r+1)(r+2)(r+3)}{1 \cdot 2 \cdot 3}$$

Therefore the coefficient of z^r in the expansion of $\dfrac{[z(1+z)]}{(1-z)^4}$ is

$$\frac{r(r+1)(r+2)}{1\cdot2\cdot3}+\frac{(r-1)r(r+1)}{1\cdot2\cdot3}=\frac{r(r+1)(2r+1)}{6}$$

that is $1^2+2^2+3^2+.....+r^2=\frac{r(r+1)(2r+1)}{6}$

Ex.40. Solve

$$a_n-5a_{n-1}-6a_{n-2}=2^n+n : n\geq2 ; a_0=a_1=1$$

Using generating function. ***CS/CT/CE : S-11***

Soln. Given : $a_n-5a_{n-1}-6a_{n-2}=2^n+n,\ n\geq2$

with the boundary conditions $a_0=1,\ a_1=1$

Since $\sum_{n=2}^{\infty} a_nz^n-5\sum_{n=2}^{\infty} a_{n-1}z^n-6\sum_{n=2}^{\infty} a_{n-2}\,z^n$

$$=\sum_{n=2}^{\infty} 2^n z^n+\sum_{n=2}^{\infty} nz^n$$

We obtain

$$A(z)-a_0-a_1z-5z\,[A(z)-a_0]\,6z^2A(z)$$

$$=\frac{4z^2}{1-2z}+z\left[\frac{1}{(1-z)^2}-1\right]$$

which simplifies to

$$A(z)=\frac{5/4}{1-z}+\frac{1/2}{(1-z)^2}-\frac{3}{1-2z}-\frac{2}{(1-2z)^2}+\frac{17/4}{1-3z}$$

[∵ By partial fractions]

$$\boxed{a_n=\frac{7}{4}+\frac{n}{2}-n2^{n+1}-5.2^n+\frac{17}{4}3^n}$$ **Ans.**

Ex.41. Solve the recurrence relation

$$a_n=a_{n-1}+2,\ n\geq2 ; a_0=3$$

CS/CT/CE : S-06,10

Soln. Given RR is $a_n=a_{n-1}+2$

Multiply both sides by x^n and take $\sum_{n=1}^{\infty}$

$$\sum_{n=1}^{\infty}a_nx^n=\sum_{n=1}^{\infty}a_{n-1}x^n+2\sum_{n=1}^{\infty}x^n$$

$$G(x)-a_0=x\,G(x)+2x\,.\frac{1}{1-x}$$

Put $a_0=3$

$$G(x)-x\,G(x)=3+\frac{2x}{1-x}=\frac{3-3x+2x}{1-x}$$

$$\Rightarrow G(x)\,(1-x)=\frac{3-x}{1-x}$$

$$\therefore G(x)=\frac{3-x}{(1-x)^2}=\frac{3}{(1-x)^2}-\frac{x}{(1-x)^2}$$

$$\therefore a_n=3(n+1)-n$$

$$\therefore a_n=3n+3-n$$

$$\therefore \boxed{a^n=2n+3}$$ **Ans.**

Ex.42. Show that there must be at least 90 ways to choose six numbers from 1 to 15 so that all the choices have same sum. ***CS/CT/CE : W-06,12,S-08***

Soln. Here $n=15,\ r=6$

$\therefore$ No. of ways $=C(n+r-1,r)$

$=C(15+6-1,6)$

$=C(20,6)$

$=\frac{20!}{6!\ 14!}$

$=38,760$

$\therefore$ $\boxed{\text{There are more than 90 ways.}}$

Ex.43. Find explicit formula for the recurrence relation

$$a_n=a_{n-1}+3,\ a_0=3$$

CS/CT/CE : W-12

Soln. By Ex (41) here we obtain

$$G(x)-x\,G(x)=3+\frac{3x}{1-x}=\frac{3-3x+3x}{1-x}$$

$$G(x)\,(1-x)=\frac{3}{1-x}$$

$$G(x)=\frac{3}{(1-x)^2}$$

$$\therefore \boxed{a_n=3(n+1)}$$ **Ans.**

Ex.44. Solve the following recurrence relation

$$a_{n+2}-6a_{n+1}+8a_n=3$$

CS/CT/CE : S-13

Soln. Given RR is $a_{n+2}-6a_{n+1}+8a_n=3$

$$\Rightarrow (E^2-6E+8)\,a_n=3$$

$\therefore$ A.E. is $m^2-6m+8=0$

$\therefore\ m=2,4$

$\therefore$ C.F. $=b_1(2)^n+b_2(4)^n$

Now P.S. $=\frac{1}{f(E)}f(n)$

$$=\frac{1}{E^2-6E+8}.3(1)^n$$

$$=3\frac{1}{1-6+8}=1$$

$\therefore$ The C.S. is $a_n=$ C.F. + P.S.

$$\therefore \boxed{a_n=b_1(2)^n+b_2(4)^n+1}$$ **Ans.**

COMPLETE LIST OF FORMULAE:

(1) $P(n, r) = \dfrac{n!}{(n-r)!}$

(2) **Permutation with Repetition :**

$P(n : n_1, n_2, ..., n_r) = \dfrac{n!}{(n!, n_2! ... n_r!)}$

(3) **Permutation with condition :**

(i) $P(n-k, r-k)$,
when k objects to be included

(ii) $P(n-k, r)$,
when k objects are never taken.

(4) $C(n, r) = \dfrac{n!}{r!(n-r)!}$

(5) **Combination with repetition :**
$C(n+r-1, r)$.

(6) **Combination with condition :**

(i) $C(n-k, r-k)$,
when k objects to be included

(ii) $C(n-k, r)$,
when k objects are never taken.

(7) $\Delta f(a) = f(a+h) - f(a)$

(8) $Ef(x) = f(x+h)$

(9) $E = 1 + \Delta$

(10) $(x)^{(n)} = x(x-1)(x-2)(x-3)...(x-(n-1))$

(11) $G(x) = \sum_{n=0}^{\infty} a_n x^n$

(12) $E(x) = \sum_{n=0}^{\infty} a_n \dfrac{x^n}{n!}$

PROBLEMS FOR PRACTICE :

Q.1. Three dice are rolled. Find the no. of possible outcomes in which atleast one die shows 4.

Ans. 152

Q.2. In how many ways can a prize winner choose three DVDS from the top to list if repetition is allowed.

Ans. 220

Q.3. Show that if 30 dictionaries in a library contain a total of 61,327 pages, then one of the dictionaries must have 2045 pages.

Q.4. Solve the RR $a_{n+2} - 5a_{n+1} + 6a_n = 2^n$.

Ans. $a_n = b_1(2)^n + b_2(3)^n - 2^{n-1}(n+2)$

Q.5. Find the generating function of the fibonalli sequence defined by $f(n) = f(n-1) + f(n-2)$ with $f(0) = f(1) = 2$.

Ans. $G(x) = \dfrac{-1}{x^2 + x - 1}$

Q.6. Solve using generating function
$a_n - 2a_{n-1} - 3a_{n-2} = 0; n \geq 2; a_0 = 3, a_1 = 1$.

Ans. $a_n = 3^n + 2(-1)^n$

WINTER EXAMINATION - 2013

B. E. IV SEMESTER (CS, CT)

DISCRETE MATHEMATICS AND GRAPH THEORY

Statistical Analysis

Unit Number	Marks
Unit - I	24 M
Unit - II	22 M
Unit - III	20 M
Unit - IV	13 M
Unit - V	34 M
Unit - VI	13 M
Questions covered in VBD	126 M (100 %)

WINTER EXAM. - 2013
QUESTION PAPER

SECTION - A

Q.1.(a) If A and B are any two sets, then show that $A - B$, $A \cap B$ and $B - A$ are pair wise disjoint. **6M**

(b) Let $X = \{1, 2, 3, \ldots\ldots, 9\}$

$R_1 = \{(x, y) / x - y$ is divisible by $3\}$

$R_2 = \{(x, y) / x, y \in X$ and x, y are in the same column of matrix A$\}$

where, $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Find R_1 and R_2. Prove that R_1 and R_2 are equivalence relations. Is $R_1 = R_2$? Justify your answer. **7M**

Q.2.(a) Let $X = \{1, 2, 3, 4, 5\}$, $R = \{\{1, 2\}, \{3, 4\}, \{2, 2\}\}$, $S = \{(4, 2), (2, 5), (3, 1), (1, 3)\}$

Find SoR, RoS, Ro(SoR). **4M**

(b) If $f : X \to Y$ and $g : Y \to Z$ both f and g are one-one and onto. Prove that :

$(gof)^{-1} = f^{-1} og^{-1}$ **5M**

(c) By principle of Mathematical Induction, prove that :

$$1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \frac{n(n+1)(2n+1)}{6}, n \geq 1.$$

5M

Q.3.(a) Determine the validity of the argument : If my brother stands first in class, I will give him a watch.

He stood first or I was out of station.

I did not give him a watch.

$\therefore$ I was out of station. **5M**

(b) Negate the following statements :

(i) If he studies, he will go to college or to art school.

(ii) He is tall but handsome. **5M**

(c) Show that: $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$.

Q.4.(a) Describe an Abelian group. Prove that the set $G = \{0, 1, 2, 3, 4, 5\}$ is a finite Abelian group under addition modulo 6 as operation. **7M**

(b) Prove that every subgroup of cyclic group is cyclic. **6M**

[Out of syllabus]

Q.5.(a) Prove that a normal subgroup H of a group G is normal if and only if $g^{-1} hg \in H$, for every $h \in H$, $g \in G$. **7M**

(b) Prove that the intersections of two subgroups of a group G is a subgroup of G. Give an example to show that the union of two subgroups is not necessarily a subgroup. **6M**

SECTION - B

Q.6.(a) Prove that the set of complex numbers a + ib where 'a' and 'b' are integers forms a ring under ordinary addition and multiplication of complex numbers. **7M**

(b) Draw Hasse Diagram of the lattice D_{20}. Write composition table. Write the complements of elements of D_{20}. **6M**

Q.7.(a) Construct switching circuit for the Boolean expression $(A + B).(B' + C) + (C' + A).(C + B)$. Simplify this and construct an equivalent simplified circuit. **6M**

(b) Draw diagraphs corresponding to adjacency matrices and show that they are isomorphic.

$$M_1 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} \text{ and}$$

$$M_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$ **7M**

Q.8.(a) Define :

(i) Directed graph.

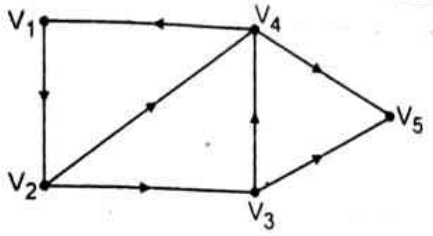
(ii) Path.

(iii) Reachability of graph. **3M**

(b) Draw binary tree for the relation given by :

$R = \{(1, 2), (1, 3), (1, 4), (2, 5), (4, 6), (4, 7)\}$ on set $A = \{1, 2, 3, 4, 5, 6, 7\}$. **5M**

(c) Find all the indegrees and outdegrees of the node of the graph. Give all elementary cycles of this graph. Obtain an acyclic diagraph by deleting one edge of the given graph. List all the nodes which are reachable from another node of the graph.

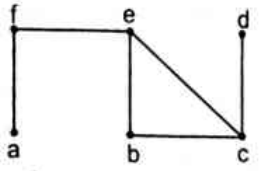


6M

Q.9.(a) Construct the tree of the following algebraic expression :

$[5(1-x)] + [\{5-(y+3)\} \times \{7+(x+4)\}]$. **4M**

(b) Find all spanning trees for the graph G shown in the figure by removing edge in simple circuit :



4M

(c) Draw a directed tree with 4 nodes at level 1, 6 nodes at level 2. Obtain the corresponding binary tree. **5M**

Q.10.(a) Solve the recurrence relation :

$a_n = a_{n-1} + 2, n \geq 2$.

subject to the initial condition $a_1 = 3$. **7M**

(b) Find the generating function of the sequence :

1, 2, 3, 4, 5, ... **6M**

WINTER EXAM. - 2013

SOLUTION

(Unit-Wise)

UNIT - I

(Total Marks : 24)

Q.1.(a) If A and B are any two sets, then show that $A - B$, $A \cap B$ and $B - A$ are pair wise disjoint. **6M**

Ans. To prove : $A - B$, $A \cap B$ and $B - A$ are pair wise disjoint

(i) We prove : $(A - B) \cap (A \cap B) = \phi$

Let $x \in (A - B) \cap (A \cap B)$

$\Rightarrow x \in A - B$ and $x \in A \cap B$

$\Rightarrow (x \in A$ and $x \notin B)$ and $(x \in A$ and $x \in B)$

$\Rightarrow (x \in A$ and $x \notin B)$ and $(x \in A$ and $x \in B)$

$\Rightarrow (x \in A$ and $x \in A)$ and $(x \in B$ and $x \notin B)$

$\Rightarrow x \in A$ and $x \in \phi$

$\therefore (A - B) \cap (A \cap B) = \phi$ .....(a)

(ii) Now we prove : $(A \cap B) \cap (B - A) = \phi$

Let $x \in (A \cap B) \cap (B - A)$

$\Rightarrow x \in A \cap B$ and $x \in B - A$

$\Rightarrow (x \in A$ and $x \in B)$ and $(x \in B$ and $x \notin A)$

$\Rightarrow (x \in A$ and $x \notin A)$ and $(x \in B$ and $x \in B)$

$\Rightarrow x \in \phi$ and $x \in B$

$\therefore (A \cap B) \cap (B - A) = \phi$ .....(b)

(iii) Finally we prove : $(A - B) \cap (B - A) = \phi$

Let $x \in (A - B) \cap (B - A)$

$\Rightarrow x \in A - B$ and $x \in B - A$

$\Rightarrow (x \in A$ and $x \notin B)$ and $(x \in B$ and $x \notin A)$

$\Rightarrow (x \in A$ and $x \notin A)$ and $(x \in B$ and $x \notin B)$

$\Rightarrow x \in \phi$ and $x \in \phi$

$\therefore (A - B) \cap (B - A) = \phi$ .....(c)

From (a), (b), (c), we get

$\therefore$ $\boxed{A - B,\ A \cap B \ \&\ B - A \text{ are pair wise disjoint}}$

Ans.

Q.2.(c) By principle of Mathematical Induction, prove that :

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}, n \geq 1.$$

5M

Ans. See P.1-45, Ex.86.

Q.3.(a) Determine the validity of the argument : If my brother stands first in class, I will give him a watch. He stood first or I was out of station. I did not give him a watch. $\therefore$ I was out of station. **5M**

Ans. Let p : My brother stands first in class.

q : I will give him a watch.

r : I was out of station.

$S_1 : p \rightarrow q$

$S_2 : p \vee r$

$S_3 : \sim q$

$\therefore S : r$

We check whether $(S_1 \wedge S_2 \ S_3) \rightarrow S$ is a tautology.

p	q	r	S_3 $\sim q$	S_1 $p \rightarrow q$	S_2 $p \vee r$	1 $S_1 \wedge S_2 \wedge S_3$	$1 \rightarrow r$
T	T	T	F	T	T	F	T
T	T	F	F	T	T	F	T
T	F	T	T	F	T	F	T
T	F	F	T	F	T	F	T
F	T	T	F	T	T	F	T
F	T	F	F	T	F	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	F	F	T

Since $(S_1 \wedge S_2 \wedge S_3) \rightarrow S$ is a tautology, hence the given argument is valid. ....**Ans.**

Q.3.(b) Negate the following statements :

(i) If the studies, he will go to college or to art school.

(ii) He is tall but handsome. **3M**

Ans.

(i) Let p : He studies.

q : He will go to college.

r : He will go to art school.

$\therefore$ Given statement is $p \rightarrow (q \vee r)$

We have $\sim (p \rightarrow q) \equiv p \wedge \sim q$

$\therefore \sim [p \rightarrow (q \vee r)] \equiv p \wedge \sim (q \vee r)$

$= p \wedge (\sim q \wedge \sim r)$

$\therefore$ The negation is

He studies and he will not go to college and art school. **Ans.**

(ii) Let p : He is tall.

q : He is handsome.

$\therefore$ Given statement is $p \vee q$

We have $\sim (p \vee q) \equiv \sim p \wedge \sim q$

$\therefore$ The negation is

He is not tall and handsome. **Ans.**

Q.3.(c) Show that : $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$. **5M**

Ans. See P.1-12, Ex.8.

UNIT - II

(Total Marks : 22)

Q.1.(b) Let X = {1, 2, 3, ..., 9}

$R_1 = \{(x, y) / x - y$ is divisible by 3$\}$

$R_2 = \{(x, y) / x, y \in X$ and x, y are in the same column of matrix A$\}$

where, $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Find R_1 and R_2. Prove that R_1 and R_2 are equivalence relations. Is $R_1 = R_2$? Justify your answer. **7M**

Ans. See P.2-17, Ex.19.

Q.2.(a) Let X = {1, 2, 3, 4, 5}, R = {(1, 2), (3, 4), (2, 2)}, S = {(4, 2), (2, 5), (3, 4), (1, 3)}

Find SoR, RoS, Ro(SoR). **4M**

Ans. See P.2-19, Ex.24.

Q.2.(b) If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ both f and g are one-one and onto. Prove that :

$(gof)^{-1} = f^{-1} og^{-1}$. **5M**

Ans. See P.2-33, Theorem(2)

Q.6.(b) Draw Hasse Diagram of the lattice D_{20}. Write composition table. Write the complements of elements of D_{20}. **6M**

Ans. See P.2-30, Ex.49.

UNIT - III

(Total Marks : 20)

Q.4.(a) Describe an Abelian group. Prove that the set G = {0, 1, 2, 3, 4, 5} is a finite Abelian group under addition modulo 6 as operation. **7M**

Ans. See P.3-10, 3-15, Ex.11(a).

Q.5.(a) Prove that a normal subgroup H of a group G is normal if and only if $g^{-1} hg \in H$, for every $h \in H$, $g \in G$. **7M**

Ans. See P.3-31, Ex.50(a).

Q.5.(b) Prove that the intersections of two subgroups of a group G is a subgroup of G. Give an example to show that the union of two subgroups is not necessarily a subgroup. **6M**

Ans. See P.3-27, Ex.42.

UNIT - IV

(Total Marks : 13)

Q.6.(a) Prove that the set of complex numbers a + ib where 'a' and 'b' are integers forms a ring under ordinary addition and multiplication of complex numbers. **7M**

Ans. See P.4-16, Ex.12.

Q.7.(a) Construct switching circuit for the Boolean expression (A + B). (B' + C) + (C' + A). (C + B). Simplify this and construct an equivalent simplified circuit. **6M**

Ans. See P.4-37, Ex.57.

UNIT - V

(Total Marks : 34)

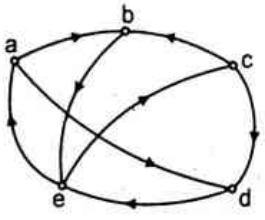
Q.7.(b) Draw diagraphs corresponding to adjacency

matrices and show that they are isomorphic.

$$M_1 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} \text{ and}$$

$$M_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$ **7M**

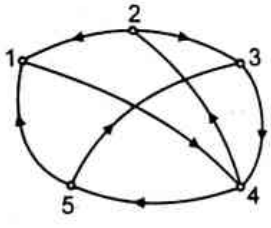
Ans.



Digraph of M_1

$V(M_1) = \{a, b, c, d, e\}$

$E(M_1) = \{(a, b), (a, d), (b, e), (c, b), (c, d), (d, e), (e, a), (e, c)\}$

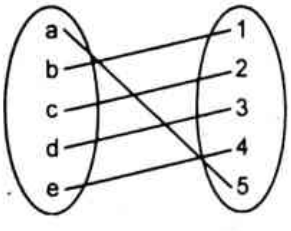


Digraph of M_2

$V(M_2) = \{1, 2, 3, 4, 5\}$

$E(M_2) = \{(1, 4), (2, 1), (2, 3), (3, 4), (4, 2), (4, 5), (5, 1), (5, 3)\}$

Let $f: V(M_1) \to V(M_2)$



$f(a) = 5$
$f(b) = 1$
$f(c) = 2$
$f(d) = 3$
$f(e) = 4$

Clearly f is one-one and onto.

Also we observe that

$(a, b) \in E(M_1) \Rightarrow (f(a), f(b)) \in E(M_2)$

$\forall a, b \in V(M_1)$

$\therefore$ Both the graphs are isomorphic. ....Ans.

Q.8.(a) Define: **(i)** Directed graph. **1M**

Ans. See P.5-13,

(ii) Path. **1M**

Ans. See P.5-22,

(iii) Reachability of graph. **1M**

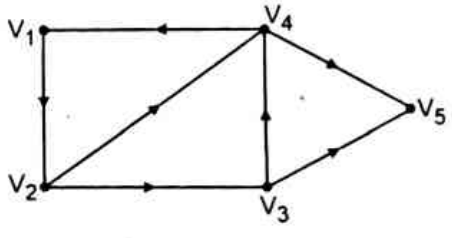
Ans. See P.5-23,

Q.8.(b) Draw binary tree for the relation given by :

$R = \{(1, 2), (1, 3), (1, 4), (2, 5), (4, 6), (4, 7)\}$ on set $A = \{1, 2, 3, 4, 5, 6, 7\}$. **5M**

Ans. See P.5-41, Ex.34.

Q.8.(c) Find all the indegrees and outdegrees of the node of the graph. Give all elementary cycles of this graph. Obtain an acyclic diagraph by deleting one edge of the given graph. List all the nodes which are reachable from another node of the graph.



6M

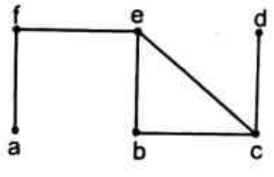
Ans. See P.5-25, Ex.10(a).

Q.9.(a) Construct the tree of the following algebraic expression :

$[5(1 - x)] + [\{5 - (y + 3)\} \times \{7 + (x + 4)\}]$. **4M**

Ans. See P.5-46, Ex.48.

Q.9.(b) Find all spanning trees for the graph G shown in the figure by removing edge in simple circuit :

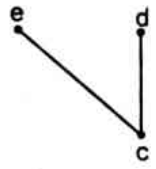


4M

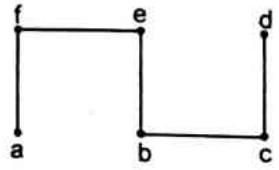
Ans.

(i) Let us select vertex c. There are three edges incident with c i.e. (c, b), (c, e), (c, d). Since the weights are not given, let us select edge (c, b).

(ii) Now there are two edges incident with c i.e. (c, e) and (c, d)



In above step if we select edge (c, e), then the minimal spanning tree would be



Q.9.(c) Draw a directed tree with 4 nodes at level 1, 6 nodes at level 2. Obtain the corresponding binary tree. **5M**

Ans. See P.5-47, Ex.49.

UNIT - VI

(Total Marks : 13)

Q.10.(a) Solve the recurrence relation :

$a_n = a_{n-1} + 2, n \geq 2$.

subject to the initial condition $a_1 = 3$. **7M**

Ans. See P.6-23, Ex.41.

Q.10.(b) Find the generating function of the sequence :

1, 2, 3, 4, 5, ... **6M**

Ans. See P.6-15, Refer Point (3).

WINTER EXAMINATION - 2013

B.E. IV SEMESTER (IT)

DISCRETE MATHEMATICS AND GRAPH THEORY

Statistical Analysis

Unit Number	Marks
Unit - I	27 M
Unit - II	39 M
Unit - III	26 M
Unit - IV	20 M
Unit - V	20 M
Unit - VI	---
Questions covered in VBD	132M (100%)

WINTER EXAM. - 2013
QUESTION PAPER

SECTION - A

Q.1.(a) Using laws of algebra of sets, prove that :

(i) $A^c \subset B$, if $A \cup B = U$

(ii) If $A \cap (B - C) = (A \cap B) - (A \cap C)$, then $A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$. **6M**

(b) Show that : $\sqrt{2}$ is irrational. **4M**

(c) Construct truth table for :

$(p \wedge q) \wedge \sim (p \vee q)$. **3M**

Q.2.(a) Cosnsider the conditional proposition $p \to q$. Write converse, inverse and contrapositive of it. Is any of them logically equivalent to $p \to q$? Verify by truth table. **6M**

(b) Show by method of induction that $n^3 + 2n$ is divisible by 3. **4M**

(c) Determine the validity of following argument : If I study, then I will not fail in Mathematics. If I do not play basketball, then I will study. But I failed in Mathematics.

Therefore, I must have played Basketball. **4M**

Q.3.(a) Let X = {ball, bed, dog, egg, let} and

R = {(x, y)}| x, y ∈ X, x R y if x and y contain some common letter}

(i) Draw the graph of R.

(ii) Write M_R.

(iii) Prove that R is compatible. **7M**

(b) Given a set S = {1, 2, 3, 4, 5}. Find the relations R_A and R_B generating the partition A = {(1, 2), (3), (4, 5)} B = {(1, 2, 3), (4, 5)}. Prove that R_A and R_B are equivalence relation. **6M**

Q.4.(a) If $f : X \to Y$ and $g : Y \to Z$ and both f and g are one-one, onto, then show that gof is also one-one and onto and $(gof)^{-1} = f^{-1} og^{-1}$. **6M**

(b) Let A be a given finite set and ρ(A) its power set. Let ⊆ be the inclusion relation on the elements of ρ(A). Draw Hasse diagram of (ρ(A), ⊆) for :

(a) A = {a, b}

(b) A = {a, b, c} **7M**

Q.5.(a) Determine whether the set of even integers with binary operation * defined by $a * b = \frac{ab}{2}$ is semigroup or monoid. Show whether it is commutative. **6M**

(b) In a group G, prove that $(ab)^2 = a^2b^2$ if and only if G is abelian. **7M**

SECTION - B

Q.6.(a) Let (G, *) and (G′, *′) be two groups and let $f : G \to G'$ be a homomorphism from G to G′.

(i) If e is identity in G and e′ is identity in G′, then f(e) = e′.

(ii) If a ∈ G, then $f(a^{-1}) = [f(a)]^{-1}$. **6M**

(b) Show that the multiplicative group G = {1, i, − 1, − i} is isomorphic to group G′ = {0, 1, 2, 3} with addition modulo 4 as composition. **7M**

Q.7.(a) Let (A, +, .) be any ring with zero, then for every a, b ∈ A, show that :

(i) 0.a = a.0 = 0

(ii) a . (− b) = (− a) . b = − (a . b)

(iii) (− a) . (− b) = a. b **6M**

(b) Show that (I, ⊕ , ⊙) is a commutative ring with indentity, where the operation ⊕ and ⊙ are defined as a ⊕ b = a + b − 1 and a ⊙ b = a + b − ab for any a, b ∈ I, set of integers. **7M**

Q.8.(a) Draw Hasse diagram of the lattices D_{20} and D_{30}. Write complements of every element. **6M**

(b) Prove that the set R = { 0, 2, 4, 6, 8} is a commutative ring under addition and multiplication modulo 10. Verify that it is an integral domain. **7M**

Q.9.(a) Draw the digraph corresponding to the matrix.

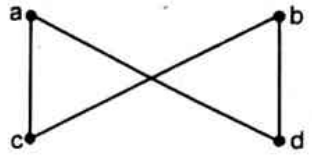
$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Find the transitive closure of the diagraph. **7M**

(b) Draw binary tree for the tree given by :

R = {(1 2), (1 3), (1 4), (2 5), (4 6), (4 7)} on set A = {1, 2, 3, 4, 5, 6, 7}. **7M**

Q.10.(a) Apply Prim's algorithm to find the spanning tree for the following graph. **6M**



(b) Draw the diagraph corresponding to the matrix :

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Interpret AA^T, A^TA and A^2.

WINTER EXAM. - 2013

SOLUTION

(Unit-Wise)

UNIT - I

(Total Marks : 27)

Q.1.(a) Using laws of algebra of sets, prove that :

(i) $A^c \subset B$ if $A \cup B = U$

(ii) If $A \cap (B - C) = (A \cap B) - (A \cap C)$ then $A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$. **6M**

Ans. See P.1-43, Ex.80.

Q.1.(b) Show that $\sqrt{2}$ is irrational. **4M**

Ans. See P.1-28, Ex.50.

Q.1.(c) Construct truth table for :

$(p \wedge q) \wedge \sim (p \vee q)$. **3M**

Ans. Given expression is $(p \wedge q) \wedge \sim (p \vee q)$.

Its TT is given below :

p	q	① $p \wedge q$	$p \vee q$	② $\sim (p \vee q)$	① $\wedge$ ②
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

Q.2.(a) Cosnsider the conditional proposition $p \to q$. Write converse, inverse and contrapositive of it. Is any of them logically equivalent to $p \to q$? Verify by truth table. **6M**

Ans. See P.1-13.

Q.2.(b) Show by method of induction that $n^3 + 2n$ is divisible by 3. **4M**

Ans. See P.1-44, Ex.85.

Q.2.(c) Determine the validity of following argument : If I study, then I will not fail in Mathematics. If I do not play basketball, then I will study. But I failed in Mathematics.

Therefore, I must have played Basketball. **4M**

Ans. See P.1-21, Ex.33.

UNIT - II

(Total Marks : 39)

Q.3.(a) Let X = {ball, bed, dog, egg, let} and $R = \{(x, y)\} \mid x, y \in X, x\ R\ y$ if x and y contain some common letter.}

(i) Draw the graph of R.

(ii) Write M_R.

(iii) Prove that R is compatible. **7M**

Ans. See P.2-23, Ex.32.

Q.3.(b) Given a set S = {1, 2, 3, 4, 5}. Find the relations R_A and R_B generating the partition

A = {(1, 2), (3), (4, 5)}, B = {(1, 2, 3), (4, 5)}. Prove that R_A and R_B are equivalence relation. **6M**

Ans. See P.2-15, Ex.12.

Q.4.(a) If $f : X \to Y$ and $g : Y \to Z$ and both f and g are one-one, onto, then show that gof is also one-one and onto and $(gof)^{-1} = f^{-1} og^{-1}$. **6M**

Ans. See P.2-33, Theorem (2).

Q.4.(b) Let A be a given finite set and $\rho(A)$ its power set. Let $\subseteq$ be the inclusion relation on the elements of $\rho(A)$. Draw Hasse diagram of $(\rho(A), \subseteq)$ for :

(a) A = {a, b}

(b) A = {a, b, c} **7M**

Ans. See P.2-30, Ex.50.

Q.8.(a) Draw Hasse diagram of the lattices D_{20} and D_{30}. Write complements of every element. **6M**

Ans. See P.2-30, Ex.49.

Q.9.(a) Draw the digraph corresponding to the matrix.

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Find the transitive closure of the diagraph. **7M**

Ans. See P.2-18, Ex.20.

UNIT - III

(Total Marks : 26)

Q.5.(a) Determine whether the set of even integers with binary operation * defined by $a * b = \frac{ab}{2}$ is semigroup or monoid. Show whether it is commutative. **6M**

Ans. See P.3-12, Ex.7.

Q.5.(b) In a group G, prove that $(ab)^2 = a^2b^2$ if and only if G is abelian. **7M**

Ans. See P.3-18, Ex.17.

Q.6.(a) Let (G, *) and (G′, *′) be two groups and let $f : G \to G'$ be a homomorphism from G to G′.

(i) If e is identity in G and e' is identity in G', then f(e) = e'.

(ii) If $a \in G$, then $f(a^{-1}) = [f(a)]^{-1}$. **6M**

Ans. See P.3-35, Ex.61.

Q.6.(b) Show that the multiplicative group

- $G = \{1, i, -1, -i\}$ is isomorphic to group $G' = \{0, 1, 2, 3\}$ with addition modulo 4 as composition. **7M**

Ans. See P.3-33, Ex.57.

UNIT - IV

(Total Marks : 20)

Q.7.(a) Let (A, +, .) be any ring with zero, then for every a, b ∈ A, show that :

(i) 0.a = a.0 = 0

(ii) a . (– b) = (– a) . b = – (a . b)

(iii) (– a) . (– b) = a. b **6M**

Ans. See P.4-11, Ex.6.

Q.7.(b) Show that (I, ⊕ , ⊙) is a commutative ring with indentity, where the operation ⊕ and ⊙ are defined as a ⊕ b = a + b – 1 and a ⊙ b = a + b – ab for any a, b ∈ I, set of integers. **7M**

Ans.

(1) We have to show (I, ⊕) is an abelian group

(i) $x \oplus (y \oplus z) = x \oplus (y + z - 1)$
$= x + y + z - 1 - 1$
$= x + y + z - 2$

$(x \oplus y) \oplus z = (x + y - 1) \oplus z$
$= x + y - 1 + z - 1$
$= x + y + z - 2$

$\therefore\ x \oplus (y \oplus z) = (x \oplus y) \oplus z$

$\therefore$ ⊕ is associative operation

(ii) Also $x \oplus y = y \oplus x$

Since $x \oplus y = x + y - 1$
$= y + x - 1$
$= y \oplus x$

$\therefore$ ⊕ is commutative.

(iii) 1 is additive identity since

$x \oplus 1 = x + 1 - 1 = x$.

(iv) Every element also has left inverse

For each $x \in I$, $z - x$ is inverse such that

$x \oplus (z - x) = x + z - x - 1 = 1$ i.e. identity

$\therefore$ (I, ⊕) is an abelian group.

(2) Now we have to show (I, ⊙) is a semi group

(i) ⊙ is closed operation

(ii) $x \odot (y \odot z) = x \odot (y + z - yz)$
$= x + y + z - yz - xy - xz + xyz$
$= x + y + z - xy - yz - xz + xyz$

Also $(x \odot y) \odot z = (x + y - xy) \odot z$
$= x + y - xy + z - xz - yz + xyz$
$= x \odot (y \odot z)$

$\therefore$ ⊙ is associative.

$\therefore$ (I, ⊙) is a semigroup.

(3) Also $x \odot (y \oplus z) = x \odot (y + z - 1)$
$= x + y + z - 1 - (xy + xz - x)$
$= x + y + z - xy - xz + x - 1$

& $x \odot y = x + y - xy$

$x \odot z = x + z - xz$

$\therefore\ (x \odot y) \oplus (x \odot z) = x + y - xy + x + z - xz - 1$
$= x + y + z - xy - xz + x - 1$
$= x \odot (y \oplus z)$

$\therefore$ ⊙ is distributive over ⊕

(4) Also $x \odot y = x + y - xy$
$= y + x - yx$
$= y \odot x$

$\therefore$ $\boxed{(I, \oplus\ \odot) \text{ is commutative ring}}$ **Ans.**

Q.8.(b) Prove that the set R = { 0, 2, 4, 6, 8} is a commutative ring under addition and multiplication modulo 10.

Verify that it is an integral domain. **7M**

Ans. See P.4-15, Ex.10(b).

UNIT - V

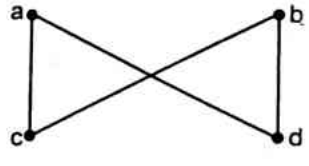
(Total Marks : 20)

Q.9.(b) Draw binary tree for the tree given by :

R = {(1 2), (1 3), (1 4), (2 5), (4 6), (4 7)} on set A = {1, 2, 3, 4, 5, 6, 7}. **7M**

Ans. See P.5-41, Ex.34.

Q.10.(a) Apply Prim's algorithm to find the spanning tree for the following graph. **6M**



Ans. See P.5-40, Ex.32.

Q.10.(b) Draw the diagraph corresponding to the matrix :

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Interpret AA^T, A^TA and A^2.

Ans. See P.5-25, Ex.8(c).

SUMMER EXAMINATION - 2014

B. E. IV SEM. (CT, CS, IT, CE) (CBS)

DISCRETE MATHEMATICS AND GRAPH THEORY

Statistical Analysis

Unit Number	Marks
Unit - I	20 M
Unit - II	36 M
Unit - III	30 M
Unit - IV	18 M
Unit - V	36 M
Unit - VI	20 M
Questions covered in VBD	160 M (100 %)

SUMMER EXAM.- 2014

QUESTION PAPER

Q.1.(a) Prove that :

(i) $(A \cap B) - C = (A - C) \cap (B - C)$

(ii) $(A \cap B)' = A' \cup B'$. **5M**

(b) Show that $((p \vee \sim q) \wedge (\sim p \vee \sim q)) \vee q$ is a tautology. **5M**

OR

Q.2.(a) In a group of students, 70 have a personal computer, 120 have personal stereo and 41 have both. How many own at least one of these devices? Draw an appropriate Venn diagram also. **4M**

(b) Show that :

$$1^2 + 2^2 + 3^2 + \ldots n^2 = \frac{n(n+1)(2n+1)}{6}, n \geq 1.$$ **5M**

Q.3.(a) Define equivalent relation. Is the relation R defined on the set of positive integers such that aRb iff $a \leq b$ is an equivalence relation? Explain if not. **6M**

(b) Let A = {1, 2, 3, 4, 5, 6}. Define a relation R on A such that R = {(x, y) | x + y is divisor of 24}.

Find :

(i) Relation matrix of R

(ii) Relation matrix of RoR

(iii) Draw digraph of R and RoR. **6M**

(c) Let $f : A \to B$ and $g : B \to C$ such that $f(a) = a - 1$ and $g(b) = b^2$. Find :

(i) $f_o g(x)$

(ii) $g_o f(x)$

(iii) $g_o g(x)$

(iv) $f_o f(x)$. **6M**

OR

Q.4.(a) The relation R defined on set A = {0, 1, 2, 3} R = {(0, 1), (1, 2), (2, 3)}. Find transitive closure of R. **6M**

(b) Let A = {1, 2, 3}, B = {a, b, c, d}. Let R and S be relations from A to B with relation matrices given by :

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix},$$

$$M_S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Find :

(i) $M_{R^{-1}}$ and $M_{S^{-1}}$

(ii) Show that : $M_{(R \cap S) o R^{-1}}$

$M_{RoR^{-1}} \wedge M_{SoR^{-1}}$ **6M**

(c) Let $f : A \to B$ defined by $f(x) = 2x^3 - 1$. Prove that f is one-to-one and onto. **6M**

Q.5.(a) Define Groupoid, Semigroup, Monoid and Group with examples. **6M**

(b) Show that the set $H = \{a + ib \in C \mid a^2 + b^2 = 1\}$ is a subgroup of (C, *), where * = multiplication of complex numbers. **6M**

OR

Q.6.(a) Define Co-sets and Normal subgroup. Prove that a subgroup H of a group G is normal, iff $g^{-1} hg \in H \; \forall h \in H$ and $g \in G$. **6M**

(b) If * is a binary operation in Q^+, defined by $a * b = \frac{ab}{3}, \; \forall a, b \in Q^+$, show that $(Q^+, *)$ form an abelian group.

Q.7.(a) If R is a set of all real numbers, then prove that (R, +, .) is an integral domain. **6M**

(b) Find the complements of every elements of the lattice $(D_{20}, \leq)$. **6M**

OR

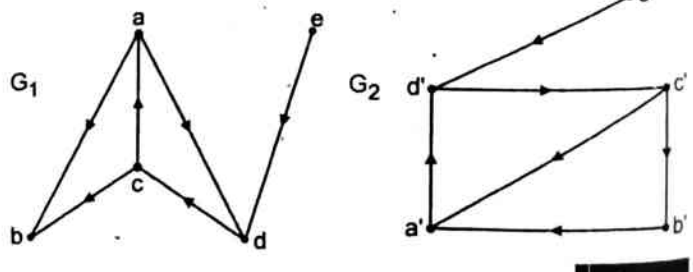
Q.8.(a) Let $R = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} : a, b \in Z \right\}$ is a ring. If f be the mapping that takes $\begin{bmatrix} a & b \\ b & a \end{bmatrix} \to a - b$. Show that f is homomorphism. Also determine the Kernel of f. **6M**

(b) Draw the switching system circuit represented by the Boolean expression :

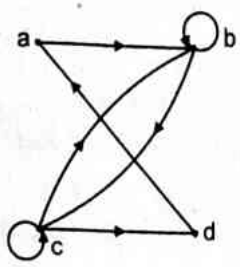
$F = (A + B)(\overline{B} + C) + (\overline{C} + A)(C + B)$

Simplify F and draw the simplified equivalent circuit. **6M**

Q.9.(a) Define isomorphic graphs. Show that the following graphs are isomorphic. **6M**



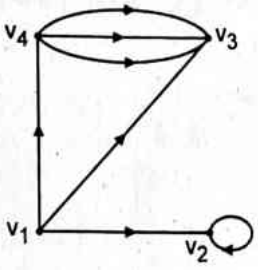
(b) Find the adjacency matrix of the following graphs and its complement : **6M**



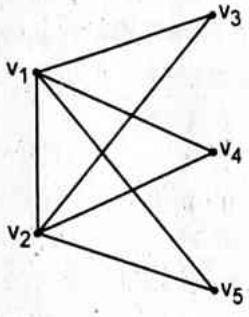
(c) Draw the tree for the algebraic expression :
$(a + 5) * [\{7 * b) + c\}/(g + d)]$ **6M**

OR

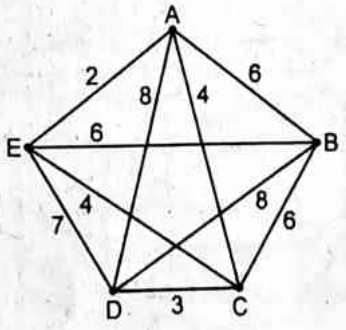
Q.10.(a) Define in-degree and out-degree of the graph. Find the in-degree and out-degree of each vertex of graph : **6M**



(b) Define Eulerian path and Eulerian circuit. Show that the graph given below is an Eulerian graph and circuit. **6M**



(c) Use Prim's algorithm to find minimal spanning tree for the graph : **6M**



Q.11.(a) Find the general solution of the recurrence relation :
$2a_n - 7a_{n-1} + 3a_{n-2} = 2^n$. **5M**

(b) Show that if any 30 people are selected, then one may choose a subset of 5 so that all five were born on the same day of the week. **5M**

OR

Q.12.(a) Apply the generating function technique to solve the recurrence relation
$a_{n+2} - 5a_{n+1} + 6a_n = 2, a_0 = 1, a_1 = 2$. **5M**

(b) Define Pigeon-hole principle. Show that if 9 books are to kept in 4 shelves, there must be at least one shelf which contains at least 3 books. **5M**

SUMMER EXAM.-2014

SOLUTION

(Unit-Wise)

UNIT - I

(Total Marks : 20)

Q.1.(a) Prove that :

(i) $(A \cap B) - C = (A - C) \cap (B - C)$

(ii) $(A \cap B)' = A' \cup B'$. **5M**

Ans. (i)See P.1-38, Ex.65

See P.1-36, De-Morgan's Law (2)

Q.1.(b)(ii) Show that $((p \vee \sim q) \wedge (\sim p \vee \sim q)) \vee q$ is a tautology. **5M**

Ans.

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
p	q	~p	~q	$p \vee \sim q$	$\sim p \vee \sim q$	(5) ^ (6)	(7) ∨ q
T	T	F	F	T	F	F	T
T	F	F	T	T	T	T	T
F	T	T	F	F	T	F	T
F	F	T	T	T	T	T	T

From table (8), it is clear that

$[(p \vee \sim q) \wedge (\sim p \vee \sim q)] \vee q$ is tautology... **Ans.**

Q.2.(a) In a group of students, 70 have a personal computer, 120 have personal stereo and 41 have both. How many own at least one of these devices? Draw an appropriate Venn diagram also. **4M**

Ans. Let A be the no. of students having personal computer.

B be the no. of students having personal stereo.

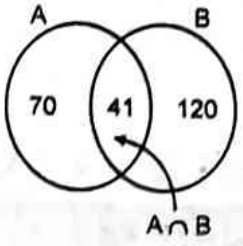
$\therefore$ $A \cap B$ be the no. of students having both.

We know that

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$= 70 + 120 - 41$

$= 149$



$\therefore$ $\boxed{\text{149 student own at least one of these devices}}$ **Ans.**

Q.2.(b) Show that :

$$1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}, n \geq 1.$$ **6M**

Ans. See P.1-45, Ex.86

UNIT - II

(Total Marks : 36)

Q.3.(a) Define equivalent relation. Is the relation R defined on the set of positive integers such that aRb iff $a \leq b$ is an equivalence relation? Explain if not. **6M**

Ans. See P.2-13, Sec.(viii)

Let X be a set of positive integers and R be the relation defined on X.

Suppose $a, b \in X$ and $a \leq b$ be an equivalue relation.

$\Rightarrow (a, a) \in R$ true for all positive integer $a \in X$.

$(a, b) \in R \Rightarrow (b, a) \in R$ as $a \leq b \Rightarrow b \leq a$ and

$(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ as $a \leq c$ true for all $a, b, c \in X$.

Conversely ,

(i) Let $a \in X \Rightarrow a \leq a \Rightarrow (a, a) \in R$

(ii) Let $a, b \in X$ s.t. $a \leq b \Rightarrow b \leq a \Rightarrow (a,b) \in R$

(iii) Let $a, b \in X$ s.t. $a \leq b$ and $b \leq c \Rightarrow a \leq c \Rightarrow (a,c) \in R$

$\therefore$ $\boxed{\text{a R b iff } a \leq b \text{ is an equivalence relation}}$ **Ans.**

Q.3.(b) Let A = {1, 2, 3, 4, 5, 6}. Define a relation R on A such that R = {(x, y) | x + y is divisor of 24}.

Find :

(i) Relation matrix of R

(ii) Relation matrix of RoR

(iii) Draw digraph of R and RoR. **6M**

Ans. Given : A = {1, 2, 3, 4, 5, 6}

R = {(x, y)| x + y is a divisor of 24}

$\therefore$ R = {(1,1), (1,2), (1,3), (1,5), (2,1), (2,2), (2,4), (2,6), (3,1), (3,3), (3,5), (4,2), (4,4), (5,1), (5,3) (6,2), (6,6)}

$\therefore$ RoR = {(1,1), (1,2), (1,3), (1,4), (1,5), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (3,1), (3,2), (3,3), (3,5), (4,1), (4,2), (4,4), (4,6), (5,1), (5,2), (5,3), (5,5), (6,1), (6,2), (6,4), (6,6)}

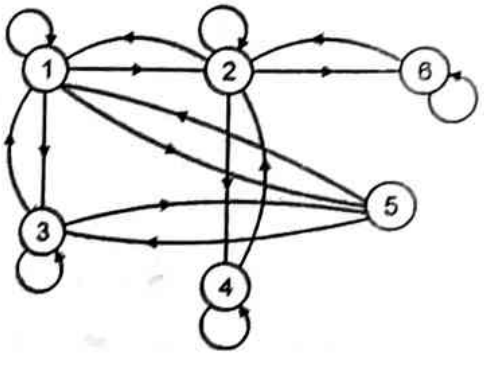
(i) Relation matrix of R :

$$\begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 & 1 \\ 3 & 1 & 0 & 1 & 0 & 1 & 0 \\ 4 & 0 & 1 & 0 & 1 & 0 & 0 \\ 5 & 1 & 0 & 1 & 0 & 0 & 0 \\ 6 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}$$

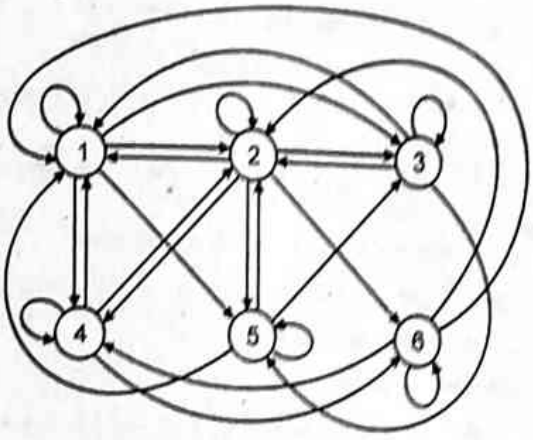
(ii) Relation matrix of RoR :

$$\begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 & 0 & 1 & 0 \\ 4 & 1 & 1 & 0 & 1 & 0 & 1 \\ 5 & 1 & 1 & 1 & 0 & 1 & 0 \\ 6 & 1 & 1 & 0 & 1 & 0 & 1 \end{array}$$

(iii) Diagraph of R :



(iv) Diagraph of RoR :



Q.3.(c) Let $f : A \to B$ and $g : B \to C$ such that $f(a) = a - 1$ and $g(b) = b^2$. Find :

(i) $f_o g(x)$

(ii) $g_o f(x)$

(iii) $g_o g(x)$

(iv) $f_o f(x)$. **6M**

Ans.

(i) $f_o g(x) = f[g(x)] = g(x) - 1 = x^2 - 1$

(ii) $g_o f(x) = g[f(x)] = [f(x)]^2 = (x-1)^2 = x^2 - 2x + 1$

(iii) $g_o g(x) = g(g(x)) = [g(x)]^2 = (x^2)^2 = x^4$

(iv) $f_o f(x) = f[f(x)] = f(x) - 1 = x - 1 - 1 = x - 2$.

Q.4.(a) The relation R define on set A = {0, 1, 2, 3} R = {(0, 1), (1, 2), (2, 3)}. Find transitive closure of R. **6M**

Ans. Given : A = {0, 1, 2, 3}

R = {(0,1), (1,2) (2,3)}

$$M_R = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

R = {(0,1) (1,2) (2,3)}

$\Rightarrow R^2 = \{(0,2)\ (1,3)\}$

$\Rightarrow R^3 = \{R^2 . R\} = \{(0,3)\}$

$R^+ = R^1 \cup R^2 \cup R^3$

$= \{(0,1)\ (1,2)\ (2,3)\} \cup \{(0,2)\ (1,3)\} \cup \{0, 3)$

$= \{(0,1)\ (0,2)\ (0,3)\ (1,2)\ (1,3)\ (2,3)\}$

$\therefore$ $\boxed{R^+ \text{ is a required transitive closure}}$ **Ans.**

Q.4.(b) Let A = {1, 2, 3}, B = {a, b, c, d}. Let R and S be relations from A to B with relation matrices given by :

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix},$$

$$M_S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Find :

(i) $M_{R^{-1}}$ and $M_{S^{-1}}$

(ii) Show that : $M_{(R \cap S)oR^{-1}} = M_{RoR^{-1}} \wedge M_{SoR^{-1}}$. **6M**

Ans. See P.2-22, Ex.30

Q.4.(c) Let $f : A \to B$ defined by $f(x) = 2x^3 - 1$. Prove that f is one-to-one and onto. **6M**

Ans. Let $f(x) = 2x^3 - 1$

To show : f is one one

Let f(x) = f(y) to show x = y

Now,

f(x) = f(y)

$\Rightarrow 2x^3 - 1 = 2y^3 - 1$

$\Rightarrow 2x^3 = 2y^3$

$\Rightarrow x^3 = y^3$

$\Rightarrow x = y$

$\therefore f(x) = f(y) \Rightarrow x = y$

$\Rightarrow$ f is one one.

(ii) Now we show f is onto

Since $x \in A\ \exists\ y \in B$ such that $f(x) = y \Rightarrow$ f is onto.

....Ans.

UNIT - III

(Total Marks : 30)

Q.5.(a) Define Groupoid, Semigroup, Monoid and Group with examples. **6M**

Ans. See P.3-7,8 & 10

Q.5.(b) Show that the set $H = \{a + ib \in C \mid a^2 + b^2 = 1\}$ is a subgroup of (C, *), where * = multiplication of complex numbers. **6M**

Ans. To prove H is a subgroup we prove G_1 and G_4

G_1 : Let x = a + ib and y = c + id $\in$ H

$\therefore x * y = (a+ib) * (c + id)$

$= ac + adi + bci - bd$

$\therefore x * y = (ac - bd) + (bc + ad) i$

Now

$$(ac-bd)^2 + (bc+ad)^2 = a^2c^2 - 2abcd + b^2d^2 + b^2c^2 + 2abcd + a^2d^2$$

$$= a^2c^2 + b^2c^2 + b^2d^2 + a^2d^2$$

$$= c^2(a^2+b^2) + d^2(a^2+b^2)$$

$$= c^2 + d^2 = 1$$

$$\left(\begin{array}{l} \because x = a + ib \in H \Rightarrow a^2 + b^2 = 1 \\ and\ y = c + id \in H \Rightarrow c^2 + b^2 = 1 \end{array}\right)$$

$\therefore x * y \in H$

$\therefore G_1$ is verified.

G_4 : **To prove :**

$x^{-1} = (a + ib)^{-1} \in H$ if $x = a + ib \in H$

Let $x = a + ib \in H \Rightarrow a^2 + b^2 = 1$

$$\Rightarrow x^{-1} = \frac{1}{x} = \frac{1}{a+ib} = \frac{1}{a+ib} \times \frac{a-ib}{a-ib} = \frac{a-ib}{a^2+b^2}$$

$$= a - ib \in H \qquad [\because a^2 + b^2 = 1]$$

$\therefore x^{-1} \in H$

$\therefore G_4$ is verified.

This show that H is a subgroup of C. ....Ans.

Q.6.(a) Define Co-sets and Normal subgroup. Prove that a subgroup H of a group G is normal, iff $g^{-1} hg \in H \ \forall h \in H$ and $g \in G$. **6M**

Ans. See P.3-28, 30 & 31, Ex.50(a)

Q.6.(b) If * is a binary operation in Q^+, defined by $a * b = \frac{ab}{3}, \forall a, b \in Q^+$, show that $(Q^+, *)$ form an abelian group. **6M**

Ans. See P.3-12, Ex.7

Q.7.(a) If R is a set of all real numbers, then prove that (R, +, .) is an integral domain. **6M**

Ans. Let R be a set of real numbers

We first prove that G is a commutative ring

G_1 : Let $a, b \in R \Rightarrow a * b \in R$ and $a + b \in R$

$\Rightarrow$ Addition and multiplication is closed

G_2 : Let $a, b, c \in R \Rightarrow a + (b + c) = (a + b) + c$ follows in a set of real numbers.

G_3 : $0 \in R$ and let $a \in R \Rightarrow a + 0 = 0 + a = a$

$\Rightarrow$ 0 is the zero element of R.

G_4 : $\forall a \in R \ \exists -a \in R$ such that

$$a + (-a) = (-a) + a = 0$$

for e.g.: $2 \in R \ \exists -2 \in R \Rightarrow (2) + (-2) = (-2) + 2 = 0$

$\Rightarrow (-a)$ is an additive inverse of a

G_5 : $\forall a, b \in R \Rightarrow a + b = b + a$ satisfies

i.e. $2, 3 \in R \Rightarrow 2 + 3 = 3 + 2 = 5$

$\therefore$ (R, +) is an abelian group.

$\Rightarrow G_5$ is verified.

$\therefore G_6$: $a, b, c \in R \Rightarrow a * (b + c) = a * b + a * c$ and $(a + b) * c = a * c + b * c$ follows

$\therefore G_6$ is verified.

G_7 : $a, b, c \in R \Rightarrow a * (b * c) = (a * b) * c$

G_7 is verified.

G_8 : $a, b \in R \Rightarrow a.b = b.a$ i.e. $2.3 = 3.2$

$\Rightarrow$ Commutivity property follows

$\therefore$ (R, +, *) is a commutative Ring.

Moreover

$a * 0 = 0 * a = 0$

$\Rightarrow$ Thus $a * b = 0 \Rightarrow$ Either $a = 0$ or $b = 0$

$\therefore$ (R, +, *) has no zero divisor

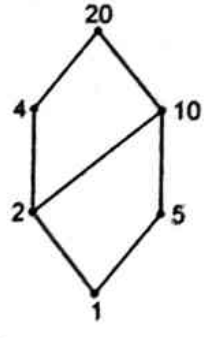
$\therefore$ $\boxed{\text{(R,+,*) is an integral domain}}$ Ans.

UNIT - IV

(Total Marks : 18)

Q.7.(b) Find the complements of every elements of the lattice $(D_{20}, \leq)$. **6M**

Ans. $(D_{20}, \leq)$ lattice is



Complement of 1 = 20

Complement of 20 = 1

Complement of 4 = 5

Complement of 5 = 4 ....Ans.

Q.8.(a) Let $R = \left\{\begin{bmatrix} a & b \\ b & a \end{bmatrix} : a, b \in Z\right\}$ is a ring. If f be the mapping that takes $\begin{bmatrix} a & b \\ b & a \end{bmatrix} \to a - b$. Show that f is homomorphism. Also determine the Kernel of f. **6M**

Ans. Given : $f\begin{bmatrix} a & b \\ b & a \end{bmatrix} \to a - b$

$$\Rightarrow f\begin{bmatrix} a & b \\ b & a \end{bmatrix} = a - b \qquad ....(1)$$

Let $x = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ and $y = \begin{bmatrix} c & d \\ d & c \end{bmatrix}$

$$\therefore\ xy = \begin{bmatrix} a & b \\ b & a \end{bmatrix}\begin{bmatrix} c & d \\ d & c \end{bmatrix}$$

$$= \begin{bmatrix} ac+bd & ad+bc \\ bc+ad & bd+ac \end{bmatrix}$$

$$\Rightarrow f(xy) = f\begin{bmatrix} ac+bd & ad+bc \\ ad+bc & ac+bd \end{bmatrix} = (ac+bd)-(ad+bc)$$

[∵ By (1)]

$$= a(c-d) - b(c-d)$$
$$= (a-b)(c-d)$$
$$= f\begin{bmatrix} a & b \\ b & a \end{bmatrix} f\begin{bmatrix} c & d \\ d & c \end{bmatrix}$$

$\therefore\ f(xy) = f(x)\,f(y)$

$\therefore$ f is a homomorphism. ....Ans.

Kernel of f :

Let $x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in R$ such that $1 - 0 = 1$

We know that

$K_f = \{x \in R \mid f(x) = 1\}$

$\therefore\ x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a Kernel of f. ...Ans.

Q.8.(b) Draw the switching system circuit represented by the Boolean Expression :

$F = (A + B)(\overline{B} + C) + (\overline{C} + A)(C + B)$

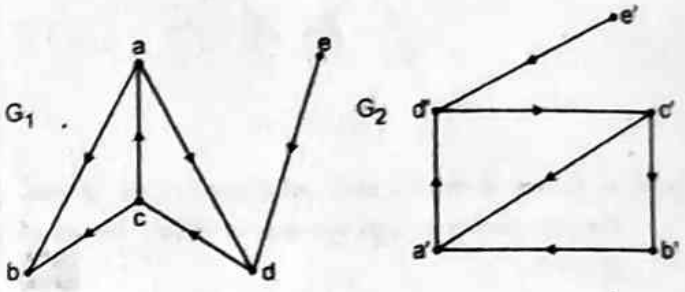
Simplify F and draw the simplified equivalent circuit. **6M**

Ans. See P.4-37, Ex.57

UNIT - V

(Total Marks : 36)

Q.9.(a) Define isomorphic graphs, (Show that the following graphs are isomorphic. **6M**



Ans. See P.5-18

To show isomorphism, we show the correspondence as follows :

$a \leftrightarrow a'$

$b \leftrightarrow b'$

$c \leftrightarrow c'$

$d \leftrightarrow d'$

$e \leftrightarrow e'$

$<a, b> \leftrightarrow <a', b'>$

$<b, c> \leftrightarrow <b', c'>$

$<c, d> \leftrightarrow <c', d'>$

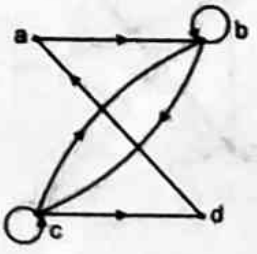
$<a, d> \leftrightarrow <a', d'>$

$<a, c> \leftrightarrow <a', c'>$

$<d, e> \leftrightarrow <d' e'>$

$\Rightarrow$ Graphs are isomorphic. ....Ans.

Q.9.(b) Find the adjacency matrix of the following graphs and its complement : **6M**

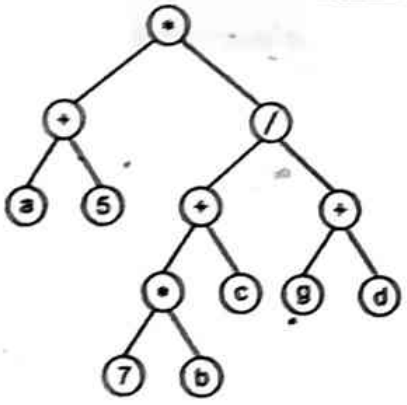


Ans. See P.5-35, Ex.28

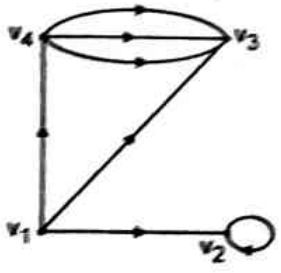
Q.9.(c) Draw the tree for the algebraic expression :

$(a + 5) * [\{(7 * b) + c\}/(g + d)]$ **6M**

Ans.



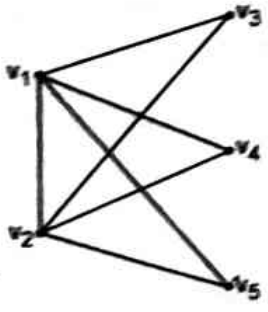
Q.10.(a) Define in-degree and out-degree of the graph. Find the in-degree and out-degree of each vertex of graph : **6M**



Ans. See P.5-15 (xiii) (3)

Node	In-degree	Out degree
V_1	0	3
V_2	2	1
V_3	4	0
V_4	1	3
	7	7

Q.10.(b) Define Eulerian path and Eulerian circuit. Show that the graph given below is an Eulerian graph and circuit. **6M**



Ans. See P.5-30

We observe that G is connected and all the vertices are having even degree.

$\deg(v_1) = 4$ $\deg(v_4) = 2$

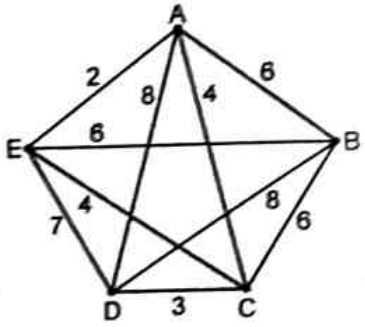
$\deg(v_2) = 4$ $\deg(v_5) = 2$

$\deg(v_3) = 2$

Thus, G has Eulerian circuit, as

$v_3 - v_1 - v_4 - v_2 - v_5 - v_1 - v_3$

Q.10.(c) Use Prim's algorithm to find minimal spanning tree for the graph : **6M**



Ans. See P.5-44, Ex.42

UNIT - VI

(Total Marks : 20)

Q.11.(a) Find the general solution of the recurrence relation :

$$2a_n - 7a_{n-1} + 3a_{n-2} = 2^n.$$ **5M**

Ans. Given equation is

$$2a_n - 7a_{n-1} + 3a_{n-2} = 2^n$$

Its A. E. is

$$2E^2 - 7E + 3 = 0$$

$$\therefore 2E^2 - 6E - E + 3 = 0$$

$$\therefore 2E(E-3) - 1(E-3) = 0$$

$$\therefore E = 3, E = 1/2$$

$$\therefore C.F. = C_1(3)^n + (1/2)^n C_2$$

$$P.I. = \frac{1}{2E^2 - 6E + 3}(2)^n$$

$$= \frac{1}{2(2)^2 - 6(2) + 3}(2)^n$$

$$= \frac{1}{8 - 12 + 3}(2)^n$$

$$\therefore P.I. = -(2)^n$$

$$\therefore a_n = C.F. + P.I.$$

$$\therefore \boxed{a_n = C_1(3)^n + C_2(1/2)^n - 2^n}$$ Ans.

which is the required solution.

Q.11.(b) Show that if any 30 people are selected, then one may choose a subset of 5 so that all five were born on the same day of the week. **5M**

Ans. See P.6-9, Ex.23

Q.12.(a) Apply the generating function technique to solve the recurrence relation :

$a_{n+2} - 5a_{n+1} + 6a_n = 2, a_0 = 1, a_1 = 2.$ **5M**

Ans. It's A.E. is

$m^2 - 5m + 6 = 0$

$\therefore (m-2)(m-3) = 0$

$\therefore m = 2, 3$

$\therefore C.F. = C_1(2)^n + C_2(3)^n$

Now,

$$P.I. = \frac{1}{E^2 - 5E + 6} 2(1)^n$$

$$= \frac{1}{1 - 5(1) + 6} 2(1)^n$$

$$= \frac{1}{2}(2)$$

$$= (1)$$

$\therefore a_n = C.F + P.I = C_1(2)^n + C_2(3)^n + 1$

Put $n = 0$

$\therefore a_0 = C_1 + C_2 + 1$

$\Rightarrow 1 = C_1 + C_2 + 1$

$\Rightarrow C_1 + C_2 = 0$ ....(1)

Put $n = 1$

$\therefore a_1 = 2C_1 + 3C_2 + 1$

$\therefore 2 = 2C_1 + 3C_2 + 1$

$\therefore 2C_1 + 3C_2 = 1$ ....(2)

$\therefore C_2 = 1$

and $C_1 = -1$

$\therefore \boxed{a_n = -(2)^n + (3)^n}$ **Ans.**

Q.12.(b) Define Pigeon-hole principle. Show that if 9 books are to kept in 4 shelves, there must be at least one shelf which contains at least 3 books. **5M**

Ans. See P.6-8 & 9, Ex.22

Notes :-

WINTER EXAMINATION - 2014
B.E. IV SEM. (CT, CS, IT, CE) (CBS)
DISCRETE MATHEMATICS AND GRAPH THEORY

Statistical Analysis

Unit Number	Marks
Unit - I	20 M
Unit - II	36 M
Unit - III	24 M
Unit - IV	24 M
Unit - V	36 M
Unit - VI	20 M
Questions covered in VBD	160 M (100 %)

WINTER EXAM.- 2014
QUESTION PAPER

Q.1.(a) Prove by mathematical induction method that the sum of the cubes of three consecutive integers are divisible by nine. **4M**

(b) Prove the logical equivalence by using algebra of proposition : **3M**

(i) $p \wedge (\sim p \vee q) \equiv p \wedge q$

(ii) $\sim(p \rightarrow q) \equiv p \wedge \sim q$

(c) Prove that : **3M**

$A - (B \cup C) = (A - B) \cap (A - C)$

OR

Q.2.(a) Is the following argument valid ?

If I study, then I will not fail in mathematics.

if I do not play basketball, then I will study.

But if failed in mathematics

Therefore I played basketball. **6M**

(b) Write the contrapositive of each of the following statement : **4M**

(i) If I have enough money, then I will buy a car and I will buy a house.

(ii) If I have time and I am not too tired, then I will go to the store.

Q.3.(a) If R be a relation in the set of integers Z defined by $R = \{(x, y) . x \in Z, y \in Z, (x - y)$ is divisible by $6\}$, then prove that R is an equivalence relation. **6M**

(b) Define transitive closure of a relation.

Let $A = \{1, 2, 3, 4\}$ and $R = \{(1, 2), (2, 3), (3, 4)\}$ be a relation on A. Find transitive closure of R and draw its digraph. **6M**

(c) List all possible functions from set $X = \{a, b, c\}$ to $Y = \{0, 1\}$. Indicate in each case whether the function is 1 – 1, onto and 1 – 1 onto. **6M**

OR

Q.4.(a) Define characteristic function. Using properties of characteristic function prove that **6M**

(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(ii) $A \cup (B \cup C) = (A \cup B) \cup C$

(iii) $(A^c)^c = A$

(b) Let $A = \{a, b, c, d\}$ and c (A) be its power set. Let $\subseteq$ be inclusion relation on the elements of c (A). Draw Hasse diagram $(c(A)), \subseteq)$. **6M**

(c) Draw tree diagram of $A \times B \times C$ where, $A = \{2, 3\}$, $B = \{1, 3, 5\}$ and $C = \{3, 4\}$ and hence find $A \times B \times C$. **6M**

Q.5.(a) Determine whether the set of integers Z, (Z,•) defined by $a \bullet b = a + b - ab$ is a monoid. **6M**

(b) Show that the set $\{1, w, w^2\}$, where is cube root of unity forms a group under multiplication. **6M**

OR

Q.6.(a) Prove that : If f is a homomorphism of a group G into a group G' with kernel K, then K is a normal subgroup of G. **6M**

(b) Let G be the group of integers under addition and let H be the set of all integral multiples of 3. Prove that H is a subgroup of G and determine all cosets of H in G. **6M**

Q.7.(a) If R is a ring such that $a^2 = a, \forall a \in R$. Prove that :

(i) $a + a = 0, \forall a \in R$

(ii) $a + b = 0 \Rightarrow a = b$

(iii) R is a commutative ring. **6M**

(b) Show that the ring of real numbers $(R, +, \cdot)$ is an integral domain. **6M**

OR

Q.8.(a) Show that there are 5 partitions of a set of three elements. Draw the diagram of the corresponding lattice. **6M**

(b) Construct switching circuit for the Boolean expression $(A . B) + [A'.(A + B + B')]$. Simplify this and construct an equivalent simplified circuit. **6M**

Q.9.(a) Define : **6M**

(i) Directed graph.

(ii) Simple path and elementary path.

(iii) Euler path.

(iv) Connected graph.

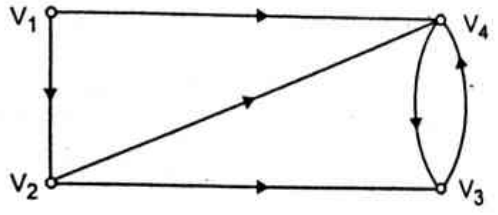
(v) Diameter of graph.

(b) Draw the diagraphs corresponding to adjacency matrices :

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

and prove that diagraphs of A and B are isomorphic. **6M**

(c) Give three different elementary paths from V_1 to V_3 for the diagraph given below. What is the shortest distance between V_1 and V_3 ? Is there any cycle in the graph? Is the diagraph transitive? **6M**



OR

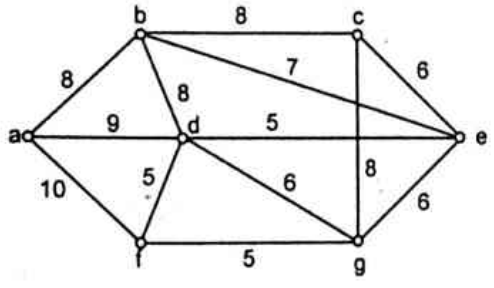
Q.10.(a) Draw tree representation for the tree given by:

A = {1, 2, 3, 4, 5, 6, 7};

R = {(1, 2), (1, 3), (1, 4), (2, 5), (4, 6), (4, 7)}

and draw corresponding binary tree. **6M**

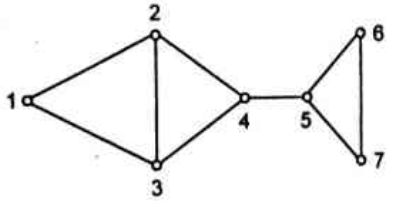
(b) Apply Prim's algorithm to construct a minimal spanning tree for the weighted graph given below: **6M**



(c) Find :

(i) Diameter. (ii) Radius and

(iii) Centre of the graph given below :



Is there any bridge? **6M**

Q.11.(a) Find a general solution for :

$a_r + a_{r-1} = (3r)2^r$. **5M**

(b) Show that if any five numbers from 1 to 8 are chosen then two of them add up to 9 using pigeon hole principle. **5M**

OR

Q.12.(a) Find the generating function for the sequence :

$1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n+1}, \ldots$ **5M**

(b) Prove that : **5M**

$C(n, r) = C(n-1, r-1) + C(n-1, r)$

WINTER EXAM.-2014

SOLUTION

(Unit-Wise)

UNIT - I

(Total Marks : 20)

Q.1.(a) Prove by mathematical induction method that the sum of the cubes of three consecutive integers are divisible by nine. **4M**

Ans. See P.1-46, Ex.88.

Q.1.(b) Prove the logical equivalence by using algebra of proposition :

(i) $p \wedge (\sim p \vee q) \equiv p \wedge q$

(ii) $\sim (p \rightarrow q) \equiv p \wedge \sim q$ **3M**

Ans.

(i) Consider L.H.S. $\equiv p \wedge (\sim p \vee q)$

$\equiv [(p \wedge \sim p) \vee (p \wedge q)]$ .....(Distributive law)

$\equiv [F \vee (p \wedge q)]$ ....(Complement law)

$\equiv (p \wedge q)$ (Identity law)

$\equiv$ R.H.S.

$\therefore$ $\boxed{p \wedge (\sim p \vee q) \equiv (p \wedge q)}$ Ans.

(ii) Consider L.H.S. $\equiv \sim (p \rightarrow q)$

$\equiv \sim (\sim p \vee q)$ .....(logical equivalence)

$\equiv \sim (\sim p) \wedge (\sim q)$ .....(De-Morgan's laws)

$\equiv p \wedge \sim q$

$\equiv$ R.H.S.

$\therefore$ $\boxed{\sim (p \rightarrow q) \equiv p \wedge \sim q}$ Ans

Q.1.(c) Prove that : **3M**

$A - (B \cup C) = (A - B) \cap (A - C)$

Ans. See P.1-37, Ex.60.

Q.2.(a) Is the following argument valid ?

If I study, then I will not fail in mathematics.

If I do not play basketball, then I will study.

But if failed in mathematics

Therefore I played basketball. **6M**

Ans. See P.1-21, Ex.33.

Q.2.(b) Write the contrapositive of each of the following statement : **4M**

(i) If I have enough money, then I will buy a car and I will buy a house.

(ii) If I have time and I am not too tired, then I will go to the store.

Ans. See P.1-15, Ex.17.

UNIT - II

(Total Marks : 36)

Q.3.(a) If R be a relation in the set of integers Z defined by $R = \{(x, y) . x \in Z, y \in Z, (x-y)$ is divisible by $6\}$, then prove that R is an equivalence relation. **6M**

Ans. See P.2-17, Ex.19.

Q.3.(b) Define transitive closure of a relation.

Let $A = \{1, 2, 3, 4\}$ and $R = \{(1, 2), (2, 3), (3, 4)\}$ be a relation on A. Find transitive closure of R and draw its digraph. **6M**

Ans. See P.2-13, for definition.

Given : $A = \{1, 2, 3, 4\}$

$R = \{(1, 2), (2, 3), (3, 4)\}$

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$R^1 = \{(1, 2)(2, 3)(3, 4)\}$

$R^2 = \{(1, 3)(2, 4)\}$

$R^3 = \{R^2 . R\} = \{(1, 4)\}$

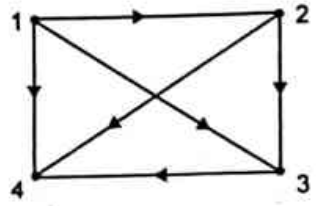
$R^* = R^1 \cup R^2 \cup R^3$

$= \{(1, 2)(2, 3)(3, 4)\} \cup \{(1, 3)(2, 4)\} \cup \{(1, 4)\}$

$\therefore R^* = \{(1, 2)(1, 3)(1, 4)(2, 3)(2, 4)(3, 4)\}$

$\therefore R^*$ is a required transitive closure. Ans.

Digraph :



Q.3.(c) List all possible functions from set $X = \{a, b, c\}$ to $Y = \{0, 1\}$. Indicate in each case whether the function is 1 – 1, onto and 1 – 1 onto. **6M**

Ans. See P.2-34, Ex.55.

Q.4.(a) Define characteristic function. Using properties of characteristic function prove that : **6M**

(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(ii) $A \cup (B \cup C) = (A \cup B) \cup C$

(iii) $(A^c)^c = A$

Ans. See P.2-38, Ex.66(i), (ii), (iv).

Q.4.(b) Let $A = \{a, b, c, d\}$ and e (A) be its power set. Let $\subseteq$ be the inclusion relation on the elements of e (A). Draw Hasse diagram of $(e(A), \subseteq)$. **6M**

Ans. See P.2-30, Ex.50.

Q.4.(c) Draw tree diagram of $A \times B \times C$ where, $A = \{2, 3\}$, $B = \{1, 3, 5\}$ and $C = \{3, 4\}$ and hence find $A \times B \times C$. **6M**

Ans. See P.2-10, Ex.1.

UNIT - III

(Total Marks : 24)

Q.5.(a) Determine whether the set of integers Z, (Z, *) defined by $a * b = a + b - ab$ is a monoid. **6M**

Ans. See P.3-14, Ex.9.

Q.5.(b) Show that the set $\{1, w, w^2\}$, where is cube root of unity forms a group under multiplication. **6M**

Ans. See P.3-11, Ex.5.

Q.6.(a) Prove that : If f is a homomorphism of a group G into a group G′ with kernel K, then K is a normal subgroup of G. **6M**

Ans. See P.3-35, Ex.62.

Q.6.(b) Let, G be the group of integers under addition and let H be the set of all integral multiples of 3. Prove that H is a subgroup of G and determine all cosets of H in G. **6M**

Ans. See P.3-27, 28, Ex.41, 43.

UNIT - IV
(Total Marks : 24)

Q.7.(a) If R is a ring such that $a^2 = a, \forall a \in R$. Prove that :

(i) $a + a = 0, \forall a \in R$ (ii) $a + b = 0 \Rightarrow a = b$

(iii) R is a commutative ring. **6M**

Ans. See P.4-11, Ex.5.

Q.7.(b) Show that the ring of real numbers (R, +, ·) is an integral domain. **6M**

Ans. See P.PS-16, Q.7.(a), Summer-2014.

Q.8.(a) Show that there are 5 partitions of a set if three elements. Draw the diagram of the corresponding lattice. **6M**

Ans. See P.4-29, Ex.41.

Q.8.(b) Construct switching circuit for the Boolean expression (A . B) + [A′.(A + B + B′)]. Simplify this and construct an equivalent simplified circuit. **6M**

Ans. See P.4-33, Ex.50.

UNIT - V
(Total Marks : 36)

Q.9.(a) Define : **6M**

(i) Directed graph.

(ii) Simple path and elementary path.

(iii) Euler path.

(iv) Connected graph.

(v) Diameter of graph.

Ans. See P.5-13, 22, 30, 28.

(v) **Diameter of a Graph :** The distance d(u, v) from vertex u to vertex v in a graph is the minimum length from u to v. The eccentricity of a vertex v is $E_G(v) = \text{Max}_{x \in V(G)} d(v, x)$. The diameter and radius of a graph are the maximum and minimum of its vertex eccentricities.

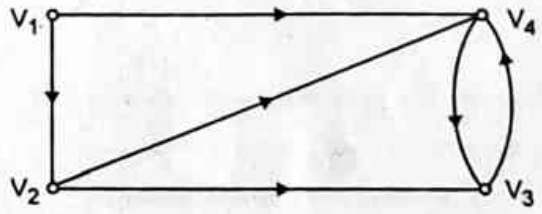
Q.9.(b) Draw the diagraphs corresponding to adjacency matrices :

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

and prove that diagraphs of A and B are isomorphic. **6M**

Ans. See P.5-32, Ex.18.

Q.9.(c) Give three different elementary paths from V_1 to V_3 for the diagraph given below. What is the shortest distance between V_1 and V_3 ? Is there any cycle in the graph? Is the diagraph transitive? **6M**



Ans. See P.5-25, Ex.9.

Q.10.(a) Draw tree representation for the tree given by:

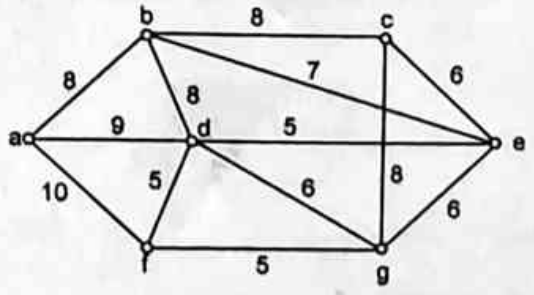
A = {1, 2, 3, 4, 5, 6, 7};

R = {(1, 2), (1, 3), (1, 4), (2, 5), (4, 6), (4, 7)}

and draw corresponding binary tree. **6M**

Ans. See P.5-41, Ex.34.

Q.10.(b) Apply Prim's algorithm to construct a minimal spanning tree for the weighted graph given below: **6M**



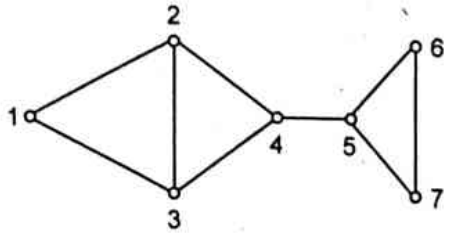
Ans. See P.5-46, Ex.47.

Q.10.(c) Find :

(i) Diameter.

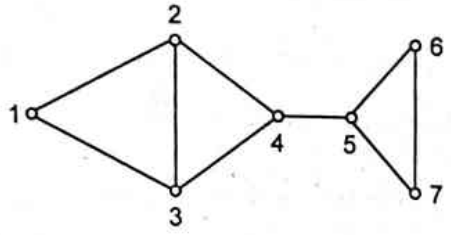
(ii) Radius and

(iii) Centre of the graph given below :



Is there any bridge? **6M**

Ans. Given graph is



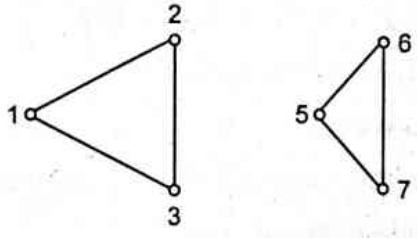
(i) Diameter = Max. eccentricity (from 1 to 7) = 6

(ii) Radius = Min. eccentricity (from 1 to 7) = 4

(iii) Centre = Set. of vertices of minimum eccentricity

$\therefore$ Centre = {4, 5}

After removing vertex 4 and the edges incident with vertex 4, the graph becomes disconnected.



$\therefore$ Vertex 4 is cut vertex and edge (4, 5) is the bridge. **.....Ans.**

UNIT - VI

(Total Marks : 20)

Q.11.(a) Find a general solution for :

$a_r + a_{r-1} = (3r)2^r$. **5M**

Ans. **See P.6-21, Ex.36.**

Q.11.(b) Show that if any five numbers from 1 to 8 are chosen then two of them add up to 9 using pigeon hole principle. **5M**

Ans. **See P.6-10, Ex.25.**

Q.12.(a) Find the generating function for the sequence :

$1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n+1}, \ldots$ **5M**

Ans. **See P.6-21, Ex.35.**

Q.12.(b) Prove that : **5M**

$$C(n, r) = C(n-1, r-1) + C(n-1, r)$$

Ans. **To Prove :** $C(n, r) = C(n-1, r-1) + C(n-1, r)$

Method 1 : The above relation can be proved in several ways. By straight forward algebraic manipulation, we obtain

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

$$= \frac{n!}{r!(n-r)!}\left(\frac{n-r}{n} + \frac{r}{n}\right)$$

$$= \frac{(n-1)!}{r!(n-r-1)!} + \frac{(n-1)!}{(r-1)!(n-r)!}$$

$$= C(n-1, r) + C(n-1, r-1)$$

Method 2 :

Here we observe that the LHS is the coefficient of x^k in the expansion of $(1+x)^n$. Hence in order to prove the identity, we show that the RHS is also the coefficient of x^k in the expansion of $(1+x)^n$.

Now, $(1+x)^n = (1+x)^{n-1}(1+x)$

$$= (1+x)^{n-1} + x(1+x)^{n-1}$$

i.e. the coefficient of x^k in $(1+x)^n$

= coefficient of x^k in $(1+x)^{n-1}$ + coeff. of x^k in $x(1+x)^{n-1}$

= coefficient of x^k in $(1+x)^{n-1}$ + coeff. of x^{k-1} in $(1+x)^{n-1}$

$$\therefore \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

i.e. $C(n, r) = C(n-1, r) + C(n-1, r-1)$

SUMMER EXAMINATION - 2015
B.E. IV SEM. (CT, CS, IT, CE) (CBS)
DISCRETE MATHEMATICS AND GRAPH THEORY

Statistical Analysis

Unit Number	Marks
Unit - I	20 M
Unit - II	36 M
Unit - III	24 M
Unit - IV	24 M
Unit - V	36 M
Unit - VI	20 M
Questions covered in VBD	160 M (100 %)

SUMMER EXAM.- 2015

QUESTION PAPER

Q.1.(a) Let U = {a, b, c, d, e, f} and A = {a, b, c}, B = {b, e, f}, find :

(i) $B^c - A^c$ (ii) $(A \cap B)'$. **4M**

(b) Prove the logical equivalence by using algebra of proposition :

$[p \wedge (\sim p \vee q)] \vee [q \wedge \sim (p \wedge q)] = q$. **4M**

OR

(c) Write the negation of the following statement :

"If the sky is cloudy then it rains and if it rains then the sky is cloudy." **2M**

OR

Q.2.(a) Is the following argument valid?

"If Roli has completed B.E., then she is assured for a good job. If Roli is assured of a good job, she is happy. Roli is not happy. So Roli has not completed B.E." **6M**

(b) Prove by mathematical induction method that $3^{4n+2} + 5^{2n+1}$ is a multiple of 14. **4M**

Q.3.(a) The relation R defined on set A = {0, 1, 2, 3} is R = {(0, 1), (1, 2), (2, 3)}. Find the transitive closure of R. **6M**

(b) Define characteristic function and prove that :

(i) $f_{A'}(x) = 1 - f_A(x)$

(ii) $f_{A \cup B}(x) = f_A(x) + f_B(x) - f_A(x) . f_B(x)$ for all x. **6M**

(c) Let X = {1, 2, 3, 6, 12, 24, 36} and the relation $\leq$ be such that $x \leq y$ if "x divides y". Draw the Hasse diagram. **6M**

OR

Q.4.(a) Let F be the set of all 1–1 onto mapping from x onto x where x = {1, 2, 3}. Find all elements of F and write the inverse of each element. **6M**

(b) Let X = {ball, bed, dog, egg, let} and the relation R = {(x, y) : x, y $\in$ X, xRy if x & y contain some common letter}. Prove that R is compatible and write the relation matrix. **6M**

(c) If A = {1, 2}, B = {2, 3} and C = {3, 4}, then prove that :

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(ii) $(A - B) \times C = (A \times C) - (B \times C)$. **6M**

Q.5.(a) Show that the set of matrices :

$$A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, \alpha \in R \text{ forms a monoid.}$$ **6M**

(b) Prove that any two right cosets of a subgroup H are either disjoint or identical. **6M**

OR

Q.6.(a) Prove that the order of each subgroup of a finite group is a divisor of the order of the group. **6M**

(b) Prove that the set {0, 1, 2, 3, 4} is a finite abelian group of order 5 under addition modulo 5 as a composition. **6M**

Q.7.(a) Prove that the set S = {0, 1, 2, 3, 4} is a ring with respect to the operation of addition and multiplication modulo 5. **6M**

(b) Define a lattice. Draw the Hasse diagram of lattices D_{20} and D_{30}. **6M**

OR

Q.8.(a) Show that $S = \{a + b\sqrt{2} ; a, b \in Z\}$ for the operation +, × is an integral domain. **6M**

(b) Construct switching circuit for the Boolean expression (A . B) + C + (A′.C′). Simplify this and construct an equivalent simplified circuit. **6M**

Q.9.(a) Draw the digraph corresponding to adjacency matrices :

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Are these two digraphs isomorphic? **6M**

(b) Define :

(a) Pseudo graph

(b) Reachable node

(c) Pendent node

(d) Strongly connected graph

(e) Euler path

(f) Forest. **6M**

(c) Construct the tree of the following expression :

$[5(1-x)] \div [\{5-(y+3)\} . \{7+(x \div y)\}]$.

Also, draw the corresponding binary tree. **6M**

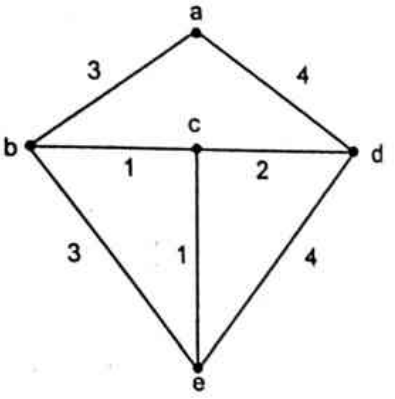
OR

Q.10.(a) Draw the digraph corresponding to matrix :

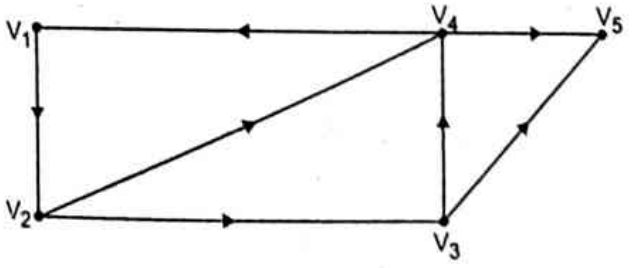
$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

and interpret AA^T, A^TA and A^2. **6M**

(b) Apply Kruskal's algorithm to construct a minimal spanning tree for the weighted graph given below : **6M**



(c) Find all indegrees and outdegrees of the graph given below :



Give all elementary cycles of this graph. Is there any source or sink? **6M**

Q.11.(a) Solve the following recurrence relation :

$a_n - 9a_{n-1} + 20a_{n-2} = 0, \ a_0 = -3, a_1 = -10.$ **5M**

(b) Find the minimum number of elements that one needs to take from the set $S = \{1, 2, 3, \ldots\ldots, 9\}$ to be sure that two of the numbers add up to 10. **5M**

OR

Q.12.(a) Prove that : $C(n+1, r) = C(n, r) + C(n, r-1)$. **5M**

(b) Find the generating function of the sequence $\{a_k\}$, if $a_k = 2 + 3k$. **5M**

SUMMER EXAM.-2015

SOLUTION

(Unit-Wise)

UNIT - I

(Total Marks : 20)

Q.1.(a) Let $U = \{a, b, c, d, e, f\}$ and $A = \{a, b, c\}$, $B = \{b, e, f\}$, find :

(i) $B' - A'$

(ii) $(A \cap B)'$. **4M**

Ans. See P.1-41, Ex.73.

Q.1.(b) Prove the logical equivalence by using algebra of proposition :

$[p \wedge (\sim p \vee q)] \vee [q \wedge \sim (p \wedge q)] = q$. **4M**

Ans. See P.1-19, Ex.29(iii).

Q.1.(c) Write the negation of the following statement :

"If the sky is cloudy then it rains and if it rains then the sky is cloudy." **2M**

Ans. Let p : The sky is cloudy.

q : It rains.

$\therefore$ Given statement is $(p \rightarrow q) \wedge (q \rightarrow p)$

Its negation is $\sim[(p \rightarrow q) \wedge (q \rightarrow p)]$

$= \sim (p \rightarrow q) \vee \sim (q \rightarrow p)$

$= p \wedge \sim q \vee q \wedge \sim p$

$\therefore$ The sky is cloudy and it does not rain or it rains and the sky is not cloudy. ....Ans.

Q.2.(a) Is the following argument valid?

"If Roli has completed B.E., then she is assured for a good job. If Roli is assured of a good job, she is happy. Roli is not happy. So Roli has not completed B.E." **6M**

Ans. See P.1-28, Ex.48.

Q.2.(b) Prove by mathematical induction method that $3^{4n+2} + 5^{2n+1}$ is a multiple of 14. **4M**

Ans. See P.1-44, Ex.85(vii).

UNIT - II

(Total Marks : 36)

Q.3.(a) The relation R defined on set A = {0, 1, 2, 3} is R = {(0, 1), (1, 2), (2, 3)}. Find the transitive closure of R. **6M**

Ans. Given : A = {0, 1, 2, 3}

$R = \{(0, 1), (1, 2), (2, 3)\}$

$\therefore R^2 = R \circ R = \{(0, 1), (1, 2), (2, 3)\} \circ \{(0, 1), (1, 2), (2, 3)\}$

$\therefore R^2 = \{(0, 2), (1, 3)\}$

Now, $R^3 = R^2 \circ R = \{(0, 2), (1, 3)\} \circ \{(0, 1), (1, 2), (2, 3)\}$

$\therefore R^3 = \{(0, 3)\}$

Also $R^4 = R^3 \circ R = \{\phi\}$

$\therefore R^* = R \cup R^2 \cup R^3 \cup R^4 \cup \ldots\ldots$

$\therefore R^* = \{(0,1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3)\}$

which is the required transitive closure ....Ans.

Q.3.(b) Define characteristic function and prove that :

(i) $f_{A'}(x) = 1 - f_A(x)$

(ii) $f_{A \cap B}(x) = f_A(x) \cdot f_B(x) - f_A(x) \cdot f_B(x)$

for all x **6M**

Ans. See P.2-38, Ex.66.

(i) We know that

$f_{A'}(x) = 1 \Leftrightarrow x \in A'$

$\Leftrightarrow x \notin A$

$\Leftrightarrow f_A(x) = 0$

$\Leftrightarrow 1 - f_A(x) = 1$

Thus $\boxed{f_{A'}(x) = 1 - f_A(x)}$ Ans.

Q.3.(c) Let X = {1, 2, 3, 6, 12, 24, 36} and the relation ≤ be such that x ≤ y if "x divides y". Draw the Hasse diagram. **6M**

Ans. See P.2-29, Ex.46.

Q.4.(a) Let F be the set of all 1–1 onto mapping from X onto X where X = {1, 2, 3}. Find all elements of F and write the inverse of each element. **6M**

Ans. See P.2-34, Ex.56.

Q.4.(b) Let X = {ball, bed, dog, egg, let} and the relation R = {(x, y) : x, y ∈ X, xRy if x & y contain some common letter}. Prove that R is compatible and write the relation matrix. **6M**

Ans. See P.2-23, Ex.32.

Q.4.(c) If A = {1, 2}, B = {2, 3} and C = {3, 4}, then prove that :

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(ii) $(A - B) \times C = (A \times C) - (B \times C)$. **6M**

Ans. Given : A = {1, 2}, B = {2, 3}, C = {3, 4}

(i) We have $B \cup C = \{2, 3, 4\}$

$\therefore A \times (B \cup C) = \{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4)\}$ ....(1)

Also,

$A \times B = \{(1, 2), (1, 3), (2, 2), (2, 3)\}$

& $A \times C = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$

$\therefore (A \times B) \cup (A \times C) = \{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4)\}$ ....(2)

$\therefore$ From (1) and (2), we get

$\boxed{A \times (B \cup C) = (A \times B) \cup (A \times C)}$ Ans.

(ii) We have $A - B = \{1\}$

$\therefore (A - B) \times C = \{(1, 3), (1, 4)\}$ ....(3)

Also, $B \times C = \{(2, 3), (2, 4), (3, 3), (3, 4)\}$

$\therefore (A \times C) - (B \times C) = \{(1, 3), (1, 4)\}$ ....(4)

$\therefore$ (3) & (4) $\Rightarrow$

$\boxed{(A - B) \times C = (A \times C) - (B \times C)}$ Ans.

UNIT - III

(Total Marks : 24)

Q.5.(a) Show that the set of matrices :

$$A_a = \begin{bmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{bmatrix}, a \in R \text{ forms a monoid.}$$ **6M**

Ans. See P.3-8, Ex.1.

Q.5.(b) Prove that any two right cosets of a subgroup H are either disjoint or identical. **6M**

Ans. Let G be a group and H be a subgroup of G. Again let Ha and Hb are two right cosets of H in G, such that $Ha \cap Hb \neq \phi$ i.e. they are disjoint.

$\Rightarrow$ At least one element say $y \in Ha \cap Hb$

$\Rightarrow y \in Ha$ and $y \in Hb$

Let $y = h_1 a$ and $y = h_2 b$ for some $h_1, h_2 \in H$

$\Rightarrow h_1 a = h_2 b$

$\Rightarrow h_1^{-1} h_1 a = h_1^{-1} h_2 b$

$\Rightarrow ea = h_1^{-1} h_2 b$

$\Rightarrow a = h_1^{-1} h_2 b$ ....(i)

Since H is a subgroup of G, so if $h_1, h_2 \in H$

$\Rightarrow h_1^{-1} h_2 \in H$

So for any $h \in H$, we have

$ha = h(h_1^{-1} h_2 b)$

$= (h h_1^{-1} h_2 b)$ $[\because hh_1^{-1}, h_2 \in H]$

$\Rightarrow ha \in Hb$

So $Ha \subseteq Hb$ (as $ha \in Ha$)

Similarly, we can show that $Hb \subseteq Ha$. So by the two, $Ha = Hb$. So if $Ha \cap Hb \neq \phi$, then $Ha = Hb$. ....Ans.

Q.6.(a) Prove that the order of each subgroup of a finite group is a divisor of the order of the group. **6M**

Ans. See P.3-28, Ex.45.

Q.6.(b) Prove that the set {0, 1, 2, 3, 4} is a finite abelian group of order 5 under addition modulo 5 as a composition. **6M**

Ans. See P.3-14, Ex.10.

UNIT - IV

(Total Marks : 24)

Q.7.(a) Prove that the set S = {0, 1, 2, 3, 4} is a ring with respect to the operation of addition and multiplication modulo 5. **6M**

Ans. Given : S = {0, 1, 2, 3, 4}

Composition table for $(R, +_5)$:

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

R_1 : We prove that $(R, +)$ is an abelian group.

G_1 : From table, we observe that $a + b \in S, \forall a, b \in S$

$\therefore$ G_1 is verified.

G_2 : Consider $1 + (2 + 4) = (1 + 2) + 4$

$1 + 1 = 3 + 4$

$2 = 2$

$\therefore a + (b + c) = (a + b) + c, \forall a, b, c \in S$

$\therefore$ G_2 is verified.

G_3 : Here $0 \in S$ s.t. $0 + a = a = a + 0, \forall a \in S$

$\therefore$ 0 is the identity element.

G_4 : Inverses of 0, 1, 2, 3, 4 are 0, 4, 3, 2, 1 resp.

$\therefore$ G_4 is verified.

G_5 : Consider $2 + 4 = 4 + 2$

$1 = 1$

$\therefore a + b = b + a, \forall a, b \in S$

$\therefore$ G_5 is verified

$\therefore$ R_1 is verified.

Composition table for $(R, \times_5)$:

$\times_5$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

R_2 : Consider $2 \times (3 \times 4) = (2 \times 3) \times 4$

$2 \times 2 = 1 \times 4$

$4 = 4$

$\therefore a \times (b \times c) = (a \times b) \times c, \forall a, b, c \in S$

Also $a, b \in S \Rightarrow ab \in S$

$\therefore$ R_2 is verified

R_3 : We observe that

$a . (b + c) = a . b + a . c$

and $(b + c) . a = b . a + c . a, \forall a, b, c \in S$

$\therefore$ R_3 is verified

$\therefore$ $\boxed{(R, +_5, \times_5) \text{ is a ring.}}$ Ans.

Q.7.(b) Define a lattice. Draw the Hasse diagram of lattices D_{20} and D_{30}. **6M**

Ans. See P.4-24, 28, Ex.39.

Q.8.(a) Show that $S = \{a + b\sqrt{2} ; a, b \in Z\}$ for the operation +, × is an integral domain. **6M**

Ans. See P.4-15, Ex.11.

Q.8.(b) Construct switching circuit for the Boolean expression $(A . B) + C + (A' . C')$. Simplify this and construct an equivalent simplified circuit. **6M**

Ans. See P.4-34, Ex.52.

UNIT - V

(Total Marks : 36)

Q.9.(a) Draw the digraph corresponding to adjacency matrices:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Are these two digraphs isomorphic? **6M**

Ans. $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

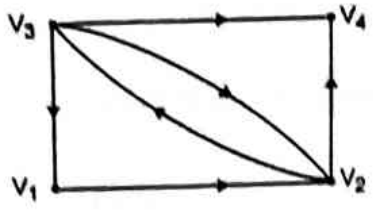


Fig.

$$B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

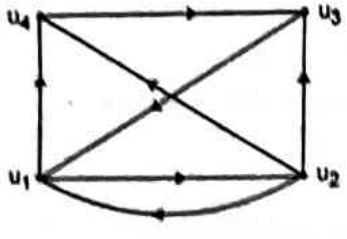


Fig.

We define the correspondence as :

$u_1 \to V_2, u_2 \to V_3$

$u_3 \to V_4, u_4 \to V_1$

Also, $(u_1, u_2) \to (V_2, V_3)$

By the above correspondence these two diagraphs are not isomorphic in nature. .....Ans.

Q.9.(b) Define :

(a) Pseudo graph **1M**

Ans. See P.5-14, Point (xi).

(b) Reachable node **1M**

Ans. See P.5-23.

(c) Pendent node **1M**

Ans. See P.5-15, Point (4).

(d) Strongly connected graph **1M**

Ans. See P.5-29, Point(iii).

(e) Euler path **1M**

Ans. See P.5-30.

(f) Forest **1M**

Ans. Collection of tree or set of n > = 0 disjoint sets is known as forest.

Q.9.(c) Construct the tree of the following expression :

$[5(1 - x)] + [\{5 - (y + 3)\} . \{7 + (x + y)\}]$.

Also, draw the corresponding binary tree. **6M**

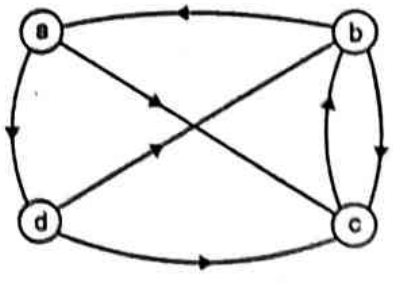
Ans. See Ex.48, P.5-46.

Q.10.(a) Draw the digraph corresponding to matrix :

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

and interpret AA^T, A^TA and A^2. **6M**

Ans.



$$A^T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \; AA^T = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \; AA^T = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

Vertex	Indegree	outdegree
a	1	2
b	2	2
c	3	1
d	1	2

Diagonal of AA^T is same as outdegree of a, b, c, d respectively.

$$\text{Now } A^TA = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

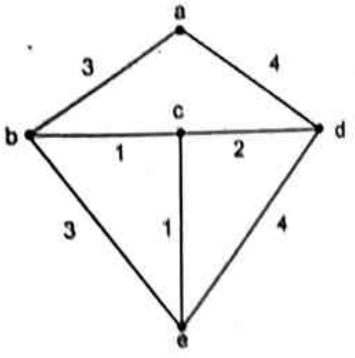
$$\therefore \; A^TA = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 1 & 3 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Diagonal of A^TA is same as indegree of a, b, c, d respectively.

$$\text{Now } A^2 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\therefore \; A^2 = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \text{ .....Ans.}$$

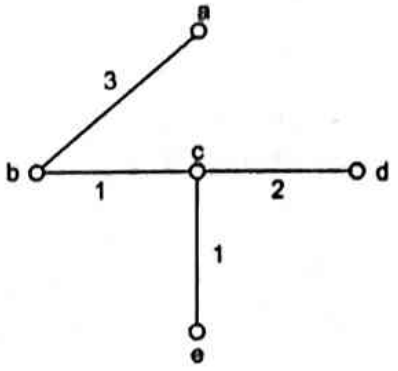
Q.10.(b) Apply Kruskal's algorithm to construct a minimal spanning tree for the weighted graph given below : **6M**



Ans.

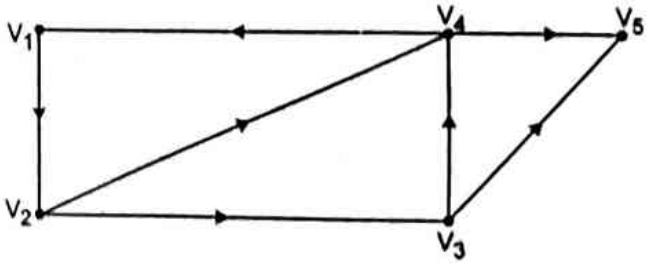
Edge	Weight
(b, c)	1
(c, e)	1
(c, d)	2
(b, a)	3
(b, e)	3
(a, d)	4
(d, e)	4

∴ Required minimal spanning tree is



Total weight = 3 + 1 + 2+ 1 = 7

Q.10.(c) Find all indegrees and outdegrees of the graph given below :



Give all elementary cycles of this graph. Is there any source or sink? **6M**

Ans.

Vertex	Indegree	Outdegree
V_1	1	1
V_2	1	2
V_3	1	2
V_4	2	1
V_5	2	0

Cycles : (1) $V_1 - V_2 - V_4 - V_1$

(2) $V_1 - V_2 - V_3 - V_4 - V_1$

(3) $V_2 - V_4 - V_1 - V_2$

(4) $V_2 - V_3 - V_4 - V_1 - V_2$

(5) $V_3 - V_4 - V_1 - V_2 - V_3$

(6) $V_4 - V_1 - V_2 - V_4$

(7) $V_4 - V_1 - V_2 - V_3 - V_4$

Source : A vertex with indegree zero is called a source. There is no source.

Sink : A vertex with outdegree zero is called a sink. V_5 is the sink.

....Ans.

UNIT - VI

(Total Marks : 20)

Q.11.(a) Solve the following recurrence relation :

$a_n - 9a_{n-1} + 20a_{n-2} = 0, \quad a_0 = -3, a_1 = -10.$ **5M**

Ans. See P.6-18, Ex.33(a).

Q.11.(b) Find the minimum number of elements that one needs to take from the set S = {1, 2, 3, ..., 9} to be sure that two of the numbers add up to 10. **5M**

Ans. **Generalized Pigeon-Hole Principle :**

If m pigeons are placed into n nesting holes, then one of the nesting holes must contain atleast $\left\lfloor \frac{m-1}{n} \right\rfloor + 1$ pigeons.

Here m = 9, n = 2

∴ By generalized pigeon-hole principle

$$\left\lfloor \frac{m-1}{n} \right\rfloor + 1 = \left\lfloor \frac{9-1}{2} \right\rfloor + 1 = 4 + 1 = 5$$

∴ $\boxed{\text{Required number of elements is 5}}$ **Ans.**

Q.12.(a) Prove that : C(n + 1, r) = C(n, r) + C(n, r – 1). **5M**

Ans. **To Prove :** C(n + 1, r) = C(n, r) + C(n, r – 1)

We prove that $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$

See P.PS-26, Q.12.(b), Winter-14.

Now replacing n by n + 1, we get

$$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$$

∴ $\boxed{C(n+1, r) = C(n, r) + C(n, r-1)}$ **Ans.**

Q.12.(b) Find the generating function of the sequence $\{a_k\}$ if $a_k = 2 + 3k$. **5M**

Ans. See P.6-16, Ex.31.

WINTER EXAMINATION - 2015
B.E. IV SEM. (CT, CS, IT, CE) (CBS)
DISCRETE MATHEMATICS AND GRAPH THEORY

Statistical Analysis

Unit Number	Marks
Unit - I	20 M
Unit - II	36 M
Unit - III	24 M
Unit - IV	24 M
Unit - V	36 M
Unit - VI	20 M
Questions covered in VBD	160 M (100 %)

WINTER EXAM.- 2015
QUESTION PAPER

Q.1.(a) Determine the validity of following argument by using truth table :

"If I try hard and I have a talent, then I will become a scientist. If I become scientist, then I will be happy. Therefore, if I will not be happy, then I did not try hard or I do not have talent." **5M**

(b) Prove that $\sqrt{3}$ is irrational. **5M**

OR

Q.2.(a) Using mathematical induction show that, for all positive integers n, $(n^3 - 4n + 6)$ is divisible by 3. **5M**

(b) Using laws of algebra of sets, prove that :

$A - B = A \cap B^c$.

where A and B are subsets of universal set. **5M**

Q.3.(a) If R is an equivalene relation on set A, prove that R^{-1} is an equivalence relation. **6M**

(b) Let A = {1, 2, 3, 4} and let R = {(1, 2), (2, 3), (3, 4), (2, 1)}, find the transitive closure of R. **6M**

(c) If $f : X \to Y$ and $g : Y \to Z$ and both f and g are one-one onto, then show that $g \circ f$ is also one-one onto and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. **6M**

OR

Q.4.(a) Let A be the set of non-zero integes and let R be the relation on A × A defined by :

$(a, b) R (c, d) \Leftrightarrow ad = bc$

Show that R is an equivalence relation. **6M**

(b) Let A be given finite set and P(A) be its power set. Let " $\subseteq$ " be the relation of P(A). Draw Hasse diagram of $(P(A), \subseteq)$ for (i) A = {a, b}, (ii) A = {a, b, c} **6M**

(c) Let R and S be the relations on {1, 2, 3, 4} defined by :

R = {(1, 1), (1, 2), (3, 4), (4, 2)} and

S = {(1, 1), (2, 1), (3, 1), (4, 4), (2, 2)}

Find RoS, M_{RoS}. Draw diagraph of RoS. **6M**

Q.5.(a) Prove that the set {0, 1, 2, 3, 4} is a finite abelian group of order 5 under addition modulo 5 as a composition. **6M**

(b) Does the following table defined a semigroup or a monoid? **6M**

*	a	b	c
a	c	b	a
b	b	c	b
c	a	b	c

OR

Q.6.(a) Prove that $f : R \to R_+$ denoted by $f(x) = e^x$ an isomorphism of R onto R_+. Here R is the additive group of real numbers and R_+ is the multi-plicative group of positive real numbers. **6M**

(b) Prove that any two right cosets of a subgroup H are either disjoint or identical. **6M**

Q.7.(a) Show that $(z_6, +_6, \times_6)$ is a ring. Explain whether it is an integral domain. **7M**

(b) Show that the intersection of two subrings of a ring R is a subring. **5M**

OR

Q.8.(a) Show that (I, ⊕, ⊙) is a commutative ring with identity where the operations ⊕ and ⊙ are defined as :

for any a, b ∈ I, a ⊕ b = a + b − 1

and a and ⊙ b = a + b − ab

where I is set of integers. **7M**

(b) Construct a circuit for the Boolean expression $(A.B)+[A'.(A+B+B')]$. Simplify and construct the equivalent circuit. **5M**

Q.9.(a) Define :

(i) Complete graph

(ii) Regular graph.

(iii) Isomorphic graph. **6M**

(b) Draw the diagraph corresponding to the adjacency matrix :

$$M_g = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

find transitive closure of the diagraph. **6M**

(c) Draw a directed tree with 4 nodes at level 1, six nodes at level 2. Obtain the corresponding binary tree. **6M**

OR

Q.10.(a) Define :

(i) Diagraph

(ii) Indegree and outdegree of a node

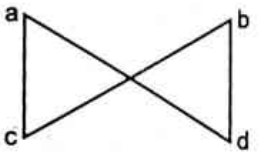
(iii) Path and cycle. **6M**

(b) Draw diagraph corresponding to :

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

and interpret AA^T, A^TA and A^2 **6M**

(c) Use Prim's algorithm to find the spanning tree for the following graph : **6M**



Q.11.(a) Find the generating function for the sequence $1, a, a^2, \ldots$, where a is a fixed constant. **5M**

(b) Find the minimum number of students in a class to be sure that four out of them are born in the same month. (Use Pigeonhole principle). **3M**

(c) Three persons enter into car, where there are 5 seats. In how many ways can they take up their seats? **2M**

OR

Q.12.(a) Show that if any five integers from 1 to 8 are chosen, then at least two of them will have a sum 9. **4M**

(b) Solve $a_n = a_{n-1} + 2a_{n-2}, n \geq 2$ with initial conditions $a_0 = 0, a_1 = 1$. **6M**

WINTER EXAM.-2015

SOLUTION

(Unit-Wise)

UNIT - I

(Total Marks : 20)

Q.1.(a) Determine the validity of following argument by using truth table :

"If I try hard and I have a talent, then I will become a scientist. If I become scientist, then I will be happy. Therefore, if I will not be happy, then I did not try hard or I do not have talent." **5M**

Ans. **See P.1-26, Ex.45.**

Q.1.(b) Prove that $\sqrt{3}$ is irrational. **5M**

Ans. **See P.1-28, Ex.50.**

Q.2.(a) Using mathematical induction show that, for all positive integers n, $(n^3 - 4n + 6)$ is divisible by 3. **5M**

Ans. Let $p(n): n^3 - 4n + 6$

$p(0) = 0 - 0 + 6 = 6$, which is divisible by 3.

$\therefore$ p(0) is true

Let $p(m): m^3 - 4m + 6$ be true.

We prove that p(m + 1) is also true

Now, $P(m+1) = (m+1)^3 - 4(m+1) + 6$

$= m^3 + 3m^2 + 3m + 1 - 4m - 4 + 6$

$= (m^3 - 4m + 6) + (3m^2 + 3m - 3)$

$= (m^3 - 4m + 6) + 3(m^2 + m - 1)$

Both $(m^3 - 4m + 6)$ (i.e. p(m)) and $3(m^2 + m - 1)$ are divisible by 3

$\therefore$ p(m + 1) is also divisible by 3

$\therefore$ $\boxed{\text{By principle of M.I. p(n) is true for all n}}$ **Ans.**

Q.2.(b) Using laws of algebra of sets, prove that :

$A - B = A \cap B^c$

where A and B are subsets of universal set. **5M**

Ans. To prove : $A - B = A \cap B^c$

Let $x \in A - B$

$\Rightarrow x \in A$ and $x \notin B$

$\Rightarrow x \in A$ and $x \in B^c$

$\Rightarrow x \in A \cap B^c$

$\Rightarrow A - B \subseteq A \cap B^c$ ....(1)

Again Let $x \in A \cap B^c$

$\Rightarrow x \in A$ and $x \in B^c$

$\Rightarrow x \in A$ and $x \notin B$

$\Rightarrow x \in A - B$

$\therefore A \cap B^c \subseteq A - B$ .....(2)

From (1) and (2), we get

$\boxed{A - B = A \cap B^c}$ **Ans.**

UNIT - II

(Total Marks : 36)

Q.3.(a) If R is an equivalene relation on Set A, prove that R^{-1} is an equivalence relation. **6M**

Ans. Given : R is an equivalence relation on A

To prove : R^{-1} is also an equivalence relation.

Let A = {a, b, c}

Define R = {(a, a), (a, b), (b, a), (b, b), (c, c), (a, c), (c, a)}

Clearly R is an equivalence relation

$R^{-1} = \{(a, a), (b, a), (a, b), (b, b), (c, c), (c, a), (a, c)\}$

We observe that $(x, x) \in R^{-1}, \forall x \in A$

$\therefore$ R^{-1} is reflexive.

Also $(x, y) \in R^{-1} \Rightarrow (y, x) \in R^{-1}, \forall x, y \in A$

$\therefore$ R^{-1} is symmetric

Also $(x, y), (y, z) \in R^{-1} \Rightarrow (x, z) \in R^{-1}, \forall x, y, z \in A$

$\therefore$ R^{-1} is transitive

$\therefore$ $\boxed{R^{-1} \text{ is an equivalence relation.}}$

Q.3.(b) Let A = {1, 2, 3, 4} and let R = {(1, 2), (2, 3), (3, 4), (2, 1)} find the transitive closure of R. **6M**

Ans. See P.2-22, Ex.31.

Q.3.(c) If $f : X \to Y$ and $g : Y \to Z$ and both f and g are one-one onto, then show that gof is also one-one onto and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. **6M**

Ans. See P.2-33, Theorem.

Q.4.(a) Let A be the set of non-zero integes and let R be the relation on A × A defined by:

$(a, b) R (c, d) \Leftrightarrow ad = bc$

Show that R is an equivalence relation. **6M**

Ans. See P.2-17, Ex.18.

Q.4.(b) Let A be given finite set and P(A) be its power set. Let " $\subseteq$ " be the relation of P(A). Draw Hasse diagram of $(P(A), \subseteq)$ for (i) A = {a, b}, (ii) A = {a, b, c} **6M**

Ans. See P.2-30, Ex.50.

Q.4.(c) Let R and S be the relations on {1, 2, 3, 4} defined by :

R = {(1, 1), (1, 2), (3, 4), (4, 2)} and

S = {(1, 1), (2, 1), (3, 1), (4, 4), (2, 2)}

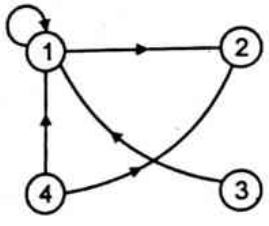
Find RoS, M_{RoS}. Draw diagraph of RoS. **6M**

Ans. Given : R = {(1, 1), (1, 2), (3, 4), (4, 2)}

S = {(1, 1), (2, 1), (3, 1), (4, 4), (2, 2)}

$\therefore$ RoS = {(1, 1), (1, 2), (3, 4), (4, 1), (4, 2)}

$$M_{RoS} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 1 & 1 & 0 & 0 \end{array}$$



Graph of RoS

UNIT - III

(Total Marks : 24)

Q.5.(a) Prove that the set {0, 1, 2, 3, 4} is a finite abelian group of order 5 under addition modulo 5 as a composition. **6M**

Ans. See P.3-14, Ex.10.

Q.5.(b) Does the following table defined a semigroup or a monoid? **6M**

*	a	b	c
a	c	b	a
b	b	c	b
c	a	b	c

Ans. G1 : From the table, we observe that

$a * b \in G, \forall a, b \in G$

$\therefore$ G1 is verified.

G2 : We observe that

$(a * b) * c = b * c = b$

Also $a * (b * c) = a * b = b$

$\therefore$ $(a * b) * c = a * (b * c), \forall a, b, c \in G$

$\therefore$ G2 is verified

G3 : Here identity element does not exist.

Such that $a * e = a = e * a, \forall a \in G$

$\therefore$ G3 does not hold

$\therefore$ $\boxed{\text{G is a semigroup.}}$ **Ans.**

Q.6.(a) Prove that $f : R \to R_+$ denoted by $f(x) = e^x$ an isomorphism of R onto R_+. Here R is the additive group of real numbers and R_+ is the multiplicative group of positive real numbers. **6M**

Ans. **See P.3-38, Ex.71.**

Q.6.(b) Prove that any two right cosets of a subgroup H are either disjoint or identical. **6M**

Ans. **See P.PS-30, Q.5(b).**

UNIT - IV

(Total Marks : 24)

Q.7.(a) Show that $(z_6, +_6, \times_6)$ is a ring. Explain whether it is an integral domain. **7M**

Ans. **See P.4-10, Ex.4(a).**

Q.7.(b) Show that the intersection of two subrings of a ring R is a subring. **5M**

Ans. **See P.4-19, Ex.15.**

Q.8.(a) Show that $(I, \oplus, \circ)$ is a commutative ring with identity where the operations $\oplus$ and $\circ$ are defined as :

for any $a, b \in I$, $a \oplus b = a + b - 1$

and a and $\circ$ b = a + b − ab

where I is set of integers. **7M**

Ans. **See P.PS-10, Q.7(b).**

Q.8.(b) Construct a circuit for the Boolean expression $(A.B)+[A'.(A+B+B')]$. Simplify and construct the equivalent circuit. **5M**

Ans. **See P.4-33, Ex.50.**

UNIT - V

(Total Marks : 36)

Q.9.(a) Define (i) Complete graph

(ii) Regular graph.

(iii) Isomorphic graph. **6M**

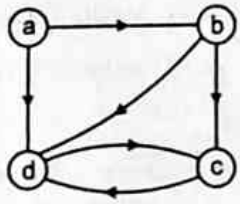
Ans. **See P.5-20(1),(3), 18.**

Q.9.(b) Draw the diagraph corresponding to the adjacency matrix :

$$M_g = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

find transitive closure of the diagraph. **6M**

Ans.



Here R = {(a, b), (a, d), (b, c), (b, d), (c, d), (d, c)}

$R^2 = R \circ R = \{(a, b), (a, d), (b, c), (b, d), (c, d), (d, c)\}$

$\circ \{(a, b), (a, d), (b, c), (b, d), (c, d), (d, c)\}$

$\therefore R^2 = \{(a, c), (a, d), (b, d), (b, c), (c, c), (d, d)\}$

$R^3 = R^2 \circ R = \{(a, c), (a, d), (b, d), (b, c), (c, c), (d, d)\}$

$\circ \{(a, b), (a, d), (b, c), (b, d), (c, d), (d, c)\}$

$\therefore R^3 = \{(a, d), (a, c), (b, c), (b, d), (c, d), (d, c)\}$

$R^4 = R^3 \circ R = \{(a, d), (a, c), (b, c), (b, d), (c, d), (d, c)\}$

$\circ \{(a, b), (a, d), (b, c), (b, d), (c, d), (d, c)\}$

$\therefore R^4 = \{(a, c), (a, d), (b, d), (b, c), (c, c), (d, d)\}$

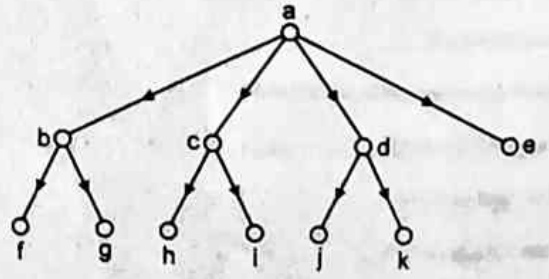
$\therefore$ Transitive closure

$R^+ = R \cup R^2 \cup R^3 \cup R^4 \cup \ldots\ldots$

$\therefore R^+ = \{(a, b), (a, c), (a, d), (b, c), (b, d), (c, c), (c, d), (d, c), (d, d)\}$

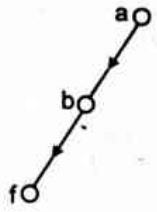
Q.9.(c) **Draw a directed tree with 4 nodes at level 1, six nodes at level 2. Obtain the corresponding binary tree.** **6M**

Ans.

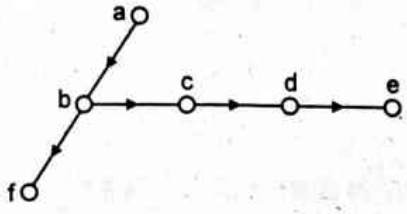


Corresponding Binary Tree :

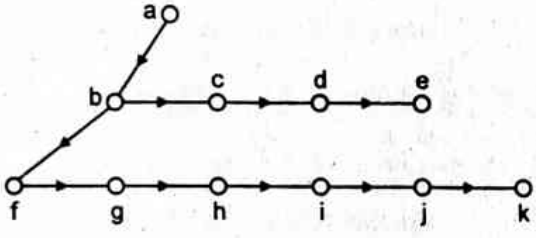
Step (1) :



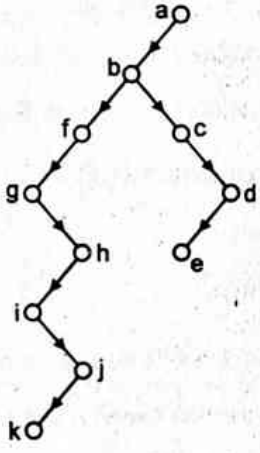
Step (2) :



Step (3) :



Step (4) :



Q.10.(a) Define : **6M**

(i) Diagraph

Ans. **See P.5-13, (ii)**

(ii) Indegree and outdegree of a node

Ans. **See P.5-15, (xiii)(3)**

(iii) Path and cycle

Ans. **See P.5-22, (i),(vii)**

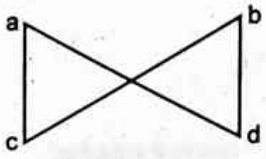
Q.10.(b) Draw diagraph corresponding to

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

and interpret AA^T, A^TA and A^2. **6M**

Ans. **See P.5-23, Ex.8(a).**

Q.10.(c) Use Prim's algorithm to find the spanning tree for the following graph : **6M**



Ans. **See P.5-40, Ex.32.**

UNIT - VI

(Total Marks : 20)

Q.11.(a) Find the generating function for the sequence 1, a, a^2, ..., where a is a fixed constant. **5M**

Ans. **See P.6-21, Ex.34.**

Q.11.(b) Find the minimum number of students in a class to be sure that four out of them are born in the same month. (Use Pigeonhole principle). **3M**

Ans. **See P.6-9, Ex.24.**

Q.11.(c) Three persons enter into car, where there are 5 seats. In how many ways can they take up their seats? **2M**

Ans. The first person has a choice of 5 seats and can sit in any one of those 5 seats. So there are 5 ways of occupying the first seat. The second person has a choice of 4 seats. Similarly the third person has a choice of 3 seats. Hence the required number of ways in which all the three persons can take up their seats is

$^5P_3 = 5 \times 4 \times 3 = 60$.

Q.12.(a) Show that if any five integers from 1 to 8 are chosen, then at least two of them will have a sum 9. **4M**

Ans. **See P.6-10, Ex.25.**

Q.12.(b) Solve $a_n = a_{n-1} + 2a_{n-2}, n \geq 2$ with initial conditions $a_0 = 0, a_1 = 1$. **6M**

Ans. The given recurrence relation is

$$a_n = a_{n-1} + 2a_{n-2},\ n \geq 2,\ a_0 = 0,\ a_1 = 1$$

$$\Rightarrow a_n - a_{n-1} - 2a_{n-2} = 0 \qquad \text{.....(1)}$$

which is a second order linear homogeneous recurrence relation with constant coefficients.

Let $a_n = r^n$ is a solution of (1),

The characteristic equation is

$$r^2 - r - 2 = 0$$

$$\Rightarrow (r-2)(r+1) = 0$$

$\Rightarrow$ $r = 2, -1$ are distinct real roots.

So the general solution is

$$a_n = b_1(2)^n + b_2(-1)^n$$

Again $a_0 = 0$ implies $b_1 + b_2 = 0$

and $a_1 = 1$ implies $2b_1 - b_2 = 1$

On solving above two equations, we get

$b_1 = \frac{1}{3}$ and $b_2 = -\frac{1}{3}$. Hence the explicit solution is given by

$$\therefore \boxed{a_n = \left(\frac{1}{3}\right)2^n - \left(\frac{1}{3}\right)(-1)^n} \text{ Ans.}$$

Notes :-

SUMMER EXAMINATION - 2016
B.E. IV SEM. (CT, CS, IT, CE) (CBS)
DISCRETE MATHEMATICS AND GRAPH THEORY

Statistical Analysis

Unit Number	Marks
Unit - I	20 M
Unit - II	36 M
Unit - III	24 M
Unit - IV	24 M
Unit - V	36 M
Unit - VI	20 M
Questions covered in VBD	160 M (100 %)

SUMMER EXAM. - 2016
QUESTION PAPER

Q.1.(a) If U = {0, 1, 2, 3, 4, 5, 6, 7, 8}, A = {1, 5, 6, 7, 8}, B = {0, 1, 6, 7}, C = {1, 2, 3, 5, 8}, verify that :

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(ii) $(A \cap B)' = A' \cup B'$ **5M**

(b) Using mathematical induction, show that for all positive integers n, $n^3 - 4n + 6$ is divisible by 3. **5M**

OR

Q.2.(a) Check the validity of the following argument by truth table :

"If Roli has completed MBA, then she is assured of a good job. If Roli is assured of a good job, she is happy. Roli is not happy, so Roli has not completed M.B.A. **5M**

(b) In a group of students, 70 have a personal computer, 120 have a personal stereo and 41 have both. How many own at least one of these devices? Draw an appropriate Venn diagram also. **5M**

Q.3.(a) Let A be a set of non-zero integers and let R be a relation on A × A defined by (a, b) R (c, d) ⇔ ad = bc, show that R is an equivalence relation. **7M**

(b) Let A = {0, 1, 2, 3},

(a) Write the matrices of the relation

$R = \{(x, y) : x + y = 3\}$ and $S = \{(x, y) : x + y \le 4\}$

(b) Find : $M_R \odot M_S$

(c) Show that $(M_R \odot M_S)' \ne M'_S \odot M'_R$ **6M**

(c) Consider A = B = C = R and let $f : A \to B$, $g : B \to C$ be defined by $f(x) = x + 9$ and $g(y) = y^2 + 3$, find $(f \circ g)(3)$, $(g \circ f)(3)$, $(f \circ g)(3)$, $(g \circ g)(3)$. **5M**

OR

Q.4.(a) Let A = {a, b, c} and P(A) be its power set. Let ⊆ be the partial order relation on it. Draw Hasse diagram of (P(A), ⊆). **6M**

(b) Consider a relation R defined on A as R = {(1, 2), (2, 3), (3, 4), (2, 1)} and A = {1, 2, 3, 4}. Find transitive closure of R. Also draw diagraph of transitive A closure relation. **6M**

(c) List all possible functions from set X = {a, b, c} to the set Y = {0, 1}. Indicate in each case whether the function is one-one, onto or both. **6M**

Q.5.(a) Prove that the set of matrices of the form $\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}, a \ne 0, a \in R$ is a group with respect to matrix multiplication. Is the group abelian? **6M**

(b) Consider a mapping $f : G \to G$ defined by $f(x) = x^4$, where G is a multiplicative group of non-zero complex numbers. Show that the mapping is homomorphism with kernel f = {1, – 1, i, –i}. **6M**

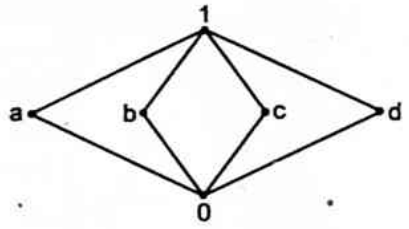
OR

Q.6.(a) Prove that cube roots of unity forms a group under multiplication. **6M**

(b) A homomorphism f from G into G′ with Kernel K is an isomorphism of G into G′ iff K = {e}. **6M**

Q.7.(a) Show that the set of integers with the composition '∘' and '*' defined by $a \circ b = a + b + 1$ and $a * b = ab + a + b$ is a ring. **6M**

(b) Define complemented, modular and distributive lattice. Show that the following lattice is complemented, modular but not distributive. **6M**



OR

Q.8.(a) Construct the switching circuit for the Boolean expression and simplify. Construct equivalent simplified circuit also $(A \cdot B) + [A' \cdot (A + B + B')]$ **6M**

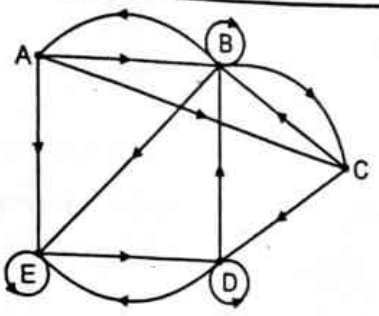
(b) If R is ring such that $a^2 = a, \forall a \in R$, then show that :

(i) $a + a = 0, \forall a \in R$

(ii) $a + b = 0 \Rightarrow a = b, \forall a, b \in R$

(iii) R is commutative ring. **6M**

Q.9.(a) Define in-degree and out-degree of vertex. Find in-degree and out-degree of each vertex of the following directed graph : **6M**



(b) Draw a digraph corresponding to the following adjacency matrix and interpret the results:

$AA^T, A^TA, A^2, A^3, A^4$. **6M**

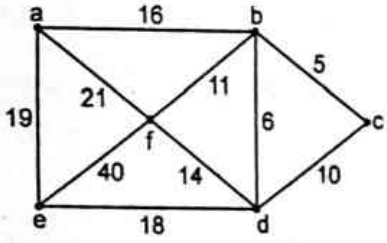
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

(c) Draw the binary tree of the expression :

$((5 * a) + (3 - (6 * a))) + (a - (3 * b))$ **6M**

OR

Q.10.(a) Use Prim's algorithm to find a minimal spanning tree for the given graph: **6M**



(b) Define :

(i) Null graph

(ii) Trail

(iii) Reachable node

(iv) Tree

(v) Height of the tree

(vi) Radius of a graph **6M**

(c) Draw the digraph corresponding to matrix:

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Find its complement and write the matrix of that complemented graph. **6M**

Q.11.(a) How many people must you have guarantee that at least g of them will have birthday in the same day of the week? **5M**

(b) Solve the recurrence relation using generating function :

$a_n = 3a_{n-1} + 2, a_0 = 1$. **5M**

OR

Q.12.(a) Prove that : $C(n+1, r) = C(n, r) + C(n, r-1)$ **5M**

(b) Use generating function technique to solve following recurrence relation :

$a_n - 9a_{n-1} + 20a_{n-2} = 0$ given $a_0 = -3, a_1 = -10$ **5M**

SUMMER EXAM. - 2016

SOLUTION

(Unit-Wise)

UNIT - I

(Total Marks : 20)

Q.1.(a) If U = {0, 1, 2, 3, 4, 5, 6, 7, 8}, A = {1, 5, 6, 7, 8}, B = {0, 1, 6, 7}, C = {1, 2, 3, 5, 8}, verify that :

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(ii) $(A \cap B)' = A' \cup B'$ **5M**

Ans. Given : U = {0, 1, 2, 3, 4, 5, 6, 7, 8},

A = {1, 5, 6, 7, 8},

B = {0, 1, 6, 7},

C = {1, 2, 3, 5, 8}

(i) Now, $B \cap C = \{1\}$

$\Rightarrow A \cup (B \cap C) = \{1, 5, 6, 7, 8\}$ .... (i)

Also, $A \cup B = \{0, 1, 5, 6, 7, 8\}$

and $(A \cup C) = \{1, 2, 3, 5, 6, 7, 8\}$

$\Rightarrow (A \cup B) \cap (A \cup C) = \{1, 5, 6, 7, 8\}$ .... (ii)

From (i) and (ii), we get

$\therefore \boxed{A \cup (B \cap C) = (A \cup B) \cap (A \cup C)}$ **Ans.**

(ii) Now, $A \cap B = \{1, 6, 7\}$

$(A \cap B)' = \{0, 2, 3, 4, 5, 8\}$ ...(iii)

Also, $A' = \{0, 2, 3, 4\}$

and $B' = \{2, 3, 4, 5, 8\}$

$\Rightarrow A' \cup B' = \{0, 2, 3, 4, 5, 8\}$ .... (iv)

From (iii) and (iv), we get

$\therefore \boxed{(A \cap B)' = A' \cup B'}$ Ans.

Q.1.(b) Using mathematical induction, show that for all positive integers n, $n^3 - 4n + 6$ is divisible by 3. **5M**

Ans. Let $P(n): n^3 - 4n + 6$ is divisible by 3

$P(0)$ 6 is divisible by 3

This is true

$\therefore P(0)$ is true

Let $P(m)$ is true i.e.

$P(m): m^3 - 4m + 6$ is divisible by 3

We prove that $P(m + 1)$ is also true.

Now, $P(m+1) = (m+1)^3 - 4(m+1) + 6$

$= m^3 + 3m^2 + 3m + 1 - 4m - 4 + 6$

$= (m^3 - 4m + 6) + (3m^2 + 3m - 3)$

$= (m^3 - 4m + 6) + 3(m^2 + m - 1)$

Both $(m^3 - 4m + 6)$ (i.e. $P(m)$) and $3(m^2 + m - 1)$ are divisible by 3

$\therefore P(m + 1)$ is divisible by 3

$\therefore P(m + 1)$ is true

i.e. $P(m)$ is true $\Rightarrow P(m + 1)$ is also true

$\therefore$ By principle of mathematical induction

$P(n)$ is true for all $n \in N$

$\therefore \boxed{n^3 - 4n + 6 \text{ is divisible by } 3}$ Ans.

Q.2.(a) Check the validity of the following argument by truth table :

"If Roli has completed MBA, then she is assured of a good job. If Roli is assured of a good job, she is happy. Roli is not happy, so Roli has not completed M.B.A. **5M**

Ans. See P.1-28, Ex.48.

Q.2.(b) In a group of students, 70 have a personal computer, 120 have a personal stereo and 41 have both. How many own at least one of these devices? Draw an appropriate Venn diagram also. **5M**

Ans. See P.PS-13, Q.2(a).

UNIT - II

(Total Marks : 36)

Q.3.(a) Let A be a set of non-zero integers and let R be a relation on $A \times A$ defined by (a, b) R (c, d) $\Leftrightarrow$ ad = bc, show that R is an equivalence relation. **7M**

Ans. See P.2-17, Ex.18.

[Replace (x, y) R (u, v) by (a, b) R (c, d)]

Q.3.(b) Let A = {0, 1, 2, 3}

(a) Write the matrices of the relation

$R = \{(x, y) : x + y = 3\}$ and $S = \{(x, y) : x + y \le 4\}$

(b) Find : $M_R \odot M_S$

(c) Show that $(M_R \odot M_S)' \neq M'_S \odot M'_R$ **6M**

Ans. See P.2-21, Ex.29.

Q.3.(c) Consider A = B = C = R and let $f : A \to B$, $g : B \to C$ be defined by $f(x) = x + 9$ and $g(y) = y^2 + 3$, find $(f \circ g)(3)$, $(g \circ f)(3)$, $(f \circ g)(3)$, $(g \circ g)(3)$. **5M**

Ans. Given : $f(x) = x + 9$ and $g(y) = y^2 + 3$ and f and g are mappings as

$f : A \to B$ and

$g : B \to C$

Now, $(f \circ g)(3) = f[g(3)]$

$= f(3^2 + 3) = f(12)$

$\therefore \boxed{(f \circ g)(3) = 21}$ Ans.

$(g \circ f)(3) = g[f(3)] = g[3 + 9]$

$= g[12] = (12)^2 + 3 = 144 + 3$

$\therefore \boxed{(g \circ f)(3) = 147}$ Ans.

$(f \circ f)(3) = f[f(3)] = f[3 + 9]$

$= f[12] = (12) + 9$

$\therefore \boxed{(f \circ f)(3) = 21}$ Ans.

$(g \circ g)(3) = g[g(3)] = g[(3)^2 + 3]$

$= g[12] = (12)^2 + 3$

$\therefore \boxed{(g \circ f)(3) = 147}$ Ans.

Q.4.(a) Let A = {a, b, c} and P(A) be its power set. Let $\subseteq$ be the partial order relation on it. Draw Hasse diagram of (P(A), $\subseteq$). **6M**

Ans. See P.2-30, Ex.50.

Q.4.(b) Consider a relation R defined on A as R = {(1, 2), (2, 3), (3, 4), (2, 1)} and A = {1, 2, 3, 4}. Find transitive closure of R. Also draw diagraph of transitive closure relation. **6M**

Ans. See P.2-22, Ex.31.

Q.4.(c) List all possible functions from set X = {a, b, c} to the set Y = {0, 1}. Indicate in each case whether the function is one-one, onto or both. **6M**

Ans. See P.2-34, Ex.55.

UNIT - III

(Total Marks : 24)

Q.5.(a) Prove that the set of matrices of the form $\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}, a \neq 0, a \in R$ is a group with respect to matrix multiplication. Is the group abelian? **6M**

Ans. Let $G = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \middle| a \in R, a \neq 0 \right\}$

G1 : Let $A = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}, B = \begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix} \in G$

Then, $AB = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix} = \begin{bmatrix} ab & 0 \\ 0 & a^{-1}b^{-1} \end{bmatrix}$

Clearly all element of AB are real

$\therefore$ A, B$\in$G$\Rightarrow$AB$\in$G

$\therefore$ G1 is verified

G2 : Since associative property holds in matrices.

$\therefore$ It holds in G

$\therefore$ G2 is verified.

G3: Let $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Then AI = A = IA, $\forall A \in G$

$\therefore$ Here I is the identity element (matrix)

$\therefore$ G3 is verified

G4: Let $A = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$

Then $A^{-1} = \frac{1}{1-0} \begin{bmatrix} a^{-1} & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^{-1} & 0 \\ 0 & a \end{bmatrix} \in G$

as all entries are real.

Also, $AA^{-1} = I = A^{-1}A$

Therefore for every A $\in$ G there exists multiplicative inverse A^{-1}

$\therefore$ G4 holds in G

G5: Let $A = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}, B = \begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix} \in G$

$\Rightarrow AB = \begin{bmatrix} ab & 0 \\ 0 & a^{-1}b^{-1} \end{bmatrix}$ and

$BA = \begin{bmatrix} ba & 0 \\ 0 & b^{-1}a^{-1} \end{bmatrix} = \begin{bmatrix} ab & 0 \\ 0 & a^{-1}b^{-1} \end{bmatrix} = AB$ *[$\because a, b \in R$]*

$\therefore$ G5 holds true

$\therefore$ $\boxed{\text{G is an abelian group}}$ **Ans.**

Q.5.(b) Consider a mapping $f : G \to G$ defined by $f(x) = x^4$, where G is a multiplicative group of non-zero complex numbers. Show that the mapping is homomorphism with kernel f = {1, – 1, i, –i}. **6M**

Ans. Given : $f : G \to G$ s. t. $f(x) = x^4$

Also, f = {1, – 1, i, – i}

We know that fourth roots of unity forms an abelian group under multiplication.

$\therefore$ G is a group

Now, $f(xy) = (xy)^4$

$= x^4y^4$ *[$\because$ G is abelian]*

$= f(x) . f(y)$

$\therefore$ $\boxed{f : G \to G \text{ is a homomorphism}}$ **Ans.**

Q.6.(a) Prove that cube roots of unity forms a group under multiplication. **6M**

Ans. See P.3-11, Ex.5.

Q.6.(b) A homomorphism f from G into G' with Kernel K is an isomorphism of G into G' iff K = {e}. **6M**

Ans. See P.3-36, Ex.63.

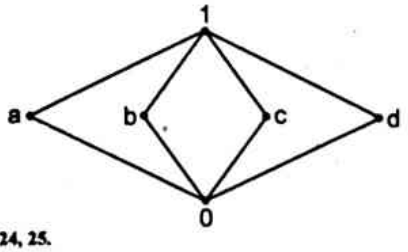
UNIT - IV

(Total Marks : 24)

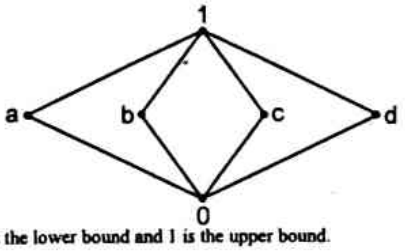
Q.7.(a) Show that the set of integers with the composition 'O' and '*' defined by $a \circ b = a + b + 1$ and $a * b = ab + a + b$ is a ring. **6M**

Ans. See P.4-23, Ex.25.

Q.7.(b) Define complemented, modular and distributive lattice. Show that the following lattice is complemented, modular but not distributive **6M**



Ans. See P.4-24, 25.



Here 0 is the lower bound and 1 is the upper bound.

Also $a \vee d = 1$ and $a \wedge d = 0$

$\therefore$ a and d are complements of each other

Similarly $b \vee c = 1$ and $b \wedge c = 0$

$\therefore$ b and c are complements of each other

Hence the given lattice is a complemented lattice. We also observe that $a \vee (b \wedge c) = (a \vee b) \wedge c$

$\therefore$ The given lattice is modular.

We observe that $a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$

$\therefore$ The given lattice is not distributive. ...Ans.

Q.8.(a) Construct the switching circuit for the Boolean expression and simplify. Construct equivalent simplified circuit also $(A \cdot B) + [A' \cdot (A + B + B')]$ **6M**

Ans. See P.4-33, Ex.50.

Q.8.(b) If R is ring such that $a^2 = a, \forall a \in R$, then show that :

(i) $a + a = 0, \forall a \in R$

(ii) $a + b = 0 \Rightarrow a = b, \forall a, b \in R$

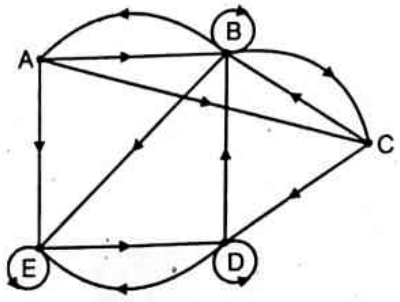
(iii) R is commutative ring. **6M**

Ans. See P.4-11, Ex.5.

UNIT - V

(Total Marks : 36)

Q.9.(a) Define in-degree and out-degree of vertex. Find in-degree and out-degree of each vertex of the following directed graph : **6M**



Ans. See P.5-15.

Vertex	In-degree	Out degree
A	1	3
B	4	4
C	2	2
D	3	3
E	4	2

Q.9.(b) Draw a digraph corresponding to the following adjacency matrix and interprete results :

$AA^T, A^TA, A^2, A^3, A^4$. **6M**

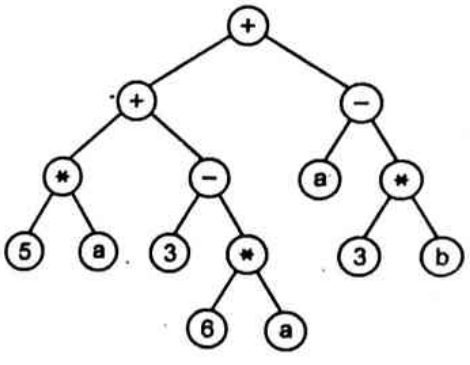
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Ans. See P.5-23, Ex.8(a).

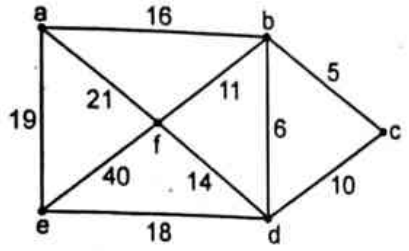
Q.9.(c) Draw the binary tree of the expression :

$((5 * a) + (3 - (6 * a))) + (a - (3 * b))$ **6M**

Ans.

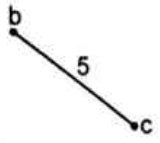


Q.10.(a) Use Prim's algorithm to find a minimal spanning tree for the given graph : **6M**

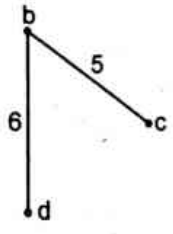


Ans.

(1) Let us select vertex b as edge (b, c) has minimum weight.

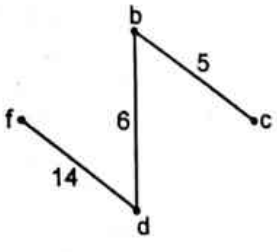


(2) Now there are 3 edges incident with b i.e. (b, d), (b, a), (b, f) and one edge incident with c i.e. (c, d) and w(b, d) = 6, w(b, a) = 16, w(b, f) = 11 and w(c, d) = 10. Hence minimum weight is 6.

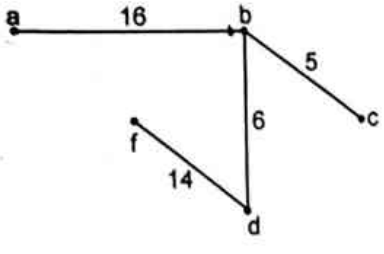


(3) There are two edges incident with d and w(d, f) = 14, w(d, e) = 18

∴ Minimum weight is 14

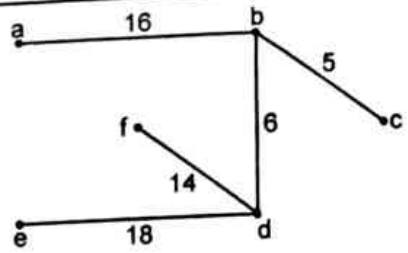


(4) Again choosing minimum weight



(5) Now, all the edges have been traversed except d.

∴ we choose (d, e) and join it.



This is the required minimal spanning tree.

Q.10.(b) Define :

(i) Null graph (ii) Trail (iii) Reachable node

(iv) Tree (v) Height of the tree

(vi) Radius of a graph **6M**

Ans. **See P.5-14, 22, 23, 36.**

(vi) Radius of a graph : Let G be a graph and v be a vertex of G. The eccentricity of the vertex v is the maximum distance from v to any vertex.

i.e. $e(v) = \max \{d(v, w) : w \text{ in } V(G)\}$

The radius of G is the minimum eccentricity among the vertices of G.

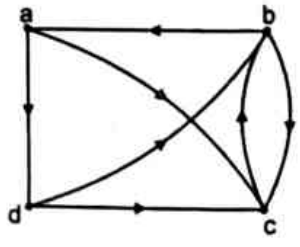
$\therefore$ Radius (G) = min $\{e(v)\}$

Q.10.(c) Draw the digraph corresponding to matrix:

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Find its complement and write the matrix of that complemented graph. **6M**

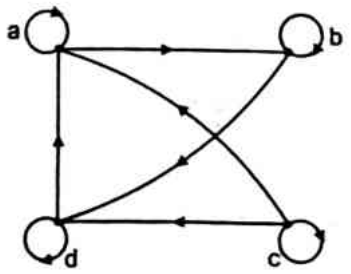
Ans. Digraph :



Complement :

$$A' = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Complemented graph :



UNIT - VI

(Total Marks : 20)

Q.11.(a) How many people must you have guarantee that at least g of them will have birthday in the same day of the week? **5M**

Ans. Here $n = 7$, $m = ?$

By pigeon-hole principle we know that if m pigeons are placed into n nesting holes, then one of nesting holes must contains

$$\left[\frac{m-1}{n}\right]+1$$

Here we consider required number of persons as pigeons (m) and 7 days of week as nesting holes.

$$\Rightarrow \left[\frac{m-1}{n}\right]+1=g$$

$$\Rightarrow \left[\frac{m-1}{7}\right]=g-1$$

$$\Rightarrow m-1=(g-1)7$$

$$\Rightarrow m-1=7g-7$$

$$\therefore m=79-6$$

$\therefore$ $\boxed{\text{Required number of peoples are } 7g-6}$ **Ans.**

Q.11.(b) Solve the recurrence relation using generating function : $a_n = 3a_{n-1} + 2,\ a_0 = 1$. **5M**

Ans. **See P.6-18, Ex.33(c).**

Q.12.(a) Prove that : $C(n+1, r) = C(n, r) + C(n, r-1)$ **5M**

Ans. **See P.PS-34, Q.12(a).**

Q.12.(b) Use generating function technique to solve following recurrence relation :

$a_n - 9a_{n-1} + 20a_{n-2} = 0$ given $a_0 = -3,\ a_1 = -10$ **5M**

Ans. **See P.6-18, Ex.33(a).**

Notes :-

WINTER EXAMINATION - 2016
B.E. IV SEM. (CT, CS, IT, CE) (CBS)
DISCRETE MATHEMATICS AND GRAPH THEORY

Statistical Analysis

Unit Number	Marks
Unit - I	20 M
Unit - II	36 M
Unit - III	24 M
Unit - IV	24 M
Unit - V	36 M
Unit - VI	20 M
Questions covered in VBD	160 M (100 %)

WINTER EXAM. - 2016
QUESTION PAPER

Q.1.(a) Prove that : $(A \cap B)' = A' \cup B'$ and $(A \cup B)' = A' \cap B'$ **6M**

(b) Construct the truth table for the following formula : $(\sim q \to p) \vee q$. **2M**

(c) Write the negation of "If the sky is cloudy, then it rains and if rains then the sky is cloudy." **2M**

OR

Q.2.(a) Test the validity of the following argument :

"If I study, then I will not fail in Mathematics. If I do not play basket ball, then I will study. But I failed in Mathematics ..... therefore I played basket ball. **5M**

(b) Show by mathematical induction that $n^3 + 2n$ is divisible by 3. **3M**

(c) Write the contrapositive of "If x is a boss, then x is bad". **2M**

Q.3.(a) Let A = {1, 2, 3, 4} and R = {(1, 2), (2, 3), (3, 4)} be a relation on A. Find transitive closure of R and draw its digraph. **6M**

(b) Let A = {a, b, c} and P(A) be its power set. Let $\subseteq$ be the partial order relation on it. Draw Hasse diagram of $(P(A), \subseteq)$. **6M**

(c) Prove that : $A \times (B \cup C) = (A \times B) \cup (A \times C)$. **6M**

OR

Q.4.(a) If $f : x \to y$ and $g : y \to z$ and both f and g are one-one and onto. **6M**

Show that :

(i) $g \circ f$ is one-one and onto.

(ii) $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

(b) Let A = {1, 2, 3, 4, 5, 6, 7} and R be a relation on A given by R = {(x, y) : x − y is divisible by 3}. Prove that R is an equivalence relation. **6M**

(c) Using properties of characteristic function, prove that : **6M**

(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(ii) $(A^c)^c = A$

Q.5.(a) Prove that fourth roots of unity forms an abelian group under multiplication. **6M**

(b) Show that the set of matrices $A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, $\alpha \in R$ forms a monoid. **6M**

OR

Q.6.(a) Define normal subgroup and prove that every subgroup of an abelian group is a normal subgroup. **6M**

(b) Show that the set A = {1, 2, 3} under multiplication modulo 4 is not a group, but B = {1, 2, 3, 4} under multiplication modulo 5 is a group. **6M**

Q.7.(a) Let (L_1, D_6) and $L_2 = (P(S), S)$ be two lattices where D_6 = {1, 2, 3, 6} and S = {a, b}. Then show that L_1 and L_2 are isomorphic. **6M**

(b) If R is a ring such that $a^2 = a, \forall a \in R$, prove that : **6M**

(i) $a + a = 0$

(ii) $a + b = 0 \Rightarrow a = b$

(iii) R is a commutative ring.

OR

Q.8.(a) Show that the set S of all matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, $a, b \in R$ is a field with respect to matrix addition and multiplication. **6M**

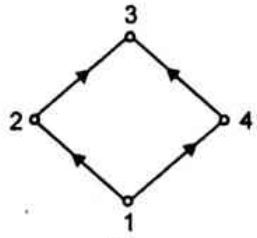
(b) Construct switching circuit for the Boolean polynomial $(A \cdot B) + (A \cdot B') + (A' \cdot B')$. Simplify this and construct an equivalent circuit. **6M**

Q.9.(a) Draw a digraph corresponding to the adjacency matrix : **6M**

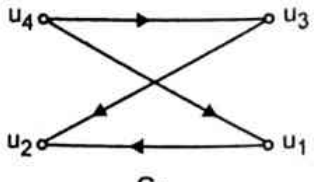
$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Give all possible paths of length 2. Is there any cycle of length 2?

(b) Show that the two graph G_1 and G_2 given below are isomorphic : **6M**



G_1



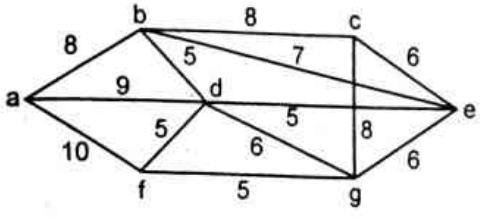
G_2

(c) Draw tree and corresponding binary tree for the relation given by " **6M**

R = {(1, 2), (1, 3), (1, 4), (2, 5), (4, 6), (4, 7)} on set A = {1, 2, 3, 4, 5, 6, 7}

OR

Q.10.(a) Apply Prim's algorithm to construct a minimal spanning tree for the weighted graph given below : **6M**



(b) Define :

(i) Null graph

(ii) Trail

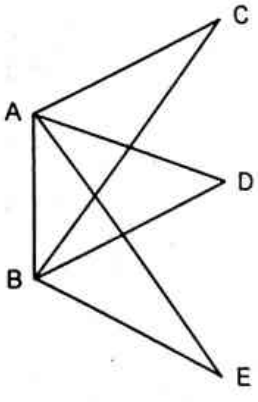
(iii) Reachable node

(iv) Tree

(v) Height of the tree

(vi) Radius of a graph.

(c) Define Eulerian path and Eulerian circuit. Show that the graph given below is an Eulerian graph and circuit. **6M**



Q.11.(a) Solve the following recurrence relation by using generating function : **5M**

$a_n = 3a_{n-1} + 2, a_o = 1$.

(b) Show that if seven numbers from 1 to 12 are chosen then two of them will add upto 13. **5M**

OR

Q.12.(a) Find the generating function of $n^2, n \geq 0$. **5M**

(b) Let n be a positive integer, prove that : **5M**

$$\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\ldots\ldots=\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots$$

WINTER EXAM. - 2016

SOLUTION

(Unit-Wise)

UNIT - I

(Total Marks : 20)

Q.1.(a) Prove that : $(A \cap B)' = A' \cup B'$ and $(A \cup B)' = A' \cap B'$ **6M**

Ans. See P. 1-36, De-Morgan's laws.

Q.1.(b) Construct the truth table for the following formula : $(\sim q \rightarrow p) \vee q$. **2M**

Ans.

p	q	~ q	~ q→p	(~ q→p) ∨ q
T	T	F	T	T
T	F	T	T	T
F	T	F	T	T
F	F	T	F	F

Q.1.(c) Write the negation of "If the sky is cloudy, then it rains and if it rains then the sky is cloudy." **2M**

Ans. See P.PS-29, Q.1(c).

Q.2.(a) Test the validity of the following argument :

"If I study, then I will not fail in Mathematics. If I do not play basket ball, then I will study. But I failed in Mathematics ..... therefore I played basket ball. **5M**

Ans. See P.1-21, Ex.33.

Q.2.(b) Show by mathematical induction that $n^3 + 2n$ is divisible by 3. **3M**

Ans. See P.1-44, Ex.85(iii).

Q.2.(c) Write the contrapositive of "If x is a boss, then x is bad". **2M**

Ans. Let p : x is boss

q : x is bad

Contrapositive of $(p \rightarrow q)$ is $\sim q \rightarrow \sim p$

i.e.If x is not bad, then x is not a boss. ...Ans.

UNIT - II

(Total Marks : 36)

Q.3.(a) Let $A = \{1, 2, 3, 4\}$ and $R = \{(1, 2), (2, 3), (3, 4)\}$ be a relation on A. Find transitive closure of R and draw its digraph. **6M**

Ans. See P.PS-24, Q.3(b).

Q.3.(b) Let $A = \{a, b, c\}$ and $P(A)$ be its power set. Let $\subseteq$ be the partial order relation on it. Draw Hasse diagram of $(P(A), \subseteq)$. **6M**

Ans. See P.2-30, Ex.50.

Q.3.(c) Prove that : $A \times (B \cup C) = (A \times B) \cup (A \times C)$. **6M**

Ans. See P.2-9, Prop.(3).

Q.4.(a) If $f : x \to y$ and $g : y \to z$ and both f and g are one-one and onto. Show that : **6M**

(i) $g \circ f$ is one-one and onto.

(ii) $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Ans. See P.2-33, Theorem 2.

Q.4.(b) Let $A = \{1, 2, 3, 4, 5, 6, 7\}$ and R be a relation on A given by $R = \{(x, y) : x - y \text{ is divisible by } 3\}$. Prove that R is an equivalence relation. **6M**

Ans. See P.2-19, Ex.23.

Q.4.(c) Using properties of characteristic function, prove that : **6M**

(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(ii) $(A^c)^c = A$

Ans. See P.2-38, Ex.66.

UNIT - III

(Total Marks : 24)

Q.5.(a) Prove that fourth roots of unity forms an abelian group under multiplication. **6M**

Ans. See P.3-12, Ex.6.

Q.5.(b) Show that the set of matrices $A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, $\alpha \in R$ forms a monoid. **6M**

Ans. See P.3-8, Ex.1.

Q.6.(a) Define normal subgroup and prove that every subgroup of an abelian group is a normal subgroup. **6M**

Ans. See P.3-30, 32, Ex.53.

Q.6.(b) Show that the set $A = \{1, 2\ 3\}$ under multiplication modulo 4 is not a group, but $B = \{1, 2, 3, 4\}$ under multiplication modulo 5 is a group. **6M**

Ans. See P.3-13, Ex.8.

UNIT - IV

(Total Marks : 24)

Q.7.(a) Let (L_1, D_6) and $L_2 = (P(S), S)$ be two lattices where $D_6 = \{1, 2, 3, 6\}$ and $S = \{a, b\}$. Then show that L_1 and L_2 are isomorphic. **6M**

Ans. See P.4-27, Ex.33.

Q.7.(b) If R is a ring such that $a^2 = a$, $\forall a \in R$, prove that : **6M**

(i) $a + a = 0$

(ii) $a + b = 0 \Rightarrow a = b$

(iii) R is a commutative ring.

Ans. See P.4-11, Ex.5.

Q.8.(a) Show that the set S of all matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, $a, b \in R$ is a field with respect to matrix addition and multiplication. **6M**

Ans. See P.4-17, Ex.13.

Q.8.(b) Construct switching circuit for the Boolean polynomial $(A \cdot B) + (A \cdot B') + (A' \cdot B')$. Simplify this and construct an equivalent circuit. **6M**

Ans. See P.4-34, Ex.51.

UNIT - V

(Total Marks : 36)

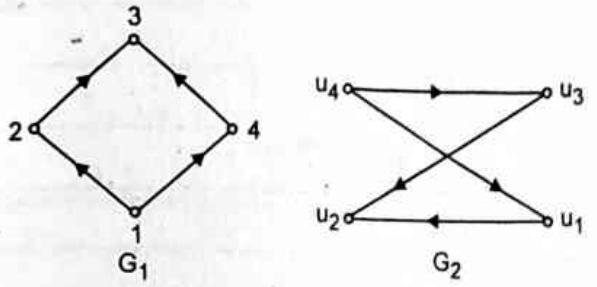
Q.9.(a) Draw a digraph corresponding to the adjacency matrix : **6M**

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Give all possible paths of length 2. Is there any cycle of length 2?

Ans. See P.5-34, Ex.24.

Q.9.(b) Show that the two graph G_1 and G_2 given below are isomorphic : **6M**



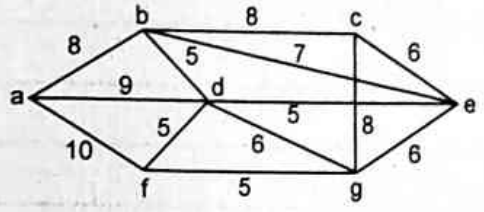
Ans. See P.5-34, Ex.25.

Q.9.(c) Draw tree and corresponding binary tree for the relation given by : **6M**

R = {(1, 2), (1, 3), (1, 4), (2, 5), (4, 6), (4, 7)} on set A = {1, 2, 3, 4, 5, 6, 7}

Ans. See P.5-41, Ex.34.

Q.10.(a) Apply Prim's algorithm to construct a minimal spanning tree for the weighted graph given below : **6M**



Ans. See P. 5-46, Ex.47.

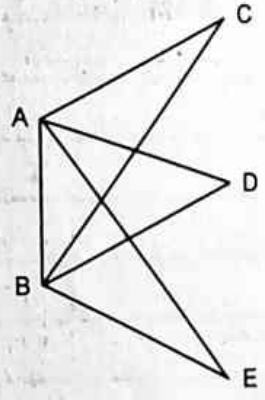
Q.10.(b) Define : **6M**

(i) Null graph (ii) Trail (iii) Reachable node

(iv) Tree (v) Height of the tree (vi) Radius of a graph.

Ans. See P. 5-14, 22, 23, 36, PS-49, Q.10(b).

Q.10.(c) Define Eulerian path and Eulerian circuit. Show that the graph given below is an Eulerian graph and circuit. **6M**



Ans. See P. 5-30 and P.PS-18, Q.10(b).

UNIT - VI

(Total Marks : 20)

Q.11.(a) Solve the following recurrence relation by using generating function :

$a_n = 3a_{n-1} + 2, a_o = 1$. **5M**

Ans. See P.6-18, Ex.33(c).

Q.11.(b) Show that if seven numbers from 1 to 12 are chosen then two of them will add upto 13. **5M**

Ans. We construct 6 different sets each containing 2 integers that will give a sum 13.

$A_1 = \{1,12\}$; $A_2 = \{2,11\}$; $A_3 = \{3,10\}$;

$A_4 = \{4,9\}$; $A_5 = \{5,8\}$, $A_6 = \{6,7\}$

Here m = 7, n = 6 (m > n)

Hence by Pigeon Hole principle two of the integers chosen will give a sum 13. ...Ans.

Q.12.(a) Find the generating function of n^2 , $n \geq 0$. **5M**

Ans. Given : $a_n = n^2$

We have $G(x) = \sum_{n=0}^{\infty} a_n x^n$

$$= \sum_{n=0}^{\infty} n^2 x^n$$

$$= 0x^0 + 1^2 x^1 + 2^2 x^2 + 3^2 x^3 + 4^2 x^4 + ...$$

$$= x + 4x^2 + 9x^3 + 16x^4 + ...$$

$$= x(1 + 4x + 9x^2 + 16x^3 + ...)$$

$$\therefore G(x) = \frac{x(1+x)}{(1-x)^3}$$

which is the required generating function. ...Ans.

Q.12.(b) Let n be a positive integer, prove that :

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + ... = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + ...$$ **5M**

Ans. By Binomial theorem, we have

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$

Put x = 1, y = – 1

$$\therefore\ 0 = \sum_{r=0}^{n} \binom{n}{r} (1)^{n-r} (-1)^r$$

$$\Rightarrow \sum_{r=0}^{n} \binom{n}{r} (-1)^r = 0$$

$$\Rightarrow \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \binom{n}{4} - \binom{n}{5} + \ldots = 0$$

$$\Rightarrow \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots$$ ...Ans.

Notes :-

SUMMER EXAMINATION - 2017
B.E. IV SEM. (CT, CS, IT, CE) (CBS)
DISCRETE MATHEMATICS AND GRAPH THEORY

Statistical Analysis

Unit Number	Marks
Unit - I	20 M
Unit - II	39 M
Unit - III	24 M
Unit - IV	21 M
Unit - V	36 M
Unit - VI	20 M
Questions covered in VBD	160 M (100 %)

SUMMER EXAM.- 2017

QUESTION PAPER

Q.1.(a) Prove that for any two non-empty sets A and B.

$A - (B \cap C) \subseteq (A - B) \cup (A - C)$ **3M**

(b) Prove the logical equivalence by truth table : **3M**

$p \Leftrightarrow q = (p \Rightarrow q) \wedge (q \Rightarrow p)$

(c) Prove that $\sqrt{2}$ is irrational by giving a proof by method of contradiction. **4M**

OR

Q.2.(a) Show that $n^2 > (2n + 1)$, for $n \geq 3$ by method of mathematical induction. **4M**

(b) Determine the validity of following argument :

"If I drive to work, then I will arrive in time.

I do not drive to work".

Therefore, I will not arrive in time. **4M**

(c) State De-Morgan's law of set theory. **2M**

Q.3.(a) Prove that : **6M**

(i) $f_{A \cap B} = f_A f_B$ and

(ii) $f_{A \cup B} = f_A + f_B - f_A f_B$,

where f is characteristic function.

(b) Let X = {bull, bed, dog, egg, let} and the relation

R = {(x, y) : x, y ∈ X, xRy if x and y contains some common letter}

Prove that R is compatible and write the relation matrix. **6M**

(c) Let A = {1, 2, 3, 4} and R = {(1, 2), (2, 3), (3, 4)} be a relation on A. Find transitive closure of R and draw its diagraph. **6M**

OR

Q.4.(a) If $f : X \to Y$ and $g : Y \to Z$ and both f and g are one-one onto, then show that gof is also one-one onto and $(gof)^{-1} = f^{-1} o g^{-1}$. **6M**

(b) Let X = {1, 2, 3, 6, 12, 24, 36} and the relation "≤" be such that $x \leq y$ if "x divides y". Draw the Hasse diagram. **6M**

(c) If $f : R^+ \to R^+$ and $g : R^+ \to R^+$ be functions defined by $f(x) = \sqrt{x}$ and $g(x) = 3x + 1$ for all $x \in R^+$. Find fog and gof. Is fog = gof? **6M**

Q.5.(a) Prove that the set {0, 1, 2, 3, 4} is a finite abelian group of order 5 under addition modulo 5 as a composition. **6M**

(b) Determine whether the set of integers Z, with binary operation * defined by a * b = a + b – ab is a monoid. **6M**

OR

Q.6.(a) Define normal subgroup and prove that every subgroup of an abelian group is a normal subgroup. **6M**

(b) Let G be a set of integers under addition and let H be the set of all integral multiples of 3. Prove that H is a subgroup of G and determine all cosets of H in G. **6M**

Q.7.(a) Show that $(Z_6, +_6, x_6)$ is a ring. Explain whether it is an integral domain. **7M**

(b) Construct a switching circuit for Boolean expression (A . B) + C + (A'. C'). Simplify the circuit and construct an equivalent simplified circuit. **5M**

OR

Q.8.(a) Define lattice. Draw the Hasse diagram of lattices D_{20} and D_{30}. **5M**

(b) Show that (I, ⊖, ⊙) is a commutative ring with identity where operation ⊖ and ⊙ are defined as,

for any a, b ∈ I, a ⊖ b = a + b – 1

and a ⊙ b = a + b – ab

where I is set of integers. **7M**

Q.9.(a) Define :

(i) Diagraph (ii) Path

(iii) Cycle (iv) Euler path

(v) Forest **5M**

(b) Draw a diagraph corresponding to matrix - **7M**

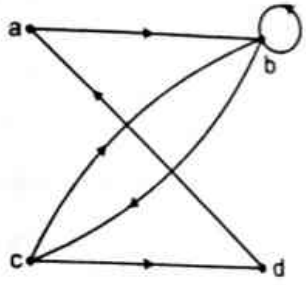
$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

and interprete AA^T, A^TA and A^2.

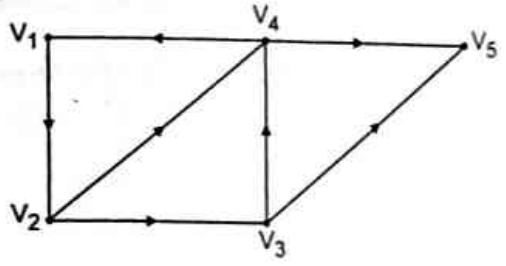
(c) Construct a tree and corresponding binary tree for relation given by : R = {(1, 2), (1, 3), (1, 4), (2, 5), (4, 6), (4, 7)} on set A = {1, 2, 3, 4, 5, 6, 7}. **6M**

OR

Q.10.(a) Find the adjacency matrix of following graph and its complement : **5M**

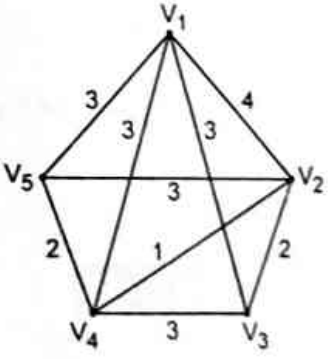


(b) Find all indegree and outdegree of graph given below : **6M**



Find all elementary cycles of this graph.

(c) Apply Prim's algorithm to find minimal spanning tree from the weighted graph given below ; **7M**



Q.11.(a) Solve the following recurrence relation :

$a_n - 9a_{n-1} + 20a_{n-2} = 0;$

$a_0 = -3, a_1 = -10$ **5M**

(b) Prove Pascal's identity **5M**

$c(n+1, r) = c(n, r) + c(n, r-1)$

where n and r are +ve integers with $r \le n$.

OR

Q.12.(a) Find the minimum number of students in a class to be sure that four of them are born in the same month. **5M**

(b) Find the generating function for the sequence 1, a, a^2,..... where a is fixed constant. **5M**

SUMMER EXAM.-2017

SOLUTION

(Unit-Wise)

UNIT - I

(Total Marks : 20)

Q.1.(a) Prove that for any two non-empty sets A and B.

$A - (B \cap C) \subseteq (A - B) \cup (A - C)$ **3M**

Ans. See P.1-37, Ex.61.

Q.1.(b) Prove the logical equivalence by truth table :

$p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$ **3M**

Ans. See P.1-20, Ex.30(ii).

Q.1.(c) Prove that $\sqrt{2}$ is irrational by giving a proof by method of contradiction. **4M**

Ans. See P.1-28, Ex.50.

Q.2.(a) Show that $n^2 > (2n+1)$, for $n \ge 3$ by method of mathematical induction. **4M**

Ans. Let P (n) : $n^2 > (2n+1)$

P (3) : $3^2 > 2(3) + 1 = 6 + 1$

i.e. $9 > 7$

$\therefore$ P (3) is true

Let P (n) be true for n = m

$\therefore$ P (m) be true

We prove that P (m + 1) is also true

Now, P (m) : $m^2 > (2m+1)$

$\Rightarrow m^2 + 2m + 1 > 2m + 1 + 2m + 1$

$\Rightarrow (m+1)^2 > 4m + 2$

$\Rightarrow (m+1)^2 > (2m+3) + (2m-1)$

$\Rightarrow (m+1)^2 > 2(m+1) + 1 + (2m-1)$

$\therefore$ P (m + 1) is also true as $m \geq 3$

$\therefore$ By Principle of mathematical induction P (n) is true for all $n \geq 3$.

Q.2.(b) Determine the validity of following argument.

"If I drive to work, then I will arrive in time.

I do not drive to work".

Therefore, I will not arrive in time. **4M**

Ans. Let p : I drive to work

q : I will arrive in time

$S_1 : p \rightarrow q$

$S_2 : \sim p$

$S : \sim q$

We verify whether $(S_1 \wedge S_2) \rightarrow S$ is a tautology.

		S_2	S	S_1	(1)	(1) $\rightarrow$ S
p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$S_1 \wedge S_2$	$S_1 \wedge S_2 \rightarrow S$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	T	F
F	F	T	T	T	T	T

We observe that $(S_1 \wedge S_2) \rightarrow S$ is not a tautology.

$\therefore$ The given argument is invalid. .....Ans.

Q.2.(c) State De-Morgan's law of set theory. **2M**

Ans. See P.1-18, Ex.28.

UNIT - II

(Total Marks : 39)

Q.3.(a) Prove that : **6M**

(i) $f_{A \cap B} = f_A . f_B$ and (ii) $f_{A \cup B} = f_A + f_B - f_A . f_B$,

where f is characteristic function.

Ans. See P.2-40, 39, Ex.70, 68.

Q.3.(b) Let X = {ball, bed, dog, egg, let} and the relation

R = {(x, y) : x, y $\in$ X, xRy if x and y contains some common letter}

Prove that R is compatible and write the relation matrix. **6M**

Ans. See P.2-23, Ex.32.

Q.3.(c) Let A = {1, 2, 3, 4} and R = {(1, 2), (2, 3), (3, 4)} be a relation on A.

Find transitive closure of R and draw its diagraph. **6M**

Ans. See P.PS-24, Q.3(b).

Q.4.(a) If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ and both f and g are one-one onto, then show that gof is also one-one onto and $(gof)^{-1} = f^{-1} o\, g^{-1}$. **6M**

Ans. See P.2-33, Thm(2).

Q.4.(b) Let X = {1, 2, 3, 6, 12, 24, 36} and the relation "$\leq$" be such that $x \leq y$ if "x divides y". Draw the Hasse diagram. **6M**

Ans. See P.2-29, Ex.46.

Q.4.(c) **If $f : R^+ \rightarrow R^+$ and $g : R^+ \rightarrow R^+$ be functions defined by $f(x) = \sqrt{x}$ and $g(x) = 3x + 1$ for all $x \in R^+$. Find fog and gof. Is fog = gof?** **6M**

Ans. Given : $f(x) = \sqrt{x}$, $g(x) = 3x + 1$

$fog(x) = f[g(x)]$

$= f[3x + 1]$

$\therefore fog(x) = \sqrt{3x+1}$ ....(i)

Also $gof(x) = g[f(x)] = g[\sqrt{x}] = 3\sqrt{x} + 1$ ....(ii)

$\therefore$ From (i) and (ii), we have $fog \neq gof$

Q.8.(a) Draw the Hasse diagram of lattices D_{20} and D_{30}. **3M**

Ans. See P.2-30, Ex.49.

UNIT - III

(Total Marks : 24)

Q.5.(a) Prove that the set {0, 1, 2, 3, 4} is a finite abelian group of order 5 under addition modulo 5 as a composition. **6M**

Ans. See P.3-14, Ex.10.

Q.5.(b) Determine whether the set of integers Z, with binary operation * defined by $a * b = a + b - ab$ is a monoid. **6M**

Ans. See P.3-14, Ex.9.

Q.6.(a) Define normal subgroup and prove that every subgroup of an abelian group is a normal subgroup. **6M**

Ans. See P.3-30, 32, Ex.53.

Q.6.(b) Let G be a set of integers under addition and let H be the set of all integral multiples of 3. Prove that H is a subgroup of G and determine all cosets of H in G. **6M**

Ans. Given : $G = \{0, \pm 1, \pm 2, \pm 3, ...\}$

$H = \{0, \pm 3, \pm 9, \pm 12, \pm 15, ...\}$

H is subgroup of G since :

(i) H is non-empty.

(ii) Let $x, y \in H$, then there exists $p, q \in G$ such that

$x = 3p, y = 3q$

Now $x + y^{-1} = 3p - 3q$

$= 3(p - q)$ where $p - q \in G$

Thus $x + y^{-1} \in H$

$\therefore$ H is a subgroup of G.

For cosets : **See P.3-28, Ex.43.**

UNIT - IV

(Total Marks : 21)

Q.7.(a) Show that $(Z_6, +_6, \times_6)$ is a ring. Explain whether it is an integral domain. **7M**

Ans. **See P.4-10, Ex.4(a)**

Q.7.(b) Construct a switching circuit for Boolean expression $(A . B) + C + (A' . C')$. **5M**

Ans. **See P.4-34, Ex.52.**

Q.8.(a) Define lattice. **2M**

Ans. **See P.4-23, Definition**

Q.8.(b) Show that $(I, \oplus, \odot)$ is a commutative ring with identity where operation $\oplus$ and $\odot$ are defined as,

for any $a, b \in I$, $a \oplus b = a + b - 1$

and $a \odot b = a + b - ab$

where I is set of integers. **7M**

Ans. **See P.PS-10, Ex.7(b).**

UNIT - V

(Total Marks : 36)

Q.9.(a) Define :

(i) Diagraph (ii) Path

(iii) Cycle (iv) Euler path

(v) Forest **5M**

Ans. **See P.5-13, 22, 30, PS-32, Q.9(b).**

Q.9.(b) Draw a diagraph corresponding to matrix -

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

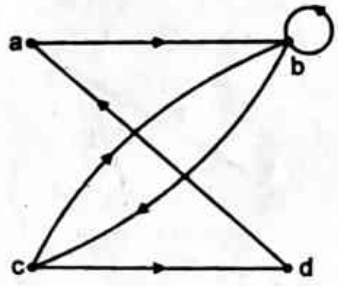
and interprete AA^T, A^TA and A^2. **7M**

Ans. **See P.PS-32, Q.10(a).**

Q.9.(c) Construct a tree and corresponding binary tree for relation given by $R = \{(1, 2), (1, 3), (1, 4), (2, 5), (4, 6), (4, 7)\}$ on set $A = \{1, 2, 3, 4, 5, 6, 7\}$. **6M**

Ans. **See P.5-41, Ex.34.**

Q.10.(a) Find the adjacency matrix of following graph and its complement. **5M**

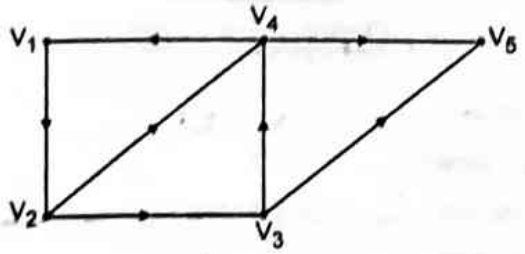


Ans.

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array} \text{ and}$$

$$A' = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

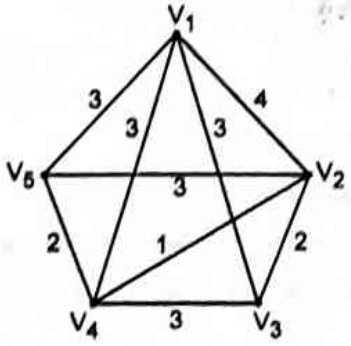
Q.10.(b) Find all indegree and outdegree of graph given below : **6M**



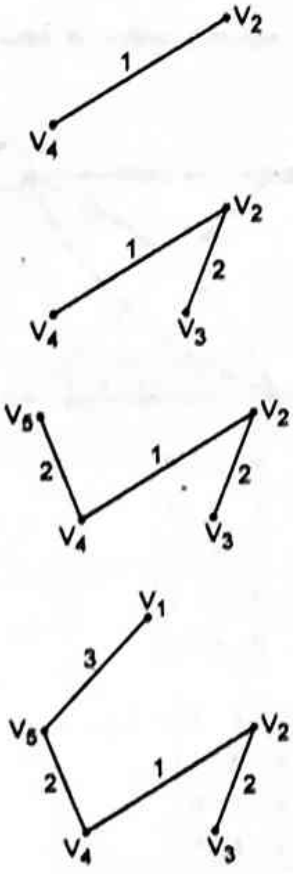
Find all elementary cycles of this graph.

Ans. See P.PS-33, Q.10(c).

Q.10.(c) Apply Prim's algorithm to find minimal spanning tree from the weighted graph given below : **7M**



Ans.



Minimum weight = 3 + 2 + 1 + 2 = 8

UNIT - VI

(Total Marks : 20)

Q.11.(a) Solve the following recurrence relation

$$a_n - 9a_{n-1} + 20a_{n-2} = 0,$$

$$a_0 = -3, a_1 = -10$$ **5M**

Ans. See P.6-18, Ex.33(a)

Q.11.(b) Prove Pascals identity **5M**

$$c(n+1, r) = c(n, r) + c(n, r-1)$$

where n and r are +ve integers with $r \leq n$.

Ans. See P.PS-26, 34, Q.12(b), 12(a).

Q.12.(a) Find the minimum number of students in a class to be sure that four of them are born in the same month. **5M**

Ans. See P.6-9, Ex.24.

Q.12.(b) Find the generating function for the sequence 1, a, a^2,..... where a is fixed constant. **5M**

Ans. See P.6-21, Ex.34.

WINTER EXAMINATION - 2017

B.E. IV SEM. (CT, CS, IT, CE) (CBS)

DISCRETE MATHEMATICS AND GRAPH THEORY

Statistical Analysis

Unit Number	Marks
Unit - I	20 M
Unit - II	36 M
Unit - III	24 M
Unit - IV	24 M
Unit - V	36 M
Unit - VI	20 M
Questions covered in VBD	160 M (100 %)

WINTER EXAM.- 2017
QUESTION PAPER

Q.1.(a) Prove by mathematical induction method that $3^{4n+2} + 5^{2n+1}$ is a multiple of 14 for $n \geq 0$. **5M**

(b) Prove the logical equivalence by using algebra of proposition :

$[p \wedge (\sim p \vee q)] \vee [q \wedge \sim(p \wedge q)] = q$ **5M**

OR

Q.2.(a) Check the validity of the argument.

"If the labour market is perfect, then the wages of all persons in a particular employment will be equal. But it is always the case that wages for such persons are not equal. Therefore, the labour market is not perfect. **5M**

(b) Prove that : $[p \wedge (p \rightarrow q)] \rightarrow q$ is a tautology. **5M**

Q.3.(a) Let X = {ball, bed, dog, let, egg}

and R = {(x, y) | x, y ∈ x, xRy if x and y contain common letter}

Prove that R is compatible. Is R an equivalence relation? **6M**

(b) Let $A = Z^+$, the set of positive integers and let $R = \{(a, b) \in A \times A | a \text{ divides } b\}$.

Is R symmetric, asymmetric or antisymmetric? **6M**

(c) If $f : x \rightarrow y$ and $g : y \rightarrow z$ and both f and g are one-one and onto, then show that gof is also one-one onto and $(g \circ f)^{-1} = f^{-1} o g^{-1}$. **6M**

OR

Q.4.(a) By using the properties of characteristic function, prove that :

$f_{A \oplus B}(x) = f_A(x) + f_B(x) - 2f_A(x) . f_B(x)$ for all x. **6M**

(b) Let A ={a,b,c} and ρ (A) be its power set. Let ⊆ be the inclusion relation on the elements of ρ (A). Draw Hasse diagram of (ρ (A), ⊆). **6M**

(c) Let A ={1, 2, 3, 4} and R = {(1, 2), (2, 3), (3, 4), (2, 1)}.

Find the matrix of relation and transitive closure of R. **6M**

Q.5.(a) Show that the fourth root of unity form an abelian group with respect to multiplication. **6M**

(b) Prove that order of each subgroup of a finite group is a divisor of the order of the group. **6M**

OR

Q.6.(a) Show that the set of matrices :

$$A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, \alpha \in R.$$

forms a monoid. **6M**

(b) Prove that any two right or left cosets of a subgroup are either disjoint or identical. **6M**

Q.7.(a) Prove that the set R = {0, 2, 4, 6, 8} is a commutative ring under addition and multiplication modulo 10. **7M**

(b) Define lattice. Draw the Hasse diagram of lattices D_{24} and D_{30}. **5M**

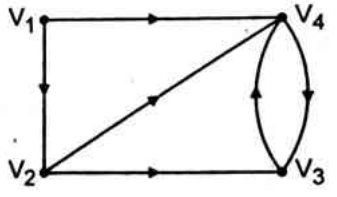
OR

Q.8.(a) Construct switching circuit for the Boolean expression (A . B) + (A . B′) + (A′. B′).

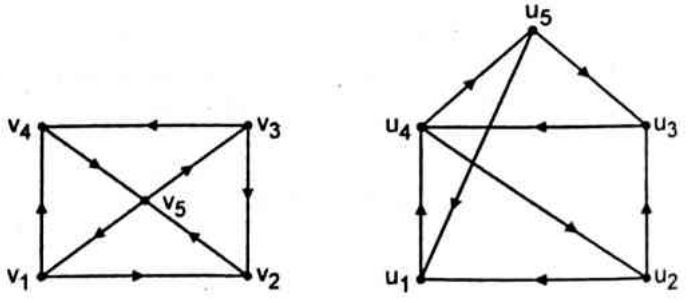
Simplify this and construct an equivalent simplified circuit. **6M**

(b) Prove that every field is an integral domain. **6M**

Q.9.(a) Give three different elementary paths from v_1 to v_3 for the digraph given below. What is the shortest distance between v_1 and v_3 ? Is there any cycle in the graph? **6M**



(b) Show that the following graphs are isomorphic : **6M**



(c) Draw the digraph corresponding to matrix :

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Find AA^T, A^TA and A^2 and interpret the result. **6M**

OR

Q.10.(a) Draw tree representation for the tree given by

R = {(1, 2), (1, 3), (1, 4), (2, 5), (4, 6), (4, 7)} on set A = {1, 2, 3, 4, 5, 6, 7} and draw corresponding binary tree. **6M**

(b) Define :

(i) Complete graph.

(ii) Euler's path.

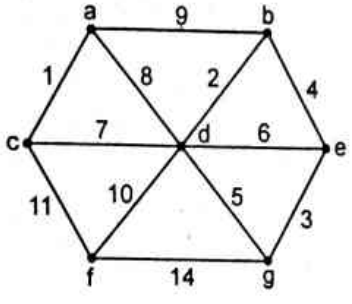
(iii) Bridge for connected graph.

(iv) Forest.

(v) Root of the tree.

(vi) Regular graph. **6M**

(c) Apply Kruskal's algorithm to construct a minimal spanning tree for the weighted graph given below : **6M**



Q.11.(a) Prove that :

$c(n+1, r) = c(n, r) + c(n, r-1)$. **5M**

(b) Find generating function of $n^2, n \geq 0$. **5M**

OR

Q.12.(a) Find the general solution of the recurrence relation :

$a_{n+2} - 2a_{n+1} + a_n = 2^n, a_0 = 2, a_1 = 1$. **5M**

(b) Show that if seven numbers from 1 to 12 are chosen then two of them will add upto 13. **5M**

WINTER EXAM.-2017

SOLUTION

(Unit-Wise)

UNIT - I

(Total Marks : 20)

Q.1.(a) Prove by mathematical induction method that $3^{4n+2} + 5^{2n+1}$ is a multiple of 14 for $n \geq 0$. **5M**

Ans. See P.1-44, Ex.85(vii).

Q.1.(b) Prove the logical equivalence by using algebra of proposition :

$[p \wedge (\sim p \vee q)] \vee [q \wedge \sim (p \wedge q)] = q$ **5M**

Ans. See P.1-19, Ex.29(iii).

Q.2.(a) Check the validity of the argument.

"If the labour market is perfect, then the wages of all persons in a particular employment will be equal. But it is always the case that wages for such persons are not equal. Therefore, the labour market is not perfect. **5M**

Ans. See P.1-23, Ex.39.

Q.2.(b) Prove that : $[p \wedge (p \to q)] \to q$ is a tautology. **5M**

Ans.

		(1)	(2)	
p	q	$p \to q$	$p \wedge (1)$	$(2) \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

From the last column, we observe that

$[p \wedge (p \to q)] \to q$ is a tautology.....**Ans.**

UNIT - II

(Total Marks : 36)

Q.3.(a) Let X = {ball, bed, dog, let, egg}

and R = {(x, y) | x, y ∈ x, xRy if x and y contain common letter}

Prove that R is compatible. Is R an equivalence relation? **6M**

Ans. See P.2-23, Ex.32.

(R is not an equivalence relation)

Q.3.(b) Let $A = Z^+$, the set of positive integers and let R = {(a, b) ∈ A x A|a divides b}.

Is R symmetric, asymmetric or anti symmetric? **6M**

Ans. Given : R = {(a, b) ∈ A × A | a divides b},

$A = Z^+$

Symmetric : We check aRb ⇒ bRa

Now aRb ⇒ a divides b

bRa ⇒ b divides a

If a divides b, it does not necessarily mean b divides a

$\therefore aRb \not\Rightarrow bRa$

$\therefore$ R is not symmetric.

Asymmetric : From above part, we see that

aRb but b $\not R$ a.

$\therefore$ R is asymmetric

Anti symmetric : We check (a, b) R (b, a) $\Rightarrow$ a = b

If a divides b and b divides a, then a = b

$\therefore$ (a, b) R (b, a) $\Rightarrow$ a = b

$\therefore$ R is anti symmetric.

Q.3.(c) If $f : x \to y$ and $g : y \to z$ and both f and g are one-one and onto, then show that gof is also one-one onto and $(g \circ f)^{-1} = f^{-1} o g^{-1}$. **6M**

Ans. See P.2-33, Theorem 2, P.2-38, Ex.65.

Q.4.(a) By using the properties of characteristic function, prove that :

$$f_{A \oplus B}(x) = f_A(x) + f_B(x) - 2f_A(x).f_B(x) \text{ for all } x.$$ **6M**

Ans. Consider $f_{A \oplus B}(x)$

$= f_{(A-B) \cup (B-A)}(x)$

$= f_{A-B}(x) + f_{B-A}(x) - f_{(A-B) \cap (B-A)}(x)$

$= f_{A-B}(x) + f_{B-A}(x) - f_{(A-B)} \; f_{(B-A)}(x)$ ...(1)

Let $x \in A - B \Rightarrow x \in A, x \notin B$

$\therefore f_A(x) = 1, f_B(x) = 0$

Then $f_A(x) + f_B(x) - 2.f_A(x).f_B(x) = 1 + 0 - 0 = 1$

Also (1) $\Rightarrow f_{A \oplus B}(x) = 1 + 0 - 0 = 1$

$\therefore f_{A \oplus B}(x) = f_A(x) + f_B(x) - 2f_A(x).f_B(x)$

Let $x \in B - A \Rightarrow x \in B, x \notin A$

$\therefore f_B(x) = 1, f_A(x) = 0$

Then $f_A(x) + f_B(x) - 2.f_A(x).f_B(x) = 0 + 1 - 0 = 1$

Also (1) $\Rightarrow f_{A \oplus B}(x) = 1 + 0 - 0 = 1$

$\therefore f_{A \oplus B}(x) = f_A(x) + f_B(x) - 2.f_A(x).f_B(x)$

Q.4.(b) Let A = {a,b,c} and ρ(A) be its power set. Let $\subseteq$ be the inclusion relation on the elements of ρ(A). Draw Hasse diagram of (ρ(A), $\subseteq$). **6M**

Ans. See P.2-30, Ex.50.

Q.4.(c) Let A = {1, 2, 3, 4} and R = {(1, 2), (2, 3), (3, 4), (2, 1)}. Find the matrix of relation and transitive closure of R. **6M**

Ans. See P.2-22, Ex.31.

UNIT - III
(Total Marks : 24)

Q.5.(a) Show that the fourth root of unity form an abelian group with respect to multiplication. **6M**

Ans. See P.3-12, Ex.6.

Q.5.(b) Prove that order of each subgroup of a finite group is a divisor of the order of the group. **6M**

Ans. See P.3-28, Ex.45.

Q.6.(a) Show that the set of matrices :

$$A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, \alpha \in R.$$

forms a monoid. **6M**

Ans. See P.3-8, Ex.1.

Q.6.(b) Prove that any two right or left cosets of a subgroup are either disjoint or identical. **6M**

Ans. See P.PS-30, Q.5(b).

UNIT - IV
(Total Marks : 24)

Q.7.(a) Prove that the set R = {0, 2, 4, 6, 8} is a commutative ring under addition and multiplication modulo 10. **7M**

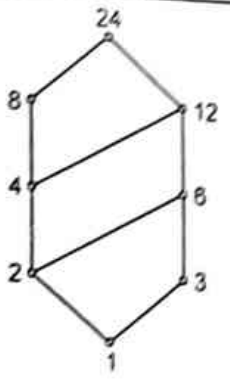
Ans. See P.4-15, Ex.10(b).

Q.7.(b) Define lattice. Draw the Hasse diagram of lattices D_{24} and D_{30}. **5M**

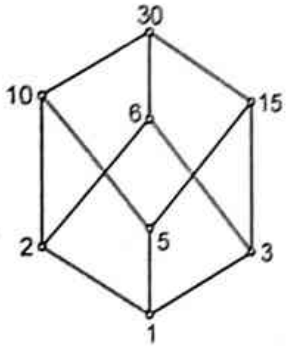
Ans. P.4-23

We have D_{24} = {1, 2, 3, 4, 6, 8, 12, 24}

D_{30} = {1, 2, 3, 5, 6, 10, 15, 30}



D_{24}



D_{30}

Q.8.(a) Construct switching circuit for the Boolean expression (A . B) + (A . B′) + (A′.B′).

Simplify this and construct an equivalent simplified circuit. **6M**

Ans. See P.4-34, Ex.51.

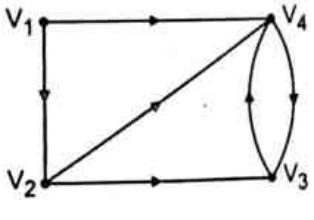
Q.8.(b) Prove that every field is an integral domain. **6M**

Ans. See P.4-13, Ex.9.

UNIT - V

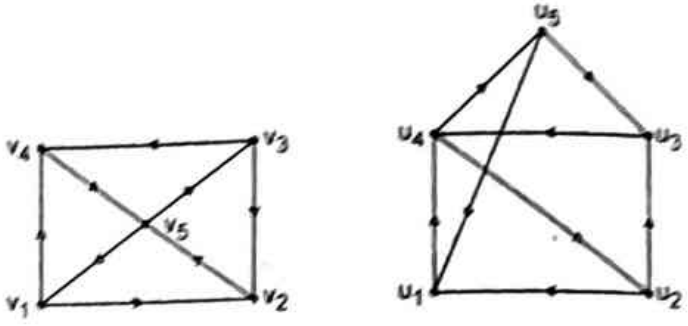
(Total Marks : 36)

Q.9.(a) Give three different elementary paths from v_1 to v_3 for the digraph given below. What is the shortest distance between v_1 and v_3? Is there any cycle in the graph? **6M**



Ans. See P.5-25, Ex.9.

Q.9.(b) Show that the following graphs are isomorphic. **6M**



Ans. See P.5-19, Ex.6(c).

Q.9.(c) Draw the digraph corresponding to matrix :

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Find AA^T, A^TA and A^2 and interpret the result. **6M**

Ans. See P.5-23, Ex.8(a).

Q.10.(a) Draw tree representation for the tree given by

R = {(1, 2), (1, 3), (1, 4), (2, 5), (4, 6), (4, 7)} on set A = {1, 2, 3, 4, 5, 6, 7} and draw corresponding binary tree. **6M**

Ans. See P.5-41, Ex.34.

Q.10.(b) Define :

(i) Complete graph.

(ii) Euler's path.

(iii) Bridge for connected graph.

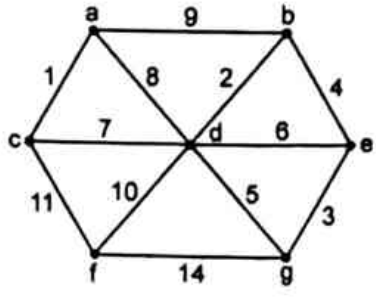
(iv) Forest.

(v) Root of the tree.

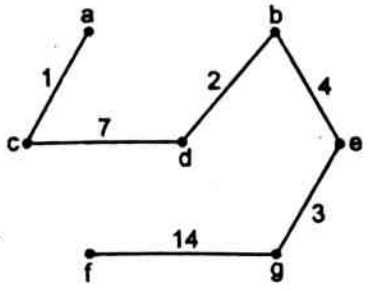
(vi) Regular graph. **6M**

Ans. See P.5-20, 30, 29, 41, 36, P.PS-32, Q.9(b)(f).

Q.10.(c) Apply Kruskal's algorithm to construct a minimal spanning tree for the weighted graph given below : **6M**



Ans.



UNIT - VI

(Total Marks : 20)

Q.11.(a) Prove that :

$c(n+1, r) = c(n, r) + c(n, r-1)$. **5M**

Ans. See P.PS-34, Ex.12(a).

Q.11.(b) Find generating function of $n^2, n \geq 0$. **5M**

Ans. See P.PS-55, Q.12(a).

Q.12.(a) Find the general solution of the recurrence relation :

$a_{n+2} - 2a_{n+1} + a_n = 2^n, a_0 = 2, a_1 = 1.$ **5M**

Ans. Given : $a_{n+2} - 2a_{n+1} + a_n = 2^n, a_0 = 2, a_1 = 1$

Multiply by x^n and take $\sum_{n=0}^{\infty}$

$$\sum_{n=0}^{\infty} a_{n+2}x^n - 2\sum_{n=0}^{\infty} a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 2^n x^n$$

$$\Rightarrow \frac{1}{x^2}[G(x) - a_0 - a_1 x] - \frac{2}{x}[G(x) - a_0] + G(x) = \frac{1}{1-2x} \text{ Using}$$

$a_0 = 2, a_1 = 1$, we get

$$G(x) = \frac{7x^2 - 7x + 2}{(1-2x)(x-1)^2} = \frac{A}{1-2x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

On solving, we get $A = 1, B = -3, C = -2$

$$\therefore\ G(x) = \frac{1}{1-2x} - \frac{3}{x-1} - \frac{2}{(x-1)^2}$$

$$G(x) = \frac{1}{1-2x} + \frac{3}{1-x} - \frac{2}{(1-x)^2}$$

$$\therefore \sum_{n=0}^{\infty} a_n x^n = 1 + 2x + ... + 2^n x^n + ... + 3(1 + x + x^2 + ... + x^n)$$

$$-2(1 + 2x + 3x^2 + ... + (n+1)x^n)$$

$$\therefore\ \boxed{a_n = 2^n + 3 - 2(n+1)} \text{ Ans.}$$

Q.12.(b) Show that if seven numbers from 1 to 12 are chosen then two of them will add upto 13. **5M**

Ans. See P.PS-55, Q.11(b).

SUMMER EXAMINATION - 2018
B.E. IV SEM. (CT, CS, IT, CE) (CBS)
DISCRETE MATHEMATICS AND GRAPH THEORY

Statistical Analysis

Unit Number	Marks
Unit - I	17 M
Unit - II	39 M
Unit - III	25 M
Unit - IV	24 M
Unit - V	36 M
Unit - VI	20 M
Questions covered in VBD	160 M (100 %)

SUMMER EXAM.- 2018

QUESTION PAPER

Q.1.(a) Prove that $A-(B\cap C)=(A-B)\cup(A-C)$. **4M**

(b) Write inverse, contrapositive and converse of the statement :

"Oxygen in air decreases as you go up" **3M**

(c) Write the negation of the following statements :

(i) He is tall but handsome

(ii) If she work, she will earn money and

(iii) If he studies, he will go to college or to art school. **3M**

OR

Q.2.(a) Prove by method of induction $1+2^n<3^n$ for $n\geq 2$. **3M**

(b) Test the validity of the following statement :

"If I like discrete mathematics, then I will study. Either I study discrete mathematics or I failed the course. Therefore, if I fail the course, then I don't like discrete mathematics." **4M**

(c) Prove that **3M**

$A\times(B\cap C)=(A\times B)\cap(A\times C)$

Q.3.(a) Let A be the set of non-zero integers and let R be the relation on $A\times A$ defined by $(a, b)\ R\ (c, d) \Leftrightarrow ad=bc$.

Show that R is an equivalence relation. **6M**

(b) Let $A=\{a, b, c\}$ and $P(A)$ be its power set. Let "$\subseteq$" be the relation defined on $P(A)$. Draw Hasse diagram of the Poset $(P(A),\subseteq)$. **6M**

(c) If $f: X\to Y$ and $g: Y\to Z$ are one-one and onto functions, then show that gof is also one-one and onto and $(gof)^{-1}=f^{-1}og^{-1}$. **6M**

Q.4.(a) Let $X=$ {ball, bed, dog, egg, let} and

$R=\{(x, y)\mid x, y\in X, x\ R\ y$, if x and y contain some common letter$\}$

Write M_R. Draw the graph of relation R prove that R is compatible but not transitive **6M**

(b) Prove that **6M**

(i) $f_{A\cap B}=f_A\, f_B$ and

(ii) $f_{A\cup B}=f_A+f_B-f_A\, f_B$

where f is the characteristic function

(c) List all possible functions from the set $X=\{a, b, c\}$ to the se $Y=\{0, 1\}$. Indicate in each case whether the function is one-one or onto or both. **6M**

Q.5.(a) Show that the fourth roots of unity forms an abelian group with respect to multiplication. **6M**

(b) Determine whether the set of even integers with binary operation * defined by $a*b=\frac{ab}{2}$ is semigroup or monoid. Show whether it is commutative. **6M**

OR

Q.6.(a) Show that the intersection of any two normal subgroups of a group G is a normal subgroup of G. **6M**

(b) Let T be the set of all even integers. Show that the semigroup $(Z, +)$ and $(T, +)$ are isomorphic. **7M**

Q.7.(a) Prove that the set $S=\{0, 1, 2, 3, 4\}$ is a ring w.r.t. the operations of addition and multiplication modulo 5. **7M**

(b) Construct the switching circuit for the Boolean expression.

$(A.B)+C+(A'.C')$.

Simplify this and construct an equivalent simplified circuit. **5M**

OR

Q.8.(a) Show that $S=\{a+b\sqrt{2}\mid a,b\in Z\}$ is an integral domain w.r.t. the addition "+" and multiplication "×". **7M**

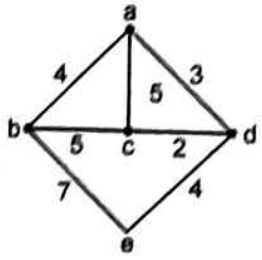
(b) Define a lattice. Draw Hasse diagram of the lattices D_{20} and D_{30}. **5M**

Q.9.(a) Draw the digraph corresponding to the matrix :

$$A=\begin{bmatrix}1&0&1&1\\1&0&1&0\\0&1&0&0\\0&1&1&0\end{bmatrix}$$

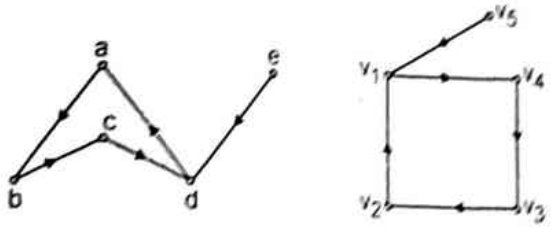
and interpret AA^T, A^TA and A^2. **6M**

(b) Apply Kruskal's algorithm to construct a minimal spanning tree for the weighted graph given below : **6M**



Also find the minimum weight of this spanning tree.

(c) Define isomorphic graphs. Show that following two graphs are isomorphic : **6M**



OR

Q.10.(a) Define : **4M**

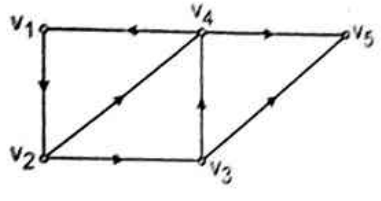
(i) Weighted graph

(ii) Euler's path

(iii) Complete binary tree

(iv) Complete graph

(b) Find in-degree and out-degree of each node of the graph given below and give all elementary cycles of this graph **6M**



(c) Construct the tree for the following expression :

$(5(1-x)+(5-(y+3)))\cdot(7+(x+y))$

Also, draw the corresponding binary tree **8M**

Q.11.(a) State extended pigeonhole principle. Show that if any 30 people are selected, then we may choose a subset of 5 so that all 5 were born on the same day of the week. **5M**

(b) Solve the following recurrence relation : **5M**

$a_r = 3a_{r-1} + 2$, given $a_0 = 1$.

OR

Q.12.(a) Prove that : $C(n, r) = C(n-1, r-1) + C(n-1, r)$ **5M**

(b) Find the generating functions for the sequence :

$1, a, a^2, \text{------}$, where a is a fixed constant. **5M**

SUMMER EXAM. - 2018

SOLUTION

(Unit-Wise)

UNIT - I

(Total Marks : 17)

Q.1.(a) Prove that : **4M**

$A-(B\cap C)=(A-B)\cup(A-C)$

Ans. See P.1-37, Ex.61.

Q.1.(b) Write inverse, contrapositive and converse of the statement :

"Oxygen in air decreases as you go up" **3M**

Ans. See P.1-14, Ex.16.

Q.1.(c) Write the negation of the following statements :

(i) He is tall but handsome.

(ii) If she work, she will earn money and

(iii) If he studies, he will go to college or to art school. **3M**

Ans. (i) See P.PS-4, Q.3(b)(ii).

(ii) P.1-14, Ex.13(i).

(iii) P.PS-4, Q.3(b)(i).

Q.2.(a) Prove by method of induction : **3M**

$1+2^n < 3^n$ for $n \geq 2$.

Ans. Let $P(n): 1+2^n < 3^n$, $n \geq 2$

Consider

$P(2): 1+2^2 < 3^2$

i.e. $5 < 9$

$\therefore$ P(2) is true.

Let the result be true for $n = m \geq 2$ i.e.

$3^m > 1+2^m$ ...(1)

We prove that $P(m+1)$ is also true.

For $n = m+1$, we have

$3^{m+1} = 3.3^m > 3(1+2^m)$ [$\because$ By (1)]

$= 2(1+2^m)+(1+2^m)$

$= 2+2^{m+1}+1+2^m$

$= (1+2^{m+1})+(3+2^m)$

$\therefore\ 3^{m+1} > 1+2^{m+1}$ $\quad [\because 3+2^m > 0 \text{ as } m \geq 2]$

Hence the result is true for $n = m+1$

if it is true for $n = m$ i.e.

$P(m+1)$ is true if $P(m)$ is true.

$\therefore$ By principle of mathematical induction $P(n)$ is true for all $n \geq 2$.

...Ans.

Q.2.(b) Test the validity of the following statement :

"If I like discrete mathematics, then I will study. Either I study discrete mathematics or I failed the course. Therefore, if I fail the course, then I don't like discrete mathematics." **3M**

Ans. Let p : I like DMGT

q : I will study

r : I failed the course

$\therefore\ S_1 : p \to q$

$S_2 : q \vee r$

$S : r \to \sim p$

$\therefore$ The given argument will be valid if

$(S_1 \wedge S_2) \to S$ is a tautology

i.e. $[(p \to q) \wedge (q \vee r)] \to (r \to \sim p)$ is a tautology

p	q	r	(1) $\sim p$	(2) $p \to q$	(3) $q \vee r$	(4) (2) $\wedge$ (3)	(5) $r \to$ (1)	(4) $\to$ (5)
T	T	T	F	T	T	T	F	F
T	T	F	F	T	T	T	T	T
T	F	T	F	F	T	F	F	T
T	F	F	F	F	F	F	T	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	F	F	T	T

Since $(S_1 \wedge S_2) \to S$ is not a tautology, hence the given argument is invalid. ...Ans.

UNIT - II

(Total Marks : 39)

Q.2.(c) Prove that : $A \times (B \cap C) = (A \times B) \cap (A \times C)$ **3M**

Ans. See P.2-10, Ex.4.

Q.3.(a) Let A be the set of non-zero integers and let R be the relation on $A \times A$ defined by $(a, b)\ R\ (c, d) \Leftrightarrow ad = bc$.

Show that R is an equivalence relation. **6M**

Ans. See P.2-17, Ex.18.

Q.3.(b) Let $A = \{a, b, c\}$ and $P(A)$ be its power set. Let "$\subseteq$" be the relation defined on $P(A)$. Draw Hasse diagram of the Poset $(P(A), \subseteq)$. **6M**

Ans. See P.2-30, Ex.50.

Q.3.(c) If $f : X \to Y$ and $g : Y \to Z$ are one-one and onto functions, then show that gof is also one-one and onto and $(gof)^{-1} = f^{-1}og^{-1}$. **6M**

Ans. See P.2-33, Theorem 2.

Q.4.(a) Let X = {ball, bed, dog, egg, let} and

$R = \{(x, y) \mid x, y \in X, x\ R\ y$, if x and y contain some common letter$\}$.

Write M_R. Draw the graph of relation R prove that R is compatible but not transitive. **6M**

Ans. See P.2-23, Ex.32.

Q.4.(b) Prove that : **6M**

(i) $f_{A \cap B} = f_A . f_B$ and

(ii) $f_{A \cup B} = f_A + f_B - f_A . f_B$

where f is the characteristic function.

Ans. See P.2-39, 40, Ex.68, 70.

Q.4.(c) List all possible functions from the set $X = \{a, b, c\}$ to the set $Y = \{0, 1\}$. Indicate in each case whether the function is one-one or onto or both. **6M**

Ans. See P.2-34, Ex.55.

UNIT - III

(Total Marks : 25)

Q.5.(a) Show that the fourth roots of unity forms an abelian group with

respect to multiplication. **6M**

Ans. See P.3-12, Ex.6.

Q.5.(b) Determine whether the set of even integers with binary operation * defined by $a*b=\frac{nb}{2}$ is semigroup or monoid. Show whether it is commutative. **6M**

Ans. See P.3-12, Ex.7.

Q.6.(a) Show that the intersection of any two normal subgroups of a group G is a normal subgroup of G. **6M**

Ans. See P.3-31, Ex.51.

Q.6.(b) Let T be the set of all even integers. Show that the semigroup (Z, +) and (T, +) are isomorphic. **7M**

Ans. See P.3-34, Ex.57(a).

UNIT - IV

(Total Marks : 24)

Q.7.(a) Prove that the set S = {0, 1, 2, 3, 4} is a ring w.r.t. the operations of addition and multiplication modulo 5. **7M**

Ans. See P.PS-31, Q.7(a).

Q.7.(b) Construct the switching circuit for the Boolean expression :

$(A.B)+C+(A'.C')$.

Simplify this and construct an equivalent simplified circuit. **5M**

Ans. See P.4-34, Ex.52.

Q.8.(a) Show that $S=\{a+b\sqrt{2}\mid a,b\in Z\}$ is an integral domain w.r.t. the addition "+" and multiplication "×". **7M**

Ans. See P.4-15, Ex.11.

Q.8.(b) Define a lattice. Draw Hasse diagram of the lattices D_{20} and D_{30}. **5M**

Ans. See P.4-23, 28, Ex.39.

UNIT - V

(Total Marks : 36)

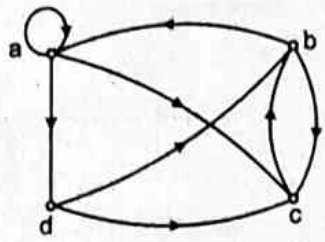
Q.9.(a) Draw the digraph corresponding to the matrix :

$$A=\begin{bmatrix}1&0&1&1\\1&0&1&0\\0&1&0&0\\0&1&1&0\end{bmatrix}$$

and interpret AA^T, A^TA and A^2. **6M**

Ans.

$$A=\begin{array}{c}\\a\\b\\c\\d\end{array}\begin{array}{c}\begin{array}{cccc}a&b&c&d\end{array}\\\begin{bmatrix}1&0&1&1\\1&0&1&0\\0&1&0&0\\0&1&1&0\end{bmatrix}\end{array}$$



$$A^T=\begin{bmatrix}1&1&0&0\\0&0&1&1\\1&1&0&1\\1&0&0&0\end{bmatrix}$$

$$A.A^T=\begin{bmatrix}1&0&1&1\\1&0&1&0\\0&1&0&0\\0&1&1&0\end{bmatrix}\begin{bmatrix}1&1&0&0\\0&0&1&1\\1&1&0&1\\1&0&0&0\end{bmatrix}=\begin{bmatrix}3&2&0&1\\2&2&0&1\\0&0&1&1\\1&1&1&2\end{bmatrix}$$

The diagonal elements of AA^T are the out degrees of a, b, c, d.

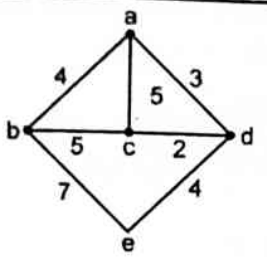
$$A^T.A=\begin{bmatrix}1&1&0&0\\0&0&1&1\\1&1&0&1\\1&0&0&0\end{bmatrix}\begin{bmatrix}1&0&1&1\\1&0&1&0\\0&1&0&0\\0&1&1&0\end{bmatrix}=\begin{bmatrix}2&0&2&1\\0&2&1&0\\2&1&3&1\\1&0&1&1\end{bmatrix}$$

The diagonal elements of A^T are the in degrees of a, b, c, d.

$$\text{Now } A^2=\begin{bmatrix}1&0&1&1\\1&0&1&0\\0&1&0&0\\0&1&1&0\end{bmatrix}\begin{bmatrix}1&0&1&1\\1&0&1&0\\0&1&0&0\\0&1&1&0\end{bmatrix}$$

$$\therefore\ A^2=\begin{bmatrix}1&2&2&1\\1&1&1&1\\1&0&1&0\\1&1&1&0\end{bmatrix}$$ .....Ans.

Q.9.(b) Apply Kruskal's algorithm to construct a minimal spanning tree for the weighted graph given below : **6M**

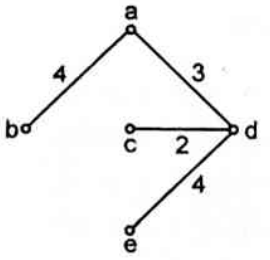


Also find the minimum weight of this spanning tree.

Ans.

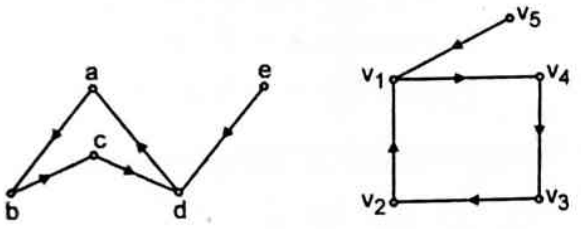
Edge	Weight
(c, d)	2
(a, d)	3
(b, a)	4
(d, e)	4
(a, c)	5
(b, c)	5
(b, e)	7

∴ Required minimal spanning tree is,

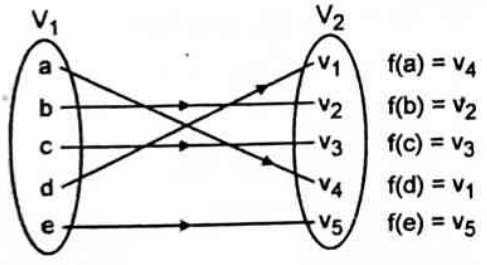


Minimum weight = 4 + 3 + 2 + 4 = 13.

Q.9.(c) Define isomorphic graphs. Show that following two graphs are isomorphic : **6M**



Ans. See P.5-18 for definition.



Clearly $f: V_1 \to V_2$ is one-one and onto.

Also $(a, b) \in E_1 \Rightarrow f(a), f(b) \in E_2$

∴ Both the graphs are isomorphic. ....Ans.

Q.10.(a) Define : **4M**

(i) Weighted graph.

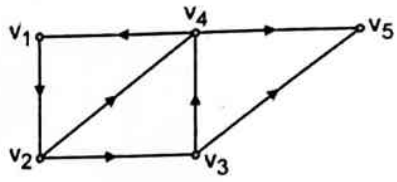
(ii) Euler's path.

(iii) Complete binary tree.

(iv) Complete graph.

Ans. **See P.5-14, 30. 41, 20.**

Q.10.(b) Find in-degree and out-degree of each node of the graph given below and give all elementary cycles of this graph. **6M**



Ans. **See P.5-25, Ex.10(a).**

Q.10.(c) Construct the tree for the following expression :

$$(5(1-x)+(5-(y+3))).(7+(x+y))$$

Also, draw the corresponding binary tree. **8M**

Ans. **See P.5-46, Ex.48.**

UNIT - VI

(Total Marks : 20)

Q.11.(a) State extended pigeonhole principle. Show that if any 30 people are selected, then we may choose a subset of 5 so that all 5 were born on the same day of the week. **5M**

Ans. **See P.6-8, 9, Ex.23.**

Q.11.(b) Solve the following recurrence relation : **5M**

$a_r = 3a_{r-1} + 2$, given $a_o = 1$.

Ans. **See P.6-18, Ex.33(c).**

Q.12.(a) Prove that : $C(n, r) = C(n-1, r-1) + C(n-1, r)$ **5M**

Ans. **See P.PS-26, Q.12(b).**

Q.12.(b) Find the generating functions for the sequence :

$1, a, a^2, ——$, where a is a fixed constant. **5M**

Ans. **See P.6-21, Ex.34.**

WINTER EXAMINATION - 2018

B.E. IV SEM. (CT, CS, IT, CE) (CBS)

DISCRETE MATHEMATICS AND GRAPH THEORY

Statistical Analysis

Unit Number	Marks
Unit - I	20 M
Unit - II	36 M
Unit - III	24 M
Unit - IV	24 M
Unit - V	36 M
Unit - VI	20 M
Questions covered in VBD	160 M (100 %)

WINTER EXAM.- 2018
QUESTION PAPER

Q.1. (a) State and prove De Morgan's laws. **5M**

(b) Prove that : $A \cap (B - C) = (A \cap B) - (A \cap C)$. **5M**

OR

Q.2. (a) Test the validity of the following argument :

If I study, then I will not fail in Mathematics :

If I do not play basketball, then I will study.

But I failed in Mathematics.

Therefore, I must have played basketball. **5M**

(b) Prove by mathematical induction that :

$3^{4n+2} + 5^{2n+1}$ is a multiple of 14. **5M**

Q.3. (a) Draw the tree diagram of $(A \times B \times C)$

where $A=\{2, 3\}$, $B=\{1, 3, 5\}$, $C=\{3, 4\}$ and hence find $(A \times B \times C)$. **6M**

(b) Let A be a given set and P(A) be its power set. Let $\leq$ be the inclusion relation on P (A).

Draw the Hasse diagram of $(P(A), \leq)$ for $A=\{a,b\}$ and $A=\{a,b,c\}$. **6M**

(c) Using the properties of characteristic function, prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. **6M**

OR

Q.4.(a) For the relation $R=\{(1,2),(2,1),(2,3),(3,4)\}$ find the relation matrix and draw the graph. Also find the transitive closure of R. **6M**

(b) Let X = {ball, bed, egg, dog, let} and $R=\{(x, y) \mid x, y \in X, xRy$ if x and y contain some common letter$\}$. Draw the graph of R. Prove that R is compatible but not transitive. **6M**

(c) Let $f : A \to B$ and $g : B \to C$ such that $f(a) = a - 1$ and $g(b) = b^2$.

Find : (i) $f \circ g\ (x)$ (ii) $g \circ f\ (x)$ (iii) $g \circ g\ (x)$ (iv) $f \circ f\ (x)$. **6M**

Q.5.(a) Prove that fourth roots of unity forms a group under multiplication. **6M**

(b) Prove that the order of each subgroup of a finite group is a divisor of the order of the group. **6M**

OR

Q.6.(a) Prove that monoid homomorphism preserves associativity, identity, commutativity and invertibility. **6M**

(b) If f is a homomorphism of G into G' with kernel K, then prove that K is a normal subgroup of G. **6M**

Q.7. (a) If R is a ring, then show that for all $a,b \in R$:

(i) $a \cdot 0 = 0 = 0 \cdot a$

(ii) $a \cdot (-b) = (-a) \cdot b = -(a \cdot b)$.

(iii) $(-a) \cdot (-b) = a \cdot b$. **6M**

(b) Prove that every chain is a distributive lattice. **6M**

OR

Q.8.(a) If $f : R \to R'$ is a ring homomorphism, then prove that :

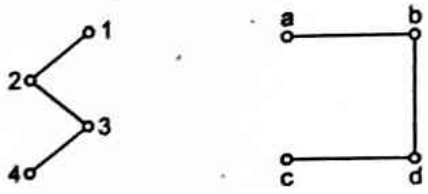
(i) $f(0) = 0$ (ii) $f(-a) = -f(a)$, for all $a \in R$. **6M**

(b) Construct the switching circuit for the Boolean expression

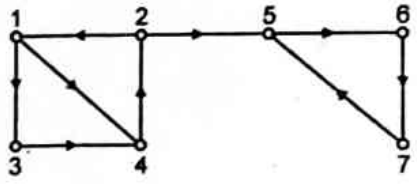
$(A \cdot B) + [A' \cdot (A + B + B')]$

Simplify this expression and also draw the equivalent switching circuit. **6M**

Q.9.(a) Show that the following graphs are isomorphic. **6M**



(b) Find the node base of the following digraph : **6M**



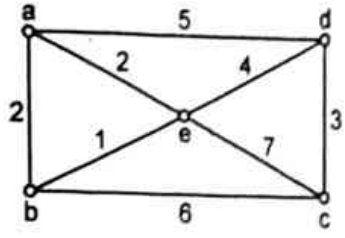
(c) Draw a directed tree with two nodes at level 1, five nodes at level 2, three nodes at level 3. Obtain the corresponding binary tree. **6M**

OR

Q.10.(a) Draw a digraph corresponding to the following adjacency matrix and interprete the results AA^T, A^TA, A^2 : **6M**

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

(b) Find the minimal spanning tree of the following graph using Prim's algorithm : **6M**



(c) Represent the following algebraic expressions using binary tree : **6M**

(i) $(x + (y + z)) - (a \times (b + c))$.

(ii) $(2x + (3 - 4x)) + (x - (3 \times 11))$.

Q.11.(a) Show that if five integers are chosen from 1 to 8, then two of them will have a sum 9. **5M**

(b) Solve the following recurrence relation using generating function $a_n = 3a_{n-1} + 2$ with $a_0 = 1$. **5M**

OR

Q.12.(a) Find the generating function of $n^2, n \geq 0$. **5M**

(b) Prove that : $C(n+1, r) = C(n, r) + C(n, r-1)$. **5M**

WINTER EXAM. - 2018

SOLUTION

(Unit-Wise)

UNIT - I

(Total Marks : 20)

Q.1.(a) State and prove De Morgan's laws. **5M**

Ans. See P.1-18, Ex.28.

Q.1.(b) Prove that : $A \cap (B - C) = (A \cap B) - (A \cap C)$. **5M**

Ans. We prove that L.H.S. $\subseteq$ R.H.S. and

R.H.S. $\subseteq$ L. H. S.

Let $x \in A \cap (B - C)$

$\Rightarrow x \in A$ and $x \in B - C$

$\Rightarrow x \in A$ and $x \in B$ and $x \notin C$

$\Rightarrow (x \in A$ and $x \in B)$ and $(x \in A$ and $x \notin C)$

$\Rightarrow x \in A \cap B$ and $x \notin A \cap C$

$\therefore x \in A \cap (B - C) \Rightarrow x \in (A \cap B) - (A \cap C)$

So $A \cap (B - C) \subseteq (A \cap B) - (A \cap C)$

$\therefore$ L.H.S. $\subseteq$ R. H. S. ...(1)

Let $y \in (A \cap B) - (A \cap C)$

$\Rightarrow y \in A \cap B$ and $y \notin A \cap C$

$\Rightarrow (y \in A$ and $y \in B)$ and $(y \in A$ and $y \notin C)$

$\Rightarrow y \in A$ and $y \in B$ and $y \notin C$

$\Rightarrow y \in A$ and $y \in (B - C)$

$\Rightarrow y \in A \cap (B - C)$

So $(A \cap B) - (A \cap C) \subseteq A \cap (B - C)$

$\therefore$ R.H.S. $\subseteq$ L. H. S. ...(2)

From (1) and (2), we get

$\boxed{A \cap (B - C) = (A \cap B) - (A \cap C)}$ Ans.

Q.2.(a) Test the validity of the following argument :

If I study, then I will not fail in Mathematics :

If I do not play basketball, then I will study.

But I failed in Mathematics.

Therefore, I must have played basketball. **5M**

Ans. See P.1-21, Ex.33.

Q.2.(b) Prove by mathematical induction that :

$3^{4n+2} + 5^{2n+1}$ is a multiple of 14. **5M**

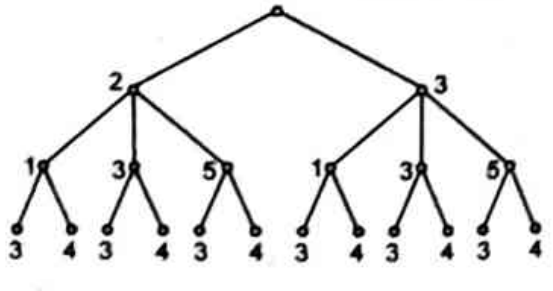
Ans. See P.1-44, Ex.85(vii).

UNIT - II

(Total Marks : 36)

Q.3.(a) Draw the tree diagram of $(A \times B \times C)$ where $A = \{2, 3\}$, $B = \{1, 3, 5\}$, $C = \{3, 4\}$ and hence find $(A \times B \times C)$. **6M**

Ans.



$A \times B \times C = \{(2, 1, 3), (2, 1, 4), (2, 3, 3), (2, 3, 4), (2, 5, 3), (2, 5, 4), (3, 1, 3), (3, 1, 4), (3, 3, 3), (3, 3, 4), (3, 5, 3), (3, 5, 4)\}$

Q.3.(b) Let A be a given set and P (A) be its power set. Let $\leq$ be the inclusion relation on P (A).

Draw the Hasse diagram of (P (A), $\leq$) for

A = {a,b} and A = {a,b,c}. **6M**

Ans. See P.2-30, Ex.50.

Q.3.(c) Using the properties of characteristic function, prove that

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. **6M**

Ans. See P.2-38, Ex.66(i).

Q.4.(a) For the relation R = {(1,2),(2,1),(2,3),(3,4)}, find the relation matrix and draw the graph. Also find the transitive closure of R. **6M**

Ans. See P.2-22, Ex.31.

Q.4.(b) Let X = {ball, bed, egg, dog, let} and R = {(x, y) | x, y ∈ X, xRy if x and y contain some common letter}. Draw the graph of R. Prove that R is compatible but not transitive. 6M

Ans. See P.2-23, Ex.32.

Q.4.(c) Let $f : A \to B$ and $g : B \to C$ such that $f(a) = a - 1$ and $g(b) = b^2$.

Find : (i) $f \circ g$ (x) (ii) $g \circ f$ (x) (iii) $g \circ g$ (x) (iv) $f \circ f$ (x). **6M**

Ans. See P.PS-15, Q.3(c).

UNIT - III

(Total Marks : 24)

Q.5.(a) Prove that fourth roots of unity forms a group under multiplication. **6M**

Ans. See P.3-12, Ex.6.

Q.5.(b) Prove that the order of each subgroup of a finite group is a divisor of the order of the group. **6M**

Ans. See P.3-28, Ex.45.

Q.6.(a) Prove that monoid homomorphism preserves associativity, identity, commutativity and invertibility. **6M**

Ans. See P.3-9, Ex.2.

Q.6.(b) If f is a homomorphism of G into G' with kernel K, then prove that K is a normal subgroup of G. **6M**

Ans. See P.3-35, Ex.62.

UNIT - IV

(Total Marks : 24)

Q.7.(a) If R is a ring, then show that for all $a, b \in R$:

(i) $a \cdot 0 = 0 = 0 \cdot a$

(ii) $a \cdot (-b) = (-a) \cdot b = -(a \cdot b)$.

(iii) $(-a) \cdot (-b) = a \cdot b$. **6M**

Ans. See P.4-11, Ex.6.

Q.7.(b) Prove that every chain is a distributive lattice. **6M**

Ans. See P.4-27, Ex.35.

Q.8.(a) If $f : R \to R'$ is a ring homomorphism, then prove that :

(i) $f(0) = 0$ (ii) $f(-a) = -f(a)$, for all $a \in R$. **6M**

Ans. See P.4-19, Ex.17.

Q.8.(b) Construct the switching circuit for the Boolean expression

$(A \cdot B) + [A' \cdot (A + B + B')]$

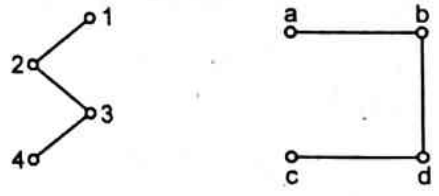
Simplify this expression and also draw the equivalent switching circuit. **6M**

Ans. See P.4-33, Ex.50.

UNIT - V

(Total Marks : 36)

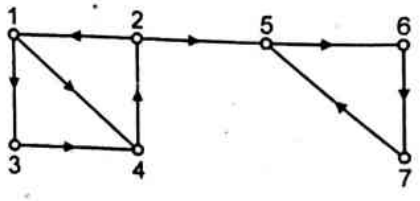
Q.9.(a) Show that the following graphs are isomorphic :



6M

Ans. See P.5-18, Ex.6(a).

Q.9.(b) Find the node base of the following digraph :



6M

Ans. See P.5-26, Ex.11(i).

Q.9.(c) Draw a directed tree with two nodes at level 1, five nodes at level 2, three nodes at level 3. Obtain the corresponding binary tree. **6M**

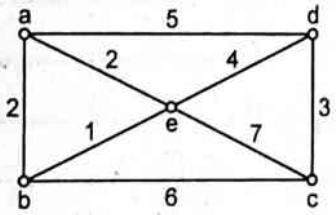
Ans. **See P.5-42, Ex.36.**

Q.10.(a) Draw a digraph corresponding to the following adjacency matrix and interprete the results AA^T, A^TA, A^2 : **6M**

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Ans. **See P.5-23, Ex.8(a).**

Q.10.(b) Find the minimal spanning tree of the following graph using Prim's algorithm : **6M**



Ans. **See P.5-38, Ex.30.**

Q.10.(c) Represent the following algebraic expressions using binary tree : **6M**

(i) $(x + (y + z)) - (a \times (b + c))$.

(ii) $(2x + (3 - 4x)) + (x - (3 \times 11))$.

Ans. **See P.5-43, 44, Ex.39(iv), 40(ii).**

UNIT - VI

(Total Marks : 20)

Q.11.(a) Show that if five integers are chosen from 1 to 8, then two of them will have a sum 9. **5M**

Ans. **See P.6-10, Ex.25.**

Q.11.(b) Solve the following recurrence relation using generating function $a_n = 3a_{n-1} + 2$ with $a_0 = 1$. **5M**

Ans. **See P.6-18, Ex.33(c).**

Q.12.(a) Find the generating function of $n^2, n \geq 0$. **5M**

Ans. **See P.PS-55, Q.12(a).**

Q.12.(b) Prove that : $C(n + 1, r) = C(n, r) + C(n, r - 1)$. **5M**

Ans. **See P.PS-34, Q.12(a) and**

See P.PS-26, Q.12(b).

Notes :-

SUMMER EXAMINATION - 2019

B.E. IV SEM. (CT, CS, IT, CE) (CBS)

DISCRETE MATHEMATICS AND GRAPH THEORY

Statistical Analysis

Unit Number	Marks
Unit - I	20 M
Unit - II	36 M
Unit - III	24 M
Unit - IV	24 M
Unit - V	36 M
Unit - VI	20 M
Questions covered in VBD	160 M (100 %)

SUMMER EXAM.- 2019

QUESTION PAPER

Q.1.(a) For any two non-empty sets A and B, prove that :

$A-(B\cup C)=(A-B)\cap(A-C)$ **3M**

(b) Write the inverse, converse and contra positive of "If x is boss then x is bad". **3M**

(c) Using principle of mathematical induction, show that for all positive integer 'n' n^3+2n is divisible by 3. **4M**

OR

Q.2.(a) If my brother stands first in the class then

I will give him a watch.

He stood first or I was out of station.

I did not give him a watch.

Therefore, I was out of station

Check the validity of the above argument. **5M**

(b) If $U=\{0,1,2,3,4,5,6,7,8\}$,

$A=\{1,5,6,7,8\}$, $B=\{0,1,6,7\}$,

$C=\{1,2,3,5,8\}$, then verify that :

(i) $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$

(ii) $(A\cap B)'=A'\cup B'$ **5M**

Q.3.(a) Let R be the relation defined on the set of ordered pairs of positive integers defined by (x, y) R (u, v) iff xv = yu, then show that 'R' is an equivalence relation. **6M**

(b) For the relation $R=\{(1,2),(2,3),(3,4),(2,1)\}$. Find the relation matrix of R, draw the graph of R and also find the transitive closure of R. **6M**

(c) Given $M_R=\begin{bmatrix}1&0&1\\1&1&0\\1&1&1\end{bmatrix}$, $M_S=\begin{bmatrix}1&0&0&1&0\\1&0&1&0&1\\0&1&0&1&0\end{bmatrix}$

Find R, S, $R\circ S$, $\tilde{R}$, $\tilde{S}$, $S\circ R$ and

prove that $M_{\tilde{R}\circ\tilde{S}}=M_{\tilde{S}\circ\tilde{R}}$. **6M**

OR

Q.4.(a) Let X = {hall, bed, dog, egg, let} and R = {(x, y) / x, y ∈ X, xRy if x and y contains some common letter}. Find M_R, draw the graph of R and prove that R is compatible. **6M**

(b) Using properties of characteristic function, prove that :

(i) $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$

(ii) $(A')'=A$ **6M**

(c) If $f:X\to Y$ and $g:Y\to Z$, then show that :

(i) If $g\circ f$ is one-one then f is one-one

(ii) If $g\circ f$ is onto then g is onto **6M**

Q.5.(a) Prove that fourth roots of unity forms an abelian group under multiplication. **6M**

(b) Show that the set of matrices :

$A_\alpha=\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$, where $\alpha\in R$, forms a monoid. **6M**

OR

Q.6.(a) Prove that the intersection of two normal subgroups of a group G is a normal subgroup of G. **6M**

(b) A non-empty subset H of a group G is a subgroup of G iff :

(i) $a,b\in H\Rightarrow a.b\in H$

(ii) $a\in H\Rightarrow a^{-1}\in H$ **6M**

Q.7.(a) If R is a ring such that $a^2=a, \forall a\in R$, then show that :

(i) $a+a=0, \forall a\in R$

(ii) $a+b=0\Rightarrow a=b, \forall a,b\in R$

(iii) R is a commutative ring. **6M**

(b) Show that the set S of all matrices of the form

$\begin{bmatrix}a&b\\-b&a\end{bmatrix}$ where a, b ∈ R is a field w.r.to matrix addition and multiplication. **6M**

OR

Q.8.(a) Construct switching circuit for the Boolean expression

$A\cdot B+A\cdot B'+A'\cdot B'$ simplify this and construct an equivalent circuit. Verify the equivalent circuit by truth table. **6M**

(b) Let (L_1, D_6) and $L_2=(\rho(s), s)$ be two letters, where $D_6=\{1,2,3,6\}$ and $S=\{a,b\}$ then show that L_1 and L_2 are isomorphic. **6M**

Q.9.(a) Draw the digraphs corresponding to adjacency matrices : **6M**

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Prove that diagraphs of A and B are isomorphic.

(b) Draw a diagraph corresponding to the adjacency matrix

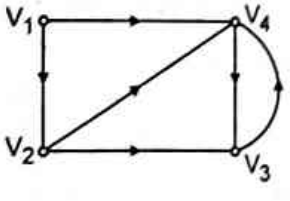
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$ and interpret the results $AA^T, A^TA, A^2, A^3, A^4$

6M

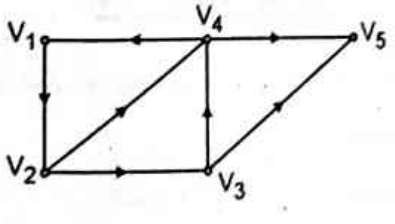
(c) Draw a tree for the relation R = {(1, 2), (1, 3), (1, 4), (2, 5), (4, 6), (4, 7)} on a set A = {1, 2, 3, 4, 5, 6, 7}. Also give the corresponding binary tree. **6M**

OR

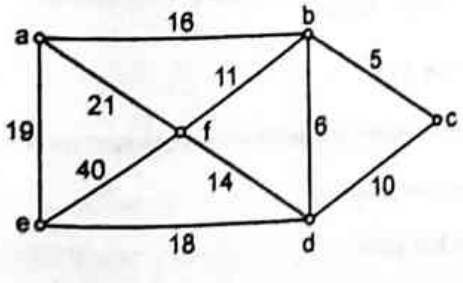
Q.10.(a) Give three different elementary paths from V_1 to V_3 for the diagraph given below. What is the shortest distance between V_1 and V_3 ? Is there any cycle in the graph? **6M**



(b) Find indegree and outdegree of each node of the graph given below. Give all elementary cycles of this graph. List all the nodes which are reachable from other nodes. **6M**



(c) Using Prim's algorithm, find a minimal spanning tree for the given graph : **6M**



Q.11.(a) Solve the recurrence relation :

$a_n = 3a_{n-1} + 2, a_0 = 1$ **5M**

(b) Find the minimum number of students in a class to be sure that four of them are born in the same month. **5M**

OR

Q.12.(a) Find the generating function of $n^2, n \geq 0$. **5M**

(b) Prove that : $C(n, r) = C(n-1, r-1) + C(n-1, r)$ **5M**

SUMMER EXAM. - 2019

SOLUTION

(Unit-Wise)

UNIT - I

(Total Marks : 20)

Q.1.(a) For any two non-empty sets A and B, prove that :

$A - (B \cup C) = (A - B) \cap (A - C)$ **3M**

Ans. See P.1-37, Ex.60.

Q.1.(b) Write the inverse, converse and contra positive of "If x is boss then x is bad". **3M**

Ans. Let p : x is boss

q : x is bad

Converse ($q \rightarrow p$) : If x is bad, then x is boss.

Inverse ($\sim p \rightarrow \sim q$) : If x is not boss, then x is not bad.

Contrapositive ($\sim q \rightarrow \sim p$) : If x is not bad, then x is not boss.

Q.1.(c) Using principle of mathematical induction, show that for all positive integer 'n' $n^3 + 2n$ is divisible by 3. **4M**

Ans. See P.1-44, Ex.85(iii).

Q.2.(a) If my brother stands first in the class then

I will give him a watch.

He stood first or I was out of station.

I did not give him a watch.

Therefore, I was out of station

Check the validity of the above argument. **5M**

Ans. See P.PS-3, Q.3(a).

Q.2.(b) If $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$,

$A = \{1, 5, 6, 7, 8\}$, $B = \{0, 1, 6, 7\}$,

$C = \{1, 2, 3, 5, 8\}$, then verify that :

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(ii) $(A \cap B)' = A' \cup B'$ **5M**

Ans. Given : $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$

$A = \{1, 5, 6, 7, 8\}$

$B = \{0, 1, 6, 7\}$ and

$C = \{1, 2, 3, 5, 8\}$

(i) $B \cap C = \{1\}$

$\therefore A \cup (B \cap C) = \{1, 5, 6, 7, 8\}$

$A \cup B = \{0, 1, 5, 6, 7, 8\}$ and

$A \cup C = \{1, 2, 3, 5, 6, 7, 8\}$

Now, $(A \cup B) \cap (A \cup C) = \{1, 5, 6, 7, 8\}$

$\therefore \boxed{A \cup (B \cap C) = (A \cup B) \cap (A \cup C)}$ **Ans.**

(ii) $A \cap B = \{1, 6, 7\}$

$\Rightarrow (A \cap B)' = \{0, 2, 3, 4, 5, 8\}$

Also $A' = \{0, 2, 3, 4\}$ and

$B' = \{2, 3, 4, 5, 8\}$

$\Rightarrow A' \cup B' = \{0, 2, 3, 4, 5, 8\}$

$\therefore \boxed{(A \cap B)' = A' \cup B'}$ **Ans.**

UNIT - II

(Total Marks : 36)

Q.3.(a) Let R be the relation defined on the set of ordered pairs of positive integers defined by (x, y) R (u, v) iff xv = yu, then show that 'R' is an equivalence relation. **6M**

Ans. See P.2-17, Ex.18.

Q.3.(b) For the relation $R = \{(1, 2), (2, 3), (3, 4), (2, 1)\}$. Find the relation matrix of R, draw the graph of R and also find the transitive closure of R. **6M**

Ans. See P.2-22, Ex.31.

Q.3.(c) Given

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, M_S = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Find $R, S, R \circ S, \tilde{R}, \tilde{S}, S \circ R$ and

prove that $M_{\tilde{R} \circ \tilde{S}} = M_{\tilde{S} \circ \tilde{R}}$. **6M**

Ans. See P.2-27, Ex.41.

Q.4.(a) Let X = {ball, bed, dog, egg, let} and $R = \{(x, y) / x, y \in X, xRy$ if x and y contains some common letter$\}$. Find M_R, draw the graph of R and prove that R is compatible. **6M**

Ans. See P.2-23, Ex.32.

Q.4.(b) Using properties of characteristic function prove that :

(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(ii) $(A')' = A$ **6M**

Ans. See P.2-38, Ex.66.

Q.4.(c) If $f : X \to Y$ and $g : Y \to Z$, then show that :

(i) If $g \circ f$ is one-one, then f is one-one.

(ii) If $g \circ f$ is onto, then g is onto. **6M**

Ans. See P.2-37, Ex.61.

UNIT - III

(Total Marks : 24)

Q.5.(a) Prove that fourth roots of unity forms an abelian group under multiplication. **6M**

Ans. See P.3-12, Ex.6.

Q.5.(b) Show that the set of matrices :

$$A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, \text{ where } \alpha \in R, \text{ forms a monoid.}$$ **6M**

Ans. See P.3-8, Ex.1.

Q.6.(a) Prove that the intersection of two normal subgroups of a group G is a normal subgroup of G. **6M**

Ans. See P.3-31, Ex.51.

Q.6.(b) A non-empty subset H of a group G is a sub-group of G iff :

(i) $a, b \in H \Rightarrow a . b \in H$

(ii) $a \in H \Rightarrow a^{-1} \in H$ **6M**

Ans. **See P.3-26, Ex.39.**

UNIT - IV

(Total Marks : 24)

Q.7.(a) If R is a ring such that $a^2 = a, \forall a \in R$, then show that :

(i) $a + a = 0, \forall a \in R$

(ii) $a + b = 0 \Rightarrow a = b, \forall a, b \in R$

(iii) R is a commutative ring. **6M**

Ans. **See P.4-11, Ex.5.**

Q.7.(b) Show that the set S of all matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ where a, b $\in$ R is a field w.r.to matrix addition and multiplication. **6M**

Ans. **See P.4-17, Ex.13.**

Q.8.(a) Construct switching circuit for the Boolean expression $A \cdot B + A \cdot B' + A' \cdot B'$ simplify this and construct an equivalent circuit. Verify the equivalent circuit by truth table. **6M**

Ans. **See P.4-34, Ex.51.**

Q.8.(b) Let (L_1, D_6) and $L_2 = (\rho(s), s)$ be two letters, where $D_6 = \{1, 2, 3, 6\}$ and $S = \{a, b\}$, then show that L_1 and L_2 are isomorphic. **6M**

Ans. **See P.4-27, Ex.33.**

UNIT - V

(Total Marks : 36)

Q.9.(a) Draw the digraphs corresponding to adjacency matrices : **6M**

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Prove that diagraphs of A and B are isomorphic.

Ans. **See P.5-32, Ex.18.**

Q.9.(b) Draw a diagraph corresponding to the adjacency matrix

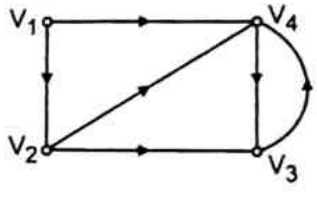
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$ and interpret the results $AA^T, A^TA, A^2, A^3, A^4$. **6M**

Ans. **See P.5-23, Ex.8(a).**

Q.9.(c) Draw a tree for the relation R = {(1, 2), (1, 3), (1, 4), (2, 5), (4, 6), (4, 7)} on a set A = {1, 2, 3, 4, 5, 6, 7}. Also give the corresponding binary tree. **6M**

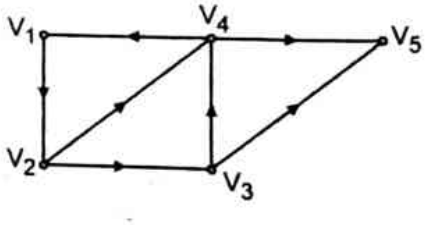
Ans. **See P.5-41, Ex.34.**

Q.10.(a) Give three different elementary paths from V_1 to V_3 for the diagraph given below. What is the shortest distance between V_1 and V_3 ? Is there any cycle in the graph? **6M**



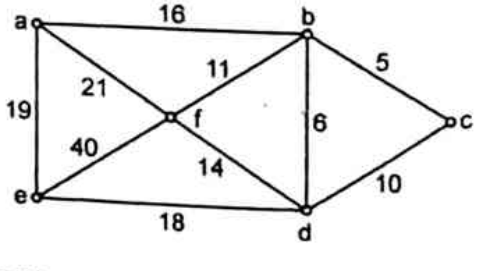
Ans. **See P.5-25, Ex.9.**

Q.10.(b) Find indegree and outdegree of each node of the graph given below. Give all elementary cycles of this graph. List all the nodes which are reachable from other nodes. **6M**



Ans. **See P.5-25, Ex.10(a).**

Q.10.(c) Using Prim's algorithm, find a minimal spanning tree for the given graph : **6M**



Ans. **See P.PS-49, Q.10(a).**

UNIT - VI

(Total Marks : 20)

Q.11.(a) Solve the recurrence relation :

$a_n = 3a_{n-1} + 2, a_0 = 1$ **5M**

Ans. See P.6-18, Ex.33(C).

Q.11.(b) Find the minimum number of students in a class to be sure that four of them are born in the same month. **5M**

Ans. See P.6-9, Ex.24.

Q.12.(a) Find the generating function of $n^2, n \geq 0$. **5M**

Ans. See P.PS-55, Q.12(a).

Q.12.(b) Prove that : $C(n, r) = C(n-1, r-1) + C(n-1, r)$ **5M**

Ans. See P.PS-26, Ex.12(b).

Notes :-

WINTER EXAMINATION - 2019
B.E. IV SEM. (CT, CS, IT, CE) (CBS)
DISCRETE MATHEMATICS AND GRAPH THEORY

Statistical Analysis

Unit Number	Marks
Unit - I	20 M
Unit - II	36 M
Unit - III	24 M
Unit - IV	24 M
Unit - V	36 M
Unit - VI	20 M
Questions covered in VBD	160 M (100 %)

WINTER EXAM.- 2019
QUESTION PAPER

Q.1.(a) Prove that: $A \cap (B - C) = (A \cap B) - (A \cap C)$ **5M**

(b) Prove by Mathematical induction that $n^3 + 2n$ is divisible by 3. **5M**

OR

Q.2.(a) Check the validity of the following argument: If it rains today, then we will not have a party today. If we do not have a party today, then we will have a party tomorrow

Therefore, if it rains today, then we will have a party tomorrow. **5M**

(b) Prove that $\sqrt{2}$ is an irrational number. **5M**

Q.3.(a) If Z is a set of integers and a, b $\in$ Z, a R b if $b = a^r$ for some positive integer r, show that R is a partial order relation. **6M**

(b) Let A = {1, 2, 3, 4, 5, 6}. Define a relation R on A such that **6M**

R = {(x, y) / x + y is a divisor of 24}

Find:

(i) Relation matrix R

(ii) Relation matrix of $R \circ R$

(iii) Draw digraph of R and $R \circ R$

(c) If $f : A \to B$ and $g : B \to C$ both f and g are one-one and onto, then show that $g \circ f$ is also one-one and onto and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. **6M**

OR

Q.4.(a) Let A = R × R. A relation R on A is defined as (a, b) R (c, d) iff $a^2 + b^2 = c^2 + d^2$. Show that R is an equivalence relation. **6M**

(b) For any integer n, S_n denotes the set of all divisors of n and D denotes the division operator.

Draw the Hasse diagram for. (i) n= 30 (ii) n = 36. **6M**

(c) Using characteristic function, show that:

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. **6M**

Q.5.(a) Prove that fourth roots of unity forms a group under multiplication. **6M**

(b) Prove that the intersection of two subgroups of a group G is a subgroup of G. **6M**

OR

Q.6.(a) Show that the multiplication group G = {1, i, −1, −i} is isomorphic to the group G′ = {0, 1, 2, 3} with addition modulo 4 as composition. **6M**

(b) Show that if f is homomorphism of G into G′ with kernel K, then Kernel K is normal subgroup of G. **6M**

Q.7.(a) Show that $(I, \oplus, \bigcirc)$ is a commutative ring with identity whose operation $\oplus$ and $\bigcirc$ are defined as

for any $a, b \in I$, $a \oplus b = a + b - 1$

and $a \bigcirc b = a + b - ab$

where I is a set of integer **6M**

(b) Construct the switching circuit for the Boolean expression:

$(A \cdot B) + C + (A' \cdot C')$

Simplify this and construct an equivalent simplified circuit. **6M**

OR

Q.8.(a) Define lattice. Draw the Hasse diagram of lattices D_{24} and D_{30}. **6M**

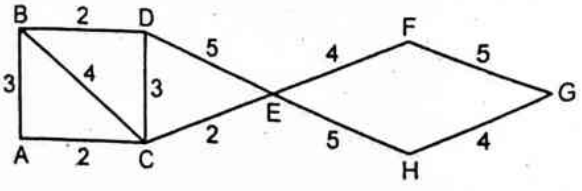
(b) Show that the set {0, 1, 2, 3, 4, 5, 6} is a commutative ring w. r. to addition modulo 7 and multiplication modulo 7 as the composition. **6M**

Q.9.(a) Draw the digraph corresponding to adjacency matrix A, B, A^T and B^T where

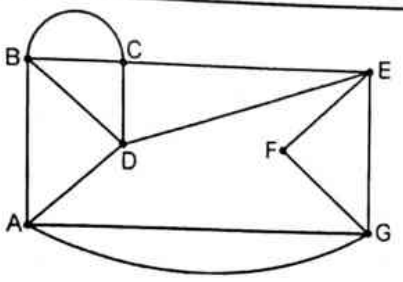
$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Show that the digraph corresponding to A^T and B^T are isomorphic. **6M**

(b) Find the minimal spanning tree of the following graph using Prim's algorithm: **6M**



(c) Define Eulerian path and Eulerian circuit. Show that the graph given below is an Eulerian graph and circuit: **6M**



OR

Q.10.(a) Draw the tree for the algebraic expressions:

(i) $(a + 5) * [\{(7 + b) + c\} / (g + d)]$

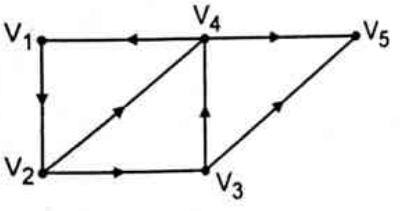
(ii) $(2x + (3 - 4x)) + (x - (3 * 11))$ **6M**

(b) Draw the diagraph represented by the given adjacency matrix and interpreter. **6M**

AA^T, A^TA, A^2

$$\text{where } A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

(c) Find the indegrees and outdegrees of the node of the graph. Give all elementary cycles of this graph. Obtain an acyclic digraph by deleting the edge of the given graph. List all the nodes which are reachable from another node of the graph. **6M**



Q.11.(a) Show that if any five numbers from 1 to 8 are chosen, then two of them add up to 9 using pigeon hole principle. **5M**

(b) Solve the recurrence relation:

$2a_n - 7a_{n+1} + 3a_{n-2} = 2^n$ **5M**

OR

Q.12.(a) Find the generating function for the sequence

1, 0, – 1, 0, 1, 0, – 1, 0, 1, ------ **5M**

(b) How many words of 3 different letters can be formed from the letters of word: **5M**

(i) MAST

(ii) COMPUTER

WINTER EXAM. - 2019

SOLUTION

(Unit-Wise)

UNIT - I

(Total Marks : 20)

Q.1.(a) Prove that: $A \cap (B - C) = (A \cap B) - (A \cap C)$ **5M**

Ans. See P.PS-77, Q.1(b).

Q.1.(b) Prove by Mathematical induction that $n^3 + 2n$ is divisible by 3. **5M**

Ans. See P.1-44, Ex.85(iii).

Q.2.(a) Check the validity of the following argument: If it rains today, then we will not have a party today. If we do not have a party today, then we will have a party tomorrow.

Therefore, if it rains today, then we will have a party tomorrow. **5M**

Ans. See P.1-24, Ex.41.

Q.2.(b) Prove that $\sqrt{2}$ is an irrational number. **5M**

Ans. See P.1-28, Ex.50.

UNIT - II

(Total Marks : 36)

Q.3.(a) If Z is a set of integers and a, b $\in$ Z, a R b if $b = a^r$ for some positive integer r, show that R is a partial order relation. **6M**

Ans. See P.2-14, Ex.9.

Q.3.(b) Let A = {1, 2, 3, 4, 5, 6}. Define a relation R on A such that **6M**

R = {(x, y) / x + y is a divisor of 24}

Find:

(i) Relation matrix R

(ii) Relation matrix of $R \circ R$

(iii) Draw digraph of R and $R \circ R$

Ans. See P.PS-14, Q.3(b).

Q.3.(c) If $f : A \to B$ and $g : B \to C$ both f and g are one-one and onto, then show that $g \circ f$ is also one-one and onto and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. **6M**

Ans. See P.2-33, Theorem 2.

Q.4.(a) Let $A = R \times R$. A relation R on A is defined as (a, b) R (c, d) iff $a^2 + b^2 = c^2 + d^2$ Show that R is an equivalence relation. **6M**

Ans. See P.2-16, Ex.13.

Q.4.(b) For any integer n, S_n denotes the set of all divisors of n and D denotes the division operator

Draw the Hasse diagram for: (i) n= 30 (ii) n = 36. **6M**

Ans. See P.2-30 Ex.49.

Q.4.(c) Using characteristic function, show that

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. **6M**

Ans. See P.2-38, Ex.66(i).

UNIT - III

(Total Marks : 24)

Q.5.(a) Prove that fourth roots of unity forms a group under multiplication. **6M**

Ans. See P.3-12, Ex.6.

Q.5.(b) Prove that the intersection of two subgroups of a group G is a subgroup of G. **6M**

Ans. See P.3-27, Ex.42.

Q.6.(a) Show that the multiplication group $G = \{1, i, -1, -i\}$ is isomorphic to the group $G' = \{0, 1, 2, 3\}$ with addition modulo 4 as composition. **6M**

Ans. See P.3-33, Ex.57.

Q.6.(b) Show that if f is homomorphism of G into G' with kernel K, then Kernel K is normal subgroup of G. **6M**

Ans. See P.3-35, Ex.62.

UNIT - IV

(Total Marks : 24)

Q.7.(a) Show that $(I, \oplus, \bigcirc)$ is a commutative ring with identity whose operation $\oplus$ and $\bigcirc$ are defined as

for any $a, b \in I$ $a \oplus b = a + b - 1$

and $a \bigcirc b = a + b - ab$

where I is a set of integer **6M**

Ans. See P.PS-10, Q.7(b)

Q.7.(b) Construct the switching circuit for the Boolean expression:

$(A \cdot B) + C + (A' \cdot C')$

Simplify this and construct an equivalent simplified circuit. **6M**

Ans. See P.4-34, Ex.52.

Q.8.(a) Define lattice, Draw the Hasse diagram of lattices D_{24} and D_{30}. **6M**

Ans. See P.4-23, Definition & P.4-25, 26, Q.29, 31.

and P.PS-66, Q.7(b).

Q.8.(b) Show that the set {0, 1, 2, 3, 4, 5, 6} is a commutative ring w. r. to addition modulo 7 and multiplication modulo 7 as the composition. **6M**

Ans. See P.4-9, Ex.4.

UNIT - V

(Total Marks : 36)

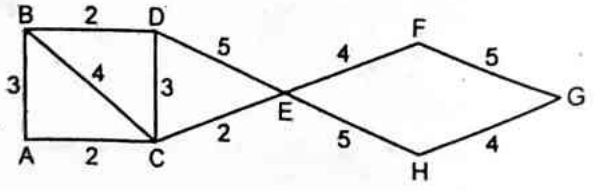
Q.9.(a) Draw the digraph corresponding to adjacency matrix A, B, A^T and B^T where

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

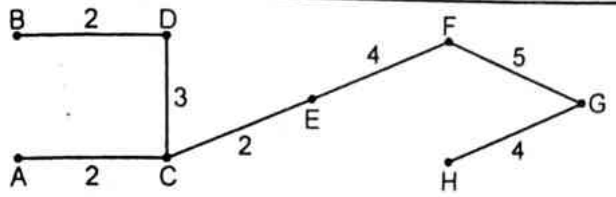
Show that the digraph corresponding to A^T and B^T are isomorphic. **6M**

Ans. See P.5-33, Ex.22.

Q.9.(b) Find the minimal spanning tree of the following graph using Prim's algorithm: **6M**

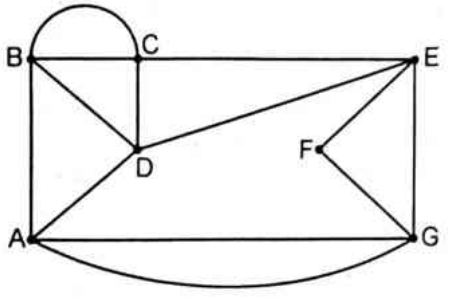


Ans. The minimal spanning free is



Its weight = 2 + 3 + 2 + 2 + 4 + 5 + 4 = 22

Q.9.(c) Define Eulerian path and Eulerian circuit. Show that the graph given below is an Eulerian graph and circuit: **6M**



Ans. **See P.5-30, Definition.**

We observe that, this graph contains a Euler path which is a circuit, hence it is an Eulerian circuit.

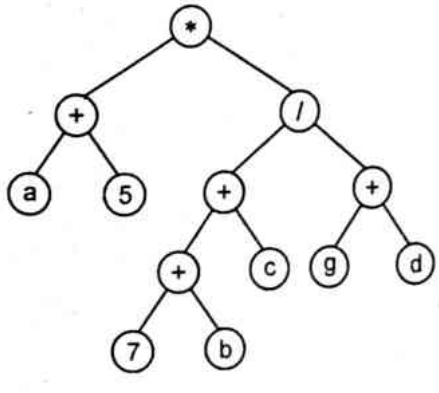
Since it contains an Eulerian circuit, it is a Euler graph.

Q.10.(a) Draw the tree for the algebraic expressions:

(i) (a + 5) * [{(7 + b) + c} / (g + d)]

(ii) (2x + (3 − 4x)) + (x −(3 × 11)) **6M**

Ans. **(i) (a + 5) * [{(7 + b) + c} / (g + d)]**



(ii) See P.5-44, Ex.40(ii).

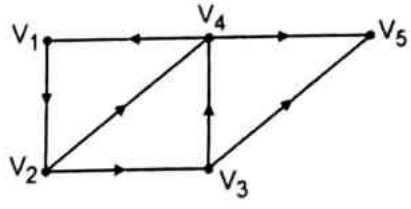
Q.10.(b) Draw the diagraph represented by the given adjacency matrix and interpreter. **6M**

AA^T, A^TA, A^2

$$\text{where } A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Ans. **See P.5-24, Ex.8(b).**

Q.10.(c) Find the indegrees and outdegrees of the node of the graph. Give all elementry cycles of this graph. Obtain an acyclic digraph by deleting the edge of the given graph. List all the nodes which are reachable from another node of the graph. **6M**



Ans. **See P.5-25, Ex.10(a).**

UNIT - VI

(Total Marks : 20)

Q.11.(a) Show that if any five numbers from 1 to 8 are chosen, then two of them add up to 9 using pigeon hole principle. **5M**

Ans. **See P.6-10, Ex.25.**

Q.11.(b) Solve the recurrence relation: $2a_n - 7a_{n-1} + 3a_{n-2} = 2^n$ **5M**

Ans. **See P.PS-18, Q.11(a).**

Q.12.(a) Find the generating function for the sequence

1, 0, − 1, 0, 1, 0, − 1, 0, 1, ——— **5M**

Ans. **See P.6-15, Ex.30(c).**

Q.12.(b) How many words of 3 different letters can be formed from the letters of word: **5M**

(i) MAST

(ii) COMPUTER

Ans. **(i)** Here n = 4

$$P(n, r) = \frac{n!}{(n-r)!}$$

$$\therefore\ P(4, 3) = \frac{4!}{1!} = 24 \text{ ...Ans.}$$

(ii) Here n = 8

$$P(n, r) = \frac{n!}{(n-r)!}$$

$$\therefore\ P(8, 3) = \frac{8!}{5!} = 8 \times 7 \times 6 = 336 \text{ ....Ans.}$$

Notes:-

APPENDIX

UNIT - II
FUZZY SET AND FUZZY LOGIC

INTRODUCTION

Fuzzy sets or fuzzy models are mathematical means of representing vagueness and imprecise information. These models have the capability of recongnising, representing, manipulating, interpreting and utilizing data and information that are vague and lack certainty. Fuzzy logic is based on the observation that people make decisions based on imprecise and non-numerical information. Fuzzy logic is a form of many-valued logic in which the truth value of variables may be nay real number between 0 and 1. It is employed to handle the concept of partial truth where the truth value may range between completely true and completely false.

FUZZY SET

Let X be a non-zero set. A fuzzy set 'A' of X is defined as

$A = \{(\mu_A(x), x)\}, x \in X$

Where $\mu_A : X \to [0, 1]$ is a function called as the membership function of 'A' and $\mu_A(x)$ is the grade of membership or degree of belongingness or degree of membership of $x \in X$ in 'A'.

The fuzzy set 'A' can also be written as

$$A = \{(\mu_A(x_1), x_1), (\mu_A(x_2), x_2)) \ldots\ldots\ldots (\mu_A(x_n), x_n))\}$$

Symbolically, we can write

$$A = \left\{\frac{\mu_A(x_1)}{x_1}, \frac{\mu_A(x_2)}{x_2}, \ldots\ldots, \frac{\mu_A(x_n)}{x_n}\right\}$$

CRISP SET

Crisp set is often referred as a classical or ordinary set in the context of fuzzy set theory.

A set defined using a characteristic function that assigns a value of either 0 or 1 to each element of the universe, thereby discriminating between members and non-members is called a crisp set.

The membership function can be used to define a set 'A' given by

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

The function χ (read as 'chi') is known as crisp membership function.

The set of natural numbers, real numbers, integers, rational numbers are all examples of crisp set.

OPERATIONS AND COMBINATIONS ON FUZZY SETS

(1) Equality of Two Fuzzy Sets:

Two fuzzy sets A and B are said to be equal written as A = B, if and only if $\mu_A(x) = \mu_B(x)$, which means that their membership values must be equal.

Let X = {a, b, c}

Consider the fuzzy sets A, B, C of X defined by

A = {(0.3, a), (0.6, b), (0.7, c)}

B = {(0.6, b), (0.3, a), (0.7, c)}

C = {(0.4, a), (0.5, b), (0.9, c)}

Here A = B, but $B \neq C$, $A \neq C$

(2) Complement or Absolute Complement of a Fuzzy set:

The complement or absolute complement of a fuzzy set 'A' is the set $\overline{A}$ with degree of the membership of an element $\overline{A}$ is equal to one minus the degree of membership of this element in A.

Let $A = \left\{\frac{0.2}{3}, \frac{0.7}{4}, \frac{0.8}{6}, \frac{0.9}{7}\right\}$

$\overline{A} = 1 - \mu_A(x)$

$\therefore \overline{A} = \left\{\frac{0.8}{3}, \frac{0.3}{4}, \frac{0.2}{6}, \frac{0.1}{7}\right\}$

(3) Union of Two Fuzzy Sets:

The union of two fuzzy sets A and B is the fuzzy set $A \cup B$, where the degree of membership of an element in $A \cup B$ is the maximum of the degrees of membership of this element in A and in B.

Let $A = \left\{\frac{0.3}{a}, \frac{0.9}{b}, \frac{0.6}{c}, \frac{0.4}{d}\right\}$ and

$$B = \left\{\frac{0.4}{a}, \frac{0.8}{b}, \frac{0.2}{c}, \frac{0.7}{d}\right\}$$

$$A \cup B = \max\{\mu_A(x), \mu_B(x)\}$$

$$\therefore\ A \cup B = \left\{\frac{0.4}{a}, \frac{0.9}{b}, \frac{0.6}{c}, \frac{0.7}{d}\right\}$$

(4) Intersection of Two Fuzzy Sets:

The intersection of two fuzzy sets A and B is the fuzzy set $A \cap B$, where the degree of membership of an element in $A \cap B$ is the minimum of the degrees of membership of this element in A and in B.

Let $A = \left\{\frac{0.3}{a}, \frac{0.9}{b}, \frac{0.6}{c}, \frac{0.4}{d}\right\}$ and

$$B = \left\{\frac{0.4}{a}, \frac{0.8}{b}, \frac{0.2}{c}, \frac{0.7}{d}\right\}$$

$$A \cap B = \min\{\mu_A(x), \mu_B(x)\}$$

$$\therefore\ A \cap B = \left\{\frac{0.3}{a}, \frac{0.8}{b}, \frac{0.2}{c}, \frac{0.4}{d}\right\}$$

(5) Difference of Two Fuzzy Sets:

The difference of two fuzzy sets A and B is defined as the intersection of first fuzzy set A and complement of second fuzzy set B. It is denoted by A/B or A – B.

$$A - B = A \cap \overline{B}$$

Let $A = \left\{\frac{1}{1}, \frac{0.15}{1.5}, \frac{0.3}{2}, \frac{0.5}{2.5}, \frac{0}{3}\right\}$

$$B = \left\{\frac{1}{1}, \frac{0.6}{1.5}, \frac{0.2}{2}, \frac{0.1}{2.5}, \frac{0}{3}\right\}$$

Here $\overline{B} = \left\{\frac{0}{1}, \frac{0.4}{1.5}, \frac{0.8}{2}, \frac{0.9}{2.5}, \frac{1}{3}\right\}$

$$\therefore\ A - B = A \cap \overline{B} = \left\{\frac{0}{1}, \frac{0.15}{1.5}, \frac{0.3}{2}, \frac{0.5}{2.5}, \frac{0}{3}\right\}$$

The relative complement of A with B is B – A

i.e. $B \cap \overline{A}$

The relative complement of B with A is A – B

i.e. $A \cap \overline{B}$

(6) Algebraic Sum of Two Fuzzy Sets:

Let A and B be any two fuzzy sets, then their algebraic sum is denoted and defined by

$$\mu_{A+B}(x) = [\mu_A(x) + \mu_B(x)] - [\mu_A(x) \cdot \mu_B(x)]$$

Let $A = \left\{\frac{1}{1} + \frac{0.5}{1.5} + \frac{0.3}{2} + \frac{0.4}{2.5}\right\}$ and

$$B = \left\{\frac{0.4}{1} + \frac{0.2}{1.5} + \frac{0.7}{2} + \frac{0.1}{2.5}\right\}$$

$$\therefore\ \mu_{A+B}(1) = (1 + 0.4) - (1 \times 0.4) = 1$$

$$\mu_{A+B}(1.5) = (0.5 + 0.2) - (0.5 \times 0.2) = 0.6$$

$$\mu_{A+B}(2) = (0.3 + 0.7) - (0.3 \times 0.7) = 0.79$$

$$\mu_{A+B}(2.5) = (0.4 + 0.1) - (0.4 \times 0.1) = 0.46$$

$$\therefore\ A + B = \left\{\frac{1}{1} + \frac{0.6}{1.5} + \frac{0.79}{2} + \frac{0.46}{2.5}\right\}$$

(7) Algebraic Product of Two Fuzzy Sets:

Let A and B be any two fuzzy Sets. Then their algebraic product is denoted and defined by

$$\mu_{A \cdot B}(x) = \mu_A(x) \cdot \mu_B(x)$$

Consider the above two fuzzy sets.

$$\therefore\ A \cdot B = \left\{\frac{0.4}{1} + \frac{0.1}{1.5} + \frac{0.21}{2} + \frac{0.04}{2.5}\right\}$$

(8) Bounded Sum of Two Fuzzy Sets:

The bounded sum of two fuzzy sets A and B is denoted and defined by

$$\mu_{A \oplus B} = \min[1, (\mu_A(x) + \mu_B(x))]$$

Let $A = \left\{\frac{1}{1} + \frac{0.5}{1.5} + \frac{0.3}{2} + \frac{0.4}{2.5}\right\}$ and

$$B = \left\{\frac{0.4}{1} + \frac{0.2}{1.5} + \frac{0.7}{2} + \frac{0.1}{2.5}\right\}$$

$$\therefore\ \mu_{A \oplus B}(1) = \min[1, 0.4] = 0.4$$

$$\mu_{A \oplus B}(1.5) = \min[1, 0.7] = 0.7$$

$$\mu_{A \oplus B}(2) = \min[1, 1] = 1$$

$$\mu_{A \oplus B}(2.5) = \min[1, 0.5] = 0.5$$

$$\therefore\ A \oplus B = \left\{\frac{0.4}{1} + \frac{0.7}{1.5} + \frac{1}{2} + \frac{0.5}{2.5}\right\}$$

(9) Bounded Difference of Two Fuzzy Sets:

The bounded difference of two fuzzy sets A and B is denoted defined by

$\mu_{A \ominus B}(x) = \max[0, (\mu_A(x) - \mu_B(x))]$

Consider the above two fuzzy sets.

$\mu_{A \ominus B}(1) = \max[0, 0.6] = 0.6,$

$\mu_{A \ominus B}(2) = \max[0, -0.4] = 0$

$$\therefore A \ominus B = \left\{\frac{0.6}{1} + \frac{0.3}{1.5} + \frac{0}{2} + \frac{0.3}{2.5}\right\}$$

SOME DEFINITIONS

(1) Let X be a non-empty set. Let $A = \{(\mu_A(x), x)\}, x \in X$ be the fuzzy set on X, where $\mu_A : X \to [0, 1]$ is the membership function of 'A'. Let x be an element of X, then x is called:

(a) Not included in the fuzzy set, if $\mu_A(x) = 0$ (no member)

(b) Fully included in the fuzzy set if $\mu_A(x) = 1$ (full member)

(c) Partially included in the fuzzy set if $0 < \mu_A(x) < 1$ (fuzzy member)

(2) **Crisp Sets Related to a Fuzzy Set:**

For any fuzzy set $A = \{\mu_A(x), x\}$ and $\alpha \in [0, 1]$ the following crisp sets are defined:

(a) $A^{\geq \alpha} = A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}$ is called α-cut.

(b) $A^{> \alpha} = A'_\alpha = \{x \in X \mid \mu_A(x) > \alpha\}$ is called strong α-cut.

(c) $S(A) = Supp(A) = A^{>0} = \{x \in X \mid \mu_A(x) > 0\}$ is called support.

(d) $C(A) = Core(A) = A^{=1} = \{x \in X \mid \mu_A(x) = 1\}$ is called core.

(3) A fuzzy set $A = \{\mu_A(x), x\}$ is called empty iff $\mu_A(x) = 0, \ \forall x \in X$

(4) A fuzzy set 'A' is included in a fuzzy set $B (A \subseteq B)$ iff $\mu_A(x) \leq \mu_B(x), \forall x \in X$.

(5) For any fuzzy set A, an element $x \in X$ that satisfies $\mu_A(x) = 0.5$, is called a crossover point.

(6) For a fuzzy set A, $\alpha \in [0, 1]$, for which

$A^{=\alpha} = \{x \in X \mid \mu_A(x) = \alpha\}$ is non-empty, is called level of A.

(7) The level set of A is the set of all levels $\alpha \in [0, 1]$ representing distinct cuts. It is the image of μ_A.

$\Lambda_A = \{\alpha \in [0, 1] : A^{=\alpha} \neq \phi\} = \mu_A(X)$.

(8) For a fuzzy set A, its height is given by

$Hgt(A) = Sup\{\mu_A(x) \mid x \in X\}$

(9) A fuzzy set A is said to be normalized iff

$Hgt(A) = 1$

(10) Two fuzzy sets A and B are said to be disjoint iff

$\mu_A(x) \wedge \mu_B(x) = 0, \ \forall x \in X$

Equivalently $\min\{\mu_A(x), \mu_B(x)\} = 0, \forall x \in X$

For any disjoint fuzzy sets A and B,

$\mu_{A \cup B}(x) = \mu_A(x) + \mu_B(x)$

(11) **Scalar Cardinality:**

For a fuzzy set A with finite support Supp(A), its cardinality (Scalar cardinality or Sigma count) is given by

$$Card(A) = sc(A) = |A| = \sum_{x \in X} \mu_A(x).$$

If X itself is a finite set, the relative cardinality is given by

$Rel\ Card(A) = \|A\| = |A| / |X|$

(12) **Fuzzy Measure:**

Let X be a universe of discourse (Universal Set), C be a class of subsets of X and $E, F \in C$,

then a function $g : C \to R$ where

(i) $\phi \in C \Rightarrow g(\phi) = 0$

(ii) $E \subseteq F \Rightarrow g(E) \leq g(F)$

is called a fuzzy measure. Here R is set of real numbers.

A fuzzy measure is called normalized or regular if $g(X) = 1$.

IMPORTANT FORMULAE :

(1) A fuzzy set 'A' of a non-zero set 'X' can be symbolically written as

$A = \{(\mu_A(x_1), x_1), (\mu_A(x_2), x_2)), \ldots\ldots, (\mu_A(x_n), x_n)\}$

$$\text{or } A = \left\{\frac{\mu_A(x_1)}{x_1}, \frac{\mu_A(x_2)}{x_2}, \ldots\ldots, \frac{\mu_A(x_n)}{x_n}\right\}$$

or $A = \left\{ \frac{\mu_A(x_1)}{x_1} + \frac{\mu_A(x_2)}{x_2} + \ldots\ldots + \frac{\mu_A(x_n)}{x_n} \right\}$

(2) The crisp membership function χ of a set A is

$$\chi_A(x) = \begin{cases} 1, \text{ if } x \in A \\ 0, \text{ if } x \notin A \end{cases}$$

(3) $\overline{A} = 1 - \mu_A(x)$

(4) $A \cup B = \max\{\mu_A(x), \mu_B(x)\}$

(5) $A \cap B = \min\{\mu_A(x), \mu_B(x)\}$

(6) $A - B = A \cap \overline{B}$

(7) Relative Complement of A with B = $B - A = B \cap \overline{A}$

(8) Relative Complement of B with A = $A - B = A \cap \overline{B}$

(9) $\mu_{A+B}(x) = [\mu_A(x) + \mu_B(x)] - [\mu_A(x) \cdot \mu_B(x)]$

(10) $\mu_{A \cdot B}(x) = \mu_A(x) \cdot \mu_B(x)$

(11) $\mu_{A \oplus B}(x) = \min[1, (\mu_A(x) + \mu_B(x))]$

(12) $m_{A \ominus B}(x) = \max[0, (\mu_A(x) - \mu_B(x))]$

(13) $A^{\geq \alpha} = A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}$ is called α - cut.

(14) $A^{> \alpha} = A'_\alpha = \{x \in X \mid \mu_A(x) > \alpha\}$ is called strong α - cut.

(15) $S(A) = \text{Supp}\ (A) = A^{>0} = \{x \in X \mid \mu_A(x) > 0\}$ is called support.

(16) $C(A) = A^{=1} = \{x \in X \mid \mu_A(x) = 1\}$ is called core.

(17) $\text{Card}\ (A) = |A| = \sum_{x \in X} \mu_A(x)$

(18) $\text{Rel Card}\ (A) = \|A\| = |A|/|X|$

SOLVED EXAMPLES:

Ex.1. Let X = {a, b, c}. The fuzzy sets A and B of X are defined as

$$A = \left\{ \frac{0.3}{a}, \frac{0.5}{b}, \frac{0.7}{c} \right\}, \qquad B = \left\{ \frac{0.4}{a}, \frac{0.5}{b}, \frac{0.8}{c} \right\}.$$

Find: (i) $A \cup B$ (ii) $A \cap B$ (iii) $\overline{A}, \overline{B}$

(iv) Check whether $\overline{A} \cap A \neq \phi$ and $\overline{A} \cup A \neq X$ (v) $A - B$

Soln. Given: X = {a, b, c},

$$A = \left\{ \frac{0.3}{a}, \frac{0.5}{b}, \frac{0.7}{c} \right\}.$$

$$B = \left\{ \frac{0.4}{a}, \frac{0.5}{b}, \frac{0.8}{c} \right\}$$

(i) $A \cup B = \max\{\mu_A(x), \mu_B(x)\} = \left\{ \frac{0.4}{a}, \frac{0.5}{b}, \frac{0.8}{c} \right\}$

(ii) $A \cap B = \min\{\mu_A(x), \mu_B(x)\} = \left\{ \frac{0.3}{a}, \frac{0.5}{b}, \frac{0.7}{c} \right\}$

(iii) $\overline{A} = 1 - \mu_A(x) = \left\{ \frac{0.7}{a}, \frac{0.5}{b}, \frac{0.3}{c} \right\}$

$\overline{B} = 1 - \mu_B(x) = \left\{ \frac{0.6}{a}, \frac{0.5}{b}, \frac{0.2}{c} \right\}$

(iv) $\overline{A} \cap A = \min\{\mu_{\overline{A}}(x), \mu_A(x)\}$

$= \left\{ \frac{0.3}{a}, \frac{0.5}{b}, \frac{0.3}{c} \right\} \neq \phi$

$\overline{A} \cup A = \max\{\mu_{\overline{A}}(x), \mu_A(x)\}$

$= \left\{ \frac{0.7}{a}, \frac{0.5}{b}, \frac{0.7}{c} \right\} \neq X$

(v) $A - B = A \cap \overline{B} = \left\{ \frac{0.3}{a}, \frac{0.5}{b}, \frac{0.3}{c} \right\}$ .....Ans.

Ex.2. Consider the fuzzy sets A, B, C defined on the interval X = [0, 10] of real numbers by the membership grade function

$$\mu_A(x) = \frac{x}{x+2}, \quad \mu_B(x) = 2^{-x}, \quad \mu_C(x) = \frac{1}{1 + 10(x-2)^2}.$$

Determine the mathematical formula for:

(i) $\overline{A}$ (ii) $A \cup B \cup C$

Soln.

x	$\mu_A(x) = \frac{x}{x+2}$	$\mu_B(x) = 2^{-x}$	$\mu_C(x) = \frac{1}{1+10(x-2)^2}$
0	0	1	0.024
1	0.33	0.5	0.09
2	0.5	0.25	1
3	0.6	0.125	0.09
4	0.66	0.0625	0.0243
5	0.71	0.0312	0.0109

6	0.75	0.0156	0.0062
7	0.77	0.0078	0.0039
8	0.8	0.0039	0.0027
9	0.818	0.0019	0.002
10	0.833	0.0009	0.0015

Here $A = \left\{\frac{0}{0}, \frac{0.33}{1}, \frac{0.5}{2}, \frac{0.6}{3}, \frac{0.66}{4}, \frac{0.71}{5}, \frac{0.75}{6}, \frac{0.77}{7}, \frac{0.8}{8}, \frac{0.818}{9}, \frac{0.833}{10}\right\}$

$\therefore \overline{A} = \left\{\frac{1}{0}, \frac{0.67}{1}, \frac{0.5}{2}, \frac{0.4}{3}, \frac{0.34}{4}, \frac{0.29}{5}, \frac{0.25}{6}, \frac{0.23}{7}, \frac{0.2}{8}, \frac{0.182}{9}, \frac{0.167}{10}\right\}$

& $A \cup B \cup C = \left\{\frac{1}{0}, \frac{0.5}{1}, \frac{1}{2}, \frac{0.6}{3}, \frac{0.66}{4}, \frac{0.71}{5}, \frac{0.75}{6}, \frac{0.77}{7}, \frac{0.8}{8}, \frac{0.818}{9}, \frac{0.833}{10}\right\}$ **..Ans.**

Ex.3. Let A be a fuzzy set defined by

$A = \frac{0.4}{v} + \frac{0.2}{w} + \frac{0.5}{x} + \frac{0.4}{y} + \frac{1}{z}$. **Find the α cut set A_α for $\alpha = 0.2, 0.5, 0.9, 1$.**

Soln. An α cut of fuzzy set A is a crisp set A_α that contains all the elements of the universal set X that have a membership grade in A greater than or equal to the specified value of α.

i.e. $A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}$

Given: $A = \frac{0.4}{v} + \frac{0.2}{\omega} + \frac{0.5}{x} + \frac{0.4}{y} + \frac{1}{z}$

$\therefore A_\alpha$ at $\alpha = 0.2$ will be a set of all x such that $\mu_A(x) \geq 0.2$.

$\therefore A_{0.2} = \{0.4, 0.2, 0.5, 0.4, 1\}$

$= \{v, \omega, x, y, z\}$

At $\alpha = 0.5$, we have

$A_{0.5} = \{0.5, 1\} = \{x, z\}$

At $\alpha = 0.9$,

$A_{0.9} = \{1\} = \{z\}$

At $\alpha = 1$,

$A_1 = \{1\} = \{z\}$ **....Ans.**

Ex.4. Given that $A = \frac{0.8}{3} + \frac{0.9}{4} + \frac{0.6}{5}$ and $B = \frac{0.7}{3} + \frac{0.3}{5}$.

Verify De -Morgan's Law $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Ans. Given: $A = \frac{0.8}{3} + \frac{0.9}{4} + \frac{0.6}{5}$ and

$B = \frac{0.7}{3} + \frac{0.3}{5}$

$\therefore A \cup B = \frac{0.8}{3} + \frac{0.9}{4} + \frac{0.6}{5}$

$\Rightarrow \overline{A \cup B} = \frac{0.2}{3} + \frac{0.1}{4} + \frac{0.4}{5}$

Also $\overline{A} = \frac{0.2}{3} + \frac{0.1}{4} + \frac{0.4}{5}$ and

$\overline{B} = \frac{0.3}{3} + \frac{0.7}{5}$

$\Rightarrow \overline{A} \cap \overline{B} = \frac{0.2}{3} + \frac{0.1}{4} + \frac{0.4}{5}$

$\therefore \boxed{\overline{A \cup B} = \overline{A} \cap \overline{B}}$ **Hence verified.**

Ex.5. Let age in years = X = {5, 10, 20, 30, 40, 50, 60, 70}

Let A, B, C be the fuzzy sets defined by

Infant = A = {0, 0, 0, 0, 0, 0, 0, 0}

Young = B = {1, 1, 0.8, 0.5, 0.2, 0.1, 0, 0}

Old = C = {0, 0, 0.1, 0.2, 0.4, 0.6, 0.8, 1}

Find: (i) $A \cup B$ (ii) $A \cap B$ (iii) $A \cup (B \cap C)$

(iv) $B \cup C$ (v) $B \cap C$

Soln. Here X = {5, 10, 20, 30, 40, 50, 60, 70}

$A = \left\{\frac{0}{5}, \frac{0}{10}, \frac{0}{20}, \frac{0}{30}, \frac{0}{40}, \frac{0}{50}, \frac{0}{60}, \frac{0}{70}\right\}$

$B = \left\{\frac{1}{5}, \frac{1}{10}, \frac{0.8}{20}, \frac{0.5}{30}, \frac{0.2}{40}, \frac{0.1}{50}, \frac{0}{60}, \frac{0}{70}\right\}$

$C = \left\{\frac{0}{5}, \frac{0}{10}, \frac{0.1}{20}, \frac{0.2}{30}, \frac{0.4}{40}, \frac{0.6}{50}, \frac{0.8}{60}, \frac{1}{70}\right\}$

(i) $A \cup B = \max\{\mu_A(x), \mu_B(x)\}$

$= \left\{\frac{1}{5}, \frac{1}{10}, \frac{0.8}{20}, \frac{0.5}{30}, \frac{0.2}{40}, \frac{0.1}{50}, \frac{0}{60}, \frac{0}{70}\right\}$

(ii) $A \cap B = \min\{\mu_A(x), \mu_B(x)\}$

$$= \left\{\frac{0}{5}, \frac{0}{10}, \frac{0}{20}, \frac{0}{30}, \frac{0}{40}, \frac{0}{50}, \frac{0}{60}, \frac{0}{70}\right\}$$

(iii) $B \cap C = \min\{\mu_B(x), \mu_C(x)\}$

$$= \left\{\frac{0}{5}, \frac{0}{10}, \frac{0.1}{20}, \frac{0.2}{30}, \frac{0.2}{40}, \frac{0.1}{50}, \frac{0}{60}, \frac{0}{70}\right\}$$

$$\therefore\ A \cup (B \cap C) = \left\{\frac{0}{5}, \frac{0}{10}, \frac{0.1}{20}, \frac{0.2}{30}, \frac{0.2}{40}, \frac{0.1}{50}, \frac{0}{60}, \frac{0}{70}\right\}$$

(iv) $B \cup C = \left\{\frac{1}{5}, \frac{1}{10}, \frac{0.8}{20}, \frac{0.5}{30}, \frac{0.4}{40}, \frac{0.6}{50}, \frac{0.8}{60}, \frac{1}{70}\right\}$

(v) $B \cap C = \left\{\frac{0}{5}, \frac{0}{10}, \frac{0.1}{20}, \frac{0.2}{30}, \frac{0.2}{40}, \frac{0.1}{50}, \frac{0}{60}, \frac{0}{70}\right\}$ **..Ans.**

Ex.6. If universe of discourse is X = {0, 1, 2,.....10}, then write fuzzy set A whose membership grade function is $\mu_A(x) = \frac{x}{x+1}$. Also find scalar cardinality of A and α cut of A for $\alpha = 0.5$

Ans.

x	$\mu_{A(x)} = \frac{x}{x+1}$
0	0
1	0.5
2	0.66
3	0.75
4	0.8
5	0.833
6	0.857
7	0.875
8	0.888
9	0.9
10	0.91

The scalar cardinality of A

$$= 0.5 + 0.66 + 0.75 + 0.8 + 0.833 + 0.857 + 0.875 + 0.888 + 0.9 + 0.91 = 7.98$$

Here $\alpha = 0.5$

$\therefore$ α cut for A = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}**..Ans.**

Q.7. If $A = \left\{\frac{1}{2}, \frac{0.5}{3}, \frac{0.3}{4}, \frac{0.2}{5}\right\}$ and $B = \left\{\frac{0.5}{2}, \frac{0.7}{3}, \frac{0.2}{4}, \frac{0.4}{5}\right\}$ are fuzzy sets on universe X = {1, 2, 3, 4, 5}, then find $\overline{A}$, $\overline{B}$, $A \cup \overline{A}$ and $B \cap \overline{B}$.

Ans. Given:

X	A	B
1	0	0
2	1	0.5
3	0.5	0.7
4	0.3	0.2
5	0.2	0.4

$$\therefore\ \overline{A} = \left\{\frac{1}{1}, \frac{0}{2}, \frac{0.5}{3}, \frac{0.7}{4}, \frac{0.8}{5}\right\} \text{ and } \overline{B} = \left\{\frac{1}{1}, \frac{0.5}{2}, \frac{0.3}{3}, \frac{0.8}{4}, \frac{0.6}{5}\right\}$$

$$\Rightarrow A \cup \overline{A} = \frac{1}{1} + \frac{1}{2} + \frac{0.5}{3} + \frac{0.7}{4} + \frac{0.8}{5} = 2 \text{ and}$$

$$B \cap \overline{B} = \frac{0}{1} + \frac{0.5}{2} + \frac{0.3}{3} + \frac{0.2}{4} + \frac{0.4}{5} = 0.48 \text{ ....Ans.}$$

Ex.8. If universe of discourse is X = {0, 1, 2, 3, 4, 5} then write the fuzzy set A whose membership grade function is $\mu_A(x) = \frac{1}{1+10(x-2)^2}$.

Find the scalar cardinality and 0.02 cut of A. Is A normalized?

Ans.

x	$\mu_A(x) = \frac{1}{1+10(x-2)^2}$
0	0.024
1	0.090
2	1
3	0.090
4	0.024
5	0.0109

$\therefore$ The Scalar cardinality of A

= 1.2389 **....Ans.**

Here $\alpha = 0.02$

$\therefore$ α cut of A = {0, 1, 2, 3, 4, 5}.

We know that $Hgt(A) = Sup\{\mu_A(x) \mid x \in X\}$

Here $Hgt(A) = 1$

$\therefore$ $\boxed{\text{A is normalized}}$ **Ans.**

Ex.9. If X = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} and

A = {(1, 0.), (2, 0.1), (3, 0.2), (4, 0.5), (5, 0.3), (6, 0.1), (7, 0), (8, 0), (9, 0), (10, 0)}, then find:

(i) S(A) (ii) $A_{0.1}$ (iii) h(A)

Soln. Here X = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

$$A = \left\{\frac{0}{1}, \frac{0.1}{2}, \frac{0.2}{3}, \frac{0.5}{4}, \frac{0.3}{5}, \frac{0.1}{6}, \frac{0}{7}, \frac{0}{8}, \frac{0}{9}, \frac{0}{10}\right\}$$

(i) $S(A) = Supp(A) = \{x \in X \mid \mu_A(x) > 0\}$

$= \{0.1, 0.2, 0.5, 0.3\}$

(ii) $A_{0.1} = \{x \in X \mid \mu_A(x) \geq 0.1\}$

$= \{2, 3, 4, 5, 6\}$

(iii) $h(A) = Hgt(A) = \sup\{\mu_A(x) \mid x \in X\} = 0.5$

Here $Hgt(A) \neq 1$

$\therefore$ A is not normalized ....**Ans.**

Ex.10. Let A = {(0, 0.4), (1, 0.5), (2, 0.6), (3, 0.7), (4, 0.3) and B = {(0, 0.5), (1, 0.6), (2, 0.8), (3, 0.9), (4, 0.7)}

Find: (i) $A \cap B$ (ii) $A \cup B$ (iii) $\overline{A}$

(iv) Relative complement of B with respect to A.

Soln. Here $A = \left\{\frac{0.4}{0}, \frac{0.5}{1}, \frac{0.6}{2}, \frac{0.7}{3}, \frac{0.3}{4}\right\}$ and

$$B = \left\{\frac{0.5}{0}, \frac{0.6}{1}, \frac{0.8}{2}, \frac{0.9}{3}, \frac{0.7}{4}\right\}$$

(i) $A \cap B = \min\{\mu_A(x), \mu_B(x)\}$

$$= \left\{\frac{0.4}{0}, \frac{0.5}{1}, \frac{0.6}{2}, \frac{0.7}{3}, \frac{0.3}{4}\right\}$$

(ii) $A \cup B = \max\{\mu_A(x), \mu_B(x)\}$

$$= \left\{\frac{0.5}{0}, \frac{0.6}{1}, \frac{0.8}{2}, \frac{0.9}{3}, \frac{0.7}{4}\right\}$$

(iii) $\overline{A} = 1 - \mu_A(x) = \left\{\frac{0.6}{0}, \frac{0.5}{1}, \frac{0.4}{2}, \frac{0.3}{3}, \frac{0.7}{4}\right\}$

(iv) $\overline{B} = 1 - \mu_B(x) = \left\{\frac{0.5}{0}, \frac{0.4}{1}, \frac{0.2}{2}, \frac{0.1}{3}, \frac{0.3}{4}\right\}$

$\therefore$ $A - B = A \cap \overline{B}$

$$= \left\{\frac{0.4}{0}, \frac{0.4}{1}, \frac{0.2}{2}, \frac{0.1}{3}, \frac{0.3}{4}\right\}$$ ....**Ans.**

Ex.11. If the universe of discourse is X = {0, 1, 2, 3, 4, 5}, then write the fuzzy set A whose membership grade function is $\mu_A(x) = \frac{x}{x+2}$.

Also find $\overline{A}$, Scalar cardinality of A, 0.2 cut of A.

Ans. Given: X = {0, 1, 2, 3, 4, 5}. $\mu_A(x) = \frac{x}{x+2}$

x	$\mu_A(x) = \frac{x}{x+2}$
0	0
1	0.33
2	0.5
3	0.6
4	0.66
5	0.71

$$A = \left\{\frac{0}{0}, \frac{0.33}{1}, \frac{0.5}{2}, \frac{0.6}{3}, \frac{0.66}{4}, \frac{0.71}{5}\right\}$$

$$\therefore \overline{A} = \left\{\frac{1}{0}, \frac{0.67}{1}, \frac{0.5}{2}, \frac{0.4}{3}, \frac{0.34}{4}, \frac{0.29}{5}\right\}$$

Scalar cardinality of A = 0 + 0.33 + 0.5 + 0.6 + 0.66 + 0.71 = 2.8

0.2 cut of A,

$A_{0.2} = \{1, 2, 3, 4, 5\}$ ....**Ans.**

Ex.12. Given a fuzzy set $A = \frac{0}{x_1} + \frac{0.6}{x_2} + \frac{0.5}{x_3}$

Find the measure of fuzziness of set A.

Soln. Given: $A = \frac{0}{x_1} + \frac{0.6}{x_2} + \frac{0.5}{x_3}$

We have

Measure of fuzziness

$$F_Z(A) = \frac{Card(A \cap \overline{A})}{Card(A \cup \overline{A})} \quad \text{....(1)}$$

Now,

$$\overline{A} = \frac{1}{x_1} + \frac{0.4}{x_2} + \frac{0.5}{x_3}$$

$\Rightarrow A \cap \overline{A} = \min\{\mu_A(x), \mu_{\overline{A}}(x)\}$

$$= \frac{0}{x_1} + \frac{0.4}{x_2} + \frac{0.5}{x_3}$$

$\therefore$ Card $(A \cap \overline{A}) = 0 + 0.4 + 0.5 = 0.9$

Also $A \cup \overline{A} = \max\{\mu_A(x), \mu_{\overline{A}}(x)\}$

$$= \left\{\frac{1}{x_1} + \frac{0.6}{x_2} + \frac{0.5}{x_3}\right\}$$

$\therefore$ Card $(A \cup \overline{A}) = 0 + 0.6 + 0.5 = 2.1$

$\therefore$ (I) $\Rightarrow F_Z(A) = \frac{0.9}{2.1} = 0.4286$ ....**Ans.**

Ex.13. Find the measure of fuzziness for the fuzzy set $A = \frac{0.2}{x_1} + \frac{0.4}{x_2}$

Soln. Given: $A = \frac{0.2}{x_1} + \frac{0.4}{x_2}$

$$\Rightarrow \overline{A} = \frac{0.8}{x_1} + \frac{0.6}{x_2}$$

$\therefore A \cap \overline{A} = \frac{0.2}{x_1} + \frac{0.4}{x_2}$

$\therefore$ Card $(A \cap \overline{A}) = 0.2 + 0.4 = 0.6$

Also $A \cup \overline{A} = \frac{0.8}{x_1} + \frac{0.6}{x_2}$

$\therefore$ Card $(A \cup \overline{A}) = 0.8 + 0.6 = 1.4$

Now, Measure of fuzziness

$$F_Z(A) = \frac{\text{Card}(A \cap \overline{A})}{\text{Card}(A \cup \overline{A})}$$

$\therefore F_Z(A) = \frac{0.6}{1.4} = 0.4286$ ....**Ans.**

Ex.14. Find the measure of fuzziness E(A),

if A = [0.3, 0.8]

Soln. Given: A = [0.3, 0.8]

$\Rightarrow \overline{A} = [0.7, 0.2]$

$\therefore A \cap \overline{A} = [0.3, 0.2]$

$\therefore$ Card $(A \cap \overline{A}) = 0.3 + 0.2 = 0.5$

Also $A \cup \overline{A} = [0.7, 0.8]$

$\therefore$ Card $(A \cup \overline{A}) = 0.7 + 0.8 = 1.5$

$\therefore$ Measure of fuzziness $E(A) = \frac{\text{Card}(A \cap \overline{A})}{\text{Card}(A \cup \overline{A})}$

$\therefore E(A) = \frac{0.5}{1.5} = 0.3333$ ....**Ans.**

Ex.15. Find $A \cap B$, $A \cup B$, if

$$A = \frac{0.9}{1} + \frac{0.7}{3} + \frac{0.2}{4} + \frac{0.3}{6} \text{ and } B = \frac{0.1}{2} + \frac{0.4}{3} + \frac{0.5}{4} + \frac{0.8}{5}$$

Soln. Given: $A = \frac{0.9}{1} + \frac{0.7}{3} + \frac{0.2}{4} + \frac{0.3}{6}$ and

$$B = \frac{0.1}{2} + \frac{0.4}{3} + \frac{0.5}{4} + \frac{0.8}{5}$$

$\therefore A \cap B = \min\{\mu_A(x), \mu_B(x)\}$

$$= \frac{0.9}{1} + \frac{0.1}{2} + \frac{0.4}{3} + \frac{0.2}{4} + \frac{0.8}{5} + \frac{0.3}{6} \text{ and}$$

$A \cup B = \max\{\mu_A(x), \mu_B(x)\}$

$$= \frac{0.9}{1} + \frac{0.1}{2} + \frac{0.7}{3} + \frac{0.5}{4} + \frac{0.8}{5} + \frac{0.3}{6}$$ ....**Ans.**

Ex.16. Let U = {a, b, c, d} be the domain and A and B be fuzzy sets on U as given below:

	a	b	c	d
A	0.5	0.8	0.0	0.3
B	0.2	1.0	0.1	0.7

Find:

(i) $A \cup B$ (ii) $A \cap B$ (iii) A' (iv) B'.

Also prove that (v) $(A \cup B)' = A' \cap B'$.

Soln. Given:

	a	b	c	d
A	0.5	0.8	0.0	0.3
B	0.2	1.0	0.1	0.7

(i) $A \cup B(a) = \max[A(a), B(a)]$

$= \max(0.5, 0.2) = 0.5$

Similarly, $A \cup B(b) = 1.0$,

$A \cup B(c) = 0.1$

and $A \cup B(d) = 0.7$

Thus finally we obtain

	a	b	c	d
$A \cup B$	0.5	1.0	0.1	0.7

Note that this table could have been arrived at from the first table by taking the maximum of each column separately. In a similar way, performing the minimum of each column separately, we obtain $A \cap B$ as follows :

(ii)

	a	b	c	d
$A \cap B$	0.2	0.8	0.0	0.3

(iii) Subtracting each entry of A from 1 in the first table, we obtain A'

	a	b	c	d
A'	0.5	0.2	1.0	0.7

(iv) Similarly,

	a	b	c	d
B'	0.8	0	0.9	0.3

(v) Subtracting each entry of $A \cup B$ from 1, we obtain $(A \cup B)'$ as

	a	b	c	d
$(A \cup B)'$	0.5	0.0	0.9	0.3

Also taking minimum of each colum in (iii) & (iv) above we obtain $A' \cap B'$ as

	a	b	c	d
$A' \cap B'$	0.5	0.0	0.9	0.3

Thus $(A \cup B)' = A' \cap B'$ ..... **Ans.**

Ex.17. Let $A = \frac{0.5}{3} + \frac{1}{5} + \frac{0.6}{7}$, $B = \frac{0.7}{3} + \frac{0.6}{5}$ **be two fuzzy sets.**

Find: (i) A + B (ii) AB (iii) $A \oplus B$ (iv) $A \ominus B$

Soln. Given: $A = \frac{0.5}{3} + \frac{1}{5} + \frac{0.6}{7}$ and $B = \frac{0.7}{3} + \frac{0.6}{5} + \frac{0}{7}$

(i) $\mu_{A+B}(x) = [\mu_A(x) + \mu_B(x)] - [\mu_A(x) \cdot \mu_B(x)]$

$$\therefore \quad A + B = \frac{0.85}{3} + \frac{1}{5} + \frac{0.6}{7}$$

(ii) $\mu_{A \cdot B}(x) = \mu_A(x) \cdot \mu_B(x)$

$$\therefore \quad A \cdot B = \frac{0.35}{3} + \frac{1}{5} + \frac{0}{7}$$

(iii) $\mu_{A \oplus B} = \min[1, (\mu_A(x) + \mu_B(x))]$

$$\therefore \quad A \oplus B = \frac{1}{3} + \frac{1}{5} + \frac{0.6}{7}$$

(iv) $\mu_{A \ominus B}(x)$ $A = \left\{\frac{0.2}{x_1}, \frac{0.3}{x_2}, \frac{0.5}{x_3}\right\}$ and $B = \left\{\frac{0.5}{y_1}, \frac{0.6}{y_2}, \frac{0.7}{y_3}\right\}$

$$\therefore \quad A \ominus B = \frac{0}{3} + \frac{0.4}{5} + \frac{0.6}{7} \text{ ....Ans.}$$

Ex.18. For the fuzzy sets $A = \frac{0.2}{x_1} + \frac{0.5}{x_2} + \frac{0.6}{x_3}$ **and**

$B = \frac{0.1}{x_1} + \frac{0.4}{x_2} + \frac{0.5}{x_3}$, **find: (i) $A \oplus B$ (ii) $A \ominus B$**

Soln. Given: $A = \frac{0.2}{x_1} + \frac{0.5}{x_2} + \frac{0.6}{x_3}$ and $B = \frac{0.1}{x_1} + \frac{0.4}{x_2} + \frac{0.5}{x_3}$

(i) $\mu_{A \oplus B}(x) = \min[1, (\mu_A(x) + \mu_B(x))]$

$$\therefore \quad A \oplus B = \frac{0.3}{x_1} + \frac{0.9}{x_2} + \frac{1}{x_3}$$

(ii) $\mu_{A \ominus B}(x) = \max[0, (\mu_A(x) - \mu_B(x))]$

$$\therefore \quad A \ominus B = \frac{0.1}{x_1} + \frac{0.1}{x_2} + \frac{0.1}{x_3} \text{ ....Ans.}$$

CARTESIAN PRODUCT OF FUZZY SETS

Let A and B be any two fuzzy sets defined on the universe of discourse X and Y resp. such that

$X = \{x_1, x_2, x_3 \ldots\ldots x_n\}$

$Y = \{y_1, y_2, y_3 \ldots\ldots y_n\}$

and fuzzy sets

$$A = \left\{\frac{\mu_A(x_1)}{x_1}, \frac{\mu_A(x_2)}{x_2}, \ldots\ldots, \frac{\mu_A(x_n)}{x_n}\right\} \text{ and}$$

$$B = \left\{\frac{\mu_B(y_1)}{y_1}, \frac{\mu_B(y_2)}{y_2}, \ldots\ldots, \frac{\mu_B(y_m)}{y_m}\right\}$$

then the Cartesian product of A and B is defined and denoted by

$$A \times B = \left\{ \frac{\mu_{A\times B}(x_1, y_1)}{(x_1, y_1)}, \frac{\mu_{A\times B}(x_1, y_2)}{(x_1, y_2)}, \frac{\mu_{A\times B}(x_1, y_3)}{(x_1, y_3)} \right.$$

$$\ldots \frac{\mu_{A\times B}(x_2, y_1)}{(x_2, y_1)}, \frac{\mu_{A\times B}(x_2, y_2)}{(x_2, y_2)},$$

$$\left. \ldots \frac{\mu_{A\times B}(x_n, y_m)}{(x_n, y_m)} \right\}.$$

Where, $\mu_{A\times B}(x_i, y_j) = \min\{\mu_A(x_i), \mu_B(y_j)\}$,

$1 \le i \le n,\ 1 \le j \le m$

Example:

Let $A = \left\{ \frac{0.1}{a}, \frac{0.3}{b}, \frac{0.7}{c} \right\}$ and $B = \left\{ \frac{0.2}{x}, \frac{0.4}{y} \right\}$,

be two fuzzy sets, then we have

$\mu_{A\times B}(a, x) = \min\{\mu_A(a), \mu_B(x)\}$

$= \min\{0.1, 0.2\} = 0.1$

$\mu_{A\times B}(a, y) = \min\{\mu_A(a), \mu_B(y)\}$

$= \min\{0.1, 0.4\} = 0.1$

$\mu_{A\times B}(b, x) = \min\{\mu_A(b), \mu_B(x)\}$

$= \min\{0.3, 0.2\} = 0.2$

$\mu_{A\times B}(b, y) = \min\{\mu_A(b), \mu_B(y)\}$

$= \min\{0.3, 0.4\} = 0.3$

$\mu_{A\times B}(c, x) = \min\{\mu_A(c), \mu_B(x)\}$

$= \min\{0.7, 0.2\} = 0.2$

$\mu_{A\times B}(c, y) = \min\{\mu_A(c), \mu_B(y)\}$

$= \min\{0.7, 0.4\} = 0.4$

$\therefore$ The Cartesian product A and B is given by

$$A \times B = \left\{ \frac{\mu_{A\times B}(a, x)}{(a, x)}, \frac{\mu_{A\times B}(a, y)}{(a, y)}, \frac{\mu_{A\times B}(b, x)}{(b, x)}, \frac{\mu_{A\times B}(b, y)}{(b, y)}, \right.$$

$$\left. \frac{\mu_{A\times B}(c, x)}{(c, x)}, \frac{\mu_{A\times B}(c, y)}{(c, y)} \right\}$$

$$\therefore \quad A \times B = \left\{ \frac{0.1}{(a,x)}, \frac{0.1}{(a,y)}, \frac{0.2}{(b,x)}, \frac{0.3}{(b,y)}, \frac{0.2}{(c,x)}, \frac{0.4}{(c,y)} \right\}$$

Notes:

(i) If fuzzy set A and B contain elements m and n resp. then $A \times B$ have $(m \times n)$ no. of elements.

(ii) The above cross product can also be written as

$$A \times B = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0.3 \\ 0.2 & 0.4 \end{bmatrix} \end{array}$$

FUZZY RELATIONS

Let R be the relation defined on two fuzzy sets say A and B on the universe of discourse X and Y resp. with $\mu_A(x)/x \in X$ and $\mu_B(y)/y \in Y$, then $R = A \times B$, where, R has a membership function given by $\mu_{A\times B}(x, y) = \min\{\mu_A(x), \mu_B(y)\}$

Example:

Let $A = \left\{ \frac{0.1}{X_1}, \frac{0.4}{X_2}, \frac{0.7}{X_3} \right\}$ and $B = \left\{ \frac{0.2}{Y_1}, \frac{0.4}{Y_2}, \frac{0.9}{Y_3} \right\}$

Let R be the relation on A and B, then

$$R = \begin{array}{c} \\ X_1 \\ X_2 \\ X_3 \end{array} \begin{array}{c} \begin{array}{ccc} Y_1 & Y_2 & Y_3 \end{array} \\ \begin{bmatrix} 0.1 & 0.1 & 0.1 \\ 0.2 & 0.4 & 0.4 \\ 0.2 & 0.4 & 0.7 \end{bmatrix} \end{array}$$

Composition of Fuzzy Relation:

Let R and S be two relations defined on the set X to Y and Y to Z resp. given by

$$R = \begin{array}{c} \\ X_1 \\ X_2 \end{array} \begin{array}{c} \begin{array}{cc} Y_1 & Y_2 \end{array} \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \end{array} \text{ and } S = \begin{array}{c} \\ Y_1 \\ Y_2 \end{array} \begin{array}{c} \begin{array}{cc} Z_1 & Z_2 \end{array} \\ \begin{bmatrix} x & y \\ z & w \end{bmatrix} \end{array},$$

then composition of R and S is denoted by $R \circ S$ from X to Z and is given by

$R \circ S(X_1, Z_1) = \text{Max}[\min(a,x), \min(b,z)]$

$R \circ S(X_1, Z_2) = \text{Max}[\min(a,y), \min(b,w)]$

$R \circ S(X_2, Z_1) = \text{Max}[\min(c,x), \min(d,z)]$

$R \circ S(X_2, Z_2) = \text{Max}[\min(c,y), \min(d,w)]$

Union of Fuzzy Relations:

Let R and S be two Fuzzy relations defined on the $A \times B$, then

$\mu_{R\cup S}(a, b) = \text{Max}\{\mu_R(a, b), \mu_S(a, b)\}$

Intersection of Fuzzy Relations:

Let R and S be two Fuzzy relations defined on the $A \times B$, then

$\mu_{R\cap S}(a, b) = \text{Min}\{\mu_R(a, b), \mu_S(a, b)\}$

Complement of Fuzzy Relations:

Let R be the relations defined on the set A × B, then complement of R is denoted by $\overline{R}$ or R^C and is given by

$$\mu_{\overline{R}}(a) = \mu_{R^c}(a) = 1 - \mu_R(a)$$

Note: The union and intersection of Fuzzy relation exists only if they are defined on same universe of discourse.

Ex.19. Let $X = \{X_1, X_2, X_3\}$, $Y = \{Y_1, Y_2, Y_3, Y_4, Y_5\}$ and $Z = \{Z_1, Z_2, Z_3, Z_4\}$.

Relations R and S are defined as:

R	Y_1	Y_2	Y_3	Y_4	Y_5
X_1	0.1	0.2	0	1	0.7
X_2	0.3	0.5	0	0.2	1
X_3	0.8	0	1	0.4	0.3

and

S	Z_1	Z_2	Z_3	Z_4
Y_1	0.9	0	0.3	0.4
Y_2	0.2	1	0.8	0
Y_3	0.8	0	0.7	1
Y_4	0.4	0.2	0.3	0
Y_5	0	1	0	0.8

, then find $R \circ S$.

Soln. Let

$$T = R \circ S = \begin{array}{c} \\ X_1 \\ X_2 \\ X_3 \end{array}\begin{array}{c} \begin{array}{cccc} Z_1 & Z_2 & Z_3 & Z_4 \end{array} \\ \begin{bmatrix} d_1 & d_2 & d_3 & d_4 \\ e_1 & e_2 & e_3 & e_4 \\ f_1 & f_2 & f_3 & f_4 \end{bmatrix} \end{array} \quad \ldots (1)$$

Where, d_1 = Max {min (0.1, 0.9), min (0.2, 0.2), min(0, 0.8), min(1, 0.4), min(0.7, 0)}

= Max {0.1, 0.2, 0, 0.4, 0} = 0.4

d_2 = Max {min(0.1, 0), min(0.2, 1), min(0, 0), min(1, 0.2), min(0.7, 1)}

= Max {0, 0.2, 0, 0.2, 0.7} = 0.7

Similarly,

d_3 = Max {0.1, 0.2, 0.3, 0.3, 0} = 0.3

d_4 = Max {0.1, 0, 0, 0, 0.7} = 0.7

e_1 = Max {0.3, 0.2, 0, 0.2, 0} = 0.3

e_2 = Max {0, 0.5, 0, 0.2, 1} = 0.5

e_3 = Max {0.3, 0.5, 0, 0.2, 0} = 0.5

e_4 = Max {0.3, 0, 0, 0, 0.8} = 0.8

f_1 = Max {0.8, 0, 0.8, 0.4, 0} = 0.8

f_2 = Max {0, 0, 0, 0.2, 0.3} = 0.3

f_3 = Max {0.3, 0, 0.7, 0.3, 0} = 0.7

f_4 = Max {0.4, 0, 1, 0, 0.3} = 1

Now putting the above values in equation (1), we get

$$T = R \circ S = \begin{bmatrix} 0.4 & 0.7 & 0.3 & 0.7 \\ 0.3 & 0.5 & 0.5 & 0.8 \\ 0.8 & 0.3 & 0.7 & 1 \end{bmatrix} \ldots\ldots \textbf{Ans.}$$

Ex.20. If $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$. The fuzzy set A of X and fuzzy set B of Y are defined as follows:

$$A = \left\{\frac{0.2}{x_1}, \frac{0.3}{x_2}, \frac{0.5}{x_3}\right\} \text{ and } B = \left\{\frac{0.5}{y_1}, \frac{0.6}{y_2}, \frac{0.7}{y_3}\right\},$$

then find A × B.

Soln. We have

$$\mu_{(A \times B)}(x, y) = \min\{\mu_A(x), \mu_B(y)\}$$

$$\therefore A \times B = \left\{\frac{0.2}{(x_1, y_1)}, \frac{0.2}{(x_1, y_2)}, \frac{0.2}{(x_1, y_3)}, \frac{0.3}{(x_2, y_1)}, \frac{0.3}{(x_2, y_2)}, \frac{0.3}{(x_2, y_3)}, \frac{0.5}{(x_3, y_1)}, \frac{0.5}{(x_3, y_2)}, \frac{0.5}{(x_3, y_3)}\right\}$$

In matrix form we can write

$$A \times B = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}\begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.3 & 0.3 & 0.3 \\ 0.5 & 0.5 & 0.5 \end{bmatrix} \end{array} \ldots\ldots \textbf{Ans.}$$

Ex.21. If fuzzy sets $A = \left\{\frac{0.3}{30} + \frac{0.7}{60} + \frac{1}{100}\right\}$, $B = \left\{\frac{0.2}{20} + \frac{0.4}{40} + \frac{0.6}{60}\right\}$ and $C = \left\{\frac{0.3}{50} + \frac{0.6}{100}\right\}$.

Find fuzzy relation R = A × B, S = B × C.

Soln. We have

R = A × B and

$$\mu_{A \times B}(x, y) = \mu_R(x, y) = \min\{\mu_A(x), \mu_B(y)\}$$

$$\therefore R = A \times B = \left\{\frac{0.2}{(30, 20)} + \frac{0.3}{(30, 40)} + \frac{0.3}{(30, 60)} \frac{0.2}{(60, 20)} + \frac{0.4}{(60, 40)} + \frac{0.6}{(60, 60)} + \frac{0.2}{(100, 20)} + \frac{0.4}{(100, 40)} + \frac{0.6}{(100, 60)}\right\}$$

In matrix form

$$R = \begin{array}{c} \\ 30 \\ 60 \\ 100 \end{array}\begin{array}{c} \begin{array}{ccc} 20 & 40 & 60 \end{array} \\ \begin{bmatrix} 0.2 & 0.3 & 0.3 \\ 0.2 & 0.4 & 0.6 \\ 0.2 & 0.4 & 0.6 \end{bmatrix} \end{array}$$

Now $S(x, y) = (B \times C)\ (x, y)$

We have

$$\mu_S(x, y) = \mu_{B \times C}(x, y) = \text{Min}\{\mu_B(x), \mu_C(y)\}$$

$$S = B \times C = \left\{\frac{0.2}{(20, 50)} + \frac{0.2}{(20, 100)} + \frac{0.3}{(40, 50)} \frac{0.4}{(40, 100)} + \frac{0.3}{(60, 50)} + \frac{0.6}{(60, 100)}\right\}$$

In matrix form

$$S = B \times C = \begin{array}{c} \\ 20 \\ 40 \\ 60 \end{array}\begin{array}{c} \begin{array}{cc} 50 & 100 \end{array} \\ \begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.4 \\ 0.3 & 0.6 \end{bmatrix} \end{array} \text{ ....Ans.}$$

Ex.22. Let $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$ and $C = \{c_1, c_2, c_3\}$ the fuzzy relation R from A to B is given by

$$R = \left\{\frac{0.1}{a_1, b_1} + \frac{0.2}{(a_1, b_2)} + \frac{0.4}{(a_2, b_1)}\right\}$$

and the relation S from B to C is given by

$$S = \frac{0.2}{(b_1, c_1)} + \frac{0.4}{(b_1, c_2)} + \frac{0.2}{(b_2, c_1)}$$

Find the composition relation $R \circ S$ from A to C.

Soln. The relation R and S in matrix form can be written as

$$R = \begin{array}{c} \\ a_1 \\ a_2 \end{array}\begin{array}{c} \begin{array}{cc} b_1 & b_2 \end{array} \\ \begin{bmatrix} 0.1 & 0.2 \\ 0.4 & 0 \end{bmatrix} \end{array} \text{ and } S = \begin{array}{c} \\ b_1 \\ b_2 \end{array}\begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \begin{bmatrix} 0.2 & 0.4 & 0 \\ 0.2 & 0 & 0 \end{bmatrix} \end{array}$$

Now, we have

$$R \circ S(a_1, c_1) = \text{Max}\{\min[(a_1, b_1), (b_1, c_1)], \min[(a_1, b_2), (b_2, c_1)]\}$$

$$= \text{Max}\{\min(0.1, 0.2), \min(0.2, 0.2)\}$$

$$= \text{Max}\{0.1, 0.2\} = 0.2$$

Similarly,

$$R \circ S(a_1, c_2) = \text{Max}\{0.1, 0\} = 0.1$$

$$R \circ S(a_1, c_3) = \text{Max}\{0, 0\} = 0$$

$$R \circ S(a_2, c_1) = \text{Max}\{0.2, 0\} = 0.2$$

$$R \circ S(a_2, c_2) = \text{Max}\{0.4, 0\} = 0.4$$

$$R \circ S(a_2, c_3) = \text{Max}\{0, 0\} = 0$$

$\therefore$ The composition

$$R \circ S = \begin{array}{c} \\ a_1 \\ a_2 \end{array}\begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \begin{bmatrix} 0.2 & 0.1 & 0 \\ 0.2 & 0.4 & 0 \end{bmatrix} \end{array}$$

$$\therefore\ R \circ S = \frac{0.2}{(a_1, c_1)} + \frac{0.1}{(a_1, c_2)} + \frac{0.2}{(a_2, c_1)} + \frac{0.4}{(a_2, c_2)} \text{ ..... Ans.}$$

Ex.23. Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4\}$.

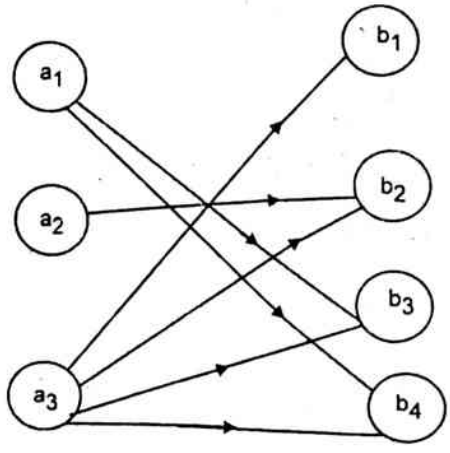
Let R be the relation from A to B given by $\frac{0.1}{(a_1, b_3)} + \frac{0.8}{(a_1, b_4)} + \frac{0.8}{(a_2, b_2)} + \frac{0.1}{(a_3, b_1)} + \frac{0.8}{(a_3, b_2)} + \frac{1}{(a_3, b_3)} + \frac{0.8}{(a_3, b_4)}$

Write fuzzy matrix M_R and draw graph.

Soln. We know that, the relation R from A to B in matrix form is denoted by M_R and is known as fuzzy matrix which can be written as follows:

$$M_R = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array}\begin{array}{c} \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0.1 & 0.8 \\ 0 & 0.8 & 0 & 0 \\ 0.1 & 0.8 & 1 & 0.8 \end{bmatrix} \end{array}$$

Graph: In graphical form, the above relation can be shown as



Ex.24. If $A = \left\{\frac{0.3}{30}, \frac{0.7}{60}, \frac{1}{100}, \frac{0.2}{120}\right\}$ and $B = \left\{\frac{0.2}{20}, \frac{0.4}{40}, \frac{0.6}{60}, \frac{0.8}{80}\right\}$ are fuzzy sets, then find the relation $R = A \times B$. Also find R^2.

Soln. We have, $R = A \times B$ and

$$\Rightarrow \mu_{A\times B}(x, y) = \min\{\mu_A(x), \mu_B(y)\}$$

$$\therefore \quad R = \left\{\frac{0.2}{(30,20)}, \frac{0.3}{(30,40)}, \frac{0.3}{(30,60)}, \frac{0.3}{(30,80)}, \frac{0.2}{(60,20)}, \frac{0.4}{(60,40)}, \frac{0.6}{(60,60)}, \frac{0.7}{(60,80)}, \frac{0.2}{(100,20)}, \frac{0.4}{(100,40)}, \frac{0.6}{(100,60)}, \frac{0.8}{(100,80)}, \frac{0.2}{(120,20)}, \frac{0.2}{(120,40)}, \frac{0.2}{(120,60)}, \frac{0.2}{(120,80)}\right\}$$

Now, In matrix form

$$R = M_R = \begin{array}{c} \\ 30 \\ 60 \\ 100 \\ 120 \end{array} \begin{array}{c} \begin{array}{cccc} 20 & 40 & 60 & 80 \end{array} \\ \begin{bmatrix} 0.2 & 0.3 & 0.3 & 0.3 \\ 0.2 & 0.4 & 0.6 & 0.7 \\ 0.2 & 0.4 & 0.6 & 0.8 \\ 0.2 & 0.2 & 0.2 & 0.2 \end{bmatrix} \end{array}$$

Now

$$R^2 = R \circ R = \begin{array}{c} \\ 30 \\ 60 \\ 100 \\ 120 \end{array} \begin{array}{c} \begin{array}{cccc} 20 & 40 & 60 & 80 \end{array} \\ \begin{bmatrix} 0.2 & 0.3 & 0.3 & 0.3 \\ 0.2 & 0.4 & 0.6 & 0.7 \\ 0.2 & 0.4 & 0.6 & 0.8 \\ 0.2 & 0.2 & 0.2 & 0.2 \end{bmatrix} \end{array}$$

$$\circ \begin{array}{c} \\ 30 \\ 60 \\ 100 \\ 120 \end{array} \begin{array}{c} \begin{array}{cccc} 20 & 40 & 60 & 80 \end{array} \\ \begin{bmatrix} 0.2 & 0.3 & 0.3 & 0.3 \\ 0.2 & 0.4 & 0.6 & 0.7 \\ 0.2 & 0.4 & 0.6 & 0.8 \\ 0.2 & 0.2 & 0.2 & 0.2 \end{bmatrix} \end{array}$$

$$\therefore \; R \circ R = \begin{bmatrix} 0.2 & 0.3 & 0.3 & 0.3 \\ 0.2 & 0.4 & 0.6 & 0.6 \\ 0.2 & 0.4 & 0.6 & 0.6 \\ 0.2 & 0.2 & 0.2 & 0.2 \end{bmatrix} \text{ ..... Ans.}$$

Ex.25. Define union and intersection of fuzzy set also find: (i) $R \cap S$ (ii) $R \cup S$ if

R	Y_1	Y_2	Y_3	Y_4
X_1	0.3	0.2	1	0
X_2	0.8	1	0	0.2
X_3	0.5	0.4	0.4	0

S	Y_1	Y_2	Y_3	Y_4
X_1	0.3	0	0.7	0
X_2	0.1	0.8	1	1
X_3	0.6	0.9	0.3	0.1

Soln. For definition see introduction section

(i) We have $\mu_{R\cap S} = \min\{\mu_R, \mu_S\}$

From table, we have

$$R = \left\{\frac{0.3}{(x_1,y_1)}, \frac{0.2}{(x_1,y_2)}, \frac{1}{(x_1,y_3)}, \frac{0}{(x_1,y_4)}, \frac{0.8}{(x_2,y_1)}, \frac{1}{(x_2,y_2)}, \frac{0}{(x_2,y_3)}, \frac{0.2}{(x_2,y_4)}, \frac{0.5}{(x_3,y_1)}, \frac{0.4}{(x_3,y_2)}, \frac{0.4}{(x_3,y_3)}, \frac{0}{(x_3,y_4)}\right\}$$

$$\text{and } S = \left\{\frac{0.3}{(x_1,y_1)}, \frac{0}{(x_1,y_2)}, \frac{0.7}{(x_1,y_3)}, \frac{0}{(x_1,y_4)}, \frac{0.1}{(x_2,y_1)}, \frac{0.6}{(x_3,y_1)}, \frac{0.9}{(x_3,y_2)}, \frac{0.8}{(x_2,y_2)}, \frac{1}{(x_2,y_3)}, \frac{1}{(x_2,y_4)}, \frac{0.3}{(x_3,y_3)}, \frac{0.1}{(x_3,y_4)}\right\}$$

Now,

$$\therefore \; R\cap S = \left\{\frac{0.3}{(x_1,y_1)}, \frac{0}{(x_1,y_2)}, \frac{0.7}{(x_1,y_3)}, \frac{0}{(x_1,y_4)}, \frac{0.1}{(x_2,y_1)}, \frac{0.8}{(x_2,y_2)}, \frac{0}{(x_2,y_3)}, \frac{0.2}{(x_2,y_4)}, \frac{0.5}{(x_3,y_1)}, \frac{0.4}{(x_3,y_2)}, \frac{0.3}{(x_3,y_3)}, \frac{0}{(x_3,y_4)}\right\}$$

$R\cap S$	Y_1	Y_2	Y_3	Y_4
X_1	0.3	0	0.7	0
X_2	0.1	0.8	0	0.2
X_3	0.5	0.4	0.3	0

Now, $(R \cup S) = \max(\mu_R, \mu_S)$

$$\therefore \; R\cup S = \left\{\frac{0.3}{(x_1,y_1)}, \frac{0}{(x_1,y_2)}, \frac{1}{(x_1,y_3)}, \frac{0}{(x_1,y_4)}, \frac{0.8}{(x_2,y_1)}, \frac{1}{(x_2,y_2)}, \frac{1}{(x_2,y_3)}, \frac{1}{(x_2,y_4)}, \frac{0.6}{(x_3,y_1)}, \frac{0.9}{(x_3,y_2)}, \frac{0.4}{(x_3,y_3)}, \frac{0.1}{(x_3,y_4)}\right\}$$

$R\cup S$	Y_1	Y_2	Y_3	Y_4
X_1	0.3	0.2	1	0
X_2	0.8	1	1	1
X_3	0.6	0.9	0.4	0.1

.....Ans.

Ex.26. If R and S are fuzzy relations given by

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.5 & 0.1 \\ 0.2 & 0.9 \\ 0.8 & 0.6 \end{bmatrix} \end{array} \text{ and } S = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.6 & 0.5 \\ 0.4 & 0.8 \\ 0.7 & 0.9 \end{bmatrix} \end{array}$$

then find $R\cup S$, $R\cap S$ and $\overline{R}$

Soln. We know that $\mu_{R\cup S} = \text{Max}\{\mu_R, \mu_S\}$ and

$\mu_{R\cap S} = \min\{\mu_R, \mu_S\}$, $\mu_{\overline{R}} = 1 - \mu_R$

Now, from matrix table

$$R = \left\{\frac{0.5}{(x_1, y_1)}, \frac{0.1}{(x_1, y_2)}, \frac{0.2}{(x_2, y_1)}, \frac{0.9}{(x_2, y_2)}, \frac{0.8}{(x_3, y_1)}, \frac{0.6}{(x_3, y_2)}\right\}$$

$$S = \left\{\frac{0.6}{(x_1, y_1)}, \frac{0.5}{(x_1, y_2)}, \frac{0.4}{(x_2, y_1)}, \frac{0.8}{(x_2, y_2)}, \frac{0.7}{(x_3, y_1)}, \frac{0.9}{(x_3, y_2)}\right\}$$

Now,

$$R \cap S = \left\{\frac{0.5}{(x_1, y_1)}, \frac{0.1}{(x_1, y_2)}, \frac{0.2}{(x_2, y_1)}, \frac{0.8}{(x_2, y_2)}, \frac{0.7}{(x_3, y_1)}, \frac{0.6}{(x_3, y_2)}\right\}$$

$\therefore$

$R \cap S$	y_1	y_2
x_1	0.5	0.1
x_2	0.2	0.8
x_3	0.7	0.6

Now,

$$R \cup S = \left\{\frac{0.6}{(x_1, y_1)}, \frac{0.5}{(x_1, y_2)}, \frac{0.4}{(x_2, y_1)}, \frac{0.9}{(x_2, y_2)}, \frac{0.8}{(x_3, y_1)}, \frac{0.9}{(x_3, y_2)}\right\}$$

$\therefore$

$R \cup S$	y_1	y_2
x_1	0.6	0.5
x_2	0.4	0.9
x_3	0.8	0.9

and

$$\overline{R} = \left\{\frac{0.5}{(x_1, y_1)}, \frac{0.9}{(x_1, y_2)}, \frac{0.8}{(x_2, y_1)}, \frac{0.1}{(x_2, y_2)}, \frac{0.2}{(x_3, y_1)}, \frac{0.4}{(x_3, y_2)}\right\}$$

$$\therefore \overline{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.5 & 0.9 \\ 0.8 & 0.1 \\ 0.2 & 0.4 \end{bmatrix} \end{array} \quad \text{.....Ans.}$$

Ex.27. If $A = \frac{0.3}{30} + \frac{0.7}{60} + \frac{1}{100}$, $B = \frac{0.2}{20} + \frac{0.4}{40} + \frac{0.6}{60}$, **and** $C = \frac{0.3}{50} + \frac{0.6}{100}$, **then find** $R = A \times B$, $S = B \times C$. **Also find** $R \circ S$

Soln. We have $\mu_{R=A\times B} = \min(\mu_A, \mu_B)$

$\mu_{S=B\times C} = \min(\mu_B, \mu_C)$

$$\therefore R = \begin{array}{c} \\ 30 \\ 60 \\ 100 \end{array} \begin{array}{c} \begin{array}{ccc} 20 & 40 & 60 \end{array} \\ \begin{bmatrix} 0.2 & 0.3 & 0.3 \\ 0.2 & 0.4 & 0.6 \\ 0.2 & 0.4 & 0.6 \end{bmatrix} \end{array} \text{ and } S = \begin{array}{c} \\ 20 \\ 40 \\ 60 \end{array} \begin{array}{c} \begin{array}{cc} 50 & 100 \end{array} \\ \begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.4 \\ 0.3 & 0.6 \end{bmatrix} \end{array}$$

Now

Let $T = R \circ S$

We have

$R \circ S\,(30, 50) = \text{Max}\,\{\min\{0.2, 0.2\}, \min\{0.3, 0.3\}, \min\{0.3, 0.3\}\}$

$= \text{Max}\,\{0.2, 0.3, 0.3\} = 0.3$

$R \circ S\,(30, 100) = \text{Max}\,\{\min\{0.2, 0.2\}, \min\{0.3, 0.4\}, \min\{0.3, 0.6\}\}$

$= \text{Max}\,\{0.2, 0.3, 0.3\} = 0.3$

$R \circ S\,(60, 50) = \text{Max}\,\{\min\{0.2, 0.2\}, \min\{0.4, 0.3\}, \min\{0.6, 0.3\}\}$

$= \text{Max}\,\{0.2, 0.3, 0.3\} = 0.3$

$R \circ S\,(60, 100) = \text{Max}\,\{\min\{0.2, 0.2\}, \min\{0.4, 0.4\}, \min\{0.6, 0.6\}\}$

$= \text{Max}\,\{0.2, 0.4, 0.6\} = 0.6$

$R \circ S\,(100, 50) = \text{Max}\,\{\min\{0.2, 0.2\}, \min\{0.4, 0.3\}, \min\{0.6, 0.3\}\}$

$= \text{Max}\,\{0.2, 0.3, 0.3\} = 0.3$

$R \circ S\,(100, 100) = \text{Max}\,\{\min\{0.2, 0.2\}, \min\{0.4, 0.4\}, \min\{0.6, 0.6\}\}$

$= \text{Max}\,\{0.2, 0.4, 0.6\} = 0.6$

$$\therefore T = R \circ S = \begin{array}{c} \\ 30 \\ 60 \\ 100 \end{array} \begin{array}{c} \begin{array}{cc} 50 & 100 \end{array} \\ \begin{bmatrix} 0.3 & 0.3 \\ 0.3 & 0.6 \\ 0.3 & 0.6 \end{bmatrix} \end{array} \quad \text{.....Ans.}$$

Ex.28. If

R	Y_1	Y_2	Y_3	Y_4
X_1	0.4	0.5	0.1	0
X_2	0.8	0.9	1.0	0
X_3	0.2	0.1	0.5	0.6

and

S	Y_1	Y_2	Y_3	Y_4
X_1	0.5	0.2	0.6	0.8
X_2	0.9	1	0.1	0.1
X_3	0.8	0.9	0.6	0.7

Then find (i) $R \cup S$ **(ii)** $R \cap S$ **(iii)** $\overline{R}$

Soln. We have

(i) $\mu_{R\cup S} = \text{Max}\{\mu_R, \mu_S\}$

$\therefore$

$R \cup S$	Y_1	Y_2	Y_3	Y_4
X_1	0.5	0.5	0.6	0.8
X_2	0.9	1	1	0.1
X_3	0.8	0.9	0.6	0.7

(ii) $\mu_{R\cap S} = \min\{\mu_R, \mu_S\}$

$\therefore$

$R \cap S$	Y_1	Y_2	Y_3	Y_4
X_1	0.4	0.2	0.1	0
X_2	0.8	0.9	0.1	0
X_3	0.2	0.1	0.5	0.6

(iii) $\mu_{\overline{R}} = 1 - \mu_R$

$\therefore$

$\overline{R}$	Y_1	Y_2	Y_3	Y_4
X_1	0.6	0.5	0.9	1
X_2	0.2	0.1	0	1
X_3	0.8	0.9	0.5	0.4

.....Ans.

FUZZY LOGIC

- The word fuzzy refers to the fuzziness means the things that are not clear, means, sometimes there are few statements, where it is very difficult for someone to come to the conclusion whether the given statement has the validity either true or false (yes or no).
- At that time, this concept provides many values between the true and false and gives the flexibility to find the best solution to that problem.

Example of fuzzy logic as Compairing to Boolean Logic:

Consider statement

Deepika is beautiful

In Boolean logic, it has only two possibilities either 0 (for No) and 1 (for Yes).

But in fuzzy logic, it has a multiple possibilities like, many people think, she is 0.5 i.e. 50% beautiful, for some people she is 0.2 i.e. 20% beautiful and so on in short, the range of beautiness of Deepika varies person to person, in such case fuzzy logic comes in action.

- Fuzzy logic contains the multiple logical values and these values are the truth values of a variable or a problem between 0 and 1.

This concept was introduced by Lotfi Zadeh in 1965 while working on the problem of computer understanding of natural language.

Fuzzy Logic in Computer Science and AI:

In CS and AI, Fuzzy logic is used to imitate human reasoning and cognition. As a result, fuzzy logic is well suited for the following.

- Engineering for decision without clear certainties and uncertainties or with the imprecise data such as with natural Language Processing Technologies.
- Resulting and controlling machine outputs, according to multiple inputs/input variables such as with temperature control system.
- Apart from this, various types of AI system and technologies use fuzzy logic, this includes vehicle intelligence, consumer elements, medicines, software chemical and aerospace.

Connectives in Fuzzy Logic with Truth Values:

Let A and B are two fuzzy sets with membership grade $u_A(x)$, $u_B(x)$ respectively defined on the same universe of discourse, then these two sets A and B can be connected with following connectives:

(1) **Conjunction ($\wedge$):**

$$\mu_A(x) \wedge \mu_B(x) = \text{Min}\{\mu_A(x), \mu_B(x)\}$$

(2) **Disjunction ($\vee$):**

$$\mu_A(x) \vee \mu_B(x) = \text{Max}\{\mu_A(x), \mu_B(x)\}$$

(3) **Negation or Complement (–):**

$$\overline{\mu}_A(x) = 1 - \mu_A(x)$$

(4) **Implication or Condition ($\rightarrow$):**

$$\mu_A(x) \rightarrow \mu_B(x) = \min(1, 1 + b - a)$$

(5) **Bicondition ($\Leftrightarrow$ or $\leftrightarrow$) :**

$$\mu_A(x) \leftrightarrow \mu_B(x) = 1 - |a - b|$$

Ex.29. Construct the truth table for $A \wedge (A \rightarrow B) \rightarrow B$ if

$$A = \left\{\frac{0.3}{a}, \frac{0.3}{b}, \frac{0.7}{c}, \frac{0.7}{d}\right\} \text{ and } B = \left\{\frac{0.2}{a}, \frac{0.8}{b}, \frac{0.2}{c}, \frac{0.8}{d}\right\}$$

Ans. Truth table:

A	B	$A \rightarrow B$	$A \wedge (A \rightarrow B)$	$[A \wedge (A \rightarrow B)] \rightarrow B$
0.3	0.2	0.9	0.3	0.9
0.3	0.8	1	0.3	1
0.7	0.2	0.5	0.5	0.7
0.7	0.8	1	0.7	1

COMPLETE LIST OF FORMULAE:

(1) A fuzzy set 'A' of a non-zero set 'X' can be symbolically written as

$$A = \{(\mu_A(x_1), x_1), (\mu_A(x_2), x_2)), \ldots\ldots, (\mu_A(x_n), x_n)\}$$

$$\text{or } A = \left\{\frac{\mu_A(x_1)}{x_1}, \frac{\mu_A(x_2)}{x_2}, \ldots\ldots, \frac{\mu_A(x_n)}{x_n}\right\}$$

$$\text{or } A = \left\{\frac{\mu_A(x_1)}{x_1} + \frac{\mu_A(x_2)}{x_2} + \ldots\ldots + \frac{\mu_A(x_n)}{x_n}\right\}$$

(2) The crisp membership function χ of a set A is

$$\chi_A(x) = \begin{cases} 1, \text{ if } x \in A \\ 0, \text{ if } x \notin A \end{cases}$$

(3) $\overline{A} = 1 - \mu_A(x)$

(4) $A \cup B = \max\{\mu_A(x), \mu_B(x)\}$

(5) $A \cap B = \min\{\mu_A(x), \mu_B(x)\}$

(6) $A - B = A \cap \overline{B}$

(7) Relative Complement of A with $B = B - A = B \cap \overline{A}$

(8) Relative Complement of B with $A = A - B = A \cap \overline{B}$

(9) $\mu_{A+B}(x) = [\mu_A(x) + \mu_B(x)] - [\mu_A(x) \cdot \mu_B(x)]$

(10) $\mu_{A \cdot B}(x) = \mu_A(x) \cdot \mu_B(x)$

(11) $\mu_{A \oplus B}(x) = \min[1, (\mu_A(x) + \mu_B(x))]$

(12) $\mu_{A \ominus B}(x) = \max[0, (\mu_A(x) - \mu_B(x))]$

(13) $A^{\geq \alpha} = A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}$ is called α - cut.

(14) $A^{> \alpha} = A'_\alpha = \{x \in X \mid \mu_A(x) > \alpha\}$ is called strong α - cut.

(15) $S(A) = \text{Supp}\ (A) = A^{>0} = \{x \in X \mid \mu_A(x) > 0\}$ is called support.

(16) $C(A) = A^{=1} = \{x \in X \mid \mu_A(x) = 1\}$ is called core.

(17) $\text{Card}(A) = |A| = \sum_{x \in X} \mu_A(x)$

(18) $\text{Rel Card}(A) = \|A\| = |A|/|X|$

PROBLEMS FOR PRACTICE:

Q.1. If A and B are fuzzy sets on $X = \{p, q, r\}$

defined by $A = \left\{\frac{0.2}{p}, \frac{0.7}{q}, \frac{0.8}{r}\right\}$, $B = \left\{\frac{0.3}{p}, \frac{0.4}{q}, \frac{0.7}{r}\right\}$

then find: (i) $A \cup B$ (ii) $A \cap B$ (iii) $\overline{A}, \overline{B}$

Ans. (i) $\left\{\frac{0.3}{p}, \frac{0.7}{q}, \frac{0.8}{r}\right\}$

(ii) $\left\{\frac{0.2}{p}, \frac{0.4}{q}, \frac{0.7}{r}\right\}$

(iii) $\overline{A} = \left\{\frac{0.8}{p}, \frac{0.3}{q}, \frac{0.2}{r}\right\}$, $\overline{B} = \left\{\frac{0.7}{p}, \frac{0.6}{q}, \frac{0.3}{r}\right\}$

Q.2. If $X = \{1, 2, 3, 4, 5\}$ and $A = \{(1, 0.2), (2, 0.4), (3, 0.5), (4, 0), (5, 0.3)\}$, then find (i) $S(A)$ (ii) $A_{0.3}$

Ans. (i) $\{0.2, 0.3, 0.4, 0.5\}$ (ii) $\{2, 3, 5\}$

Q.3. If $A = \left\{\frac{0.3}{1} + \frac{0.7}{2} + \frac{0.8}{3}\right\}$, $B = \left\{\frac{0.1}{1} + \frac{0.5}{2} + \frac{0.7}{3}\right\}$,

then find the relative complement of A with b.

Ans. $\left\{\frac{0.1}{1} + \frac{0.3}{2} + \frac{0.2}{3}\right\}$

Q.4. If $A = \frac{0.3}{1} + \frac{0.7}{2} + \frac{0.5}{3}$, $B = \frac{0.5}{1} + \frac{0.2}{2} + \frac{0.7}{3}$,

then find: (i) $A + B$ (ii) AB

Ans. (i) $\frac{0.65}{1} + \frac{0.76}{2} + \frac{0.85}{3}$ (ii) $\frac{0.15}{1} + \frac{0.14}{2} + \frac{0.49}{3}$

Q.5. Given fuzzy sets

$$A = \left\{\frac{0.2}{x_1} + \frac{0.5}{x_2} + \frac{1}{x_3}\right\}, \quad B = \left\{\frac{0.3}{y_1} + \frac{0.9}{y_2}\right\}$$

Find the relation R from A to B

Ans. $R = A \times B = \begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.5 \\ 0.3 & 0.9 \end{bmatrix}$

Q.6. Given that

$X = \{x_1, x_2\}$, $Y = \{y, y_2\}$, $Z = \{z_1, z_2, z_3\}$

Let R and S be the relation defined on X to Y and Y to Z respectively such that

$$R = \begin{bmatrix} 0.7 & 0.5 \\ 0.8 & 0.4 \end{bmatrix} \text{ and } S = \begin{bmatrix} 0.9 & 0.6 & 0.2 \\ 0.1 & 0.7 & 0.5 \end{bmatrix}$$

Find the relation $T = R \circ S$

Ans. $T = \begin{bmatrix} 0.7 & 0.6 & 0.5 \\ 0.8 & 0.6 & 0.4 \end{bmatrix}$

Q.7. Given that the Fuzzy sets

$$A = \frac{0.3}{30} + \frac{0.7}{60} + \frac{1.0}{100} + \frac{0.2}{120}$$

$$B = \frac{0.2}{20} + \frac{0.4}{40} + \frac{0.6}{60} + \frac{0.8}{80} + \frac{1.0}{100} + \frac{0.1}{120} \text{ and}$$

$$C = \frac{0.33}{500} + \frac{0.67}{1000} + \frac{1.0}{1500} + \frac{0.15}{1800}$$

Find R from A to B and S from B to C Also find $T = R \circ S$

Ans. $R = \begin{bmatrix} 0.2 & 0.3 & 0.3 & 0.3 & 0.3 & 0.1 \\ 0.2 & 0.4 & 0.6 & 0.7 & 0.7 & 0.1 \\ 0.2 & 0.4 & 0.6 & 0.8 & 1 & 0.1 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.1 \end{bmatrix}$

$$S = \begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.15 \\ 0.33 & 0.4 & 0.4 & 0.15 \\ 0.33 & 0.6 & 0.6 & 0.15 \\ 0.33 & 0.67 & 0.8 & 0.15 \\ 0.33 & 0.67 & 1.0 & 0.15 \\ 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix}$$

$$T = R \circ S = \begin{bmatrix} 0.3 & 0.3 & 0.3 & 0.15 \\ 0.33 & 0.67 & 0.7 & 0.15 \\ 0.33 & 0.67 & 1 & 0.15 \\ 0.2 & 0.2 & 0.2 & 0.15 \end{bmatrix}$$

Q.9. Given the fuzzy relations as

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 0.8 & 0.1 & 0.1 & 0.7 \\ 0 & 0.8 & 0 & 0 \\ 0.9 & 1.0 & 0.7 & 0.8 \end{bmatrix} \end{array}$$

$$S = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 0.4 & 0 & 0.9 & 0.6 \\ 0.9 & 0.4 & 0.5 & 0.7 \\ 0.3 & 0 & 0.8 & 0.5 \end{bmatrix} \end{array}$$

Find: (1) $R \cup S$ (2) $R \cap S$ (3) $\overline{R}$

Ans. (1)
$$R \cup S = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 0.8 & 0.1 & 0.9 & 0.7 \\ 0.9 & 0.8 & 0.5 & 0.7 \\ 0.9 & 1.0 & 0.8 & 0.8 \end{bmatrix} \end{array}$$

(2)
$$R \cap S = \begin{bmatrix} 0.4 & 0 & 1 & 0.6 \\ 0 & 0.4 & 0 & 0 \\ 0.3 & 0 & 0.7 & 0.5 \end{bmatrix}$$

(3)
$$\overline{R} = \begin{bmatrix} 0.2 & 0.9 & 0.9 & 0.3 \\ 1.0 & 0.2 & 1.0 & 1.0 \\ 0.1 & 0 & 0.3 & 0.2 \end{bmatrix}$$

UNIT - IV
GRAPH THEORY

QUOTIENT GRAPH

Q.1. What is Quotient Graph?

Ans. Let G = (V, E) be a graph, Let ~ be an equivalence relation on V. The Quotient graph of G with respect to ~ is a graph whose vertex set is the quotient set V/~ and two equivalence classes [u], [v] form an edge if and only if uv forms an edge in G.

Example: Let G = (V, E), where $V = \{v_1, v_2, \ldots, v_8\}$. E is defined by

$v_i v_j \in E \Leftrightarrow |i - j| \geq 3.$

See Fig.1(a) Let ~ be a relation on V defined by

$v_i \sim v_j \Leftrightarrow 5 \mid (i - j)$

Then ~ is an equivalence relation. The quotient graph G/~ is demonstrated in Fig.1(b).

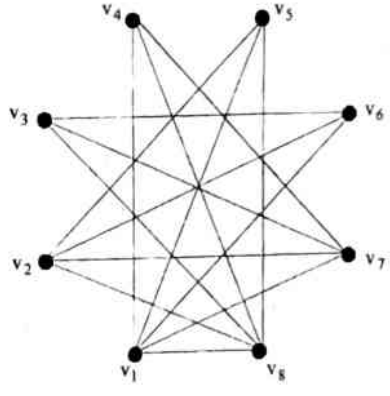
Example: Let G = (V, E), where $V = \{v_1, v_2, \ldots, v_8\}$. E is defined by

$v_i v_j \in E \Leftrightarrow 3 \leq |i - j| \leq 5.$

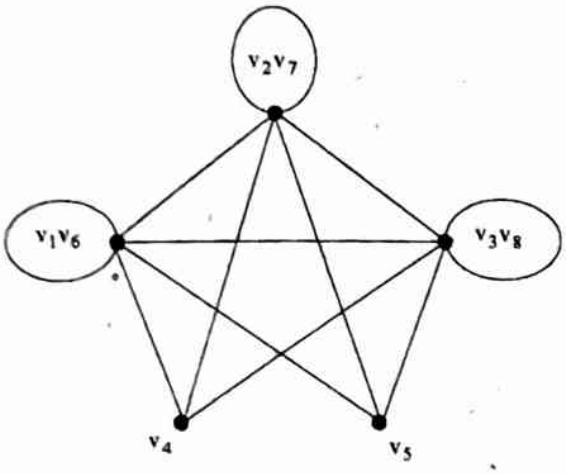
See Fig.2(a). Let ~ be a relation on V defined by

$v_i \sim v_j \Leftrightarrow 6 \mid (i - j).$

Then ~ is an equivalence relation and the quotient graph G/~ is given in Fig.2(b).

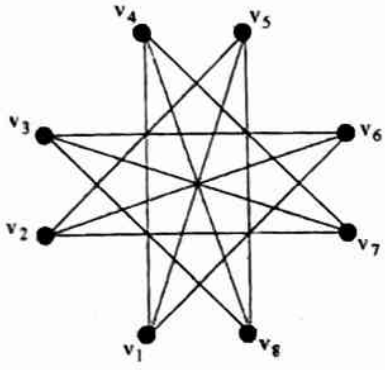


(a) G

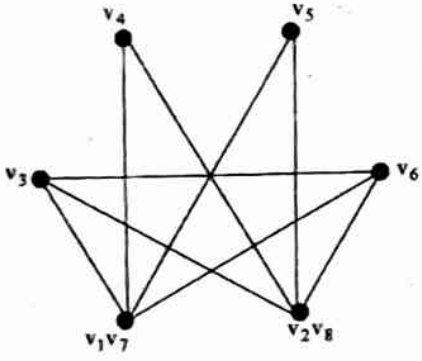


(b) G/~

Fig.1. Constructiosn of a quotient graph.



(a) G



(b) G/~

Fig.2. Construction of a quotient graph.

UNDIRECTED TREE

Q.2. What is a Undirected Tree?

Ans. Undirected Tree:

- An undirected graph tree is one in which the pair of vertices in an edge is unordered.
- An undirected graph is a tree if any two of the following three properties are true:

(a) It is connected.

(b) There are no cycles.

(c) There are n – 1 edges, where n is the number of nodes.

- An undirected graph is a tree if there is exactly one simple path between any pair of vertices.

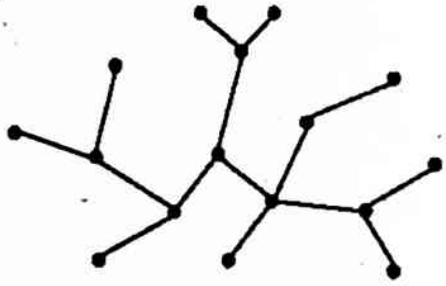


Fig. Undirected Tree

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