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Complex numbers

1.) A Complex number 'z' consist of an ordered pair of real (ordinary) numbers (a, b) that obeys particular algebraic rules.

$$z = (a, b) \neq (b, a)$$

$$(a, b) = (c, d)$$

$$\Rightarrow a = c, \quad b = d.$$

2.) Algebraic Rules of Complex Number

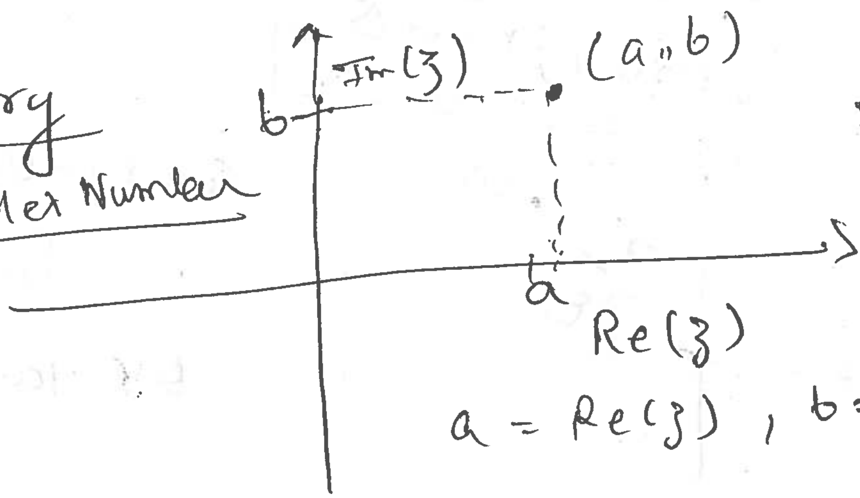
a) $c \cdot z = c \cdot (a, b) = (c \cdot a, c \cdot b)$, c is a scaler

b) $z_1 + z_2 = (a_1, b_1) + (a_2, b_2)$
$= \cancel{a_1 + a_2)} \; (a_1 + a_2, b_1 + b_2)$

c) $z_1 \times z_2 = (a_1 a_2 - b_1 b_2, \; a_1 b_2 + a_2 b_1)$

d) $\dfrac{1}{z} = \left(\dfrac{a}{a^2 + b^2}, \; \dfrac{-b}{a^2 + b^2} \right)$

3) Geometry of Complex Number



$z = (a + jb)$

$j = \sqrt{-1}$

$a = Re(z), \; b = Im(z)$

$$z_1 \times z_2 = (a_1 + jb_1)\cdot(a_2 + jb_2)$$

$$= (a_1 a_2 - b_1 b_2 + j(a_1 b_2 + b_1 a_2)$$

$$\frac{1}{z} = \frac{1}{a+jb}\,\frac{(a-jb)}{(a-jb)} = \frac{a}{a^2+b^2} - j\frac{b}{a^2+b^2}$$

4. <u>Euler's formulae</u>

$$e^{cx} = 1 + cx + \frac{c^2x^2}{2!} + \frac{(cx)^3}{3!} \cdots$$

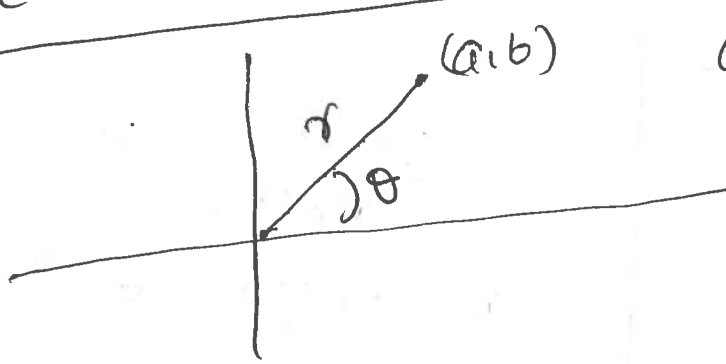
what if $c = j$

$$= 1 + jx - \frac{x^2}{2!} - j\frac{x^3}{3!} + \frac{x^4}{4!}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + j\left(x - \frac{x^3}{3!} \cdots\right)$$

$$e^{jx} = \cos x + j\sin x$$

or $\boxed{re^{j\theta} = r\cos\theta + jr\sin\theta}$ - Polar form.



$a = r\cos\theta,\ b = r\sin\theta$

$r = \sqrt{a^2+b^2}$

$\theta = \tan^{-1}\left(\frac{b}{a}\right)$

$z = a + jb$ - Cartesian form

$z = re^{j\theta}$ Polar form.

Certain calculation are easier in one form than other, so easy conversion is needed betn forms.

<u>Complex Conjugate</u> - z^* flips the sign of the imaginary part.

$$\left.\begin{aligned} z^* &= a - jb \\ &= re^{-j\theta} \end{aligned}\right\}$$

$$zz^* = (a+jb)(a-jb) = a^2 + b^2$$

$$= re^{j\theta} re^{-j\theta} = r^2$$

$$|z|^2 = z \cdot z^* \text{ or } |z| = r$$

<u>Real part</u> - $Re\{z\} = \frac{z + z^*}{2}$

$$Im\{z\} = \frac{z - z^*}{2j}$$

Example1 - $\frac{z_1}{z_2} = \frac{a_1+jb_1}{a_2+jb_2} = \frac{(a_1+jb_1)(a_2-jb_2)}{(a_2+jb_2)(a_2-jb_2)}$

$$= \frac{(a_1a_2+b_1b_2)+j(a_2b_1-a_1b_2)}{(a_2^2+b_2^2)}$$

$$\frac{z_1}{z_2} = \frac{r_1e^{j\theta_1}}{r_2e^{j\theta_2}} = \left(\frac{r_1}{r_2}\right)e^{j(\theta_1-\theta_2)}$$

$$= re^{j\theta} \quad ; \; r = \frac{r_1}{r_2}$$

$$\theta = \theta_1 - \theta_2$$

Example-2

Solve $\frac{s+2}{s^2+s+1}$; $s = j2\pi f$

$$= \frac{j2\pi f+2}{-4\pi^2f^2+j2\pi f+1}$$

$$= \frac{\sqrt{(4\pi^2f^2+4)}\; e^{j\arctan(\pi f)}}{\sqrt{(1-4\pi^2f^2)^2+4\pi^2f^2}\; e^{j\arctan\left(\frac{2\pi f}{1-4\pi^2f^2}\right)}}$$

$$= \frac{\sqrt{4\pi^2f^2+4}}{\sqrt{(1-4\pi^2f^2)^2+4\pi^2f^2}} \; e^{j\tan^{-1}(\pi f) - j\tan^{-1}\left(\frac{2\pi f}{1-4\pi^2f^2}\right)}$$

→ ampl phase

Trignometric identities and Euler

$e^{ix} = \cos x + i\sin x$; $\cos^2 x + \sin^2 x = 1$

(1) $\cos(-x) + i\sin(-x) = e^{-ix} = \dfrac{1}{e^{ix}}$

$$= \frac{1}{\cos x + i\sin x} = \frac{\cos x - i\sin x}{\cos^2 x + \sin^2 x}$$

$$= \cos x - i\sin x$$

$\Rightarrow \cos(-x) = \cos x$

$\sin(-x) = -\sin x$

(2) $\cos(x+y) + i\sin(x+y) = e^{i(x+y)}$

$$= e^{ix} \cdot e^{iy}$$

$$= (\cos x + i\sin x)(\cos y + i\sin y)$$

$$= \cos x\cos y - \sin x\sin y + i(\sin x\cos y + \cos x\sin y)$$

$\Rightarrow \cos(x+y) = \cos x\cos y - \sin x\sin y$

$\sin(x+y) = \sin x\cos y + \cos x\sin y$

(3) $\cos(2x) + i\sin(2x) = e^{i2x} = (e^{ix})^2$

$= [\cos x + i\sin x]^2 = (\cos x + i\sin x)(\cos x + i\sin x)$

$= \underline{\cos^2 x - \sin^2 x} + i\,\underline{2\sin x\cos x}$

$\Rightarrow \cos 2x = \cos^2 x - \sin^2 x$

$\sin 2x = 2\sin x\cos x$

(4) $\cos x - \cos(y) + i[\sin x - \sin y]$

$= (\cos x + i\sin x) - [\cos y + i\sin y]$

$= e^{ix} - e^{iy}$

$= e^{ix + \frac{iy}{2} - \frac{iy}{2}} - e^{iy + \frac{ix}{2} - \frac{ix}{2}}$

$= e^{i\left(\frac{x+y}{2}\right)}\left[e^{i\left(\frac{x-y}{2}\right)} - e^{i\left(\frac{y-x}{2}\right)}\right]$

$= \cancel{\cos(x+y)}$

$= \cancel{e^{i\left(\frac{x+y}{2}\right)}}\left[2j\sin\left(\frac{x-y}{2}\right)\right]\left[\cos\frac{x+y}{2} + j\sin\frac{x+y}{2}\right]$

$= \underline{j2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)} - \underline{2\sin\left(\frac{x-y}{2}\right)\sin\frac{x+y}{2}}$

Example - 3

$$\cos x \cos y = \left[\frac{e^{ix}+e^{-ix}}{2}\right]\left[\frac{e^{iy}+e^{-iy}}{2}\right]$$

$$= \frac{\left[e^{i(x+y)} + e^{i(x-y)} + e^{i(y-x)} + e^{-i(x+y)}\right]}{4}$$

$$= \frac{1}{2}\left[\frac{e^{i(x+y)}+e^{-i(x+y)}}{2} + \frac{e^{i(x-y)}+e^{i(y-x)}}{2}\right]$$

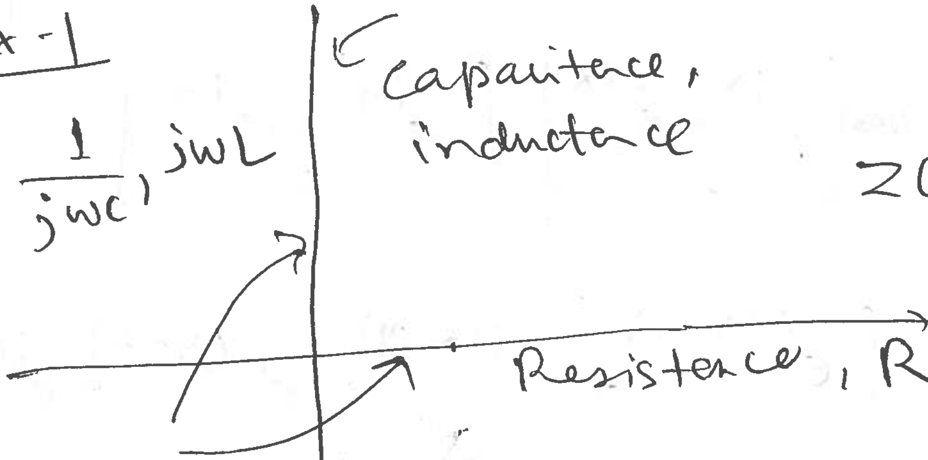
$$= \frac{1}{2}\left[\cos(x+y) + \cos(x-y)\right]$$

Why complex number?

'i' is an operator, an operator that takes you from real ~~plat~~ line to imaginary line.

$i^2 = i(i) = -1$ i.e. when you apply this operator twice, ~~you reach to~~ it is equivalent to multiplying by -1.

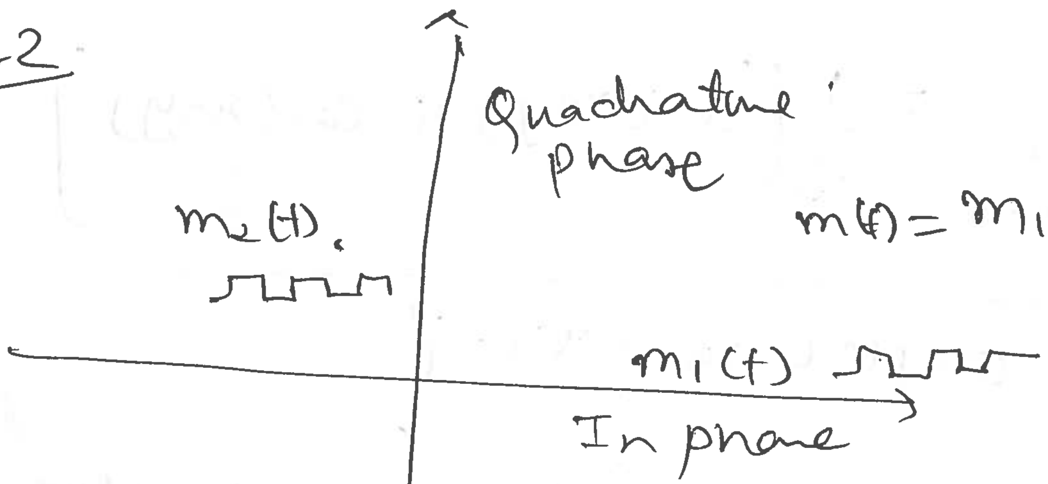
Ex-1



$$Z(t) = R + j\omega L + \frac{1}{j\omega C}$$

they behave differently, an we want to Keep things separate.

Ex-2



$$m(t) = m_1(t) + j m_2(t)$$

Ex-3 Changing the phase of ~~sinuso~~ Signal

$$\sin(t) \longrightarrow \boxed{e^{j\frac{\pi}{2}}} \longrightarrow ?$$

$$\sin t = \operatorname{Im}\{e^{jt}\} =$$

$$\text{output} = e^{jt} e^{j\frac{\pi}{2}} = \operatorname{Im}\{e^{j(\frac{\pi}{2}+t)}\}$$

$$= \sin\left(\frac{\pi}{2}+t\right) = \cos(t)$$

Ex-4)