STAT 626
Homework 5

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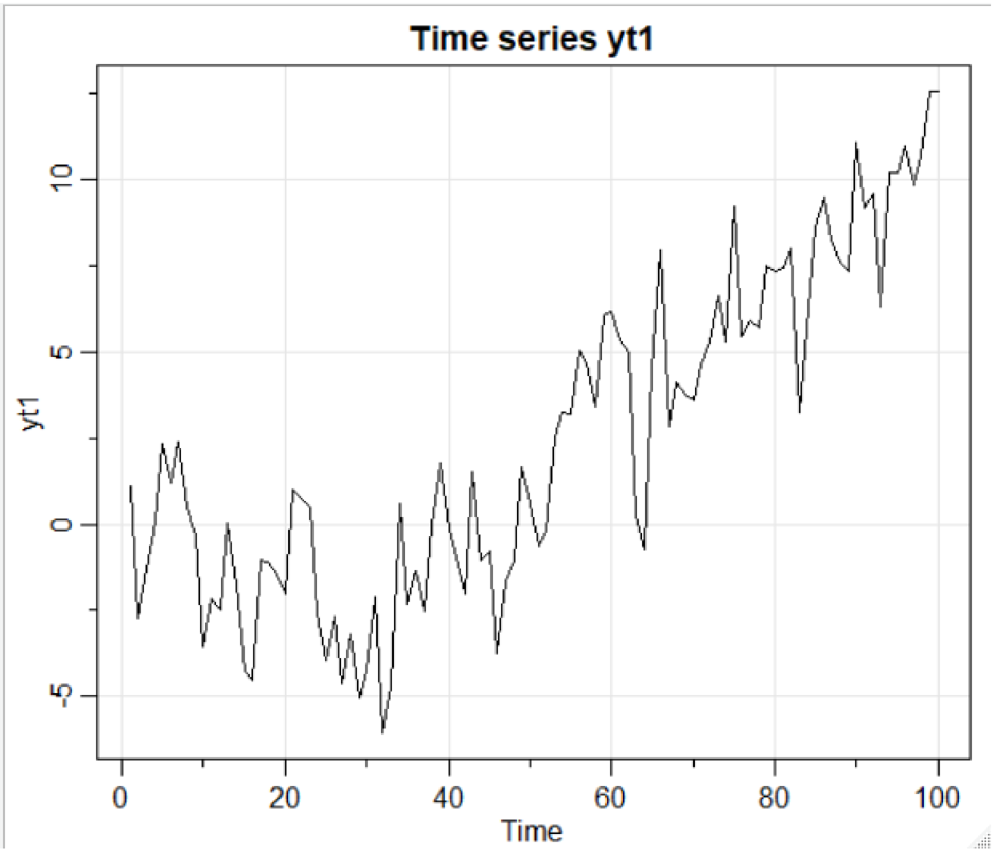
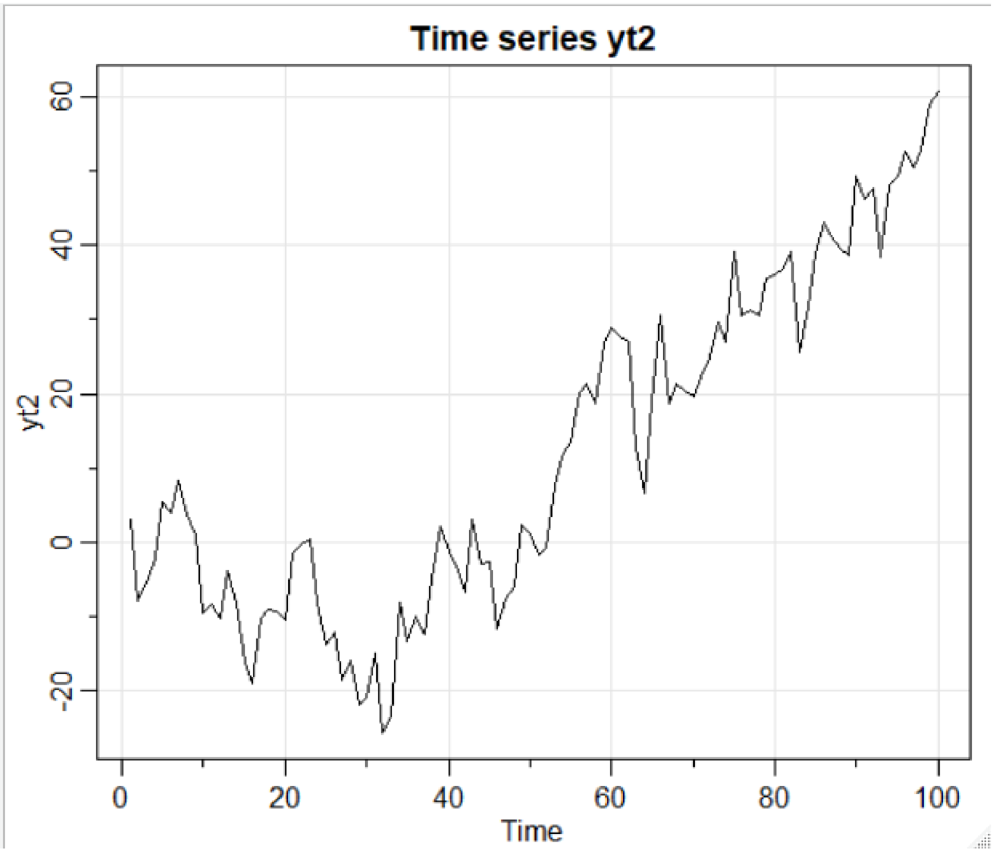
I.

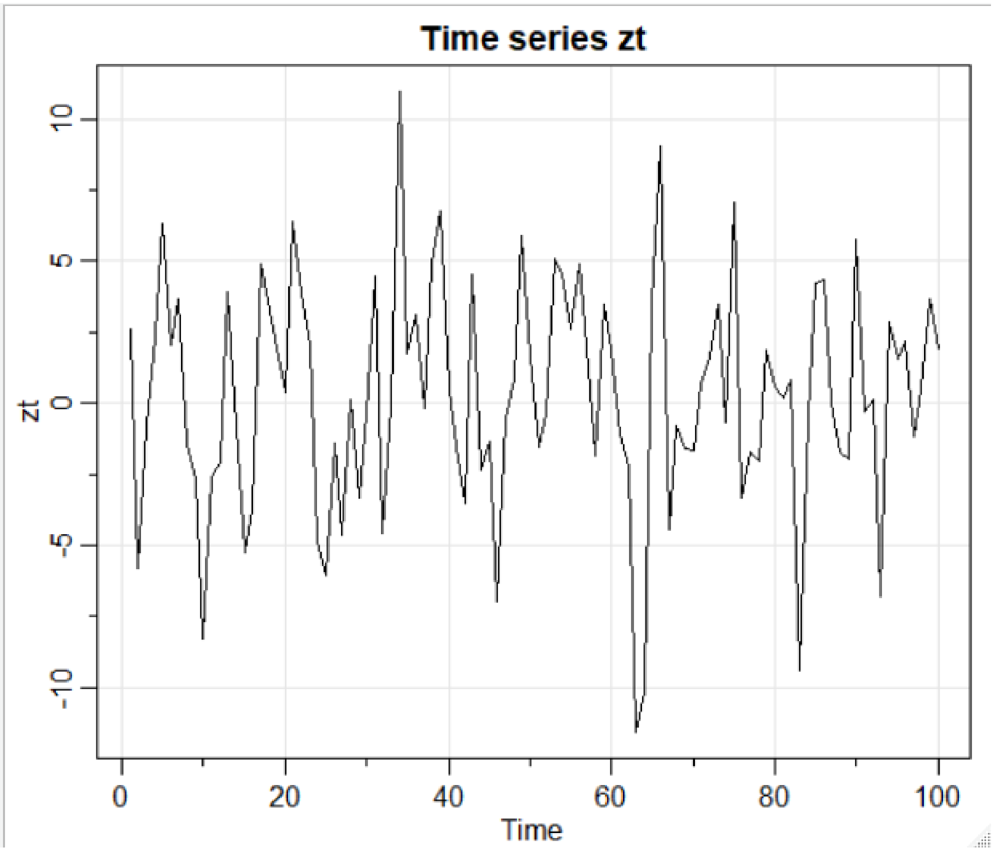
a)



Time series yt1
yt1
10
5
0
-5
0
20
40
60
80
100
Time



Time series yt2
yt2
60
40
20
0
-20
0
20
40
60
80
100
Time



From the plots above, y_{t1} and y_{t2} appear to be non-stationary (mean not constant), whereas z_t appears to be stationary (constant mean and variance).

HW 5

I

b]

$$E[y_{t,1}] = E[x_{t,1}] + E\left[\sum_{j=1}^{t} w_j\right]$$

$$= 0 \quad \because E[x_{t,1}] = 0 \quad AR(1)$$

$$\& \; E[w_t] = 0$$

$$\gamma_{y_{t,1}}(0) = \gamma_{x_{t,1}}(0) + \mathrm{Var}\left(\sum_{j=1}^{t} w_j\right)$$

$$= \frac{1}{1-0.5^2} + t \times (1)^2$$

$$= 1.33 + t$$

$$\gamma_{y_{t,1}}(h) = \gamma_{x_{t,1}}(h) + \mathrm{cov}\left(\sum_{j=1}^{t+h} w_j, \sum_{i=1}^{t} w_i\right)$$

$$= \phi^h \gamma_x(0) + \min\{t+h, t\}\sigma_w^2$$

$$\underline{\gamma_{y_{t,1}}(h) = 0.5^h \times 1.33 + t \times \sigma_w^2}$$

Since autocovariance of $y_{t,1}$ is dependent on time, it is <u>not stationary</u>.

c.]

$$z_t = 5\,y_{t_1} - y_{t_2}$$

$$E(z_t) = 5\,E(y_{t_1}) - E(y_{t_2})$$

$$= \underline{\underline{0}}$$

$$\gamma_{z_t}(h) = 5\,\gamma_{y_{t_1}}(h) - \gamma_{y_{t_2}}(h)$$

$$= 5 \times 1.33 \times 0.5^{h} - \frac{0.9^{h}}{1-0.9^{2}}$$

$$= \underline{\underline{6.65 \times (0.5)^{h} - 5.26 \times (0.9)^{h}}}$$

Mean of z_t is constant and $\gamma_{z_t}(h)$ is independent of time, so z_t is <u>stationary</u>.

d.]

For y_{t_1}, y_{t_2}:

→ $E(y_{t_1}) = E(y_{t_2}) = 0$

→ $\gamma_{y_{t_1}}(h)$ & $\gamma_{y_{t_2}}(h)$ both are time dependent. It increases with time which can be observed in part a) plot.

For z_t:

PTO

→ [illegible] stationary (y_{t_1}, y_{t_2}), we need y_{t_1} and y_{t_2} each to be stationary which is not the case, [illegible]

For z_t:,

→ $E(z_t) = 0$ and $\gamma_{z_t}(h)$ is not time dependent and it depends only on lag.

→ Therefore, z_t is stationary. The plot also shows the same.

e.] CCF of y_{t_1}, y_{t_2}.

→ To be jointly stationary (y_{t_1}, y_{t_2}), we need y_{t_1} and y_{t_2} each to be stationary which is not the case, so they are not jointly stationary.

→ Since they are not jointly stationary, CCF cannot be computed.

II Problem 3.5

$$x_t = \mu_t + y_t$$
$$x_{t-1} = \mu_{t-1} + y_{t-1}$$

$$x_t - x_{t-1} = (\mu_t + y_t) - (\mu_{t-1} + y_{t-1})$$

$$\mu_t = \delta + \mu_{t-1} + w_t$$

$$x_t - x_{t-1} = \delta + \cancel{\mu_{t-1}} + w_t + y_t - \cancel{\mu_{t-1}} - y_{t-1}$$

$$= \delta + w_t + y_t - y_{t-1}$$

Given that y_t is stationary:

$$E[x_t - x_{t-1}] = E[\delta + w_t + y_t - y_{t-1}]$$
$$= \delta + 0 + E[y_t] - E[y_{t-1}]$$
$$= \delta$$

[Since y_t is a stationary process, $E[y_t] = E[y_{t-1}]$]

Let $y_t - y_{t-1} = z_t$

$x_t - x_{t-1} = \delta + w_t + z_t$

So to prove $x_t - x_{t-1}$ stationary, we need to prove z_t is stationary.

$\gamma_z(h) = cov(z_{t+h}, z_t)$

$= cov(y_{t+h} - y_{t+h-1}, y_t - y_{t-1})$

$= E[y_{t+h}\, y_t - y_{t+h}\, y_{t-1} - y_{t+h-1}\, y_t + y_{t+h-1}\, y_{t-1}]$

$= E[y_{t+h}\, y_t - y_{t+h+1}\, y_t - y_{t+h-1}\, y_t + y_{t+h}\, y_t]$

$= E[2 y_{t+h}\, y_t - y_{t+h+1}\, y_t - y_{t+h-1}\, y_t]$

$= 2 E[y_{t+h}\, y_t] - E[y_{t+h+1}\, y_t] - E[y_{t+h-1}\, y_t]$

$= 2\gamma_y(h) - \gamma_y(h+1) - \gamma_y(h-1)$

$\gamma_y(h)$ does not depend on time, as y_t is stationary.

$\therefore$ $\gamma_z(h)$ is independent of time.

& $E[z_t] = 0$ [constant].

$\therefore$ z_t is stationary.

$\therefore$ $\nabla x_t = x_t - x_{t-1}$ is also stationary.

II.

Problem 3.6

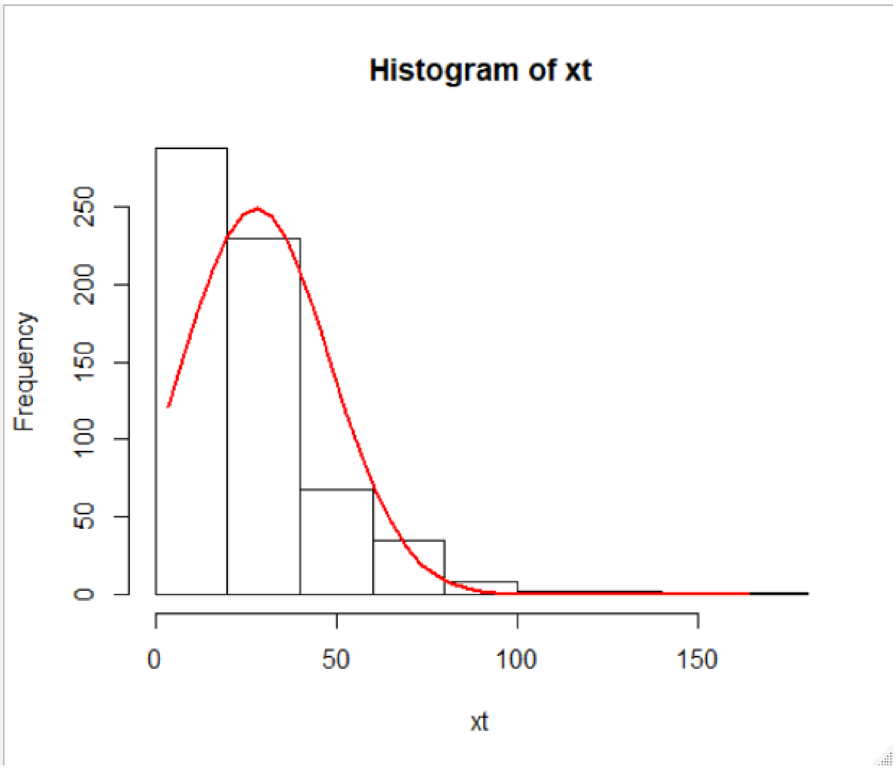
a)

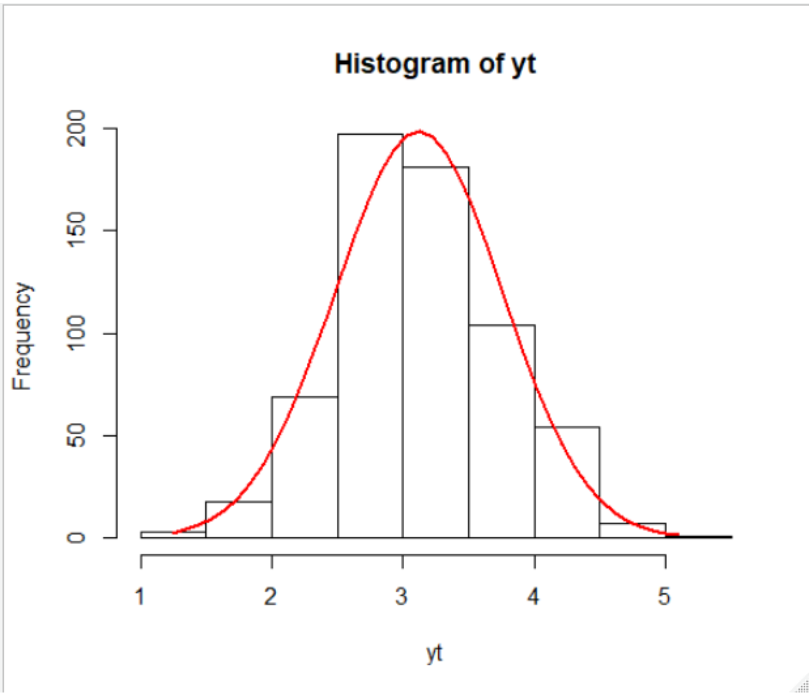
```
> v1=varve[1:(length(varve)/2)]
> v2=varve[(1+length(varve)/2):length(varve)]
> var(v1)    # Variance of first half of the data
[1] 133.4574
> var(v2)    # Variance of the second half of the data
[1] 594.4904
```

From the above screenshot, we notice that the sample variance drastically increases from first half of to second half of the data (xt). This shows that xt exhibits heteroscedasticity.

```
> v=log(varve)
> v1=v[1:length(v)/2]
> v2=v[(1+length(v)/2):length(v)]
> var(v1)    # Variance of first half of the data
[1] 0.269403
> var(v2)    # Variance of the second half of the data
[1] 0.451371
```

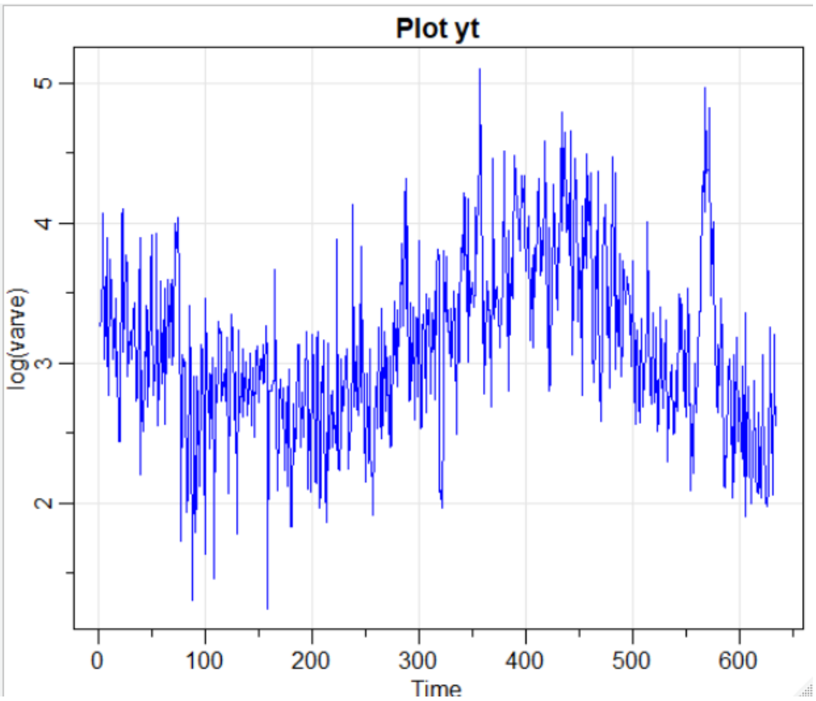
The sample variance of yt is almost the same across both the samples.





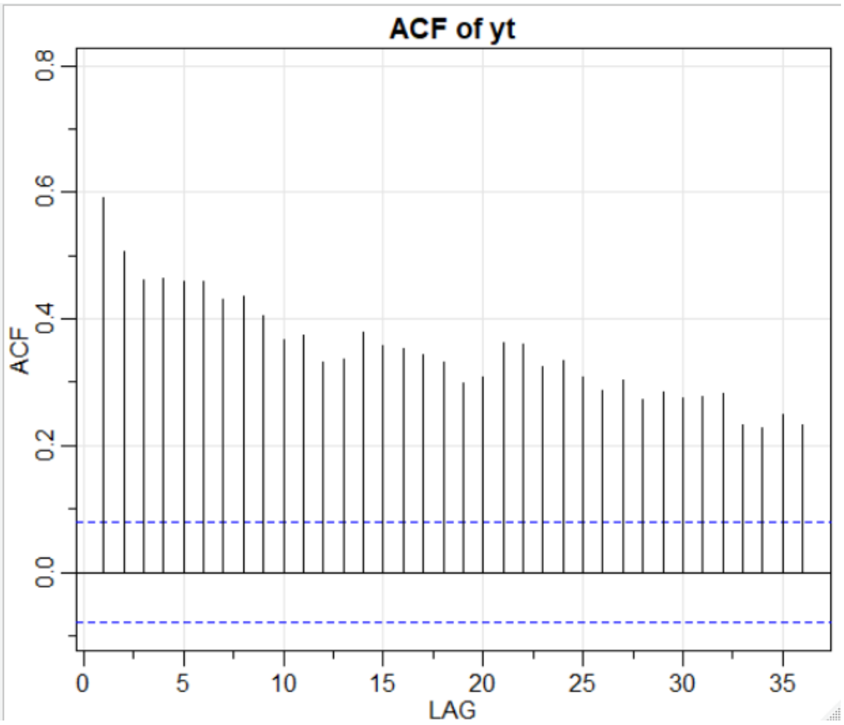
The red line the graphs is the normal curve. As it can be seen from the above two figures, yt is closer to normality than xt.

b)



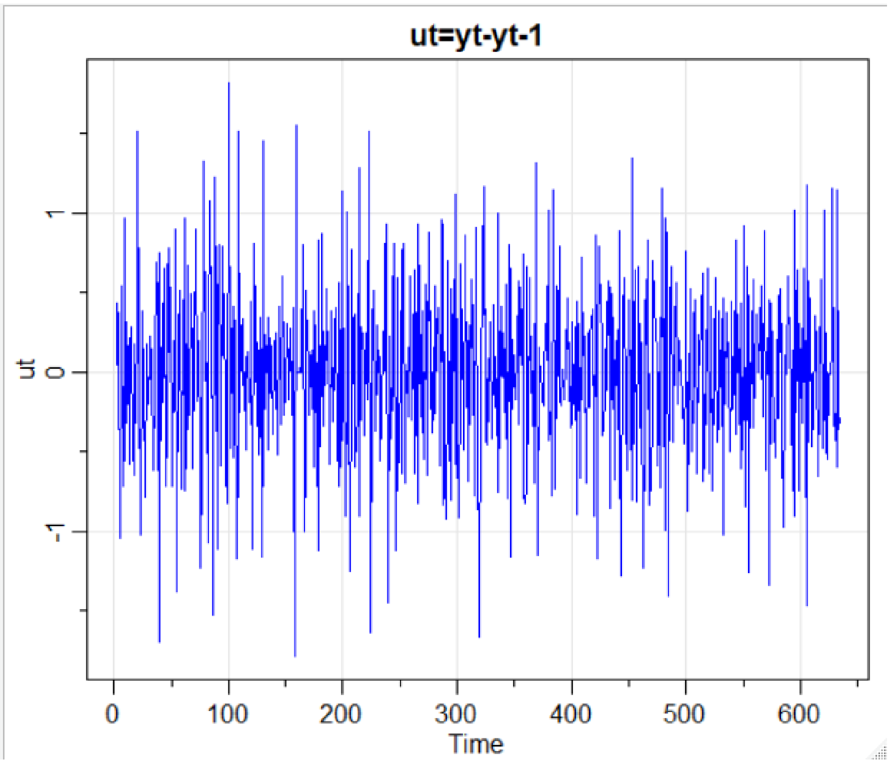
The period between 200 and 300 years of yt time series is comparable to land surface global temperature records.

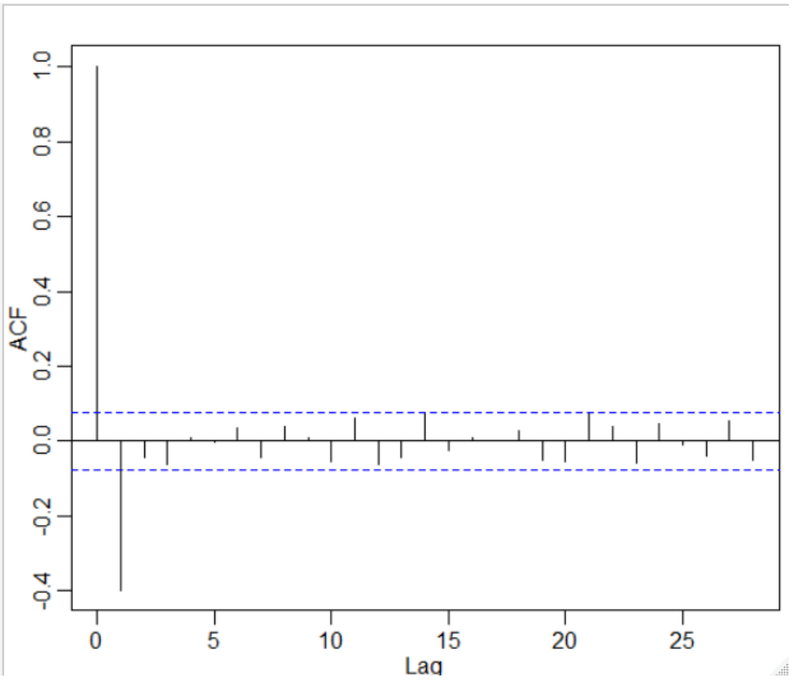
c)



Auto correlation is decreasing slowly to zero, indicating yt is not stationary.

d)

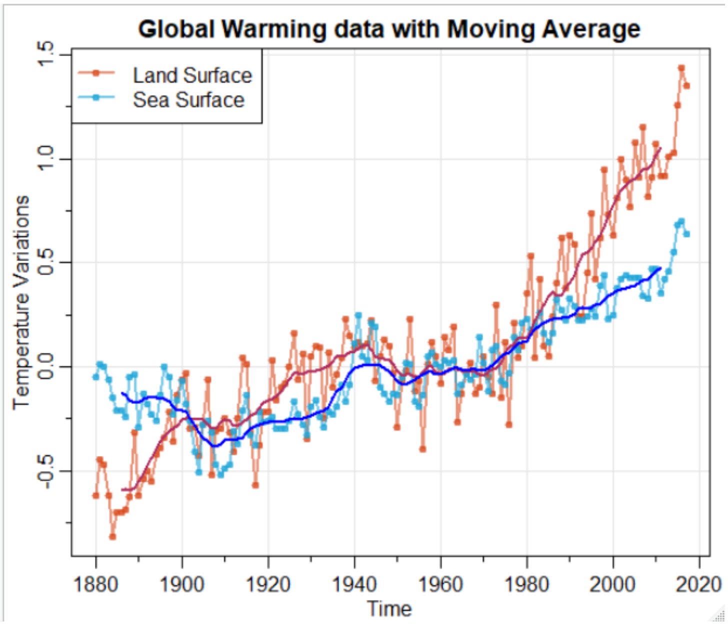


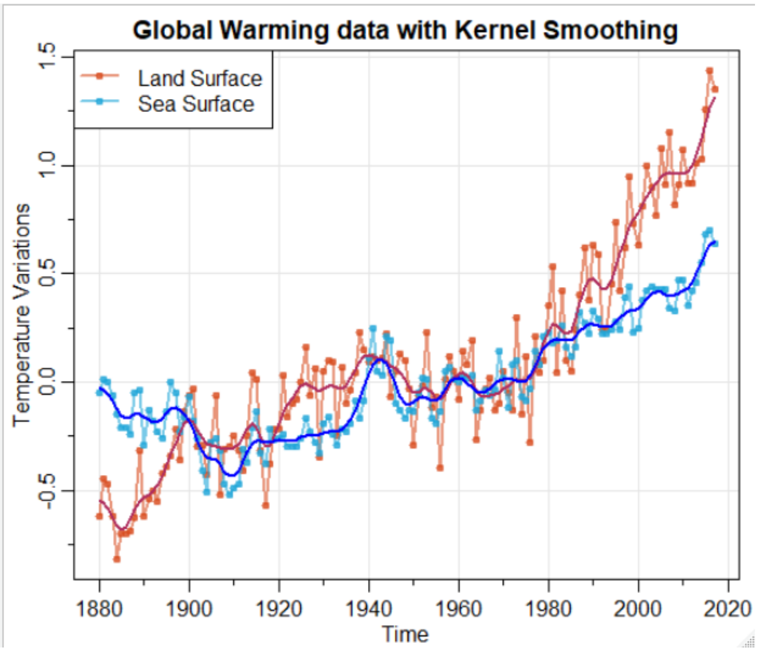


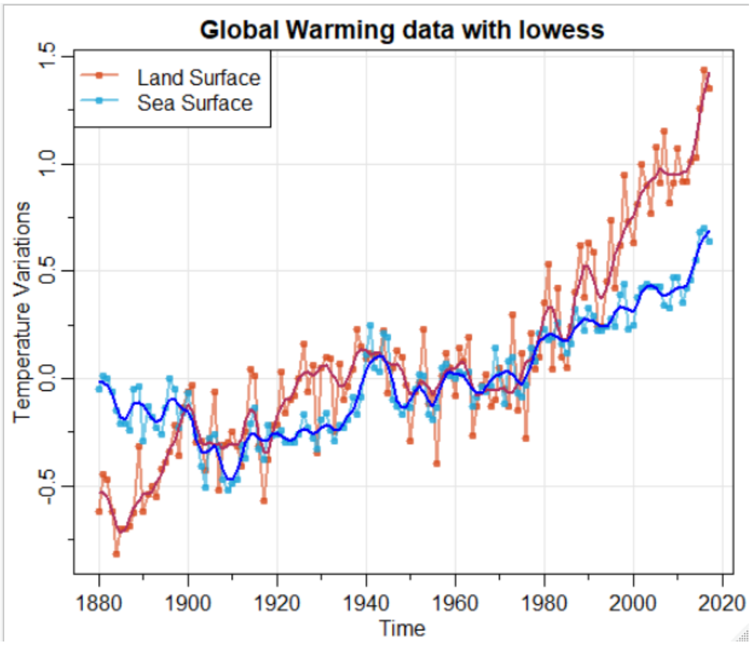
From the time series plot, we notice that the mean is constant, and the variance is fixed across time. From the above two plots, we can conclude that differencing the log varve data makes the series stationary.

Practical interpretation for ut: Calculating the growth of a stock per day which is done by taking the difference between the stock price’s opening and closing values in a day.

Problem 3.7


Global Warming data with Moving Average
Land Surface
Sea Surface
Temperature Variations
-0.5
0.0
0.5
1.0
1.5
1880
1900
1920
1940
1960
1980
2000
2020
Time


Global Warming data with Kernel Smoothing
Land Surface
Sea Surface
Temperature Variations
-0.5
0.0
0.5
1.0
1.5
1880
1900
1920
1940
1960
1980
2000
2020
Time



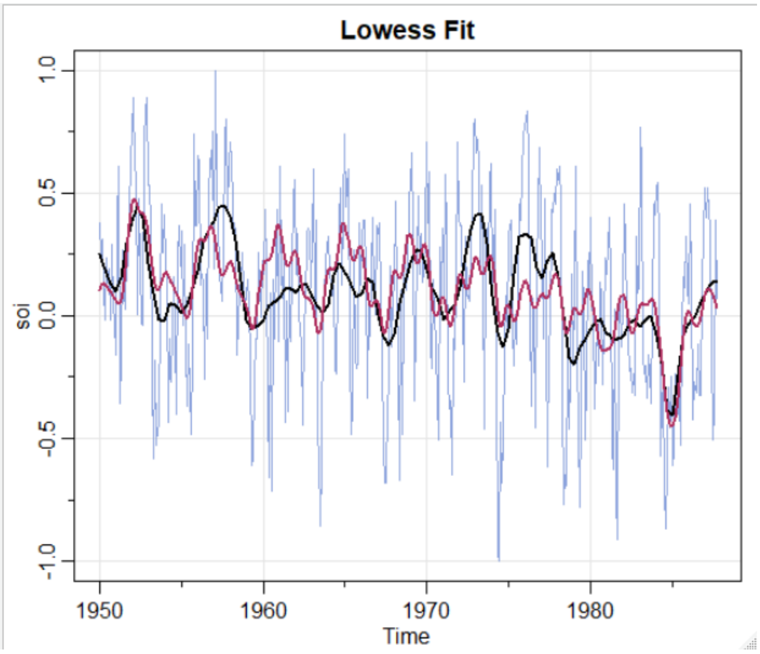
From the above plots, Moving Average is most smooth curve, whereas Lowess fit is the least smooth curve

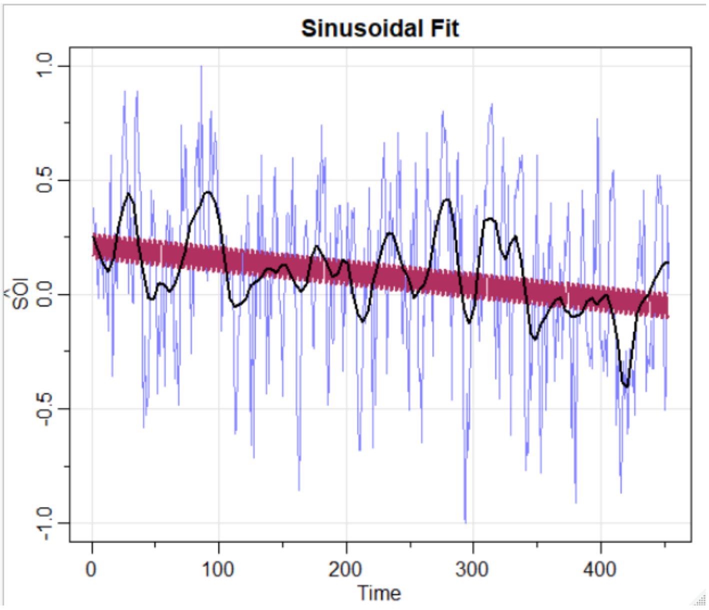
Moving average doesn’t fluctuate on short term fluctuations as it depends on average of past data points. So, Moving average fit has low sensitivity and end points are not used for the fit. But it tracks the long-term trend.

Kernel smoothing uses normally distributed weights. So, we consider the end points. Kernel curve follows the fluctuations better than Moving average curve.

Lowess fit uses K-nearest neighbors. It follows the fluctuations better than Kernel smoothing curve.

Problem 3.8





From the above plots,

Both the fits track the downward trend in the data.

Lowess fit tracks trend and cyclic behavior of the data. Its able to capture the fluctuations better than that by the Sinusoidal fit. Amplitude of lowess fit varies with the fluctuation in data.

Sinusoidal fit has constant amplitude throughout the time range. And has a cycle of 4 years.

II.] i. 2 Problem 4.2

$$x_t = \phi x_{t-1} + w_t \qquad t = 1, 2, \ldots$$
$$|\phi| < 1$$

a] At $t = 1$

$x_1 = \phi x_0 + w_1$

Since $x_0 = w_0$

$x_1 = \phi w_0 + w_1$

At $t = 2$

$x_2 = \phi x_1 + w_2$

$= \phi(\phi w_0 + w_1) + w_2$

$= \phi^2 w_0 + \phi w_1 + w_2$

At $t = 3$

$x_3 = \phi x_2 + w_3$

$= \phi(\phi^2 w_0 + \phi w_1 + w_2) + w_3$

$= \phi^3 w_0 + \phi^2 w_1 + \phi w_2 + w_3$

Similarly,

At $t = 4$,

$\vdots$

$$x_t = \sum_{j=0}^{t} \phi^j w_{t-j}, \quad t = 0, 1, \ldots$$

Hence Proved.

b.] $$E(x_t) = E\left[\sum_{j=0}^{t} \phi^j w_{t-j}\right]$$

$$= \sum_{j=0}^{t} \phi^j E(w_{t-j})$$

$$= 0$$

c.] $$Var(x_t) = Var\left[\sum_{j=0}^{t} \phi^j w_{t-j}\right]$$

$$= \sum_{j=0}^{t} \phi^{2j} Var(w_{t-j})$$

$$= \sigma^2_w \sum_{j=0}^{t} \phi^{2j}$$

Since $$\sum_{i=0}^{K} a^i = \frac{1 - a^{K+1}}{1 - a}$$

$$\therefore Var(x_t) = \sigma^2_w \left[\frac{1 - \phi^{2(t+1)}}{1 - \phi^2}\right]$$

Hence Proved.

d] $cov(x_{t+h}, x_t) = cov\left(\sum_{j=0}^{\infty} \phi^j w_{t+h-j}, \sum_{j=0}^{\infty} \phi^k w_{t-k}\right)$

$= cov[w_{t+h} + \phi w_{t+h-1} + \phi^h w_t + \phi^{h+1} w_{t-1} + \cdots, \phi^0 w_t + \phi w_{t-1} + \phi^2 w_{t-2} + \cdots]$

$= \sigma^2_w \sum_{j=0} \phi^{h+j} \phi^j$

$= \sigma^2_w \phi^h \sum_{j=0}^{\infty} \phi^{2j}$

$= \dfrac{\sigma^2_w \phi^h}{1-\phi^2}$ — (1)

As $Var(x_t) = \phi^2 Var(x_{t-1}) + Var(w_t)$
$Var(x_t) = Var(x_{t-1})$

$\therefore Var(x_t) = \dfrac{\sigma^2_w}{1-\phi^2}$ — (2)

From (1) & (2)
$cov(x_{t+h}, x_t) = Var(x_t) \cdot \phi^h$

e.] From b & d,
Mean = 0 (constant) and
covariance is function of h.

$\therefore$ x_t is stationary.

f.] From c,
$$Var(x_t) = \frac{\sigma^2_w}{1-\phi^2}\left[1-\phi^{2(t+1)}\right]$$
As $t \to \infty$ $\phi^{2(t+1)} \longrightarrow 0$ as $|\phi|<1$

$$\therefore Var = \frac{\sigma^2_w}{1-\phi^2}$$
$\hookrightarrow$ Fixed independent of time

g.] From f.] we know that as $t \to \infty$, the series becomes stationary. Thus, one can generate more than [illegible] observations and discard the beginning [[illegible] [avoiding startup problem] and consider the series as stationary.

h.] $$x_0 = \frac{w_0}{\sqrt{1-\phi^2}}$$

$$x_t = \phi^t x_0 + \sum_{j=0}^{t-1} \phi^j w_{t-j}$$

$$\therefore x_t = \frac{\phi^t w_0}{\sqrt{1-\phi^2}} + \sum_{j=0}^{t-1} \phi^j w_{t-j}$$

$$\therefore \text{var}(x_t) = \text{var}\left(\frac{\phi^t w_0}{\sqrt{1-\phi^2}}\right) + \text{var}\left(\sum_{j=0}^{t-1} \phi^j w_{t-j}\right)$$

$$= \frac{\sigma^2_w}{1-\phi^2}\phi^{2t} + \sigma^2_w \sum_{j=0}^{t-1} \phi^{2j}$$

$$= \frac{\sigma^2_w}{1-\phi^2}\phi^{2t} + \sigma^2_w \left(1 + \phi^2 + \cdots \phi^{2(t-1)}\right)$$

$$= \frac{\sigma^2_w \phi^{2t}}{1-\phi^2} + \sigma^2_w \left[\frac{1-\phi^{2t}}{1-\phi^2}\right]$$

$$= \sigma^2_w \left[\frac{\phi^{2t} + 1 - \phi^{2t}}{1-\phi^2}\right]$$

$$\text{var}(x_t) = \frac{\sigma^2_w}{1-\phi^2}$$

PTO

$\therefore$ variance is independent of time t.
4 process is stationary.

III.]

$$x_t = \sum_{j=0}^{\infty} \phi^j w_{t-j} - \frac{\phi}{1-\phi^2} w_{t+1}, \quad |\phi| < 1,$$

$$E[x_t] = E\Big[\phi^0 w_t + \phi^1 w_{t-1} + \phi^2 w_{t-2} + \cdots - \frac{\phi}{1-\phi^2} w_{t+1} \Big]$$

$$= E(w_t) + \phi\, E(w_{t-1}) + \cdots - \frac{\phi}{1-\phi^2} E(w_{t+1})$$

$$= 0 \quad [\because E(w_t) = 0]$$

[Assuming $\{w_t\}$ is $WN(0, \sigma_w^2)$]

$$\gamma(h) = \mathrm{cov}(x_{t+h}, x_t)$$

$h = 0$

$$\mathrm{Var}(x_t) = \mathrm{Var}\Big[\phi w_t + \phi w_{t-1} + \phi^2 w_{t-2} + \cdots - \frac{\phi}{1-\phi^2} w_{t+1} \Big]$$

$$= \sigma_w^2 + \phi^2 \sigma_w^2 + \phi^4 \sigma_w^2 + \cdots + \frac{\phi^2}{(1-\phi^2)^2} \sigma_w^2$$

$$= \sigma_w^2 \Big[1 + \phi^2 + \phi^4 + \cdots + \frac{\phi^2}{(1-\phi^2)^2} \Big]$$

$$= \frac{\sigma^2_w}{1-\phi^2} + \frac{\phi^2 \cdot \sigma^2_w}{(1-\phi^2)^2}$$

$$= \frac{(1-\phi^2)\sigma^2_w + \phi^2 \sigma^2_w}{(1-\phi^2)^2}$$

$$= \frac{1}{1-\phi^2} \sigma^2_w$$

$h = 1$

$$\gamma(1) = cov(x_{t+1}, x_t)$$

$$= E\left[\left(w_{t+1} + \phi w_t + \phi^2 w_{t-1} + \cdots - \frac{\phi}{1-\phi^2} w_{t+2}\right)\left(w_t + \phi w_{t-1} + \phi^2 w_{t-2} + \cdots - \frac{\phi}{1-\phi^2} w_{t+1}\right)\right]$$

$$= \frac{-\phi}{1-\phi^2} \sigma^2_w + \left(\phi \sigma^2_w + \phi^3 \sigma^2_w + \phi^5 \sigma^2_w + \cdots\right)$$

$$= \frac{-\phi}{1-\phi^2} \sigma^2_w + (\phi + \phi^3 + \phi^5 + \cdots) \sigma^2_w$$

$$= \frac{-\phi}{1-\phi^2} \sigma^2_w + \frac{\phi}{1-\phi^2} \sigma^2_w$$

$$= 0$$

Similarly for $h = 2, 3, \ldots$
$\gamma(h) = 0$

$$\therefore \gamma(h) = \begin{cases} \dfrac{\sigma_w^2}{(1-\phi^2)^2}, & h = 0 \\ 0, & h \neq 0 \end{cases}$$

$\therefore$ $\gamma(h)$ does not depend on t, it only depends on lag h.

$\therefore$ x_t is stationary.

From the equation given, x_t depends on w_{t+1} i.e. future value of white noise.

$\therefore$ x_t is not causal.

IV.] Standard AR(p) model with coefficients $\phi_1, \phi_2 \cdots \phi_p$ is:

$$x_t = \sum_{j=1}^{p} \phi_j x_{t-j} + w_t$$

$$= \phi_1 x_{t-1} + \phi_2 x_{t-2} + \phi_3 x_{t-3} + \cdots \phi_p x_{t-p} + w_t$$

$$= [\phi_1 + \phi_2 + \cdots + \phi_p] x_{t-1} - [\phi_2 + \phi_3 + \cdots \phi_p] x_{t-1} +$$
$$[\phi_2 + \phi_3 + \cdots \phi_p] x_{t-2} - [\phi_3 + \phi_4 + \cdots \phi_p] x_{t-2}$$
$$\cdots$$
$$\cdots$$
$$+ [\phi_{p-1} \ \phi_p] x_{t-(p-1)} - \phi_p x_{t-(p-1)}$$
$$+ \phi_p x_{t-p} + w_t$$

$$= \left[\sum_{j=1}^{p} \phi_j\right] x_{t-1} - \left[\sum_{j=2}^{p} \phi_j\right] (x_{t-1} - x_{t-2})$$
$$- \left[\sum_{j=3}^{p} \phi_j\right] (x_{t-2} - x_{t-3})$$
$$- \cdots$$
$$\cdots$$
$$- \left[\sum_{j=p}^{p} \phi_j\right] [x_{t-(p-1)} - x_{t-p}] + w_t$$

Subtracting x_{t-1} from both sides

$$x_t - x_{t-1} = \left[\sum_{j=1}^{p} \phi_j - 1\right] x_{t-1} + \sum_{j=1}^{p-1} \psi_j \nabla x_{t-j} + w_t$$

$$\nabla x_t = \gamma x_{t-1} + \sum_{j=1}^{p-1} \psi_j \nabla x_{t-j} + w_t$$

where,

$$\gamma = \sum_{j=1}^{p} \phi_j - 1$$

$$\psi_j = - \sum_{i=j}^{p} \phi_i \quad , \quad j = 2, \ldots, p.$$