STAT 626
Homework 6

Name-Mayank Jaggi

UIN-526005299

Stat 626

Homework 6

Problem 4.1

MA(1) $\Rightarrow$ $x_t = \theta w_{t-1} + w_t$

$E(x_t) = 0$

$\gamma(1) = cov(x_{t+1}, x_t)$

$= cov((w_{t+1} + \theta w_t), (w_t + \theta w_{t-1}))$

$= E[w_t w_{t+1} + \theta w_{t+1} w_{t-1} + \theta w_t^2 + \theta^2 w_t w_{t-1}]$

$= E[\theta w_t^2]$

$\gamma(1) = \theta \sigma^2_w$

$\gamma(0) = var(w_t + \theta w_{t-1}) = \sigma^2 w + \theta^2 \sigma^2_w$

$\rho(1) = \dfrac{\gamma(1)}{\gamma(0)} = \dfrac{\theta}{1+\theta^2}$

Taking derivative on Both Sides w.r.t. is θ

$$\frac{d}{d\theta} \rho = \frac{d}{d\theta}\left(\frac{\theta}{1+\theta^2}\right) = 0$$

$$\Rightarrow (1+\theta^2)^{-1} + \frac{(-1)\,\theta \cdot 2\theta}{(1+\theta^2)^2} = 0$$

$$\Rightarrow \frac{1+\theta^2 - 2\theta^2}{(1+\theta^2)^2} = 0$$

$$\Rightarrow 1-\theta^2 = 0$$

$$\Rightarrow \theta = \pm 1$$

When $\theta = 1$, $\rho(1) = \frac{1}{1+1} = \frac{1}{2}$

(Max value)

when $\theta = -1$, $\rho(1) = \frac{-1}{1+1} = \frac{-1}{2}$

(Min value)

$\therefore |\rho(1)| \leq 1/2$ [From max and min values]

4.] Problem 4.3

i, $x_t = 0.8x_{t-1} - 0.15x_{t-2} + w_t - 0.3w_{t-1}$

$\Rightarrow x_t = 0.8Bx_t - 0.15B^2x_t + w_t - 0.3Bw_t$

$\Rightarrow (1 - 0.8B - 0.15B^2)x_t = w_t(1-0.3B)$

$\Rightarrow (1 - 0.3B - 0.5B - 0.15B^2)x_t = w_t(1-0.3B)$

$\Rightarrow (1-0.3B)(1-0.5B)x_t = w_t(1-0.3B)$

$\Rightarrow (1-0.5B)x_t = w_t$

$(1-0.3B)$ is the common term. Thus parameter redundancy is present.

b.] Reduced form of Model

$(1-0.5B)x_t = w_t$

$x_t - 0.5x_{t-1} = w_t$

$x_t = 0.5x_{t-1} + w_t \Rightarrow AR(1)$

$\therefore$ Reduced form is $AR(1)$

$(1-0.5B)x_t = w_t$

$\phi(B) = 1-0.5B$

Roots are $z = \frac{1}{|\phi|} = \frac{1}{(0.5)} = 2$

$z = 2 > 1$, so outside the unit circle

$\therefore$ Reduced form of model given is <u>Causal</u>.

ii) $x_t = x_{t-1} - 0.5x_{t-2} + w_t - w_{t-1}$

a.] $\Rightarrow (1-B-0.5B^2)x_t = w_t(1-B)$

Here no common term. Thus there is no parameter redundancy present.

b.] $\phi(B) = 1-B-0.5B^2$

Roots $= \frac{-(-1) \pm \sqrt{(-1)^2 - 4\times1\times0.5}}{2\times0.5}$

$= 1 \pm \sqrt{-1}$

$= 1 \pm i$

$|z| = \sqrt{2} = 1.414 > 1$

$\therefore$ Model is causal

$$\theta(B) = 1 - B$$

$$|z| = \left|\frac{1}{\theta}\right| = 1$$

Not greater than 1, so Model is not invertible.

Problem 4.3

c)

```
> round(ARMAtoMA(ar=0.5,ma=0,50),3)
 [1] 0.500 0.250 0.125 0.062 0.031 0.016 0.008 0.004 0.002 0.001 0.000 0.0
00 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000
[23] 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.0
00 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000 0.000
[45] 0.000 0.000 0.000 0.000 0.000 0.000

> round(ARMAtoAR(ar=0.5,ma=0,50),3)
 [1] -0.5  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
[27]  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
```

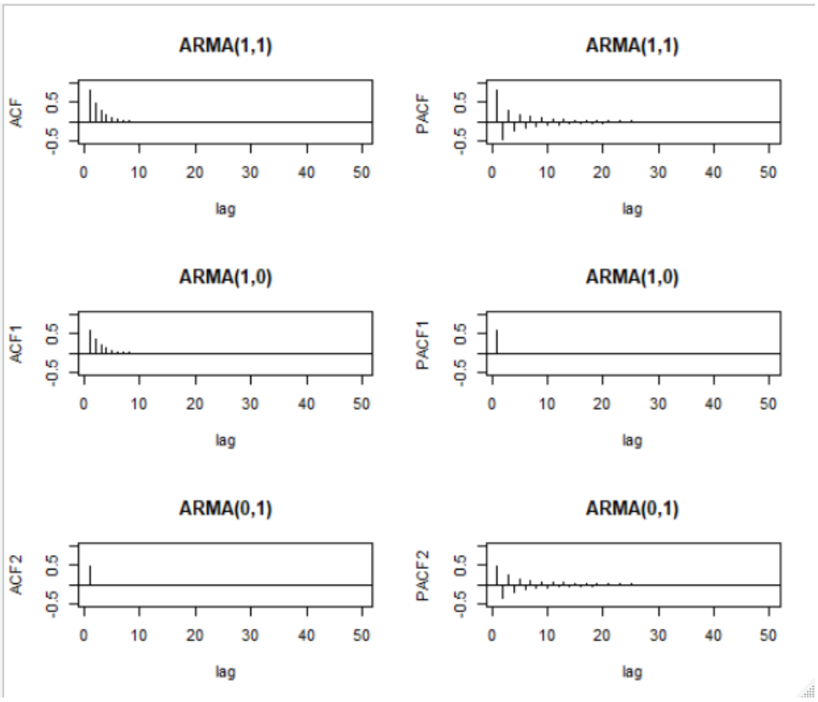
In Model 1, after cancelling redundant terms, we have AR(1) model. Coefficients for MA representation falls to zero at a fast rate since the model has causality. AR(1) model is in the form of xt. Therefore, AR coefficients are all zero except for the one shown in the equation.

```
> round(ARMAtoMA(ar=c(1,-0.5),ma=-1,50),3)
 [1]  0.000 -0.500 -0.500 -0.250  0.000  0.125  0.125  0.062  0.000 -0.031
-0.031 -0.016  0.000  0.008  0.008  0.004  0.000 -0.002 -0.002
[20] -0.001  0.000  0.000  0.000  0.000  0.000  0.000  0.000  0.000  0.000
0.000  0.000  0.000  0.000  0.000  0.000  0.000  0.000  0.000
[39]  0.000  0.000  0.000  0.000  0.000  0.000  0.000  0.000  0.000  0.000
0.000  0.000

> round(ARMAtoAR(ar=c(1,-0.5),ma=-1,50),3)
 [1] 0.0 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0
.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5
[34] 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5
```

In Model 2, coefficients of MA fall off to zero at a high rate since the model has causality. Model is noninvertible; therefore, AR coefficients doesn’t fall off to zero.

Problem 4.4

a)

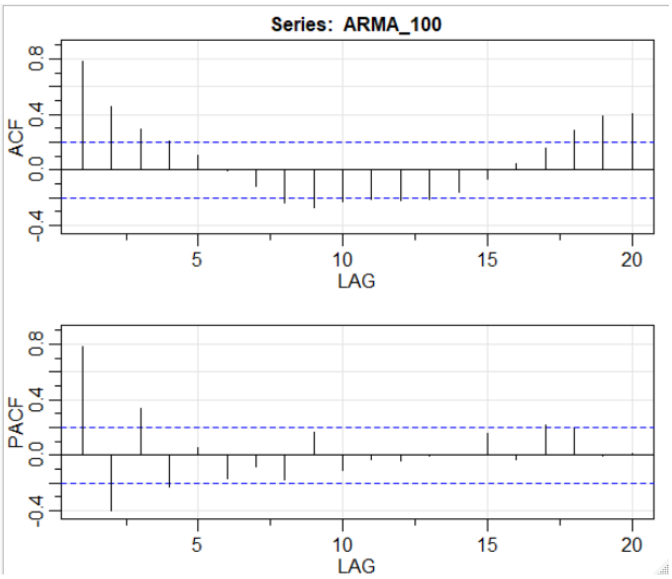


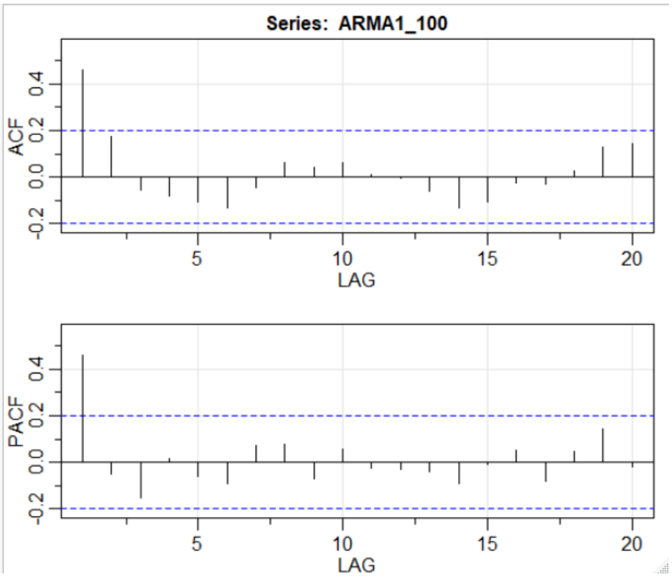
For Row 1 plots, we cannot find the order of models from ACF and PACF.

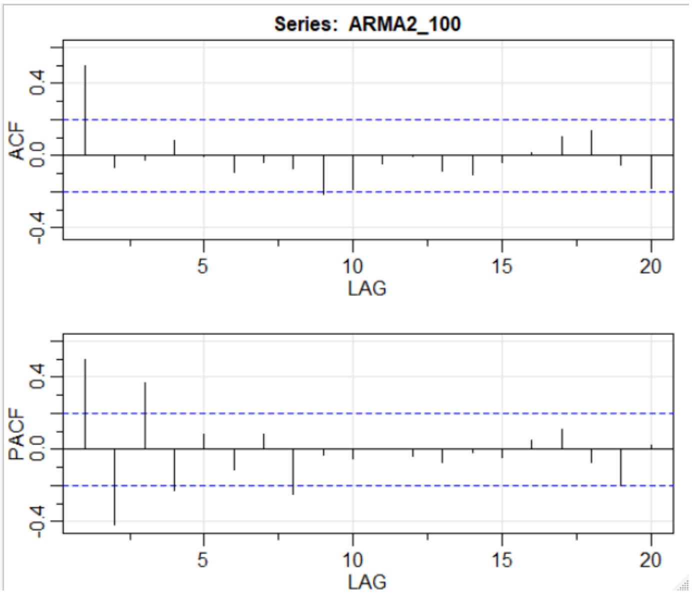
For Row 2 plots, PACF cuts off to zero after lag 1 and ACF doesn't tell any order of model. Thus, the model is AR(1)

For Row 3 plots , ACF cuts off at lag 1 to zero and PACF fails to tell any order of model. Thus, from model is MA(1).

b)




Series: ARMA1_100
ACF
0.4
0.2
0.0
-0.2
5
10
15
20
LAG
PACF
0.4
0.2
0.0
-0.2
5
10
15
20
LAG


Series: ARMA2_100
ACF
0.4
0.0
-0.4
5
10
15
20
LAG
PACF
0.4
0.0
-0.4
5
10
15
20
LAG

Theoretical ACF and PACF:

Problem 4.4

e.] ACF of ARMA (1,1) for $n=100$

$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{(\phi+\theta)(1+\phi\theta)}{1+2\phi\theta+\theta^2}$$

$$= \frac{(0.6+0.9)(1+0.6\times0.9)}{1+2\times0.6\times0.9+0.9^2}$$

$$= 0.8$$

ACF of AR(1)

$$\rho(1) = \phi = 0.6$$

ACF of MA (1)

$$\rho(1) = \frac{\theta}{1+\theta^2} = 0.497$$

PACF of AR(1) $= \rho(1) = \phi = 0.6$

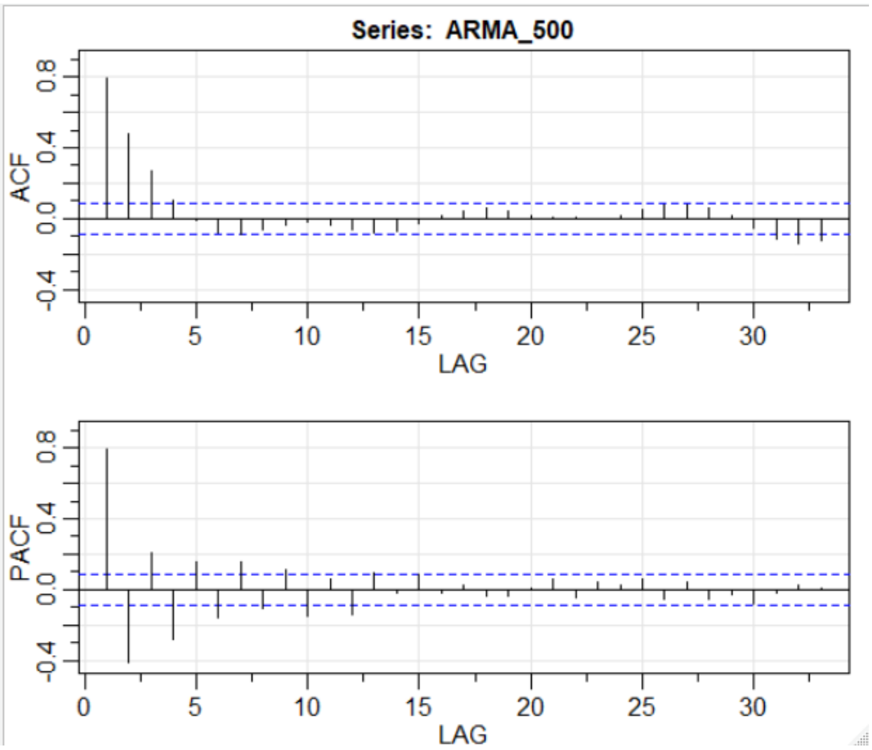
PACF of MA(q) $= \phi_{kk} = \dfrac{\theta^k(1-\theta^2)}{1-\theta^{2(k+1)}}, \; k \geq 1$

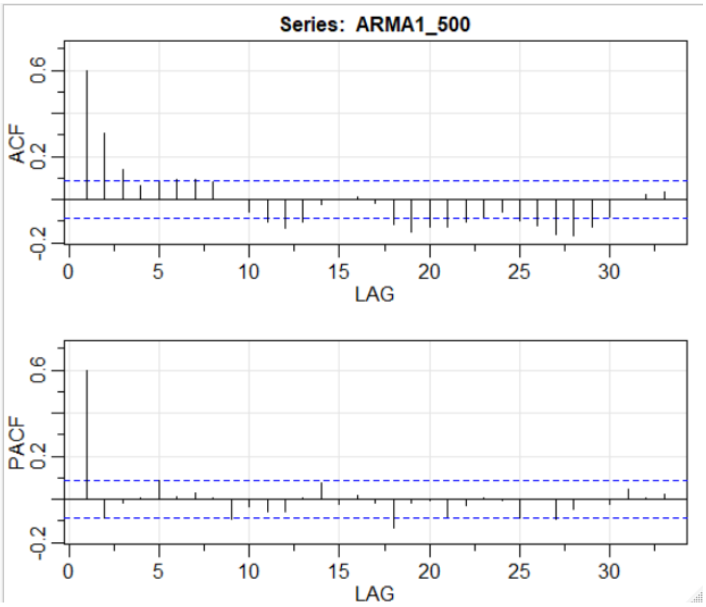
PACF of MA(1) $= \phi_{11} = 0.497$

Computed ACF and PACF:

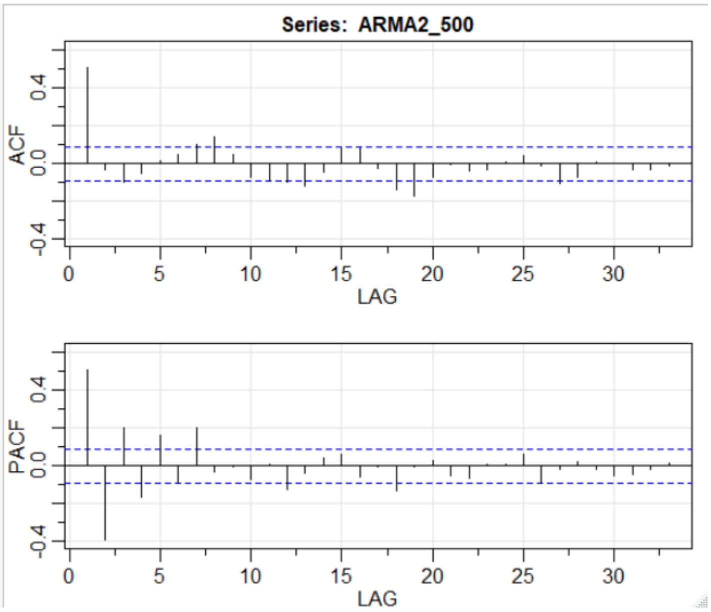
```
> acf1(ARMA_100,plot=FALSE)[1]
[1] 0.78
> acf1(ARMA1_100,plot=FALSE)[1]
[1] 0.46
> acf1(ARMA2_100,plot=FALSE)[1]
[1] 0.5
> pacf(ARMA_100,plot=FALSE)[1]

Partial autocorrelations of series 'ARMA_100', by lag

    1
0.783
> pacf(ARMA1_100,plot=FALSE)[1]

Partial autocorrelations of series 'ARMA1_100', by lag

   1
0.46
> pacf(ARMA2_100,plot=FALSE)[1]

Partial autocorrelations of series 'ARMA2_100', by lag

    1
0.498
```

c)





Series: ARMA1_500
ACF
LAG
PACF



Series: ARMA2_500
ACF
LAG
PACF

Theoretical ACF and PACF

c] ACF of ARMA (1,1) = $\rho(1)$ = 0.8

ACF of AR(1) = $\rho(1)$ = 0.6

ACF of MA (1) = $\rho(1) = \frac{\theta}{1+\theta^2}$ = 0.497

PACF of AR(1) = $\rho(1) = \phi_{11}$ = 0.6

PACF of MA(1) = $\rho(1) = \phi_{11}$ = 0.497

As we increase this sample observation, ACF & PACF values are close to theoretical values.

Behaviour of ACF & PACF for ARMA (1,1)
↳ ACF → tails off
↳ PACF → tails off

AR(1)
↳ ACF → tails off
↳ PACF → cuts off at lag 1

MA(1)
↳ ACF → cuts off at lag 1
↳ PACF → tails off

Computed ACF and PACF:

```
acf1(ARMA_500,plot=FALSE)[1]
[1] 0.79
> acf1(ARMA1_500,plot=FALSE)[1]
[1] 0.6
> acf1(ARMA2_500,plot=FALSE)[1]
[1] 0.51
>
> pacf(ARMA_500,plot=FALSE)[1]

Partial autocorrelations of series ‘ARMA_500’, by lag

    1
0.794
> pacf(ARMA1_500,plot=FALSE)[1]

Partial autocorrelations of series ‘ARMA1_500’, by lag

    1
0.599
> pacf(ARMA2_500,plot=FALSE)[1]

Partial autocorrelations of series ‘ARMA2_500’, by lag

    1
0.507
```

9.] Problem 4.10

ARMA (p, q) models are stationary when
(p, q) $|\phi| < 1$

Assumption : $|\phi| < 1$

Consider ARMA(1, 1) : $p = 1, q = 1$

$x_t = \phi x_{t-1} + w_t + \theta w_{t-1}$

$x_{n-1} = \phi x_{n-2} + w_{n-1} + \theta w_{n-2}$ ---- (1)

$x_n = \phi x_{n-1} + w_n + \theta w_{n-1}$ ---- (2)

$x_{n+1} = \phi x_n + w_{n+1} + \theta w_n$ ---- (3)

Estimating

$\hat{x}_{n+1} = \phi x_n + \underbrace{E(w_{n+1})}_{0} + \theta w_n$

All future w's have an Expected value of 0.

$\therefore \hat{x}_{n+1} = \phi x_n + \theta w_n$ ---- (4)

(P 70)

For $t = n+2$

$$x_{n+2} = \phi x_{n+1} + w_{n+2} + \theta w_{n+1} \quad \text{--- (5)}$$

Estimating,

$$\hat{x}_{n+2} = \phi \hat{x}_{n+1} + 0 + 0$$

$$= \phi [\phi x_n + \theta w_n] \quad [\text{From (4)}]$$

We can further back substitute x_n using equation (2), (1) and so on.

Since we have infinite history and our model is stationary, using Wold Decomposition Theorem, future values of x_t can be represented as linear ~~comp~~ combination of white noise.

So the white noise terms will be in the range

$$w_{-\infty} \cdots \cdots w_0, w_1, \cdots w_n.$$

All future w's ~~have~~ Expected value $= 0$.

$$\therefore \hat{x}_{n+m} = \sum_{j=m}^{\infty} \psi_j \, w_{m+n-j}$$

Which is true for any value of p & q.
Where ψ_j is a constant.

So,

$$\hat{x}_{n+m} = \sum_{j=m}^{\infty} \psi_j \, w_{m+n-j}$$

$$= \psi_m w_n + \psi_{m+1} w_{n+1-1} + \psi_{m+2} w_{n-2-2}$$

$$+ \cdots \psi_{m+n} w_0 + \psi_{m+n+1} w_{-1} \cdots\cdots$$

In case of x_{m+n} values, similarly using equation ①, ②, ③, ⑤ we ~~prove~~ show

$$x_{n+m} = \sum_{j=0}^{\infty} \psi_j \, w_{m+n-j}$$

Using the above results,

$$x_{n+m} - \hat{x}_{n+m} = \sum_{j=0}^{\infty} \psi_j \, w_{m+n-j} - \sum_{j=m}^{\infty} \psi_j \, w_{m+n-j}$$

$$\Rightarrow x_{n+m} - x^{n}_{n+m} = \sum_{j=0}^{m-1} \psi_j w_{m+n-j}$$

[Common terms get cancelled out.]

$$E[x_{n+m} - x^{n}_{n+m}]^2 = E\left[\sum_{j=0}^{m-1} \psi_j w_{m+n-j}\right]^2$$

$$= E[\psi_0 w_{m+n} + \psi_1 w_{m+n-1} + \psi_2 w_{m+n-2} + \cdots$$

$$\psi_{m-1} w_{n+1}]^2$$

$$= Var[\psi_0 w_{m+n} + \cdots \psi_{m-1} w_{n+1}]$$

$[\because E(w_n) = 0]$

$$= \psi_0^2 Var(w_{m+n}) + \cdots \psi^2_{m-1} Var(w_{n+1})$$

$$= \psi_0^2 \sigma^2_w + \cdots \psi^2_{m-1} \sigma^2_w$$

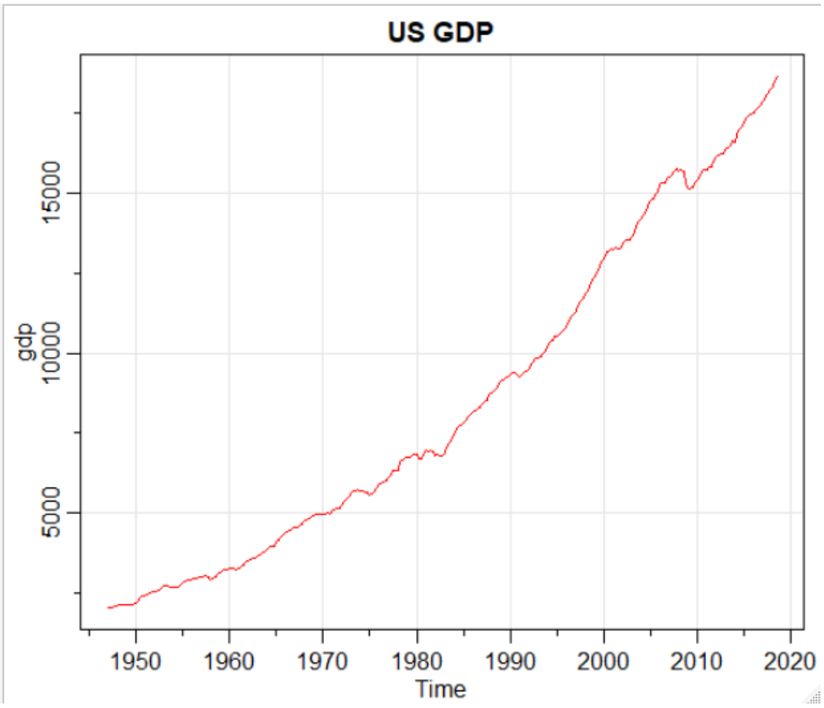
[white noise are uncorrelated]

$$= \sigma^2_w \sum_{j=0}^{m-1} \psi_j$$

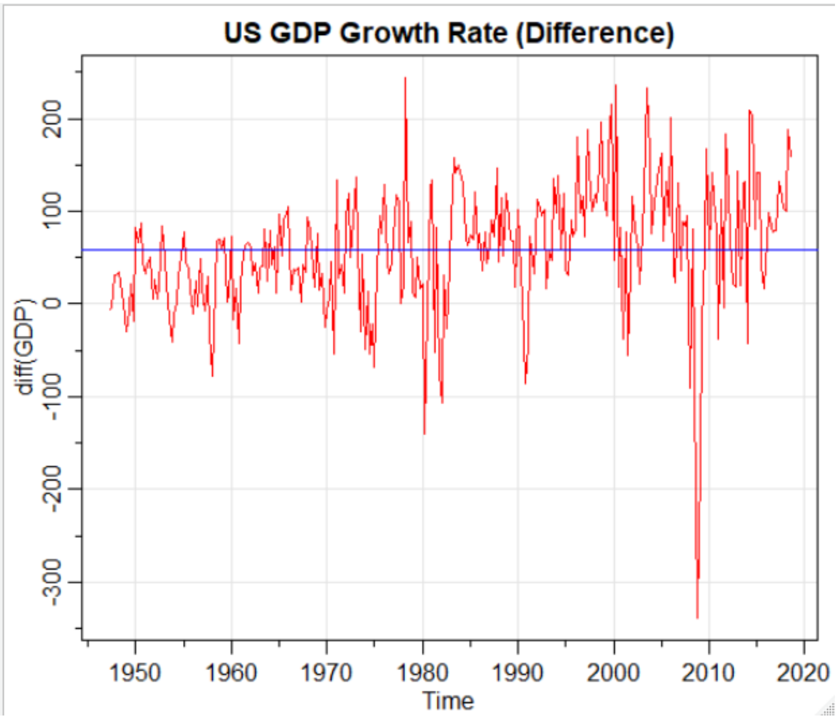
Hence Proved.

Problem 5.2

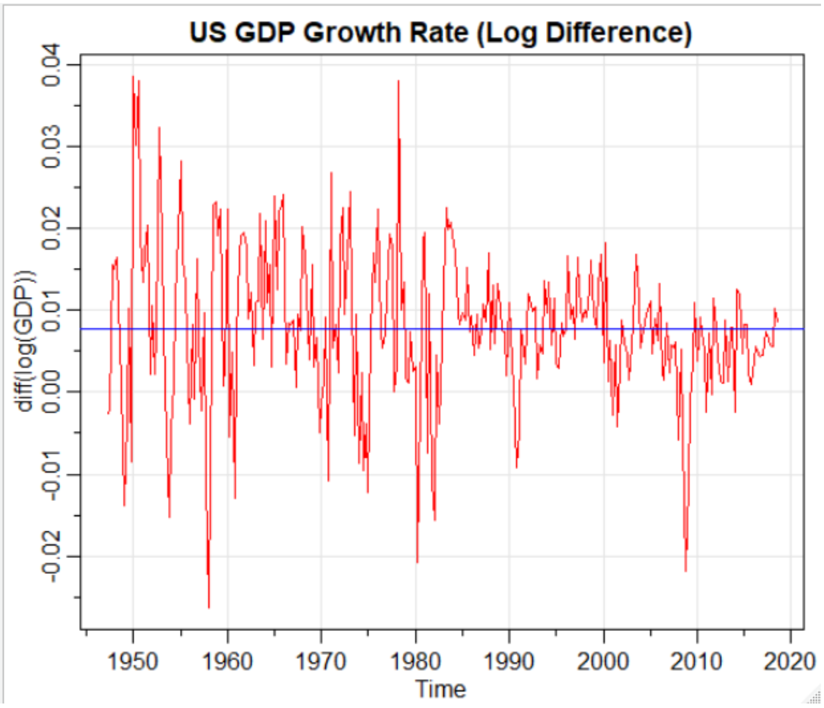
Plot the Data



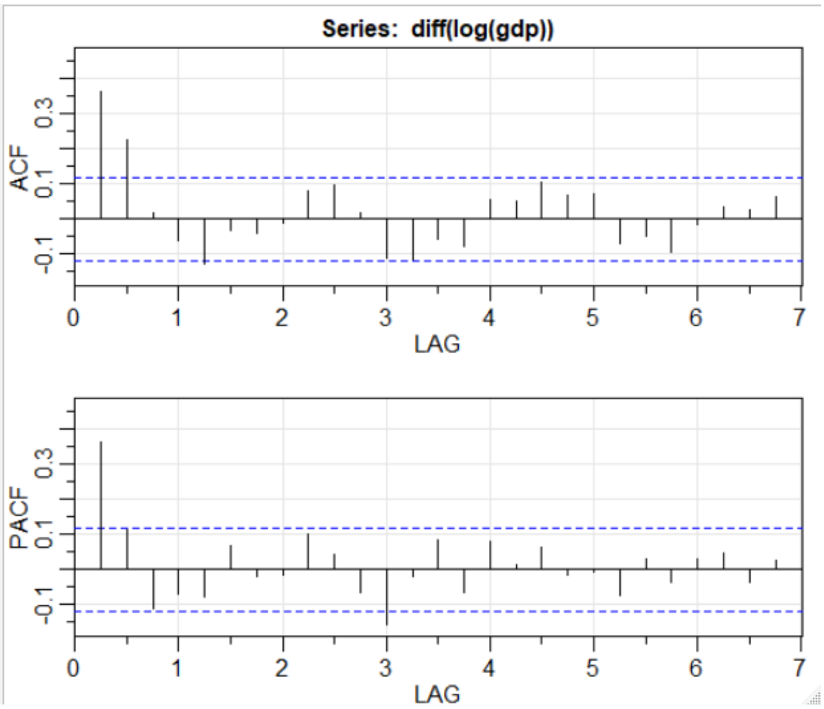
To understand the data, we use differencing it.



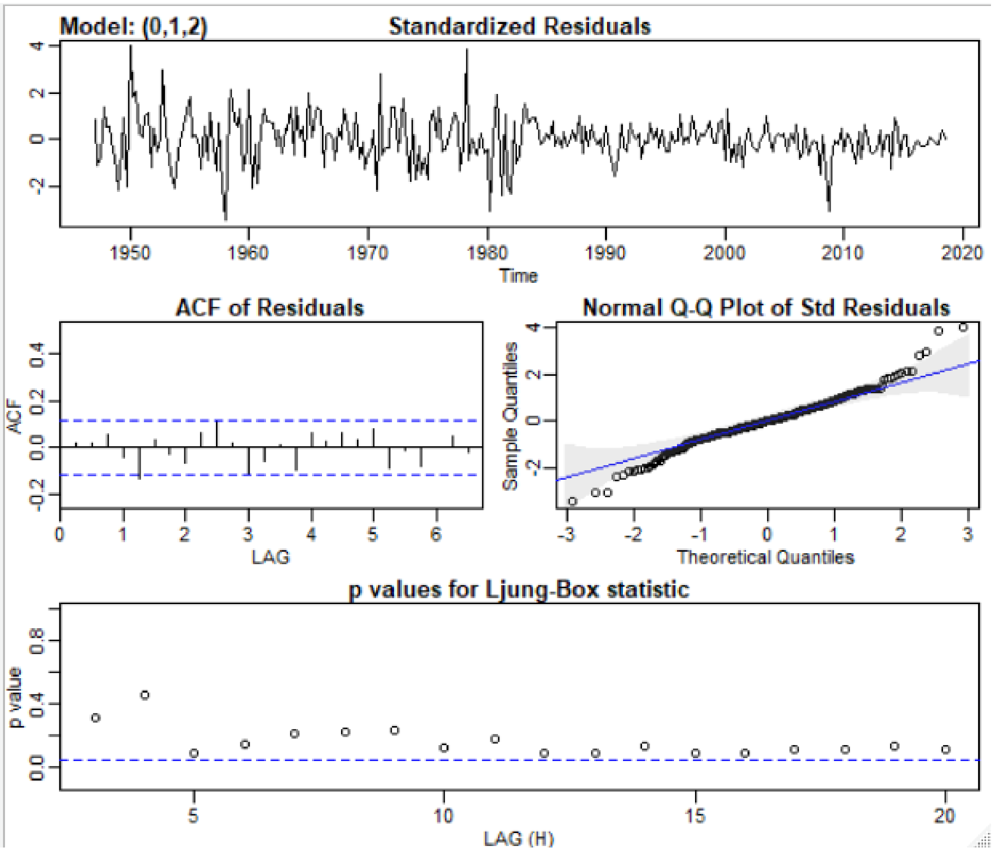
Here the mean is not constant, and the variance is increasing (heteroskedasticity). To remove heteroskedasticity we take log of the series and then difference to avoid any negative terms in log.

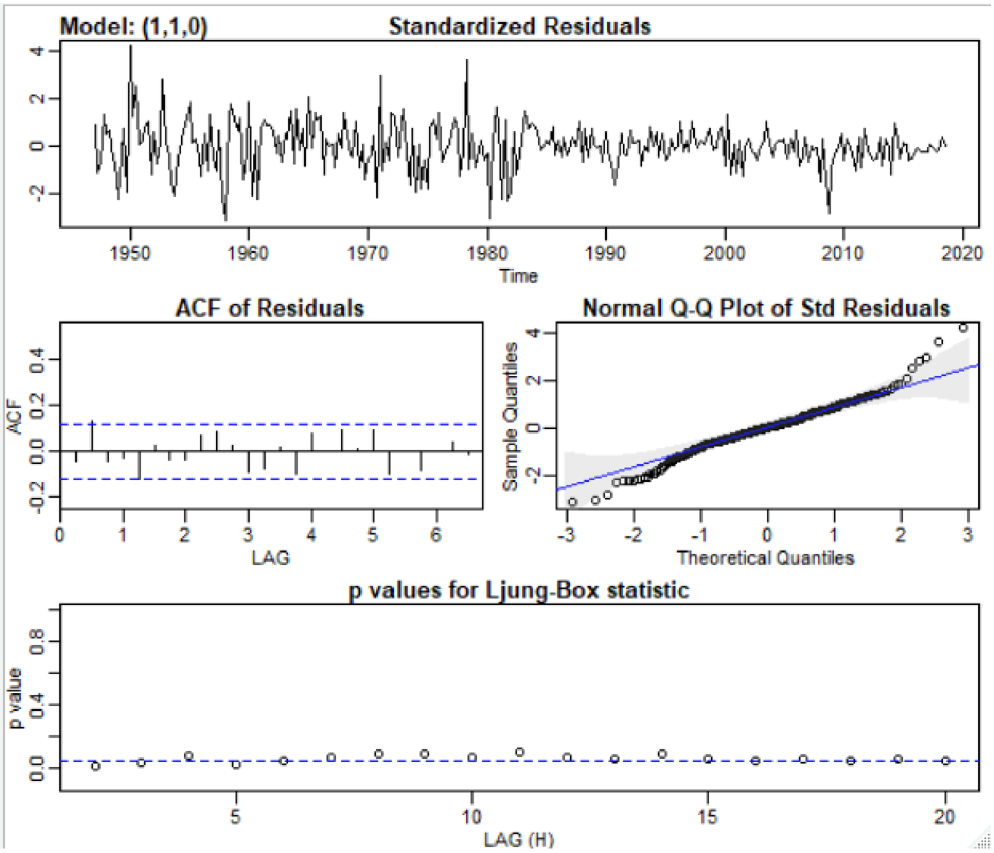


The log-differenced transformed series seems to have significant improvement and appears to have constant mean and variance.



Based on the above plots, it can be either ARIMA(0,1,2) or ARIMA(1,1,0)


Model: (0,1,2)
Standardized Residuals
Time
ACF of Residuals
ACF
LAG
Normal Q-Q Plot of Std Residuals
Sample Quantiles
Theoretical Quantiles
p values for Ljung-Box statistic
p value
LAG (H)


Model: (1,1,0)
Standardized Residuals
Time
ACF of Residuals
ACF
LAG
Normal Q-Q Plot of Std Residuals
Sample Quantiles
Theoretical Quantiles
p values for Ljung-Box statistic
p value
LAG (H)

The AIC and BIC for both the models are almost equal. However from the Q statistics we observer residuals are not above 0.05 in ARIMA(1,1,0).

So, we select ARIMA(0,1,2) model.

From ARIMA(0,1,2) residual plots, following is observed:

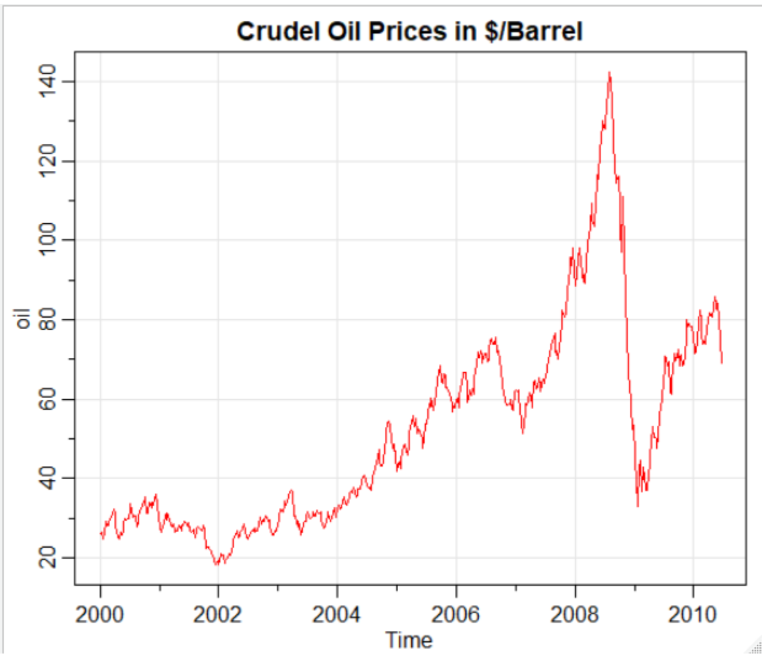
- From the Standardized Residuals plot, it looks like white noise with 0 mean and almost constant variance.
- Residuals are uncorrelated, so they can be considered as white noise.
- ACF of the residuals are below the significant level.
- Residuals are approximately normally distributed

Going for a higher model, ARIMA(0,1,3)

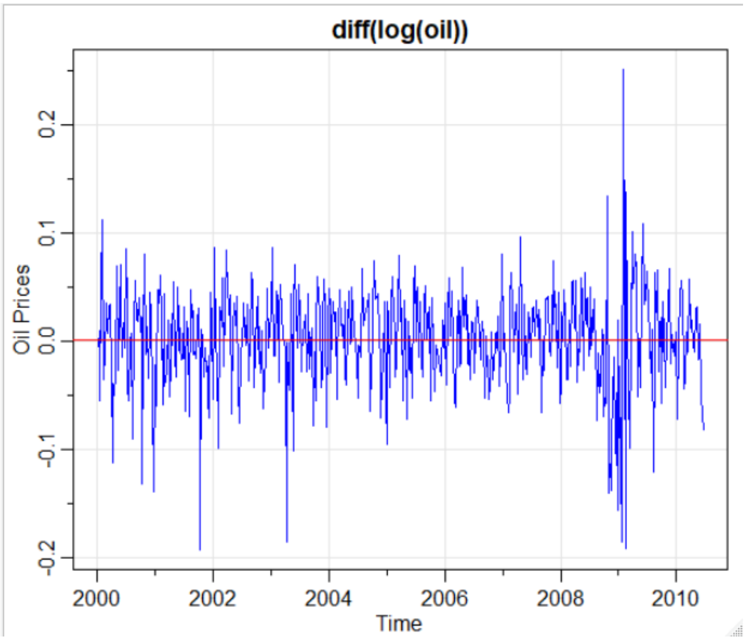
```
$ttable
      Estimate     SE t.value p.value
ma1     0.3235 0.0581  5.5718  0.0000
ma2     0.2756 0.0635  4.3440  0.0000
ma3     0.0925 0.0595  1.5555  0.1209
xmean   0.0077 0.0009  8.9754  0.0000
```

From the above table, its clear that ma3 value is not significant. Therefore, addition of parameters does not help.

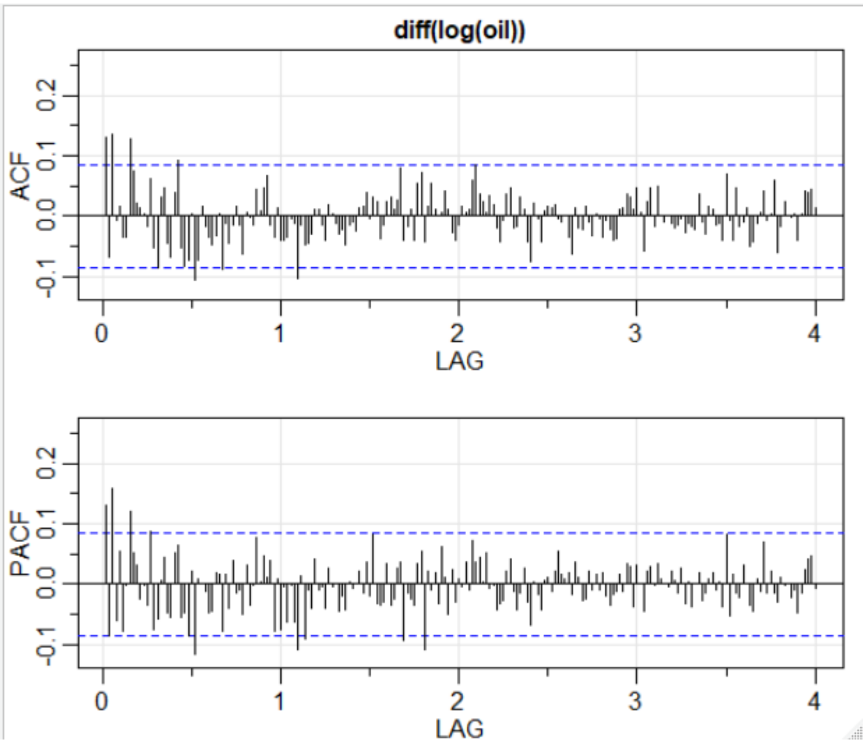
Problem 5.3



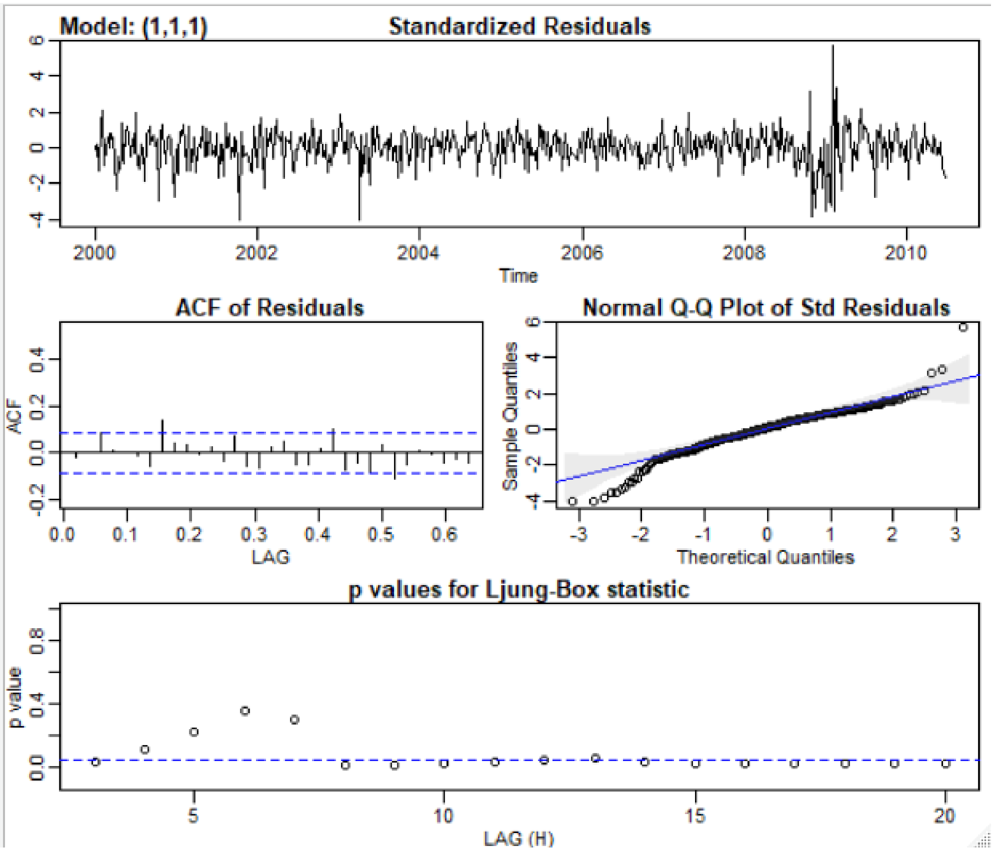
We difference and take log of the data to remove heteroskedasticity and non-constant mean.



The log-differenced transformed series seems to have significant improvement and appears to have constant mean and variance.



From the ACF plot, we observe ACF cuts off at lag 1. So, q=1. From PACF plot, we observe PACF cuts off at lag 1. So, p=1.



Overfitting:

When we take ARIMA(1,1,2)

```
$ttable
         Estimate     SE t.value p.value
ar1       -0.4501 0.1373 -3.2786  0.0011
ma1        0.6251 0.1373  4.5530  0.0000
ma2       -0.0446 0.0560 -0.7970  0.4258
constant   0.0018 0.0021  0.8249  0.4098
```

We get ma2 parameter insignificant.

Similarly for ARIMA(2,1,1):

```
$ttable
         Estimate     SE t.value p.value
ar1       -0.5031 0.0957 -5.2598  0.0000
ar2       -0.0467 0.0494 -0.9452  0.3450
ma1        0.6736 0.0865  7.7863  0.0000
constant   0.0018 0.0021  0.8397  0.4015
```

We get ar2 parameter insignificant.

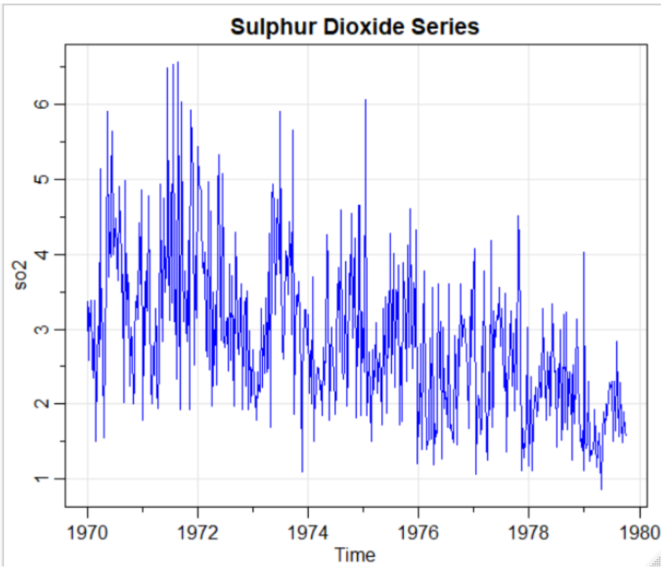
So we choose ARIMA(1,1,1).

From ARIMA(1,1,1) residual plots, following is observed:

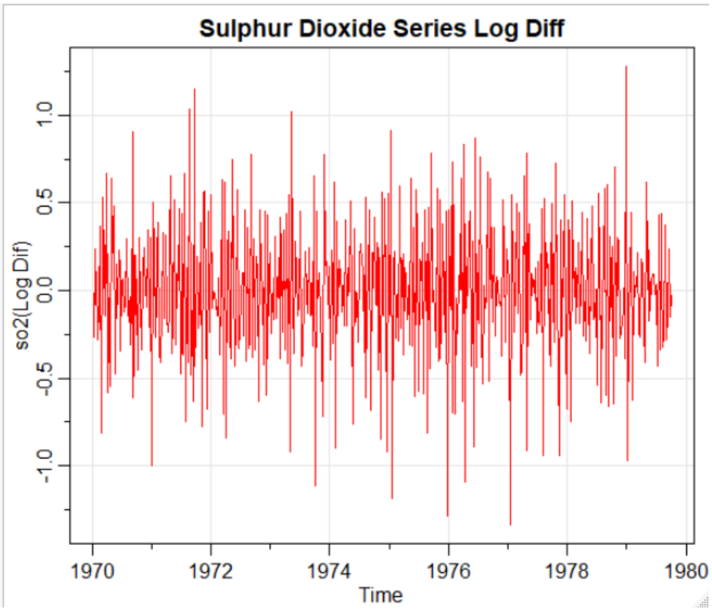
- From the Standardized Residuals plot, it looks like white noise with 0 mean and almost constant variance.
- Residuals seem to be correlated as some values are below 0.05 in Q statistic plot.
- ACF of the residuals are below the significant level (approximately).
- Residuals are approximately normally distributed.

Problem 5.6

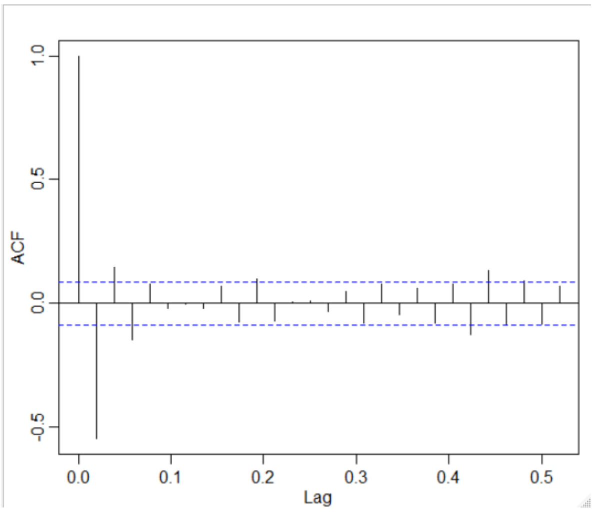
a)

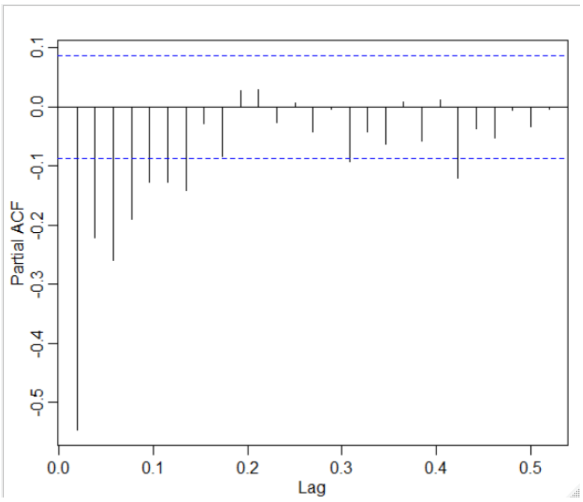


Data is transformed to stationarity by taking log transformation and then differencing.

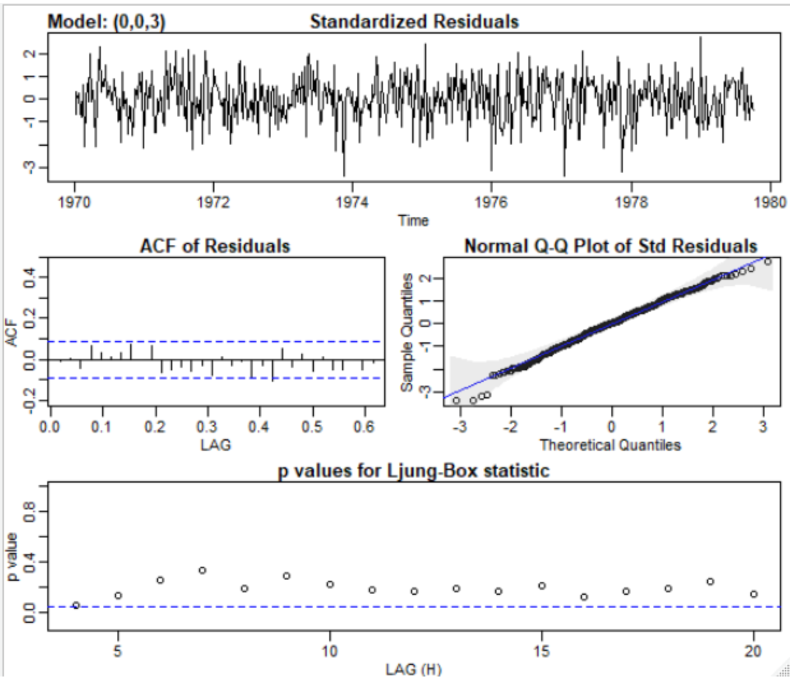


The log-differenced transformed series seems to have significant improvement and appears to have constant mean and variance.





From the ACF plot, we observe ACF is cuts off to zero at lag 3. From PACF plot, we can say that it tails off after 7 lags. Thus, the suitable model is MA(3)



From ARIMA(0,0,3) residual plots, following is observed:

- From the Standardized Residuals plot, it looks like white noise with 0 mean and almost constant variance.
- Residuals are uncorrelated, so they can be considered as white noise.
- ACF of the residuals are below the significant level.
- Residuals are approximately normally distributed.

Overfitting:

For our model ARIMA(0,0,3), we get the following AIC, BIC values

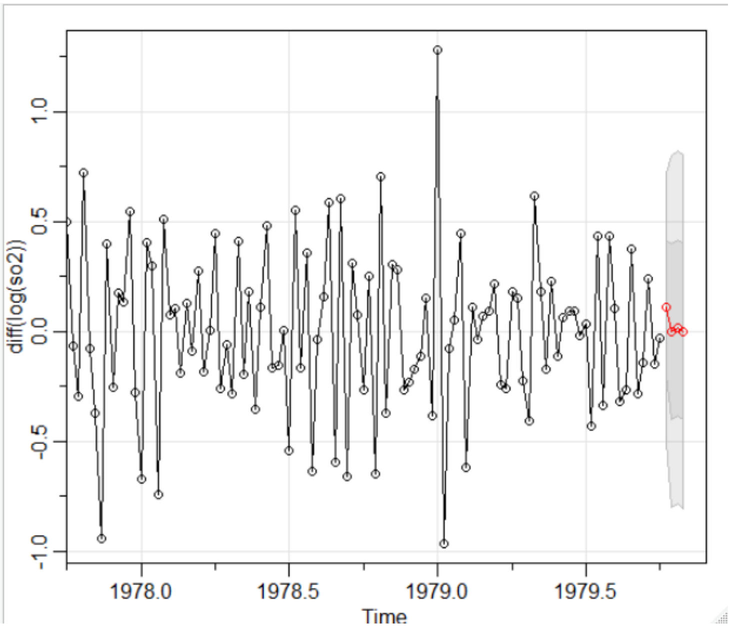
```
$AIC
[1] 0.4949513

$BIC
[1] 0.5366526
```

For ARIMA(0,0,4), we get the following AIC, BIC values

```
$AIC
[1] 0.4940845

$BIC
[1] 0.5441261
```

We prefer BIC as we have a large sample size. Based on BIC values for 2 models, our fitted model ARIMA(0,0,3) has lower BIC. Also, ARIMA(0,0,4) has an ma4 parameter which is insignificant.

Forecasting ahead for a period of 4 weeks (1 month)



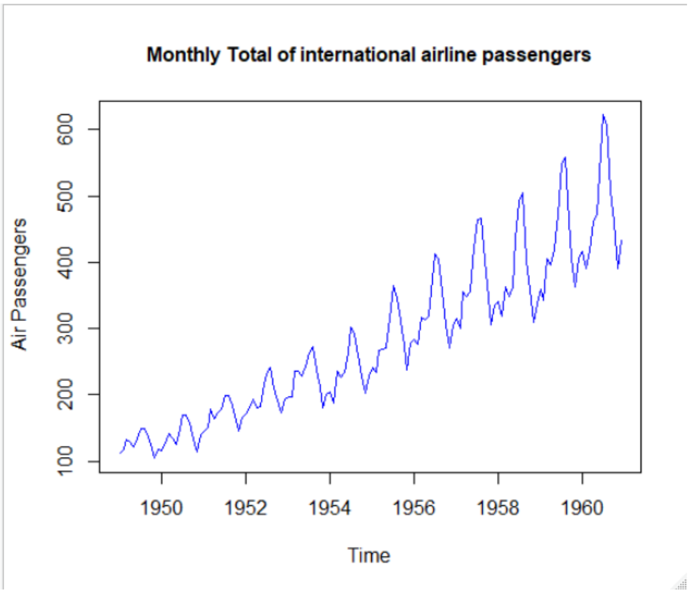
Estimated values in the forecasting period:

```
$pred
Time Series:
Start = c(1979, 41)
End = c(1979, 44)
Frequency = 52
[1]  0.111282917 -0.002104940  0.016708439 -0.001033706

$se
Time Series:
Start = c(1979, 41)
End = c(1979, 44)
Frequency = 52
[1] 0.3064603 0.4006287 0.4014948 0.4031390
```

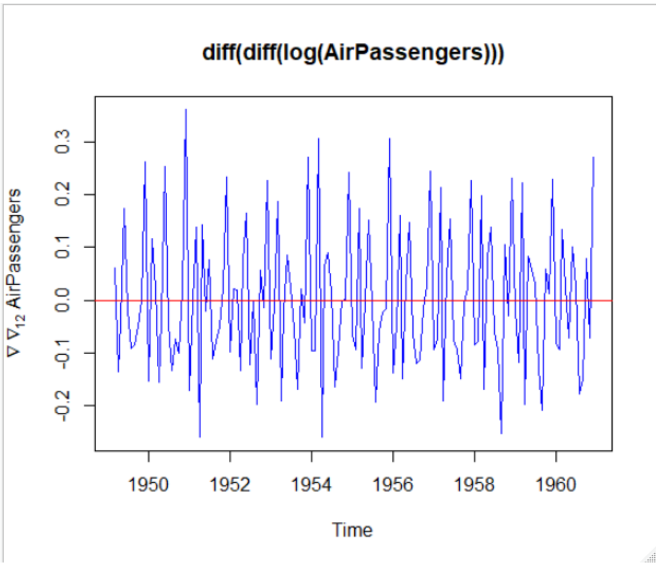
Upper and Lower limit of 95% prediction interval:

```
> upper <- pred$pred + 1.96*(pred$se)
> upper
Time Series:
Start = c(1979, 41)
End = c(1979, 44)
Frequency = 52
[1] 0.7119451 0.7831273 0.8036382 0.7891188
> lower <- pred$pred - 1.96*(pred$se)
> lower
Time Series:
Start = c(1979, 41)
End = c(1979, 44)
Frequency = 52
[1] -0.4893793 -0.7873372 -0.7702213 -0.7911862
```

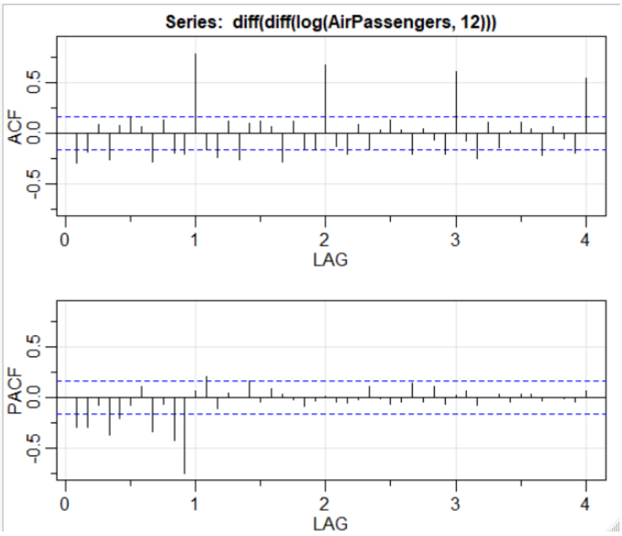
Problem 5.7



Data is transformed to stationarity by taking log transformation and then double differencing



The log-differenced transformed series seems to have significant improvement and appears to have constant mean and variance.



From the ACF plot, we observe ACF cuts off at lag 2. So, q=2. Also, there is seasonality that can be seen in ACF. P=1 based on the tailing off observation at seasonal level in ACF. From PACF plot, we observe PACF cuts off at lag 2. So, p=2.

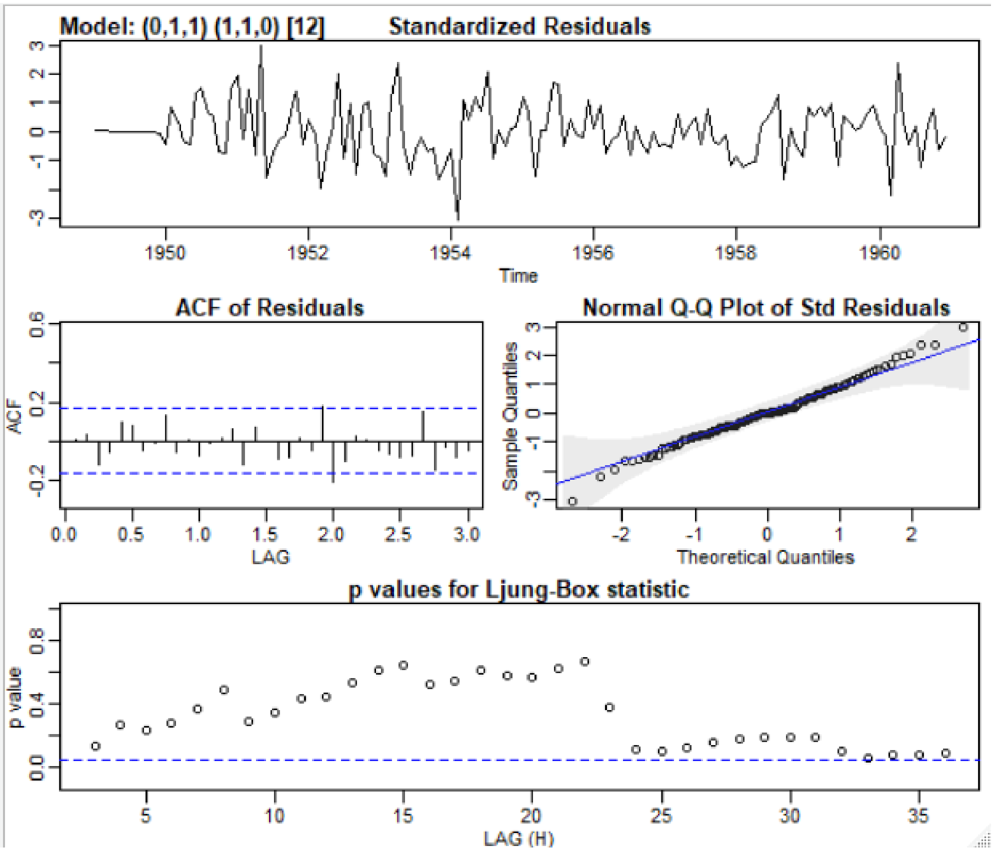
Considering a ARIMA(2,1,2)*$(1,1,0)_{12}$

```
$ttable
     Estimate     SE t.value p.value
ar1   -0.1158 0.5671 -0.2041  0.8386
ar2    0.1838 0.1698  1.0827  0.2810
ma1   -0.3195 0.5648 -0.5658  0.5725
ma2   -0.2011 0.2806 -0.7168  0.4748
sar1  -0.4651 0.0814 -5.7161  0.0000
```

We get ar1,ar2,ma1,ma2 insignificant

When ARIMA(2,1,1)*$(1,1,0)_{12}$ is considered, we still get insignificant terms. Finally, at ARIMA(0,1,1)*$(1,1,0)_{12}$, we get significant parameters only as shown below:

```
$ttable
     Estimate     SE t.value p.value
ma1   -0.4423 0.0832 -5.3173       0
sar1  -0.4743 0.0798 -5.9422       0
```

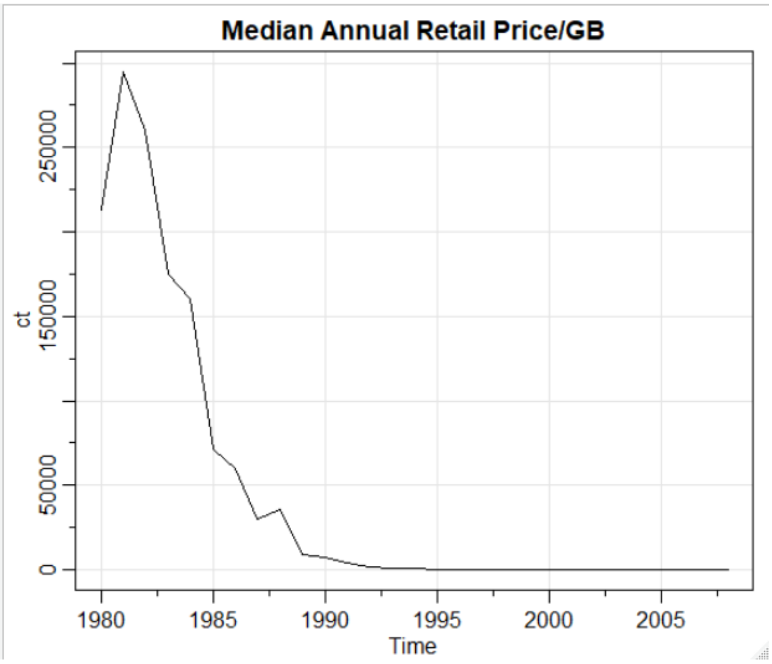


From ARIMA(0,1,1)*$(1,1,0)_{12}$ residual plots, following is observed:

- From the Standardized Residuals plot, mean is almost 0 and variance seems to be almost constant.
- Residuals are uncorrelated, so they can be considered as white noise.
- ACF of the residuals are below the significant level.
- Residuals are approximately normally distributed.

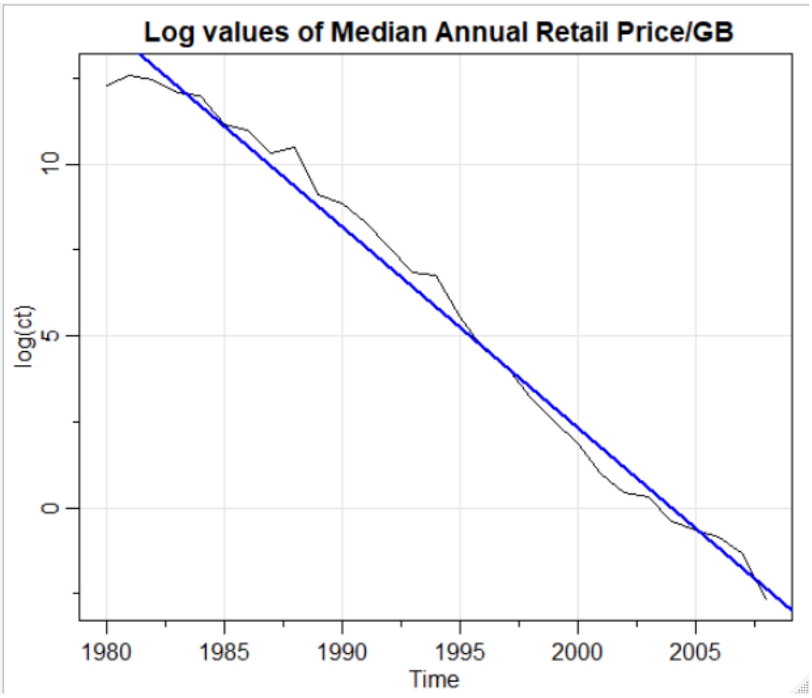
Problem 5.14

a)



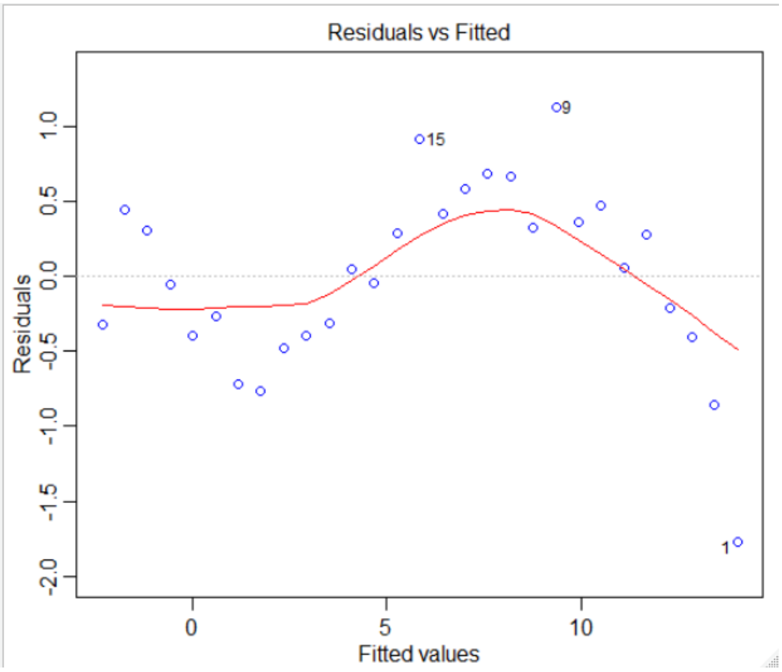
The Time series model has a decreasing Trend.

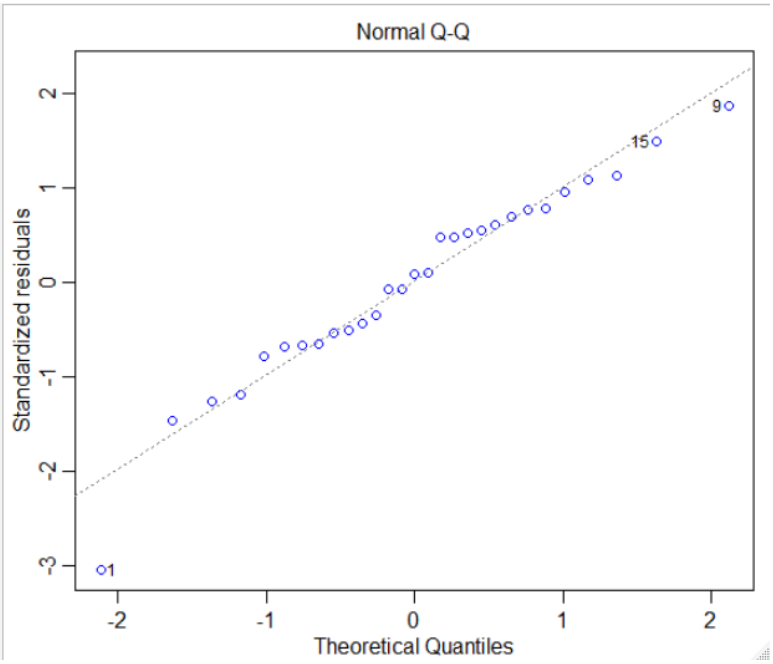
b)



The blue is a fitted line for a linear regression of log(ct) on time. From the above plot, it is clear that ct has an exponential relation with Time.

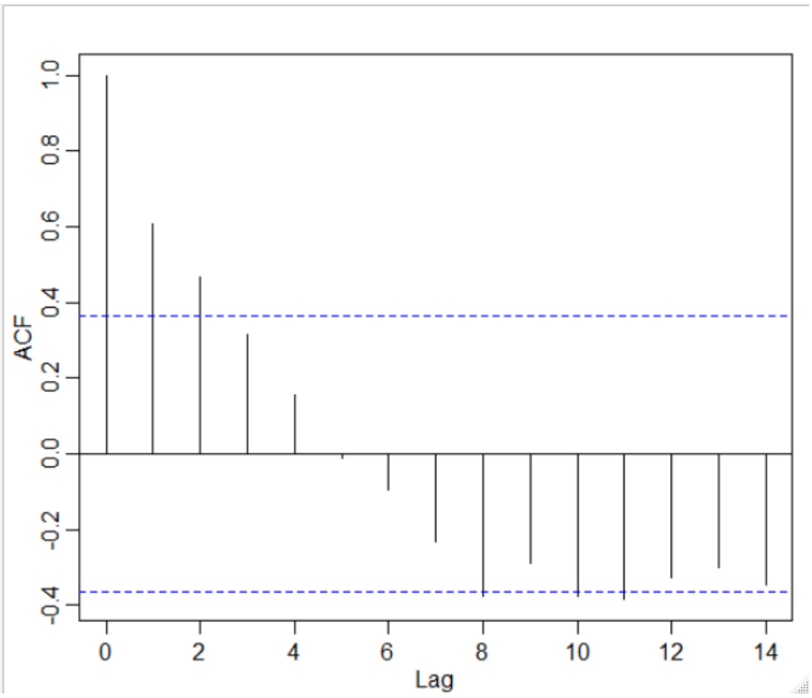
c)

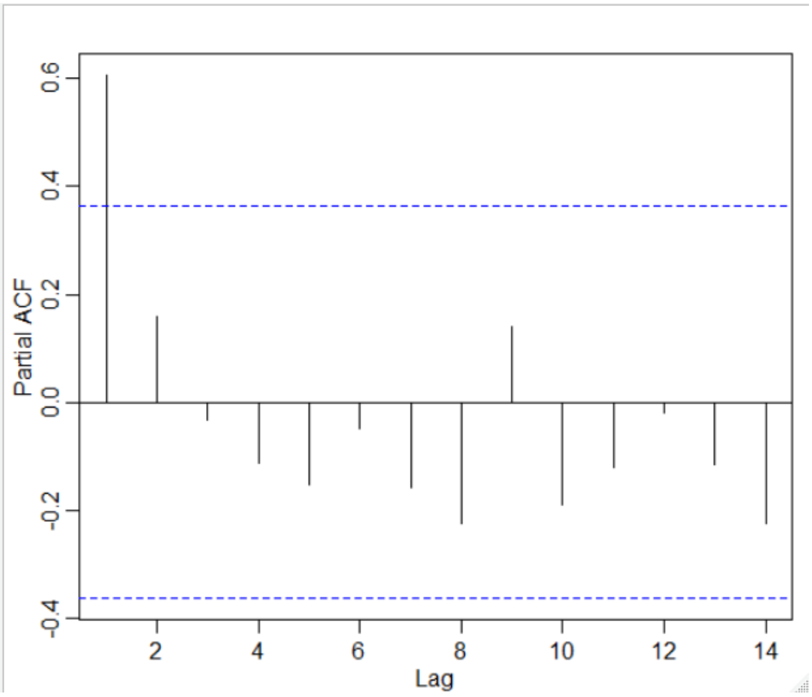




The above plots show that the residuals have a downward trend and have a normalized curve.

d)





As shown in the above plots, ACF values cuts off at lag=2 and PACF values cut off at lag=1. This indicates that the residuals follow a MA(2) (as per ACF) or AR(1) (as per PACF) model. We choose the simpler model which is AR(1).

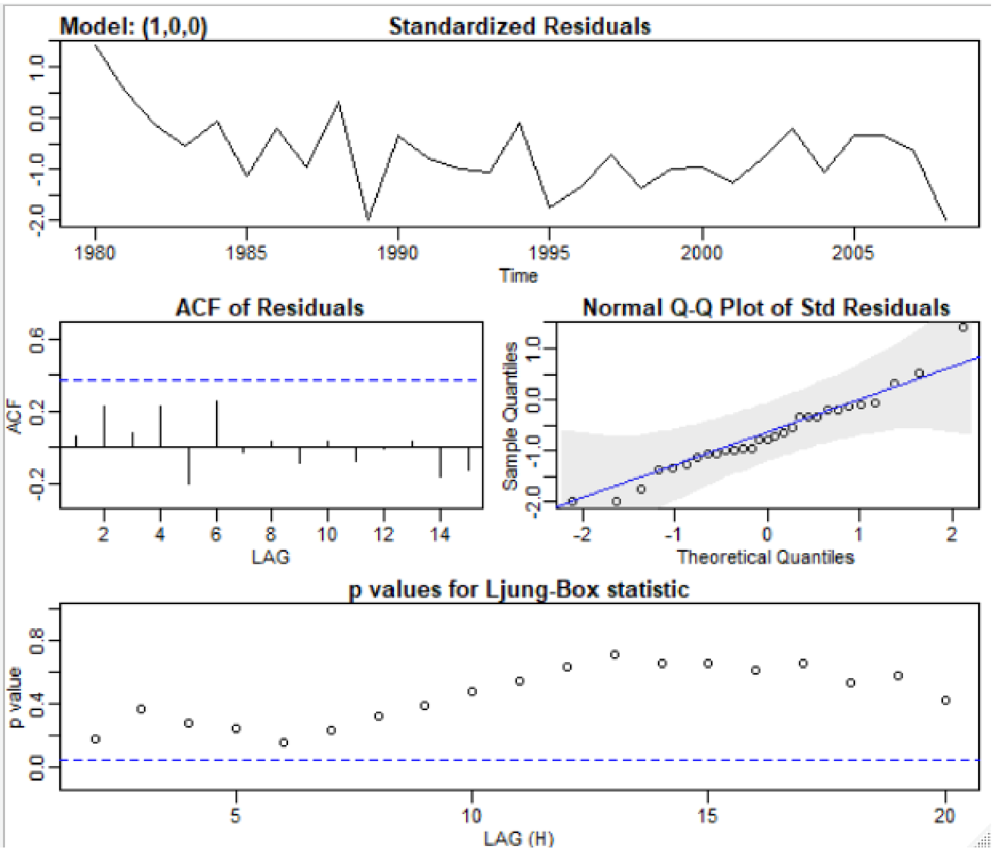
Additionally, I have calculated AIC and BIC of the two regression models. It gives the same result as well (ie. **AR(1) is a better model**)

Model: (1,0,0) ---- AR(1)

```
$AIC
[1] 2.374685

$BIC
[1] 2.468982
```

Model: (0,0,2) ---- MA(2)

```
$AIC
[1] 4.858715

$BIC
[1] 5.00016
```



The above plots show that-

- Residuals are uncorrelated, so they can be considered as white noise.
- ACF of the residuals are below the significant level.
- Residuals are normally distributed.