# STAT 626
# Homework 3

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**Problem 2.1**

Importance of Stationarity:

- It simplifies the time series model, thus making it easier to analyze.
- Can perform statistics on a stationary time series model, i.e. estimate the mean and variance of the time series data.

## Problem 2.2

$$x_t = \beta_0 + \beta_1 t + w_t$$

a] $E[x_t] = E[\beta_0 + \beta_1 t + w_t]$

$= E[\beta_0 + \beta_1 t] + E[w_t]$

$= \beta_0 + \beta_1 t \qquad [\because E(w_t) = 0]$

Since $E(x_t]$ is not constant as it depends on $t$, $x_t$ is not stationary.

b] $y_t = x_t - x_{t-1}$

$= \beta_0 + \beta_1 t + w_t - \beta_0 - \beta_1(t-1) - w_{t-1}$

$= \beta_1 + w_t - w_{t-1}$

$E[y_t] = E[\beta_1 + w_t - w_{t-1}]$

$= \beta_1 + 0 - 0$

$= \beta_1$

$\therefore$ constant Mean

$cov(y_t, y_{t+h})$

$\gamma$ At $h = 0$, $\gamma(0) = cov(y_t, y_t)$

$= Var(y_t)$

$Var(y_t) = Var(\beta_0 + w_t - w_{t-1})$

$= 0 + \sigma^2 w + \sigma^2 w$

$= 2\sigma^2 w \quad [\because Var(-x) = Var(x)]$

At $h = 1$

$\gamma(h) = cov(y_{t+1}, y_t)$

$= E[y_{t+1} - \beta_1][y_t - \beta_1]$

$= E[[\beta_1 + w_{t+1} - w_t - \beta_1][\beta_1 + w_t - w_{t-1} - \beta_1]]$

$= E[(w_{t+1} - w_t)(w_t - w_{t-1})]$

$= E[w_t w_{t+1} - w_{t+1} w_{t-1} - w_t^2 + w_t w_{t-1}]$

$= -E[w_t^2]$

$= -[Var(w_t) - (E(w_t))^2]$

$= -\sigma^2 w$

At $h = 2$, $\gamma(2) = cov(y_{t+2}, y_t)$

cov

$= cov(\beta_0 + w_{t+2} - w_{t+1},$

$\beta_0 + w_t - w_{t-1})$

$= 0$

$\therefore \gamma(h)$ varies only with $h$ and does not depend on $t$.

$\therefore$ $y_t$ is stationary.

c.] $v_t = \frac{1}{3}(x_{t-1} + x_t + x_{t+1})$

$= \frac{1}{3}[\beta_0 + \beta_1(t-1) + w_{t-1} + \beta_0 + \beta_1 t + w_t$

$+ \beta_0 + \beta_1(t+1) + w_{t+1}]$

$= \frac{1}{3}[3\beta_0 + 3\beta_1 t + w_{t-1} + w_t + w_{t+1}]$

$= \beta_0 + \beta_1 t + \frac{1}{3}[w_{t-1} + w_t + w_{t+1}]$

$E[v_t] = E[\beta_0 + \beta_1 t] + \frac{1}{3} E[w_{t-1} + w_t + w_{t+1}]$

$$= \beta_0 + \beta_1 t + \frac{1}{3}(0+0+0)$$

$$\therefore \quad E(v_t) = \beta_0 + \beta_1 t$$

Hence Proved.

Problem 2.3

$$x_t = \frac{1}{4}(w_{t-1} + 2w_t + w_{t+1})$$

$$\gamma(s, t) = cov(x_s, x_t)$$

At $h = 0$, $s = t$

$$\gamma(0) = cov(x_t, x_t)$$

$$= Var(x_t)$$

$$= \frac{1}{16} Var(w_{t-1} + w_{t+1} + 2w_t)$$

$$= \frac{1}{16}\{Var(w_{t-1}) + Var(w_{t+1}) + 4 Var(w_t)\}$$

$[\because w_t$ is independent & uncorrelated$]$

$$= \frac{1}{16}[\sigma^2_w + \sigma^2_w + 4\sigma^2_w]$$

$$= \frac{\cancel{6}^3}{\cancel{16}_8} \sigma^2_w$$

$\therefore \gamma(0) = \frac{3}{8}\sigma^2_w$

$h=1$

$\gamma(1) = cov(x_{t+1}, x_t)$

$= cov\left(\frac{1}{4}(w_t + 2w_{t+1} + w_{t+2}),\right.$

$\left.\frac{1}{4}(w_{t-1} + 2w_t + w_{t+1})\right)$

$= \frac{1}{16} E(2w_t^2 + 2w_{t+1}^2)$

$= \frac{1}{8}\left[E(w_t^2) + E(w_{t+1}^2)\right]$

$= \frac{1}{8}\left[\sigma^2_w + \sigma^2_w\right]$

$= \frac{1}{4}\sigma^2_w$

$h = 2$

$\gamma(2) = cov(x_{t+2}, x_t)$

$= cov\left(\frac{1}{4}(w_{t+1} + 2w_{t+2} + w_{t+3}),\right.$

$\left.\frac{1}{4}(w_{t-1} + 2w_t + w_{t+1})\right)$

$$= \frac{1}{16} E(w_{t+1}^2)$$

$$= \frac{1}{16} \sigma^2_w$$

when $h = 3$, $\gamma(3) = cov(x_{t+3}, x_t)$

$$= cov\left(\frac{1}{4}(w_{t+2} + 2w_{t+3} + w_{t+4}), \frac{1}{4}(w_{t-1} + 2w_t + w_{t+1})\right)$$

No overlapping terms, so,

$= 0$

$h = 4$ $\gamma(4) = 0$

∴ Auto covariance function is

$$\gamma(h) = \begin{cases} \frac{3}{8} \sigma^2_w & h = 0 \\ \frac{1}{4} \sigma^2_w & h = 1 \\ \frac{1}{16} \sigma^2_w & h = 2 \\ 0 & h > 2 \end{cases}$$

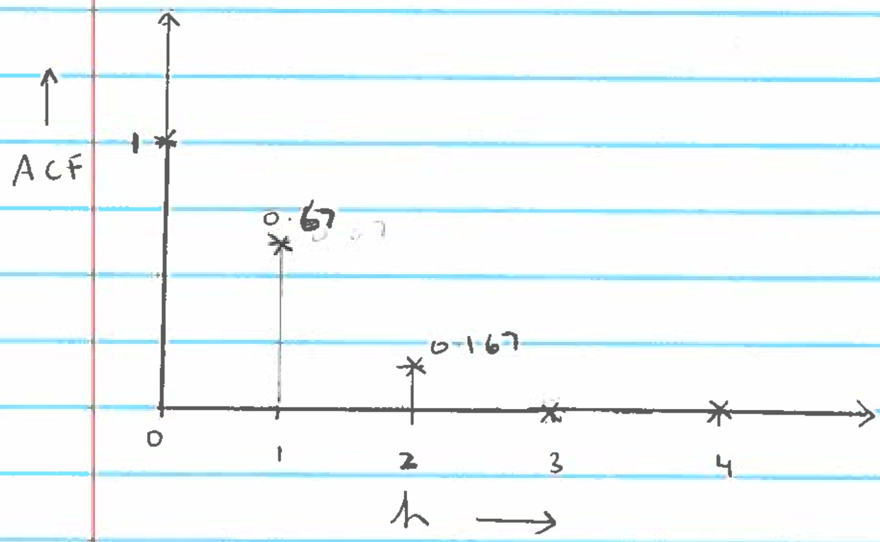
Auto correlation function = $\rho(h)$

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)}$$

$$\therefore \rho(h) = \begin{cases} 1, & h = 0 \\ \frac{2}{3}, & h = 1 \\ \frac{1}{6}, & h = 2 \\ 0, & h > 2 \end{cases}$$

$\frac{1}{4} \times \frac{8}{3}$

$\frac{1}{16} \times \frac{8}{3}$



Problem 2.4

a] $x_t = \phi x_{t-1} + w_t \quad , \quad w_t \sim wn(0,1)$

Let $E[x_t] = \mu_{xt}$

$E[x_t] = \phi E[x_{t-1}] + E(w_t)$

$\mu_{xt} = \phi \mu_{xt} + 0$

$(1-\phi)\mu_{xt} = 0$

$\therefore E(x_t) = \mu_{xt} = 0$

b.] $Var(x_t) = Var(\phi x_{t-1} + w_t)$

$= Var(\phi x_{t-1}) + Var(w_t) + 2 cov(\phi x_{t-1}, w_t)$

Given: $x_{t-1}$ is uncorrelated with $w_t$

$\therefore Var(x_t) = \phi^2 Var(x_{t-1}) + Var(w_t) + 0$

Since $x_t$ is stationary,
$Var(x_t) = \gamma_x(0)$

And $var(x_t) = var(x_{t-1})$

$$\gamma_x(0) = \phi^2 \gamma_x(0) + \sigma^2_w$$

$$\gamma_x(0) = \frac{\sigma^2_w}{1-\phi^2}$$

$$= \frac{1}{1-\phi^2} \qquad \because w_t \sim wn(0,1)$$

Hence Proved

c.] Variance is ~~positive~~ ~~pdf~~ finite

$$|\phi| < 1$$

$$\therefore \phi \in (-1, 1)$$

d.] $\gamma_x(1) = cov(x_t, x_{t-1})$

$$= cov(\phi x_{t-1} + w_t, x_{t-1})$$

$$= cov(\phi x_{t-1}, x_{t-1})$$

$$= E[\phi x_{t-1} x_{t-1}]$$

$$= \phi E[x^2_{t-1}]$$

$$= \phi [Var(x_{t-1}) - 0]$$

$$\rho_x(1) = \frac{\gamma_x(1)}{\gamma_x(0)}$$

$$= \frac{\gamma\ \phi\ Var(x_{t-1})}{\gamma_x(0)}$$

$$= \phi \frac{\gamma_x(0)}{\gamma_x(0)} \quad \text{[From } a\text{]}$$

$$= \phi$$

$$\therefore \underline{\underline{\rho_x(1) = \phi}}$$

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Problem 2.5

a] $x_t = \delta + x_{t-1} + w_t$

At $t = 1$, $x_1 = \delta + x_0 + w_1$

$x_0 = 0$, [Given]

$\therefore\ x_1 = \delta + w_1$

At $t = 2$, $x_2 = \delta + x_1 + w_2$

$= 2\delta + w_1 + w_2$

At $t = 3$, $n_3 = \delta + n_2 + \omega_3$

$$= \delta + 2\delta + \omega_1 + \omega_2 + \omega_3$$

$$= 3\delta + \omega_1 + \omega_2 + \omega_3$$

$\vdots$

$\therefore$ at $t = t$

$$n_t = \delta \quad t\delta + \omega_1 + \omega_2 + \cdots \omega_t$$

$$= t\delta + \sum_{k=1}^{t} \omega_k$$

Hence proved.

Sol.)

$$n_t = t\delta + \sum_{k=1}^{t} \omega_k$$

$$E[n_t] = E(t\delta) + E\left[\sum_{k=1}^{t} \omega_k\right]$$

$$= t\delta + \Sigma 0$$

$$= t\delta$$

$$\therefore E[n_t] = t\delta$$

PTO

At $h = 0$, $\gamma(0) = cov(x_t, x_t)$

$= var(x_t)$

$Var(x_t) = Var(\delta t) + Var(\sum_{k=1}^{t} w_t)$

$= 0 + t\sigma^2_w$

$= t\sigma^2_w$

At $h = 1$, $\gamma(1) = cov(x_{t+1}, x_t)$

$= cov(\delta + x_t + w_{t+1}, x_t)$

$= Var(x_t)$

$= t\sigma^2_w$ (From Above)

$\therefore$ For all values of $h$,

$cov(x_{t+h}, x_t) = Var(x_t)$
$= t\sigma^2_w$

$\Rightarrow$ In general $cov(x_a, x_b) = b\sigma^2_w$
when $b < a$.

c.] From part b]

$$E(x_t) = t\delta$$

$$\gamma(h) = t\sigma^2_w$$

Since $E(x_t)$ is not constant & a function of $t$ and $\gamma(h)$ is a function of $t$, $x_t$ is not stationary.

d.] $$\rho(t-1, t) = \frac{\text{cov}(x_{t-1}, x_t)}{\sqrt{\text{Var}(x_{t-1})\,\text{Var}(x_t)}}$$

Here,

$$\text{cov}(x_{t-1}, x_t) = (t-1)\sigma^2_w$$

($\because$ Number of overlapping terms is $t-1$)

$$\text{Var}(x_{t-1}) = (t-1)\sigma^2_w$$

$$\text{Var}(x_t) = t\sigma^2_w$$

$$\therefore \rho(t-1, t) = \frac{(t-1)\sigma^2_w}{\sqrt{(t-1)\,t}\cdot\sigma^2_w}$$

$$= \sqrt{\frac{t-1}{t}}$$

$$\lim_{t \to \infty} \sqrt{\frac{t-1}{t}}$$

$t-1 \simeq t$ as $t \to \infty$

$$= \sqrt{\frac{t}{t}}$$

$$= 1$$

$\therefore \delta(t-1, t) = 1$ as $t \to \infty$.

This implies that as $t \to \infty$ (takes bigger values), the two consecutive series varies in tandem [as corr=1]

2.] We consider, taking difference of two consecutive terms in the series.

Let $y_t = n_t - n_{t-1}$

$$= \delta + n_{t-1} + w_t - n_{t-1}$$

$$y_t = \delta + w_t$$

$\therefore$ Transformed series is

$$y_t = \delta + w_t$$

$$E(y_t) = E(\delta) + E(w_t)$$

$$= \delta \qquad -\ 1$$

constant and independent of $t$.

At $h=0$

$$\gamma(0) = \text{cov}(y_t, y_t)$$

$$= \text{var}(y_t)$$

$$= \sigma^2_w$$

At $h=1$

$$\gamma(1) = \text{cov}(y_{t+1}, y_t)$$

$$= \text{cov}(\delta + w_{t+1}, \delta + w_t)$$

since there are no overlapping terms,

$$\gamma(1) = 0$$

This is true for all values of $h$ except $h=0$.

$$\therefore \gamma(h) = \begin{cases} \sigma^2_w & h=0 \\ 0 & h\neq 0 \end{cases} \qquad -\ 2$$

$\gamma(h)$ is constant & doesn't depend on $t$.

$\therefore$ From ① & ② the transformed series $y_t$ is stationary.

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Problem 2.7

$$x_t = U_1 \sin 2\pi w_0 t + U_2 \cos(2\pi w_0 t)$$

$$E(x_t) = E\left(U_1 \sin(2\pi w_0 t) + U_2 \cos(2\pi w_0 t)\right]$$

$$= E\left[U_1 \sin(2\pi w_0 t)\right] + E\left[U_2 \cos(2\pi w_0 t)\right]$$

$$= \sin(2\pi w_0 t)\, E(U_1) + \cos(2\pi w_0 t)\, E(U_2)$$

Given $E(U_1) = E(U_2) = 0$

$\therefore$ $E(x_t) = 0$ — ①

$$\gamma(h) = \operatorname{cov}(x_{t+h}, x_t)$$

$$= \operatorname{cov}\left(U_1 \sin(2\pi w_0(t+h)) + U_2 \cos(2\pi w_0(t+h)),\ U_1 \sin 2\pi w_0 t + U_2 \cos 2\pi w_0 t\right)$$

PTO

**Problem 2.6**

The global temperature graph Figure 1.2 is **non-stationary**. The land surface and sea surface temperature both have a positive trend and thus its mean is increasing as time passes. Since the mean is not constant, the data is **non-stationary.**

$\therefore$ From (1) & (2) the transformed series $y_t$ is stationary.

Problem 2.7

$x_t = U_1 \sin 2\pi w_0 t + U_2 \cos(2\pi w_0 t)$

$E(x_t) = E\left(U_1 \sin(2\pi w_0 t) + U_2 \cos(2\pi w_0 t)\right]$

$= E\left[U_1 \sin(2\pi w_0 t)\right] + E\left[U_2 \cos(2\pi w_0 t)\right]$

$= \sin(2\pi w_0 t)\, E(U_1) + \cos(2\pi w_0 t)\, E(U_2)$

Given $E(U_1) = E(U_2) = 0$

$\therefore E(x_t) = 0 \quad -(1)$

$\gamma(h) = \operatorname{cov}(x_{t+h}, x_t)$

$= \operatorname{cov}\left(U_1 \sin(2\pi w_0 (t+h)) + U_2 \cos(2\pi w_0 (t+h)),\ U_1 \sin 2\pi w_0 t + U_2 \cos 2\pi w_0 t\right)$

PTO

$$= \sin(2\pi w_0(t+h)) \sin 2\pi w_0 t \ \overset{\sigma^2}{\cancel{\operatorname{cov}(U_1, U_1)}}$$

$$+ \sin(2\pi w_0(t+h)) \cos(2\pi w_0 t) \ \overset{0}{\cancel{\operatorname{cov}(U_1, U_2)}}$$

$$+ \cos(2\pi w_0(t+h)) \sin(2\pi w_0 t) \ \overset{0}{\cancel{\operatorname{cov}(U_2, U_1)}}$$

$$+ \cos(2\pi w_0(t+h)) \cos(2\pi w_0 t) \ \overset{\sigma^2}{\cancel{\operatorname{cov}(U_2, U_2)}}$$

$$= \sigma^2 \left[ \sin(2\pi w_0(t+h)) \sin 2\pi w_0 t + \cos(2\pi w_0(t+h)) \cos 2\pi w_0 t \right]$$

$$= \sigma^2 \cos\left[ 2\pi w_0(t+h) - 2\pi w_0 t \right]$$

$$= \sigma^2 \cos 2\pi w_0 h$$

Hence $\therefore$ $\gamma(h) = \sigma^2 \cos 2\pi w_0 h$

Hence Proved.

Problem 2.9

$x_t = w_t \, w_{t-1}$

$E(x_t) = E(w_t \, w_{t-1})$

$= E(w_t) \, E(w_{t-1})$ since white noise

$= 0 \times 0$ white noise mean $= 0$

$= 0$ — (1)

$\gamma(h) = cov(x_{t+h}, x_t)$

At $h = 0$

$\gamma(0) = cov(x_t, x_t)$

$= Var(x_t)$

$= Var(w_t \, w_{t-1})$

$= E(w_t^2 \, w_{t-1}^2) - (E(w_t) \, E(w_{t-1}))^2$

$= E(w_t^2 \, w_{t-1}^2)$

$= E(w_t^2) \cdot E(w_{t-1}^2)$

$= \sigma^2_w \cdot \sigma^2_w$ [As $E(w_t) = 0$]

$= \sigma^4_w$

$\therefore \gamma(0) = \sigma^4_w$

At $h = \pm 1, \pm 2, \ldots$ no overlap

So $\gamma(h) = 0$

$$\therefore \gamma(h) = \begin{cases} \sigma^4_w & , h = 0 \\ 0 & , h \neq 0 \end{cases} \quad -(2)$$

From (1) & (2) Mean is constant & $\gamma(h)$ is not a function of time (Independent of time)

$\therefore x_t$ is stationary.

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Problem 2.10

$$x_t = \mu + w_t + \theta w_{t-1}$$

a) $E(x_t) = E(\mu + w_t + \theta w_{t-1})$

$= E(\mu) + E(w_t) + E(\theta w_{t-1})$

$= \mu + 0 + \theta \times 0$

$= \mu$

Hence Proved.

b] $\gamma(x_t, x_{t+h}) = E(x_t \cdot x_{t+h}) - E(x_t) \, E(x_{t+h})$

$= E\left[(\mu + w_t + \theta w_{t-1})(\mu + w_{t+h} + \theta w_{t+h-1}) - \mu^2\right]$

At $h = 0$,

$\gamma(0) = E\left[(\mu + w_t + \theta w_{t-1})(\mu + w_t + \theta w_{t-1}) - \mu^2\right]$

$= E\left[\mu^2 + w_t^2 + \theta^2 w_{t-1}^2\right] - \mu^2$

$\gamma(0) = \mu^2 + \sigma^2_w + \theta^2 \sigma^2_w - \mu^2$

$= \sigma^2_w (1 + \theta^2)$

At $h = 1$,

$\gamma(1) = E\left[(\mu + w_t + \theta w_{t-1})(\mu + w_{t+1} + \theta w_t) - \mu^2\right]$

$= E\left[\mu^2 + \theta w_t^2\right] - \mu^2$

$= \theta \sigma^2_w$

PTO

At $h = -1$

$$\gamma(-1) = E\left[(\mu + w_t + \theta w_{t-1})(\mu + w_{t-1} + \theta w_{t-2}) - \mu^2\right]$$

$$= E\left[\mu^2 + \theta w_{t-1}^2\right] - \mu^2$$

$$= \theta \sigma^2 w$$

At $h = 2$,

$$\gamma(2) = E\left[(\mu + w_t + \theta w_{t-1})(\mu + w_{t+2} + \theta w_{t+1}) - \mu^2\right]$$

$$= E\left[\mu^2\right] - \mu^2$$

$$= 0$$

for higher values of $h$ ($h \geq 2$) $\gamma(h) = 0$.

Therefore,

$$\gamma(h) = \begin{cases} \sigma^2 w (1+\theta^2) & h = 0 \\ \theta \sigma^2 w & h = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

c] From a & b, mean = constant and $\gamma(h)$ ~~is not~~ ~~the~~ function of $t$

$\therefore$ $x_t$ is stationary.

d.] $$Var(\bar{x}) = \frac{1}{n} \sum_{h=-n}^{n} \left(1 - \frac{|h|}{n}\right) \gamma_x(h)$$

i, $\theta = 1$

$\gamma(0) = \sigma^2_w (1+\theta)$

$= 2\sigma^2_w$

$\gamma(1) = \gamma(-1) = \sigma^2_w$

$\gamma(h) = 0,$ ~~at~~ for $h \geq 2$

$$Var(\bar{x}) = \frac{1}{n}\left[\left(1 - \frac{|-n|}{n}\right) \gamma(-n) + \cdots \right.$$

$$\left(1 - \frac{|-1|}{n}\right)\gamma(-1) + \left(1 - \frac{0}{n}\right)\gamma(0)$$

$$+ \left(1 - \frac{|1|}{n}\right)\gamma(1) + \cdots\cdots$$

$$\left. + \left(1 - \frac{|n|}{n}\right) \gamma(n)\right]$$

$$= \cancel{\frac{1}{n}\Big[ 0 + \cdots \left(1 - \frac{1}{n}\right)\sigma^2_w + 2\sigma^2_w +}$$

PTO

$$= \frac{1}{n}\Big[ 0 + \cdots \left(1-\frac{1}{n}\right)\sigma^2 w + 2\sigma^2 w + \left(1-\frac{1}{n}\right)\sigma^2 w$$

$$+ \cdots\cdots 0 \Big]$$

$$= \frac{1}{n}\left[\left(1-\frac{1}{n}\right)\sigma^2 w + 2\sigma^2 w + \left(1-\frac{1}{n}\right)\sigma^2 w\right]$$

$$= \frac{1}{n}\left[\frac{2(n-1)\sigma^2 w + 2n\sigma^2 w}{n}\right]$$

$$= \frac{2}{n}\sigma^2 w\left[1-\frac{1}{n}+1\right]$$

$$= \frac{2}{n}\sigma^2 w\left[2-\frac{1}{n}\right]$$

$$Var(\bar{x}) = \frac{2}{n}\sigma^2 w\left[2-\frac{1}{n}\right]$$

ii) $\theta = 0$

$\gamma(0) = \sigma^2 w$

$\gamma(1) = \gamma(-1) = 0 \quad w$

$\gamma(h) = 0 \quad , h \geq 2.$

$$Var(\bar{x}) = \frac{1}{n}\left[\left(1-\frac{|0|}{n}\right)\gamma(0)\right]$$

$$= \frac{1}{n}\gamma(0)$$

$$= \frac{\sigma^2_w}{n}$$

iii) $\theta = -1$

$\gamma(0) = 2\sigma^2_w$

$\gamma(1) = \gamma(-1) = -\sigma^2_w$

$\gamma(h) = 0 \quad , \quad h \geq 2$

$$Var(\bar{x}) = \frac{1}{n}\left[\left(1-\frac{1}{n}\right)\gamma(-1) + \left(1-\frac{0}{n}\right)\gamma(0) + \left(1-\frac{1}{n}\right)\gamma(1)\right]$$

$$= \frac{1}{n}\left[-2\left(1-\frac{1}{n}\right)\sigma^2_w + 2\sigma^2_w\right]$$

$$= \frac{1}{n}\sigma^2_w\left[\cancel{2} - 2 + \frac{2}{n}\right]$$

$$= \frac{2\sigma^2_w}{n^2}$$

$$\therefore \; Var(\bar{x}) = \begin{cases} \frac{2\sigma^2_w}{n}\left(2-\frac{1}{n}\right), & \theta=1 \\ \frac{\sigma^2_w}{n}, & \theta=0 \\ \frac{2\sigma^2_w}{n^2}, & \theta=-1 \end{cases}$$

e.] When sample size $n$ is large then $\frac{n-1}{n} \approx 1$

$$Var(\bar{x}) = \begin{cases} \frac{2\sigma^2_w}{n}\left(1+\frac{n-1}{n}\right), & \theta=1 \\ \frac{\sigma^2_w}{n}, & \theta=0 \\ \frac{2\sigma^2_w}{n^2}\left(1-\frac{n-1}{n}\right), & \theta=-1 \end{cases}$$

~~Now when~~ $n$ is large

$\frac{n-1}{n} \approx 1$

$$\therefore \; Var(\bar{x}) = \begin{cases} \frac{4\sigma^2_w}{n}, & \theta=1 \\ \frac{\sigma^2_w}{n}, & \theta=0 \\ 0, & \theta=-1 \end{cases}$$

As the value of $\theta$ decreases from 1 to -1, the variance decreases. As the variance decreases, the accuracy of mean estimator increases.

$\theta \downarrow$ variance $\downarrow$ accuracy of mean $\uparrow$

## Bonus Question

$w_t \sim \text{iid } N(0, \sigma_w^2)$

$x_t = w_{t-1} \cdot w_{t-2} \; (w_{t-1} + w_t + t)$

$E(x_t)$ = Mean of $x_t$

$= E[w_{t-1} w_{t-2} (w_{t-1} + w_t + t)]$

$= E[w_{t-1}^2 \; w_{t-2} + w_t \; w_{t-1} \; w_{t-2} +$
$t \; w_{t-1} \; w_{t-2}]$

$= E[w_{t-1}^2 \; w_{t-2}] + E[w_t \; w_{t-1} \; w_{t-2}] +$
$t \; E[w_{t-1} \; w_{t-2}]$

$\{$ $E[XYZ] = E[X] \, E[Y] \, E[Z]$ if $X, Y, Z$ are independent random variables

Also, if $E(XY) = E(X) \, E(Y)$ is true
$X, Y$ independent random variables
Then $E[f(X) \, g(Y)] = E[f(X)] \, E[g(Y)]$

$\}$

$= E[w_{t-1}^2] \, E[w_{t-2}] + E[w_t] \, E[w_{t-1}] \, E[w_{t-2}]$
$+ \; t \; E[w_{t-1}] \, E[w_{t-2}]$

$= E[w_{t-1}^2] \cdot 0 + 0 + 0 = 0$

$\therefore \underline{\underline{E(x_t) = 0}}$

$\gamma(h) = \operatorname{cov}(x_{t+h}, x_t)$

At $h=0$

$\gamma(0) = \operatorname{cov}(x_t, x_t) = \operatorname{Var}(x_t)$

$\gamma(0) = E[x_t^2] - (\cancel{E(x_t)})^2$ , $0$

$= E[x_t^2]$

$= E[(w_{t-1}^2 w_{t-2} + w_t w_{t-1} w_{t-2} + t\, w_{t-1} w_{t-2})^2]$

$= E[w_{t-1}^4]\, E[w_{t-2}^2] + E[w_t^2]\, E[w_{t-1}^2]\, E[w_{t-2}^2] + t^2 E(w_{t-1}^2)\, E(w_{t-2}^2)$
$+ 2E[w_{t-1}^3] \cdot E[w_{t-2}^2] \cdot E[w_t] + 2t\, E[w_{t-1}^3]\, E[w_{t-2}^2] + 2t\, E[w_{t-2}^2] \cdot E[w_{t-1}^2]\, E[w_t]$

We know that $w_t$ is iid

$E[\ldots^m] = \begin{cases} 0 & \text{if } m \text{ is odd} \\ 2^{-m/2} \dfrac{m!}{(m/2)!} & \text{if } m \text{ is even} \end{cases}$

~~W~~ Where $X \sim N(0,1)$

Assuming $\sigma^2_w = 1$

$w_t \sim N(0,1)$

$$\therefore \gamma(0) = E[w^4_{t-1}] \; E[w^2_{t-2}] + E[w_t^2] \; E[w_{t-1}^2] \; E[w_{t-2}^2] + t^2 E[w^2_{t-1}] + E[w^2_{t-2}]$$

$$= 2^{-2} \frac{4!}{2!} \cdot 2^{-1} \frac{2!}{1!} + \left(2^{-1} \frac{2!}{1!}\right)^3 + t^2 \left(2^{-1} \frac{2!}{1!}\right)^2$$

$$= \frac{3}{2} + 1 + t^2$$

$$= \underline{\underline{4 + t^2}}$$

$\gamma(0)$ depends on $t$ ~~so~~ $\gamma(0)$ is not constant.

$\therefore$ ~~$x$~~$_t$ is <u>not stationary</u>.

$h=1, \quad \gamma(1) = \operatorname{cov}(x_{t+1}, x_t)$

$= E[x_{t+1} \cdot x_t]$

$= E\Big[\{ w_t w_{t-1} (w_t + w_{t+1} + t+1) \} \{ w_{t-1} w_{t-2}$

$(w_{t-1} + w_t + t) \} \Big]$

$= E\Big[ w_t w_{t-1}^2 w_{t-2} ( w_t w_{t-1} + w_t^2 + t w_t$

$+ w_{t+1} w_{t-1} + w_{t+1} w_t + t w_{t+1}$

$+ t w_{t-1} + t w_t + t^2 + w_{t-1} + w_t + t) \Big]$

$= E\Big[ w_t w_{t-1}^2 w_{t-2} ( w_t w_{t-1} + w_t^2 + 2 w_t$

$+ w_{t+1} w_{t-1} + w_{t+1} w_t + t w_{t+1} + t w_{t-1}$

$+ t^2 + w_{t-1} + w_t + t) \Big]$

$= E\Big[ w_t^2 w_{t-1}^3 w_{t-2} + w_t^3 w_{t-1}^2 w_{t-2}$

$+ 2 w_t^2 w_{t-1}^2 w_{t-2} + w_t w_{t-1}^3 w_{t-2} w_{t+1}$

$+ w_t^2 w_{t-1}^2 w_{t-2} w_{t+1} + t w_t w_{t-1}^2 w_{t-2}$

$+$

$w_{t+1} + t w_t w_{t+1} w_{t-1}^2 w_{t-2} +$

$t \, w_t \, w_{t-1}^3 \, w_{t-2} + t^2 \, w_t \, w_{t-1} \, w_{t-2}$

$+ \, w_t \, w_{t-1}^3 \, w_{t-2} + w_t^2 \, w_{t-1}^2 \, w_{t-2}$

$+ \, w_t^t \, w_t \, w_{t-1}^2 \, w_{t-2} ]$

Since each term has at atleast one odd power $E(x)$ so the value will be

$= 0$

This is true for all $h$

$\gamma(h) = cov(x_{t+h}, x_t) = 0 \quad , h > 0$

$\therefore$ All term would have at least one odd power $E(x^m)$ term.

$\therefore$ Autocovariance function

$$= \begin{cases} 4 + t^2 & \text{if } h = 0 \\ 0 & \text{if } h > 0 \end{cases}$$