Discrete Structures Assignment - 2

Ans-1 Given :- $a_r = \begin{cases} 1, & r=0 \\ 3, & r=1 \\ 5, & r=2 \\ 0, & r \geq 3 \end{cases}$

and $c_r = 7^r$ for all r

Also given that $c = a * b$

To derive :- b.

we know in convolution of a and b

$$c_r = a * b_r$$

$$c_r = \sum_{i=0}^{r} a_i b_{r-i}$$

So $c_0 = a_0 b_0 = 7^0 = 1$

$\Rightarrow b_0 = 1$ — 1)

$c_1 = a_0 b_1 + a_1 b_0 = 7^1 = 7$

$1 \times b_1 + 3 \times 1 = 7$

$b_1 = 4$ — 2)

$c_r = a_0 b_r + a_1 b_{r-1} + a_2 b_{r-2} + a_3 b_{r-3} + \cdots a_r b_0 = 7^r$

given $a_r = 0$ for $r \geq 3$

$\Rightarrow a_0 b_r + a_1 b_{r-1} + a_2 b_{r-2} = 7^r$

Now putting the values of a_0, a_1 and a_2.

$$b_r + 3 b_{r-1} + 5 b_{r-2} = 7^r$$

Now multiply both sides by z^r and taking summation

$$\sum_{r=2}^{\infty} b_r z^r + 3 \sum_{r=2}^{\infty} b_{r-1} z^r + 5 \sum_{r=2}^{\infty} b_{r-2} = \sum_{r=2} 7^r z^r$$

$$(B(z) - b_0 - b_1 z) + 3z(B(z) - b_0) + 5z^2 B(z) = \frac{1}{1-7z} - 1 - 7z$$

Now putting values of b_0 and b_1 from ① and ②

$$B(z) - 1 - 4z + 3z(B(z) - 1) + 5z^2 B(z) = \frac{1}{1-7z} - 1 - 7z$$

$$B(z) = \frac{1}{(1-7z)(5z^2+3z+1)}$$

Let $5z^2+3z+1 = 5(x-\alpha)(x-\beta)$

where α and β are found to be complex roots.

$$\frac{1}{5(1-7z)(z-\beta)(z-\alpha)} = B(z)$$

Now using partial fractions,

$$B(z) = \frac{A}{5(1-7z)} + \frac{B}{5(z-\alpha)} + \frac{C}{5(z-\beta)}$$

$$= \frac{A}{5(1-7z)} + \frac{(-1)\beta}{5\alpha(1-\frac{z}{\alpha})} - \frac{C}{5\beta(1-\frac{z}{\beta})}$$

let $\frac{A}{5} = k_1$, $\frac{-\beta}{5\alpha} = k_2$ and $-\frac{C}{5\beta} = k_3$

$$\frac{1}{5(1-7z)(z-\alpha)(z-\beta)} = \frac{k_1}{1-7z} + \frac{k_2}{1-\frac{z}{\alpha}} + \frac{k_3}{1-\frac{z}{\beta}}$$

where A, B and C and k_1, k_2 and k_3 can found on putting values of α and β and 3 distinct (any) values of z.

Now $B(z) = \frac{k_1}{1-7z} + \frac{k_2}{1-\frac{z}{\alpha}} + \frac{k_3}{1-\frac{z}{\beta}}$

we know $B(z) = \sum_{n=0}^{\infty} b_n z^n$

$$B(z) = \sum_{n=0}^{\infty} k_1 7^n z^n + \sum_{n=0}^{\infty} k_2 \left(\frac{1}{\alpha}\right)^n z^n + \sum_{n=0}^{\infty} k_3 \left(\frac{1}{\beta}\right)^n z^n$$

$$= \sum_{n=0}^{\infty} \left(k_1 7^n + k_2 \left(\frac{1}{\alpha}\right)^n + k_3 \left(\frac{1}{\beta}\right)^n \right) z^n$$

$$\Rightarrow b_n = k_1 7^n + k_2 \left(\frac{1}{\alpha}\right)^n + k_3 \left(\frac{1}{\beta}\right)^n \quad \text{Ans}$$

Ans-2 To show :- The solution of the recurrence solution $a_k - 7a_{k-1} + 10a_{k-2} = 3^k$ with $a_0 = 0$ and $a_1 = 1$ is $a_k = \frac{8}{3} 2^k - \frac{9}{2} 3^k + \frac{11}{6} 5^r$

firstly we multiply both sides by z^k.

$$a_k z^k - 7a_{k-1} z^k + 10 a_{k-2} z^k = 3^k z^k$$

Then taking summation from $k=2$ to ∞ on both sides,

$$\sum_{k=2}^{\infty} a_k z^k - 7z \sum_{k=2}^{\infty} a_{k-1} z^{k-1} + 10z^2 \sum_{k=2}^{\infty} a_{k-2} z^{k-2} = \sum_{k=2}^{\infty} 3^k z^k$$

we know $A(z) = \sum_{k=0}^{\infty} a_k z^k$

$$\Rightarrow (A(z) - a_0 - a_1 z) - 7z(A(z) - a_0) + 10z^2 A(z) = \frac{1}{1-3z} - 1 - 3z$$

Now putting the values of a_0 and a_1 in above equation we get $A(z) = \dfrac{6z^2+z}{(1-3z)(1-5z)(1-2z)}$

Now using partial fraction,

$$\frac{6z^2+z}{(1-3z)(1-5z)(1-2z)} = \frac{A}{(1-3z)} + \frac{B}{(1-5z)} + \frac{C}{1-2z}$$

$$= \frac{A(1-5z)(1-2z) + B(1-3z)(1-2z) + C(1-3z)(1-5z)}{(1-3z)(1-5z)(1-2z)}$$

$$6z^2+z = A+B+C + z^2(10A+6B+15C) + z(-7A-5B-8C)$$

Now comparing coefficients of LHS and RHS

$A+B+C=0$, $10A+6B+15C=6$ and $7A+5B+8C=-1$

on simultaneously solving these equations we get $A = -\frac{9}{2}$, $B = \frac{11}{6}$ and $C = \frac{8}{3}$

So $A(z) = -\frac{9}{2}\left(\frac{1}{1-3z}\right) + \frac{11}{6}\left(\frac{1}{1-5z}\right) + \frac{8}{3}\left(\frac{1}{1-2z}\right)$

$$= -\frac{9}{2}\sum_{r=0}^{\infty} 3^r z^r + \frac{11}{6}\sum_{r=0}^{\infty} 5^r z^r + \frac{8}{3}\sum_{r=0}^{\infty} 2^r z^r$$

$$= \sum_{r=0}^{\infty}\left(-\frac{9}{2}\times 3^r + 5^r \times \frac{11}{6} + \frac{8}{3}\times 2^r\right) z^r$$

we know $A(z) = \sum_{r=0}^{\infty} a_r z^r$

$\Rightarrow a_r = \frac{8}{3}\times 2^r + 5^r \times \frac{11}{6} - \frac{9}{2}\times 3^r$

Hence proved.

Ans-8 Given :- $b_r - b_{r-1} = K(a_r - b_{r-1})$

where K is a proportional constant.

a) if $a_r = 100\left(\frac{3}{2}\right)^2$, $K = 2$ and $b_0 = 0$

$b_r - b_{r-1} = 2\left(100\left(\frac{9}{4}\right) - b_{r-1}\right)$

$b_r + b_{r-1} = 450$

Now multiply each side by z^r

$b_r z^r + b_{r-1} z^r = 450 z^r$

Now taking $\sum^{\infty}$ on ~~both~~ sides,

$$\sum_{r=1}^{\infty} b_r z^r + \sum_{r=1}^{\infty} b_{r-1} z^r = \sum_{r=1}^{\infty} 450 z^r$$

$\Rightarrow (B(z) - b_0) + zB(z) = \frac{450z}{1-z}$

$B(z) = \frac{450z}{(1-z)(1+z)}$

Now using partial fractions

$\frac{450z}{(1-z)(1+z)} = \frac{A}{1+z} + \frac{B}{1-z}$

$A + B = 0$ (on putting $z=0$) —1)

$-300 = -A + B$ (on putting $z=2$) —2)

on solving ① and ②

$A = -225$ and $B = +225$

$\Rightarrow B(z) = \frac{-225}{1+z} + \frac{225}{1-z}$

$= \sum_{n=0}^{\infty} 225 z^n - \sum_{n=0}^{\infty} 225 (-1)^n z^n$

$= 225 \sum_{n=0}^{\infty} (1-(-1)^n) z^n$

we know $B(z) = \sum_{n=0}^{\infty} b_n z^n$

$\Rightarrow b_n = 225(1-(-1)^n) \; n \geq 0$

Ans

b) for $0 \leq n \leq 9$, $a_n = 100\left(\frac{3}{2}\right)^n$

$b_n - b_{n-1} = 2\left(100\left(\frac{3}{2}\right)^n - b_{n-1}\right)$

$b_n + b_{n-1} = 200\left(\frac{3}{2}\right)^n$

Now multiplying by z^n and taking $\sum_{n=1}^{\infty}$ on both sides

$\sum_{n=1}^{\infty} b_n z^n + \sum_{n=1}^{\infty} b_{n-1} z^n = 200 \sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n z^n$

$B(z) - b_0 + zB(z) = 200\left(\frac{1}{1-\frac{3z}{2}} - 1\right)$

$B(z) = \frac{600z}{(2-3z)(1+z)} = \frac{300z}{(1+z)(1-3z/2)}$

Now using partial fractions,

$\frac{300z}{(1+z)(1-3z/2)} = \frac{A}{1+z} + \frac{B}{1-3z/2}$

on putting $z=0$, $A + B = 0$ —1)

on putting $z=1$, $A - 2B = -300$ —2)

on solving ① and ②

we get $A = -120$ and $B = 120$

$B(z) = \frac{120}{1-3z/2} - \frac{120}{1+z}$

$= 120 \sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n z^n - 120 \sum_{n=0}^{\infty} (-1)^n z^n$

$$= \sum_{r=0}^{\infty} \left(120\left(\frac{3}{2}\right)^r - 120(-1)^r \right) z^r$$

we know $B(z) = \sum_{r=0}^{\infty} b_r z^r$

$$\Rightarrow b_r = 120\left(\left(\frac{3}{2}\right)^r - (-1)^r \right) \text{ Ans}$$

For $r \geq 10$, $a_r = 100\left(\frac{3}{2}\right)^{10}$

$$b_r + b_{r-1} = 200\left(\frac{3}{2}\right)^{10}$$

Now multiplying by z^r and taking Σ on both sides

$$(B(z) - b_0) + zB(z) = 200\left(\frac{3}{2}\right)^{10} \frac{z}{1-z}$$

$$B(z) = 200\left(\frac{3}{2}\right)^{10} \frac{z}{(1+z)(1-z)}$$

Using partial fractions,

$$200\left(\frac{3}{2}\right)^{10} \frac{z}{(1+z)(1-z)} = 200\left(\frac{3}{2}\right)^{10} \left[\frac{A}{1+z} + \frac{B}{1-z} \right]$$

on putting $z=0$, $A+B=0$ —1)

on putting $z=2$, $\frac{A}{3} - B = -\frac{2}{3}$ —2)

On solving 1 and 2

$A = -\frac{1}{2}$, $B = \frac{1}{2}$

$$B(z) = 200\left(\frac{3}{2}\right)^{10} \left[\frac{-1}{2(1+z)} + \frac{1}{2(1-z)} \right]$$

$$= 100\left(\frac{3}{2}\right)^{10} \left[\sum_{r=0}^{\infty} z^r - \sum_{r=0}^{\infty} (-1)^r z^r \right]$$

we know $B(z) = \sum_{r=0}^{\infty} b_r z^r$

$$\Rightarrow b_r = 100\left(\frac{3}{2}\right)^{10} \left[1 - (-1)^r \right]$$

$$\Rightarrow b_r = \begin{cases} 120\left(\left(\frac{3}{2}\right)^r - (-1)^r \right) & 0 \leq r \leq 9 \\ 100\left(\frac{3}{2}\right)^{10} \left[1 - (-1)^r \right] & r \geq 10 \end{cases}$$

Ans.

Ans-4 To find :- The sum of $1^3+2^3+3^3+\cdots+n^3$ using Generating function.

Let $a = (0^3, 1^3, 2^3, 3^3 \cdots r \cdots)$

$A(z)$ of $a = 0+1^3 z + 2^3 \cdot z^2 + 3^3 \cdot z^3 + 4^3 \cdot z^4 + \cdots$

we know $\frac{1}{1-z} = 1+z+z^2+z^3+\cdots$

Diff both sides and multiply by z,

$\frac{z}{(1-z)^2} = z+2z^2+3z^3+4z^4+\cdots$

Again diff and multiply by z on both sides,

$\frac{(1+z)z}{(1-z)^3} = z+2^2 z^2+3^2 z^3+4^2 z^4+\cdots$

Again diff and multiply by z on both sides,

$\frac{z(z^2+4z+1)}{(1-z)^4} = 1+2^3 z^2+3^3 \cdot z^3+4^3 \cdot z^4+\cdots$

$= A(z)$

Also let $B(z) = \frac{1}{1-z}$

Now $C(z)$

$= A(z) \cdot B(z) = (1^3)z + (1^3+2^3)z^2 + (1^3+2^3+3^3)z^3 + \cdots$

$= \frac{z(z^2+4z+1)}{(1-z)^5}$ — (1)

So in order to find sum of cubes, we need coefficient of z^n in $C(z) \Rightarrow$ in (1)

$= (z^3+4z^2+z)(1-z)^{-5}$

we know coefficient of x^r in $(1-x)^{-n}$

$= {}^{r+n-1}C_{n-1}$

$= 1 \times {}^{n+1}C_4 + 4 \cdot {}^{n+2}C_4 + {}^{n+3}C_4$

$$= \frac{(n+1)n(n-1)(n-2)}{24} + 4\cdot\frac{(n+2)(n+1)n(n-1)}{24} + \frac{(n+3)(n+2)(n+1)n}{24}$$

$$= \frac{n(n+1)}{24}\left[(n-1)(n-2) + (n+2)(n-1)\times 4 + (n+3)(n+2)\right]$$

$$= \frac{n(n+1)}{24}\left[n^2-3n+2 + 4n^2+4n-8 + n^2+5n+6\right]$$

$$= \frac{n(n+1)}{24}\left[6n^2+6n\right]$$

$$= \frac{n(n+1)(n)(n+1)}{4} = \left(\frac{n(n+1)}{2}\right)^2$$ Ans

Ans-5 Given : Increase in assets during the r^{th} year

$\rightarrow a_r - a_{r-1}$

Also given $a_r - a_{r-1} = 5(a_{r-1} - a_{r-2})$

$a_r + 5a_{r-2} = 6a_{r-1}$

Now multiply both sides by z^r and taking Σ

$$\sum_{r=2}^{\infty} a_r z^r + 5\sum_{r=2}^{\infty} a_{r-2} z^r = 6\sum_{r=2}^{\infty} a_{r-1} z^r$$

$$(A(z) - a_0 - a_1 z) + 5z^2 A(z) = 6(A(z) - a_0)$$

given $a_0 = 3$ and $a_1 = 7$, putting them in above equation,

$$A(z) - 3 - 7z + 5z^2 A(z) = 6(A(z) - 3)$$

$$A(z) = \frac{3-11z}{5z^2-6z+1}$$

Using partial fraction,

$$\frac{3-11z}{5z^2-6z+1} = \frac{A}{z-1} + \frac{B}{(5z-1)}$$

putting $z=0$, $-A-B=3$

$A+B=-3$ —①

putting $z=-1$, $-\frac{A}{2} - \frac{B}{6} = \frac{14}{12}$ —②

On solving ① and ②, we get $A=-2$ and $B=-1$

$$A(z) = \frac{-2}{z-1} - \frac{1}{5z-1}$$

$$A(z) = \frac{2}{1-z} + \frac{1}{1-5z}$$

$$= 2\sum_{r=0}^{\infty} z^r + \sum_{r=0}^{\infty} 5^r z^r$$

$$= \sum_{r=0}^{\infty} (2+5^r)\, z^r$$

$\Rightarrow a_r = 5^r + 2$ Ans

Ans-6 Given series :- $\sum_{r=0}^{\infty} \left(\frac{5n-4n^3}{9n^3+2}\right)^n$

To find :- whether it is convergent or divergent.

Using Cauchy's Root Test,

Let $\sum u_n = \sum_{r=0}^{\infty} \left(\frac{5n-4n^3}{9n^3+2}\right)^n$

$$L = \lim_{n\to\infty} |u_n|^{1/n}$$

$$L = \lim_{n\to\infty} \left|\frac{5n-4n^3}{9n^3+2}\right|^{n\times 1/n}$$

Now dividing numerator and denominator by n^3

$$L = \lim_{n\to\infty} \left|\frac{\frac{5}{n^2}-4}{9+\frac{2}{n^3}}\right|$$

as $n\to\infty$, $\frac{5}{n}\to 0$ and $\frac{2}{n^3}\to 0$

$\Rightarrow |L| = \frac{4}{9}$

Since $L < \frac{4}{9}$

By Root test, we can see that the series above is convergent.

Ans-7 To show :- $\frac{1}{a\cdot1^2+b} + \frac{2}{a\cdot2^2+b} + \frac{3}{a\cdot3^2+b} + \cdots \frac{n}{an^2+b}$ is divergent.

let $\sum u_n = \frac{1}{a\cdot1^2+b} + \frac{2}{a\cdot2^2+b} + \frac{3}{a\cdot3^2+b} + \cdots \frac{n}{an^2+b}$

$\Rightarrow u_n = \frac{n}{an^2+b}$

let $v_n = \frac{1}{an}$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \left(\frac{n}{an^2+b}\right) \times an = \frac{an^2}{an^2+b} = 1$

As limit is coming a finite, positive, unique non-zero number and also $\sum v_n$ is divergent by cauchy's Integral Test :- let $f(x)$ be a positive monotonic decreasing integrable function, Then if $\int^{\infty} f(x)dx$ is finite and unique then $\sum v_n$ is convergent otherwise divergent

we can see that $\int^{\infty} \frac{dx}{ax} = \frac{1}{a} \log x \Big|_1^{\infty} \to \infty$

$\Rightarrow \sum v_n$ is divergent.

So by Comparison test, $\sum u_n$ is also proved to be divergent.

Ans-8 Given series :-

$$1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \cdot 2\, \gamma(\gamma+1)} x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1 \cdot 2 \cdot 3\, \gamma(\gamma+1)(\gamma+2)} x^3 + \cdots +$$

where α, β, γ and x are positive values.

a) To show :- The series converges for $x<1$ and diverges for $x>1$ using D'Alembert's Ratio Test.

$\Rightarrow$ According to D'Alembert's Ratio Test,

if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$

Then the series is convergent for $l<1$ and divergent for $l>1$.

Now we can see,

$$u_n = \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-2)\,\beta(\beta+1)\cdots(\beta+n-2)\,x^{n-1}}{\gamma(\gamma+1)(\gamma+2)\cdots(\gamma+n-2)\,(n-1)!}$$

Also $$u_{n+1} = \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\,\beta(\beta+1)\cdots(\beta+n-1)\,x^n}{\gamma(\gamma+1)\cdots(\gamma+n-1)\,n!}$$

Now $$\frac{u_{n+1}}{u_n} = \frac{(\alpha+n-1)(\beta+n-1)\,x}{(\gamma+n-1)\,n}$$

Now $$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(\alpha+n-1)(\beta+n-1)\,x}{(\gamma+n-1)\,n}$$

Now dividing numerator and denominator by n^2

$$\lim_{n\to\infty} \frac{\left(\frac{\alpha}{n}+1-\frac{1}{n}\right)\left(\frac{\beta}{n}+1-\frac{1}{n}\right)x}{\left(\frac{\gamma}{n}+1-\frac{1}{n}\right)1} = x$$

$\Rightarrow$ if $x>1$, then series will be divergent and if $x<1$, then series will be convergent by D'Alembert Ratio Test

b) If $x=1$ then the above D'Alembert's Ratio Test gets failed.

So in order to prove the series to be convergent or divergent, we use Raabe's Test.

⇒ Raabe's test states that if-
if $\sum u_n$ be a series of infinite positive terms
then $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = L$

and if $L > 1$, series will be convergent
and if $L \leq 1$, series will be divergent.
As $\frac{u_{n+1}}{u_n} = \frac{(\alpha+n-1)(\beta+n-1)x}{(\gamma+n-1)n}$ (From previous part a)

$$\lim_{n\to\infty} n\left(\frac{(\gamma+n-1)n}{(\alpha+n-1)(\beta+n-1)} - 1\right)$$

$$= n\left(\frac{n\gamma+\cancel{n^2}-\cancel{n}-\alpha\beta-\alpha n+\alpha-n\beta-\cancel{n^2}+\cancel{n}+\beta+n-1}{(\alpha\beta+n\alpha-\alpha+n\beta+n^2-n-\beta-n+1)}\right)$$

$$= \lim_{n\to\infty} n\left(\frac{n(\gamma-\alpha-\beta+1)-\alpha\beta+\alpha+\beta-1}{n^2+n(\alpha+\beta-2)+1-\beta-\alpha+\alpha\beta}\right)$$

$$= \lim_{n\to\infty} \frac{n^2(\gamma-\alpha-\beta+1)+n(\alpha+\beta-1-\alpha\beta)}{n^2+n(\alpha+\beta-2)-1-\beta-\alpha+\alpha\beta}$$

Now dividing numerator and denominator by n^2

$$= \lim_{n\to\infty} \frac{(\gamma-\alpha-\beta+1) + \cancel{\frac{\alpha+\beta-1-\alpha\beta}{n}}^{0}}{1+\cancel{\frac{\alpha+\beta-2}{n}}^{0} + \cancel{\frac{1-\beta-\alpha+\alpha\beta}{n^2}}^{0}}$$

$$= \gamma-\alpha-\beta+1 = L$$

So for series to be convergent
$L > 1$

$\gamma-\alpha-\beta+\cancel{1} > \cancel{1}$

$\Rightarrow \gamma-\alpha-\beta > 0$ Ans

and for series to be divergent, $L \leq 1$

$\gamma-\alpha-\beta+\cancel{1} \leq \cancel{1}$

$\Rightarrow \gamma-\alpha-\beta \leq 0$ Ans

Hence showed.