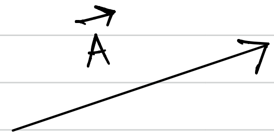
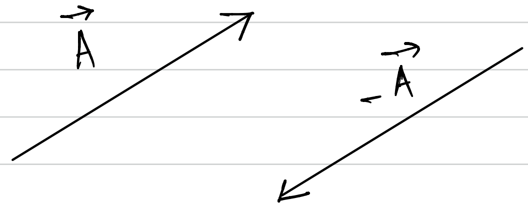
Vector: Has direction & magnitude. Class 1. 22/12/22

Example: Displacement

Written as: Denoted as $\vec{A}$. The magnitude is $|\vec{A}|$

In diagrams: Shown by an arrow. Length of the arrow is proportional to the magnitude & the arrow indicates direction

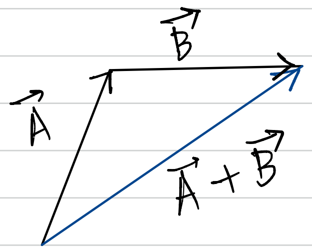




* A vector that is opposite to $\vec{A}$ is $-\vec{A}$

  Location is not important

* Addition:



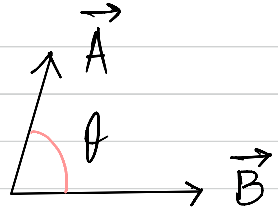
Addition is commutative: $\vec{A}+\vec{B} = \vec{B}+\vec{A}$

* Multiplication by scalar: If multiplied by scalar a then the magnitude increased by factor a, direction unchanged.

Multiplication is distributive : $a(\vec{A}+\vec{B}) = a\vec{A} + a\vec{B}$

* Dot product of two Vectors $\vec{A}$ & $\vec{B}$:

$$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}| \cos\theta$$



Geometrically: $|\vec{A}|$ times the projection of $\vec{B}$ along $\vec{A}$

This is a scalar. Also called Scalar product

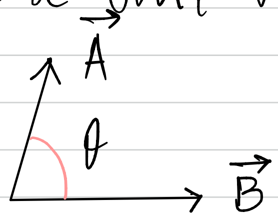
Dot/Scalar product is commutative : $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$

and distributive : $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$

Prove this Exercise!

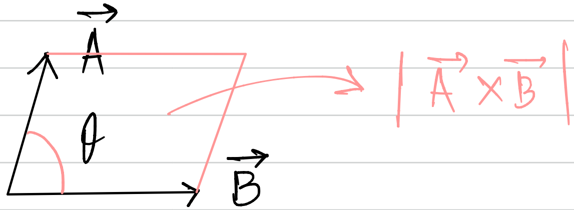
* Cross Product of two vectors $\vec{A}$ & $\vec{B}$: Defined as $\vec{A}\times\vec{B} = |\vec{A}||\vec{B}| \sin\theta \, \hat{n}$

$\hat{n}$ is a unit vector the direction of which is determined by the righ-hand rule.



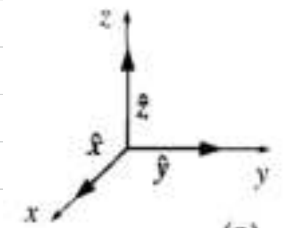
In this figure $\hat{n}$ points onto the page, & $\vec{B}\times\vec{A}$ points out of the page.

Geometrically $|\vec{A} \times \vec{B}|$ is the area of parallelogram generated by $\vec{A}$ & $\vec{B}$



* Component Form: Cartesian coordinate system with unit/basis vectors

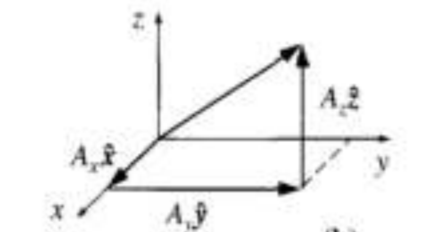
$\hat{x}, \hat{y}, \hat{z}$



(a)

$\hat{x}.\hat{x} = 1$, $\hat{y}.\hat{y} = 1$ & $\hat{z}.\hat{z} = 1$

$\hat{x}.\hat{y} = 0$, $\hat{x}.\hat{z} = 0$ & $\hat{y}.\hat{z} = 0$



(b)

Any vector $\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$

A_x, A_y & A_z are the components of $\vec{A}$ - projection of $\vec{A}$ on to the coordinate axes.

* Addition: $\vec{A} + \vec{B} = (A_x + B_x)\hat{x} + (A_y + B_y)\hat{y} + (A_z + B_z)\hat{z}$

* Multiplication by scalar: $a\vec{A} = (aA_x)\hat{x} + (aA_y)\hat{y} + (aA_z)\hat{z}$

Dot/Scalar Product: $\vec{A}.\vec{B} = A_x B_x + A_y B_y + A_z . B_z$

$$\vec{A}.\vec{A} = A_x^2 + A_y^2 + A_z^2$$

$$|A| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$ is the magnitude of the vector $\vec{A}$

Cross Product: From the definition of cross product we find for the unit vectors

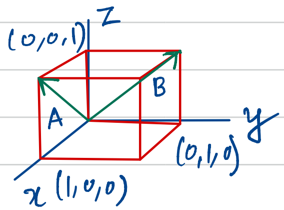
$$\begin{aligned} \hat{\mathbf{x}} \times \hat{\mathbf{x}} = \quad \hat{\mathbf{y}} \times \hat{\mathbf{y}} &= \hat{\mathbf{z}} \times \hat{\mathbf{z}} = 0, \\ \hat{\mathbf{x}} \times \hat{\mathbf{y}} = -\hat{\mathbf{y}} \times \hat{\mathbf{x}} &= \hat{\mathbf{z}}, \\ \hat{\mathbf{y}} \times \hat{\mathbf{z}} = -\hat{\mathbf{z}} \times \hat{\mathbf{y}} &= \hat{\mathbf{x}}, \\ \hat{\mathbf{z}} \times \hat{\mathbf{x}} = -\hat{\mathbf{x}} \times \hat{\mathbf{z}} &= \hat{\mathbf{y}}. \end{aligned}$$

Hence $\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\hat{x} + (A_z B_x - A_x B_z)\hat{y} + (A_x B_y - A_y B_x)\hat{z}$

The following is easy to remember

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}.$$

Find the angle between face diagonal of a cube

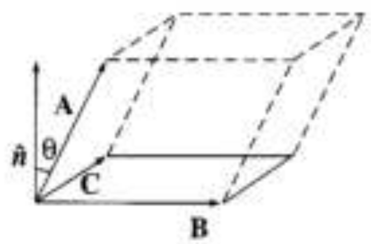


$\vec{A} = 1\hat{x} + 0\hat{y} + 1\hat{z}$; $\vec{B} = 0\hat{x} + 1\hat{y} + 1\hat{z}$

$\Rightarrow \vec{A}.\vec{B} = 1 = |A||B|\cos\theta$

$= \sqrt{2}\sqrt{2}\cos\theta \quad \Rightarrow \theta = \cos^{-1} 1/2 = 60°$

Scalar triple Product: Geometrically $\vec{A}\cdot(\vec{B}\times\vec{C})$ is the volume of a parallelepiped formed by $\vec{A}$, $\vec{B}$ & $\vec{C}$



$$\vec{A}\cdot(\vec{B}\times\vec{C}) = |\vec{A}||\vec{B}\times\vec{C}|\cos\theta$$

$|\vec{B}\times\vec{C}|$ is the area, $|\vec{A}|\cos\theta$ is the height.

It is evident that $\vec{A}\cdot\vec{B}\times\vec{C} = \vec{B}\cdot\vec{C}\times\vec{A} = \vec{C}\cdot\vec{A}\times\vec{B}$

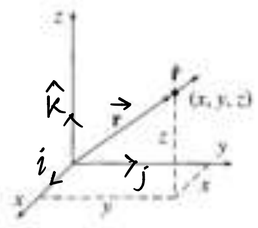
Easy to remember expression for the triple product.

$$\vec{\mathbf{A}}\cdot(\vec{\mathbf{B}}\times\vec{\mathbf{C}}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}.$$

Vector triple Product: $\vec{A}\times\vec{B}\times\vec{C} = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$ this is BAC−CAB rule.

Lecture 2, 24/12/22

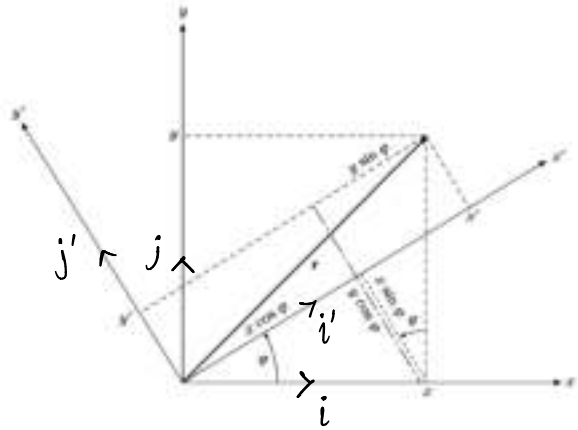
* Transformation of Vectors: Consider the position vector $\vec{r}$ of a point (x, y, z) in the Cartesian coordinate



$$\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$

Consider rotation of the coordinate about the z-axis by angle ϕ

In the new coordinate the position is (x', y', z') and the position vector



$$r' = x'\,\hat{i}' + y'\,\hat{j}' + z'\,\hat{k}'$$

From the diagram we find

$$\left.\begin{aligned} x' &= x\cos\varphi + y\sin\varphi, \\ y' &= -x\sin\varphi + y\cos\varphi. \\ z' &= z \end{aligned}\right\}$$

Any three component object (A_x, A_y, A_z) that transforms like the components of a position vector $\vec{r}$ is defined as a vector. In other words under the above coordinate transformation the components A_x, A_y & A_z transform as

$$A'_x = A_x \cos\varphi + A_y \sin\varphi,$$
$$A'_y = -A_x \sin\varphi + A_y \cos\varphi,$$
$$A'_z = A_z$$

It is better to write the above in matrix form

$$\begin{pmatrix} A'_x \\ A'_y \\ A_z' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$ *

* Magnitude of a vector remains invariant (unchanged) under rotation

$$A'_x = A_x \cos\phi + A_y \sin\phi, \quad A'_y = -A_x \sin\phi + A_y \cos\phi, \quad A'_z = A_z$$

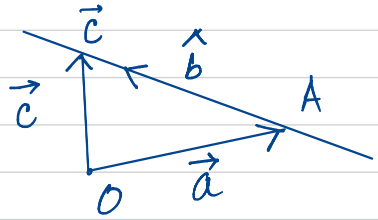
Magnitude of the new vector $A' = \hat{\imath} A'_x + \hat{\jmath} A'_y + \hat{k} A'_z$ is given by

$$|A'| = \sqrt{(A'_x)^2 + (A'_y)^2 + (A'_z)^2}$$

= Substitute these equations

$$= \sqrt{A_x^2 + A_y^2 + A_z^2} = |A|$$ Hence proved.

Some Applications:



Equation of a line: With reference to the the figure, we want to find the equation of the line going through the point A & R.

Position vector of point A is $\vec{a}$, w.r.t. the point O

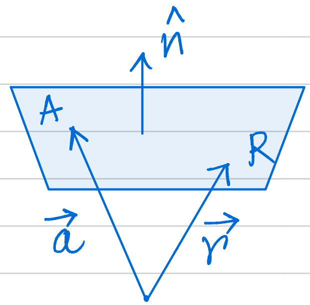
" " " " C is $\vec{c}$ " " " " "

Unit vector along the line is $\hat{b} = \dfrac{\vec{c}-\vec{a}}{|\vec{c}-\vec{a}|}$

For any scalar λ, the vector $\vec{a}+\lambda\hat{b}$ represents the position vector of a point on the line.

Position vector of a general point $\vec{r} = \vec{a} + \lambda\hat{b} = \vec{a} + \lambda \dfrac{\vec{c}-\vec{a}}{|\vec{c}-\vec{a}|}$

Equation of a plane: Can be expressed in many ways.



* With reference to the figure $\hat{n}$ is a unit vector normal to the plane.

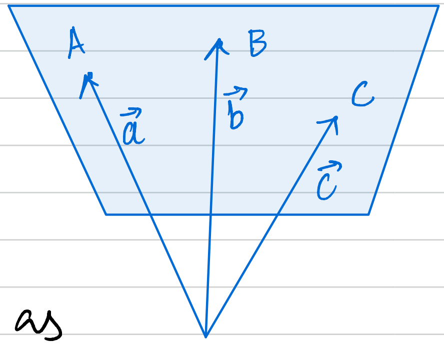
A is any point on the plane with position vector $\vec{a}$.

If $\vec{r}$ is the position vector of any arbitrary point R on the plane then

$$(\vec{r}-\vec{a}).\hat{n} = 0 \Rightarrow \vec{r}.\hat{n} = \vec{a}.\hat{n}$$

represents the equation of the plane.

* w.r.to the figure, A, B & C are three points on a plane and they are not on a line. Their position vectors are $\vec{a}$, $\vec{b}$, & $\vec{c}$.



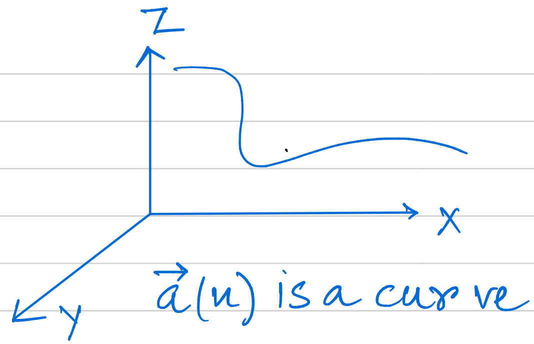
Position vector of any point on the plane can be written as

$$\vec{r} = \vec{a} + \lambda \frac{\vec{b}-\vec{a}}{|\vec{b}-\vec{a}|} + \mu \frac{\vec{c}-\vec{a}}{|\vec{c}-\vec{a}|}$$

this is then the equation of the plane.

Vector Calculus: Differentiation:

A vector $\vec{a}$ that is a function of scalar variable u

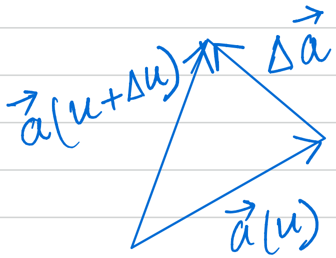


$$\vec{a}(u) = \hat{i}\, a_x(u) + \hat{j}\, a_y(u) + \hat{k}\, a_z(u)$$

Here $a_x(u)$, $a_y(u)$ & $a_z(u)$ are scalar function of u, and represent the component of the vector $\vec{a}$ along the x, y and z coordinate axes. We assume that the variable u is continuous and so are the functions $a_x(u)$, $a_y(u)$, $a_z(u)$

A small change Δu in the variable u leads to a small change in the vector.

$$\Delta\vec{a} = \vec{a}(u+\Delta u) - \vec{a}(u)$$



Derivative of $\vec{a}(u)$ w.r.to u is then defined as

$$\frac{d\vec{a}}{du} = \lim_{\Delta u \to 0} \frac{\vec{a}(u+\Delta u) - \vec{a}(u)}{\Delta u}$$

$$= \hat{i} \lim_{\Delta u \to 0} \frac{a_x(u+\Delta u) - a_x(u)}{\Delta u} + \hat{j} \lim_{\Delta u \to 0} \frac{a_y(u+\Delta u) - a_y(u)}{\Delta u}$$

$$+ \hat{k} \lim_{\Delta u \to 0} \frac{a_z(u+\Delta u) - a_z(u)}{\Delta u}$$

Assuming the limits exist, we can write

$$\frac{d\vec{a}}{du} = \hat{\imath}\,\frac{da_x}{du} + \hat{\jmath}\,\frac{da_y}{du} + \hat{k}\,\frac{da_z}{du}$$

This means that, at least in the Cartesian coordinate we can do the derivative component by component

* Note that the derivative $d\vec{a}/du$ of a vector $\vec{a}(u)$ is a vector that is not in general parallel to $\vec{a}(u)$

Example! Position of a particle in a Cartesian coordinate is

$$\vec{r} = \hat{\imath}\,x(t) + \hat{\jmath}\,y(t) + \hat{k}\,z(t)$$

the velocity of the particle is given by

$$\vec{v}(t) = \hat{\imath}\,\frac{dx(t)}{dt} + \hat{\jmath}\,\frac{dy(t)}{dt} + \hat{k}\,\frac{dz(t)}{dt}$$

and the acceleration is

$$\vec{a}(t) = \hat{\imath}\,\frac{d^2x(t)}{dt^2} + \hat{\jmath}\,\frac{d^2y(t)}{dt^2} + \hat{k}\,\frac{d^2z(t)}{dt^2}$$

Suppose $\vec{r} = 2t^2\hat{\imath} + (3t-2)\hat{\jmath} + (3t^2-1)\hat{k}$

then $\vec{v}(t) = 4t\hat{i} + 3\hat{j} + 6t\hat{k}$

$\vec{a}(t) = 4\hat{i} + 6\hat{k}$

Composite Vectors:

$$\frac{d}{du}(\phi\vec{a}) = \phi\frac{d\vec{a}}{du} + \frac{d\phi}{du}\vec{a} \quad ; \quad \phi \text{ is a scalar}$$

$$\frac{d}{du}(\vec{a}\cdot\vec{b}) = \vec{a}\cdot\frac{d\vec{b}}{du} + \frac{d\vec{a}}{du}\cdot\vec{b}$$

$$\frac{d}{du}(\vec{a}\times\vec{b}) = \vec{a}\times\frac{d\vec{b}}{du} + \frac{d\vec{a}}{du}\times\vec{b}$$

In this case the ordering is important.

* **Differential of a Vector:** In the previous example $\Delta\vec{a}$ is the change in $\vec{a}(u)$ due to change du in the variable u. In the limit $\Delta u \to 0$ the change $\Delta\vec{a}$ is infinitesimally small and is called the differential $d\vec{a}$

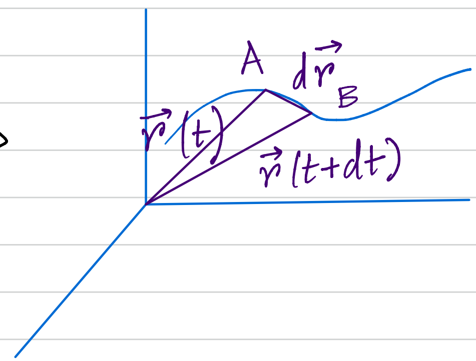
$$d\vec{a} = \frac{d\vec{a}}{du}\,du \quad \to \text{this is also a vector.}$$

Example: If $d\vec{r}$ is infinitesimal change in position vector $\vec{r}$ then

$$d\vec{r} = \frac{d\vec{r}}{dt}dt = \vec{v}\,dt \quad ; \quad \vec{v}(t) \to \text{velocity}$$

Example!

Geometrical Meaning; Tangent Vector: Derivative of a vector is a vector that is tangent. Consider a particle that is moving in a Cartesian coordinate. The position vector of the particle at any time t is $\vec{r}(t)$



Position vector at t and $t+dt$ are $\vec{r}(t)$ and $\vec{r}(t+dt)$. Here the displacement vector $d\vec{r}$ is tangent to the curve at $\vec{r}(t)$. Hence the derivative $d\vec{r}/dt$ is tangent to the curve at $\vec{r}(t)$.

* Direction of the tangent vector $d\vec{r}/dt$ is independent of the parameter of the curve (in this case t)
* Magnitude of $d\vec{r}/dt$ depends on the parameter t.

Arc Length: Tangent vector can be used to calculate the length of a curve. In the above example, the distance between the nearby points A & B is

$$ds = |\vec{r}(t+dt) - \vec{r}(t)| = |d\vec{r}| + O(|d\vec{r}|^2) + \cdots$$

{ using Taylor series expansion

Neglecting the terms of the order $|d\vec{r}|^2$

$$ds = |d\vec{r}| = \pm\left|\frac{d\vec{r}}{dt}dt\right| = \pm\left|\frac{d\vec{r}}{dt}\right|dt$$

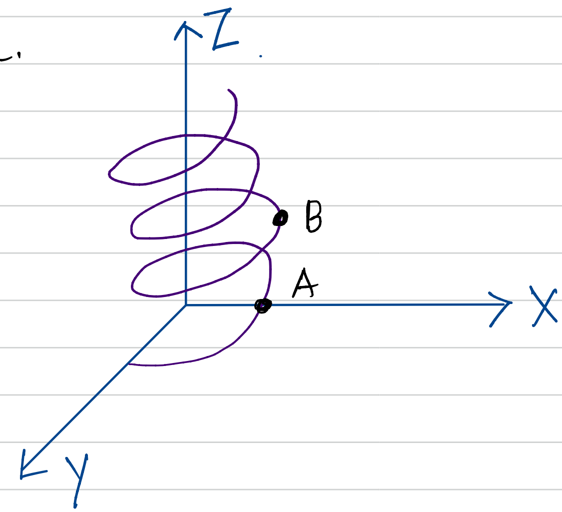
I put + sign for distance measured in the direction of increasing t and − sign for the direction of decreasing t.

So the total length of the curve from time t_1 & t_2 is

$$s = \pm\int_{t_1}^{t_2}\left|\frac{d\vec{r}}{dt}\right|dt \quad \begin{cases} + \text{ sign for } t_2 > t_1 \\ - \text{ sign for } t_1 > t_2 \end{cases}$$

Example: The position vector of a particle in a Cartesian coordinate is given by $\vec{r} = \hat{i}\cos t + \hat{j}\sin t + \hat{k}\,t$ where t is the time.

The curve that the particle follows is a helix.



$$ds = \left|\frac{d\vec{r}}{dt}\right|dt$$

$$= \left|-\hat{i}\sin t + \hat{j}\cos t + \hat{k}\right|dt$$

$$= \sqrt{2}\,dt$$

So the length from time $t_1 = 0$ and $t_2 = t$ is $s = \int_0^{t_2}\sqrt{2}dt = \sqrt{2}t_2 = 2t$

For $t_1 = 0$ and $t_2 = 2\pi$ the particle makes one full rotation - shown by points A & B on the plot. Total arc length is given by $s = \int_0^{2\pi} \sqrt{2}\, dt = 2\sqrt{2}\pi$.

Lecture 3, 27/12/22

For $t_1 = 0$ and $t_2 = 2\pi$ the particle makes one full rotation - shown by points A & B on the plot. Total arc length is given by $s = \int_0^{2\pi} \sqrt{2}\,dt = 2\sqrt{2}\pi$.

Integration of Vectors: Two points to remember

1. The integral and the integrand have the same nature
2. For a indefinite integral, the constant of integration has the same nature as the integral.

For indefinite integral

$$\int \vec{a}(u)\,du = \vec{A}(u) + \vec{b}$$

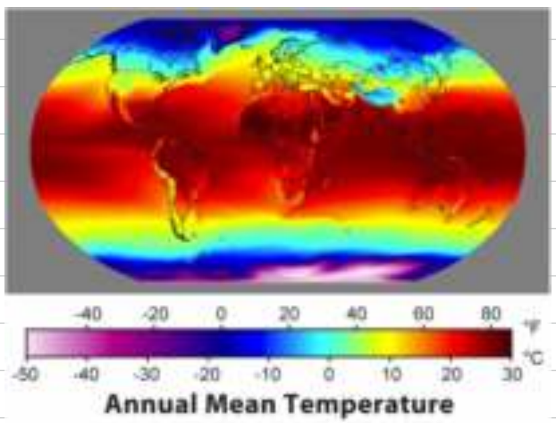
$\left\{ \begin{array}{l} \text{Assuming that} \\ \vec{a}(u) = d\vec{A}(u)/u \end{array} \right.$

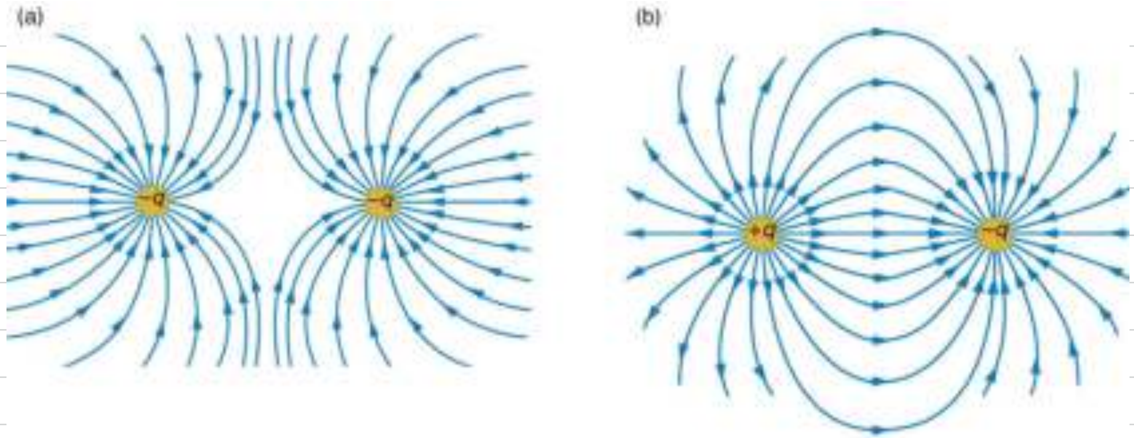
For definite integral

$$\int_{u_1}^{u_2} \vec{a}(u)\,du = \vec{A}(u_2) - \vec{A}(u_1)$$

Scalar and Vector Field: A scalar field, that is defined in a given region of space, is a continuous function that gives a number at each point in that region. We denote a scalar field by $\phi(x, y, z)$ which is a function of the coordinates x, y, z. Example of scalar field is temperature $T(x,y,z)$ in a given region, ~~say~~ inside a room, surface of earth etc.

A vector field is a continuous function that is defined in a given region of space, and associates a vector quantity at each point in that region. We denote a vector field by $\vec{v}(x,y,z)$. Example of vector field is velocity vector field in a flowing river or ocean, electric field created by a static charge, magnetic field of a magnet or flowing electric current, etc.





Line Integral in Scalar Field: There is a scalar field $\phi(x,y,z)$ and we want to integrate $\phi(x,y,z)$ along a line or a curve that is parametrized by a scalar variable u. The line/curve is defined by the position vector

$$\vec{r}(u) = \hat{i}\, r_x(u) + \hat{j}\, r_y(u) + \hat{k}\, r_z(u)$$

The integral is given by

$$\int \phi(x,y,z)\,|d\vec{r}(u)| = \int \phi(x,y,z)\left|\frac{d\vec{r}(u)}{du}\right| du = \int \phi(x,y,z)\,\frac{ds}{du}\,du = \int \phi(x,y,z)\,ds$$

see class 2, 24/12/22

Here $|d\vec{r}(u)/du| = ds/du$ is the length of the tangent vector to the curve.

Example: You want to integrate the temperature function on the surface of the earth along the coast line of India.

Example: Find the line integral of $\phi(x,y,z) = (x^2 + y^2 + z)$ over a curve defined by $\vec{r}(u) = \hat{i}\cos u + \hat{j}\sin u + u$, for $0 < u < 2\pi$

Soln: The parameter of the curve is u. First we express the integrand in terms of u: $\phi(x,y,z) = x^2+y^2+z = (\cos u)^2 + (\sin u)^2 + u = 1+u$.

$$\frac{ds}{du} = \left|\frac{d\vec{r}}{du}\right| = \sqrt{(-\sin u)^2+(\cos u)^2+1} = \sqrt{2}.$$

Hence substituting in the formula

$$\int (x^2+y^2+z)\,ds = \int_0^{2\pi} (1+u)\sqrt{2}\,du = \left[\sqrt{2}u + \frac{\sqrt{2}u^2}{2}\right]_0^{2\pi} = 2\pi\sqrt{2}(1+\pi)$$

Example: Consider a semicircle of radius a on X-Y plane. Find the length.

Soln: The semi-circle can be parametrized by $x = a\cos\theta$, $y = a\sin\theta$ where $0<\theta<\pi$.

The length is $\int 1\,|d\vec{r}(\theta)| = \int 1\,ds = \int_0^{\pi} 1\,\frac{ds}{d\theta}\,d\theta$

Now $\vec{r} = \hat{i}\,a\cos\theta + \hat{j}\,a\sin\theta \Rightarrow \frac{ds}{d\theta} = \sqrt{a^2\sin^2\theta + a^2\cos^2\theta} = a$

Hence length $= \int_0^{\pi} a\,d\theta = a\pi$.

Line Integral in Vector Field: Consider a force field $\vec{F}(x,y,z)$. The line integral of $\vec{F}(x,y,z)$ along a line $\vec{r}(u)$, where u is a scalar that parametrizes the line is given by

$$\int \vec{F}(x,y,z)\cdot d\vec{r}(u) = \int \vec{F}\cdot \frac{d\vec{r}}{du}\, du$$

Example where such line integral may arise: You want to calculate the work done to move a point charge q in a given electric potential along a given path.

Example: Let $\vec{F} = \hat{i}\, xe^{y} + \hat{j}\, z^2 + \hat{k}\, xy$. The line along which the integration to be performed is $\vec{r}(u) = \hat{i}\, u + \hat{j}\, u^2 + \hat{k}\, u^3$ for $0<u<1$.

Soln: The force field along the curve is $\vec{F}(\vec{r}(u)) = \hat{i}\, ue^{u^2} + \hat{j}\, u^6 + \hat{k}\, u^3$

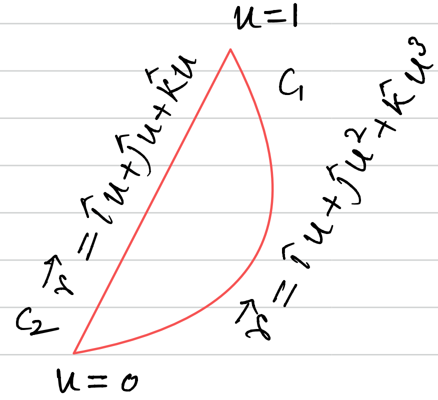
So the integral is

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}\cdot\frac{d\vec{r}}{du}\, du = \int_0^1 \left(\hat{i}\, ue^{u^2} + \hat{j}\, u^6 + \hat{k}\, u^3\right)\cdot\left(\hat{i} + \hat{j}\, 2u + 3u^2\right) du$$

$$= \int_0^1 \left(ue^{u^2} + 2u^7 + 3u^5\right) du = \frac{1}{4}(1+3e)$$

Example: We do the previous integral again but for curve

$$\vec{r}(u) = \hat{i}u + \hat{j}u + \hat{k}u \quad \text{for } 0<u<1$$



The answer is given by

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^1 \left(\hat{i}\, ue^u + \hat{j}\, u^2 + \hat{k}\, u^2\right) \cdot \left(\hat{i} + \hat{j} + \hat{k}\right) du$$

$$= \int_0^1 \left(ue^u + 2u^2\right) = 5/3.$$

* Note that the integration result is different for different curves. This is because in case of vector fields the integration depends on the path along which the integration is done.

* Are there vector fields in which the line integral is independent of the path and only depends on the end points? - To answer this question we need the concepts of (i) partial derivatives (ii) gradiant of a scalar field.

Partial Derivatives: A function $f = f(x, y)$ is function of x & y.
Partial Derivative w.r.to x is defined as

$$\frac{\partial f}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x, y)}{\delta x}$$

A similar definition for $\partial f/\partial y$.

Example: $f(x, y) = 2x^3y^2 + y^2$ Then we have

$$\frac{\partial f}{\partial x} = 6x^2y^2 \quad ; \quad \frac{\partial f}{\partial y} = 4x^3y + 2y$$

2nd derivatives: $$\frac{\partial^2 f}{\partial x^2} = 12xy^2 \quad ; \quad \frac{\partial^2 f}{\partial y^2} = 4x^3 + 2$$

Mixed derivatives: $$\frac{\partial^2 f}{\partial y \partial x} = 12x^2y \quad ; \quad \frac{\partial^2 f}{\partial x \partial y} = 12x^2y = \frac{\partial^2 f}{\partial y \partial x}$$

This is generalized to functions of more than two variables.

Chain Rule: Let a function $f = f(x, y)$. The total differential of the function is given by

$$df = f(x+dx, y+dy) - f(x, y)$$

using partial derivative we can write

$$df = [f(x+dx, y+dy) - f(x, y+dy)] + [f(x, y+dy) - f(x,y)]$$

$$= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$\partial f/\partial x$ & $\partial f/\partial y$ evaluated at some point between x & $x+dx$, y & $y+dy$

If $f = f(x(u), y(u))$ then the derivative w.r.t. to u can be written as

$$\frac{df}{du} = \frac{\partial f}{\partial x}\frac{dx}{du} + \frac{\partial f}{\partial y}\frac{dy}{du}$$

If x & y depend on two parameters r, θ, ie for $f = f(x(r,\theta), y(r,\theta))$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} \quad \& \quad \frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}$$

Example: Given that $\frac{dx}{dt} = u(t, x(t))$. Determine $\frac{d^2x}{dt^2}$ in terms of u and its partial derivatives.

Soln: $$\frac{d^2x}{dt^2} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}$$

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Gradiant of a scalar field, $\vec{\nabla}$: Consider a scalar function $\phi = \phi(x,y,z)$ and we want to know the derivative at a point say x, y, z. Example: let say scalar function $T(x,y,z)$ gives the temperature in a room. The derivative is supposed to tell how fast the temperature changes at a given point — obviously it changes differently in different directions — it means that the derivative of the scalar function must be a vector.

Using partial derivative

$$d\phi(x,y,z) = \frac{\partial \phi}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

$$= \left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right) \cdot \left(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz\right)$$

$$= \left(\vec{\nabla}\phi\right) \cdot d\vec{r} \quad \left\{ \vec{r} = \hat{i}x + \hat{j}y + \hat{k}z \text{ is the position vector at } x, y, z \right.$$

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$ is the gradiant operator

$\vec{\nabla}\phi$ is called the gradiant of scalar field ϕ.

$\vec{\nabla}$ is called "del" or "nabla"

Geometrical Interpretation of gradiant: For a change in $\vec{r}$ from $\vec{r}$ to $d\vec{r}$

$$d\phi = \vec{\nabla}\phi \cdot d\vec{r} = |\vec{\nabla}\phi||d\vec{r}|\cos\theta$$

Fix $|d\vec{r}|$ and search in various direction to find the maximum change.

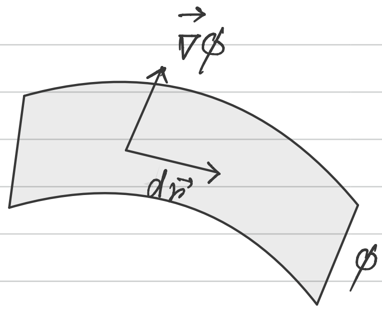
Maximum change happens when $\theta = 0$.

$\Rightarrow$ The gradiant $\vec{\nabla}\phi$ points to the maximum change of ϕ

The magnitude $|\vec{\nabla}\phi|$ gives the rate of change along the maximum direction.

* **Unit vector normal to a surface:** Let us consider a surface that is given by a scalar function $\phi \equiv \phi(x,y,z)$. From above

$$d\phi = \vec{\nabla}\phi \cdot d\vec{r}$$



If $d\vec{r}$ is the infinitesimal separation between two points on the surface then $d\phi = 0$ { on the surface ϕ does not change

$\Rightarrow \vec{\nabla}\phi \cdot d\vec{r}$ or $\vec{\nabla}\phi$ is perpendicular to $d\vec{r}$ ie, the surface at $\vec{r}$

$$\text{Unit vector } \hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}.$$

Example: Given the magnitude of position vector $|\vec{r}| = (x^2+y^2+z^2)^{1/2}$ find the gradiant

Soln:
$$\vec{\nabla}|\vec{r}| = \hat{i}\frac{\partial|\vec{r}|}{\partial x} + \hat{j}\frac{\partial|\vec{r}|}{\partial y} + \hat{k}\frac{\partial|\vec{r}|}{\partial z}$$

$$= \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}}\hat{i} + \frac{1}{2}\frac{2y}{\sqrt{x^2+y^2+z^2}}\hat{j} + \frac{1}{2}\frac{2z}{\sqrt{x^2+y^2+z^2}}\hat{k}$$

$$= \frac{\hat{i}x+\hat{j}y+\hat{k}z}{\sqrt{x^2+y^2+z^2}} = \frac{\vec{r}}{|\vec{r}|} = \hat{r}$$

This example tells you that distance increases most rapidly along the radial direction

Example: For the function $\phi = x^2y + yz$ at the point $(1,2,-1)$, find the rate of change with distance in the direction $\vec{a} = \hat{i}+2\hat{j}+3\hat{k}$. At this point what is the greatest rate of change and in which direction does it occour?

Soln: At $(1,2,-1)$ the gradiant is

$$\vec{\nabla}\phi\Big|_{(1,2,-1)} = \left(2xy\,\hat{i} + (x^2+z)\hat{j} + yk\right)\Big|_{(1,2,-1)}$$
$$= 4\hat{i} + 2\hat{k}$$

Unit vector along $\vec{a}$ is $\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{\sqrt{14}}(\hat{i} + 2\hat{j} + 3\hat{k})$

Hence the rate of change

$$\frac{d\phi}{ds} = \vec{\nabla}\phi \cdot \hat{a} = \frac{1}{\sqrt{14}}(4+6) = \frac{10}{\sqrt{14}}.$$

The greatest rate is along the direction of $\vec{\nabla}\varphi$ and the value $|\vec{\nabla}\varphi| = \sqrt{20}$

* How scalar field changes on a curve: ϕ is a scalar field and $\vec{r}(u)$ is a curve.

The change of the field along the curve is given by

$$\frac{d\phi}{du} = \frac{\partial\phi}{\partial x}\cdot\frac{dx}{du} + \frac{\partial\phi}{\partial y}\frac{d\varphi}{dy} + \frac{\partial\varphi}{\partial z}\cdot\frac{dz}{du}$$

$$= \vec{\nabla}\phi \cdot \frac{d\vec{r}}{du}.$$

* Exact differential: $d\phi = \vec{\nabla}\phi \cdot d\vec{r}$ at any point $\vec{r}$

* Back to Integration: Consider a vector field $\vec{F} = \vec{\nabla}\phi$ where ϕ is scalar.

The line integral of $\vec{F}$ over a curve curve $C: \vec{r} \equiv \vec{r}(u)$; $u_1 < u < u_2$

$$\int \vec{F} \cdot d\vec{r} = \int_C \vec{\nabla}\phi \cdot d\vec{r}$$

$$= \int_C \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right) \cdot (\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= \int_C \left(\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz\right)$$

$$= \int_{u_1}^{u_2} \left(\frac{\partial \phi}{\partial x}\frac{dx}{du} + \frac{\partial \phi}{\partial y} \cdot \frac{dy}{du} + \frac{\partial \phi}{\partial z} \cdot \frac{dz}{du}\right) du$$

$$= \int_{u_1}^{u_2} \frac{d}{du}\phi(\vec{r}(u))\, du = \phi(\vec{r}(u))\Big|_{u_1}^{u_2}$$

$$= \phi(\vec{r}(u_2)) - \phi(\vec{r}(u_1))$$

So the result only depends on the end point and is independent of the path of the integration. If the curve is closed ie $u_1 = u_2$ then $\oint \vec{F} \cdot d\vec{r} = 0$

The vector $\vec{F}$ that is equal to the gradiant of a scalar field is called a conservative vector field.

* In a conservative vector field the line integral around a closed curve vanishes

* The opposite is also true: if the line integral in a vector field is independent of path then the field can be expressed as the gradiant of a scalar field.

* To check if a vector field can be constructed from a scalar:

Let $\vec{F} = F_x \hat{i} + \hat{j} F_y + \hat{k} F_z$ and assume that $\vec{F} = \vec{\nabla}\phi$

Then

$$F_x = \frac{\partial \phi}{\partial x}, \quad F_y = \partial\phi/\partial y, \quad F_z = \partial\phi/\partial z$$

For suitable well behaved function ϕ, the order of partial derivative does not matter

$$\Rightarrow \frac{\partial \phi}{\partial x \partial y} = \frac{\partial \phi}{\partial y \partial x} \quad \Rightarrow \frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x} \quad \text{or} \quad \frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i} \begin{cases} x_i, x_j = x, y, z \\ F_i, F_j = F_x, F_y, F_z \\ \text{for } i, j = 1, 2, 3 \end{cases}$$

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Example: Let $\vec{F} = \hat{i}\, 3x^2 y \sin z + \hat{j}\, x^3 \sin z + \hat{k}\, x^3 y \cos z$. Is this conservative?

Soln: $\dfrac{\partial F_2}{\partial x_1} = \dfrac{\partial F_y}{\partial x} = 3x^2 \sin z$; $\dfrac{\partial F_1}{\partial x_2} = \dfrac{\partial F_x}{\partial y} = 3x^2 \sin z$

Similarly $\dfrac{\partial F_3}{\partial x_1} = 3x^2 y \cos z = \dfrac{\partial F_1}{\partial x_3}$

& $\dfrac{\partial F_3}{\partial x_2} = x^3 \cos z = \dfrac{\partial F_2}{\partial x_3}$

Hence it is conservative.

* $\vec{\nabla}$ operator: The gradiant $\vec{\nabla} = \hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}$ looks like a vector but is not a vector in the usual sense. It is a vector operator.

* Divergence of a vector: The divergence of a vector $\vec{V} = \hat{i}\, v_x + \hat{j}\, v_y + \hat{k}\, v_z$ is given by

$$\vec{\nabla}\cdot\vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}\, v_x + \hat{j}\, v_y + \hat{k}\, v_z\right)$$

$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Example: For a force field $\vec{F} = \vec{r}/|\vec{r}|^3$ (for all $\vec{r} \neq 0$) find $\vec{\nabla}\cdot\vec{F}$.

Soln: We can write $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$$= \hat{i}\,\frac{x}{(x^2+y^2+z^2)^{3/2}} + \hat{j}\,\frac{y}{(x^2+y^2+z^2)^{3/2}} + \hat{k}\,\frac{z}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial F_1}{\partial x} = \frac{(x^2+y^2+z^2)^{3/2} - 3x^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3} = \frac{1}{|\vec{r}|^3} - \frac{3x^2}{|\vec{r}|^5}$$

We can similarly calculate $\partial F_2/\partial y$ & $\partial F_3/\partial z$ and the results are easy to guess. Hence

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{3}{|\vec{r}|^3} - \frac{3(x^2+y^2+z^2)}{|\vec{r}|^5} = 0$$

The $\vec{\nabla}\cdot\vec{F} = 0$ for all $\vec{r}$ apart from $\vec{r} = 0$, where it diverges. Meaning of this result is little tricky – we may come back to this later.

Examples: Consider two vectors $\vec{v} = \hat{k}$, $\vec{v} = z\hat{k}$, $v = \frac{1}{z}\hat{k}$ Calculate the divergences.

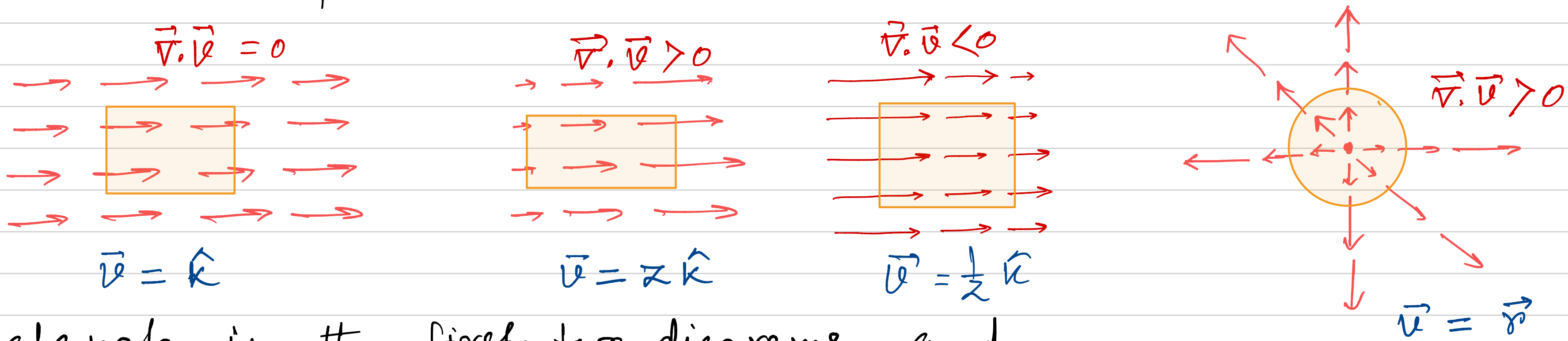
Soln: First : $\vec{\nabla}\cdot\vec{v} = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(1) = 0$

2nd : $\vec{\nabla}\cdot v = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(z) = 1$; For third : $\vec{\nabla}\cdot\vec{v} < 0$

Example: Let $\vec{v} = \hat{i}x + \hat{j}y + \hat{k}z$. Then calculate the divergence

$$\vec{\nabla}\cdot\vec{v} = 3$$

Physical Interpretation: In the previous examples we have considered four vectors. These are plotted below



The rectangle in the first two diagrams, and the circle in the third diagram corresponds to infinitesimal volumes. The vectors, which are denoted by the arrows indicate flow of some quantity - let say water. The divergence correspond to the rate (w.r.to coordinates) of flow of water through the volume.

* Concrete Interpretation: See Arfken.

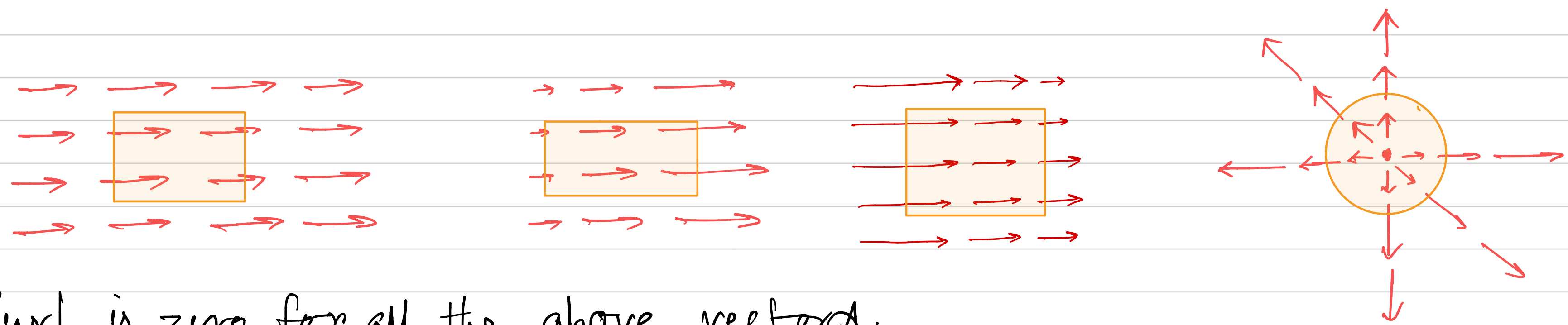
* Solenoidal Vector: Any vector, say $\vec{B}$, for which $\vec{\nabla}\cdot\vec{B} = 0$.

Curl of a Vector: Defined by

$$\vec{\nabla}\times\vec{V} = \hat{\imath}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \hat{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

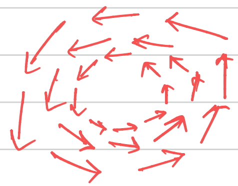
Interpretation: The curl of a vector is a measure of how much the vector curls around a point. For example, the following vectors have zero curl

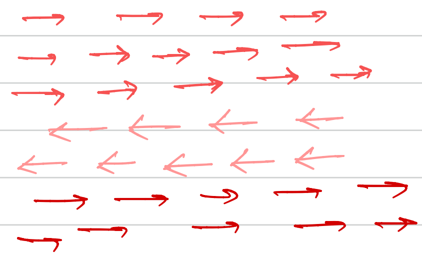


Curl is zero for all the above vectors.

Following Vector fields have non-zero curl

* Vector $\vec{v}$ for which $\vec{\nabla} \times \vec{v} = 0$ is called irrotational





Example: Consider flow of water in a river. If you drop a paper-boat and it starts to circle or rolate in the water then the velocity vector of the water has curl.

* Example: $\vec{F} = \hat{i}y - \hat{j}x$. Calculate the divergence and curl.

Soln: $\vec{\nabla} \cdot \vec{F} = 0$ and $\vec{\nabla} \times \vec{F} = -2\hat{k}$. So the vector swirl. The direction of curl is perpendicular to the plane of the swirl.

* Properties of gradiant, divergence and curl:

Linear differential operator properties: $\vec{\nabla}(a\phi + \psi) = a\vec{\nabla}\phi + \nabla\psi$; $a \to$ constant

$$\vec{\nabla} \cdot (a\vec{F} + \vec{G}) = a\,\vec{\nabla} \cdot \vec{F} + \vec{\nabla} \cdot \vec{G} \quad ; \quad \vec{\nabla} \times (a\vec{F} + G) = a\vec{\nabla} \times \vec{F} + \vec{\nabla} \times \vec{G}$$

Leibeniz Properties:

$$\vec{\nabla}(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$$

$$\nabla\cdot(\phi\vec{F}) = (\vec{\nabla}\phi)\cdot\vec{F} + \phi(\vec{\nabla}\cdot\vec{F})$$

$$\vec{\nabla}\times(\phi\vec{F}) = (\vec{\nabla}\phi)\times\vec{F} + \phi(\vec{\nabla}\times\vec{F})$$

$$\vec{\nabla}\cdot(\vec{F}\times\vec{G}) = (\vec{\nabla}\times\vec{F})\cdot\vec{G} - \vec{F}\cdot(\vec{\nabla}\times\vec{G})$$

* More identities involving two operations of $\vec{\nabla}$ can be found in books

There are also two other properties for $\vec{\nabla}(\vec{F}\cdot\vec{G})$ & $\vec{\nabla}\times(\vec{F}\times\vec{G})$.

Theorem: A vector field that is conservative is irrotational. In other words if ϕ is a scalar field & $\vec{F}$ is a vector field then

$\vec{F} = \vec{\nabla}\phi$ implies that $\vec{\nabla}\times\vec{F} = 0$ and vice versa

Proof (incomplete): For $\vec{F} = \vec{\nabla}\phi$ the components are

$$F_x = \partial\phi/\partial x, \quad F_y = \partial\phi/\partial y \quad \& \quad F_z = \partial\phi/\partial z.$$

Now

$$\vec{\nabla}\times\vec{F} = \hat{\imath}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \hat{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

$$= \hat{\imath}\left(\frac{\partial\phi}{\partial y\partial z} - \frac{\partial\phi}{\partial z\partial y}\right) + \hat{\jmath}\left(\frac{\partial\phi}{\partial z\partial x} - \frac{\partial\phi}{\partial x\partial z}\right) + \hat{k}\left(\frac{\partial\phi}{\partial x\partial y} - \frac{\partial\phi}{\partial y\partial x}\right) = 0$$

Theorem: Any divergence free field can be written as a curl of another vector field ie if $\vec{\nabla}.\vec{F}=0$ then $\vec{F}=\vec{\nabla}\times\vec{A}$ and vice versa.

Proof: Same as the previous theorem (incomplete) Exercise

* The previous two theorem imply

1. Curl of gradiant is zero: $\vec{\nabla}\times(\vec{\nabla}\phi)=0$
2. Divergence of a curl is zero: $\vec{\nabla}.(\vec{\nabla}\times\vec{A})=0$

* The Laplacian: The operator $\vec{\nabla}$ is first order derivative operator w.r.to the coordinates. By operating $\vec{\nabla}$ twice. second-order derivative can be obtained.

For example

$$\vec{\nabla}.\vec{\nabla}\phi = \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2} \equiv \nabla^2\phi$$

$$\nabla^2 = \vec{\nabla}.\vec{\nabla} = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$ is Called the Laplacian.

* It can be shown that there are just two types of fundamental second

derivative - the Laplacian and gradiant-of-divergence (ie $\vec{\nabla}(\nabla\cdot\vec{V})$)

Here note that $\vec{\nabla}(\vec{\nabla}\cdot\vec{V}) \neq \nabla^2\vec{V}$

* Identities: Two important identities involving Laplacian are given below

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{V}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{V}) - \nabla^2 V$$

Fore more identities see Arfken or Reily-Hobson-Bence.

(ii) If the unit vectors vary as the values of the coordinates change (i.e. are not constant in direction throughout the whole space) then the derivatives of these vectors appear as contributions to $\nabla^2\mathbf{a}$.

Cartesian coordinates are an example of the first case in which each component satisfies $(\nabla^2\mathbf{a})_i = \nabla^2 a_i$. In this case (10.41) can be applied to each component separately:

$$[\nabla \times (\nabla \times \mathbf{a})]_i = [\nabla(\nabla \cdot \mathbf{a})]_i - \nabla^2 a_i. \tag{10.42}$$

However, cylindrical and spherical polar coordinates come in the second class. For them (10.41) is still true, but the further step to (10.42) cannot be made.

More complicated vector operator relations may be proved using the relations given above.

►*Show that*

$$\nabla \cdot (\nabla\phi \times \nabla\psi) = 0,$$

where ϕ and ψ are scalar fields.

From the previous section we have

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b}).$$

If we let $\mathbf{a} = \nabla\phi$ and $\mathbf{b} = \nabla\psi$ then we obtain

$$\nabla \cdot (\nabla\phi \times \nabla\psi) = \nabla\psi \cdot (\nabla \times \nabla\phi) - \nabla\phi \cdot (\nabla \times \nabla\psi) = 0, \tag{10.43}$$

since $\nabla \times \nabla\phi = 0 = \nabla \times \nabla\psi$, from (10.37). ◄

10.9 Cylindrical and spherical polar coordinates

The operators we have discussed in this chapter, i.e. grad, div, curl and ∇^2, have all been defined in terms of Cartesian coordinates, but for many physical situations other coordinate systems are more natural. For example, many systems, such as an isolated charge in space, have spherical symmetry and spherical polar coordinates would be the obvious choice. For axisymmetric systems, such as fluid flow in a pipe, cylindrical polar coordinates are the natural choice. The physical laws governing the behaviour of the systems are often expressed in terms of the vector operators we have been discussing, and so it is necessary to be able to express these operators in these other, non-Cartesian, coordinates. We first consider the two most common non-Cartesian coordinate systems, i.e. cylindrical and spherical polars, and go on to discuss general curvilinear coordinates in the next section.

10.9.1 Cylindrical polar coordinates

As shown in figure 10.7, the position of a point in space P having Cartesian coordinates x, y, z may be expressed in terms of cylindrical polar coordinates

ρ, ϕ, z, where

$$x = \rho\cos\phi, \qquad y = \rho\sin\phi, \qquad z = z, \tag{10.44}$$

and $\rho \geq 0$, $0 \leq \phi < 2\pi$ and $-\infty < z < \infty$. The position vector of P may therefore be written

$$\mathbf{r} = \rho\cos\phi\,\mathbf{i} + \rho\sin\phi\,\mathbf{j} + z\,\mathbf{k}. \tag{10.45}$$

If we take the partial derivatives of $\mathbf{r}$ with respect to ρ, ϕ and z respectively then we obtain the three vectors

$$\mathbf{e}_\rho = \frac{\partial\mathbf{r}}{\partial\rho} = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}, \tag{10.46}$$

$$\mathbf{e}_\phi = \frac{\partial\mathbf{r}}{\partial\phi} = -\rho\sin\phi\,\mathbf{i} + \rho\cos\phi\,\mathbf{j}, \tag{10.47}$$

$$\mathbf{e}_z = \frac{\partial\mathbf{r}}{\partial z} = \mathbf{k}. \tag{10.48}$$

These vectors lie in the directions of increasing ρ, ϕ and z respectively but are not all of unit length. Although $\mathbf{e}_\rho$, $\mathbf{e}_\phi$ and $\mathbf{e}_z$ form a useful set of basis vectors in their own right (we will see in section 10.10 that such a basis is sometimes the *most* useful), it is usual to work with the corresponding *unit* vectors, which are obtained by dividing each vector by its modulus to give

$$\hat{\mathbf{e}}_\rho = \mathbf{e}_\rho = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}, \tag{10.49}$$

$$\hat{\mathbf{e}}_\phi = \frac{1}{\rho}\,\mathbf{e}_\phi = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}, \tag{10.50}$$

$$\hat{\mathbf{e}}_z = \mathbf{e}_z = \mathbf{k}. \tag{10.51}$$

These three unit vectors, like the Cartesian unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, form an orthonormal triad at each point in space, i.e. the basis vectors are mutually orthogonal and of unit length (see figure 10.7). Unlike the fixed vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, however, $\hat{\mathbf{e}}_\rho$ and $\hat{\mathbf{e}}_\phi$ change direction as P moves.

The expression for a general infinitesimal vector displacement $d\mathbf{r}$ in the position of P is given, from (10.19), by

$$\begin{aligned} d\mathbf{r} &= \frac{\partial\mathbf{r}}{\partial\rho}\,d\rho + \frac{\partial\mathbf{r}}{\partial\phi}\,d\phi + \frac{\partial\mathbf{r}}{\partial z}\,dz \\ &= d\rho\,\mathbf{e}_\rho + d\phi\,\mathbf{e}_\phi + dz\,\mathbf{e}_z \\ &= d\rho\,\hat{\mathbf{e}}_\rho + \rho\,d\phi\,\hat{\mathbf{e}}_\phi + dz\,\hat{\mathbf{e}}_z. \end{aligned} \tag{10.52}$$

This expression illustrates an important difference between Cartesian and cylindrical polar coordinates (or non-Cartesian coordinates in general). In Cartesian coordinates, the distance moved in going from x to $x + dx$, with y and z held constant, is simply $ds = dx$. However, in cylindrical polars, if ϕ changes by $d\phi$, with ρ and z held constant, then the distance moved is *not* $d\phi$, but $ds = \rho\,d\phi$.

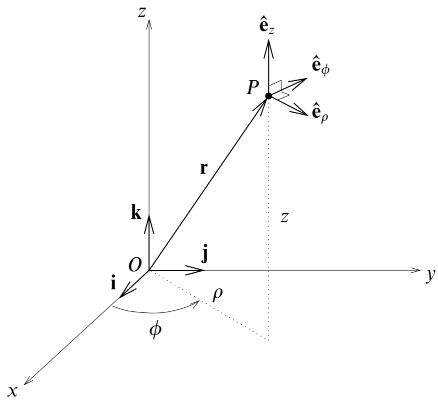


Figure 10.7 Cylindrical polar coordinates ρ, ϕ, z.

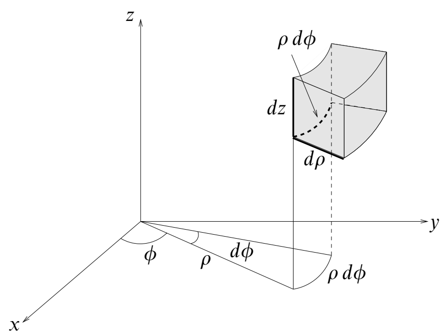


Figure 10.8 The element of volume in cylindrical polar coordinates is given by $\rho\, d\rho\, d\phi\, dz$.

Factors, such as the ρ in $\rho\, d\phi$, that multiply the coordinate differentials to give distances are known as *scale factors*. From (10.52), the scale factors for the ρ-, ϕ- and z- coordinates are therefore 1, ρ and 1 respectively.

The magnitude ds of the displacement $d\mathbf{r}$ is given in cylindrical polar coordinates by

$$(ds)^2 = d\mathbf{r} \cdot d\mathbf{r} = (d\rho)^2 + \rho^2 (d\phi)^2 + (dz)^2,$$

where in the second equality we have used the fact that the basis vectors are orthonormal. We can also find the volume element in a cylindrical polar system (see figure 10.8) by calculating the volume of the infinitesimal parallelepiped

$$
\begin{aligned}
\nabla\Phi &= \frac{\partial\Phi}{\partial\rho}\hat{\mathbf{e}}_\rho + \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\hat{\mathbf{e}}_\phi + \frac{\partial\Phi}{\partial z}\hat{\mathbf{e}}_z \\
\nabla\cdot\mathbf{a} &= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho a_\rho) + \frac{1}{\rho}\frac{\partial a_\phi}{\partial\phi} + \frac{\partial a_z}{\partial z} \\
\nabla\times\mathbf{a} &= \frac{1}{\rho}\begin{vmatrix} \hat{\mathbf{e}}_\rho & \rho\hat{\mathbf{e}}_\phi & \hat{\mathbf{e}}_z \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial z} \\ a_\rho & \rho a_\phi & a_z \end{vmatrix} \\
\nabla^2\Phi &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}
\end{aligned}
$$

Table 10.2 Vector operators in cylindrical polar coordinates; Φ is a scalar field and $\mathbf{a}$ is a vector field.

defined by the vectors $d\rho\,\hat{\mathbf{e}}_\rho$, $\rho\,d\phi\,\hat{\mathbf{e}}_\phi$ and $dz\,\hat{\mathbf{e}}_z$:

$$dV = |d\rho\,\hat{\mathbf{e}}_\rho\cdot(\rho\,d\phi\,\hat{\mathbf{e}}_\phi\ \times\ dz\,\hat{\mathbf{e}}_z)| = \rho\,d\rho\,d\phi\,dz,$$

which again uses the fact that the basis vectors are orthonormal. For a simple coordinate system such as cylindrical polars the expressions for $(ds)^2$ and dV are obvious from the geometry.

We will now express the vector operators discussed in this chapter in terms of cylindrical polar coordinates. Let us consider a vector field $\mathbf{a}(\rho,\phi,z)$ and a scalar field $\Phi(\rho,\phi,z)$, where we use Φ for the scalar field to avoid confusion with the azimuthal angle ϕ. We must first write the vector field in terms of the basis vectors of the cylindrical polar coordinate system, i.e.

$$\mathbf{a} = a_\rho\,\hat{\mathbf{e}}_\rho + a_\phi\,\hat{\mathbf{e}}_\phi + a_z\,\hat{\mathbf{e}}_z,$$

where a_ρ, a_ϕ and a_z are the components of $\mathbf{a}$ in the ρ-, ϕ- and z- directions respectively. The expressions for grad, div, curl and ∇^2 can then be calculated and are given in table 10.2. Since the derivations of these expressions are rather complicated we leave them until our discussion of general curvilinear coordinates in the next section; the reader could well postpone examination of these formal proofs until some experience of using the expressions has been gained.

►*Express the vector field* $\mathbf{a} = yz\,\mathbf{i} - y\,\mathbf{j} + xz^2\,\mathbf{k}$ *in cylindrical polar coordinates, and hence calculate its divergence. Show that the same result is obtained by evaluating the divergence in Cartesian coordinates.*

The basis vectors of the cylindrical polar coordinate system are given in (10.49)–(10.51). Solving these equations simultaneously for $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ we obtain

$$
\begin{aligned}
\mathbf{i} &= \cos\phi\,\hat{\mathbf{e}}_\rho - \sin\phi\,\hat{\mathbf{e}}_\phi \\
\mathbf{j} &= \sin\phi\,\hat{\mathbf{e}}_\rho + \cos\phi\,\hat{\mathbf{e}}_\phi \\
\mathbf{k} &= \hat{\mathbf{e}}_z.
\end{aligned}
$$

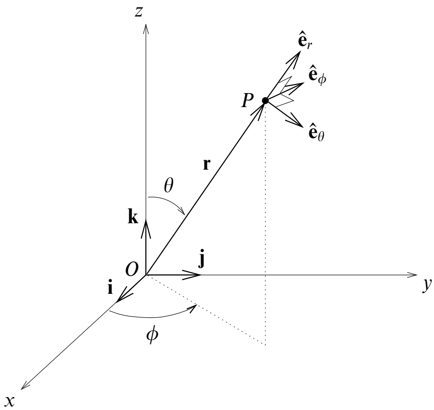


Figure 10.9 Spherical polar coordinates r, θ, ϕ.

Substituting these relations and (10.44) into the expression for $\mathbf{a}$ we find

$$\begin{aligned}\mathbf{a} &= z\rho\sin\phi\,(\cos\phi\,\hat{\mathbf{e}}_\rho - \sin\phi\,\hat{\mathbf{e}}_\phi) - \rho\sin\phi\,(\sin\phi\,\hat{\mathbf{e}}_\rho + \cos\phi\,\hat{\mathbf{e}}_\phi) + z^2\rho\cos\phi\,\hat{\mathbf{e}}_z \\ &= (z\rho\sin\phi\cos\phi - \rho\sin^2\phi)\,\hat{\mathbf{e}}_\rho - (z\rho\sin^2\phi + \rho\sin\phi\cos\phi)\,\hat{\mathbf{e}}_\phi + z^2\rho\cos\phi\,\hat{\mathbf{e}}_z.\end{aligned}$$

Substituting into the expression for $\nabla\cdot\mathbf{a}$ given in table 10.2,

$$\begin{aligned}\nabla\cdot\mathbf{a} &= 2z\sin\phi\cos\phi - 2\sin^2\phi - 2z\sin\phi\cos\phi - \cos^2\phi + \sin^2\phi + 2z\rho\cos\phi \\ &= 2z\rho\cos\phi - 1.\end{aligned}$$

Alternatively, and much more quickly in this case, we can calculate the divergence directly in Cartesian coordinates. We obtain

$$\nabla\cdot\mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} = 2zx - 1,$$

which on substituting $x = \rho\cos\phi$ yields the same result as the calculation in cylindrical polars. ◄

Finally, we note that similar results can be obtained for (two-dimensional) polar coordinates in a plane by omitting the z-dependence. For example, $(ds)^2 = (d\rho)^2 + \rho^2(d\phi)^2$, while the element of volume is replaced by the element of area $dA = \rho\,d\rho\,d\phi$.

10.9.2 Spherical polar coordinates

As shown in figure 10.9, the position of a point in space P, with Cartesian coordinates x, y, z, may be expressed in terms of spherical polar coordinates r, θ, ϕ, where

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta, \tag{10.53}$$

and $r \geq 0$, $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$. The position vector of P may therefore be written as

$$\mathbf{r} = r \sin\theta \cos\phi \, \mathbf{i} + r \sin\theta \sin\phi \, \mathbf{j} + r \cos\theta \, \mathbf{k}.$$

If, in a similar manner to that used in the previous section for cylindrical polars, we find the partial derivatives of $\mathbf{r}$ with respect to r, θ and ϕ respectively and divide each of the resulting vectors by its modulus then we obtain the unit basis vectors

$$\begin{aligned}
\hat{\mathbf{e}}_r &= \sin\theta \cos\phi \, \mathbf{i} + \sin\theta \sin\phi \, \mathbf{j} + \cos\theta \, \mathbf{k}, \\
\hat{\mathbf{e}}_\theta &= \cos\theta \cos\phi \, \mathbf{i} + \cos\theta \sin\phi \, \mathbf{j} - \sin\theta \, \mathbf{k}, \\
\hat{\mathbf{e}}_\phi &= -\sin\phi \, \mathbf{i} + \cos\phi \, \mathbf{j}.
\end{aligned}$$

These unit vectors are in the directions of increasing r, θ and ϕ respectively and are the orthonormal basis set for spherical polar coordinates, as shown in figure 10.9.

A general infinitesimal vector displacement in spherical polars is, from (10.19),

$$d\mathbf{r} = dr \, \hat{\mathbf{e}}_r + r \, d\theta \, \hat{\mathbf{e}}_\theta + r \sin\theta \, d\phi \, \hat{\mathbf{e}}_\phi; \tag{10.54}$$

thus the scale factors for the r-, θ- and ϕ- coordinates are 1, r and $r \sin\theta$ respectively. The magnitude ds of the displacement $d\mathbf{r}$ is given by

$$(ds)^2 = d\mathbf{r} \cdot d\mathbf{r} = (dr)^2 + r^2 (d\theta)^2 + r^2 \sin^2\theta (d\phi)^2,$$

since the basis vectors form an orthonormal set. The element of volume in spherical polar coordinates (see figure 10.10) is the volume of the infinitesimal parallelepiped defined by the vectors $dr \, \hat{\mathbf{e}}_r$, $r \, d\theta \, \hat{\mathbf{e}}_\theta$ and $r \sin\theta \, d\phi \, \hat{\mathbf{e}}_\phi$ and is given by

$$dV = |dr \, \hat{\mathbf{e}}_r \cdot (r \, d\theta \, \hat{\mathbf{e}}_\theta \times r \sin\theta \, d\phi \, \hat{\mathbf{e}}_\phi)| = r^2 \sin\theta \, dr \, d\theta \, d\phi,$$

where again we use the fact that the basis vectors are orthonormal. The expressions for $(ds)^2$ and dV in spherical polars can be obtained from the geometry of this coordinate system.

We will now express the standard vector operators in spherical polar coordinates, using the same techniques as for cylindrical polar coordinates. We consider a scalar field $\Phi(r, \theta, \phi)$ and a vector field $\mathbf{a}(r, \theta, \phi)$. The latter may be written in terms of the basis vectors of the spherical polar coordinate system as

$$\mathbf{a} = a_r \, \hat{\mathbf{e}}_r + a_\theta \, \hat{\mathbf{e}}_\theta + a_\phi \, \hat{\mathbf{e}}_\phi,$$

where a_r, a_θ and a_ϕ are the components of $\mathbf{a}$ in the r-, θ- and ϕ- directions respectively. The expressions for grad, div, curl and ∇^2 are given in table 10.3. The derivations of these results are given in the next section.

As a final note, we mention that, in the expression for $\nabla^2\Phi$ given in table 10.3,

$$\nabla\Phi = \frac{\partial\Phi}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\mathbf{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\hat{\mathbf{e}}_\phi$$

$$\nabla\cdot\mathbf{a} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 a_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ a_\theta) + \frac{1}{r\sin\theta}\frac{\partial a_\phi}{\partial\phi}$$

$$\nabla\times\mathbf{a} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\mathbf{e}}_r & r\hat{\mathbf{e}}_\theta & r\sin\theta\,\hat{\mathbf{e}}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ a_r & ra_\theta & r\sin\theta\ a_\phi \end{vmatrix}$$

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$

Table 10.3 Vector operators in spherical polar coordinates; Φ is a scalar field and $\mathbf{a}$ is a vector field.

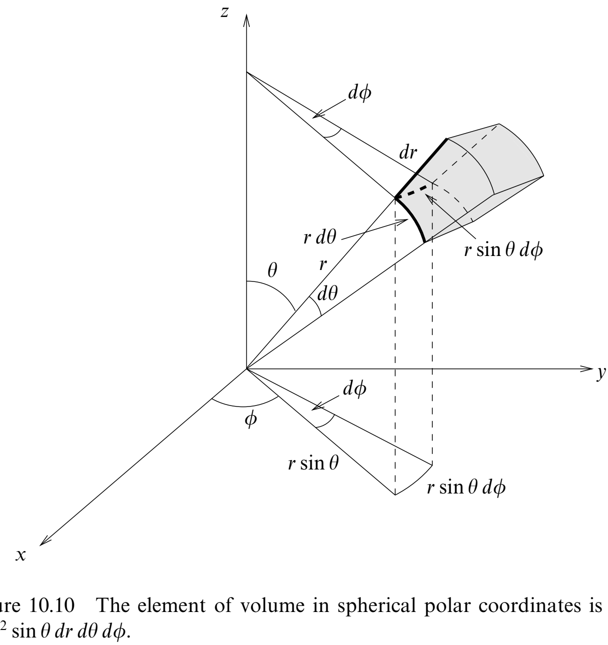


Figure 10.10 The element of volume in spherical polar coordinates is given by $r^2\sin\theta\,dr\,d\theta\,d\phi$.

we can rewrite the first term on the RHS as follows:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi),$$

which can often be useful in shortening calculations.

→ Follow Ashok DAS

Electromagnetic wave in classical Physics: In the free space the Maxwell's equations are

$$\left.\begin{array}{llll} \text{(i)} & \vec{\nabla}\cdot\vec{\mathbf{E}} = 0, & \text{(iii)} & \vec{\nabla}\times\vec{\mathbf{E}} = -\dfrac{\partial\vec{\mathbf{B}}}{\partial t}, \\ \text{(ii)} & \vec{\nabla}\cdot\vec{\mathbf{B}} = 0, & \text{(iv)} & \vec{\nabla}\times\vec{\mathbf{B}} = \mu_0\epsilon_0\dfrac{\partial\vec{\mathbf{E}}}{\partial t}. \end{array}\right\}$$

Applying curl operator to (iii) we get

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}\cdot(\vec{\nabla}\cdot\vec{E}) - \vec{\nabla}^2\vec{E} = -\mu_0\epsilon_0\frac{\partial^2 E}{\partial t^2}$$ Since $\vec{\nabla}\cdot\vec{E} = 0$ we get

$$\boxed{\vec{\nabla}^2\vec{E} = \mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2}}$$

Similarly, applying $\vec{\nabla}$ to the equation (iv) we get

$$\boxed{\vec{\nabla}^2 B = \mu_0\epsilon_0\frac{\partial^2\vec{B}}{\partial t^2}}$$

These are the two wave equations for the the electric & the magnetic fields. Here $c = \dfrac{1}{\sqrt{\mu_0\epsilon_0}}$ is the velocity of the electromagnetic wave - speed of light.

Lecture 6, 3/2/23

* ~~Vector Operators in Cylindrical Polar & Spherical Polar Coordinates:~~

See Riley-Hobson-Bence – Uploaded in Moodle.

* A complete Example - Electromagnetic Wave:

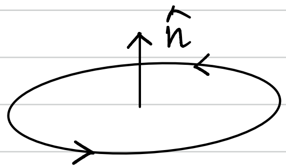
Note added seperately –

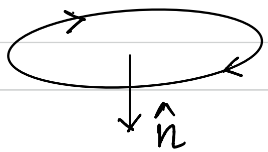
Lecture 7, 5/1/23

Surface & Volume Integrals: An integral over a surface is given as

$$\int \phi \, d\vec{s} \quad \text{or} \quad \int \vec{V} \cdot d\vec{s} \quad \text{or} \quad \int \vec{V} \times d\vec{s}$$

Now the differential $d\vec{s} = \hat{n}\, dA$, where $dA = |d\vec{s}|$ is the magnitude and $\hat{n}$ is the unit vector that is normal to the surface. For a closed surface, the direction of the vector is outward normal. For open surface, the direction of $\hat{n}$ determined by the direction of the perimeter – by the right-hand thumb rule



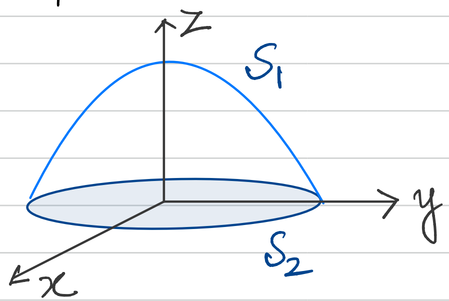


Volume integrals are of the type $\int \vec{v}\, dv$ or $\int \phi\, dv$ where dv is a scalar quantity.

* Gauss's Divergence Theorem: It relates the closed surface integral of a vector to the volume integral of the divergence of the vector. The theorem states that

$$\oiint_S \vec{F} \cdot \vec{ds} = \iiint_V \vec{\nabla} \cdot \vec{F}\, dv$$

Example: Given a vector field $\vec{F} = \hat{k}(z+R)$ and a solid hemisphericall ball, $x^2+y^2+z^2 \leq R$ check the Gauss's theorem.



Soln: $\iiint_V \vec{\nabla} \cdot \vec{F}\, dv = \iiint_V dv = \frac{2}{3}\pi R^3$

Now, the left hand side of the Gauss's theorem is given by

$$\oiint \vec{F} \cdot \vec{ds} = \iint \vec{F} \cdot \vec{ds_1} + \iint \vec{F} \cdot \vec{ds_2}$$

$$= \iint \vec{F} \cdot \hat{n}_1\, ds_1 + \iint \vec{F} \cdot \hat{n}_2\, ds_2$$

On S_1 the unit vector $\hat{n}_1 = \frac{1}{R}(\hat{i}x + \hat{j}y + \hat{k}z)$

$$\vec{F}.\hat{n}_1 = \frac{z(z+R)}{R} = R\cos\theta(1+\cos\theta)$$

{ Spherical Polar coordinate
$z = R\cos\theta$

Hence $\int \vec{F}.\hat{n}_1 ds_1 = \int \vec{F}.\hat{n}_1 (R d\theta)(R\sin\theta\, d\phi)$

$$= \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \,(R^2 \sin\theta)\, R\cos\theta\,(1+\cos\theta)$$

{ dr
$r d\theta$
$r\sin\theta\, d\phi$.

$$= 2\pi R^3 \left[-\frac{1}{3}\cos^3\theta - \frac{1}{2}\cos\theta\right]_0^{\pi/2} = \frac{5}{3}\pi R^3$$

For the second surface $\hat{n}_2 = -\hat{k}$

$$\int \vec{F}.\hat{n}_2\, ds_2 = \int \vec{F}.\hat{n}_2\, dr . r d\theta$$

$$= \int_0^R r dr \int_0 d\phi \,(-R)$$

$$= \frac{R^2}{2} . 2\pi(-R) = -\pi R^2$$

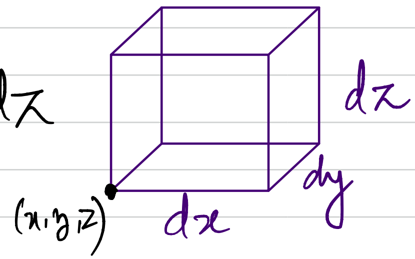
{ In the S_2 plane the area element is given by $dr . r\sin\frac{\pi}{2}\, d\phi$
$= r dr\, d\phi$

Adding the two results $= \frac{2}{3}\pi R^3$.

Proof of Divergence theorem: Theorem Statement $\int_V \vec{\nabla}\cdot\vec{F}\,dv = \oint \vec{F}\cdot ds$

Any finite volume V that is closed by the surface S can be divided into infinitesimally small cubes of sides dx, dy & dz.

Consider infinitesimally small volume of sides dx, dy & dz



Lets discuss the R.H.S of the theorem

$$\int \vec{F}\cdot ds = \sum \vec{F}\cdot\hat{n}_i\,\Delta S_i$$

the sum is over all the point (x, y, z) around which infinitesimal volume is considered. First consider vector through the dydz surface. Contribution from the two sides given by

$$\left[F_x(x+dx, y, z) - F(x, y, z)\right] dy\,dz$$

$$= \frac{\partial F_x}{\partial x}\,dx\,dy\,dz$$

The sign is negative because the surface normal are opposite on either sides.

Similar results hold for the other two surfaces - can be summed. Now we have to sum these up for all the volume element consider - The contributions from the adjacent walls cancel each other leaving only the contribution from the overall surface.

So
$$\oint \vec{F}\cdot\hat{n}\,ds = \sum \vec{F}\cdot\hat{n}_i \Delta S_i$$
$$= \sum\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)dx\,dy\,dz$$
$$= \int_V \vec{\nabla}\cdot\vec{F}\,dV \qquad \text{proven!}$$

<u>Example</u>: Given $\vec{F} = r^2\hat{r}$ test the divergence theorem for a sphere of radius R

<u>Soln</u>: Using the $\vec{\nabla}$ operator for spherical polars
$$\vec{\nabla}\cdot\vec{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^4\right) = 4r$$

$dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$. So $\displaystyle\int_V \vec{\nabla}\cdot\vec{F}\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} 4r^3 \sin\theta\, dr\, d\theta\, d\phi = 4\pi R^4$

For RHS: On the surface $\vec{F} = R^2\hat{r}$ & $d\vec{s} = \hat{r}R^2\sin\theta\, d\theta\, d\phi \Rightarrow \displaystyle\oint \vec{F}\cdot d\vec{s} = \int_0^{2\pi}\int_0^{\pi} R^4 \sin\theta\, d\theta\, d\phi = 4\pi R^4$

Green's theorem in Plane: It relates the line integral over a closed curve c to a double integral over the surface enclosed by the curve. This is a 2D version of Stoke's theorem that we will study next.

Statement: Let $\vec{F} = F_x(x,y)\hat{i} + F_y(x,y)\hat{j}$ be any differentiable function in the two dimension in x-y plane in an area S that is bounded by a curve c.

Then the theorem states that → the path c is anti-clockwise

$$\iint_S \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) ds = \oint_c \left(F_x\, dx + F_y\, dy \right)$$

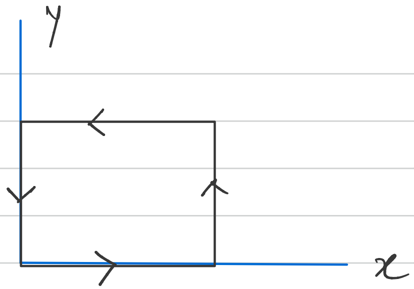
Proof: Very easy to prove in a rectangle (Exercise)

Example: Prove Green's theorem for $\vec{r} = -\hat{i}y + \hat{j}x$ for a square of side L in the first quadrant with vertex at the origin

Soln: The integrand of the surface integral is $\quad \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 2$

Then the result of the integral $\quad \oiint 2\, dA = 2L^2$

For the R.H.S of the equation $\oint (F_x\,dx + F_y\,dy)$ the contour is shown in the figure.



$$\oint (F_x\,dx + F_y\,dy) = \int_0^L 0.dx + \int (-L)\,dx + \int_L^0 0.dx + \int_0^L L\,dy$$

$$= 2L^2$$

Example 2: Same as the previous problem, but now the curve C is a circle of radius R centered at the origin.

Soln: The LHS $= \iint_S 2.dA = 2\pi R^2$

To do the RHS, $x = R\cos\theta$, $y = R\sin\theta$, then $dx = -R\sin\theta$, $dy = R\cos\theta$

and the RHS is $\oint_C (-y\,dx + x\,dy) = \int_0^{2\pi} (R^2\sin^2\theta + R^2\cos^2\theta)\,d\theta = 2\pi R^2$

Stoke's Theorem: This is 3D extension of Green's theorem. For a 3D surface s closed by a simple curve c the theorem states that

$$\int_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{s} = \oint_C \vec{F} \cdot d\vec{r}$$

Example: Prove Stoke's theorem for $\vec{F} = -y\hat{i} + x\hat{j}$ for a hemisphere of radius R with $z > 0$ bounded by a circle of radius R lying in the x-y plane with the center at the origin.

Soln: $\oint \vec{F} \cdot d\vec{r} = \oint_C (F_x dx + F_y dy) = \oint_C (-y dx + x dy) = \int_0^{2\pi} (R^2 \sin^2\theta + R^2 \cos^2\theta) d\theta = 2\pi R^2$

The LHS of the Stoke's theorem is $\vec{\nabla} \times \vec{F} = 2\hat{k}$

Area differential $d\vec{s} = \hat{r} R^2 \sin\theta \, d\theta \, d\varphi$. Hence the LHS of the theorem is

$$\int_0^{2\pi} \int_0^{\pi/2} \hat{k} \cdot \hat{r} \; 2R^2 \sin\theta \, d\theta \, d\varphi \qquad \left\{ \hat{k} \cdot \hat{r} = \cos\theta . \right.$$

$$= 2R^2 \int_0^{2\pi} d\phi \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta = 2\pi R^2 \sin^2\theta \Big|_0^{\pi/2} = 2\pi R^2$$

3

Complex numbers and hyperbolic functions

This chapter is concerned with the representation and manipulation of complex numbers. Complex numbers pervade this book, underscoring their wide application in the mathematics of the physical sciences. The application of complex numbers to the description of physical systems is left until later chapters and only the basic tools are presented here.

3.1 The need for complex numbers

Although complex numbers occur in many branches of mathematics, they arise most directly out of solving polynomial equations. We examine a specific quadratic equation as an example.

Consider the quadratic equation

$$z^2 - 4z + 5 = 0. \tag{3.1}$$

Equation (3.1) has two solutions, z_1 and z_2, such that

$$(z - z_1)(z - z_2) = 0. \tag{3.2}$$

Using the familiar formula for the roots of a quadratic equation, (1.4), the solutions z_1 and z_2, written in brief as $z_{1,2}$, are

$$\begin{aligned} z_{1,2} &= \frac{4 \pm \sqrt{(-4)^2 - 4(1 \times 5)}}{2} \\ &= 2 \pm \frac{\sqrt{-4}}{2}. \end{aligned} \tag{3.3}$$

Both solutions contain the square root of a negative number. However, it is not true to say that there are no solutions to the quadratic equation. The *fundamental theorem of algebra* states that a quadratic equation will always have two solutions and these are in fact given by (3.3). The second term on the RHS of (3.3) is called an *imaginary* term since it contains the square root of a negative number;

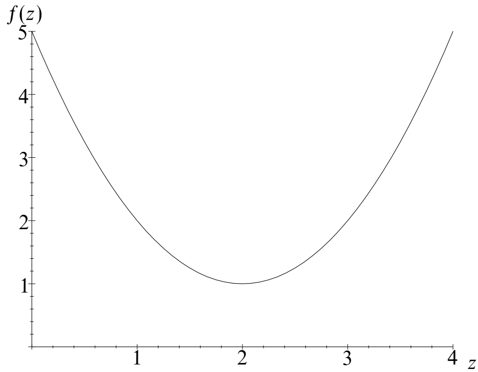


Figure 3.1 The function $f(z) = z^2 - 4z + 5$.

the first term is called a *real* term. The full solution is the sum of a real term and an imaginary term and is called a *complex number*. A plot of the function $f(z) = z^2 - 4z + 5$ is shown in figure 3.1. It will be seen that the plot does not intersect the z-axis, corresponding to the fact that the equation $f(z) = 0$ has no purely real solutions.

The choice of the symbol z for the quadratic variable was not arbitrary; the conventional representation of a complex number is z, where z is the sum of a real part x and i times an imaginary part y, i.e.

$$z = x + iy,$$

where i is used to denote the square root of -1. The real part x and the imaginary part y are usually denoted by $\operatorname{Re} z$ and $\operatorname{Im} z$ respectively. We note at this point that some physical scientists, engineers in particular, use j instead of i. However, for consistency, we will use i throughout this book.

In our particular example, $\sqrt{-4} = 2\sqrt{-1} = 2i$, and hence the two solutions of (3.1) are

$$z_{1,2} = 2 \pm \frac{2i}{2} = 2 \pm i.$$

Thus, here $x = 2$ and $y = \pm 1$.

For compactness a complex number is sometimes written in the form

$$z = (x, y),$$

where the components of z may be thought of as coordinates in an xy-plot. Such a plot is called an *Argand diagram* and is a common representation of complex numbers; an example is shown in figure 3.2.

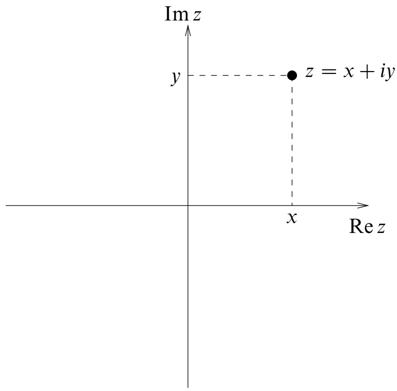


Figure 3.2 The Argand diagram.

Our particular example of a quadratic equation may be generalised readily to polynomials whose highest power (degree) is greater than 2, e.g. cubic equations (degree 3), quartic equations (degree 4) and so on. For a general polynomial $f(z)$, of degree n, the fundamental theorem of algebra states that the equation $f(z) = 0$ will have exactly n solutions. We will examine cases of higher-degree equations in subsection 3.4.3.

The remainder of this chapter deals with: the algebra and manipulation of complex numbers; their polar representation, which has advantages in many circumstances; complex exponentials and logarithms; the use of complex numbers in finding the roots of polynomial equations; and hyperbolic functions.

3.2 Manipulation of complex numbers

This section considers basic complex number manipulation. Some analogy may be drawn with vector manipulation (see chapter 7) but this section stands alone as an introduction.

3.2.1 Addition and subtraction

The addition of two complex numbers, z_1 and z_2, in general gives another complex number. The real components and the imaginary components are added separately and in a like manner to the familiar addition of real numbers:

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2),$$

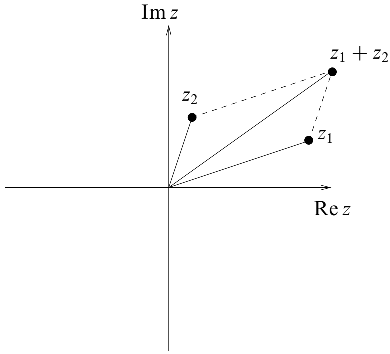


Figure 3.3 The addition of two complex numbers.

or in component notation

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).$$

The Argand representation of the addition of two complex numbers is shown in figure 3.3.

By straightforward application of the commutativity and associativity of the real and imaginary parts separately, we can show that the addition of complex numbers is itself commutative and associative, i.e.

$$z_1 + z_2 = z_2 + z_1,$$
$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3.$$

Thus it is immaterial in what order complex numbers are added.

►*Sum the complex numbers* $1 + 2i$, $3 - 4i$, $-2 + i$.

Summing the real terms we obtain

$$1 + 3 - 2 = 2,$$

and summing the imaginary terms we obtain

$$2i - 4i + i = -i.$$

Hence

$$(1 + 2i) + (3 - 4i) + (-2 + i) = 2 - i. \blacktriangleleft$$

The subtraction of complex numbers is very similar to their addition. As in the case of real numbers, if two identical complex numbers are subtracted then the result is zero.

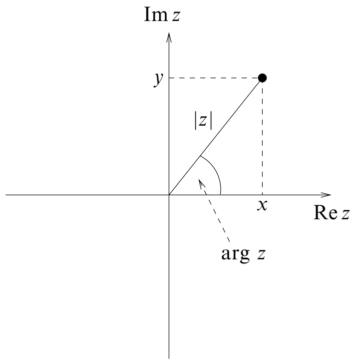


Figure 3.4 The modulus and argument of a complex number.

3.2.2 Modulus and argument

The modulus of the complex number z is denoted by $|z|$ and is defined as

$$|z| = \sqrt{x^2 + y^2}. \tag{3.4}$$

Hence the modulus of the complex number is the distance of the corresponding point from the origin in the Argand diagram, as may be seen in figure 3.4.

The argument of the complex number z is denoted by $\arg z$ and is defined as

$$\arg z = \tan^{-1}\left(\frac{y}{x}\right). \tag{3.5}$$

It can be seen that $\arg z$ is the angle that the line joining the origin to z on the Argand diagram makes with the positive x-axis. The anticlockwise direction is taken to be positive by convention. The angle $\arg z$ is shown in figure 3.4. Account must be taken of the signs of x and y individually in determining in which quadrant $\arg z$ lies. Thus, for example, if x and y are both negative then $\arg z$ lies in the range $-\pi < \arg z < -\pi/2$ rather than in the first quadrant ($0 < \arg z < \pi/2$), though both cases give the same value for the ratio of y to x.

►*Find the modulus and the argument of the complex number $z = 2 - 3i$.*

Using (3.4), the modulus is given by

$$|z| = \sqrt{2^2 + (-3)^2} = \sqrt{13}.$$

Using (3.5), the argument is given by

$$\arg z = \tan^{-1}\left(-\tfrac{3}{2}\right).$$

The two angles whose tangents equal -1.5 are -0.9828 rad and 2.1588 rad. Since $x = 2$ and $y = -3$, z clearly lies in the fourth quadrant; therefore $\arg z = -0.9828$ is the appropriate answer. ◄

3.2.3 Multiplication

Complex numbers may be multiplied together and in general give a complex number as the result. The product of two complex numbers z_1 and z_2 is found by multiplying them out in full and remembering that $i^2 = -1$, i.e.

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1x_2 + ix_1y_2 + iy_1x_2 + i^2 y_1y_2 \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2). \end{aligned} \tag{3.6}$$

►*Multiply the complex numbers $z_1 = 3 + 2i$ and $z_2 = -1 - 4i$.*

By direct multiplication we find

$$\begin{aligned} z_1 z_2 &= (3 + 2i)(-1 - 4i) \\ &= -3 - 2i - 12i - 8i^2 \\ &= 5 - 14i. \blacktriangleleft \end{aligned} \tag{3.7}$$

The multiplication of complex numbers is both commutative and associative, i.e.

$$z_1 z_2 = z_2 z_1, \tag{3.8}$$

$$(z_1 z_2) z_3 = z_1 (z_2 z_3). \tag{3.9}$$

The product of two complex numbers also has the simple properties

$$|z_1 z_2| = |z_1||z_2|, \tag{3.10}$$

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2. \tag{3.11}$$

These relations are derived in subsection 3.3.1.

►*Verify that (3.10) holds for the product of $z_1 = 3 + 2i$ and $z_2 = -1 - 4i$.*

From (3.7)

$$|z_1 z_2| = |5 - 14i| = \sqrt{5^2 + (-14)^2} = \sqrt{221}.$$

We also find

$$\begin{aligned} |z_1| &= \sqrt{3^2 + 2^2} = \sqrt{13}, \\ |z_2| &= \sqrt{(-1)^2 + (-4)^2} = \sqrt{17}, \end{aligned}$$

and hence

$$|z_1||z_2| = \sqrt{13}\sqrt{17} = \sqrt{221} = |z_1 z_2|. \blacktriangleleft$$

We now examine the effect on a complex number z of multiplying it by ± 1 and $\pm i$. These four multipliers have modulus unity and we can see immediately from (3.10) that multiplying z by another complex number of unit modulus gives a product with the same modulus as z. We can also see from (3.11) that if we

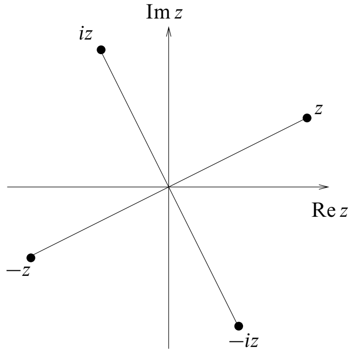


Figure 3.5 Multiplication of a complex number by ± 1 and $\pm i$.

multiply z by a complex number then the argument of the product is the sum of the argument of z and the argument of the multiplier. Hence multiplying z by unity (which has argument zero) leaves z unchanged in both modulus and argument, i.e. z is completely unaltered by the operation. Multiplying by -1 (which has argument π) leads to rotation, through an angle π, of the line joining the origin to z in the Argand diagram. Similarly, multiplication by i or $-i$ leads to corresponding rotations of $\pi/2$ or $-\pi/2$ respectively. This geometrical interpretation of multiplication is shown in figure 3.5.

►*Using the geometrical interpretation of multiplication by i, find the product* $i(1 - i)$.

The complex number $1 - i$ has argument $-\pi/4$ and modulus $\sqrt{2}$. Thus, using (3.10) and (3.11), its product with i has argument $+\pi/4$ and unchanged modulus $\sqrt{2}$. The complex number with modulus $\sqrt{2}$ and argument $+\pi/4$ is $1 + i$ and so

$$i(1 - i) = 1 + i,$$

as is easily verified by direct multiplication. ◄

The division of two complex numbers is similar to their multiplication but requires the notion of the complex conjugate (see the following subsection) and so discussion is postponed until subsection 3.2.5.

3.2.4 Complex conjugate

If z has the convenient form $x + iy$ then the complex conjugate, denoted by z^*, may be found simply by changing the sign of the imaginary part, i.e. if $z = x + iy$ then $z^* = x - iy$. More generally, we may define the complex conjugate of z as the (complex) number having the same magnitude as z that when multiplied by z leaves a real result, i.e. there is no imaginary component in the product.

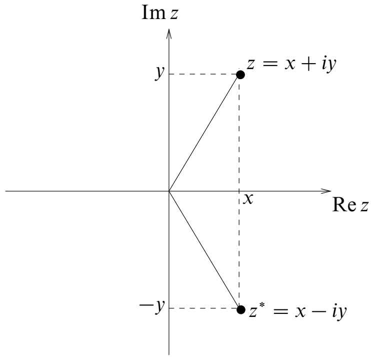


Figure 3.6 The complex conjugate as a mirror image in the real axis.

In the case where z can be written in the form $x + iy$ it is easily verified, by direct multiplication of the components, that the product zz^* gives a real result:

$$zz^* = (x + iy)(x - iy) = x^2 - ixy + ixy - i^2y^2 = x^2 + y^2 = |z|^2.$$

Complex conjugation corresponds to a reflection of z in the real axis of the Argand diagram, as may be seen in figure 3.6.

►*Find the complex conjugate of $z = a + 2i + 3ib$.*

The complex number is written in the standard form

$$z = a + i(2 + 3b);$$

then, replacing i by $-i$, we obtain

$$z^* = a - i(2 + 3b). \blacktriangleleft$$

In some cases, however, it may not be simple to rearrange the expression for z into the standard form $x + iy$. Nevertheless, given two complex numbers, z_1 and z_2, it is straightforward to show that the complex conjugate of their sum (or difference) is equal to the sum (or difference) of their complex conjugates, i.e. $(z_1 \pm z_2)^* = z_1^* \pm z_2^*$. Similarly, it may be shown that the complex conjugate of the product (or quotient) of z_1 and z_2 is equal to the product (or quotient) of their complex conjugates, i.e. $(z_1z_2)^* = z_1^*z_2^*$ and $(z_1/z_2)^* = z_1^*/z_2^*$.

Using these results, it can be deduced that, no matter how complicated the expression, its complex conjugate may *always* be found by replacing every i by $-i$. To apply this rule, however, we must always ensure that all complex parts are first written out in full, so that no i's are hidden.

►*Find the complex conjugate of the complex number $z = w^{(3y+2ix)}$, where $w = x + 5i$.*

Although we do not discuss complex powers until section 3.5, the simple rule given above still enables us to find the complex conjugate of z.

In this case w itself contains real and imaginary components and so must be written out in full, i.e.

$$z = w^{3y+2ix} = (x + 5i)^{3y+2ix}.$$

Now we can replace each i by $-i$ to obtain

$$z^* = (x - 5i)^{(3y-2ix)}.$$

It can be shown that the product zz^* is real, as required. ◄

The following properties of the complex conjugate are easily proved and others may be derived from them. If $z = x + iy$ then

$$(z^*)^* = z, \tag{3.12}$$

$$z + z^* = 2\,\mathrm{Re}\,z = 2x, \tag{3.13}$$

$$z - z^* = 2i\,\mathrm{Im}\,z = 2iy, \tag{3.14}$$

$$\frac{z}{z^*} = \left(\frac{x^2 - y^2}{x^2 + y^2}\right) + i\left(\frac{2xy}{x^2 + y^2}\right). \tag{3.15}$$

The derivation of this last relation relies on the results of the following subsection.

3.2.5 Division

The division of two complex numbers z_1 and z_2 bears some similarity to their multiplication. Writing the quotient in component form we obtain

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2}. \tag{3.16}$$

In order to separate the real and imaginary components of the quotient, we multiply both numerator and denominator by the complex conjugate of the denominator. By definition, this process will leave the denominator as a real quantity. Equation (3.16) gives

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2} \\ &= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}.\end{aligned}$$

Hence we have separated the quotient into real and imaginary components, as required.

In the special case where $z_2 = z_1^*$, so that $x_2 = x_1$ and $y_2 = -y_1$, the general result reduces to (3.15).

►*Express z in the form $x + iy$, when*

$$z = \frac{3-2i}{-1+4i}.$$

Multiplying numerator and denominator by the complex conjugate of the denominator we obtain

$$\begin{aligned} z &= \frac{(3-2i)(-1-4i)}{(-1+4i)(-1-4i)} = \frac{-11-10i}{17} \\ &= -\frac{11}{17} - \frac{10}{17}i. \end{aligned}$$ ◄

In analogy to (3.10) and (3.11), which describe the multiplication of two complex numbers, the following relations apply to division:

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \tag{3.17}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2. \tag{3.18}$$

The proof of these relations is left until subsection 3.3.1.

3.3 Polar representation of complex numbers

Although considering a complex number as the sum of a real and an imaginary part is often useful, sometimes the *polar representation* proves easier to manipulate. This makes use of the complex exponential function, which is defined by

$$e^z = \exp z \equiv 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots. \tag{3.19}$$

Strictly speaking it is the function $\exp z$ that is defined by (3.19). The number e is the value of $\exp(1)$, i.e. it is just a number. However, it may be shown that e^z and $\exp z$ are equivalent when z is real and rational and mathematicians then *define* their equivalence for irrational and complex z. For the purposes of this book we will not concern ourselves further with this mathematical nicety but, rather, assume that (3.19) is valid for all z. We also note that, using (3.19), by multiplying together the appropriate series we may show that (see chapter 24)

$$e^{z_1}e^{z_2} = e^{z_1+z_2}, \tag{3.20}$$

which is analogous to the familiar result for exponentials of real numbers.

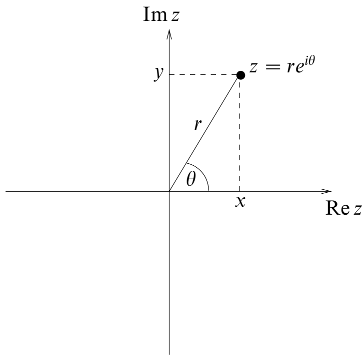


Figure 3.7 The polar representation of a complex number.

From (3.19), it immediately follows that for $z = i\theta$, θ real,

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \cdots \tag{3.21}$$

$$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \tag{3.22}$$

and hence that

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{3.23}$$

where the last equality follows from the series expansions of the sine and cosine functions (see subsection 4.6.3). This last relationship is called *Euler's equation*. It also follows from (3.23) that

$$e^{in\theta} = \cos n\theta + i\sin n\theta$$

for all n. From Euler's equation (3.23) and figure 3.7 we deduce that

$$re^{i\theta} = r(\cos\theta + i\sin\theta)$$
$$= x + iy.$$

Thus a complex number may be represented in the polar form

$$z = re^{i\theta}. \tag{3.24}$$

Referring again to figure 3.7, we can identify r with $|z|$ and θ with $\arg z$. The simplicity of the representation of the modulus and argument is one of the main reasons for using the polar representation. The angle θ lies conventionally in the range $-\pi < \theta \leq \pi$, but, since rotation by θ is the same as rotation by $2n\pi + \theta$, where n is any integer,

$$re^{i\theta} \equiv re^{i(\theta+2n\pi)}.$$

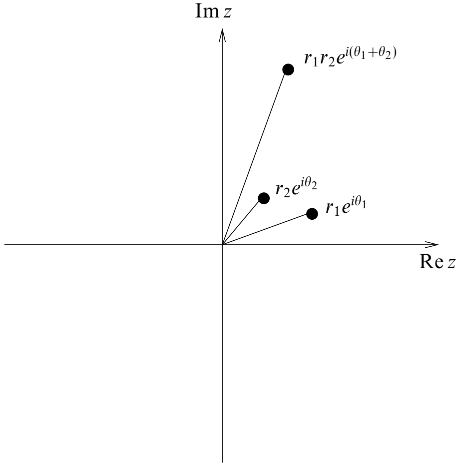


Figure 3.8 The multiplication of two complex numbers. In this case r_1 and r_2 are both greater than unity.

The algebra of the polar representation is different from that of the real and imaginary component representation, though, of course, the results are identical. Some operations prove much easier in the polar representation, others much more complicated. The best representation for a particular problem must be determined by the manipulation required.

3.3.1 Multiplication and division in polar form

Multiplication and division in polar form are particularly simple. The product of $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ is given by

$$\begin{aligned} z_1 z_2 &= r_1 e^{i\theta_1} r_2 e^{i\theta_2} \\ &= r_1 r_2 e^{i(\theta_1+\theta_2)}. \end{aligned} \tag{3.25}$$

The relations $|z_1 z_2| = |z_1||z_2|$ and $\arg(z_1 z_2) = \arg z_1 + \arg z_2$ follow immediately. An example of the multiplication of two complex numbers is shown in figure 3.8.

Division is equally simple in polar form; the quotient of z_1 and z_2 is given by

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}. \tag{3.26}$$

The relations $|z_1/z_2| = |z_1|/|z_2|$ and $\arg(z_1/z_2) = \arg z_1 - \arg z_2$ are again

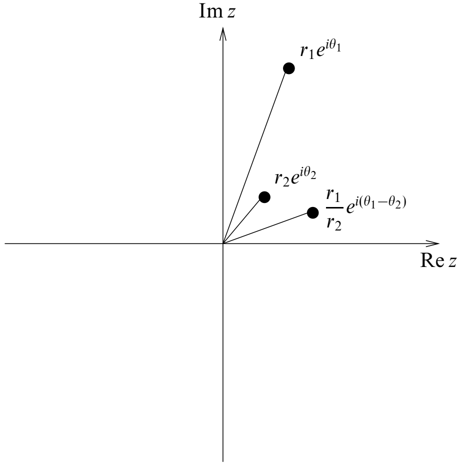


Figure 3.9 The division of two complex numbers. As in the previous figure, r_1 and r_2 are both greater than unity.

immediately apparent. The division of two complex numbers in polar form is shown in figure 3.9.

3.4 de Moivre's theorem

We now derive an extremely important theorem. Since $\left(e^{i\theta}\right)^n = e^{in\theta}$, we have

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \tag{3.27}$$

where the identity $e^{in\theta} = \cos n\theta + i\sin n\theta$ follows from the series definition of $e^{in\theta}$ (see (3.21)). This result is called *de Moivre's theorem* and is often used in the manipulation of complex numbers. The theorem is valid for all n whether real, imaginary or complex.

There are numerous applications of de Moivre's theorem but this section examines just three: proofs of trigonometric identities; finding the nth roots of unity; and solving polynomial equations with complex roots.

3.4.1 Trigonometric identities

The use of de Moivre's theorem in finding trigonometric identities is best illustrated by example. We consider the expression of a multiple-angle function in terms of a polynomial in the single-angle function, and its converse.

►*Express* $\sin 3\theta$ *and* $\cos 3\theta$ *in terms of powers of* $\cos\theta$ *and* $\sin\theta$.

Using de Moivre's theorem,

$$\begin{aligned}\cos 3\theta + i\sin 3\theta &= (\cos\theta + i\sin\theta)^3 \\ &= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\sin\theta\cos^2\theta - \sin^3\theta).\end{aligned} \tag{3.28}$$

We can equate the real and imaginary coefficients separately, i.e.

$$\begin{aligned}\cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta \\ &= 4\cos^3\theta - 3\cos\theta\end{aligned} \tag{3.29}$$

and

$$\begin{aligned}\sin 3\theta &= 3\sin\theta\cos^2\theta - \sin^3\theta \\ &= 3\sin\theta - 4\sin^3\theta. \blacktriangleleft\end{aligned}$$

This method can clearly be applied to finding power expansions of $\cos n\theta$ and $\sin n\theta$ for any positive integer n.

The converse process uses the following properties of $z = e^{i\theta}$,

$$z^n + \frac{1}{z^n} = 2\cos n\theta, \tag{3.30}$$

$$z^n - \frac{1}{z^n} = 2i\sin n\theta. \tag{3.31}$$

These equalities follow from simple applications of de Moivre's theorem, i.e.

$$\begin{aligned}z^n + \frac{1}{z^n} &= (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n} \\ &= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta) \\ &= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta \\ &= 2\cos n\theta\end{aligned}$$

and

$$\begin{aligned}z^n - \frac{1}{z^n} &= (\cos\theta + i\sin\theta)^n - (\cos\theta + i\sin\theta)^{-n} \\ &= \cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta \\ &= 2i\sin n\theta.\end{aligned}$$

In the particular case where $n = 1$,

$$z + \frac{1}{z} = e^{i\theta} + e^{-i\theta} = 2\cos\theta, \tag{3.32}$$

$$z - \frac{1}{z} = e^{i\theta} - e^{-i\theta} = 2i\sin\theta. \tag{3.33}$$

►*Find an expression for* $\cos^3\theta$ *in terms of* $\cos 3\theta$ *and* $\cos\theta$.

Using (3.32),

$$\begin{aligned}\cos^3\theta &= \frac{1}{2^3}\left(z+\frac{1}{z}\right)^3\\ &= \frac{1}{8}\left(z^3+3z+\frac{3}{z}+\frac{1}{z^3}\right)\\ &= \frac{1}{8}\left(z^3+\frac{1}{z^3}\right)+\frac{3}{8}\left(z+\frac{1}{z}\right).\end{aligned}$$

Now using (3.30) and (3.32), we find

$$\cos^3\theta = \tfrac{1}{4}\cos 3\theta + \tfrac{3}{4}\cos\theta. \blacktriangleleft$$

This result happens to be a simple rearrangement of (3.29), but cases involving larger values of n are better handled using this direct method than by rearranging polynomial expansions of multiple-angle functions.

3.4.2 Finding the nth roots of unity

The equation $z^2 = 1$ has the familiar solutions $z = \pm 1$. However, now that we have introduced the concept of complex numbers we can solve the general equation $z^n = 1$. Recalling the fundamental theorem of algebra, we know that the equation has n solutions. In order to proceed we rewrite the equation as

$$z^n = e^{2ik\pi},$$

where k is any integer. Now taking the nth root of each side of the equation we find

$$z = e^{2ik\pi/n}.$$

Hence, the solutions of $z^n = 1$ are

$$z_{1,2,\ldots,n} = 1,\quad e^{2i\pi/n},\quad \ldots,\quad e^{2i(n-1)\pi/n},$$

corresponding to the values $0, 1, 2, \ldots, n-1$ for k. Larger integer values of k do not give new solutions, since the roots already listed are simply cyclically repeated for $k = n, n+1, n+2$, etc.

►*Find the solutions to the equation* $z^3 = 1$.

By applying the above method we find

$$z = e^{2ik\pi/3}.$$

Hence the three solutions are $z_1 = e^{0i} = 1$, $z_2 = e^{2i\pi/3}$, $z_3 = e^{4i\pi/3}$. We note that, as expected, the next solution, for which $k = 3$, gives $z_4 = e^{6i\pi/3} = 1 = z_1$, so that there are only three separate solutions. ◄

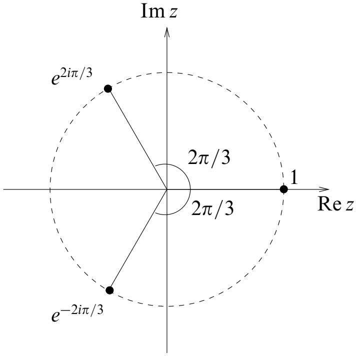


Figure 3.10 The solutions of $z^3 = 1$.

Not surprisingly, given that $|z^3| = |z|^3$ from (3.10), all the roots of unity have unit modulus, i.e. they all lie on a circle in the Argand diagram of unit radius. The three roots are shown in figure 3.10.

The cube roots of unity are often written 1, ω and ω^2. The properties $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$ are easily proved.

3.4.3 Solving polynomial equations

A third application of de Moivre's theorem is to the solution of polynomial equations. Complex equations in the form of a polynomial relationship must first be solved for z in a similar fashion to the method for finding the roots of real polynomial equations. Then the complex roots of z may be found.

►*Solve the equation* $z^6 - z^5 + 4z^4 - 6z^3 + 2z^2 - 8z + 8 = 0$.

We first factorise to give

$$(z^3 - 2)(z^2 + 4)(z - 1) = 0.$$

Hence $z^3 = 2$ or $z^2 = -4$ or $z = 1$. The solutions to the quadratic equation are $z = \pm 2i$; to find the complex cube roots, we first write the equation in the form

$$z^3 = 2 = 2e^{2ik\pi},$$

where k is any integer. If we now take the cube root, we get

$$z = 2^{1/3}e^{2ik\pi/3}.$$

To avoid the duplication of solutions, we use the fact that $-\pi < \arg z \leq \pi$ and find

$$\begin{aligned} z_1 &= 2^{1/3}, \\ z_2 &= 2^{1/3}e^{2\pi i/3} = 2^{1/3}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right), \\ z_3 &= 2^{1/3}e^{-2\pi i/3} = 2^{1/3}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right). \end{aligned}$$

The complex numbers z_1, z_2 and z_3, together with $z_4 = 2i$, $z_5 = -2i$ and $z_6 = 1$ are the solutions to the original polynomial equation.

As expected from the fundamental theorem of algebra, we find that the total number of complex roots (six, in this case) is equal to the largest power of z in the polynomial. ◄

A useful result is that the roots of a polynomial with real coefficients occur in conjugate pairs (i.e. if z_1 is a root, then z_1^* is a second distinct root, unless z_1 is real). This may be proved as follows. Let the polynomial equation of which z is a root be

$$a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = 0.$$

Taking the complex conjugate of this equation,

$$a_n^*(z^*)^n + a_{n-1}^*(z^*)^{n-1} + \cdots + a_1^* z^* + a_0^* = 0.$$

But the a_n are real, and so z^* satisfies

$$a_n(z^*)^n + a_{n-1}(z^*)^{n-1} + \cdots + a_1 z^* + a_0 = 0,$$

and is also a root of the original equation.

3.5 Complex logarithms and complex powers

The concept of a complex exponential has already been introduced in section 3.3, where it was assumed that the definition of an exponential as a series was valid for complex numbers as well as for real numbers. Similarly we can define the logarithm of a complex number and we can use complex numbers as exponents.

Let us denote the natural logarithm of a complex number z by $w = \operatorname{Ln} z$, where the notation Ln will be explained shortly. Thus, w must satisfy

$$z = e^w.$$

Using (3.20), we see that

$$z_1 z_2 = e^{w_1}e^{w_2} = e^{w_1+w_2},$$

and taking logarithms of both sides we find

$$\operatorname{Ln}(z_1 z_2) = w_1 + w_2 = \operatorname{Ln} z_1 + \operatorname{Ln} z_2, \tag{3.34}$$

which shows that the familiar rule for the logarithm of the product of two real numbers also holds for complex numbers.

We may use (3.34) to investigate further the properties of $\mathrm{Ln}\, z$. We have already noted that the argument of a complex number is multivalued, i.e. $\arg z = \theta + 2n\pi$, where n is any integer. Thus, in polar form, the complex number z should strictly be written as

$$z = re^{i(\theta+2n\pi)}.$$

Taking the logarithm of both sides, and using (3.34), we find

$$\mathrm{Ln}\, z = \ln r + i(\theta + 2n\pi), \tag{3.35}$$

where $\ln r$ is the natural logarithm of the real positive quantity r and so is written normally. Thus from (3.35) we see that $\mathrm{Ln}\, z$ is itself multivalued. To avoid this multivalued behaviour it is conventional to define another function $\ln z$, the *principal value* of $\mathrm{Ln}\, z$, which is obtained from $\mathrm{Ln}\, z$ by restricting the argument of z to lie in the range $-\pi < \theta \leq \pi$.

►*Evaluate* $\mathrm{Ln}\,(-i)$.

By rewriting $-i$ as a complex exponential, we find

$$\mathrm{Ln}\,(-i) = \mathrm{Ln}\, \left[e^{i(-\pi/2+2n\pi)}\right] = i(-\pi/2 + 2n\pi),$$

where n is any integer. Hence $\mathrm{Ln}\,(-i) = -i\pi/2,\ 3i\pi/2,\ \ldots$. We note that $\ln(-i)$, the principal value of $\mathrm{Ln}\,(-i)$, is given by $\ln(-i) = -i\pi/2$. ◄

If z and t are both complex numbers then the zth power of t is defined by

$$t^z = e^{z\,\mathrm{Ln}\, t}.$$

Since $\mathrm{Ln}\, t$ is multivalued, so too is this definition.

►*Simplify the expression* $z = i^{-2i}$.

Firstly we take the logarithm of both sides of the equation to give

$$\mathrm{Ln}\, z = -2i\,\mathrm{Ln}\, i.$$

Now inverting the process we find

$$e^{\mathrm{Ln}\, z} = z = e^{-2i\mathrm{Ln}\, i}.$$

We can write $i = e^{i(\pi/2+2n\pi)}$, where n is any integer, and hence

$$\begin{aligned}\mathrm{Ln}\, i &= \mathrm{Ln}\, \left[e^{i\left(\pi/2+2n\pi\right)}\right]\\ &= i\left(\pi/2 + 2n\pi\right).\end{aligned}$$

We can now simplify z to give

$$\begin{aligned}i^{-2i} &= e^{-2i\times i\left(\pi/2+2n\pi\right)}\\ &= e^{(\pi+4n\pi)},\end{aligned}$$

which, perhaps surprisingly, is a real quantity rather than a complex one. ◄

Complex powers and the logarithms of complex numbers are discussed further in chapter 24.

3.6 Applications to differentiation and integration

We can use the exponential form of a complex number together with de Moivre's theorem (see section 3.4) to simplify the differentiation of trigonometric functions.

► *Find the derivative with respect to x of $e^{3x}\cos 4x$.*

We could differentiate this function straightforwardly using the product rule (see subsection 2.1.2). However, an alternative method in this case is to use a complex exponential. Let us consider the complex number

$$z = e^{3x}(\cos 4x + i\sin 4x) = e^{3x}e^{4ix} = e^{(3+4i)x},$$

where we have used de Moivre's theorem to rewrite the trigonometric functions as a complex exponential. This complex number has $e^{3x}\cos 4x$ as its real part. Now, differentiating z with respect to x we obtain

$$\frac{dz}{dx} = (3+4i)e^{(3+4i)x} = (3+4i)e^{3x}(\cos 4x + i\sin 4x), \tag{3.36}$$

where we have again used de Moivre's theorem. Equating real parts we then find

$$\frac{d}{dx}\left(e^{3x}\cos 4x\right) = e^{3x}(3\cos 4x - 4\sin 4x).$$

By equating the imaginary parts of (3.36), we also obtain, as a bonus,

$$\frac{d}{dx}\left(e^{3x}\sin 4x\right) = e^{3x}(4\cos 4x + 3\sin 4x). \blacktriangleleft$$

In a similar way the complex exponential can be used to evaluate integrals containing trigonometric and exponential functions.

► *Evaluate the integral $I = \int e^{ax}\cos bx\,dx$.*

Let us consider the integrand as the real part of the complex number

$$e^{ax}(\cos bx + i\sin bx) = e^{ax}e^{ibx} = e^{(a+ib)x},$$

where we use de Moivre's theorem to rewrite the trigonometric functions as a complex exponential. Integrating we find

$$\begin{aligned}\int e^{(a+ib)x}\,dx &= \frac{e^{(a+ib)x}}{a+ib} + c\\ &= \frac{(a-ib)e^{(a+ib)x}}{(a-ib)(a+ib)} + c\\ &= \frac{e^{ax}}{a^2+b^2}\left(ae^{ibx} - ibe^{ibx}\right) + c,\end{aligned} \tag{3.37}$$

where the constant of integration c is in general complex. Denoting this constant by $c = c_1 + ic_2$ and equating real parts in (3.37) we obtain

$$I = \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c_1,$$

which agrees with result (2.37) found using integration by parts. Equating imaginary parts in (3.37) we obtain, as a bonus,

$$J = \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c_2. \blacktriangleleft$$

3.7 Hyperbolic functions

The *hyperbolic functions* are the complex analogues of the trigonometric functions. The analogy may not be immediately apparent and their definitions may appear at first to be somewhat arbitrary. However, careful examination of their properties reveals the purpose of the definitions. For instance, their close relationship with the trigonometric functions, both in their identities and in their calculus, means that many of the familiar properties of trigonometric functions can also be applied to the hyperbolic functions. Further, hyperbolic functions occur regularly, and so giving them special names is a notational convenience.

3.7.1 Definitions

The two fundamental hyperbolic functions are $\cosh x$ and $\sinh x$, which, as their names suggest, are the hyperbolic equivalents of $\cos x$ and $\sin x$. They are defined by the following relations:

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}), \tag{3.38}$$
$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}). \tag{3.39}$$

Note that $\cosh x$ is an even function and $\sinh x$ is an odd function. By analogy with the trigonometric functions, the remaining hyperbolic functions are

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \tag{3.40}$$
$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}, \tag{3.41}$$
$$\operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}, \tag{3.42}$$
$$\coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}. \tag{3.43}$$

All the hyperbolic functions above have been defined in terms of the real variable x. However, this was simply so that they may be plotted (see figures 3.11–3.13); the definitions are equally valid for any complex number z.

3.7.2 Hyperbolic–trigonometric analogies

In the previous subsections we have alluded to the analogy between trigonometric and hyperbolic functions. Here, we discuss the close relationship between the two groups of functions.

Recalling (3.32) and (3.33) we find

$$\cos ix = \tfrac{1}{2}(e^x + e^{-x}),$$
$$\sin ix = \tfrac{1}{2}i(e^x - e^{-x}).$$

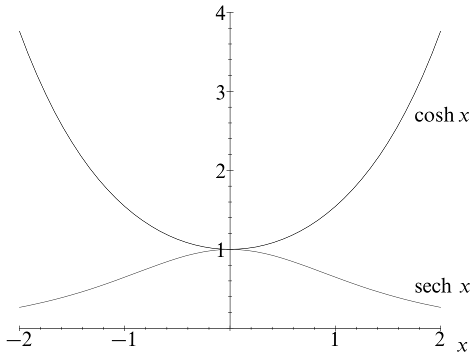


Figure 3.11 Graphs of $\cosh x$ and $\text{sech}x$.

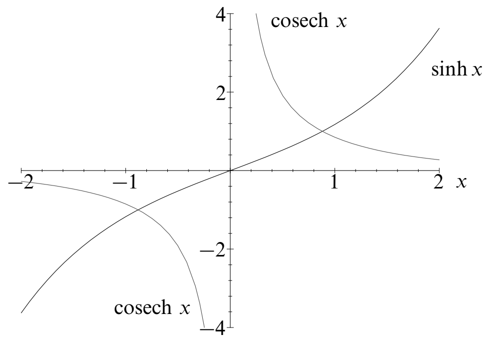


Figure 3.12 Graphs of $\sinh x$ and $\text{cosech}x$.

Hence, by the definitions given in the previous subsection,

$$\cosh x = \cos ix, \tag{3.44}$$

$$i \sinh x = \sin ix, \tag{3.45}$$

$$\cos x = \cosh ix, \tag{3.46}$$

$$i \sin x = \sinh ix. \tag{3.47}$$

These useful equations make the relationship between hyperbolic and trigono-

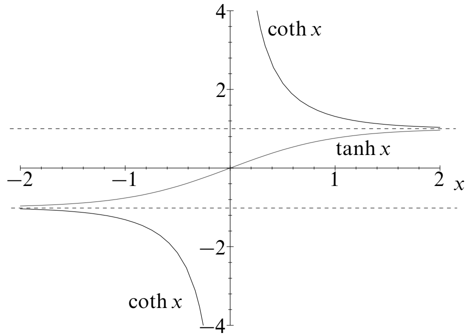


Figure 3.13 Graphs of $\tanh x$ and $\coth x$.

metric functions transparent. The similarity in their calculus is discussed further in subsection 3.7.6.

3.7.3 Identities of hyperbolic functions

The analogies between trigonometric functions and hyperbolic functions having been established, we should not be surprised that all the trigonometric identities also hold for hyperbolic functions, with the following modification. Wherever $\sin^2 x$ occurs it must be replaced by $-\sinh^2 x$, and vice versa. Note that this replacement is necessary even if the $\sin^2 x$ is hidden, e.g. $\tan^2 x = \sin^2 x/\cos^2 x$ and so must be replaced by $(-\sinh^2 x/\cosh^2 x) = -\tanh^2 x$.

►*Find the hyperbolic identity analogous to* $\cos^2 x + \sin^2 x = 1$.

Using the rules stated above $\cos^2 x$ is replaced by $\cosh^2 x$, and $\sin^2 x$ by $-\sinh^2 x$, and so the identity becomes

$$\cosh^2 x - \sinh^2 x = 1.$$

This can be verified by direct substitution, using the definitions of $\cosh x$ and $\sinh x$; see (3.38) and (3.39). ◄

Some other identities that can be proved in a similar way are

$$\operatorname{sech}^2 x = 1 - \tanh^2 x, \tag{3.48}$$

$$\operatorname{cosech}^2 x = \coth^2 x - 1, \tag{3.49}$$

$$\sinh 2x = 2 \sinh x \cosh x, \tag{3.50}$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x. \tag{3.51}$$

3.7.4 Solving hyperbolic equations

When we are presented with a hyperbolic equation to solve, we may proceed by analogy with the solution of trigonometric equations. However, it is almost always easier to express the equation directly in terms of exponentials.

►*Solve the hyperbolic equation* $\cosh x - 5\sinh x - 5 = 0$.

Substituting the definitions of the hyperbolic functions we obtain

$$\tfrac{1}{2}(e^x + e^{-x}) - \tfrac{5}{2}(e^x - e^{-x}) - 5 = 0.$$

Rearranging, and then multiplying through by $-e^x$, gives in turn

$$-2e^x + 3e^{-x} - 5 = 0$$

and

$$2e^{2x} + 5e^x - 3 = 0.$$

Now we can factorise and solve:

$$(2e^x - 1)(e^x + 3) = 0.$$

Thus $e^x = 1/2$ or $e^x = -3$. Hence $x = -\ln 2$ or $x = \ln(-3)$. The interpretation of the logarithm of a negative number has been discussed in section 3.5. ◄

3.7.5 Inverses of hyperbolic functions

Just like trigonometric functions, hyperbolic functions have inverses. If $y = \cosh x$ then $x = \cosh^{-1} y$, which serves as a definition of the inverse. By using the fundamental definitions of hyperbolic functions, we can find closed-form expressions for their inverses. This is best illustrated by example.

►*Find a closed-form expression for the inverse hyperbolic function* $y = \sinh^{-1} x$.

First we write x as a function of y, i.e.

$$y = \sinh^{-1} x \quad \Rightarrow \quad x = \sinh y.$$

Now, since $\cosh y = \frac{1}{2}(e^y + e^{-y})$ and $\sinh y = \frac{1}{2}(e^y - e^{-y})$,

$$\begin{aligned} e^y &= \cosh y + \sinh y \\ &= \sqrt{1 + \sinh^2 y} + \sinh y \\ e^y &= \sqrt{1 + x^2} + x, \end{aligned}$$

and hence

$$y = \ln(\sqrt{1 + x^2} + x). \ ◄$$

In a similar fashion it can be shown that

$$\cosh^{-1} x = \ln(\sqrt{x^2 - 1} + x).$$

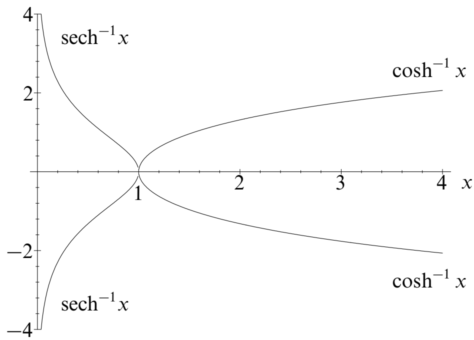


Figure 3.14 Graphs of $\cosh^{-1}x$ and $\mathrm{sech}^{-1}x$.

►*Find a closed-form expression for the inverse hyperbolic function* $y = \tanh^{-1}x$.

First we write x as a function of y, i.e.

$$y = \tanh^{-1}x \qquad \Rightarrow \qquad x = \tanh y.$$

Now, using the definition of $\tanh y$ and rearranging, we find

$$x = \frac{e^y - e^{-y}}{e^y + e^{-y}} \qquad \Rightarrow \qquad (x+1)e^{-y} = (1-x)e^y.$$

Thus, it follows that

$$\begin{aligned} e^{2y} = \frac{1+x}{1-x} \qquad \Rightarrow \qquad e^y &= \sqrt{\frac{1+x}{1-x}}, \\ y &= \ln\sqrt{\frac{1+x}{1-x}}, \\ \tanh^{-1}x &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right). \end{aligned}$$ ◄

Graphs of the inverse hyperbolic functions are given in figures 3.14–3.16.

3.7.6 Calculus of hyperbolic functions

Just as the identities of hyperbolic functions closely follow those of their trigonometric counterparts, so their calculus is similar. The derivatives of the two basic

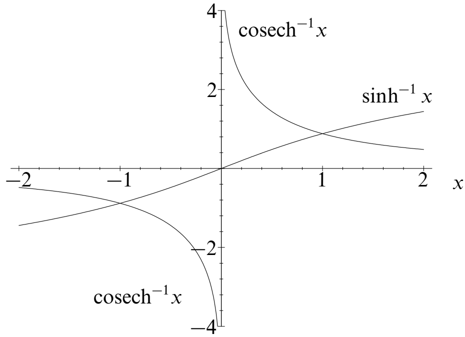


Figure 3.15 Graphs of $\sinh^{-1} x$ and $\operatorname{cosech}^{-1} x$.

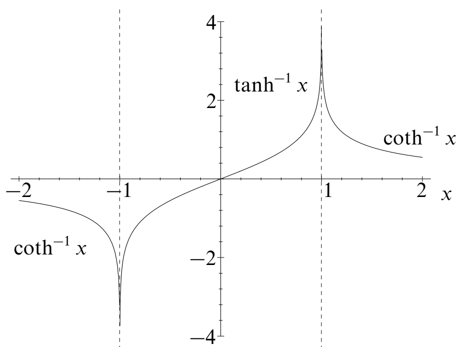


Figure 3.16 Graphs of $\tanh^{-1} x$ and $\coth^{-1} x$.

hyperbolic functions are given by

$$\frac{d}{dx}(\cosh x) = \sinh x, \tag{3.52}$$

$$\frac{d}{dx}(\sinh x) = \cosh x. \tag{3.53}$$

They may be deduced by considering the definitions (3.38), (3.39) as follows.

►*Verify the relation* $(d/dx)\cosh x = \sinh x$.

Using the definition of $\cosh x$,

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}),$$

and differentiating directly, we find

$$\begin{aligned}\frac{d}{dx}(\cosh x) &= \tfrac{1}{2}(e^x - e^{-x})\\ &= \sinh x. \blacktriangleleft\end{aligned}$$

Clearly the integrals of the fundamental hyperbolic functions are also defined by these relations. The derivatives of the remaining hyperbolic functions can be derived by product differentiation and are presented below only for completeness.

$$\frac{d}{dx}(\tanh x) = \mathrm{sech}^2 x, \tag{3.54}$$

$$\frac{d}{dx}(\mathrm{sech}\ x) = -\mathrm{sech}\ x \tanh x, \tag{3.55}$$

$$\frac{d}{dx}(\mathrm{cosech}\ x) = -\mathrm{cosech}\ x \coth x, \tag{3.56}$$

$$\frac{d}{dx}(\coth x) = -\mathrm{cosech}^2 x. \tag{3.57}$$

The inverse hyperbolic functions also have derivatives, which are given by the following:

$$\frac{d}{dx}\left(\cosh^{-1}\frac{x}{a}\right) = \frac{1}{\sqrt{x^2 - a^2}}, \tag{3.58}$$

$$\frac{d}{dx}\left(\sinh^{-1}\frac{x}{a}\right) = \frac{1}{\sqrt{x^2 + a^2}}, \tag{3.59}$$

$$\frac{d}{dx}\left(\tanh^{-1}\frac{x}{a}\right) = \frac{a}{a^2 - x^2}, \qquad \text{for } x^2 < a^2, \tag{3.60}$$

$$\frac{d}{dx}\left(\coth^{-1}\frac{x}{a}\right) = \frac{-a}{x^2 - a^2}, \qquad \text{for } x^2 > a^2. \tag{3.61}$$

These may be derived from the logarithmic form of the inverse (see subsection 3.7.5).

► *Evaluate $(d/dx)\sinh^{-1} x$ using the logarithmic form of the inverse.*

From the results of section 3.7.5,

$$\begin{aligned}\frac{d}{dx}\left(\sinh^{-1} x\right) &= \frac{d}{dx}\left[\ln\left(x+\sqrt{x^2+1}\right)\right]\\ &= \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{x}{\sqrt{x^2+1}}\right)\\ &= \frac{1}{x+\sqrt{x^2+1}}\left(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right)\\ &= \frac{1}{\sqrt{x^2+1}}. \blacktriangleleft\end{aligned}$$

3.8 Exercises

3.1 Two complex numbers z and w are given by $z = 3 + 4i$ and $w = 2 - i$. On an Argand diagram, plot

(a) $z + w$, (b) $w - z$, (c) wz, (d) z/w,
(e) $z^*w + w^*z$, (f) w^2, (g) $\ln z$, (h) $(1 + z + w)^{1/2}$.

3.2 By considering the real and imaginary parts of the product $e^{i\theta}e^{i\phi}$ prove the standard formulae for $\cos(\theta + \phi)$ and $\sin(\theta + \phi)$.

3.3 By writing $\pi/12 = (\pi/3) - (\pi/4)$ and considering $e^{i\pi/12}$, evaluate $\cot(\pi/12)$.

3.4 Find the locus in the complex z-plane of points that satisfy the following equations.

(a) $z - c = \rho\left(\dfrac{1+it}{1-it}\right)$, where c is complex, ρ is real and t is a real parameter that varies in the range $-\infty < t < \infty$.
(b) $z = a + bt + ct^2$, in which t is a real parameter and a, b, and c are complex numbers with b/c real.

3.5 Evaluate

(a) $\mathrm{Re}(\exp 2iz)$, (b) $\mathrm{Im}(\cosh^2 z)$, (c) $(-1 + \sqrt{3}i)^{1/2}$,
(d) $|\exp(i^{1/2})|$, (e) $\exp(i^3)$, (f) $\mathrm{Im}(2^{i+3})$, (g) i^i, (h) $\ln[(\sqrt{3} + i)^3]$.

3.6 Find the equations in terms of x and y of the sets of points in the Argand diagram that satisfy the following:

(a) $\mathrm{Re}\, z^2 = \mathrm{Im}\, z^2$;
(b) $(\mathrm{Im}\, z^2)/z^2 = -i$;
(c) $\arg[z/(z - 1)] = \pi/2$.

3.7 Show that the locus of all points $z = x + iy$ in the complex plane that satisfy

$$|z - ia| = \lambda|z + ia|, \quad \lambda > 0,$$

is a circle of radius $|2\lambda a/(1 - \lambda^2)|$ centred on the point $z = ia[(1 + \lambda^2)/(1 - \lambda^2)]$. Sketch the circles for a few typical values of λ, including $\lambda < 1$, $\lambda > 1$ and $\lambda = 1$.

3.8 The two sets of points $z = a$, $z = b$, $z = c$, and $z = A$, $z = B$, $z = C$ are the corners of two similar triangles in the Argand diagram. Express in terms of $a, b, \ldots, C$

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* **Limits:** $f(z)$ is a function of complex variable z. The limit of $f(z)$ as z approaches z_0 is a number w_0

$$\lim_{z \to z_0} f(z) = w_0$$

means the following. For each positive number ϵ there is a positive number δ such that

$$|f(z) - w_0| < \epsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta.$$

So if you make z arbitrarily close to z_0 then the function $f(z)$ will be close to w_0

* If limits exist then it is unique - we will not prove this - but it means the following. Suppose

$$\lim_{z \to z_0} f(z) = w_0^1 \quad \& \quad \lim_{z \to z_0} f(z) = w_0^2$$

then one can prove that $w_0^1 = w_0^2$

Example: Prove that $\lim_{z \to 1+i} (2+i)z = 1+3i$

Ans: To prove it we have to show that $\epsilon > 0$ & $\delta > 0$ exist such that whenever

$0 < |z-(1+i)| < \delta$ we will have $|(2+i)z-(1+3i)| < \epsilon$. So we have to find suitable ϵ & δ.

Start with the inequality $|(2+i)z-(1+3i)| < \epsilon$

$$\Rightarrow |2+i| \left| z - \frac{1+3i}{2+i} \right| < \epsilon$$

$$\Rightarrow \sqrt{5}\, |z-(1+i)| < \epsilon$$

$$\Rightarrow |z-(1+i)| < \epsilon/\sqrt{5}$$

Hence for any $\epsilon > 0$ we can have $\delta = \epsilon/\sqrt{5}$ and then this is valid.

* Limit independent of the path of approach: In the above examples $\underset{z\to z_0}{\mathrm{Lt}}$ means that the z is allowed to approach in arbitrary manner, not along some particular direction. In the next example we show this

Let $f(z) = z/z^*$ and we want to know if $z \to 0$ limit exist.

Let us approach $z \to 0$ along the real axis; ie $z = x + 0i$ then

$$\underset{z\to 0}{\mathrm{Lt}}\, f(z) = \frac{x+0i}{x-0i} = 1$$

If we approach along the Im(z) axis then $z = 0 + iy$

$$\lim_{z \to 0} f(z) = \frac{0 + iy}{0 - iy} = -1.$$

So there is no unique limit. Hence the limit does not exist.

* Some Theorems:

1. Suppose $f(z) = u(x, y) + i v(x, y)$; $z_0 = x_0 + i y_0$, & $w_0 = u_0 + i v_0$

Then $\lim_{z \to z_0} f(z) = w_0$ if and only if

$$\lim_{(x,y) \to (x_0, y_0)} u(x, y) = u_0 \quad \& \quad \lim_{(x,y) \to (x_0, y_0)} v(x, y) = v_0$$

2. Following can also be proven: Suppose $\lim_{z \to z_0} f(z) = w_0$ & $\lim_{z \to z_0} F(z) = W_0$

then

* $$\lim_{z \to z_0} [f(z) + F(z)] = \lim_{z \to z_0} f(z) + \lim_{z \to z_0} F(z) = w_0 + W_0$$

* $$\lim_{z \to z_0} [f(z)\, F(z)] = \lim_{z \to z_0} f(z) \lim_{z \to z_0} F(z) = w_0 W_0$$

* if $w_0 \neq 0$ then $\lim_{z \to z_0} \frac{f(z)}{F(z)} = \frac{\lim_{z \to z_0} f(z)}{\lim_{z \to z_0} F(z)} = \frac{w_0}{W_0}$

* For a Polynomial $P(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$

$$\lim_{z \to z_0} P(z) = P(z_0)$$

* Limits involving Infinity: If z_0 and w_0 are points in the z and w planes, then

$$* \lim_{z \to z_0} f(z) = \infty \text{ if and only if } \lim_{z \to z_0} f\left(\frac{1}{z}\right) = 0$$

and

$$* \lim_{z \to \infty} f(z) = w_0 \text{ if and only if } \lim_{z \to 0} f\left(\frac{1}{z}\right) = w_0$$

$$* \lim_{z \to \infty} f(z) = \infty \text{ if and only if } \lim_{z \to 0} \frac{1}{f(1/z)} = 0.$$

Example: Calculate the limit $\lim_{z \to i} \frac{(3+i)z^4 - z^2 + 2z}{z+1}$

Ans $\lim_{z \to i} \frac{(3+i)z^4 - z^2 + 2z}{z+1} = \frac{\lim_{z \to i} \left[(3+i)z^4 - z^2 + 2z\right]}{\lim_{z \to i} \left[z + i\right]}$

Now $\lim_{z\to i} z^4 = i^4 = 1$, $\lim_{z\to i} z^2 = -1$ using these we get

$$= \frac{4+3i}{1+i} = \frac{7}{2} - \frac{1}{2} i$$

Example: Compute the limit $\lim_{z\to 1+\sqrt{3}i} \frac{z^2-2z+4}{z-1-\sqrt{3}i}$

Ans: Note that if we apply the limit for the quotient then we get for the denominator $\lim_{z\to 1+\sqrt{3}i} (z-1-\sqrt{3}i) = 0$. But we can show that the limit exist

$$\lim_{z\to 1+\sqrt{3}i} \frac{z^2-2z+4}{z-1-\sqrt{3}i} = \lim_{z\to 1+\sqrt{3}i} \frac{(z-1+\sqrt{3}i)(z-1-\sqrt{3}i)}{z-1-\sqrt{3}i}$$

$$= \lim_{z\to 1+\sqrt{3}i} (z-1+\sqrt{3}i) = 2\sqrt{3}i$$

* Continuity: A function f is continuous at point z_0 if $\lim_{z\to z_0} f(z)$ exist, $f(z_0)$ exists, & $\lim_{z\to z_0} f(z) = f(z_0)$

Example: Let $f(z) = \begin{cases} z^2 & \text{for } z<0 \\ z-1 & \text{for } x\geq 0 \end{cases}$, then the $\lim_{z\to 0} f(z)$ does not exist

Since at $z=0$ the $f(z)$ is continuos.

Example: Suppose $f(z) = \frac{z^2-1}{z-1}$. Is the function continuos at $z=1$?

Ans: $\lim\limits_{z\to 1} \frac{z^2-1}{z-1} = \lim\limits_{z\to 1} (z+1) = 2.$

But at $z=1$, $f(1)$ is not defined. Hence not continuos.

* A function that is not continuos at a point is called a discontinuos function.

* Steps of checking continuity at a point: (i) Calculate the limit $\lim\limits_{z\to z_0} f(z)$ (ii) calculate $f(z_0)$. Compare the two values. If they are equal then the function is continuos.

* Derivatives: The derivative of $f(z)$ at $z=z_0$ is defined by

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$

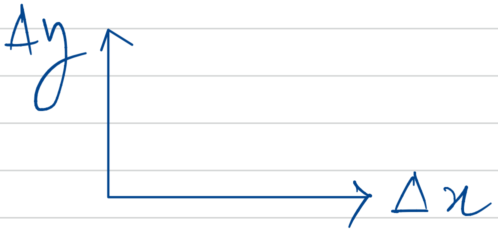
$$= \lim_{\Delta z\to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$$ provided that this limit

exists. When the derivative at z_0 exist then the function is called differentiable.

* An important about derivative about complex function is that even if there exist continuos partial derivatives of all orders of the real & imaginary parts, the derivative of the function itself may not exist.

Example: Let $f(z) = |z|^2$. In this case

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z+\Delta z)(z^* + \Delta z^*) - zz^*}{\Delta z}$$



$$= \lim_{z \to 0} \frac{\cancel{zz^*} + z\Delta z^* + z^*\Delta z + \Delta z \Delta z^* - \cancel{zz^*}}{\Delta z}$$

$$\lim_{\Delta z \to 0} \left[z^* + \Delta z^* + z \frac{\Delta z^*}{\Delta z} \right]$$

Now to find the limit we approach $\Delta z \to 0$ in two ways - along the real and imaginary axis. Now $\Delta z = \Delta x + i\Delta y$. If the approach is along the horizontal axis then

$$\Delta z^* = \Delta x + i0 = \Delta z$$

So $\lim_{\Delta z \to 0} \left[z^* + \Delta z^* + z \frac{\Delta z^*}{\Delta z} \right] = z^* + \Delta z^* + z$

For approach along the vertical axis it can be shown that

$$\lim_{\Delta z \to 0}\left[z^* + \Delta z^* + z\,\frac{\Delta z^*}{\Delta z}\right] = z^* + \Delta z^* - z$$

Since limit must be unique

$$z^* + \Delta z^* + z = z^* + \Delta z^* - z$$

$$\Rightarrow \quad z^* + z = z^* - z$$

$$\Rightarrow z = 0.$$

Hence derivative only exist at $z=0$, and not in other points.
But note that $f(z) = |z|^2 = x^2+y^2 + i0$. The $u = x^2+y^2$, $v = 0$ has derivative at all points.

* Cauchy-Riemann Equations: This is a set of equations that determines if derivative of a function exists. To obtain this set of equations we take the limit

$$\lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = \lim_{\delta z \to 0} \frac{\delta f(z)}{\delta z}$$

in two different approaches - along $\delta x = 0$ & along $\delta y = 0$

Now $\delta z = \delta x + i\delta y$. Since we can write $f(z) = u + iv$

$$\delta f = \delta u + i\delta v$$

Hence $$\frac{\delta f}{\delta z} = \frac{\delta u + i\delta v}{\delta x + i\delta y}$$

Now first we take the limit along $\delta y = 0$ with $\delta x \to 0$

$$\lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \lim_{\substack{\delta x \to 0 \\ \delta y = 0}} \left(\frac{\delta u}{\delta x} + i \frac{\delta v}{\delta x} \right) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

Similarly along the path $\delta x = 0$ with $\delta y \to 0$ we get

$$\lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \lim_{\substack{\delta x = 0 \\ \delta y \to 0}} \left(-i \frac{\delta u}{\delta y} + \frac{\delta v}{\delta x} \right) = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Since these two limits must be same, equating we get

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating the real & the imaginary parts seperately we get

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

These are the Cauchy-Riemann conditions. If these conditions are satisfied and the partial derivative of $u(x,y)$ & $v(x,y)$ exist then the derivative $\boxed{df/dz = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}}$

* CR Equations in Polar Form: $z = re^{i\theta}$ & $f(z) = u(r,\theta) + i\,v(r,\theta)$. Then the CR equations are

$$\boxed{r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \quad \& \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}}$$

Example: Determine if $f(z) = z^2$ is differentiable everywhere.

Ans: $f(z) = z^2 = x^2 - y^2 + i\,2xy \Rightarrow u = x^2 - y^2, \; v = 2xy$

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$ Hence CR satisfied.

The derivative is $$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2x + 2iy$$

* The CR conditions itself are not the sufficient conditions for $f'(z)$ to exist. It also needs some conditions about the continuity of u & v

Theorem: Let $f(z) = u(x,y) + i v(x,y)$ be defined through some region in the complex plane, and suppose the first order partial derivative of $u(x,y)$ & $v(x,y)$ with respect to x & y exist everywhere in that region. If the partial derivative satisfy the CR equations at $z_0 = x_0 + i y_0$ the the derivative $f'(z)$ exist at $z_0 = x_0 + i y_0$.