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Real Analysis Assignment H2-A2

Ans-1 Given :- $\vec{V} = (2xy + z^2)\hat{i} + (2yz + x^2)\hat{j} + (2xz + y^2)\hat{k}$

To show :- vector function $\vec{V}$ is a conservative vector field.

we know that if $\vec{V}$ is a conservative vector field then $\vec{V}$ will be irrotational

i.e $\vec{\nabla} \times \vec{V} = \vec{0}$ (Curl of $\vec{V}$ is 0).

$$\vec{\nabla} \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \vec{V_x} & \vec{V_y} & \vec{V_z} \end{vmatrix}$$

$$= \left(\frac{\partial \vec{V_z}}{\partial y} - \frac{\partial \vec{V_y}}{\partial z}\right)\hat{i} - \left(\frac{\partial \vec{V_z}}{\partial x} - \frac{\partial \vec{V_x}}{\partial z}\right)\hat{j} + \hat{k}\left(\frac{\partial \vec{V_y}}{\partial x} - \frac{\partial \vec{V_x}}{\partial y}\right)$$

As $\vec{V} = (2xy + z^2)\hat{i} + (2yz + x^2)\hat{j} + (2xz + y^2)\hat{k}$ (given)

$\Rightarrow \vec{V_x} = 2xy + z^2$, $\vec{V_y} = 2yz + x^2$, $\vec{V_z} = 2xz + y^2$

$\frac{\partial \vec{V_z}}{\partial y} = 2y$ $\qquad$ $\frac{\partial \vec{V_y}}{\partial x} = 2x$

$\frac{\partial \vec{V_y}}{\partial z} = 2y$ $\qquad$ $\frac{\partial \vec{V_x}}{\partial y} = 2x$

$\frac{\partial \vec{V_z}}{\partial x} = 2z$

$\frac{\partial \vec{V_x}}{\partial z} = 2z$

Now putting values of all in $\vec{\nabla} \times \vec{V}$,

$$= (2y - 2y)\hat{i} - (2z - 2z)\hat{j} + (2x - 2x)\hat{k}$$

$$= \vec{0}.$$

Hence it is showed that $\vec{V}$ is a conservative vector field.

To find :- Scalar potential of $\vec{V}$ (ϕ)

s.t $\vec{V} = \vec{\nabla}\phi$

we know that $\vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$

Now on comparing it with $\vec{V}$,

$$\frac{\partial \phi}{\partial x} = V_x = 2xy + z^2$$

Now on integrating it,

$$\phi(x,y,z) = x^2y + z^2x + f_1(y,z) \quad -1)$$

$$\frac{\partial \phi}{\partial y} = 2yz + x^2$$

Now on integrating it,

$$\phi(x,y,z) = y^2z + x^2y + f_2(x,z) \quad -2)$$

$$\frac{\partial \phi}{\partial z} = V_z = 2xz + y^2$$

Now on integrating it,

$$\phi(x,y,z) = xz^2 + y^2z + f_3(x,y) \quad -3)$$

Now on comparing 1, 2 and 3 simultaneously,

$f_1(y,z) = y^2z$, $f_2(x,z) = xz^2$, $f_3(x,y) = x^2y$

$\Rightarrow \phi(x,y,z) = x^2y + z^2x + y^2z + C$ Ans.

Ans-2 Given : Right circular cylinder

To prove :- Gauss Divergence theorem for the right circular cylinder i.e $\oint \vec{F}\cdot\vec{ds} = \int_V \vec{\nabla}\cdot\vec{F}\, dv$

Proof :- LHS

$\oint \vec{F}\cdot\vec{ds} = \sum \vec{F}\cdot\hat{n}\,\Delta S$

$\Rightarrow$ The sum is overall the points (ρ, ϕ, z) around which the infinitesimal volume is considered

The contribution from the $d\rho dz$ side will be given by,

$\Rightarrow F_\phi(\rho, \phi+d\phi, z)\, d\rho dz - F_\phi(\rho, \phi, z)\, d\rho dz$

$= \frac{\partial F_\phi}{\rho\,\partial\phi}\, \rho\, d\phi\, d\rho\, dz$

$\Rightarrow$ Contribution from the $d\rho\, \rho d\phi$ surface will be,

$= [F_z(\rho, \phi, z+dz) - F_z(\rho, \phi, z)]\, d\rho\, \rho d\phi$

$= \frac{\partial F_z}{\partial z}\, dz\, d\rho\, \rho d\phi$

⇒ Contribution from the $dz\,\rho d\phi$ surface,

$$F_\rho(\rho+d\rho, \phi, z)\,dz\,\rho d\phi - F_\rho(\rho, \phi, z)\,dz\,\rho d\phi$$

$$= [F_\rho(\rho+d\rho, \phi, z) - F_\rho(\rho, \phi, z)]\,dz\,\rho d\phi$$

$$= \frac{\partial F_\rho}{\partial \rho}\,d\rho\,dz\,\rho d\phi$$

So Now summing the contribution from the infinitesimal volume $= \frac{\partial F_\phi}{\rho\,\partial \phi}\,\rho d\phi\,d\rho\,dz$

$$+ \frac{\partial F_\rho}{\partial \rho}\,d\rho\,dz\,\rho d\phi + \frac{\partial F_z}{\partial z}\,dz\,\rho d\phi\,d\rho$$

$$= (\vec{\nabla}\cdot\vec{F})\,dv$$

⇒ Summing up over all the points, we get

$$\oint \vec{F}\cdot\vec{ds} = \sum \vec{F_i}\cdot ds_i = \sum \vec{F_i}\cdot\hat{n}_i\,\Delta S_i$$

$$= \sum \vec{\nabla}\cdot\vec{F}\,dv$$

$$= \int_V \vec{\nabla}\cdot\vec{F}\,dv = \text{RHS}$$

LHS = RHS

Hence proved.

Ans-3 To prove :- $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}^2\vec{A}$

Proof :- we will first calculate LHS and RHS and then prove them equal.

let $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y\right)\hat{i} - \hat{j}\left(\frac{\partial}{\partial x}A_z - \frac{\partial}{\partial z}A_x\right) + \left(\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x\right)\hat{k}$$

$\vec{\nabla}\times(\vec{\nabla}\times\vec{A})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y & -\left(\frac{\partial}{\partial x}A_z - \frac{\partial}{\partial z}A_x\right) & \frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x \end{vmatrix}$$

$$= \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x + \frac{\partial}{\partial z}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\right)\hat{i} + \left(\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x\right) - \frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y\right)\right)\hat{j}$$

$$+ \left(-\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}A_z - \frac{\partial}{\partial z}A_x\right) - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y\right)\right)\hat{k}$$

$$= \left(\frac{\partial^2 A_y}{\partial y\partial x} - \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_z}{\partial x\partial z} - \frac{\partial^2 A_x}{\partial z^2}\right)\hat{i}$$

$$+ \left(\frac{\partial^2 A_y}{\partial x^2} - \frac{\partial^2 A_x}{\partial x\partial y} - \frac{\partial^2 A_z}{\partial y\partial z} + \frac{\partial^2 A_y}{\partial z^2}\right)\hat{j}$$

$$+ \left(-\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_x}{\partial x\partial z} - \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_y}{\partial y\partial z}\right)\hat{k}$$

$= LHS$

Now solving RHS

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) = \frac{\partial}{\partial x}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\hat{i}$$

$$+ \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\hat{j} + \frac{\partial}{\partial z}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)$$

$$= \left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial x\partial y} + \frac{\partial^2 A_z}{\partial x\partial z}\right)\hat{i} + \left(\frac{\partial^2 A_x}{\partial x\partial y} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_z}{\partial y\partial z}\right)$$

$$+ \left(\frac{\partial^2 A_x}{\partial x\partial z} + \frac{\partial^2 A_y}{\partial z\partial y} + \frac{\partial^2 A_z}{\partial z^2}\right)\hat{k}$$

$$\nabla^2\vec{A} = \nabla^2 A_x\hat{i} + \nabla^2 A_y\hat{j} + \nabla^2 A_z\hat{k}$$

$$= \left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}\right)\hat{i}$$

$$+ \left(\frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2}\right)\hat{j}$$

$$+ \left(\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2}\right)\hat{k}$$

Now doing $\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$

$$= \left(\frac{\partial^2 A_y}{\partial x\partial y} + \frac{\partial^2 A_z}{\partial x\partial z} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2}\right)\hat{i}$$

$$+ \left(\frac{\partial^2 A_x}{\partial x\partial y} + \frac{\partial^2 A_z}{\partial y\partial z} - \frac{\partial^2 A_y}{\partial x^2} - \frac{\partial^2 A_y}{\partial z^2}\right)\hat{j}$$

$$+ \left(\frac{\partial^2 A_x}{\partial x\partial z} + \frac{\partial^2 A_y}{\partial z\partial y} - \frac{\partial^2 A_z}{\partial x^2} - \frac{\partial^2 A_z}{\partial y^2}\right)\hat{k}$$

we can clearly see that LHS = RHS.
Hence it is proved that

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$$

Ans-4 To show :- $\lim_{z\to\infty} \frac{2z^3-1}{z^2+1} = \infty$

In order to prove that,
let $f(z) = \frac{2z^3-1}{1+z^2}$

if we prove that $\lim_{z\to 0} \frac{1}{f(\frac{1}{z})} \to 0$ then it will automatically prove that $\lim_{z\to\infty} \frac{2z^3-1}{z^2+1} = \infty$

$$f\left(\frac{1}{z}\right) = \frac{\frac{2}{z^3}-1}{1+\frac{1}{z^2}} = \frac{2-z^3}{z(1+z^2)}$$

$$\Rightarrow \lim_{z\to 0} \frac{z(1+z^2)}{2-z^3}$$

let $z = x+iy$
As $z\to 0 \Rightarrow$ we are approaching to origin.
So we know limit only exists, if we find a unique value when approaching origin from all possible directions.

1) Approaching along Y-axis i.e $x=0$
$$\lim_{y\to 0} \frac{iy(1+(iy)^2)}{2-(iy)^3} = \frac{iy(1-y^2)}{2+iy^3} = 0$$

2) Approaching along X-axis i.e $y=0$
$$\lim_{x\to 0} \frac{x(1+x^2)}{2-x^3} = 0$$

3) Approaching along line $y=mx$ and $x\to 0$, $y\to 0$ as $z\to 0$
$$\lim_{x\to 0} \frac{(x+imx)(1+(x+imx)(x+imx))}{2-(x+imx)(x+imx)(x+imx)} = 0$$

Since we get 0 as a unique solution to the limit when we approach origin from all directions.
Hence the limit exists and is equal to zero.
So by this it is proved that
$$\lim_{z\to\infty} \frac{2z^3-1}{1+z^2} = \infty.$$

Ans-5 To show :- If $f(z) = \sqrt{r}e^{i\theta/2}$

Then $f'(z) = \frac{1}{2f(z)}$

As $f(z) = \sqrt{r}(\cos\theta/2 + i\sin\theta/2)$

$= \sqrt{r}\cos\theta/2 + i\sqrt{r}\sin\theta/2$

let $f(z) = u(r,\theta) + iv(r,\theta)$

Firstly we will check whether $f(z)$ is differentiable or not by CR equations.

For polar form, CR equations (Cauchy Riemann equations) :-

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \quad -1) \text{ and } \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \quad -2)$$

1) $\frac{\partial u}{\partial r} = \frac{1}{2\sqrt{r}}\cos\theta/2$

$\frac{\partial v}{\partial \theta} = \sqrt{r}\cos\theta/2 \times \frac{1}{2}$

$\frac{r\cos\theta/2}{2\sqrt{r}} = \sqrt{r}\cos\theta/2 \times \frac{1}{2}$

Hence proved the first one.

2) $\frac{\partial u}{\partial \theta} = \sqrt{r} - \sin\theta/2 \times \frac{1}{2}$

$\frac{\partial v}{\partial r} = \frac{1}{2\sqrt{r}}\sin\theta/2$

$-\sqrt{r}\sin\theta/2 \times \frac{1}{2} = r \times \frac{1}{2\sqrt{r}}\sin\theta/2$

Hence proved.

This proves that $f(z)$ is differentiable

Now we know by these equations that

$f'(z) = \frac{\partial u}{\partial r} - i\frac{\partial v}{\partial r}$

$= \frac{1}{2\sqrt{r}}\cos\theta/2 - i \times \frac{1}{2\sqrt{r}}\sin\theta/2$

$$= \frac{1}{2\sqrt{r}}\left(\cos\theta/_2 - i\sin\theta/_2\right) \times \frac{\cos\theta/_2 + i\sin\theta/_2}{\cos\theta/_2 + i\sin\theta/_2}$$

$$= \frac{1}{2\sqrt{r}} \times \frac{(\cos^2\theta/_2 + \sin^2\theta/_2)}{\cos\theta/_2 + i\sin\theta/_2}$$

$$= \frac{1}{2f(z)}$$

Hence Showed.

Ans-6 Given :- $f(z) = e^{-\theta}\cos(\ln r) + ie^{-\theta}\sin(\ln r)$
where $r > 0$, $0 \le r < 2\pi$

To show :- $f'(z) = \dfrac{i f(z)}{z}$

Proof :- $f(z) = e^{-\theta}(\cos(\ln r) + i\sin(\ln r))$
we know by Euler form, $\cos\phi + i\sin\phi = e^{i\phi}$

$$\Rightarrow f(z) = e^{-\theta} e^{i\ln r}$$
$$= e^{-\theta} e^{\ln r^i} = r^i e^{-\theta}$$

Let $z = re^{i\theta}$
$$z^i = r^i e^{i^2\theta} = r^i e^{-\theta}$$
$$\Rightarrow f(z) = z^i$$

Taking log on both sides,
$$\ln(f(z)) = \ln(z^i) = i\ln z$$

It is to be noted that $\ln(f(z))$ is a function, so it can differentiated but $\ln z$ is not directly differentiated and written as $\frac{1}{z}$.
We will first have to prove it.

$$d(\ln z) = \lim_{\Delta z \to 0} \frac{\ln(z+\Delta z) - \ln z}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\ln\left(\frac{z+\Delta z}{z}\right)}{\Delta z} = \frac{\ln\left(1+\frac{\Delta z}{z}\right)}{\Delta z}$$

we know that $\ln(1+x)$ when $x \to 0 \Rightarrow x$

$$\Rightarrow \lim_{\Delta z \to 0} \frac{\ln\left(1+\frac{\Delta z}{z}\right)}{\Delta z} = \frac{\Delta z}{z \times \Delta z} = \frac{1}{z}$$

$\ln(f(z)) = i \ln z$

Now differentiating on both sides,

$$\frac{f'(z)}{f(z)} = i \times \frac{1}{z}$$

$$f'(z) = i \frac{f(z)}{z}$$

Hence proved.