MA 6.101

Probability and Statistics

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Mixed Random variables

- Random variables that are neither continuous nor discrete are called as mixed random variables.
- Their CDF is partly continuous and partly piece-wise continuous.
- Example: X is a $U[0, 1]$ random variable and $Y = X$ if $X \leq 0.5$ and $Y = 0.5$ if $X > 0.5$.
- What is the CDF and PDF of Y?

Mixed Random variables

- Let $F_Y(y) = C(y) + D(y)$ where $C(y)$ corresponds to the continuous part and $D(y)$ for the discontinuous part.

- $$E[Y] = \int_{-\infty}^{\infty} xc(x)dx + \sum_{y_k} y_k P(Y = y_k)$$

  where $\{y_1, y_2, \ldots\}$ are jump points of $D(y)$ where $P(Y = y_k) > 0$.

- See section 4.3.1 from probabilitycourse.com for more examples

- Amount of workload (pending) on a server! A server on a cluster may be idle with a finite probability. If busy, the pending work is a continuous random variable.

Multiple random variables

A running example

- Consider an experiment of tossing a coin and a dice together.
- $\Omega = \{0, 1\} \times \{1, 2, 3, 4, 5, 6\}$. $\mathcal{F} = 2^{\Omega}$. $\mathbb{P}(\omega) = \frac{1}{12}$.
- Let X and Y denote the random variables depicting outcome of a coin and dice respectively.
- For $\omega = (1, 5)$ we have $X(\omega) = 1$ and $Y(\omega) = 5$.
- We are now interested in the joint PMF $p_{XY}(x, y)$ and joint CDF $F_{XY}(x, y)$ of X and Y together.

An example

- We are now interested in the joint PMF $p_{XY}(x, y)$ and joint CDF $F_{XY}(x, y)$ of X and Y together.
- $p_{XY}(x, y) := \mathbb{P}\{\omega \in \Omega : X(w) = x \text{ and } Y(\omega) = y\}$.
- $F_{XY}(x, y) := \mathbb{P}\{\omega \in \Omega : X(w) \leq x \text{ and } Y(\omega) \leq y\}$.
- We can use PMF to calculate $P((X, Y) \in A)$.
- $P((X, Y) \in A) = \mathbb{P}\{\omega \in \Omega : (X(\omega), Y(\omega)) \in A\}$. Therefore $P((X, Y) \in A) = \sum_{(x,y)\in A} p_{XY}(x, y)$.
- Suppose A is the event that you get a head and the roll is even. What is $P((X, Y) \in A)$?

Marginals

- What is $p_{XY}(1, i)$? ($= \frac{1}{12}$).
- $\sum_i p_{XY}(1, i) = \mathbb{P}\{\omega \in \Omega : X(\omega) = 1\} = \frac{1}{2} = p_X(x)$.
- Similarly, $p_{XY}(1, i) + p_{XY}(0, i) = \frac{1}{6} = p_Y(i)$.

> The marginal PMF's p_X and p_Y can be obtained from the joint PMF as follows:
> $p_X(x) = \sum_y p_{XY}(x, y)$ and $p_Y(y) = \sum_x p_{XY}(x, y)$.

This is true in general, and requires a proof.

Marginals

The marginal PMF's p_X and p_Y can be obtained from the joint PMF as follows:
$p_X(x) = \sum_y p_{XY}(x,y)$ and $p_Y(y) = \sum_x p_{XY}(x,y)$.

Proof:

$$
\begin{aligned}
P_X(x) &= \mathbb{P}\{\omega \in \Omega : X(\omega) = x\} \\
&= \mathbb{P}\{\bigcup_y \{\omega \in \Omega : X(\omega) = x, Y(\omega) = y\}\} \\
&= \sum_y \mathbb{P}\{\{\omega \in \Omega : X(\omega) = x, Y(\omega) = y\}\}
\end{aligned}
$$

Independence

- Back with the running example of coin and dice.
- Write down $p_{XY}(x, y)$ and $F_{XY}(x, y)$.
- Notice that $p_{XY}(1, i) = p_X(1)p_Y(i)$ and $F_{XY}(1, i) = F_X(1)F_Y(i)$.
- In general, if $p_{XY}(x, y) = p_X(x)p_Y(y)$ and $F_{XY}(x, y) = F_X(x)F_Y(y)$ we say X and Y are independent.

> Two random variables, X and Y are independent if the following is true:
> $p_{XY}(x, y) = p_X(x)p_Y(y)$ and $F_{XY}(x, y) = F_X(x)F_Y(y)$

Independence

Two random variables, X and Y are independent if the following is true:
$p_{XY}(x, y) = p_X(x)p_Y(y)$ and $F_{XY}(x, y) = F_X(x)F_Y(y)$

- How does this relate to $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$?
- $A = \{\omega \in \Omega : X(\omega) \leq x\}$ and $B = \{\omega \in \Omega : Y(\omega) \leq y\}$.
- $F_{XY}(x, y) := \mathbb{P}\{\omega \in \Omega : X(w) \leq x$ and $Y(\omega) \leq y\} = \mathbb{P}(A \cap B)$.

$E[XY]$

- $E[X] = \sum_x xp_X(x)$ and $E[Y] = \sum_y yp_Y(y)$
- $E[X] = \sum_x \sum_y xp_{XY}(x, y)$ and $E[Y] = \sum_x \sum_y yp_{XY}(x, y)$
- How do we define $E[XY]$?
- You want to search over all values $X \times Y$ can take $(\{1, 2, .., 6\})$ and weight it by the corresponding probabilities.
- $E[XY] = \sum_x \sum_y xyp_{XY}(x, y) = 1.75 = E[X]E[Y]$.

If X and Y are independent, $E[XY] = E[X]E[Y]$.

Example where X and Y are Dependent

- Now consider rolling a dice.
- $X = \begin{cases} 1 \text{ if outcome is odd} \\ 0 \text{ otherwise .} \end{cases}$ and $Y = \begin{cases} 1 \text{ if outcome is even} \\ 0 \text{ otherwise .} \end{cases}$
- What is $p_X(x), p_Y(y), p_{XY}(x, y)$ and $F_{XY}(x, y)$?
- What is $E[XY]$?