Simple Harmonic Oscillator

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2$$

Schrodinger equation

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + \frac{1}{2} m \omega^2 x^2 \psi(x) = E\psi$$

$$\frac{d^2 \psi}{dx^2} + \left( \frac{2mE}{\hbar^2} - \frac{m^2 \omega^2}{\hbar^2} x^2 \right) \psi = 0$$

$$\sqrt{\frac{m\omega}{\hbar}}\, x = y$$

$$\frac{d^2 \psi}{dy^2} + \left( \frac{2E}{\hbar\omega} - y^2 \right) \psi = 0$$

$y \to \infty$

$$\frac{d^2\psi}{dy^2} - y^2\psi = 0 \Rightarrow \psi \sim e^{-y^2/2}$$

$$\psi(y) = u(y)\, e^{-y^2/2}$$

$$\frac{d^2 u}{dy^2} - 2y\frac{du}{dy} + \left(\frac{2E}{\hbar\omega} - 1\right)u = 0$$

$$u(y) = \sum_{n=0}^{\infty} a_n y^n \quad — \text{(1)}$$

$$\Rightarrow \boxed{a_{n+2} = \frac{2n+1 - 2E/\hbar\omega}{(n+2)(n+1)}\, a_n}$$

For energy eigenvalues when the series in (1) is truncated to n th polynomial

$$E = \left(n + \tfrac{1}{2}\right)\hbar\omega.$$

~~Creation~~

operators.

Using ladder operators

$$\hat{H} = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

$$= \hbar\omega \left[ \frac{p^2}{2m\hbar\omega} + \frac{m\omega}{2\hbar} x^2 \right]$$

$$x_0 = \sqrt{\frac{2\hbar}{m\omega}} \qquad p_0 = \sqrt{2m\hbar\omega}$$

$$x_0 p_0 = 2\hbar$$

$$= \hbar\omega \left[ \frac{p^2}{p_0^2} + \frac{x^2}{x_0^2} \right]$$

$$= \hbar\omega \underbrace{\left[ \frac{p}{p_0} + i\frac{x}{x_0} \right]}_{a} \underbrace{\left[ \frac{p}{p_0} - i\frac{x}{x_0} \right]}_{a^+}$$

So: $$a a^+ = \frac{p^2}{p_0^2} + \frac{x^2}{x_0^2} + i\hbar [x, p]$$

$$= \frac{p^2}{p_0^2} + \frac{x^2}{x_0^2} - \frac{1}{2}$$

$$\frac{p^2}{p_0^2} + \frac{x^2}{x_0^2} = aa^\dagger + \frac{1}{2}$$

$$\mathcal{H} = \left(aa^\dagger + \frac{1}{2}\right)\hbar\omega.$$

$$[a, a^\dagger] = \left[\frac{p}{p_0} + i\frac{x}{x_0}, \frac{p}{p_0} - i\frac{x}{x_0}\right]$$

$$[a, a^\dagger] = 1.$$

$$[\hat{H}, \hat{a}] = \hbar\omega\,[a^\dagger a, a]$$

$$= \hbar\omega\left(a^\dagger a a - a a^\dagger a\right)$$

$$= \hbar\omega\left(a^\dagger a - a a^\dagger\right)a = -\hbar\omega a$$

$$[\hat{H}, \hat{a}] = -\hat{a}\,\hbar\omega$$

$$[\hat{H}, a^\dagger] = \hat{a}^\dagger\,\hbar\omega$$

Eigenstates of $\hat{H}$

$$\mathcal{H}|\phi_E\rangle = E|\phi_E\rangle$$

$$\Psi = a|\phi_E\rangle$$

$$\hat{H}\Psi = \hat{H}\, a|\phi_E\rangle$$

$$Ha - aH = -a\hbar\omega$$

$$Ha = aH - a\hbar\omega$$

$$= a(H - \hbar\omega)$$

$$H|\Psi\rangle = a(H - \hbar\omega)|\phi_E\rangle$$

$$= a(E - \hbar\omega)|\phi_E\rangle$$

$$H\, a|\phi_E\rangle = a(E - \hbar\omega)|\phi_E\rangle$$

$$H\,\overbrace{a|\Psi_E\rangle}^{\Psi} = (E - \hbar\omega)\,\overbrace{a|\phi_E\rangle}^{\Psi}$$

$$|\Psi\rangle = |\phi_{E-\hbar\omega}\rangle$$

$$|1\rangle = |E-\hbar\omega\rangle$$

$$\boxed{\begin{gathered} a\,|\phi_E\rangle = |\phi_{E-\hbar\omega}\rangle \\ \text{Similarly} \\ a^\dagger\,|\phi_E\rangle = |\phi_{E+\hbar\omega}\rangle \end{gathered}}$$

Ladder operator.

$$a\,|\phi_0\rangle = 0$$

$$\left(\frac{p}{p_0} + i\frac{x}{x_0}\right)\phi_0 = 0$$

$$i\left[-\frac{\hbar}{p_0}\frac{\partial\phi_0}{\partial x} + \frac{x}{x_0}\phi_0\right] = 0$$

$$\phi_0(x) = A\,e^{-x^2/2}$$

$$-\frac{1}{2}\frac{\partial\phi_0}{\partial x} + \frac{x}{x_0^2}\phi_0 = 0$$

$$\phi_0(x) = A\,e^{-x^2/x_0^2}$$