NT Problems

September 22, 2016

1. Let $\sigma(n)$ be the sum of the divisors of n, and let $\varphi(n)$ be the number of $0 \le d < n$ such that $(n, d) = 1$. Compute

$$\sum_{d|2016} \frac{\phi(d)}{\sigma(d)}.$$

Here, the sum goes over the divisors of n. For example, we have that $\sum_{d|15} d^2 = 1^2 + 3^2 + 5^2 + 15^2$.

2. Compute $\tau(1) + \cdots + \tau(100)$ where $\tau(n)$ is the number of divisors of n.

3. It is well known that $\sum_n n^{-2} = \frac{\pi^2}{6}$. Let $t_j(n)$ be the number of ways to write n as the product of j integers greater than 1, where the order matters. Compute $\sum_n t_2(n)n^{-2}$.

4. Call a some n *good* if there does not exist some prime whose square exactly divides it, i.e. $p^2|n \Rightarrow p^3|n$. Show that there exist arbitrarily large gaps between consecutive *good* numbers.

5. Define the Ramanujan sum $c_q(n)$ as

$$c_q(n) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(-\frac{an}{q}\right).$$

Show that

$$c_q(n) = \frac{\mu(q/(q,n))\varphi(q)}{\varphi(q/(q,n))}$$

where the Möbuis function $\mu(n)$ is 0 if n is divisible by the square of a prime, and $(-1)^{\omega(n)}$ otherwise, where $\omega(n)$ is the number of prime factors of n.

6. Let

$$\chi_3(n) = \begin{cases} 1 & n \equiv 1 \pmod 3 \\ -1 & n \equiv 2 \pmod 3 \\ 0 & n \equiv 0 \pmod 3 \end{cases}$$

be the number of divisors of n. Also, let $f(n) = \chi_3(m)$, where m is such that $n = 3^m k$ for some k not divisible by 3. Find the maximum value of $f(1) + \cdots + f(n)$ for $0 < n \leq 1000$.

7. Call a finite set $S \subset \mathbb{C}$ *good* if for all $z \in S, n \in \mathbb{Z}, z^n \in S$. Find the maximum magnitude of the sum of the elements of a *good* set of size at most 1000.