ON EISENSTEIN PRIMES

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1. Introduction and statement of results

In this paper, we prove the following result:

Theorem 1.
$$\sum\sum_{\ell^2+\ell m+m^2\leq x} \Lambda(2\ell - m)\Lambda(\ell^2 - \ell m + m^2) \sim \sigma x$$
for some $\sigma > 0$.

We shall prove Theorem 1 by following along the lines of the proof of Theorem 20.3 in [**?**], by using $\mathbb{Q}(\omega)$ rather than $\mathbb{Q}(i)$ when working with the bilinear forms that arise in Section 20.4 of [**?**]. A related result was proved by Fouvry and Iwaniec in [**?**] where it is shown that there are infinitely many primes of the form $\ell^2 + m^2$ such that ℓ is prime.

2. Preliminaries

Let $\gamma_\ell = \log \ell$ when ℓ is a prime greater than 2 and 0 otherwise. Then, let
$$a_n = \sum_{\ell^2-\ell m+m^2=n} \gamma_{2\ell-m} = \sum_{r^2+3s^2=4n} \gamma_r.$$
Let
$$A(x) = \sum_{n\leq x} a_n$$
and let
$$A_d(x) = \sum_{\substack{n\leq x\\ n\equiv 0 \pmod d}} a_n$$
Let $\rho(d) = |\{v \in \mathbb{Z}/(d) : v^2 + 3 \equiv 0 \pmod d\}|$.
We expect that $A_d(x)$ is well approximated by
$$M_d(x) = \frac{\rho(4d)}{4d} \sum_{r\leq\sqrt{4x}} \frac{1}{2}\gamma_r \sqrt{\frac{4x - r^2}{3}}$$
so we let the remainder terms $r_d(x)$ be such that
$$A_d(x) = M_d(x) + r_d(x)$$
For d even, this is clearly equal to 0, while for d odd, since $\rho(d)$ is multipicative, this is equal to
$$\frac{\rho(d)}{4d} \sum_{r\leq\sqrt{4x}} \gamma_r \sqrt{\frac{4x - r^2}{3}}$$
We then have the following:

Proposition 1. *Suppose that for some* $\sqrt{x} < D \leq x(\log x)^{-20}$,

$$R(x; D) = \sup_{y \leq x} \sum_{d \leq D} |r_d(y)| \ll A(x) \log^{-2} x \tag{2.1}$$

and let

$$T(x; D) = \sum_{\ell \leq D} \left| \sum_{\substack{\ell m \leq x \\ xD^{-1} < z \leq x^2 D^{-2}}} a_{\ell m} \mu(m) \right| \tag{2.2}$$

Then, we have that

$$\sum_{n \leq x} a_n \Lambda(n) = HA(x) \left\{1 + O((\log x)^{-1})\right\} + O(T(x, D) \log x) \tag{2.3}$$

Proof. This is Theorem 18.6 in [**?**] for our particular sequence. □

3. The remainder term

In this section, we verify that (2.1) holds. From this point on $e(\alpha) = e^{2\pi i \alpha}$. First, we study the distribution of the roots of the congruence $v^2 + 3 \equiv 0 \pmod{d}$ by studying Weyl sums related to these quadratic roots. In order to do so, we will establish a well-spacing of the points $v/d \pmod 1$. It is easy to show that for odd d, the roots to $v^2 + 3 \equiv 0 \pmod{d}$ are in a bijection with representations

$$d = r^2 + rs + s^2 = \frac{(r-s)^2 + 3(r+s)^2}{4}$$

subject to $(r, s) = 1, -r - s < r - s \leq r + s$ where $v(r - s) \equiv (r + s) \pmod{d}$.
It then follows that

$$\frac{v}{d} \equiv -\frac{4(\overline{r-s})}{r+s} + \frac{r-s}{d(r+s)} \pmod 1$$

where $\overline{r-s}$ is such that $(r - s)(\overline{r-s}) \equiv 1 \pmod{r+s}$.
Note that we then have that

$$\frac{|r-s|}{d(r+s)} < \frac{1}{2(r+s)^2}$$

Now, restrict d to the range $4D < d \leq 9D$. It then follows that $2D^{1/2} < r + s < 3D^{1/2}$, so for any two points $v_1/d_1, v_2/d_2$, $\max\left\{\frac{r_1+s_1}{r_2+s_2}, \frac{r_2+s_2}{r_1+s_1}\right\} \leq \frac{3}{2}$

$$\left\|\frac{v_1}{d_1} - \frac{v_2}{d_2}\right\| > \frac{4}{(r_1+s_1)(r_2+s_2)} - \max\left\{\frac{1}{(r_1+s_1)^2}, \frac{1}{(r_2+s_2)^2}\right\} \gg \frac{1}{D}$$

Then by the large sieve inequality of Davenport and Halberstam, we have the following

Lemma 2. *For all* $\alpha_1, \alpha_2, \dots \in \mathbb{C}$, *we have that*

$$\sum_{\substack{D < d \leq 2D \\ d \equiv 1 \ (\mathrm{mod}\ 2)}} \sum_{v^2+3 \equiv 0 \ (\mathrm{mod}\ d)} \left| \sum_{n \leq N} \alpha_n e\left(\frac{vn}{d}\right) \right|^2 \ll (D + N) \left(\sum_n \alpha_n^2 \right).$$

Applying Cauchy's inequality yields

Proposition 2. *For all $\alpha_1, \alpha_2, \dots \in \mathbb{C}$, we have that*

$$\sum_{\substack{D<d\leq 2D \\ d\equiv 1 \pmod 2}} \sum_{v^2+3\equiv 0 \pmod d} \left| \sum_{n\leq N} \alpha_n e\left(\frac{vn}{d}\right) \right| \ll D^{1/2}(D+N)^{1/2} \left(\sum_n \alpha_n^2 \right)^{1/2}. \tag{3.1}$$

Now, let

$$\rho_h(d) = \sum_{v^2+3\equiv 0 \pmod d} e\left(\frac{vh}{d}\right)$$

Then, the following holds:

Proposition 3.

$$\sum_{d\leq D} \left| \sum_{h\leq N} \alpha_h \rho_h d \right| \ll D^{1/2}(D+N)^{1/2} \left(\sum_n \alpha_n^2 \right)^{1/2}. \tag{3.2}$$

Now, we prove that (2.1) holds by proving the following:

Proposition 4. *For all $D \leq x$*

$$\sum_{d\leq D} |r_d(x)| \ll D^{1/4} x^{3/4+\epsilon} \tag{3.3}$$

Proof. Note that

$$A_d(x) = \sum_{\substack{\frac{r^2+3s^2}{4}\leq x \\ \frac{r^2+3s^2}{4}\equiv 0 \pmod d}} \gamma_r.$$

It is more convenient for now to consider only the contribution of the terms with $(r,d)=1$. To that end, note that it is possible to replace $A_d(x)$ with

$$A_d^*(x) = \sum_{\substack{\frac{r^2+3s^2}{4}\leq x \\ \frac{r^2+3s^2}{4}\equiv 0 \pmod d \\ (r,d)=1}} \gamma_r$$

since

$$\sum_{d\leq D} |A_d(x) - A_d^*(x)| \leq \sum_{d\leq D} \sum_{\ell | d} |\gamma_\ell| \sum_{\substack{r^2+3s^2\leq 4x \\ r^2+3s^2\equiv 0 \pmod{4d}}} 1$$

$$\leq \sum_{\ell} |\gamma_\ell| \sum_{\substack{r^2+3\leq 4xs^{-2} \\ r^2+3s^2\equiv 0 \pmod{4d}}} \tau(r^2+3) \ll x^{1/2+\epsilon}$$

Now, rather than approximating $A_d^*(x)$, we shall approximate

$$A_d^*(f) = \sum_{\substack{r^2+3s^2\equiv 0 \pmod{4d} \\ (r,d)=1}} \gamma_r f\left(\frac{r^2+3s^2}{4}\right)$$

for some smooth f supported on $[1,x]$ satisfying

$$f(u) = 1, \text{ for } y \leq u \leq x-y$$

$$f^{(j)}(x) \ll x^{-j}$$

where $y = \min\{x^{3/4}D^{1/4}, \frac{1}{2}x\}$. Note that bounding this is sufficient, since

$$\sum_{d\le D} |A_d^*(f) - A_d^*(x)| \le \sum_{l^2+lm+m^2\in I} \tau(l^2+lm+m^2) \ll yx^\epsilon$$

where $I = \mathbb{Z}\cap([1,y]\cup[x-y,x])$. Note that since γ_r is supported on odd primes, we have that

$$A_d^*(f) = \sum_{v^2+3\equiv 0 \pmod{4d}} \sum_{(r,d)=1} \gamma_r \sum_{s\equiv vr \pmod{4d}} f\left(\frac{r^2+3s^2}{4}\right).$$

Now, let

$$A_d(f) = \sum_{v^2+3\equiv 0 \pmod{4d}} \sum_{r} \gamma_r \sum_{s\equiv vr \pmod{4d}} f\left(\frac{r^2+3s^2}{4}\right)$$

We can replace $A_d^*(f)$ with $A_d(f)$ with an error of $O_(y\log x)$, which is small enough. We then have that by Poisson's formula

$$A_d(f) = \frac{1}{4d}\sum_r \gamma_r \sum_{k\in\mathbb{Z}} \rho_{kr}(4d) F_r\left(\frac{k}{4d}\right)$$

where

$$F_r(v) = \int_{\mathbb{R}} f\left(\frac{r^2+3t^2}{4}\right) e(-vt)dt = 2\int_0^\infty f\left(\frac{r^2+3t^2}{4}\right)\cos(2\pi vt)dt$$

Note that the the contribution from when $k=0$ is equal to $M_d(x)+O(y)$, so it is necessary and sufficient to bound the contribution from $k\neq 0$. To that end, note that by the change of variable $t = w\sqrt{x}/k$,

$$F_r\left(\frac{k}{4d}\right) = \frac{2\sqrt{x}}{k}\int_0^\infty f\left(\frac{r^2+\frac{3xw^2}{k^2}}{4}\right)\cos\left(\frac{2\pi w\sqrt{x}}{4d}\right)dw \tag{3.4}$$

Integrating by parts twice yields that this equals

$$\frac{16\sqrt{x}d^2}{\pi^2k^3}\int_0^\infty \left(f' + \frac{2w^2x}{k^2}f''\right)\left(\frac{r^2+\frac{3xw^2}{k^2}}{4}\right)\cos\left(\frac{\pi w\sqrt{x}}{2d}\right)dw \tag{3.5}$$

Now, let

$$R(f,D) = \sum_{D<d\le 2D}\left|\frac{1}{4d}\sum_r \gamma_r \sum_{k\in\mathbb{Z}\setminus\{0\}} \rho_{kr}(4d)F_r\left(\frac{k}{4d}\right)\right|$$

We then have that

$$R(f,D) \ll \frac{1}{D}\sum_{D<d\le 2D}\left|\sum\sum_{kr\neq 0}\gamma_r F_r\left(\frac{k}{4d}\right)\right|$$

To estimate this, we split this into sums with $|k|$ restricted to certain ranges. In particular, we write

$$R_k(f,D) = \frac{1}{D}\sum_{D<d\le 2D}\left|\sum_{2^k\le|k|<2^{k+1}}\sum_r \gamma_r F_r\left(\frac{k}{4d}\right)\right|$$

Then, we have that by (3.4) and Proposition 3, $R_n(f,D)$ is

$$\begin{aligned}
&\frac{1}{D}\sum_{D<d\le 2D}\left|\sum_{2^n\le|k|<2^{n+1}}\sum_r \gamma_r\rho_{kr}(d)\frac{2\sqrt{x}}{k}\int_0^\infty f\left(\frac{r^2+\frac{3xw^2}{k^2}}{4}\right)\cos\left(\frac{\pi w\sqrt{x}}{2d}\right)dw\right| \\
&\ll \frac{\sqrt{x}}{D}\int_0^{2^{n+1}}\sum_{D<d\le 2D}\left|\sum_{2^n\le|k|<2^{n+1}}\sum_r \gamma_r\rho_{kr}(d) f\left(\frac{r^2+\frac{3xw^2}{k^2}}{4}\right)\right|dw \\
&\ll \frac{x^{1/2+\epsilon}}{D}D^{1/2}(D+2^n\sqrt{x})^{1/2}(2^n\sqrt{x})^{1/2}.
\end{aligned}$$

Similarly, we also have that by (3.5) and Proposition 3 $R_n(f,D)$ is

$$\ll \frac{D\sqrt{x}}{2^{3n}}\int_0^{2^{n+1}} \sum_{D<d\leq 2D}\left|\sum_{2^n\leq |k|<2^{n+1}}\sum_r \gamma_r\rho_{kr}(d)\left(f^{'}+\frac{2w^2x}{k^2}f^{''}\right)\left(\frac{r^2+\frac{3xw^2}{k^2}}{4}\right)\right| dw$$

$$\ll \frac{x^{3/2+\epsilon}D^{3/2}}{y^2 2^{2n}}(D+2^n\sqrt{x})^{1/2}(2^n\sqrt{x})^{1/2}$$

by Proposition 2.

The desired result then follows summing over all n.

□

4. The bilinear form

Now, we shall bound the billinear form in (2.2) by estimating the following sum:

$$B_1(M,N)=\sum_{N\leq n\leq N'}\left|\sum_{M<m\leq M'} a_{mn}\mu(m)\right| \tag{4.1}$$

for some unspecified $M<M'\leq 2M, N<N'\leq 2N$ by showing the following:

Proposition 5. *For δ a sufficiently small positve number, we have that*

$$B(M,N)\ll MN(\log MN)^{-A} \tag{4.2}$$

for all $A>0$, where $M=N^\delta$.

Proof. First, note that it is sufficient to estimate

$$B_1(M,N)=\sum_{N<n\leq N'}\left|\sum_{\substack{M<m\leq M'\\ (m,n)=1}} a_{mn}\mu(m)\right| \tag{4.3}$$

since if $(m,n)=d$, if $d<M^{1/2}$, we can just transfer the factor of d to n, and otherwise use the trivial bound.

Write $\gamma(\mathfrak{a})$ to denote $\gamma_{2\,\mathrm{Re}\,\mathfrak{a}}$.

Note that we have that

$$a_n=\sum_{N\mathfrak{a}=n}\gamma(\mathfrak{a})$$

so by unique factorization in $\mathbb{Q}(\omega)$, we have that for relatively prime m,n, we have that

$$a_{mn}=\frac{1}{6}\sum_{N\mathfrak{m}=m}\sum_{N\mathfrak{n}=n}\gamma(\mathfrak{mn})$$

where the factor of $1/6$ accounts for the six units $\pm1,\pm\omega,\pm\omega^2$ in $\mathbb{Z}[\omega]$. It follows that

$$B_1(M,N)=\frac{1}{6}\sum_{N<N(\mathfrak{n})\leq N'}\left|\sum_{\substack{M<N(\mathfrak{m})\leq M'\\ (\mathfrak{m},\mathfrak{n})=1}}\gamma(\mathfrak{mn})\mu(\mathfrak{m})\right|$$

The coprimality condition can easily be dropped by a similar argument by which it was added, so it follows that it is sufficient to show that

$$B_2(M,N) = \sum_{N<N(\mathfrak{n})\le N'} \left| \sum_{M<N(\mathfrak{m})\le M'} \gamma(\mathfrak{m}\mathfrak{n})\mu(\mathfrak{m}) \right| \ll MN(\log MN)^{-A}$$

By Cauchy, we have that it is sufficient to show that

$$B_3(M,N) = \sum_{N<N(\mathfrak{n})\le N'} \left| \sum_{M<N(\mathfrak{m})\le M'} \gamma(\mathfrak{m}\mathfrak{n})\mu(\mathfrak{m}) \right|^2 \ll M^2N(\log MN)^{-A}$$

We then have that

$$B_3(M,N) = \sum_{M<N(\mathfrak{m}_1),N(\mathfrak{m}_2)\le M'} \mu(\mathfrak{m}_1)\mu(\mathfrak{m}_2)S(\mathfrak{m}_1,\mathfrak{m}_2)$$

where

$$S(\mathfrak{m}_1,\mathfrak{m}_2) = \sum_{N<N(\mathfrak{n})\le N'} \gamma(\mathfrak{n}\mathfrak{m}_1)\gamma(\mathfrak{n}\mathfrak{m}_2).$$

Now, let ℓ_1,ℓ_2 be such that

$$\mathfrak{n}\mathfrak{m}_1 + \overline{\mathfrak{n}\mathfrak{m}}_1 = \ell_1$$

$$\mathfrak{n}\mathfrak{m}_2 + \overline{\mathfrak{n}\mathfrak{m}}_2 = \ell_2$$

and let $\Delta(\mathfrak{m}_1,\mathfrak{m}_2) = \Delta = i(\mathfrak{m}_1\overline{\mathfrak{m}}_2 - \overline{\mathfrak{m}}_1\mathfrak{m}_2)$. Note that $\ell_1,\ell_2 \le 4\sqrt{MN}$. When $\Delta = 0$, note that the contribution $B_0(M,N)$ satisfies

$$B_0(M,N) \ll N(\log N)^2 \sum_{\mathrm{Im}\,\overline{\mathfrak{m}}_1\mathfrak{m}_2=0}\sum 1$$

which is clearly $\ll NM^2(\log MN)^{-A}$.

Otherwise, we have that

$$\overline{\mathfrak{n}} = \frac{i(\ell_1\mathfrak{m}_2 - \ell_2\mathfrak{m}_1)}{\Delta}$$

so it follows that

$$\ell_1\mathfrak{m}_2 \equiv \ell_2\mathfrak{m}_1 \pmod{\Delta}$$

and that

$$\Delta^2 N < N(\ell_1\mathfrak{m}_2 - \ell_2\mathfrak{m}_1) \le \Delta^2 N'.$$

It then follows that

$$S(\mathfrak{m}_1,\mathfrak{m}_2) = \sum_{\substack{\ell_1\mathfrak{m}_2\equiv\ell_2\mathfrak{m}_1 \pmod{\Delta} \\ \Delta^2N<N(\ell_1\mathfrak{m}_2-\ell_2\mathfrak{m}_1)\le\Delta^2N'}} \gamma_{\ell_1}\gamma_{\ell_2}$$

Now, we state Proposition 20.9 in [?]:

Proposition 6.

$$\sum_{q\le Q} \max_{\substack{a\in\mathbb{Z},(a,q)=1 \\ \mathfrak{a}\in\mathbb{C} \\ y\in\mathbb{R}}} \left| \sum_{\substack{\ell_1,\ell_2\le x \\ |\ell_1-\mathfrak{a}\ell_2|\le y \\ \ell_1\equiv a\ell_2 \pmod q}}\sum \gamma_{\ell_1}\gamma_{\ell_2} - \phi(q)^{-1} \sum_{\substack{\ell_1,\ell_2\le x \\ |\ell_1-\mathfrak{a}\ell_2|\le y}}\sum \gamma_{\ell_1}\gamma_{\ell_2} \right| \ll x^2(\log x)^{-A}$$

where $Q = x(\log x)^{-B}$ for some $B > 0$ that depends on A.

Now we can split up $S(\mathfrak{m}_1, \mathfrak{m}_2)$ into classes restricted to

$$\ell_1 \equiv a\ell_2 \pmod{\Delta}$$

for $a \in (\mathbb{Z}/(\Delta))^*$ such that $a\mathfrak{m}_2 \equiv \mathfrak{m}_1 \pmod{\Delta}$ and apply Proposition 6. It then follows that

$$B_0(M,N) \ll B_4(M,N) + O(NM^2(\log MN)^{-A})$$

where

$$B_4(M,N) = \sum\sum_{M<N(\mathfrak{m}_1),N(\mathfrak{m}_2)\le M'} \mu(\mathfrak{m}_1)\mu(\mathfrak{m}_2)\frac{\eta(\Delta)}{\phi(\Delta)} \sum\sum_{\substack{\ell_1,\ell_2\le x \\ \Delta^2 N<N(\ell_1\mathfrak{m}_2-\ell_2\mathfrak{m}_1)\le \Delta^2 N'}} \gamma_{\ell_1}\gamma_{\ell_2}$$

where $\eta(\Delta)$ is the total number of $a \in (\mathbb{Z}/(\Delta))^*$ such that $a\mathfrak{m}_2 \equiv \mathfrak{m}_1 \pmod{\Delta}$.
By the prime number theorem, we have that the inner sum satisfies

$$\sum\sum_{\substack{\ell_1,\ell_2\le x \\ \Delta^2 N<N(\ell_1\mathfrak{m}_2-\ell_2\mathfrak{m}_1)\le \Delta^2 N'}} \gamma_{\ell_1}\gamma_{\ell_2} = X + O(MN(\log MN)^{-A})$$

where

$$X = \int\int_{\Delta\sqrt{N}<|\ell_1\mathfrak{m}_2-\ell_2\mathfrak{m}_1|\le\Delta\sqrt{N'}} d\ell_1 d\ell_2 = |\Delta| \int\int_{N<|u+\omega v|\le N'} du dv = \frac{1}{2}\pi\sqrt{3}|\Delta|(N'-N)$$

It therefore now remains to estimate

$$S_1 = \sum\sum_{M<N(\mathfrak{m}_1),N(\mathfrak{m}_2)\le M'} \mu(\mathfrak{m}_1)\mu(\mathfrak{m}_2)\frac{\eta(\Delta)|\Delta|}{\phi(\Delta)}$$

Splitting this up for all $(\mathfrak{m}_1, \mathfrak{m}_2) = \mathfrak{d}$, we then have that

$$\begin{aligned} S_1 &= \sum_{\mathfrak{d}} \mu^2(d) \sum\sum_{\substack{M<N(\mathfrak{m}_1\mathfrak{d}),N(\mathfrak{m}_2\mathfrak{d})\le M' \\ (\mathfrak{m}_1,\mathfrak{m}_2)=(\mathfrak{m}_1\mathfrak{m}_2)=1}} \mu(\mathfrak{m}_1\mathfrak{d})\mu(\mathfrak{m}_2\mathfrak{d})\frac{\eta(\Delta N(\mathfrak{d}))|\Delta|N(\mathfrak{d})}{\phi(\Delta N(\mathfrak{d}))} \\ &= \sum_{\mathfrak{d}} \mu^2(d) \sum\sum_{\substack{M<N(\mathfrak{m}_1\mathfrak{d}),N(\mathfrak{m}_2\mathfrak{d})\le M' \\ (\mathfrak{m}_1,\mathfrak{m}_2)=(\mathfrak{m}_1\mathfrak{m}_2)=1}} \mu(\mathfrak{m}_1)\mu(\mathfrak{m}_2)\frac{\eta(\Delta N(\mathfrak{d}))|\Delta|N(\mathfrak{d})}{\phi(\Delta N(\mathfrak{d}))}. \end{aligned}$$

Note that we have that

$$\eta(\Delta N(\mathfrak{d})) = \sum_{\substack{a\in(\mathbb{Z}/(\Delta N(\mathfrak{d})))^* \\ a\equiv \mathfrak{m}_2\mathfrak{m}_1^{-1} \pmod{\bar{\mathfrak{d}}\Delta}}} 1 = N(\mathfrak{d}) \prod_{p|N(\mathfrak{d}),p\nmid\Delta} \left(1-\frac{1}{p}\right)$$

It then follows that

$$S_1 = \sum_{\mathfrak{d}} \mu^2(d)N(\mathfrak{d}) \sum\sum_{\substack{M<N(\mathfrak{m}_1\mathfrak{d}),N(\mathfrak{m}_2\mathfrak{d})\le M' \\ (\mathfrak{m}_1,\mathfrak{m}_2)=(\mathfrak{m}_1\mathfrak{m}_2)=1}} \mu(\mathfrak{m}_1)\mu(\mathfrak{m}_2)\frac{|\Delta|}{\phi(\Delta)}$$

By multiplicativity, we have that

$$\frac{|\Delta|}{\phi(\Delta)} = \sum_{d|\Delta} \mu^2(d)\phi(d)^{-1}.$$

Using this and reversing the order of summation, we have that

$$S_1 = \sum_{\mathfrak{d}} \mu^2(d)N(\mathfrak{d}) \sum\sum_{\substack{M<N(\mathfrak{m}_1\mathfrak{d}),N(\mathfrak{m}_2\mathfrak{d})\le M' \\ (\mathfrak{m}_1,\mathfrak{m}_2)=(\mathfrak{m}_1\mathfrak{m}_2)=1}} \mu(\mathfrak{m}_1)\mu(\mathfrak{m}_2) \sum_{d|\Delta} \mu^2(d)\phi(d)^{-1}$$

$$= \sum_{\mathfrak{d}} \mu^2(d) N(\mathfrak{d}) \sum_{d \le 2M} \phi(d)^{-1} \sum_{\substack{M < N(\mathfrak{m}_1\mathfrak{d}), N(\mathfrak{m}_2\mathfrak{d}) \le M' \\ (\mathfrak{m}_1,\mathfrak{m}_2)=(\mathfrak{m}_1\mathfrak{m}_2)=1 \\ \mathfrak{m}_1\overline{\mathfrak{m}}_2 \equiv \overline{\mathfrak{m}}_1\mathfrak{m}_2 \pmod d}} \sum \mu(\mathfrak{m}_1)\mu(\mathfrak{m}_2)$$

$$= \sum_{\mathfrak{d}} \mu^2(d) N(\mathfrak{d}) \sum_{d \le 2M} \phi(d)^{-1} \frac{1}{d} \sum_{\chi} \sum_{\substack{M < N(\mathfrak{m}_1\mathfrak{d}), N(\mathfrak{m}_2\mathfrak{d}) \le M' \\ (\mathfrak{m}_1,\mathfrak{m}_2)=(\mathfrak{m}_1\mathfrak{m}_2)=1}} \sum \mu(\mathfrak{m}_1)\mu(\mathfrak{m}_2)\psi(\mathfrak{m}_1)\overline{\psi}(\mathfrak{m}_2)$$

by orthogonality where χ runs over the characters of $\mathbb{Z}[\omega]/(d)$ and $\psi(\mathfrak{m}) = \chi(\mathfrak{m})\overline{\chi}(\overline{\mathfrak{m}})$

To estimate this, we use the following version of the Siegel-Walfisz Theorem that follows from the main result in [**?**]:

Proposition 7. *For any character ψ on ideals*

$$\sum_{N(\mathfrak{m}) \le x} \mu(\mathfrak{m})\psi(\mathfrak{m}) \ll_A x(\log x)^{-A}$$

for all $A > 0$

Now, let

$$S^*_{\mathfrak{d},d,\psi}(M) = \sum_{\substack{M < N(\mathfrak{m}_1\mathfrak{d}), N(\mathfrak{m}_2\mathfrak{d}) \le M' \\ (\mathfrak{m}_1,\mathfrak{m}_2)=(\mathfrak{m}_1\mathfrak{m}_2,\mathfrak{d})=1}} \sum \mu(\mathfrak{m}_1)\mu(\mathfrak{m}_2)\psi(\mathfrak{m}_1)\overline{\psi}(\mathfrak{m}_2)$$

Then, it is easy to see that

$$S^*_{\mathfrak{d},d,\psi}(M) = S_{\mathfrak{d},d,\psi}(M) + O(M^{1+\epsilon})$$

where

$$S_{\mathfrak{d},d,\psi}(M) = \sum_{\substack{M < N(\mathfrak{m}_1\mathfrak{d}), N(\mathfrak{m}_2\mathfrak{d}) \le M' \\ (\mathfrak{m}_1,\mathfrak{m}_2)=1}} \sum \mu(\mathfrak{m}_1)\mu(\mathfrak{m}_2)\psi(\mathfrak{m}_1)\overline{\psi}(\mathfrak{m}_2)$$

We then have that

$$\sum_{\mathfrak{d}_1 \in \mathbb{Z}[\omega]\setminus\{0\}} \mu^2(\mathfrak{d}_1) S_{\mathfrak{d},d,\psi}(M/N(\mathfrak{d}_1))$$

$$= \left(\sum_{M < N(m_1\mathfrak{d}) \le M'} \mu(\mathfrak{m}_1)\psi(\mathfrak{m}_1) \right) \left(\sum_{M < N(m_2\mathfrak{d}) \le M'} \mu(\mathfrak{m}_2)\overline{\psi}(\mathfrak{m}_2) \right)$$

so by a variant of Möbius inversion, we have that

$$S_{\mathfrak{d},d,\psi}(M) \ll (M/N(\mathfrak{d}))^2 (\log M/N(\mathfrak{d}))^{-A}.$$

The desired result then follows. □

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