Computational Methods, Unit 3

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October 4^{th}, 2025

System of Linear Algebraic Equations: Existence of solution, Gauss elimination method and its computational effort, concept of Pivoting, Gauss Jordan method and its computational effort, Triangular Matrix factorization methods: Dolittle algorithm, Crout's Algorithm, Cholesky method, Eigen value problem: Power method, Approximation by Spline Function: First-Degree and second degree Splines, Natural Cubic Splines, B Splines, Interpolation and Approximation

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§0 Syllabus

§1 System of Linear Algebraic Equations

A system of m equations with n unknowns is has the form:

$$\begin{aligned} A_{1,1}X_1 + A_{1,2}X_2 + A_{1,3}X_3 + \cdots + A_{1,n}X_n &= B_1 \\ A_{2,1}X_1 + A_{2,2}X_2 + A_{2,3}X_3 + \cdots + A_{2,n}X_n &= B_2 \\ A_{3,1}X_1 + A_{3,2}X_2 + A_{3,3}X_3 + \cdots + A_{3,n}X_n &= B_3 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots& \\ A_{m,1}X_1 + A_{m,2}X_2 + A_{m,3}X_3 + \cdots + A_{m,n}X_n &= B_m \end{aligned}$$

where $\{A_{i,j}\}_{(1,1)}^{(m,n)}$ and $\{B\}_1^m$ are known constants and $\{X\}_1^n$ are unknown variables such that all these equations are linearly independent of each other. Two equations are called "linearly independent" if one of them **can't** be obtained by the other. For example, $3x + 6y = 3$ is linearly dependent with $x + 2y = 1$ because one can be obtained by other(factor out 3). But $3x + 6y = 5$ and $x + 2y = 1$ are linearly independent.

More specifically,

$$\begin{aligned} A_1x_1 + B_2x_2 + A_3x_3 + \cdots + A_nx_n &= p \\ B_1x_1 + B_2x_2 + B_3x_3 + \cdots + B_nx_n &= q \end{aligned}$$

are linearly dependent iff

$$\frac{A_1}{B_1} = \frac{A_2}{B_2} = \frac{A_3}{B_3} = \cdots = \frac{A_n}{B_n} = \frac{p}{q}$$

Otherwise, they are linearly independent of each other.

§1.1 Existence of solution

In a system of simultaneous equations with n unknowns and m linearly independent equations

- $m = n$: There exists a **unique solution** set $\{X\}_1^n$
- $m < n$: There exists **many sets of solution** sets. This system is called "under determined"
- $m > n$: A **solution set may or may not exist** in this case, its called over determined system

§1.2 Gauss Elimination Method

For n unknowns, the solution set is obtained in $n-1$ steps. Each step eliminates a variable so after all the steps, only one variable is left, which constitutes a linear equation solvable by a 3^{rd} grader. We then substitute everything back. The k^{th} variable is eliminated in the k^{th} step from all equations with serial number $> k$.

Essentially, we need to convert the square matrix A in $AX = B$ into an upper triangular matrix A' in $A'X = B'$ using row operations.
To eliminate X_k from i^{th} equation($i > k$), we multiply k^{th} equation by $-A_{i,k}/A_{k,k}$ and add it to the i^{th} equation. The j^{th} coefficient of i^{th} equation is obtained via

$$A_{i,j}^{(k)} = A_{i,j} + u^* A_{k,j}$$
$$b_i^{(k)} = b_i + u^* b_k$$

for $1 \geq j \geq n$ where $u = -(A_{i,k}/A_{k,k})$

The simplified expression for solution set is

$$\{X\}_i = \left(b_i^{(i-1)} - \sum_{j=i+1}^{n} A_{i,j}^{(i-1)} X_j \right) / A_{i,i}^{(i-1)}$$

§1.2.1 Division by Zero

To eliminate X_k from i^{th} equation, we multiply k^{th} equation by $-A_{i,k}/A_{k,k}$. Sometimes, $A_{k,k}$ can be zero. This leads to unexpected(wrong) results.

§1.2.2 Number of operations

In reducing a system of m equations with n unknowns, the number of product operations involved are $\sum_{k=1}^{n-1}(n-k)(n-k+1)$. The number of subtraction operations are $\sum_{k=1}^{n-1}(n-k)(n-k+1)$ as well. And the number of division operations are $n-1$

While in the back substitution process, the total number of operations involved are $1 + 2\sum_{k=1}^{n-1} n - k + 1$

The total number of operations involved in this method are

$$
\begin{aligned}
&1 + 2\sum_{k=1}^{n-1} \left((n-k)^2 + (n-k) + (n-k+1)\right) \\
&= 1 + 4\sum_{k=1}^{n-1} \left(n^2 - 2nk + k^2 + n - k + n - k + 1\right) \\
&= 1 + 2\left(\sum_{k=1}^{n-1} k^2\right) - (4n+4)\left(\sum_{k=1}^{n-1} k\right) + (2n^2 + 2n + 2n + 2)\left(\sum_{k=1}^{n-1} 1\right) \\
&= 1 + 2(n+1)^2(n-1) - 2n(n-1)(n+1) + n(n-1)(2n-1)/6 \\
&= 2n^3/3 + n^2 + n/3 - 1
\end{aligned}
$$