Matrix Factorization

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Let the system of equation be

$$AX = B$$

We factorize A into LU which are lower and upper triangular matrices respectively. If the principal minors of $A_{n\times n}$ are non singular, then this factorization is possible as well as unique.

$$L = \begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{bmatrix}$$

and

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

$AX = B$ becomes $LUX = B$

Let $UX = Z$ So That $LZ = B$. Now Z can be found by the following

$$\left\{ \sum_{j=1}^{n} l_{ij} z_j = b_j \right\}_{i=1}^{n}$$

Then we use $UX = Z$ as

$$\left\{ \sum_{j=1}^{n} u_{ij} x_j = z_j \right\}_{i=1}^{n}$$

When $u_{ii} = 1 \forall i$, then this method is called Crout's decomposition method.
When $l_{ii} = 1 \forall i$, its called Dolittle Method.
Particularly, when $u_{ii} = l_{ii} = 1 \forall i$, its Cholesky's decomposition method.

It may be noted that for a triangular matrix, its determinant is the product of its diagonal elements.