LQR and LQG Controller For a Moving Crane

ENPM667 Final Project

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1 Introduction

We have been given a crane moving along a one-dimensional track. It behaves as a friction-less cart with mass M actuated by an external force F that constitutes the system's input. There are two loads suspended from cables attached to the crane. The loads have mass $m1$ and $m2$, and the lengths of the cables are $l1$ and $l2$, respectively. The following figure depicts the crane and associated variables used throughout this project. The workflow of this project will be as follows:

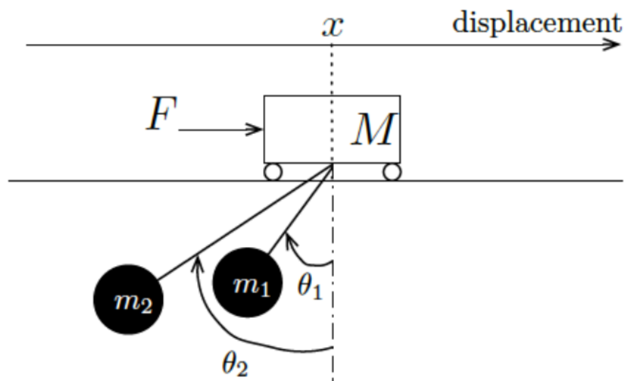

Figure 1: Diagram

1. We need to find the equations of motion for the system and the corresponding nonlinear state-space representation.

2. Obtain the linearized system around the equilibrium point specified by $x = 0$ and $\theta_1 = 0$, $\theta_2 = 0$. Write the state-space representation of the linearized system.

3. Obtain conditions on M, $m1$, $m2$, $l1$, $l2$ for which the linearized system is controllable.

4. Simulating the resulting response to initial conditions when the controller is applied to the linearized system and to the original nonlinear system. Adjust the parameters of the LQR cost until you obtain a suitable response. Use Lyapunov's indirect method to certify the stability (locally or globally) of the closed-loop system. With provided data: $M = 1000Kg$,$m_1 = m_2 = 100Kg$, $l_1 = 20m$, and $l_2 = 10m$.

5. Suppose that we select the following output vectors: $x(t)$,$(\theta_1(t), \theta_2(t))$, $(x(t), \theta_2(t))$ or $(x(t), \theta_1(t), \theta_2(t))$. Determine for which output vectors the linearized system is observable.

6. Obtain our "best" Luenberger observer for each one of the output vectors for which the system is observable and simulate its response to initial conditions and unit step input. The simulation is done for the observer applied to both the linearized system and the original nonlinear system.

7. Design an output feedback controller for your choice of the "smallest" output vector. Use the LQG method and apply the resulting output feedback controller to the original nonlinear system. Obtain our best design and illustrate its performance in simulation. In addition, how would we reconfigure our controller to asymptotically track a constant reference on x.

2 Equations of motion and non-linear state space representation

Equations of Motion for our system are given by Euler-Lagrange's method:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = F \tag{1}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \left(\frac{\partial L}{\partial \theta_1}\right) = 0 \tag{2}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \left(\frac{\partial L}{\partial \theta_2}\right) = 0 \tag{3}$$

Given the above equations, we next need to find out the Lagrange equation which is given by:

$$L = \mathrm{K.E} - \mathrm{P.E} \tag{4}$$

The kinetic energy of the system is given by :

$$\mathrm{K.E} = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m_1\dot{x}_{m_1}^2 + \frac{1}{2}m_2\dot{x}_{m_2}^2 \tag{5}$$

For equation (5) we need positions vectors of mass m_1 and m_2 and their respective velocities which are given by their differential of the position. The position of mass m_1 is given by:

$$x_{m_1} = (x - l_1 \sin(\theta_1))\,\hat{i} + (-l_1 \cos(\theta_1))\,\hat{j} \tag{6}$$

The velocity for mass m_1 is given by differentiation of the above equation.

$$\dot{x}_{m_1} = \left(\dot{x} - l_1 \cos(\theta_1)\,\dot{\theta_1}\right)\hat{i} + \left(l_1 \sin(\theta_1)\,\dot{\theta_1}\right)\hat{j} \tag{7}$$

The position of mass m_2 is given by:

$$x_{m_2} = (x - l_2 \sin(\theta_2))\,\hat{i} + (-l_2 \cos(\theta_2))\,\hat{j} \tag{8}$$

Similarly, we find the velocity vector for mass m_2:

$$\dot{x}_{m_2} = \left(\dot{x} - l_1 \cos(\theta_2)\, \dot{\theta_2}\right) \hat{i} + \left(l_2 \sin(\theta_2)\, \dot{\theta_2}\right) \hat{j} \tag{9}$$

We plug the velocities that we got from the above equations into the Kinetic energy equation which will give us:

$$\text{K.E} = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m_1 \left(\dot{x} - l_1 \dot{\theta}_1 \cos(\theta_1)\right)^2 + \frac{1}{2} m_1 \left(l_1 \dot{\theta}_1 (\sin(\theta_1))\right)^2 + \frac{1}{2} m_2 (\dot{x} - \dot{\theta}_2 l_2 \cos(\theta_2)\Big)^2 + \frac{1}{2} m_2 \left(l_2 \dot{\theta}_2 (\sin(\theta_2))\right)^2 \tag{10}$$

For calculating Lagrange(L) we also need to find the potential energy of the system(P.E) given by the equation below:

$$\text{P.E} = -m_1 g l_1 \cos(\theta_1) - m_2 g l_2 \cos(\theta_2) \tag{11}$$

Now we can calculate the Lagrange (L) by plugging P.E and K.E

$$L = \text{K.E} - \text{P.E}$$

$$= \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 \cos^2(\theta_1) - m_1 l_1 \dot{\theta}_1 \dot{x} \cos(\theta_1) + \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 \sin^2(\theta_1)$$
$$+ \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 \cos^2(\theta_2) - m_2 l_2 \dot{\theta}_2 \dot{x} \cos(\theta_2) + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 \sin^2(\theta_1) + g[m_1 l_1 \cos(\theta_1)$$
$$+ m_2 l_2 \cos(\theta_2)] \tag{12}$$

Now, we will calculate Euler-Lagrange's equation by plugging L in equations (1), (2) and (3). Let us start with plugging L in equation (1).

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = F$$
$$\frac{\partial L}{\partial \dot{x}} = M\dot{x} + (m_1 + m_2)\,\dot{x} - m_1 l_1 \dot{\theta}_1 \cos(\theta_1) - m_2 l_2 \dot{\theta}_2 \cos(\theta_2) \tag{13}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = M\bar{x} + (m_1 + m_2)\,\bar{x} - \Big[m_1 l_1 \dot{\theta}_1 \cos(\theta_1)$$
$$- m_1 l_1 \dot{\theta}_1^2 \sin(\theta_1)\Big] - \Big[m_2 l_2 \ddot{\theta}_2 \cos(\theta_2) - m_2 l_2 \dot{\theta}_2^2 \sin(\theta_2)\Big] \tag{14}$$

From the 12 we can clearly see that $\frac{\partial L}{\partial x} = 0$

We combine 13 and 15 to give:

$$[M+m_1+m_2]\ddot{x}-m_1l_1\ddot{\theta}_1\cos(\theta_1)+m_1l_1\dot{\theta}_1^2\sin(\theta 1)-m_2l_2\ddot{\theta}_2\cos(\theta_2)+m_2l_2\dot{\theta}_2^2\sin(\theta_2)=F \tag{15}$$

Moving on to plugging L in equation (2):

$$\frac{d}{dt}(\frac{\partial L}{\partial\dot{\theta}_1})-(\frac{\partial L}{\partial\theta_1})=0 \tag{16}$$

Let us solve for the first term $(\frac{\partial L}{\partial\dot{\theta}_1})$

$$\frac{\partial L}{\partial\dot{\theta}_1}=m_1l_1^2\dot{\theta}_1-m_1\dot{x}l_1\cos(\theta_1) \tag{17}$$

Differentiating the above equation with respect to time.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right)=m_1l_1^2\dot{\theta}_1-\Big(m_1l_1\ddot{x}\cos(\theta_1)-m_1\dot{x}l_1\ddot{\theta}_1\sin(\theta_1)\Big) \tag{18}$$

For the second term in equation (16) we partially differentiate L with respect to θ_1 we get:

$$\frac{\partial L}{\partial\theta_1}=m_1l_1\dot{\theta}_1\dot{x}\sin(\theta_1)-m_1l_1g\sin(\theta_1) \tag{19}$$

Since we have both the terms of equation (16) we will collectively write them as:

$$m_1l_1^2\ddot{\theta}_1-m_1\ddot{x}l_1\cos(\theta_1)+m_1\dot{\theta}_1\dot{x}l_1\sin(\theta_1)-m_1\dot{\theta}_1\dot{x}l_1\sin(\theta_1)+m_1l_1g\sin(\theta_1)=0 \tag{20}$$

Similarly we will solve our $equation3$ of Euler-Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_2}\right)-\left(\frac{\partial L}{\partial\theta_2}\right)=0 \tag{21}$$

$$\frac{\partial L}{\partial\dot{\theta}_2}=m_2l_2^2\dot{\theta}_2-m_2\dot{x}l_2\cos(\theta_2) \tag{22}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_2}\right)=m_2l_2^2\dot{\theta}_2-\Big(m_2l_2\ddot{x}\cos(\theta_2)-m_2\dot{x}l_2\ddot{\theta}_2\sin(\theta_2)\Big) \tag{23}$$

$$\frac{\partial L}{\partial\theta_2}=m_2l_2\dot{\theta}_2\dot{x}\sin(\theta_2)-m_2l_2g\sin(\theta_2) \tag{24}$$

$$m_2l_2^2\ddot{\theta}_2-m_2\ddot{x}l_2\cos(\theta_2)+m_2\dot{\theta}_2\dot{x}l_2\sin(\theta_2)-m_2\dot{\theta}_2\dot{x}l_2\sin(\theta_2)+m_2l_2g\sin(\theta_2)=0 \tag{25}$$

After cancelling terms in 25 we will get the third Lagrange's equation as:

$$m_2l_2^2\ddot{\theta}_2-m_2\ddot{x}l_2\cos(\theta_2)+m_2\dot{\theta}_2\dot{x}l_2\sin(\theta_2)=0 \tag{26}$$

Let us write the equation (15), equation (20), and equation (26) in state space form. The non-linear state space representation of the these equations are as follows:

$$\ddot{x}=\frac{1}{M+m_1+m_2}\Big(m_1l_1\ddot{\theta}_1\cos\theta_1+m_2l_2\ddot{\theta}_2\cos\theta_2-m_1l_1\dot{\theta}_1^2\sin\theta_1-m_2l_2\dot{\theta}_2^2\sin\theta_2+F\Big) \tag{27}$$

$$\ddot{\theta}_1 = \frac{\ddot{x}\cos\theta_1}{l_1} - \frac{g\sin\theta_1}{l_1} \tag{28}$$

$$\ddot{\theta}_2 = \frac{\ddot{x}\cos\theta_2}{l_2} - \frac{g\sin\theta_2}{l_2} \tag{29}$$

We can write these equations in a matrix form as:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \frac{-m_1 g\sin\theta_1\cos\theta_2 - m_2 g\sin\theta_2\cos\theta_2 - m_1 l_1\dot{\theta}_1^2\sin\theta_1 - m_2 l_2\dot{\theta}_2^2\sin\theta_2 + F}{M+m_1+m_2-m_1\cos^2\theta_1 - m_2\cos^2\theta_2} \\ \frac{-m_1 g\sin\theta_1\cos\theta_2 - m_2 g\sin\theta_2\cos\theta_2 - m_1 l_1\dot{\theta}_1^2\sin\theta_1 - m_2 l_2\dot{\theta}_2^2\sin\theta_2 + F}{(M+m_1+m_2-m_1\cos^2\theta_1 - m_2\cos^2\theta_2)l_1} - \frac{g\sin\theta_1}{l_1} \\ \theta_2 \\ \frac{-m_1 g\sin\theta_1\cos\theta_2 - m_2 g\sin\theta_2\cos\theta_2 - m_1 l_1\dot{\theta}_1^2\sin\theta_1 - m_2 l_2\dot{\theta}_2^2\sin\theta_2 + F}{(M+m_1+m_2-m_1\cos^2\theta_1 - m_2\cos^2\theta_2)l_2} - \frac{g\sin\theta_2}{l_2} \end{bmatrix} \tag{30}$$

3 Linearization of the non-linear system

We have been given that to linearize the non-linear system around the equilibrium point specified by $x = 0$, $\theta_1 = 0$, and $\theta_2 = 0$. We have chosen one type of linearization method called small angle approximation for our system. We need to re-write the state space representation of the system after linearization.
Since we have been given the equilibrium points as mentioned above we will place these conditions to our $equation$: 27, 28, and 29. We obtain the following equations:

$$\ddot{x} = \frac{1}{M + m_1 + m_2}\Big(m_1 l_1\ddot{\theta}_1.1 + 1.m_2 l_2\ddot{\theta}_2 - 0 - 0 + F\Big) \tag{31}$$

$$\ddot{\theta}_1 = \frac{\ddot{x}.1}{l_1} - \frac{g.\theta_1}{l_1} \tag{32}$$

$$\ddot{\theta}_2 = \frac{\ddot{x}.1}{l_2} - \frac{g.\theta_2}{l_2} \tag{33}$$

Please note that since $\theta_1 = 0$, $\theta_2 = 0$ and $x = 0$ we can safely assume the following:

1. $\dot{\theta}_1^2 \approx 0$
2. $\dot{\theta}_2^2 \approx 0$
3. $\sin\theta_1 \approx \theta_1$
4. $\sin\theta_2 \approx \theta_2$
5. $\cos\theta_1 \approx 1$
6. $\cos\theta_2 \approx 1$

Hence, using the assumptions we were able to write the equations (31), (32), and (33). The non-linear functions like sin, and cos have been linearized. We can write the state-space representation of the linearized equations as:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \frac{-m_1 g\theta_1 - m_2 g\theta_2 + m_1\ddot{x} + m_2\ddot{x} + F}{M + m_1 + m_2} \\ \dot{\theta}_1 \\ \frac{-m_1 g\theta_1 - m_2 g\theta_2 + F}{(M)l_1} - \frac{g\theta_1}{l_1} \\ \frac{-m_1 g\theta_1 - m_2 g\theta_2 + F}{(M)l_2} - \frac{g\theta_2}{l_2} \end{bmatrix} \tag{34}$$

$$\begin{bmatrix} \dot{x} \\ \ddot{\boldsymbol{x}} \\ \dot{\boldsymbol{\theta}}_{\boldsymbol{1}} \\ \ddot{\boldsymbol{\theta}}_1 \\ \dot{\boldsymbol{\theta}}_2 \\ \ddot{\boldsymbol{\theta}}_{\boldsymbol{2}} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-gm_1}{M} & 0 & \frac{-gm_2}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-g(M+m_1)}{Ml_1} & 0 & \frac{-gm_2}{Ml_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-gm_1}{Ml_2} & 0 & \frac{-g(M+m_2)}{Ml_2} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \dot{\boldsymbol{x}} \\ \boldsymbol{\theta_1} \\ \dot{\boldsymbol{\theta}}_1 \\ \boldsymbol{\theta_2} \\ \dot{\boldsymbol{\theta}}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{Ml_1} \\ 0 \\ \frac{1}{Ml_2} \end{bmatrix} F \tag{35}$$

4 Obtain conditions for which the linearized system is controllable

Since our system is linear time invariant(i.e, A and B matrices do not vary with time) we can find controllability of any LTI system by calculating rank of the controllability matrix (C). For the system to be controllable number of state variables should be equal to the the rank of C.

$$rank(C) = rank[B\ AB\ A^2B\ A^3B\ A^4B\ A^5B] \tag{36}$$

After computing the B, AB, A^2B, A^3B, A^4B, A^5B we calculate the C matrix shown below

$$\mathrm{C} = \begin{bmatrix} 0 & \frac{1}{M} & 0 & \kappa_3 & 0 & \kappa_6 \\ \frac{1}{M} & 0 & \kappa_3 & 0 & \kappa_6 & 0 \\ 0 & \frac{1}{Ml_1} & 0 & \kappa_1 & 0 & \kappa_4 \\ \frac{1}{Ml_1} & 0 & \kappa_1 & 0 & \kappa_4 & 0 \\ 0 & \frac{1}{Ml_2} & 0 & \kappa_2 & 0 & \kappa_5 \\ \frac{1}{Ml_2} & 0 & \kappa_2 & 0 & \kappa_5 & 0 \end{bmatrix} \tag{37}$$

where:

$$\kappa_1 = \frac{\frac{g^2 m_1 (M+m_1)}{M^2 l_1} + \frac{g^2 m_1 m_2}{M^2 l_2}}{M l_1} + \frac{\frac{g^2 m_2 (M+m_2)}{M^2 l_2} + \frac{g^2 m_1 m_2}{M^2 l_1}}{M l_2}$$
$$\kappa_2 = -\frac{g m_1}{M^2 l_1} - \frac{g m_2}{M^2 l_2}$$
$$\kappa_3 = \frac{\frac{g^2 m_1 (M+m_2)}{M^2 {l_2}^2} + \frac{g^2 m_1 (M+m_1)}{\kappa_7}}{M l_1} + \frac{\frac{g^2 (M+m_2)^2}{M^2 l_2)^2} + \frac{g^2 m_1 m_2}{\kappa_7}}{M l_2}$$
$$\kappa_4 = \frac{\frac{g^2 m_2 (M+m_1)}{M^2 {l_1}^2} + \frac{g^2 m_2 (M+m_2)}{\kappa_7}}{M l_2} + \frac{\frac{g^2 (M+m_1)^2}{M^2 {l_1}^2} + \frac{g^2 m_1 m_2}{\kappa_7}}{M l_1}$$
$$\kappa_5 = -\frac{g\,(M+m_2)}{M^2 l_2^2} - \frac{g m_1}{\kappa_7}$$
$$\kappa_6 = -\frac{g\,(M+m_1)}{M^2 l_1^2} - \frac{g m_2}{\kappa_7}$$
$$\kappa_7 = M^2 l_1 l_2$$

The determinant of the controllability matrix(C) is given by :

$$det(C) = -\frac{g^6\,(l_1 - l_2)^2}{M^6 l_1^6 l_2^6} \tag{38}$$

The system is only controllable if and only if the rank of the controllability matrix is $rank(C) = 6$ For controllability we have the following cases for our variables:

1. Case 1: $l_1 = l_2$ the determinant of C becomes zero and the rank of matrix C becomes 4. Which is less than 6. Hence the matrix is not full rank. This implies that the system is not controllable.
2. Case 2: $l_1 \neq l_2$ This gives full rank of the controllability matrix. Hence the system is controllable for $l_1 \neq l_2$

Hence the conditions for which the system is controllable is : $l_1 \neq l_2, m_1 > 0$, $m_2 > 0$, $M > 0$

5 LQR Controller

Assuming the initial conditions

1. $x = 0$
2. $\dot{x} = 0$
3. $\theta_1 = 30$
4. $\dot{\theta}_1 = 0$
5. $\theta_2 = 40$

6. $\dot{\theta}_2 = 0$

So are initial state will look like:

$$X_0 = [0, 0, 30, 0, 40, 0]$$

we choose Q and R such that, they stabilizes the system i.e:

$$Q = \begin{bmatrix} 150 & 0 & 0 & 0 & 0 & 0 \\ 0 & 150 & 0 & 0 & 0 & 0 \\ 0 & 0 & 150 & 0 & 0 & 0 \\ 0 & 0 & 0 & 150 & 0 & 0 \\ 0 & 0 & 0 & 0 & 150 & 0 \\ 0 & 0 & 0 & 0 & 0 & 150 \end{bmatrix} \tag{39}$$

$$R = 0.003 \tag{40}$$

5.1 Linearized system

For a Linearized system, the state space equation is:

$$\dot{X} = AX + BU \tag{41}$$
$$Y = CX \tag{42}$$

where,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.981 & 0 & -0.981 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -0.53955 & 0 & -0.04905 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -0.0981 & 0 & -1.0791 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ \frac{1}{1000} \\ 0 \\ \frac{1}{20000} \\ 0 \\ \frac{1}{10000} \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

And $U = -KX$ is Optimal Control for LQR. And So, K is

$$K = -R^{-1}B^TP$$

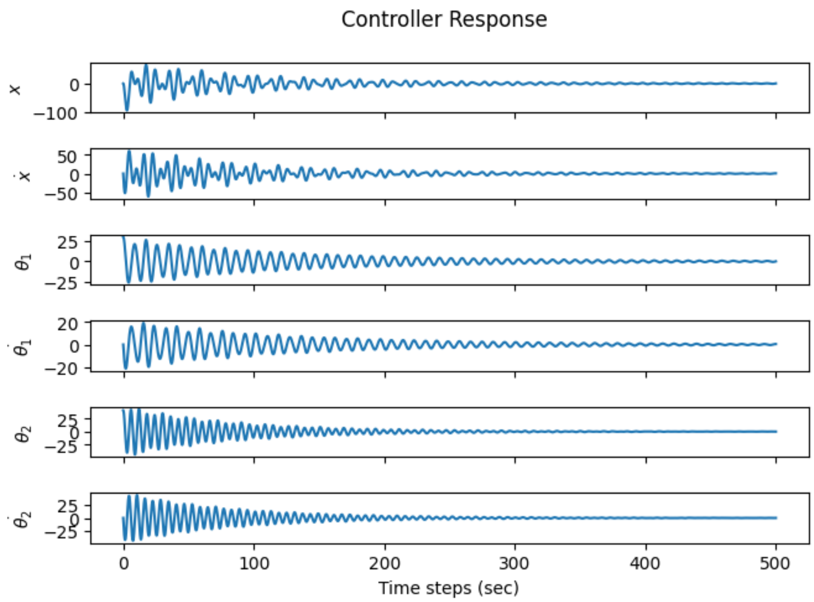

Figure 2: LQR Response For Linearized output

And P can be found using the Ricatti equation, and we get P as:

$$P = \begin{bmatrix} 576.27 & 1031.95 & -517.84 & -1610.18 & -256.01 & -854.11 \\ 1031.95 & 4043.39 & -379.26 & -7241.7 & -129.43 & -3542.27 \\ -517.84 & -379.26 & 35895.63 & 818.87 & 196.67 & -68.1 \\ -1610.18 & -7241.7 & 818.87 & 78740.52 & 951.93 & 3149.11 \\ -256.01 & -129.43 & 196.67 & 951.93 & 17275.06 & 143.47 \\ -854.11 & -3542.27 & -68.1 & 3149.11 & 143.47 & 19067.39 \end{bmatrix}$$

and So, K is

$$K = \begin{bmatrix} -173.205 & -665.416 & 69.025 & 597.953 & 13.497 & 295.616 \end{bmatrix}$$

And figure (2) who the LQR Response for all the states of linearized output.

5.2 Non Linearized system

And figure (3) who the LQR Response for all the states of non-linearized output. given by equation (30). So, we replace the states got from LTI system, in the eq (30) and we get a response.

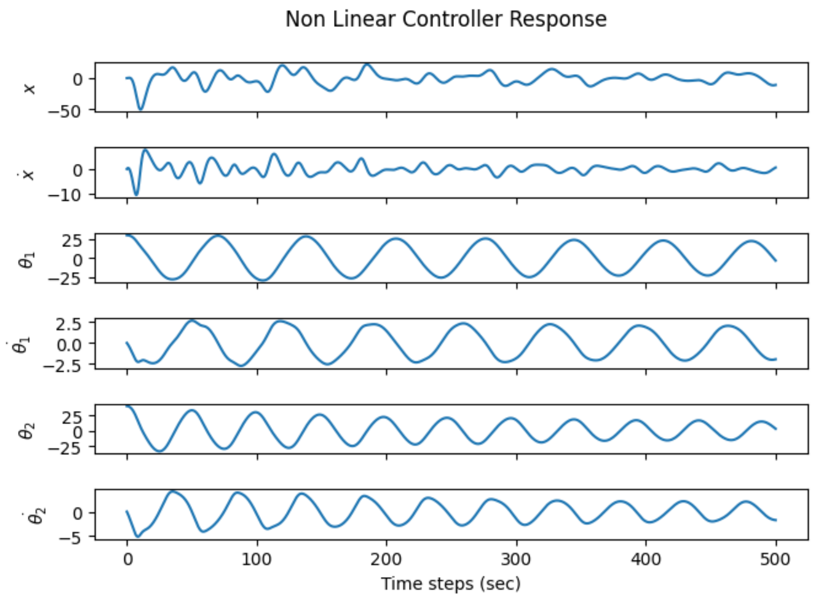

Figure 3: LQR Response for Non Linearized output

5.3 Lyapunov Stability

For getting the Lyapunov Stability using indirect method, we need to check the eigen values of the state space dynamics $A + BK$ matrix

$$\begin{aligned}
\lambda_1 &= -0.384 + 0.354i \\
\lambda_2 &= -0.384 - 0.354i \\
\lambda_3 &= -0.03 + 1.035i \\
\lambda_4 &= -0.03 - 1.035i \\
\lambda_5 &= -0.016 + 0.721i \\
\lambda_6 &= -0.016 - 0.721i
\end{aligned}$$

And we can see that all the eigenvalues lie on the left side of the plane, hence the system is stable.

6 Observability of the system

To Calculate if the system is observable, the Rank of the Observability Matrix must be $= N$ (number of states). where Observability Matrix is

$$O = \begin{bmatrix} C \\ CA \\ . \\ . \\ CA^{n-1} \end{bmatrix}$$

And the Given 4 set of output vectors conditions:

1. only x:
$$C1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
x is observable
2. (θ_1, θ_2):
$$C2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$
(θ_1, θ_2) is not observable
3. (x, θ_2):
$$C3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$
(x, θ_2) is observable
4. (x, θ_1, θ_2):
$$C4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$
(x, θ_1, θ_2) is observable

7 Luenberger Observer

Luenberger Observer is given by:

$$\hat{X}(t) = A\hat{X}(t) + BU(t) + L(Y(t) - C\hat{X}(t)) \tag{43}$$
$$Y(t) = CX(t) \tag{44}$$

Here $\hat{X}(t)$ is the desired state, and L is the Observer Gain Matrix and $L(Y(t) - C\hat{X}(t))$ is a correction term. So, Our state space equation, becomes

$$\dot{X}_e(t) = (A - LC)X_e(t) + B_D U_D(t) \tag{45}$$

Where, $X_e = X - \hat{X}$ and its called an Estimation error.

Now if we design the system with the full state feedback and Luenberger Observer, as K and L both are stable, then our system dynamic looks like this:

$$A = \begin{bmatrix} A + BK & -BK \\ 0 & A - LC \end{bmatrix}$$
$$B = \begin{bmatrix} B \\ 0 \end{bmatrix}$$
$$C = \begin{bmatrix} C & 0 \end{bmatrix}$$

For an assumed set of poles (eigenvalues) which are on the negative half-plane, the system is stable i.e

$$\lambda = [-1, -1.5, -2, -2.5, -3, -3.5]$$

The L can be found and the Ls are:

$$L_1 = \begin{bmatrix} 2.5 \\ -1.99707132797425 \cdot 10^{15} \\ 1.95631477026049 \cdot 10^{15} \\ 2.26334750503748 \cdot 10^{15} \\ -3.30671538250974 \cdot 10^{15} \\ 4.77123435635752 \cdot 10^{15} \end{bmatrix}^T \tag{46}$$

$$L_2 = \begin{bmatrix} 3.03508860861997 & 0.046934359927113 \\ -5.53898384948067 \cdot 10^{15} & -248048348358178.0 \\ 4010.61688730411 & -296.013744511451 \\ 9.23163971407697 \cdot 10^{15} & 413413912360345.0 \\ -25.9384678966168 & 6.87905418863298 \\ 4.61581985703909 \cdot 10^{15} & 206706956180179.0 \end{bmatrix}^T \tag{47}$$

$$L_3 = \begin{bmatrix} 1.64496117007468 & 0.605043188992985 & -0.0846544802781378 \\ -352680596653506.0 & 2.57701537016985 \cdot 10^{15} & 70493020208623.1 \\ 0.33284465077406 & 5.32551441238496 & 0.87094350285879 \\ 587800992190371.0 & -4.29502560063966 \cdot 10^{15} & -117488366568221.0 \\ -0.10589863743439 & 2.90567467648851 & 5.33651608011275 \\ 293900496095186.0 & -2.14751280031983 \cdot 10^{15} & -58744183284113.8 \end{bmatrix}^T \tag{48}$$

Now we will find the LQR response for this system, for the given output vectors list (6) define in the previous section.

7.1 Linearized system

7.1.1 x is a output

If only x is output and the observer gain matrix is L_1 defined in eq (46), then the system response is given by fig (4) and its step Response is given by fig (5)

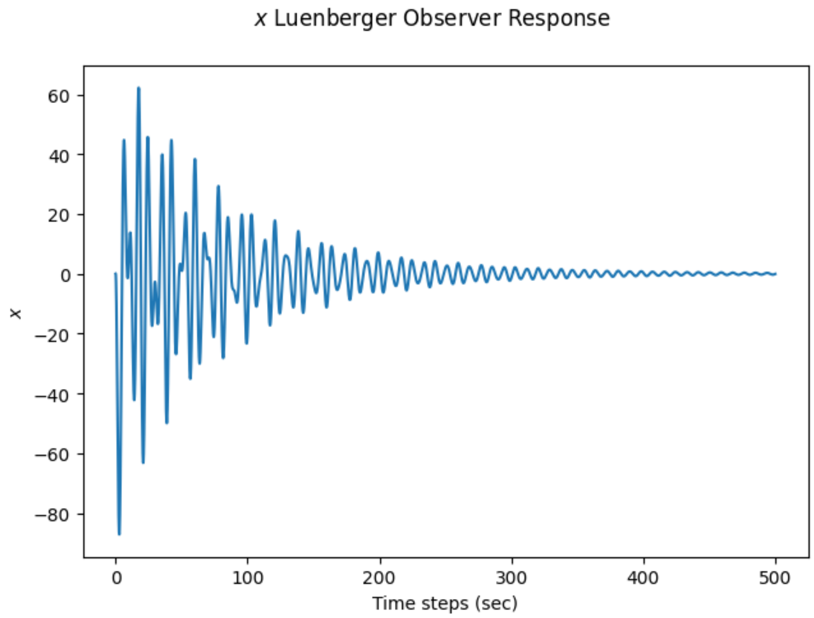

Figure 4: xLuenberger Observer Linear Response

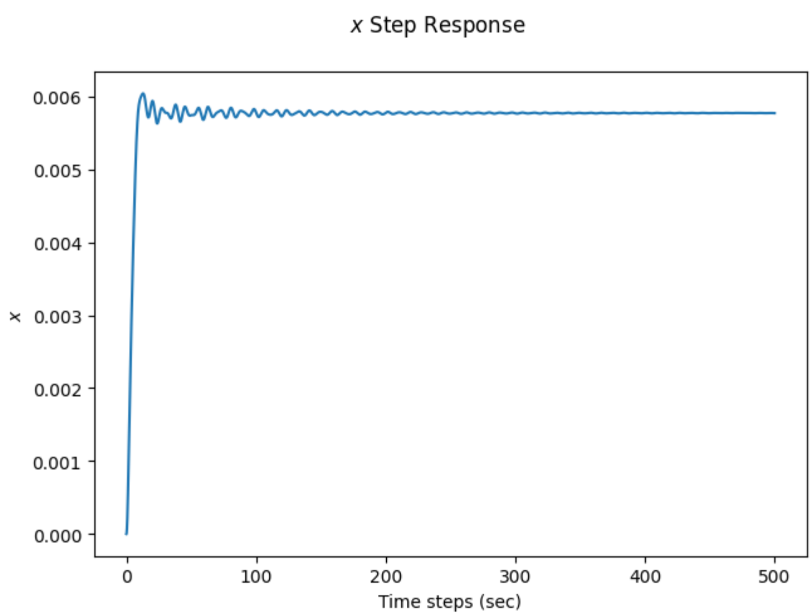

Figure 5: x Luenberger Observer Linear Step Response

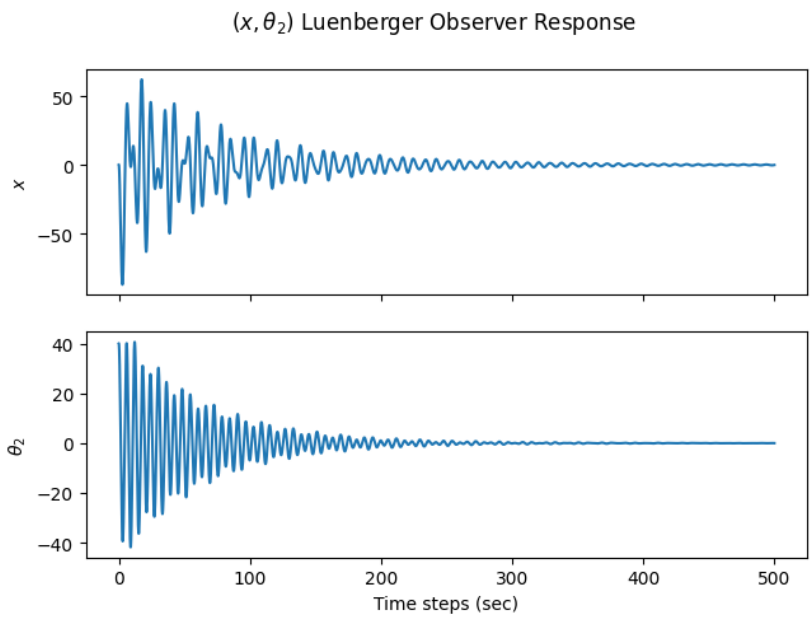

Figure 6: x and θ_2 Luenberger Observer Linear Response

7.1.2 x and θ_2 is a output

If x and θ_2 are output and the observer gain matrix is L_2 defined in eq (47), then the system response is given by fig (6) and its step Response is given by fig (7)

7.1.3 x, θ_1 and θ_2 is a output

If x, θ_1, and θ_2 are an output and the observer gain matrix is L_3 defined in eq (48), then the system response is given by fig 8 and its step Response is given by fig (**??**)

7.2 Non Linear system

7.2.1 x is a output

If only x is output and the observer gain matrix is L_1 defined in eq (46), then the system response is given by fig (10) and its step Response is given by fig (11)

7.2.2 x and θ_2 is a output

If x and θ_2 are output and the observer gain matrix is L_2 defined in eq (47), then the system response is given by fig (12) and its step Response is given by fig (13)

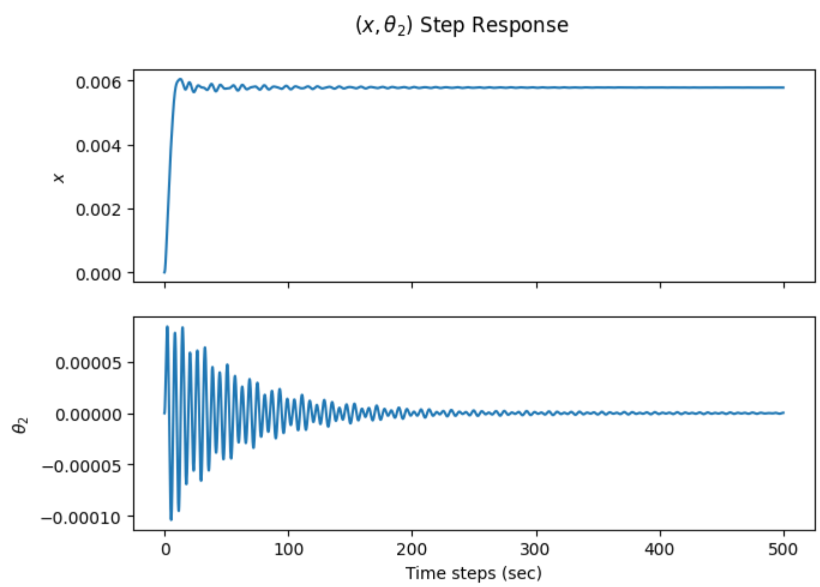

Figure 7: x and θ_2 Luenberger Observer Linear Step Response

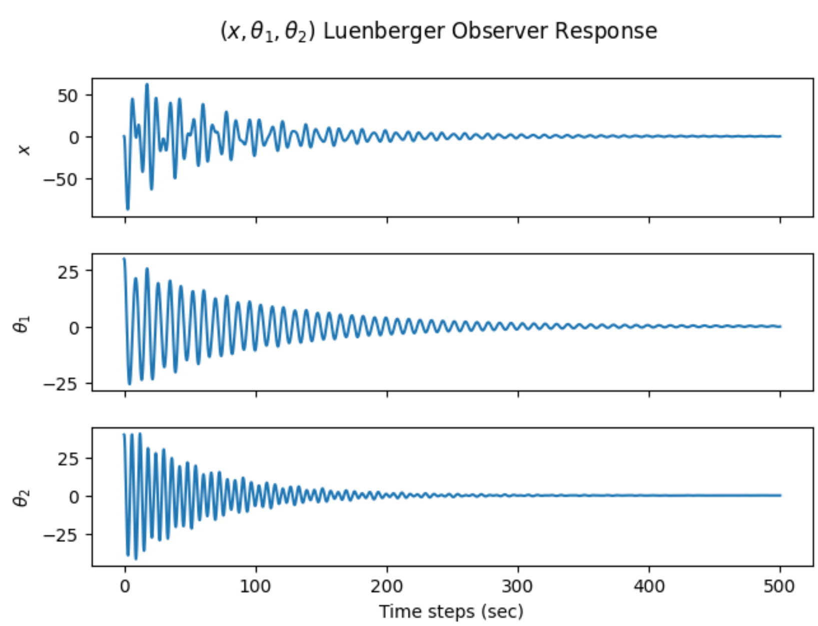

Figure 8: x, θ_1 and θ_2 Luenberger Observer Linear Response

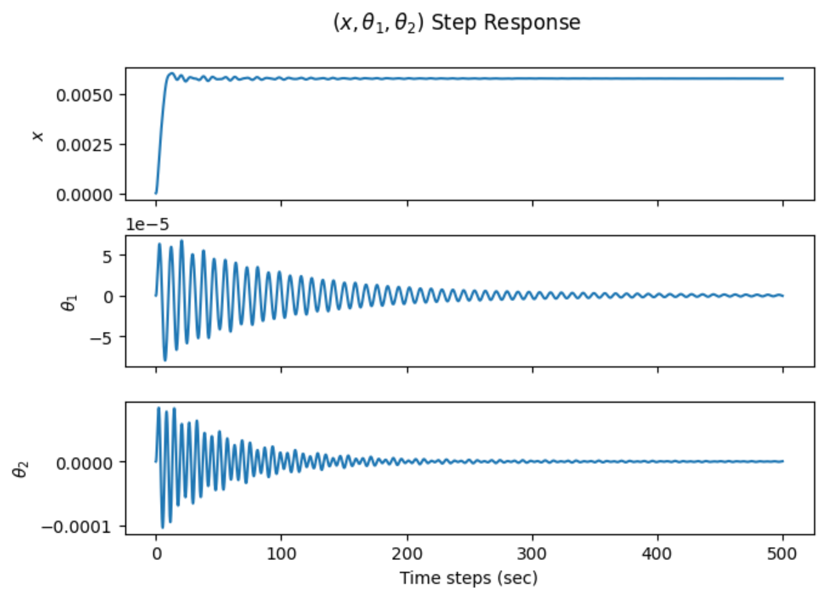

Figure 9: x, θ_1 and θ_2 Luenberger Observer Linear Step Response

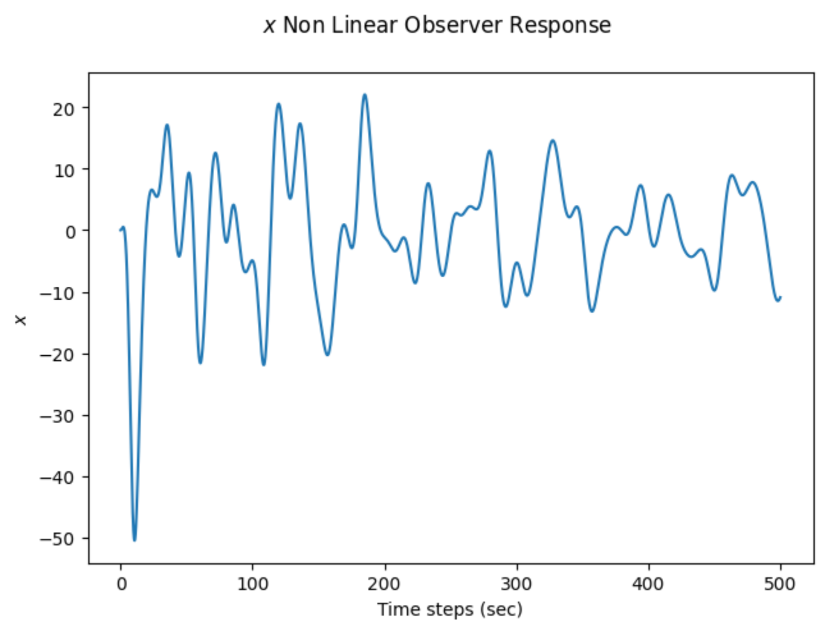

Figure 10: x Luenberger Observer Non Linear Response

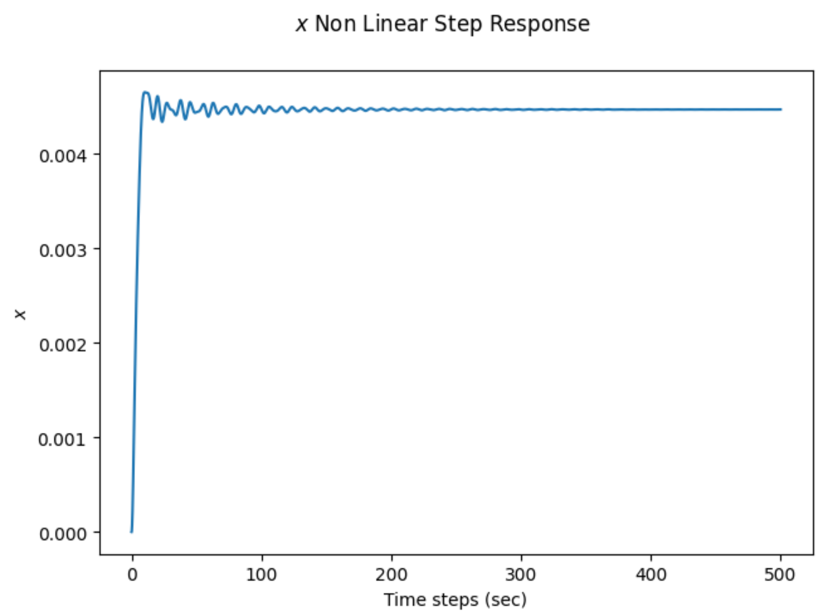

Figure 11: x Non Linear Luenberger Observer Step Response

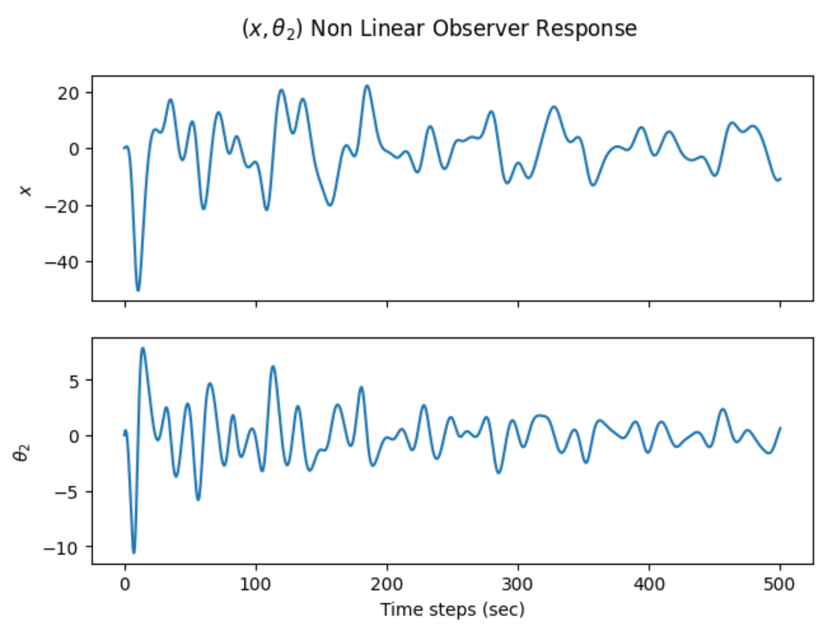

Figure 12: x and θ_2 Luenberger Oberver Non Linear Response

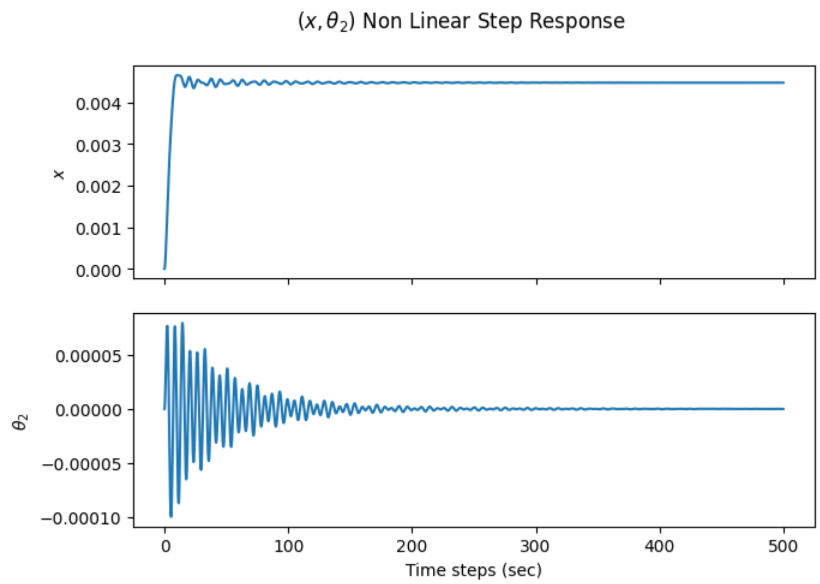

Figure 13: x, and θ_2 Non Linear Luenberger Observer Step Response

7.2.3 x, θ_1 and θ_2 is a output

If x, θ_1, and θ_2 are an output and the observer gain matrix is L_3 defined in eq (48), then the system response is given by fig (14) and its step Response is given by fig (15)

8 LQG Controller Implementation

For LQG Controller, We take few assumption such as:

1. Process Noise Covariance : $\Sigma_D = I_{6x6}$
2. Measurement Noise Covariance : $\Sigma_V = 1$

Optimal Observer gain matrix can be found using the equation:

$$L = PC^T \Sigma_V^{-1} \tag{49}$$

P can be find from following Ricatti equation

$$AP + PA^T + B_D \Sigma_D B_D^T - PC^T \Sigma_V^{-1} CP = 0 \tag{50}$$

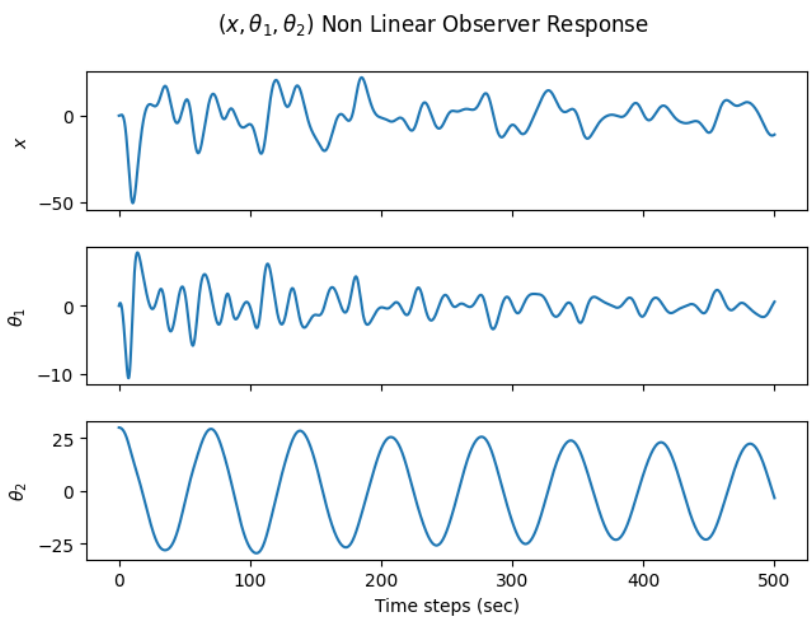

Figure 14: x, θ_1 and θ_2 Luenberger Oberver Non Linear Response

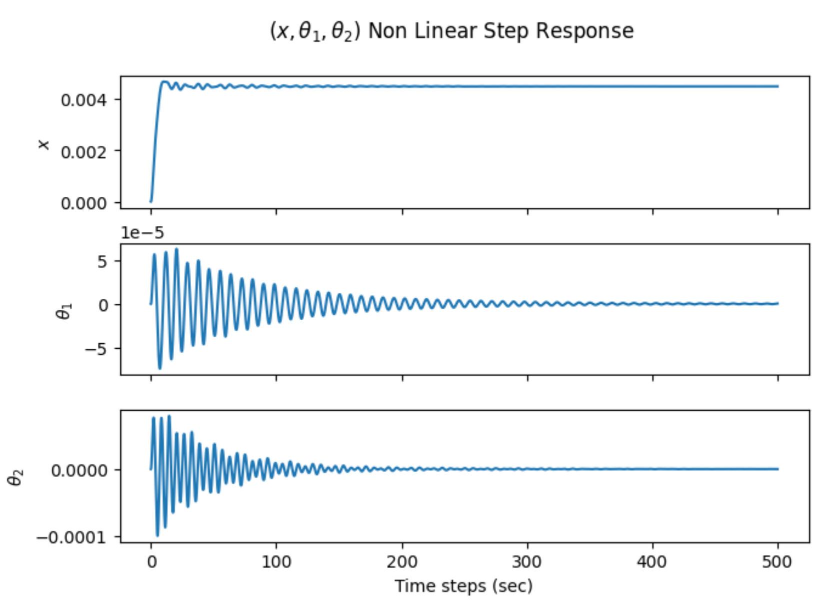

Figure 15: x, θ_1 and θ_2 Non Linear Luenberger Observer Step Response

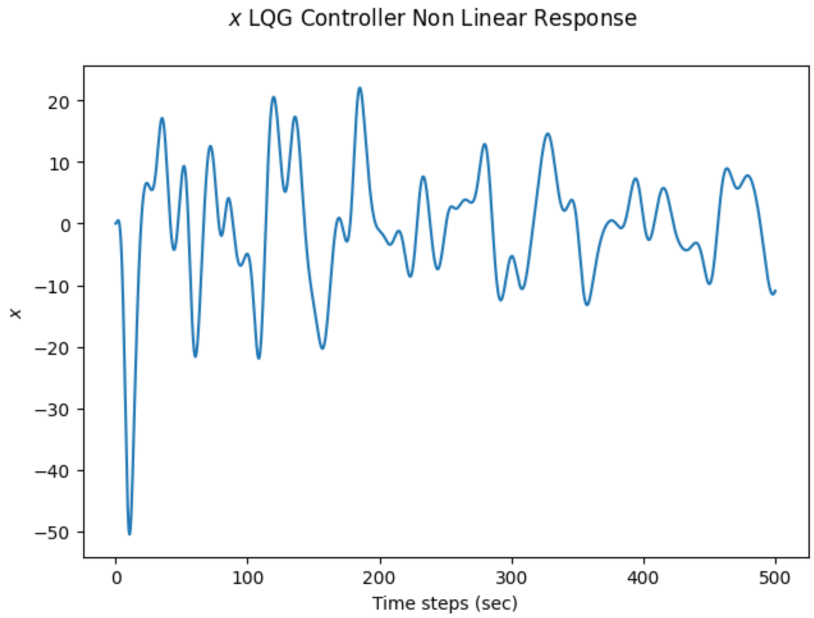

Figure 16: LQG Controller Non Linear Response

So, Optimal Observer gain matrix is:

$$L = \begin{bmatrix} 3.03576311634895 \\ 4.10792884929235 \\ -1.09170929454174 \\ 1.03899409157831 \\ -1.12798591040921 \\ 0.313810944468962 \end{bmatrix} \tag{51}$$

8.1 LQG For Non-linear System

LQG controller response for non-linear system for output vector($x(t)$) is shown in figure (16)

8.2 For performance comparison of Leunberger Observer and LQG :

Assuming the estimation error for all states are 100 and initial state conditions defined here (5), so the initial states are:

$$X = [0, 0, 30, 0, 40, 0, 100, 100, 100, 100, 100, 100]^T$$

The Luenberger Observer output is shown in (17), here L i chosen from equation (46) which is only for output vector $x(t)$.

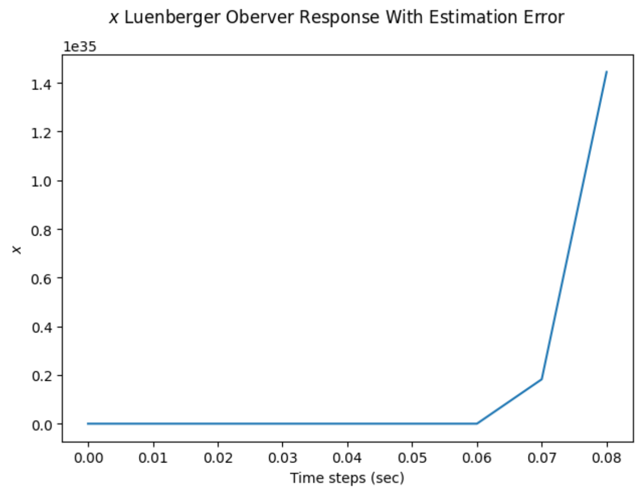

Figure 17: State x Leunberger Observer with estimation error Observation

LQG output is shown in (18) here, L is given by equation (51) which is only for output vector $x(t)$.

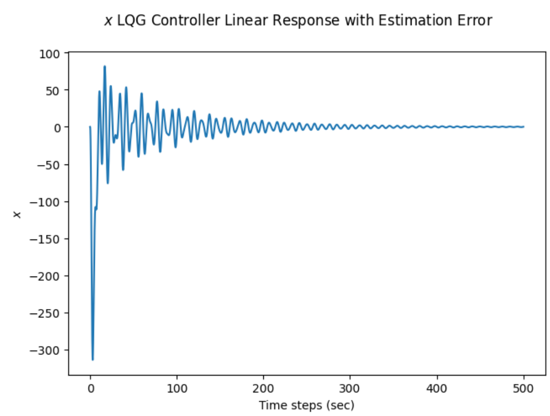

Figure 18: State x LQG Controller with estimation error Observation

From this, we can clearly say that the LQG outperforms the basic Luenberger observer gain matrix, given estimation error.
To reconfigure our system such that it asymptotically tracks a constant reference input x we will include the reference x in our state equations and this new controller will help us to asymptotically track the reference x.This new controller can take constant force disturbances applied on the Crane, provide the disturbances are Gaussian in nature.

9 Implementation Details

Please refer to github for code and Response images.