# An Optimal Control Approach to Robust Control of Robot Manipulators

ENPM667 Project - 1

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# 1 Abstract

Authors of the paper this paper [1] solves a robust control problem of robot manipulators. The robust control law was designed considering unknown load placed on manipulators and modelling the uncertainties like friction in a two-joint SCARA robot. This method takes the robust control problem and translates it to optimal control problem which proves that the solution to optimal control problem is the solution to the robust control problem.

# 2 Introduction

Optimal control problem requires that the dynamic model of the system to be perfect, whereas in reality your model is going to be nowhere close to the actual system. An incorrect model could even result in unstable system behavior. Robust control on the other hand renders your control law impervious to modelling uncertainties thereby allowing for a realistic margin of error. The problem is described as the following: we have a robot manipulator(two-joint SCARA) which is to be controlled to move any random object. To be able to control any system we need to find if there involves uncertainties, and for our case we have two uncertainties that have to taken care of. First is that we don't know the weight of the object to be picked. Second is, uncertain manipulator dynamics because of unmodeled joint parameters like friction between links. Several approaches to solve robust control problem for robots [2] to classify those method. Namely: passivity-based approach, variable-structure controllers, robust saturation approach, robust adaptive approach, and linear-multivariable approach.

The optimal control approach first proposed by Lin et al. [3], [4]to find a robust control for manipulators. In [3] a robust control problem is introduced using the state space representation, where the uncertainty is a function of state and matching condition is assumed. Furthermore, there is no uncertainty in the control input matrix. This robust control problem is then translated into an optimal control problem, where the uncertainty is reflected in the performance index. It is shown in [3] that if the solution to the optimal control problem exists, then it is also a solution to the robust control problem. In [4], the matching condition is relaxed. If it is not totally matched, the uncertainty is decomposed into a matched component and an unmatched component. The robust control problem is similarly translated into an optimal control problem, except that an augmented control is introduced to compensate for the unmatched uncertainty. In the next sections we will also prove how we can translate a robust control problem into a optimal control problem and under what conditions. Using the results in [4] authors of the implemented paper first relaxed the assumption that no uncertainty is present in the input matrix. Then, this extension can be done as long as the matrix representing the uncertainty in the control input is positively semi-definite. This new result is well suited to the robust control problem we are trying to address for robot manipulators.

For implementation and modelling the problem authors described the differential equation for the dynamics of robot manipulators and show how the uncertainties and then formulated the robust control problem for robot manipulators. Furthermore, the optimal control approach for solving the type of robust control problems

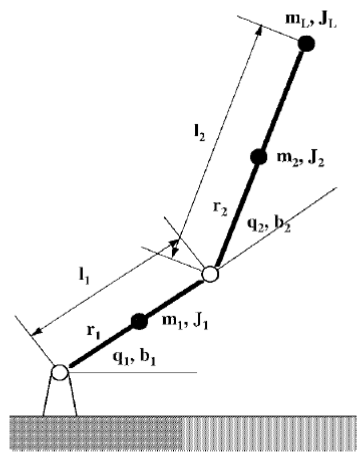


Figure 1: Two-joint SCARA Robot

proposed and discuss three cases and show how the optimal control approach can be used. Next, derive a solution for robust control problem for robot manipulator using the optimal control approach. Finally, illustrate the implementation of the approach by applying it to a two-joint SCARA robot.

# 3 Background

This Section will provide the background needed for deriving the solution for optimal control problem.

## 3.1 Modelling the dynamics of the robot manipulator

The dynamics of a robot manipulator is given by the equation:

$$M(\dot{q}, q)\ddot{q} + V(q, \dot{q}) + F(\dot{q}) + G(q) = \tau \tag{1}$$

where $q$ a matrix for joint variables, $\tau$ is the generalized forces, $M(q)$ is the inertia matrix, $V(q, \dot{q})$ is the Coriolis/centripetal vector, $G(q)$ is the gravity vector, and $F(\dot{q})$ the friction vector. The uncertainties in $M(q)$, $V(q, \dot{q})$, $F(\dot{q})$, and $G(q)$ hence the following bounds have been put on the uncertainties:

1. $M(q)$ ranges between definite matrices $M_o(q)$ and $M_{min}(q)$

$$M_o(q) \geq M(q) \geq M_{min}(q) > 0 \tag{2}$$

2. To find the bounds for the other certainties we will club those into a variable shown below:

$$N(q,\dot{q}) = V(q,\dot{q}) + F(\dot{q}) + G(q) \tag{3}$$

To assume bounds for $N(q,\dot{q})$ there exist $N_o(q,\dot{q})$ and a non-negative function $n_{max}(q,\dot{q})$ such that $n_{max}(q,\dot{q}) = 0$ and we can form the equation:

$$||N(q,\dot{q}) - N_o(q,\dot{q})|| \leq n_{max}(q,\dot{q}) \tag{4}$$

The above equation implies that the gravity vector is known at the equilibrium 0. This assumption is not difficult to satisfy because in most applications, one can select coordinates so that the gravity at $q = 0$ is canceled.

Defining the state variables for moving the robot manipulator from initial position to $(q,\dot{q}) = (0,0)$ .

$$\begin{aligned} x_1 &= q \\ x_2 &= \dot{q} \end{aligned} \tag{5}$$

$\tau$ as torque input, and treat system as double integrator system, where defining the acceleration control input as u:

$$u = M_o(q)^{-1}[\tau - N_o(q,\dot{q})] \tag{6}$$

with inverse dynamic torque control input $\tau$ as:

$$\begin{aligned} \tau &= M_0(\dot{q},q)u + V_0(q,\dot{q}) + F_0(\dot{q}) + G_0(q) \\ &= M_0(\dot{q},q)u + N_0(\dot{q},q)u \end{aligned} \tag{7}$$

where the notation $(.)_0$ represents the computed or nominal value of $(.)$ and indicates that the theoretically exact inverse dynamics control is not achievable in practice due to the uncertainties in the system. The error $\tilde{(.)} = (.)_0 - (.)$ is a measure of one's knowledge of the system parameters.

Then we calculate important equations to lead to a state equations.

$$\dot{x}_1 = \dot{q} = x_2$$

$$\dot{x}_2 = \ddot{q}$$

If we substitute Equation (7) into Equation (1) we obtain:

$$\ddot{q} = u + \eta$$

where the nonlinear function $\eta$ is given by

$$\eta = M^{-1}(\tilde{M}u + \tilde{N})$$

Now if put the $\eta$ in above equation:

$$\begin{aligned}\ddot{q} &= u + M^{-1}(\tilde{M}u + \tilde{N}) \\ &= u + M^{-1}\tilde{M}u + M^{-1}\tilde{N} \\ &= u + M^{-1}(M_0 - M)u + M^{-1}(N_0 - N) \\ &= u + (M^{-1}M_0 - I)u + M^{-1}(N_0 - N)\end{aligned}$$

Now we define uncertainties $f(x)$ and $h(x)$

$$h(x) = M(x_1)^{-1}M_o(x_1) - I \tag{8}$$

$$f(x) = M(x_1)^{-1}(N_o(x_1, x_2) - N(x_1, x_2)) \tag{9}$$

Therefore we can now define our state equation as

$$\dot{x} = Ax + B(u + h(x)u) + B(f(x)) \tag{10}$$

where,

$$\dot{x} = \begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix}; \tag{11}$$

$$A = \begin{bmatrix}0 & I \\ 0 & 0\end{bmatrix}; \tag{12}$$

$$B = \begin{bmatrix}0 \\ I\end{bmatrix}; \tag{13}$$

Here, $A \in R^{n*n}$ and $B \in R^{n*m}$. where, $m =$ number of input, and $n =$ number of states. Our goal is to control the above system under the uncertainties $f(x)$ and $h(x)$.

Our Final Control System design looks like Figure (2), where,

- Output Feedback Controller is defined by equation (6)
- Inner Feedback Controller is defined by equation (7)
- Robot state space equation is defined by equation (10)
- qd (desired angle) $= 0$.

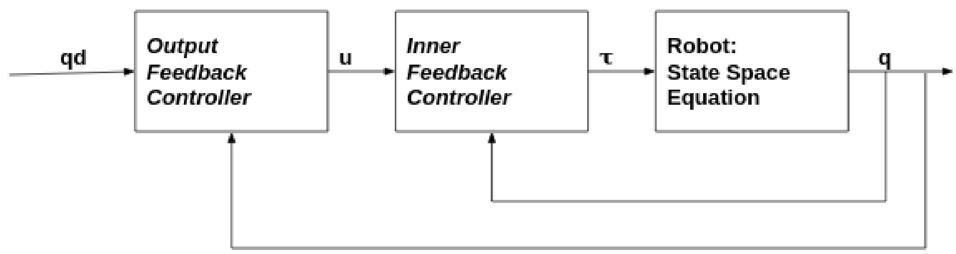


Figure 2: Controller Design

## 3.2 Theory

For linear systems, the optimal control problem becomes an LQR problem, whose solution always exists and can be easily obtained by solving an algebraic Riccati equation. However, we have a nonlinear system which we may not be able to easily compute the solution to the optimal control problem as analytical solutions may not be available, forcing us to use numerical solutions. As stated in [5] the theory for nonlinear systems in a manner similar to the theory for linear systems. We first study the case of matched uncertainty. We show that as long as the solution to the corresponding optimal control problem exists, it is a solution to the robust control problem. We then consider unmatched uncertainty. Again, we decompose the uncertainty into matched and unmatched components and introduce an augmented control for the unmatched uncertainty. A computable sufficient condition is also derived to ensure that the solution to the corresponding optimal control problem is a solution to the robust control problem. Finally, how to handle uncertainty in the input matrix will be discussed.

### 3.2.1 Matched Uncertainty

Consider the nonlinear system:

$$\dot{x} = A(x) + B(x)u \tag{14}$$

where $x \in R^n, u \in R^m$. Suppose that the function, $A(x)$, is known only up to an additive perturbation which is bounded by a known function. If this perturbation is in the range of $B(x)$, then we may seek a control that will compensate for the perturbation. The condition that the unknown perturbation be in the range of

$B(x)$ is called the matching condition, and the uncertainty is called as matched uncertainty and can be incorporated by expressing the system as:

$$\dot{x} = A(x) + B(x)f(x) + B(x)u \tag{15}$$

where $x \in R^n, u \in R^m, A(x), B(x)$ are known functions, and $f(x)$ is an unknown nonlinear function of $x$. $B(x)f(x)$ models the uncertainty in the system. Since the uncertainty is the range of $B(x)$. There are following assumptions:

1. Assumption 1: $A(0) = 0$ and $f(0) = 0$ such that $x = 0$ is an equilibrium it will be the only equilibrium if the robust control problem is solvable).

2. Assumption 2: The uncertainty $f(x)$ is bounded; that is; there exists a non-negative function $f_{max}(x)$ such that:

$$\|f(x)\| \leq f_{max}(x) \tag{16}$$

The goal is to solve the following robust control problem of stabilizing the system under uncertainty. We now have the two assumptions next, let us define the robust control problem and optimal control problem which will help us to solve the robust control problem for robotic manipulators.

I. Robust Control Problem:
To find a feedback control law $u = u_o(x)$ such that the closed loop system

$$\dot{x} = A(x) + B(x)f(x) + B(x)u_o(x) \tag{17}$$

is globally asymptotic stable for all admissible uncertainties, $f(x)$ satisfying the assumption that $\|f(x)\| \leq f_{max}(x)$. To solve the robust control problem defined here authors in [5] showed that it can indirectly translated into an optimal control problem with the help of theorem 1 explained further in the document. But, next we describe our optimal control problem and then show the proof for the theorem.

II. Optimal Control Problem:
For the nominal non linear system:

$$\dot{x} = A(x) + B(x)u \tag{18}$$

we need to find a feedback control law $u = u_0(x)$ that minimizes the following cost functions:

$$\int_0^\infty [f_{max}(x)^2 + \|x\|^2 + \gamma^2 * \|u\|^2]dt \tag{19}$$

The cost function consists of three terms: $f_{max}(x)^2$ for the cost of uncertainty; $\|x\|^2$ or $x^T x$ for the cost of regulation; $\|u\|^2$ or $u^T u$ for the cost of control; and $\gamma \neq 0$ is a design parameter.

**Theorem 1.** *If the solution to the optimal control problem exists, then it is a solution to the robust control problem.*

*Proof.* The following proof can be found in [5]. Let $u = u_0(x)$ be the solution to Optimal control problem. We start from the system of equation of the robust control problem:

$$\dot{x} = A(x) + B(x)f(x) + B(x)u_o(x) \tag{20}$$

We have to show that the above equation is is globally asymptotically stable for all uncertainties $f(x)$, satisfying $\|f(x)\| \leq f_{max}(x)$. Now let us define:

$$V(x_0) = min_{u \in R^m} \int_0^\infty \left[ f_{\max}(x)^2 + x^T x + u^T u \right] dt \tag{21}$$

To be the minimum cost of the optimal control of the nominal system from some initial state $x_0$. We would like to show that $V(x)$ is a Lyapunov function for optimal control problem. By definition, $V(x)$ must must satisfy the Hamilton–Jacobi–Bellman equation, which reduces to

$$min_{u \in R^m}(f_{\max}(x)^2 + x^T x + u^T u + V_x^T(A(x) + B(x)u)) = 0 \tag{22}$$

Since $u = u_o(x)$ is the optimal control, it must satisfy the above equation; that is:

$$f_{\max}(x)^2 + x^T x + u_0(x)^T u_0(x) + V_x^T(A(x) + B(x)u_0)) = 0 \tag{23}$$

$$2u_0(x)^T + V_x^T B(x) = 0 \tag{24}$$

Using the above two equations, we can show that $V(x)$ is a Lyapunov function for the optimal control problem. Hence we can now deduce that:

$$V(x) > 0 if x \neq 0 \tag{25}$$

$$V(x) = 0 if x = 0 \tag{26}$$

Using the above equations we need to prove that $\dot{V}(x) < 0$ for all $x \neq 0$.

$$\begin{aligned}
\dot{V}(x) = V_x^{\mathrm{T}} \dot{x} &= V_x^{\mathrm{T}} \left(A(x) + B(x) u_o(x) + B(x) f(x)\right) \\
&= V_x^{\mathrm{T}} \left(A(x) + B(x) u_o(x)\right) + V_x^{\mathrm{T}} B(x) f(x) \\
&= -f_{\max}(x)^2 - x^{\mathrm{T}} x - u_o(x)^{\mathrm{T}} u_o(x) + V_x^{\mathrm{T}} B(x) f(x) \\
&= -f_{\max}(x)^2 - x^{\mathrm{T}} x - u_o(x)^{\mathrm{T}} u_o(x) - 2 u_o(x)^{\mathrm{T}} f(x) \\
&= -f_{\max}(x)^2 + f(x)^{\mathrm{T}} f(x) - x^{\mathrm{T}} x - u_0(x)^{\mathrm{T}} u_0(x) - 2 u_0(x)^{\mathrm{T}} f(x) - f(x)^{\mathrm{T}} f(x) \\
&= -f_{\max}(x)^2 + f(x)^{\mathrm{T}} f(x) - x^{\mathrm{T}} x - \left(u_o(x) + f(x)\right)^{\mathrm{T}} \left(u_o(x) + f(x)\right) \\
&\leq -f_{\max}(x)^2 + f(x)^{\mathrm{T}} f(x) - x^{\mathrm{T}} x
\end{aligned}$$

By the second assumption $\|f(x)\| \leq f_{max}(x)$ we can now say that $f(x) \leq f_{max}(x)^2$ therefore we can write the following equations:

$$\dot{V}(x) \leq -x^T x \tag{27}$$

This equation can be split into two conditions:

$$\dot{V}(x) < 0 \quad \text{if} \quad x \neq 0 \tag{28}$$

$$\dot{V}(x) = 0 \quad \text{if} \quad x = 0 \tag{29}$$

Thus, the conditions of the Lyapunov stability theorem are satisfied. Consequently, there exists a neighborhood of $0, N = \{x : \|x\| < c\}$ for some $c > 0$ such that if $x(t)$ enters $N$, then

$$x(t) \to 0 \quad \text{as} \quad t \to \infty$$

But $x(t)$ cannot remain forever outside $N$. Otherwise,

$$\|x(t)\| \geq c$$

for all $t >$ and

$$V(x(t)) - V(x(0)) = \int_0^t \dot{V}(x(\tau)) \mathrm{d}\tau \tag{30}$$

$$\begin{aligned}
&\leq \int_0^t \left(-x^T x\right) \mathrm{d}\tau \\
&\leq -\int_0^t c^2 \, \mathrm{d}\tau \\
&\leq -c^2 t
\end{aligned} \tag{31}$$

Let $t \to \infty$, we have

$$V(x(t)) \leq V(x(0)) - c^2 t \to -\infty \tag{32}$$

which contradicts the fact that $V(x(t)) > 0$ for all $x(t)$. Therefore

$$x(t) \to 0 \quad \text{as} \quad t \to \infty$$

no matter where the trajectory begins. That is, system (6.2) is globally asymptotically stable for all admissible uncertainties. In other words, $u = u_o(x)$ is a solution to Robust Control Problem. Hence we proved the theorem 1. □

### 3.2.2 Unmatched Uncertainty

Now we assume that uncertainty is not in the range of $B(x)$. So the following non linear system looks like:

$$\dot{x} = A(x) + C(x)f(x) + B(x)u \tag{33}$$

where $f(x)$ models the uncertainty in the system dynamics and $C(x)$ can be any matrix $C(x) \in R^{n*p}$ and $C(x) \neq B(x)$. It is also possible that $C(x) = I$ then $C(x)f(x) = f(x)$ the reason authors in [5] introduced $C(x)$ is to make the definition of uncertainty $f(x)$ more flexible. We make the following two assumptions:

1. Assumption 1: $f(0) = 0$ and $A(0) = 0$, so, that $x = 0$ is an equilibrium.
2. Assumption 2: The uncertainty $f(x)$ is bounded.

We would like to solve the following robust control problem.

I. Robust Control Problem:
Find a feedback control law $u = u_o(x)$ such that the closed loop system

$$\dot{x} = A(x) + C(x)f(x) + B(x)u_0(x) \tag{34}$$

is globally asymptotic stable for all admissible uncertainties, $f(x)$. We will solve the above robust control problem indirectly by translating it into an optimal control problem.The relation between the robust control problem and the optimal control problem is shown in the Theorem (2).
where $C(x)f(x)$ is defined as combination of matched and unmatched uncertainty.

$$C(x)f(x) = B(x)f_0(x) + [C(x)f(x) - B(x)f_0(x)]$$

for some $f_0(x)$ which needs to be determined, $f_0(x)$ should be chosen so that the "energy" of $\|C(x)f(x) - B(x)f_0(x)\|$ unmatched component is minimised.

So, to the value of $f_0(x)$ should make the $B(x)f_0(x)$ equal to $C(x)f(x).$ so, that the energy minimises, hence:

$$f_0(x) = B(x)^+ C(x)f(x)$$

where, $B(x)^+$ is pseudo inverse of $B(x)$, and final decomposition will be:

$$C(x)f(x) = B(x)B(x)^+C(x)f(x) + [C(x)f(x) - B(x)B(x)^+C(x)f(x)]$$

$$C(x)f(x) = B(x)B(x)^+C(x)f(x) + [I - B(x)B(x)^+]C(x)f(x) \tag{35}$$

II. Optimal Control Problem:
For the following auxiliary non linear system, where only unmatched uncertainty have been taken:

$$\dot{x} = A(x) + B(x)u + [I - B(x)B(x)^+]C(x)v \tag{36}$$

find the feedback control law $(u_0(x), v_0(x))$ that minimizes the following cost functional:

$$\int_0^\infty [f_{max}(x)^2 + \rho^2 * g_{max}(x)^2 + \beta^2 * \|x\|^2 + \|u\|^2 + \rho^2 * \|v\|^2]dt \tag{37}$$

where $\alpha \geq 0$, $\rho \geq 0$ and $\beta \geq 0$ are design parameters. $f_{max}$,$g_{max}$ are non-negative functions such that:

$$\|B(x)^+C(x)f(x)\| \leq f_{max}(x) \tag{38}$$

$$\|\alpha^{-1}f(x)\| \leq g_{max}(x) \tag{39}$$

**Theorem 2.** *If one can choose $\alpha$, $\rho$ and $\beta$ such that the solution to the optimal control problem denoted by $(u_0(x), v_0(x))$ exists and the following condition is satisfied*

$$2\rho^2\|v_0(x)\|^2 \leq \beta'^2\|x\|^2, \;\; \forall x \in R^n$$

*for some $\beta' < \beta$ then $u_0(x)$ the $u$-component of the solution to the optimal control problem, is a solution to the robust control problem.*

*Proof.* Let $u_0(x)$, $v_0(x)$ be the solution to Optimal Control Problem. We need to show that

$$\dot{x} = A(x) + B(x)u_0(x) + C(x)f(x)$$

is globally asymptotically stable for all uncertainties $f(x)$. To prove this, we define

$$V\left(x_o\right)=\min_{u,v}\int_0^{\infty}\left(f_{\max}(x)^2+\rho^2 g_{\max}(x)^2+\beta^2\|x\|^2+\|u\|^2+\rho^2\|v\|^2\right)\mathrm{d}t \tag{40}$$

to be the minimum cost of the optimal control of the auxiliary system from some initial state $x_o$. We would like to show that $V(x)$ is a Lyapunov function for system (6.7). By definition, $V(x)$ must satisfy the Hamilton-Jacobi-Bellman equation, which reduces to

$$\begin{aligned}&\min_{u,v}\left(f_{\max}(x)^2+\rho^2 g_{\max}(x)^2+\beta^2\|x\|^2+\|u\|^2+\rho^2\|v\|^2\right.\\&\left.+V_x^{\mathrm{T}}\left(A(x)+B(x)u+\alpha\left(I-B(x)B(x)^{+}\right)C(x)v\right)\right)=0\end{aligned} \tag{41}$$

Since $u_0(x), v_0(x)$ is the optimal control, it must satisfy the above equation; that is

$$\begin{aligned}&f_{\max}(x)^2+\rho^2 g_{\max}(x)^2+\beta^2\|x\|^2+\left\|u_o(x)\right\|^2+\rho^2\left\|v_o(x)\right\|^2\\&\quad+V_x^{\mathrm{T}}\left(A(x)+B(x)u_o(x)+\alpha\left(I-B(x)B(x)^{+}\right)C(x)v_o(x)\right)=0\\&2u_o(x)^{\mathrm{T}}+V_x^{\mathrm{T}}B(x)=0\\&2\rho^2 v_o(x)^{\mathrm{T}}+V_x^{\mathrm{T}}\alpha\left(I-B(x)B(x)^{+}\right)C(x)=0\end{aligned} \tag{42}$$

With the aid of the above equations, we can show that $V(x)$ is a Lyapunov function for System. Clearly we can now show piecce wise function for $V(x)$ :

$$V(x)>0 if x\neq 0 \tag{43}$$

$$V(x)=0 if x=0 \tag{44}$$

Using all the equations above we need to show that $\dot{V}(x)<0$ for all $x\neq 0$

$$
\begin{aligned}
&\dot{V}(x) = V_x^{\mathrm{T}} \dot{x} \\
&= V_x^{\mathrm{T}} \left(A(x) + B(x)u_o(x) + C(x)f(x)\right) \\
&= V_x^{\mathrm{T}} \left(A(x) + B(x)u_o(x) + \alpha\left(I - B(x)B(x)^{+}\right) C(x)v_o(x)\right) \\
&- V_x^{\mathrm{T}} \alpha \left(I - B(x)B(x)^{+}\right) C(x)v_o(x) + V_x^{\mathrm{T}} C(x)f(x) \\
&= V_x^{\mathrm{T}} \left(A(x) + B(x)u_o(x) + \alpha\left(I - B(x)B(x)^{+}\right) C(x)v_o(x)\right) \\
&- V_x^{\mathrm{T}} \alpha \left(I - B(x)B(x)^{+}\right) C(x)V_o(x) + V_x^{\mathrm{T}} B(x)B(x)^{+}C(x)f(x) \\
&+ V_x^{\mathrm{T}} \left(I - B(x)B(x)^{+}\right) C(x)f(x) \\
&= -f_{\max}(x)^2 - \rho^2 g_{\max}(x)^2 - \beta^2\|x\|^2 - \|u_o(x)\|^2 - \rho^2\|v_o(x)\|^2 \\
&+ 2\rho^2 v_o(x)^{\mathrm{T}} v_o(x) - 2u_o(x)^{\mathrm{T}} B(x)^{+}C(x)f(x) - 2\alpha^{-1}\rho^2 v_o(x)^{\mathrm{T}} f(x) \\
&= -f_{\max}(x)^2 - \rho^2 g_{\max}(x)^2 - \beta^2\|x\|^2 - \|u_o(x)\|^2 + \rho^2\|v_o(x)\|^2 \\
&- 2u_o(x)^{\mathrm{T}} B(x)^{+}C(x)f(x) - 2\alpha^{-1}\rho^2 v_o(x)^{\mathrm{T}} f(x)
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
-\|u_o(x)\|^2 - 2u_o(x)^{\mathrm{T}} B(x)^{+}C(x)f(x) &\le \left\|B(x)^{+}C(x)f(x)\right\|^2 \le f_{\max}(x)^2 \\
-2\alpha^{-1}\rho^2 v_o(x)^{\mathrm{T}} f(x) &\le \rho^2\|v_o(x)\|^2 + \rho^2\left\|\alpha^{-1}f(x)\right\|^2 \\
&\le \rho^2\|v_o(x)\|^2 + \rho^2 g_{\max}(x)^2
\end{aligned}
$$

Therefore, if the condition $2\rho^2\|v_o(x)\|^2 \le \beta'^2\|x\|^2, \forall x \in R^n$ is satisfied

$$
\begin{aligned}
\dot{V}(x) &\le -\beta^2\|x\|^2 + 2\rho^2\|v_o(x)\|^2 \\
&= 2\rho^2\|v_o(x)\|^2 - \beta^2\|x\|^2 - \left(\beta^2 - \beta'^2\right)\|x\|^2 \\
&\le -\left(\beta^2 - \beta'^2\right)\|x\|^2
\end{aligned}
$$

In other words,

$$
\begin{aligned}
\dot{V}(x) < 0 \quad x \neq 0 \\
\dot{V}(x) = 0 \quad x = 0
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
-\|u_o(x)\|^2 - 2u_o(x)^{\mathrm{T}} B(x)^{+}C(x)f(x) &\le \left\|B(x)^{+}C(x)f(x)\right\|^2 \le f_{\max}(x)^2 \\
-2\alpha^{-1}\rho^2 v_o(x)^{\mathrm{T}} f(x) &\le \rho^2\|v_o(x)\|^2 + \rho^2\left\|\alpha^{-1}f(x)\right\|^2 \\
&\le \rho^2\|v_o(x)\|^2 + \rho^2 g_{\max}(x)^2
\end{aligned}
$$

Therefore, if the condition $2\rho^2 \left\|v_o(x)\right\|^2 \leq \beta'^2 \|x\|^2, \forall x \in R^n$ is satisfied

$$\begin{aligned}\dot{V}(x) &\leq -\beta^2 \|x\|^2 + 2\rho^2 \left\|v_o(x)\right\|^2 \\ &= 2\rho^2 \left\|v_o(x)\right\|^2 - \beta^2 \|x\|^2 - \left(\beta^2 - \beta'^2\right) \|x\|^2 \\ &\leq -\left(\beta^2 - \beta'^2\right) \|x\|^2\end{aligned}$$

In other words,

$$\begin{aligned}\dot{V}(x) < 0 \quad x \neq 0 \\ \dot{V}(x) = 0 \quad x = 0\end{aligned}$$

Thus, the conditions of The Lyapunov stability theorem are satisfied. Consequently, there exists a neighbourhood of 0 $N = x : \|x\| < c$ for $c > 0$ such that if $x(t)$ enters $N$ then

$$x(t) \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty$$

Since $x(t)$ cannot forever remain outside $N$. Otherwise,

$$\|x(t)\| \geq c \tag{45}$$

for all $t > 0$, which implies

$$\begin{aligned}V(x(t)) - V(x(0)) &= \int_0^t \dot{V}(x(\tau)) \mathrm{d}\tau \\ &\leq \int_0^t -\left(\beta^2 - \beta'^2\right) \|x\|^2 \, \mathrm{d}\tau \\ &\leq -\int_0^t \left(\beta^2 - \beta'^2\right) c^2 \, \mathrm{d}\tau \\ &\leq -\left(\beta^2 - \beta'^2\right) c^2 t\end{aligned}$$

Let $t \rightarrow \infty$, we have

$$V(x(t)) \leq V(x(0)) - \left(\beta^2 - \beta'^2\right) c^2 t \rightarrow -\infty$$

which contradicts the fact that $V(x(t)) > 0$ for all $x(t)$. Therefore

$$x(t) \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty$$

no matter where the trajectory begins. This proves that $u_o(x)$ is a solution to Robust Control Problem. □

If we compare cost function with Matched Uncertainty 3.2.1 optimal control problem, an additional condition is needed to ensure that the unmatched part of uncertainty $[I - B(x)B(x)^{+}]C(x)f(x)$ is not too high. and this condition is sufficient and can be met by proper choice of $\rho$ and $\beta$.

#### 3.2.3 Uncertainty in the Input Matrix

Now if take into account uncertainty in the control input matrix including unmatched uncertainty, the generalized non linear system, will look like:

$$\dot{x} = A(x) + C(x)f(x) + B(x)(u + h(x)u) \tag{46}$$

where, $B(x)h(x)$ models the uncertainty in the input matrix, and $C(x)f(x)$ is given by equation (35). Similar to section 3.2.1 and 3.2.2, this section also will first show the robust control problem and then its alternative optimal control problem, to get the robust control solution.
1) Robust Control Problem: Find a feedback control law $u = u_0(x)$ such that closed-loop system.

$$\dot{x} = A(x) + C(x)f(x) + B(x)(u_0(x) + h(x)u_0(x))$$

This system is globally asymptotically stable for all uncertainties $f(x)$ and $h(x)$ given certain assumptions.

1. $h(x) \geq 0$
2. Including assumptions of robust control problem of section 3.2.2
   (a) there exists a non negative function $f_{max}(x)$, such that
   $$\|B(x)^* C(x)f(x)\| \leq f_{max}(x)$$
   (b) There exists a non negative function $g_{max}(x)$, such that
   $$\|f(x)\| \leq g_{max}(x)$$

The corresponding optimal control problem is as follows.
2) Optimal Control Problem: For the following auxiliary system. Similar to section 3.2.2, optimal control problem, our equation will be same as equation (36)

$$\dot{x} = A(x) + B(x)u + [I - B(x)B(x)^{+}]C(x)v \tag{47}$$

find a feedback control law $u_0(x), v_0(x)$ that minimizes the following cost functional, the cost function is same as equation (37)

$$\int_0^\infty [f_{max}(x)^2 + \rho^2 * g_{max}(x)^2 + \beta^2 * \|x\|^2 + \|u\|^2 + \rho^2 * \|v\|^2]dt \tag{48}$$

The relation between robust control problem and optimal control problem is shown in the following theorem, which is similar to theorem (2)

**Theorem 3.** *If one can choose $\rho$ and $\beta$ such that the solution to the optimal control problem denoted by $(u_0(x), v_0(x))$ exists and the following condition is satisfied*

$$2\rho^2\|v_0(x)\|^2 \leq \beta'^2\|x\|^2, \;\; \forall x \in R^n \tag{49}$$

*for some $\beta' < \beta$ then $u_0(x)$ the $u$-component of the solution to the optimal control problem, is a solution to the robust control problem.*

*Proof.* To simplify notation, in this proof we will eliminate, when possible, the explicit reference to $x$ when we denote a function of $x$. Define

$$V(x_0) = \min_{u\in R^m, v\in R^p} \int_0^{\infty} [f_{max}(x)^2 + \rho^2 * g_{max}(x)^2 + \beta^2 * \|x\|^2 + \|u\|^2 + \rho^2 * \|v\|^2] dt$$

to be the minimum cost of the optimal control that brings the auxiliary system from the initial state $x_0$ to its equilibrium ($x = 0$). The Hamilton-Jacobi-Bellman equation gives us

$$\min_{u\in R^m, v\in R^p} [f_{max}(x)^2 + \rho^2 * g_{max}(x)^2 + \beta^2 * \|x\|^2 + \|u\|^2 + \rho^2 * \|v\|^2 + V_x^T(A + Bu + (I - BB^+)Cv)] = 0$$

where $V_x(x) = dV/dx$ Therefore, if $(u_0(x), v_0(x)$)solution to the optimal control problem, then

$$f_{max}(x)^2 + \rho^2 * g_{max}(x)^2 + \beta^2 * \|x\|^2 + \|u\|^2 + \rho^2 * \|v\|^2 + V_x^T(A + Bu + (I - BB^+)Cv) = 0$$

$$2u_0^T + V_x^T B = 0$$
$$2\rho^2 v_0^T + V_x^T[I - BB^+]C = 0$$

We now show that $u_0$ is a solution to the robust control problem, i.e., the equilibrium $x = 0$ of

$$\dot{x} = A + B(u_0 + hu_0) + Cf$$

is globally asymptotically stable for all admissible uncertainties $f(x)$ and $h(x)$. To do this, we show that $V(x)$ is a Lyapunov function. Clearly

$$V(X) > 0, \; x \neq 0;$$
$$V(x) = 0, \; x = 0;$$

To show $\dot{V}(x) = dV(x)/dt < 0$ for $x \neq 0$, we have

$$\begin{aligned}\dot{V}(x) &= V_x^T[A + Bu_0 + Cf + Bhu_0] \\ &= V_x^T[A + Bu_0 + (I - BB^+)Cv_0 \\ &+ BB^+Cf + (I - BB^+)C(f - v_0) + Bhu_0] \\ &= V_x^T[A + Bu_0 + (I - BB^+)Cv_0] \\ &+ V_x^T BB^+Cf + V_x^( I - BB^+)C(f - v_0) + V_x^T Bhu_0 \\ &= -f_{max}(x)^2 - \rho^2 g_{max}(x)^2 - \beta^2\|x\|^2 - \|u\|^2 - \\ &\rho^2 * \|v\|^2 - 2u_0^T B^+Cf - 2\rho^2 v_0^T(f - v_0) - 2u_0^T hu_0.\end{aligned}$$

Since,

$$\begin{aligned}-\|u_0\|^2 - 2u_0^T B^+Cf &= -\|u_0 + B^+Cf\|^2 + \|B^+Cf\|^2 \\ &- 2\rho^2 v_0^T f \le \rho^2(\|v_0\|^2 + \|f\|^2) - 2u_0^T hu_0 \le 0\end{aligned}$$

we then have,

$$\begin{aligned}\dot{V}(x) &= 2\rho^2\|v_0\|^2 - \beta^{‘2}\|x\|^2 - (\beta^2\|x\|^2 - \beta^{‘2}\|x\|^2) \\ &\le -(\beta^2 - \beta^{‘2})\|x\|^2 < 0.\end{aligned}$$

Thus, the conditions of the Lyapunov local stability theorem are satisfied. Consequently, there exists a neighborhood $\eta = \{x : \|x\| \le c\}$ for some $c > 0$ such that if $x(t)$ enters $\eta$, then

$$\lim_{t\to\infty} x(t) = 0$$

But $x(t)$ cannot remain forever outside $\eta$. Otherwise

$$\|x(t)\| \ge c$$

for all $t \ge 0$. Then

$$\begin{aligned}V(x(t)) - V(x(0)) &= \int_0^t \dot{V}(x(\tau))d\tau \\ &\le -(\beta^2 - \beta^{‘2})c^2 t\end{aligned}$$

Letting $t \to \infty$, we have,

$$V(x(t) \le V(x(0)) - (\beta^2 - \beta^{‘2})c^2 t \longrightarrow -\infty$$

which contradicts the fact that the $V(x(t)) \geq 0$ for all $x(t)$. Therefore

$$\lim_{t\to\infty} x(t) = 0$$

no matter where the trajectory begins. □

Hence, because of this theorem, we can translate the robust control problem into the optimal control problem.

# 4 Optimal Control Approach for Robot Manipulator

Let us now apply the above optimal control approach to robot robust control of robot manipulators. As shown from the section 3.1 the manipulator dynamics can be formulated as :

$$\dot{x} = Ax + B(u + h(x)u) + Bf(x)$$

where,

$$h(x) = M(x_1)^{-1}M_0(x) - I$$

and

$$f(x) = M(x_1)^{-1}(N_o(x_1, x_2))$$

the uncertainties $f(x)$ and $h(x)$ have the following bounds:

$$h(x) = M(x_1)^{-1}M_0(x_1) - I \geq 0$$

and

$$\begin{aligned} \|f(x)\| &= \|M(x_1)^{-1}(N_o(x_1, x_2))\| \\ &\leq \|M(x_1)^{-1}\| * \|(N_o(x_1, x_2))\| \\ &\leq \|M_{min}(x_1)\|^{-}1 n_{max}(x1, x2). \end{aligned}$$

If we define $g_{max}$ to be a bound for $\|f(x)\|$, for example, we can take $g_{max} = \|M_{min}(x_1)\|^{-1}n_{max}(x1, x2)$, then we have reformulated the robust control problem into the form studied in Section 3.2.3. Important thing to note that matching condition holds for robot manipulators, because $C(x) = B(x)$. Hence we can select $\rho = 0$ and $\beta = 1$ so that the sufficient condition, define in Theorem (3),

condition (49) is always satisfied. For above $f(x)$, we can find the largest physically feasible region of $x$ and determine a quadratic bound for $\|f(x)\|^2$ .
Assume such quadratic bound is given by

$$f(x)^T f(x) \leq x^T P x$$

for some positive definite matrix P. And its defined as matched uncertainty in section (3.2.1), we only need to solve the following LQR problem: which has cost function defined in equation (19) and for linear system the cost function is:

$$\int_0^\infty (x^T P x + x^T x + u^T u) dt$$

Note: as stated it as LQR problem, $Q = P + I$ and $R = I$ Identity matrix, and so the solution can be obtained by solving the following algebraic Riccati equation:

$$A^T S + SA + P + I - SBB^T S = 0$$

and the optimal control is given by

$$u_0 = -B^T S x$$

To solve the Riccati equation, we can take the advantage of the special structure of $A$ and $B$.

$$x = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix};$$
$$A = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix};$$
$$B = \begin{bmatrix} 0 \\ I \end{bmatrix};$$

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix};$$
$$S = \begin{bmatrix} S_1 & S_2 \\ S_2 & S_3 \end{bmatrix};$$

Then the Riccati equation reduces to

$$P_1 + I - S_2^2 = 0$$
$$S_1 + P_2 - S_2 S_3 = 0$$
$$2S_2 + P_3 + I - S_3^2 = 0$$

and the optimal control is obtained as

$$u_0 = -(P_1 + I)^{1/2} x_1^T - (2(P_1 + I)^{1/2} + P_3 + I)^{1/2} x_2^T$$

This is the robust control law for the robot manipulator.

# 5 Implementation

To implement the optimal approach using the two degree of freedom SCARA robot shown in Fig1. Using the joint variables $q1$ and $q2$ we create the inertia matrix $M(q)$

$$M(q) = \begin{bmatrix} M_{11} & M12 \\ M_{21} & M22 \end{bmatrix} \tag{50}$$

Where each of the elements can be calculated as follows:

$$M_{11} = J_1 + m_1 r_1^2 + J_2 + m_2(l_1^2 + r_2^2 + 2l_1 l_2 cosq_2) + J_L + m_L(l_1^2 + l_2^2 + 2l_1 l_1 cosq_2 \tag{51}$$

$$M_{12} = M_{21} = J_2 + m_2(r_2^2 + l_1 r_2 cosq_2) + J_L + m_L l_2^2 \tag{52}$$

$$M_{22} = J_2 + m_2 r_2^2 + J_L + m_L l_2^2 \tag{53}$$

Centripetal Vector can be defined as:

$$V(q, \dot{q}) = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \tag{54}$$

Equations for calculating $V_1$ and $V_2$ can be seen below:

$$V_1 = (m_2 l_1 + m_L l_1 l_2)(\dot{q}_2^2 + 2\dot{q}_1 \dot{q}_2) sinq_2 \tag{55}$$

$$V_2 = (m_2 l_1 + m_L l_1 l_2)\dot{q}_1^2 sinq_2 \tag{56}$$

The friction vector is given by :

$$F(q, \dot{q}) = \begin{bmatrix} b_1 \dot{q}_1 \\ b_2 \dot{q}_2 \end{bmatrix} \tag{57}$$

Without loss of generality,it is assumed that the gravity vector $G(q) = 0$ otherwise, it is easy to calculate the additional torque to balance the gravity.

Using the parameters as seen in the Fig (3) and the load is unknown, so we

$$
\begin{aligned}
m_1 &= 13.86 \text{ oz} \\
m_2 &= 3.33 \text{ oz} \\
J_1 &= 62.39 \text{ oz-in/rad/s}^2 \\
J_2 &= 16.70 \text{ oz-in/rad/s}^2 \\
r_1 &= 6.12 \text{ in} \\
r_2 &= 3.22 \text{ in} \\
l_1 &= 8 \text{ in} \\
l_2 &= 6 \text{ in} \\
b_1 &= 0.2 \text{ oz-in/rad/s} \\
b_2 &= 0.5 \text{ oz-in/rad/s.}
\end{aligned}
$$

Figure 3: Parameters [1]

assume:

$$m_L = 10\epsilon \text{ oz}$$
$$J_L = 60\epsilon^2 \text{ oz} - \text{in/rad/sec}^2$$

where $\epsilon \in [0, 0.5, 1]$ With these values, we can calculate $M(q)$ and $N(q, \dot{q}$ as follows:

$$N(q,\dot{q}) = \begin{bmatrix} (86+480\epsilon)\left(\dot{q}_2^2 + 2\dot{q}_1\dot{q}_2\right)\sin q_2 + 0.2\dot{q}_1 \\ (86+480\epsilon)\dot{q}_1^2 \sin q_2 + 0.5\dot{q}_2. \end{bmatrix}$$

$$M(q) = \begin{bmatrix} 846 + 60\epsilon^2 + 1000\epsilon + 172\cos q_2 + 960\epsilon\cos q_2 & 51 + 60\epsilon^2 + 360\epsilon + 86\cos q_2 \\ 51 + 60\epsilon^2 + 360\epsilon + 86\cos q_2 & 51 + 60\epsilon^2 + 360\epsilon \end{bmatrix}$$

Therefore, we can select the following $M_0(q)$ and $N_0(q, \dot{q})$

$$M_o(q) = \begin{bmatrix} 1906 + 1132\cos q_2 & 471 + 86\cos q_2 \\ 471 + 86\cos q_2 & 471 \end{bmatrix}$$
$$N_o(q,\dot{q}) = \begin{bmatrix} 566\left(\dot{q}_2^2 + 2\dot{q}_1\dot{q}_2\right)\sin q_2 + 0.2\dot{q}_1 \\ 566\dot{q}_1^2\sin q_2 + 0.5\dot{q}_2 \end{bmatrix}.$$

The state equation is:

$$\dot{x} = Ax + B(u + h(x)u) + Bf(x)$$

where,

$$\begin{bmatrix} x_1 x_2 x_3 x_4 \end{bmatrix}^T = \begin{bmatrix} q_1 & q_2 & \dot{q}_1 & \dot{q}_2 \end{bmatrix}^T$$

$$
\begin{aligned}
h(x) = M(q)^{-1} M_o(q) - I \geq 0 \\
f(x)^T f(x) \leq x^T P x
\end{aligned}
$$

where, $f(x)$ is:

$$
\begin{aligned}
f(x) =& M(q)^{-1} \left( N_o(q, \dot{q}) - N(q, \dot{q}) \right) \\
=& \begin{bmatrix} 846 + 60\epsilon^2 + 1000\epsilon + 172 \cos q_2 + 960\epsilon \cos q_2 & 51 + 60\epsilon^2 + 360\epsilon + 86 \cos q_2 \\ 51 + 60\epsilon^2 + 360\epsilon + 86 \cos q_2 & 51 + 60\epsilon^2 + 360\epsilon \end{bmatrix}^{-1} \\
&\cdot \begin{bmatrix} 480(1-\epsilon) \left( x_4^2 + 2x_3 x_4 \right) \sin x_2 + 0.2 x_1 \\ 480(1-\epsilon) x_3^2 \sin x_2 + 0.5 x_2 \end{bmatrix}
\end{aligned}
$$

Since $x_1$ and $x_2$ take values only in the interval $[-\pi, \pi]$ and $x_3$ and $x_4$ are bounded by the limit of the speed of the motors of the manipulator, we can find a quadratic bound for $\|f(x)\|^2$

$$
P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 218 & 24 \\ 0 & 0 & 24 & 7 \end{bmatrix}
$$

And the Optimal Control after solving the Ricatti Equation:

$$
u_0 = -\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 14.8038 & 1.3591 \\ 1.3591 & 2.8553 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}
$$

The Simulation is performed for $q_1 = 60\,\text{deg} = \pi/3$ and $q_2 = -30\,\text{deg} = \pi/6$, and for different $\epsilon$ i.e. for $\epsilon = 0$, $\epsilon = 0.5$ and $\epsilon = 1$. And, we were able to reach near to $0$ within $50$ sec. Figure (4) shows the result for $\epsilon = 0.0$. Figure (5) shows the result for $\epsilon = 0.5$. Figure (6) shows the result for $\epsilon = 1$. The code can be see on this link. The relative weights of states and control inputs in the cost function determine the response and the magnitudes of the control input.

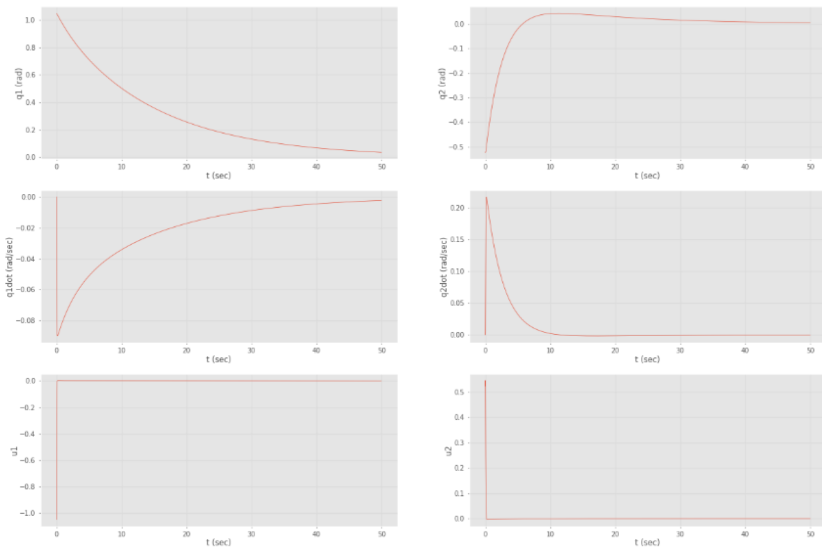


Figure 4: Simulation result for $\epsilon = 0.0$

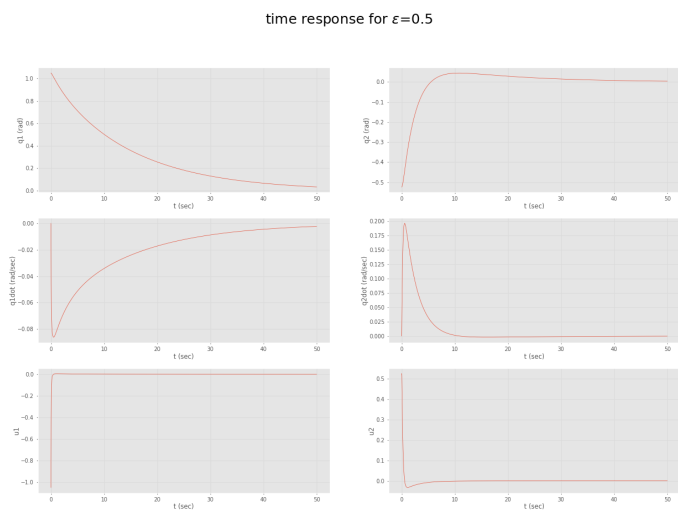


Figure 5: Simulation result for $\epsilon = 0.5$

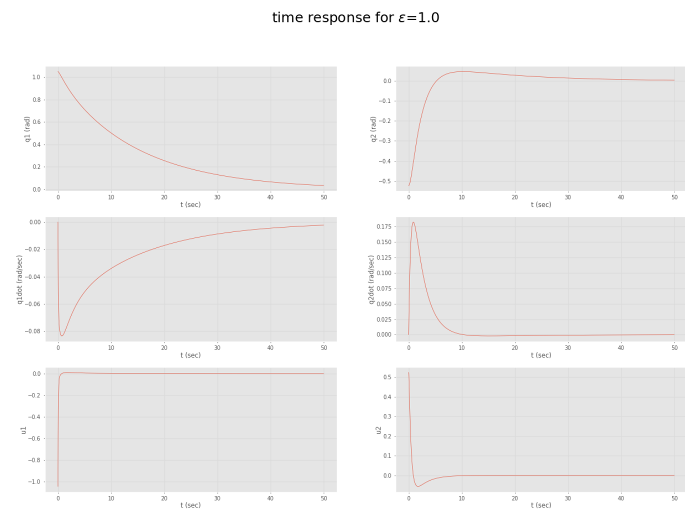


Figure 6: Simulation result for $\epsilon = 1$

# 6 Conclusion

This optimal control approach for a robust control problem has proven to be effective and efficient, additionally it's simplicity gives advantage over other complex algorithms. Also, here we used LQR, but the problem can be solved with different other optimal controller as well. Hence, this approach can be used for wide range of applications, right from robotics to aerospace engineering and many more. In future, we would like to review more such control strategies for applications in robotics and we can try to build on this paper's approach and try to improve performance i.e getting faster response times of the method described in the paper.

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