AMC 10/12 Key Strategies and Formulas

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https://www.tmasacademy.com/

2023 Update

The name of this program has changed from Explore Math to **TMAS Academy**.
Make sure to **join the discord server** to connect with other students and the owner:
https://discord.gg/xFRN3TWd

The website contains our future plans and a new book called *ACE The AMC 10 and AMC 12* that is over 200 pages long with 250+ solved problems.

Who am I?

My name is Ritvik Rustagi, and my goal is to spread mathematics to others. I produced this 50+ page handout which will be updated constantly with new theorems and deeper explanations!

My goal

I want to teach for free because some kids aren't able to learn just because of their net worth. I have always believed that the amount in your purse shouldn't be the reason for why you aren't able to learn.

Check out Part 1 of our Seminar, and we covered Geometry and Number Theory:
https://youtu.be/T1h_V6XAkqE
Check out Part 2 of our Seminar, and we covered Algebra along with Probability and Combinatorics: https://youtu.be/AuJC-jNMrDA

(Please leave a comment there regarding how the handout was and if you want more in the future. If you have any questions, then feel free to email me at weexploremath@gmail.com.

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AMC 10/12 Information That You Should Know

- 25 questions and 75 minutes
  - 3 minutes per question on average
- 6 points for all correct answers, 1.5 points for unanswered questions, and 0 points for wrong answers

- Fortunately, anyone can take the AMC 10/12 as long as they're in tenth/twelfth grade or below depending on the contest
- There is an AMC 10/12 A and B and the information above applies to both A and B
- Competition Dates
  - AMC 10/12 A: February 4
  - AMC 10/12 B: February 10
- First exam to decide IMO representatives for America
- Bring a ruler and graph paper if you're allowed to
- People who score well on the AMC 10/12 take the AIME
  - Those who do well on both and get a certain score can get into USA(J)MO and continue their journey into MOP and even IMO

Handout Information

- This handout has been written by Explore Math which consists of multiple people including a USAMO qualifier. We went through thousands of past problems to find all these theorems that are helpful on the AMC 10 and AMC 12
- All the text is original except for some images. Some of the formulas were screenshotted from The Art of Problem Solving website because it's hard to type some formulas up.

Mini Strategy

- In a lot of the past years, there have been questions related to the year you're taking the contest in. For those questions, it is **very** helpful to know the prime factorization for that year. For example, if you're taking the test in 2021, then remember that the prime factorization is 43 x 47. Deriving it in the middle of the contest can be tedious if you see it in a problem.

Strategies for a higher score

- DON'T Panic at ALL

- Leave a problem blank if you can't understand it
- If you can eliminate some answer choices, then take an educated guess
- Try to remove answer choices by finding the range of the answers. Then, you can remove the choices that are too high or low.
- In Geometry Problems
    - Draw a diagram for all the problems
    - Bring a ruler and graph paper (if it's allowed)
    - Estimate the dimensions
    - If something looks the same to you, then it probably is the same thing in geometry

Goals

Many people who are aiming to qualify AIME actually aim to make the cutoff of about 110 which has remained the average cutoff for the past few years. However, that is a big mistake. Whenever you want to do something, you should always aim a level higher. The reason is that on test day anything can happen. You might miss a problem. For example, if you aimed for DHR, but you got an extra problem wrong. You'll still be in the range of an AIME qualification since you prepared for that. However, if you prepared for a score of 100-110, then getting just one extra wrong can cost you your qualification.

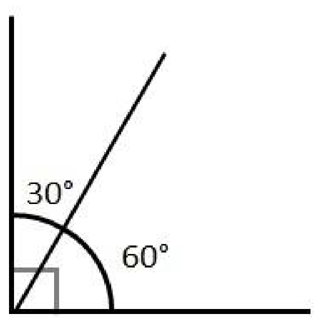
Expected Paths for AIME and USA(J)MO

For a guaranteed AIME qualification, you will need 120 out of 150. However, 110 might work, but you should always aim higher. You can get around that range by answering 18 correctly and leaving the rest blank. You can also get a score of 105 by answering exactly 15 correct, but leaving the rest blank.
For USA(J)MO, you need an average index of 240 to increase your chances to literally 99%. 240 isn't an easy target. You can either aim for 130 on the AMC 10/12 and 110 on the AIME. You could also aim to get 120 on the AMC 10/12 and 120 on the AIME. However, the first one seems a lot more reasonable because the final 5 problems on the AIME are extremely hard.

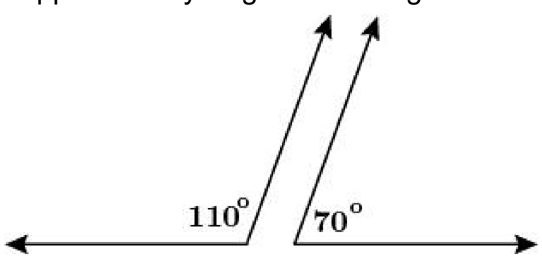
Geometry

Angles

Acute Angle: Angle measures less than 90 degrees
Right Angle: Angle measure that is 90 degrees
Obtuse Angle: Angle measure that is greater than 90 degrees
Straight Angle: Angle that is 180 degrees
Complementary Angles: Two angles that up to 90 degrees

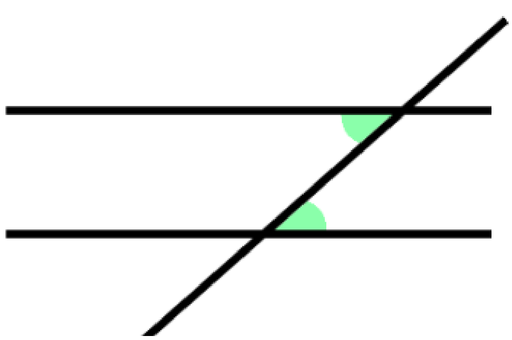


Supplementary Angles: Two angles that add up to 180 degrees

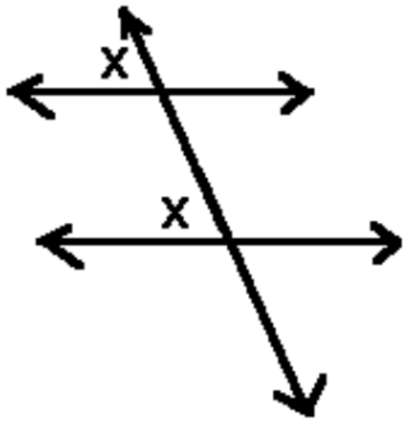


Transversals and Parallel Lines with Angles

If there are two **parallel** lines, and they are cut by a transversal (just a line interesting both of the parallel lines), then the alternate interior angles are congruent.

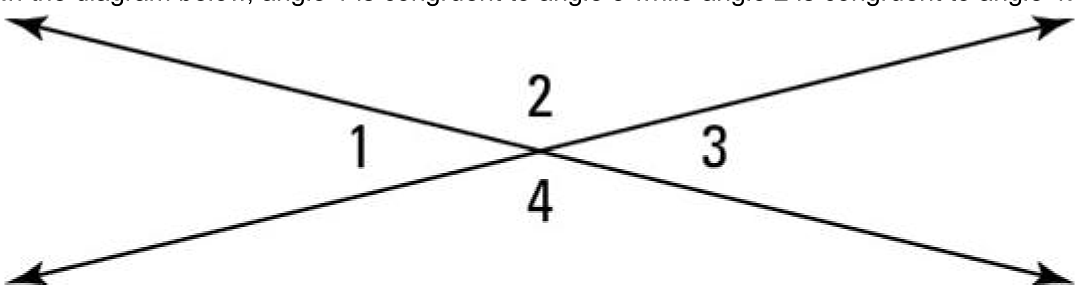


If two parallel lines are cut by a transversal, then corresponding angles are congruent.



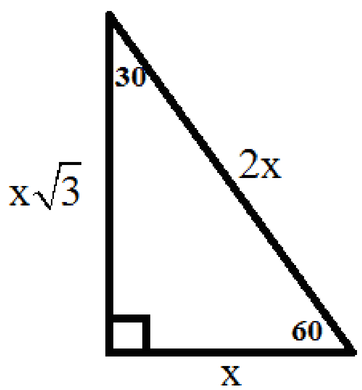
(In this diagram above, both of the angles that have X in them are congruent and have the same angle measure)

If two lines intersect, then vertical angles will be congruent.
In the diagram below, angle 1 is congruent to angle 3 while angle 2 is congruent to angle 4.

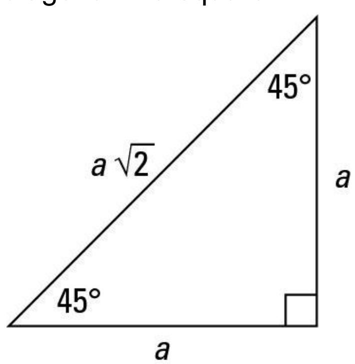


Special Right Triangles

30-60-90 Right Triangles
The diagram that you see below will apply to **All** triangles that have angles that are 30, 60, and 90. The ratio of the sides will always stay the same. The reason is that a 30 60 90 triangle is derived by dropping a perpendicular line from a vertex of an equilateral triangle to the opposite side.



45-45-90 Right Triangles
The diagram that you see below will represent the ratios of the side of a triangle that has angles 45, 45, and 90. That ratio will **ALWAYS** be the same. This triangle is derived from making a diagonal in a square.



Angles in ANY Polygon

The sum of the interior angles of any polygon is (n-2) x 180. The n represents the number of sides in the shape.

The angle measure of any angle in a **Regular** polygon:

$$\frac{(n-2) \times 180}{n}$$

Exterior Angles

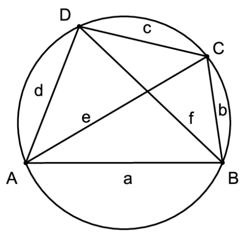
The sum of the exterior angles of ANY shape is 360 degrees. The formula to find the exterior angle of a **REGULAR** polygon is 360/n. (N represents the number of shapes.)

Cyclic Quadrilaterals

Opposite angles of a cyclic quadrilateral always sum to 180 degrees.

Ptolemy's Theorem

Ptolemy's Theorem states that if you multiply the diagonals, then that will be equivalent to the sum of the multiplication of the opposite sides. If you don't understand what I mean, then the picture and example below should help you understand it.



E x F = (C x A) + (B x D)
This statement will be true no matter what. The diagonals multiplied will equal to the opposite sides multiplied and added together.

Brahmagupta's Formula

Brahmagupta's formula is very important when t comes to finding the area of a cyclic quadrilateral which can be challenging sometimes. For any quadrilateral that has side lengths of a, b, c, and d, then the area of it is:

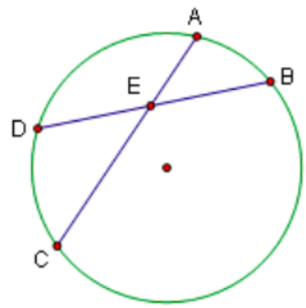
$$\sqrt{(s-a)(s-b)(s-c)(s-d)}$$

Don't forget that s stands for the semiperimeter which is $\frac{a+b+c+d}{2}$.

Power of a Point

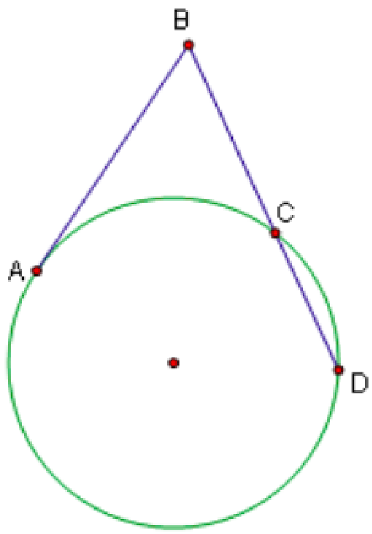
Words you need to know the definitions of to understand Power of a Point
Secant: A secant is a line that intersects a circle at two points.
Chord: This is a segment that connects any two points on the circumference of the circle.
Tangent: A line that **TOUCHES** the circle at exactly **ONE** point.

Two Intersecting Chords



Power of a point tells us that AE x CE = BE x DE.

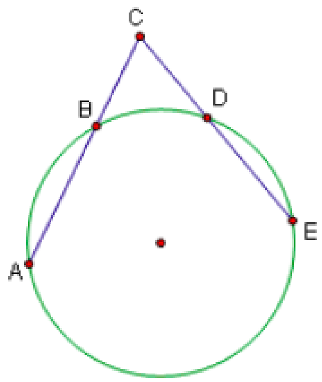
Tangent and Secants



As you can see, Segment AB is a tangent. Also, BD is a secant since it intersects the circle at two points. Now, power of a point tells us that $AB^2 = BC \times BD$.

NOTE: In the case of a tangent and a secant as you see above, remember that we got the length BC times BD and not CD times BD because one of the segments that you multiply is the part of the secant outside the circle multiplied by the entire secant. Also, the tangent and the secant need to intersect for this theorem to work.

Two Secants Intersecting

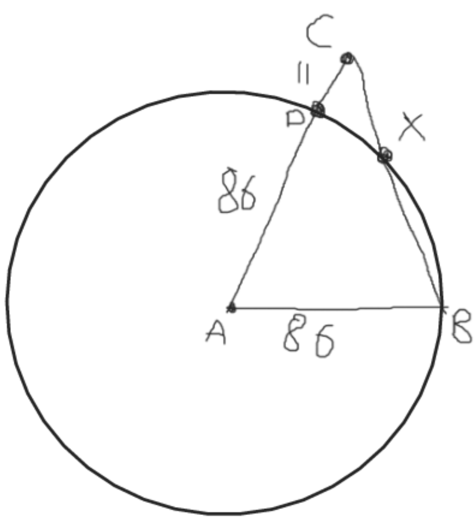


When two secants intersect outside the circle, power of a point tell us that CB x CA = CD x CE

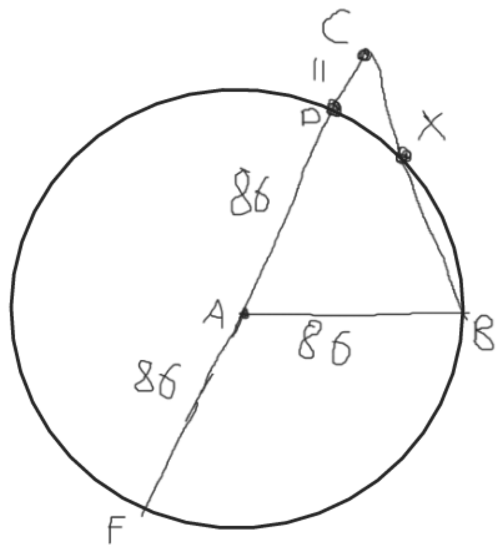
Power of a Point Problem from 2013 AMC 10A Problem 23

In $\triangle ABC$, $AB = 86$, and $AC = 97$. A circle with center A and radius AB intersects $\overline{BC}$ at points B and X. Moreover $\overline{BX}$ and $\overline{CX}$ have integer lengths. What is BC?

The first step for all Geometry problems that you do should be to read the problem and make a diagram. Diagrams can help a lot for geometry problems.



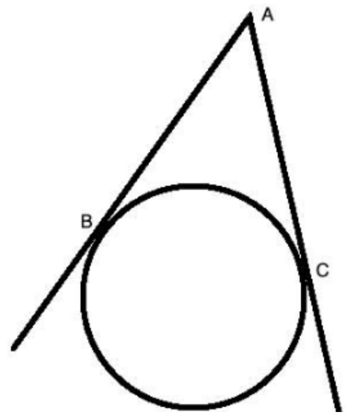
After drawing a diagram, we realize that point C is outside the circle because the segment is longer than the radius. We know this because the longest segment that you can make from the center of the circle to the circumference is the radius which is 86 in this problem. That is why we added another point which was point D. CD is 11 because AC - AD = CD. Also, after drawing the diagram, we notice that CB is just a secant which means we could use the two secants theorem if we can make another secant.



We extend DA to make a new segment which is DF which is also the diameter. We know that AF is the radius because A is the center. Also, we have made our second secant. Segment CF is a secant and so is CB, and now we can apply the two secant theorem. The two secant theorem states that CD x CF = CX x CB. We know that CD is 11, and CF is 183 (183 is 3 x 61). This means that CX x CB is 11 x 3 x 61. We also know that segment CX will be less than CB which is pretty obvious. This means that CX can be either 3, 11, or 33. The other factors will make CX > CB which is clearly not the case. Now, we just need to try in values. This job can be made easier by looking at triangle ACX. Using the triangle inequality, we know that AX + CX > AC. We know that AX is 86. Also, AC is 97. The inequality now states 86 + CX > 97. You subtract 86 from both sides to get CX > 11. The values we found of CX were 3, 11, and 33. We know it has to be greater than 11, and the only value that fits that is 33. Now you can find BC by simply dividing 33 from 11 x 3 x 61. This gives us 61 which is our final answer.

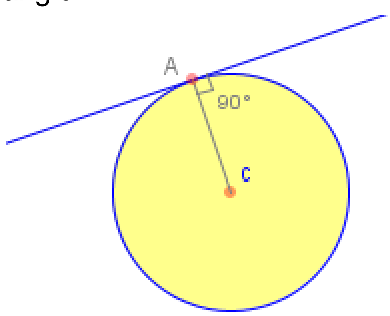
Two Tangent Theorem

If two tangents intersect at one point, then both of the tangents are congruent. (The diagram below shows us that AB and AC are two tangents that intersect at one point. This means that AB and AC are congruent. Keep in mind that the congruent segments come from the intersection of the two tangents to the tangent point.



Tangents make Right Angles

Keep in mind that if you connect the radius of a circle to the tangent point, then you make a right angle.



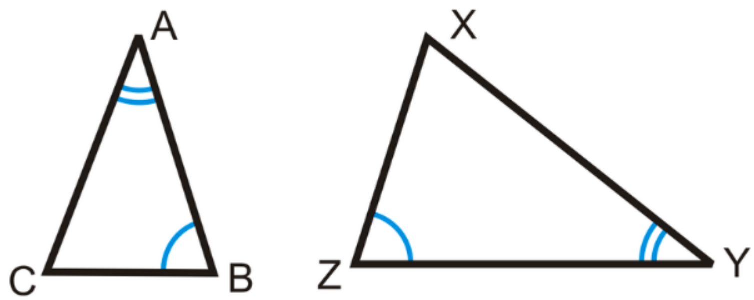
Circle Problem Tricks

On the AMC 10, it's very common to see a shape full of circles. In those problems, a common way to tackle those problems is to connect the radii to make triangles. For example, there will be some complex shapes that you might have to find the area of. Connecting the radiuses can make new shapes which are easier to work with.

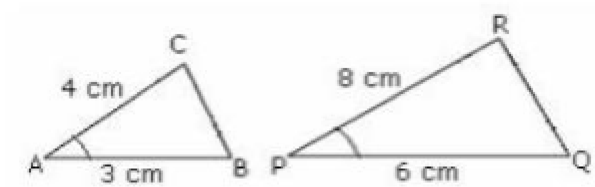
Similar Triangles

Similar triangles is a very common concept seen on the AMC 10/12. You can identify similar triangles by 3 theorems. (Similar sides are sides that are in the same ratio throughout all the sides. Also, this topic will use a lot of the angle topics from the beginning of the handout)

AA Similarity: If two angles in two triangles are similar, then the third angle is also congruent.



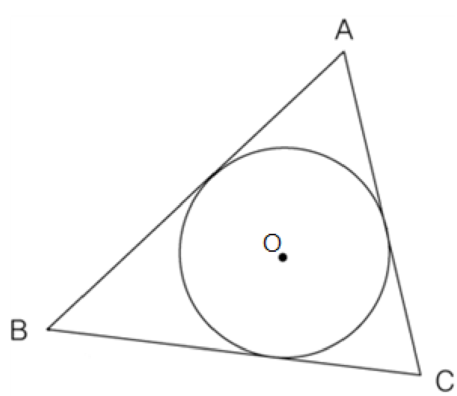
SAS Similarity: If two triangles have two similar sides and the included angle between those two sides are congruent, then the triangles are similar.



SSS Similarity: If two triangles have sides that are all in the same ratio to each other, then both of the triangles are similar.

Circles in Triangles Area Formula

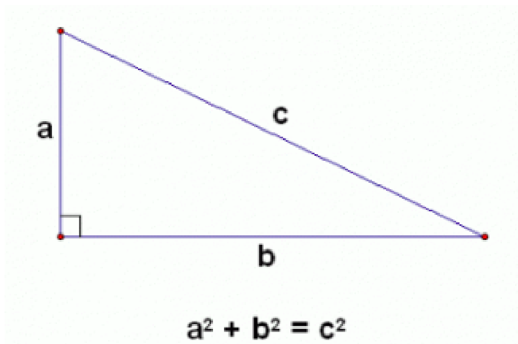
If you have a triangle and a circle inscribed in it, then there is a very important trick. For those who don't know what inscribed means, then it means that all the side lengths of the triangle are tangent to the circle that is inside the triangle. An example of what I mean is shown below.



The theorem is that the area of the triangle is also the same thing as the radius times the semiperimeter. The semiperimeter of a triangle is all the side lengths added and divided by 2. For example, the semiperimeter of a 3, 4, 5 right triangle would be 6.

Pythagorean Theorem

This theorem is very useful for right triangles. It states that $a^2 + b^2 = c^2$. The segment a and b represent the legs of the triangle while c represents the hypotenuse.



Heron’s Formula

The heron’s formula is a way to find the area of any triangle in a very easy way as long as you have all the side lengths.

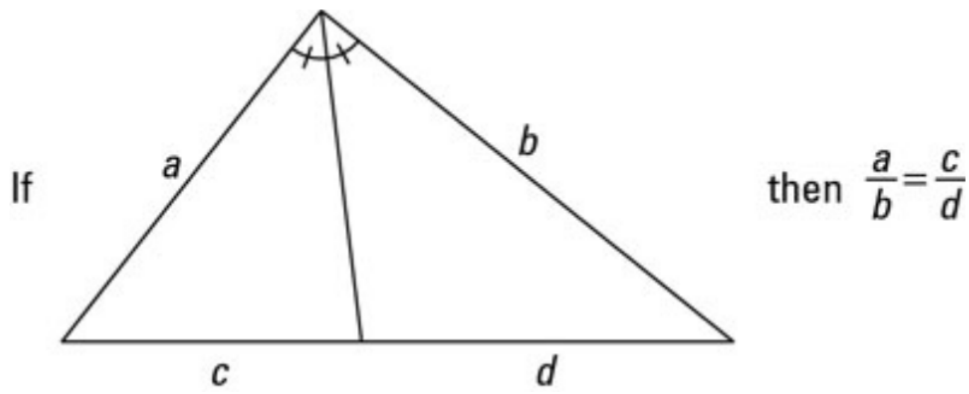
$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

The equation above represents Heron's formula. S represents the semiperimeter of the triangle. The semiperimeter is all the side lengths added divided by 2. Furthermore, a, b, and c represent the side lengths of the triangle.

Angle Bisectors and Theorem

Angle bisectors are basically a line that intersects the angle and divides it in half. For example, if a 60 degree angle is bisected, then you make 2 angles that are each 30 degrees.

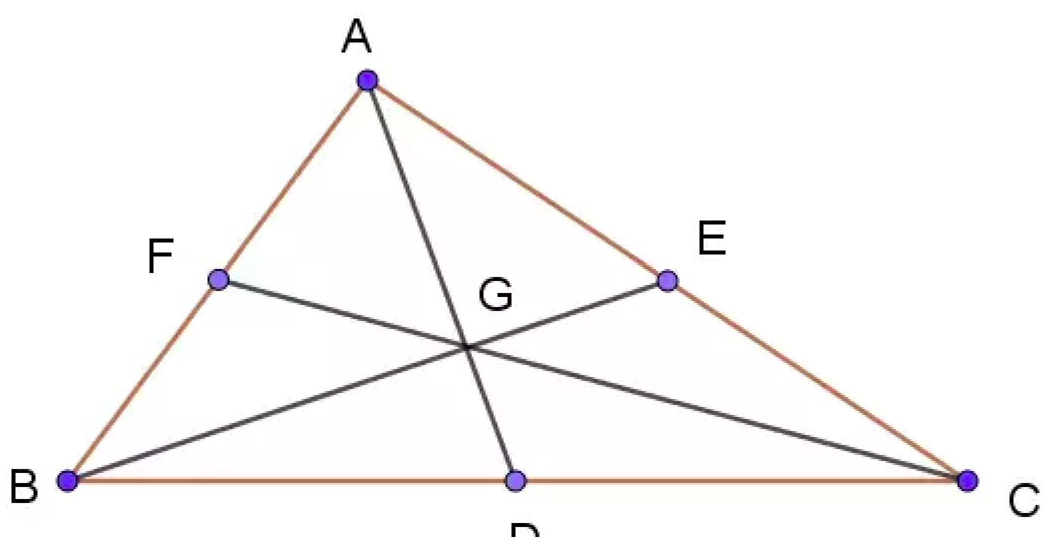
Bisector Theorem: The angle bisector theorem is all about ratios. The picture below includes the ratios when an angle is bisected, and those ratios will always be the same.



Three Medians of a Triangle

For those who don't know what a median in a triangle is, then it's just a line that is made from the vertex and divides the opposite segment in half.

In triangles, if you make 3 medians, then they will always intersect at one point. That one point is called a centroid.

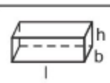
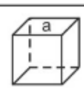
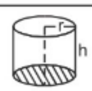
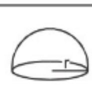


These medians also divide the triangle into six triangles of equal areas. Furthermore, the ratios of the medians become in a ratio of 2:1. For example, CG: FG, BG:GE, and AG:GD are all in the ratio of 2:1. One easy way to notice which one is in the ratio 2 and which is in the ratio of 1 is by looking at the sizes which makes it obvious.

3D Dimensional Shape Techniques

Lots of 3D shapes can be solved by looking at the cross section of the shape. This can make it significantly easy because it's a lot easier to work with 2D shapes compared to 4D.

Before reading on, look at the chart below. (The curved surface area represents the lateral surface area.)

Name	Figure	Curved Surface area	Total surface area	Volume
Cuboid		$2h(l+b)$	$2(lb+bh+lh)$	lbh
Cube		$4a^2$	$6a^2$	a^3
Right circular cylinder		$2\pi rh$	$2\pi r(r+h)$	πr^2h
Right circular cone		πrl	$\pi r(l+r)$	$\frac{1}{3}\pi r^2h$
Sphere		–	$4\pi r^2$	$\left(\frac{4}{3}\right)\pi r^3$
Hemi-sphere		$2\pi r^2$	$3\pi r^2$	$\left(\frac{2}{3}\right)\pi r^3$

Cubes

If the side length of a cube is s, then the diagonal of a cube is $s\sqrt{3}$. Also, it doesn't touch any of the faces other than the vertices. This can be one way to identify them if you don't know what diagonal means. This diagonal is also called solid diagonals and space diagonals.

There are also diagonals connecting two vertices on the same face, and the length is $s\sqrt{2}$ if the side is s.

Tetrahedron

These problems can be solved by picturing the shapes in your head. The volume of a tetrahedron is the same as the volume of a pyramid. It is $\frac{1}{3}\, X\; Area\; of\; Base\; X\; Height$.

Shoelace Theorem

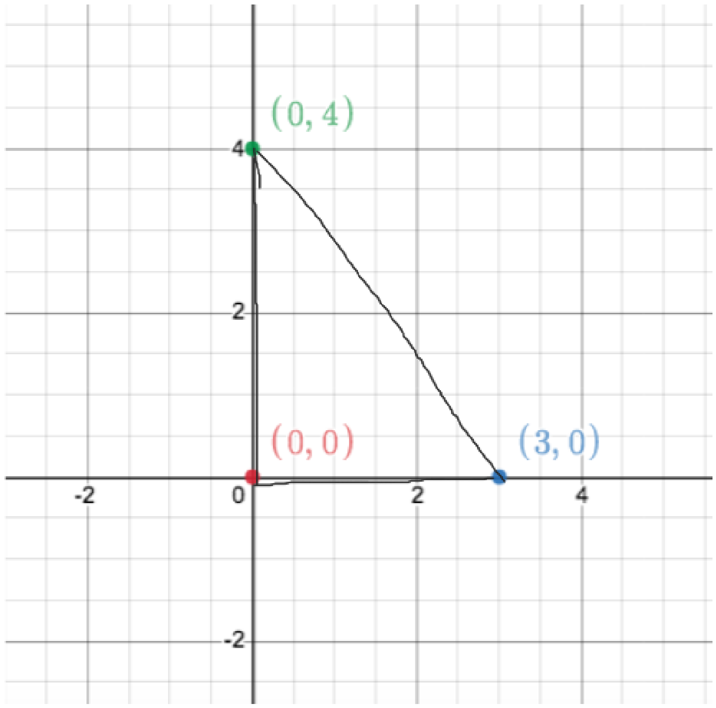
This can be very helpful when finding the area of a shape. You can sometimes plot it on a coordinate plane, and using the theorem finding the area can be very easy as long as you have the coordinates of all the vertices.

$$\text{Area} = \frac{1}{2}\left|\sum_{i=1}^{n-1} x_i y_{i+1} + x_n y_1 - \sum_{i=1}^{n-1} x_{i+1} y_i - x_1 y_n\right|$$

$$\text{Area} = \frac{1}{2}\left|x_1 y_2 + x_2 y_3 + \cdots + x_{n-1} y_n + x_n y_1 - x_2 y_1 - x_3 y_2 - \cdots - x_n y_{n-1} - x_1 y_n\right|$$

The Shoelace Formula

Understanding the theorem above can be hard, but I'll simplify it for you. Let's pretend we have a right triangle with side lengths 3, 4, and 5.



Now all you do is start at any one point and write the x coordinate and the y coordinate next to it. Go in one direction and do the same for all the vertices. After you do all the vertices of the shape, write the point you started with. In this case, we will go in the order of 0,0 and 3,0 and 0, 4 and 0,0. You could also start at the point of (3,0). However, once you start at a point and go in one direction, then you have to maintain that direction for the entire time.

X	Y
0	0
0	4
3	0
0	0

Now you will multiply and add in a shoelace pattern.

X	Y
0	0
0	4
3	0
0	0

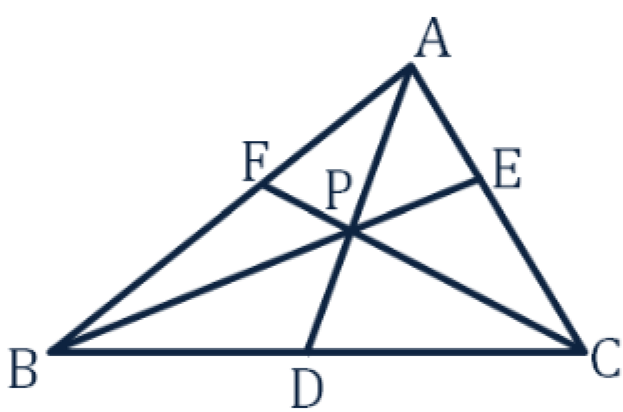
We don't connect the last x coordinate to anything yet. So using these values, the shoelace theorem states that you multiply them. (0 x 4) + (0 x 0) + (3 x 0) = 0. Now we do this again but start with the y coordinate at the top.

X	Y
0	0
0	4
3	0
0	0

Now, the shoelace theorem states that (0 x 0) + (4 x 3) + (0 x 0) = 12. Now, you subtract the second value from the first one to get 0-12 which is negative 12. You take the absolute value of the number which is 12 and divide by 2 to get the area of the triangle. We already know that the area of a 3, 4, 5 right triangle is 6 because of the area of a triangle formula (½ x base x height). We got the same answer using this theorem. Use it in problems that have complicated shapes and you have all the coordinates. If the shape is a simple square or something else, then using this theorem will be a waste of time.

Ceva's Theorem

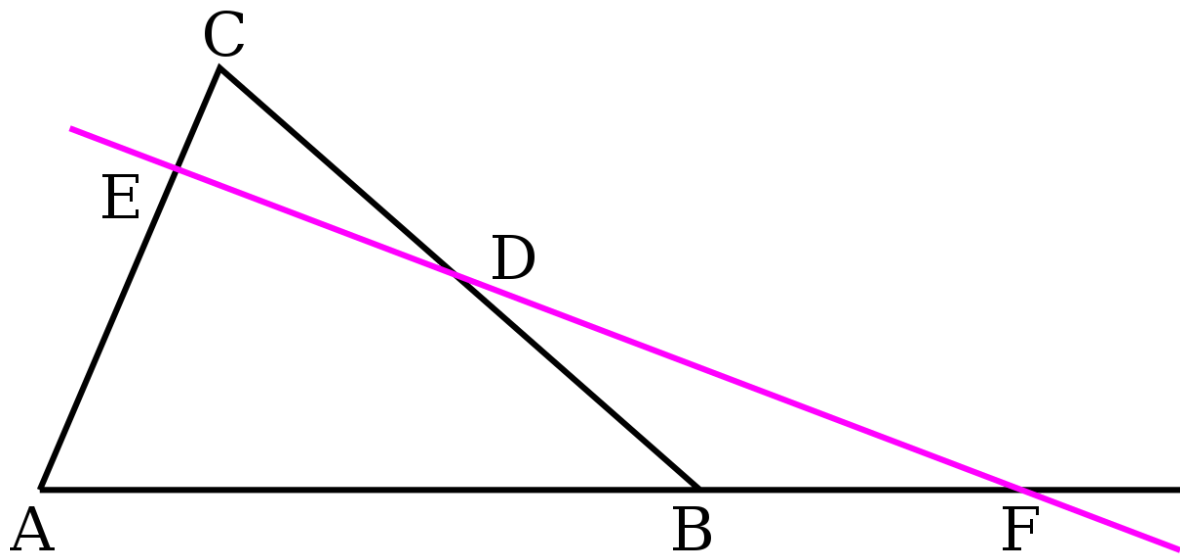
Ceva's theorem can be helpful in some situations.



If you have 3 lines drawn from all 3 vertices of a triangle, and they intersect at one point, then Ceva's Theorem tells us that $\frac{AE}{EC}$X $\frac{CD}{DB}$X $\frac{BF}{FA}$= 1. You just choose one vertex of the triangle and go in one direction the entire time. The first segment will be the numerator and the next one will be the denominator.

Menelaus Theorem

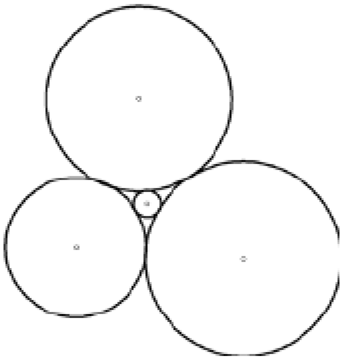
After you read the description of this theorem, you'll realize that it's very similar to Ceva's Theorem.



Menelaus Theorem works if you have a triangle and a transversal that goes through two of the sides and the other side when you extend it. FOr example, you had to extend AB for transversal EF to go through it. Now, the relationship states that $\frac{AE}{CE}$x $\frac{CD}{DB}$x $\frac{BF}{AF}$= -1. The reason that it is negative is all because of the direction. When solving a problem with this theorem, you can simply take the absolute value of both sides which will make it equal to 1 and more satisfying to work with.

Descartes Circle Theorem

This theorem has been used in some years and can be very helpful for some problems, and it's fun to know. This theorem works if we have 4 mutually tangent circles. (Below is an example of four mutually tangent circles.) Remember that if the 4th circle is internally tangent, then the curvature is negative which leads to more advanced topics.



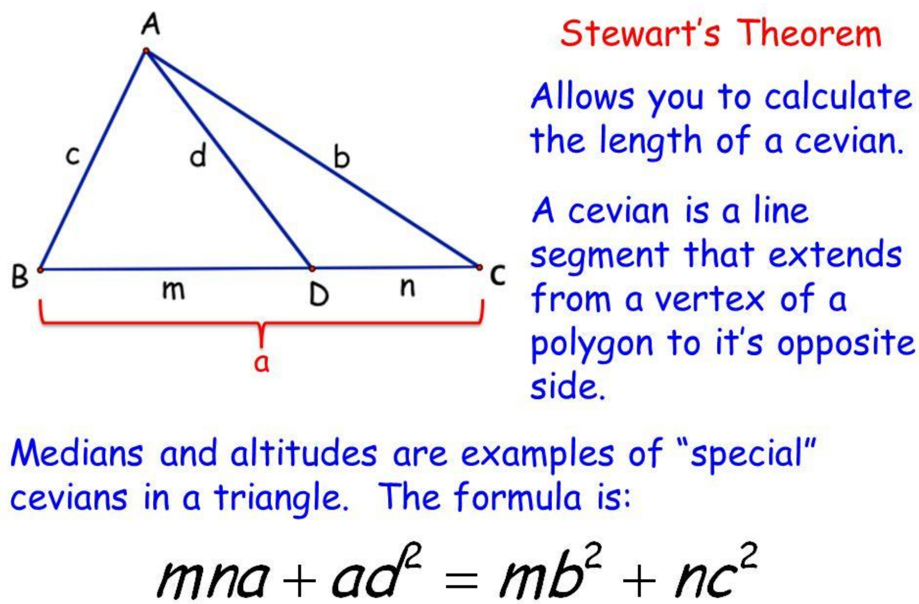
Pretend you have 3 of the radius's, but want to find the other one. This can be easily done with this theorem.

$$(a+b+c+d)^2 = 2(a^2+b^2+c^2+d^2)$$

This is the formula, and a, b, c, and d represent the curvatures of all four circles. Curvatures are just $\frac{1}{radius}$. For example, if a circle has a radius of 2, then it's curvature is ½. Using this theorem, you can easily find the radius of all circles as long as you have 3 of the radii given.

Stewart's Theorem

This is a very overpowered theorem used in triangles.



The diagram above is an example of where we can use this theorem. If you have a triangle and connect one of the vertices to the bottom, then you could use this theorem to find some side lengths.

For those who're taking the AMC 12, try proving this theorem after reading the Trigonometry section below. This proof requires Trigonometry and proving it is a lot of fun.

Pick's Theorem

Pick's theorem is another way of finding the area of a polygon, and is very useful on coordinate planes. Below is the formula. The letter I represents the number of lattice points in the interior of the polygon while B represents the number of lattice points in the boundary.

$$A = I + \frac{1}{2}B - 1$$

Euler's Polyhedral Formula

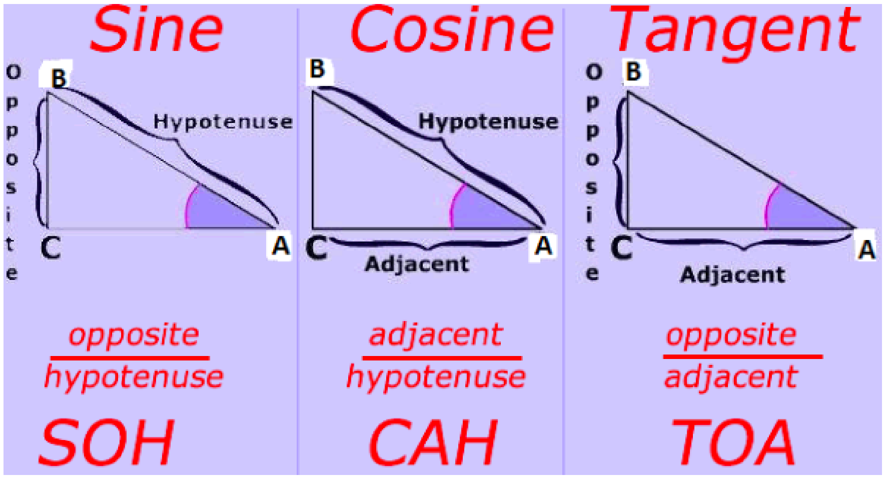
This formula states that V-E+F equals 2 in any convex polyhedron where V defines the number of vertices, E defines the number of edges, and F defines the number of faces.

Trigonometry

Note: If you're taking the AMC 12, then this information will be important for you. However, if you're taking the AMC 10, then you won't need it, but it can be used on problems to make it a bit easier.

Sine, Cosine, and Tan

Sine, Cosines, and Tans are literally just ratios in triangles. The picture below will explain it to you.

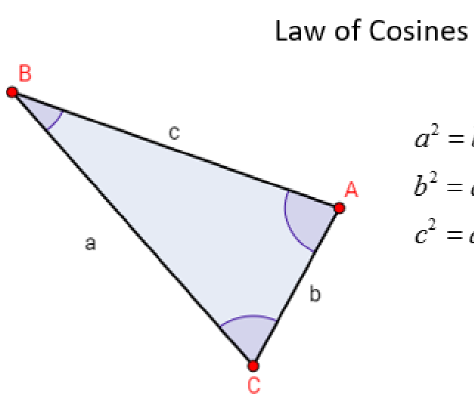


SOH CAH TOA is an easy way to remember what sine, cosine, and tangents are. I think you can tell that these ratios work for **RIGHT** triangles because it includes the hypotenuse. You are completely right if you think that, but sometimes you have to make right triangles. This can be done by drawing a perpendicular line to one of the bases. After that, you could use trigonometry.

I recommend memorizing the sin and cosines of 15, 30, 45, 60, 75, 90, and 120. Also memorize the tan of 15, 30, 45, 60, 75, and 90 degrees. This would be really helpful for people taking the AMC 12. For people taking the AMC 10, this is an optional step. However, some of the geometry problems can be solved using trigonometry. If you know trigonometry and are taking the AMC 10, then it's very helpful to memorize it because it can be your backup plan for a problem that you don't know the solution for.

Law of Cosines

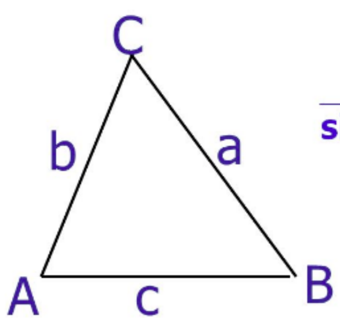
The law of cosines is very similar to the Pythagereon theorem.



$$a^2 = b^2 + c^2 - 2bcCosA$$
$$b^2 = a^2 + c^2 - 2acCosB$$
$$c^2 = a^2 + b^2 - 2abCosC$$

You can use it to find the length of any side as long as you have two sides and the included angle. Fortunately, this works on literally any triangle which makes it powerful.

<u>Law of Sines</u>



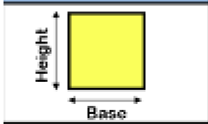
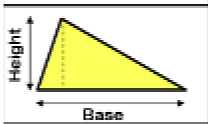
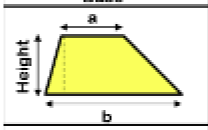
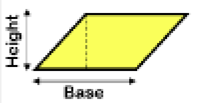
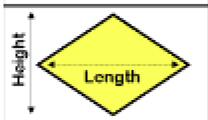
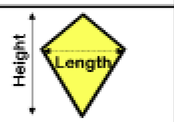
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

or

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

These will work for any triangle. In the triangle above, the lower case letters represent the side lengths while the upper case letters represent the angles. The ratio will always stay the same which makes it helpful. However, the law of cosines is used a lot more.

Area of 2D Shapes

Shape	Name	Formula for Area
	Square	Base x Height
	Rectangle	Base x Height
	Triangle	Base x Perpendicular Height ÷ 2
	Trapezium	$\frac{(a + b) \times \text{height}}{2}$
	Parallelogram	Base x Perpendicular Height
	Rhombus	Length x Height ÷ 2
	Kite	Length x Height ÷ 2

Algebra

To do well on the Algebra part, you need to practice a lot. These theorems and formulas will help you solve the problems.

Vieta's Formula

Vietas is a great way to find the sum of the roots or the multiplication of it. In any polynomial that is in the form of $ax^2 + bx + c$, Vieta says that the sum of the roots is -b/a. The multiplication of the roots in this polynomial is just c/a. (Don't forget that in vietas you don't include the variables. You just use coefficients.)

Vietas for bigger Polynomials

For any polynomial, the sum of the roots will always be -b/a. A very common mistake made is finding the sum of the roots of a polynomial in the form $ax^3 + bx + c$. Some people think that the sum of the roots is just -b/a which in this case is wrong. A better way to remember it is the term with the second highest degree divided by the term with the highest degree. For those who don't know, a degree of the term is its exponent. For example, in $3x^2 + 4x + 7$, the term $3x^2$has a degree of 2 while the term 4x has a degree of 1. In the trick problem, the sum of the roots is two. The reason is that there is an imaginary term. For example, $ax^3 + bx + c$ is the same thing as $ax^3 + 0x^2 + bx + c$. The sum of the roots is actually 0. Don't fall for this trap!

Polynomial Remainder Theorem

If you have any polynomial f(x), and you're dividing it by x-a, then the remainder can be found by replacing the variable x with the value of a throughout the entire polynomial. If that expression is equivalent to 0, then the polynomial is indeed divisible by x-a.

Conjugate Root Theorem

The AMC 10 kids can ignore this short section of conjugate roots if they want.
Theorem: If a polynomial p(x) that only has real number coefficients has a root of a+bi, then a-bi is also a root of that same polynomial. This can be extremely useful if you have a complex number root given in a polynomial.

Simon's Favorite Factoring Trick

This is just a factoring trick named after Simon Rubinstein.
You will see what this trick is with the example below.

: Find all positive integer (x, y) such that $xy + x + y = 20$.

As soon as we see that, we know that we need to factor a bit. You could factor xy and the x term. After you do so, you get x(y+1) + y = 20. Now we notice that we haven't gotten anywhere because the y is standing alone. However, SFFT is all about adding numbers to both sides to be able to factor them. This will be possible if we add 1. In that case, we'll get x(y+1)+1(y+1) = 21. We know that this is the same thing as (x+1)(y+1) = 21. This shows that we were able to factor it using this trick. It's important to number. All it takes is adding some numbers. You can use it when you have two variables that are multiplied to each other and two linear terms with those two variables.

Quadratics

The quadratic formula is usually used when you have an equation in the form $ax^2 + bx + c$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The formula above can help you factoring a quadratic. However, remember that sometimes you might see quartic polynomials that might be in the form $ax^4 + bx^2 + c$. The clever way of factoring that is to substitute x^2 with another variable. Pretend that variable is y. After you do so, you get $ay^2 + bx^2 + c$.

Discriminants

Quadratics have discriminants which tell us a lot about the roots. The discriminant of a quadratic is $b^2 - 4ac$. If that discriminant is 0, then that means there is a repeated root. For example $x^2 - 2x + 1\ has\ a\ repeated\ root\ because\ using\ the\ formula\ you'll\ get\ 0.$ You can factor it out to get $(x - 1)^2$which clearly just has one root. Furthermore, if the discriminant is positive, then there are 2 roots that are real numbers. However, if the discriminant is negative, then you'll have 2 non real roots which also involves imaginary numbers. People taking the AMC 10 won't' have to worry about the discriminant being a negative number.

Functions

Functions is a very commonly seen concept. In a lot of cases, it's just about writing out the first few terms and looking for patterns. It's pretty easy with some practice, but floor functions can be tricky.
Pretend I have a number 6.2.
[6.2] If you put a number the way I did on the left, then it means it's asking for the greatest integer less than that number which is obviously 6.
{6.2} If you put the number the way I did on the left, then it's asking for the **fractional** part of the number which is 0.2.

Remember that if there is any number x, then the [x] + {x} = x.
You can solve complicated floor functions by graphing it. That is a very common technique and can be very helpful.

AM-GM Inequality

This inequality stands for the arithmetic mean and the geometric mean. This AM-GM inequality states that if there are non negative integers, then the

$$\frac{(the\ sum\ of\ all\ the\ terms)}{(number\ of\ terms)} \geq \sqrt[number\ of\ terms]{(all\ the\ terms\ multiplied)}$$

Furthermore, a part of this inequality states that if you have two nonnegative numbers, and they are the same, then both the AM and GM will also be equal.

Arithmetic Sequences

An arithmetic sequence has a property that the difference between any two consecutive terms will always be the same no matter what. It’s also called the common difference.

An example of an arithmetic sequence is 1, 3, 5, 7, 9 in which it’s going up by 2s.

The average of the first and last terms of an arithmetic sequence is equivalent to the average of the sum of all the terms.

The sum of the numbers can be found by averaging the first and last terms and multiplying that by the number of terms in the sequence.

AN important thing to remember that the sum of 1, 2, 3, 4, 5… is $\frac{n(n+1)}{2}$. If you ever have a problem involving arithmetic sequences, you will most likely need some algebra. You can say that the first time is a and the common difference is d. Then that means the sequence will be a, a+d, a+2d, a+3d, a+4d, etc.

Geometric Sequences

A geometric sequence has a property that for any two consecutive terms, the ratios of the numbers will always be the same. An example is 2, 4, 8, 16, 32. As you can see, the numbers are multiplying by 2.

In a geometric sequence, if the first term is a and the common ratio is r, then the nth term of the sequence is ar^{n-1}.

If you have to use algebra in your geometric sequence, then a way to label the terms is a, ar, $ar^2,\ ar^3,\ etc.\ On\ the\ left,\ a\ represents\ the\ first\ term\ while\ r\ represents\ the\ number\ you\ multiply\ by.$

The sum of a geometric series is in the picture below.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

A represents the first term while r represents the ratio. The number n represents the number of terms while S of n represents the sum of the terms.

Infinite Geometric Sequences

Infinite geometric series are similar to geometric sequences, but these never end. THey will go on and on without stopping. You can find the sum of these sequences only if **the absolute value of the ratio is less than 1.** In that case, the sum will be $\frac{a}{1-r}$ where a represents the first term and r is the common ratio.

$$\sum$$

Don’t get afraid of the picture above. It just represents the sum of some numbers. The number at the bottom will be the number you start from while the number at the top will be where you end.

Algebraic Tricks and Manipulations

In problems involving algebra, try to isolate the terms that have variables and factor them out. Also, try to substitute terms into them and eliminate variables. When you have a radical, your goal should be to treat it like a variable and move it to one side of the equation. The reason is that if you square both sides, then the square root sign will be gone which will make computation a lot easier.

Commonly Seen Equations

$(x + y)^2 = x^2 + 2xy + y^2$

$(x - y)^2 = x^2 - 2xy + y^2$

$$(x+y)(x-y)=x^2-y^2$$

$$(x+a)(x+b)=x^2+(a+b)x+ab$$

$$(x+y+z)^2=x^2+y^2+z^2+2xy+2yz+2zx$$

$$(x+y)^3=x^3+y^3+3xy(x+y)=x^3+y^3+3x^2y+3xy^2$$

$$(x-y)^3=x^3-y^3-3xy(x-y)=x^3-y^3-3x^2y+3xy^2$$

$$x^3+y^3=(x+y)(x^2-xy+y^2)$$

$$x^3-y^3=(x-y)(x^2+xy+y^2)$$

Try to memorize all the factorizations above since those are super common on the AMC 10 and 12.

Sophie's Germain Identity

$$a^4 + 4b^4 = (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab)$$

If you ever see a problem in the form above, then it will be hard to factor it on the spot during the competition. However, memorizing this identity can save a lot of time if you ever come through one.

Factorials

N! Means n x (n-1) x (n-2) and so on. You multiply all the numbers including n and all the numbers less than n all the way to 1.

Pretend i have 19! which is also called 19 factorial. If a question asks me to find the highest power of 3 that divides it, then there's a very simple way to do it. All you do is divide 19 by 3. Then, there are more powers of 3 because of 3 squared which is 9. You divide that by 19 and use the integer part which is clearly 2. You disregard the remainder. You check 3 cubed which is 27, and 19/27 isn't an integer so we don't care about that. Adding 6 and 2 you got 8. That means the highest power is 8.

Logarithm Properties

People taking the AMC 10 can ignore this part, but this is important for the AMC 12 takers.

Where:

$b > 0$ but $b \neq 1$, and M, N, and k are real numbers but M and N must be positive!

Rule 1: $\log_b (M \cdot N) = \log_b M + \log_b N$

Rule 2: $\log_b \left(\frac{M}{N} \right) = \log_b M - \log_b N$

Rule 3: $\log_b \left(M^k \right) = k \cdot \log_b M$

Rule 4: $\log_b (1) = 0$

Rule 5: $\log_b (b) = 1$

Rule 6: $\log_b \left(b^k \right) = k$

Rule 7: $b^{\log_b (k)} = k$

Memorize these logarithm properties since they can be very helpful. Sometimes in problems related to logarithms, it can be a lot easier to convert them in exponential form. It makes the problem look more satisfying and you can find ideas regarding how to solve the problem a lot faster.

Number Theory

Common Mistakes

Prime numbers are numbers that are only divisible by 1 and itself. Note that 1 and 0 are NOT prime numbers. Also remember that if a question asks you for nonnegative integers, then you include all positive integers including 0. Furthermore, remember that 0 is neither negative nor positive.

Euclidean Algorithm

If you're asked to find the GCD (Greatest Common Divisor) or two numbers like 2 and 4, then you're easily gonna say 2. However, what if the numbers are huge? That's when this algorithm will help.

Pretend I have two numbers a and b in which a is less than b. THe GCD of (a, b) is equivalent to (a, b-a). You keep subtracting the numbers.

Pretend you want to find the GCD of 48 and 880. The Euclidean Algorithm states that the GCD of those two numbers is equivalent to (48, 880-48) which is equivalent to (48, 832). Instead of just subarcting this again and again, we can simply divide and use the remainder. If you divide 880 and 48, you get 18 with a remainder of 16. You disregard the 18 and only use the remainder. You then find the GCD of the remainder and the divisor which was 48 in this case. Now we need to find the GCD of 16 and 48 which is obviously 16.

Modular Arithmetic Basics

People sometimes get confused by the word mod, but it's time to clear it up. When you say 8 is divisible by 4, you can simply say 8 = 0 (mod 4). All this means is that when 8 is divisible by 4, the remainder is 0. For example, 4, 8, 12, 16, 20, 24 are all 0 in mod 4. This means that when you divide all of those numbers by 4, you always get a remainder of 0. (The equal sign that i'm using actually has another line below it like in the picture below.)

Modular Arithmetic Properties

Below are some properties of modular arithmetic which are important to know.

Theorem 4.6. If a, b, c, d, and m are integers such that $m > 0$, $a \equiv b \pmod{m}$, and $c \equiv d \pmod{m}$, then

(i) $a + c \equiv b + d \pmod{m}$,

(ii) $a - c \equiv b - d \pmod{m}$,

(iii) $ac \equiv bd \pmod{m}$.

$a^c = b^d (mod\, m)$

Euler’s Totient Function

Pretend you have a number 48, and you want to find how many integers are relatively prime to 48. Relatively prime means that the GCD of 48 and the numbers less than that are just 1. This can be done by using this function. If you don’t know this function, you’re probably gonna count by listing out 1, 5, 7, and so on. This will take a long time. However, this function just involves you to find the prime factorization of 48. The prime factorization of 48. We know that the prime factorization is just $2^4 \times 3$. This theorem states that you take each of the prime numbers that divide the number and take the reciprocal of them. After that, you subtact that fraction from 1. If we take those exact steps, we’ll get $1 - \frac{1}{2} = \frac{1}{2}$and $1 - \frac{1}{3} = \frac{2}{3}$. Now we have two fractions that are ½ and ⅔. You find the number of relatively prime numbers to 48 by multiplying those 2 fractions to 48. After you multiply 48 with ½ and ⅔, you get 16 which is indeed our answer. There is a picture of the theorem below, but it’s fine if you don’t understand the symbols since we made this hand typed explanation for that reason.
I have attached the formula with the original terminology below, but even if you don’t understand it, don’t worry because we just explained it to you.

$$\begin{aligned}
\phi(n) &= \phi({p_1}^{a_1}) \cdot \phi({p_2}^{a_2}) \cdots \phi({p_k}^{a_k}) \\
&= \left({p_1}^{a_1} - {p_1}^{a_1-1}\right) \cdot \left({p_2}^{a_2} - {p_2}^{a_2-1}\right) \cdots \left({p_k}^{a_k} - {p_k}^{a_k-1}\right) \\
&= p_1^{a_1} \cdot \left(1 - \frac{1}{p_1}\right) \cdot p_2^{a_2} \cdot \left(1 - \frac{1}{p_2}\right) \cdots p_k^{a_k} \cdot \left(1 - \frac{1}{p_k}\right) \\
&= n \cdot \left(1 - \frac{1}{p_1}\right) \cdot \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right)
\end{aligned}$$

Euler's Totient Theorem

If we have a number a and we’re dividing it by n, and a and n are both relatively prime, then $a^{\varphi(n)} \equiv 1\ (mod\, n)$. The $\varphi(n)$can be calculated using the euler totient function which is included in the handout.

Fermat’s Little Theorem

This is a theorem that can greatly help in simplifying expressions in modular arithmetic.

If p is a prime and if a is any integer,

then $a^p \equiv a \pmod{p}$.

In particular, if p does not divide a,

then $a^{p-1} \equiv 1 \pmod{p}$.

The expression at the bottom only works if a isn't a multiple of p.

Example Problem: Pretend we want to find 7^{2019} *divided by* 13. We know this is the same thing as $7^{2019} \equiv x \ (mod\ 13)$. The x on the left represents the remainder. We know that we can't directly apply this theorem. However, we know that if the divisor doesn't divide the number, then we can make the exponent the divisor minus one. We know that 13 is the divisor. We also know that $7^{12} \equiv 1 \ (mod\ 13)$. Note that I used the power of 12 because it is one less than 13 which will tell us the remainder. Furthermore, note that 7^{2019} *is the same thing as* $(7^{12})^{168} \times 7^3$. We have to find the expression at the left in mod 13. We already know that 7 to the power of 12 simplifies to 1. Then, we have 1 to the power of 168 which is obviously just one. That means all we have to do is find 7 to the power of 3 in mod 13 which we can easily compute by hand.
We know that 7 x 7 x 7 is 343. After you divide it by 13, you get 26 and a remainder of 5. Thus, our answer is 5 because we only care about the remainder. This is just an example of how you can manipulate the modular expression to apply this theorem.

Chinese Remainder Theorem

Chinese Remainder Theorem

- The solution to the following equations:

  $x = a_1 \bmod n_1$

  $x = a_2 \bmod n_2$

  $x = a_k \bmod n_k$

  where $n_1, n_2, \dots, n_k$ are relatively prime is found as follows:

  $N = n_1 n_2 \dots n_k$

  $N_i = N/n_i$

  Find s_i such that $r_i n_i + s_i N_i = 1$

  Let $e_i = s_i N_i$, then

  $x = \Sigma^k_i a_i e_i \bmod N$

The picture above can be very confusing to understand. It's just full of variables, but it's time to clear it up. The chinese remainder theorem solves a set of linear congruences. The example below will clear it up.
$x \equiv 1 \pmod 5$
$x \equiv 2 \pmod 6$
$x \equiv 3 \pmod 7$

First, make a chart. If you have 3 linear congruences, then make a 3 by 3 table. You need to have 3 columns always, but the amount of rows is the same thing as the amount of expressions you have. Designate the first row to the first expression, the second to the second one, and the third to the third one. In the middle column, multiply all of the divisors except the one that is used in the expression. For example, in the first one, I will only multiply 6 and 7, but exclude 5. Do the same for all 3 steps. In the left most column, write down all the remainders that you want. Remember that they all have to be in order. Order is what matters the most. Now our next step is to literally just use the linear statements we have at the top, and multiply the variable with the values in our middle column in **order.**
$42x \equiv 1 \pmod 5$
$35x \equiv 2 \pmod 6$
$30x \equiv 3 \pmod 7$
Now, we know that 42x is the same thing as 2x in mod 5 because 40 is a multiple of 5. We do this to all the expressions.
$2x \equiv 1 \pmod 5$
$5x \equiv 2 \pmod 6$
$2x \equiv 3 \pmod 7$
Now we will individually solve each of the x terms and put it in the rightmost column. We want to see what value of x makes $2x \equiv 1 \pmod 5$. This can be done by simple trial and error. After we do that, we find that the value of x is just 3. We do the same for all the equations to get 3, 5, and 4.

1	6 x 7 = 42	3
2	5 x 7 = 35	5
3	5 x 6 = 30	4

Our last step is to multiply the terms in each row. After we do that, we get (1 x 42 x 3), (2 x 35 x 5), and (3 x 30 x 4). Then, you add all of those terms to each other to get your answer. After we add, we get 126 + 350 + 360 = 836.

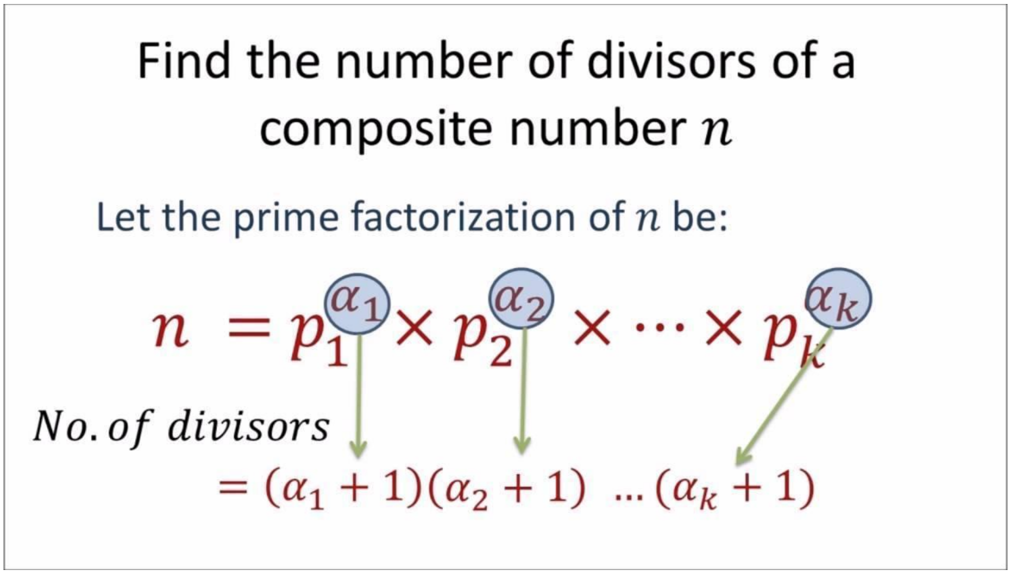
Units Digits

A lot of number theory is about recognizing patterns. This can be done by literally just looking at the units digit. For example, if a problem asks us to find the units digit of 7^{240}, we will simply look for patterns. We see that we don't need to find the exact value. We can just look at the units digit. $7^1 = 7, 7^2 = 9$, $7^3 = 3, 7^4 = 1, 7^5 = 7, 7^1 = 9$. On the left, I just wrote the units digit. Furthermore, we notice that it's repeating in a pattern. First, we got 7, 9, 3, 1, 7, 9, and it will go on and on. We see that if the exponent is a multiple of 4, our unit digit is simply 1. We see that our exponent in this situation is 240 which is a multiple of 4. Thus, the units digit is just 1.

This shows the power of looking for patterns since in many problems, they repeat.

Number of Divisors

The number of divisors of any number can be found by finding the prime factorization. After that, add one to the exponents of all the prime factors. Then, multiply them to each other to find the number of divisors.



Legendre's Formula

$$e_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor = \frac{n - S_p(n)}{p - 1}$$

The formula above works when p is a prime number and $e_p(n!)$ is the exponent of p found when you prime factorize n!. Sp(n) in the second equation is the sum of the digits of n when written in base p. I highly suggest using the formula in the middle. If you don't understand the terminology, then I'll make it a lot more clear.

$$\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right]$$

In the picture above, we can find the highest power of a prime number that divides n factorial by just using n and dividing it by p. Then you divide n by the square of p. You also take the floor function of the value you get by round down. Pretend we have 4 factorial and we want to find the highest power of 3 that divides it. We can simply do 4/3. We take the floor function of it which means we simply round down to get one which is our answer. You could square the prime number, but you'll get 4/9 which rounds down to 0. If you have a big number like 16 factorial, then you find the highest power of 2 that divides it by dividing

16 by 2, then 2 squared, then 2 cubed, then 2 to the power of 4. You need to raise the prime numbers by different powers to count the extra divisors of 2.

Relationship Between LCM and GCD

One very important thing to know is that the LCM of two numbers multiplied with the GCD of the same two numbers is literally the same thing as the multiplication of those numbers. In simple terms, I'm saying that ***a times b = GCD(a, b) times LCM (a, b).***

Wilson's Theorem

This theorem says that if we have an integer p that is greater than one, then (p-1)! + 1 will always be divisible by p if and only if p is a prime number.

Chicken McNugget Theorem

If you have two relatively primes numbers that are m and n, then the greatest integer that you cannot write in the form of am+bn is mn-m-n. Note that this works for **NONNEGATIVE** only. Furthermore, the number of numbers that can't be expressed in the form of am+bn is $\frac{(m-1)(n-1)}{2}$.

Binet's Formula

This formula is useful to find the nth term of a Fibonacci Sequence. You should know it because you never know when there will be a question regarding the Fibonacci Sequence. In the formula below, n represents the term you want.

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

Probability and Combinations.

Formula for Permutations and Combinations

Number of permutations (order matters) of *n* things taken *r* at a time:	Number of combinations (order does not matter) of *n* things taken *r* at a time:
$P(n,r)=\frac{n!}{(n-r)!}$	$C(n,r)=\frac{n!}{(n-r)!r!}$

Casework

This is a method used to find the number of total cases that you want. It is used a lot in combinatorics. However, sometimes casework can be very bad when there is a very simple solution. In those problems, it is considered brute force.

Overcounting

Complementary counting is a very common approach to solving combinatorics problems. What you do is you count more than what you need. After that, you subtract the “extras.”

Constructive Counting

This method is used when you want to count the number of numbers in a list. It should only be used for hard combinatorial problems because this method can be time consuming.

Pascal’s Triangle

You’ve probably seen the image below.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 200 252 200 120 45 10 1

Another way of rewriting all the terms in the above picture is shown below.

Row 0: $\binom{0}{0}$

Row 1: $\binom{1}{0}$ $\binom{1}{1}$

Row 2: $\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2}$

Row 3: $\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$

Row 4: $\binom{4}{0}$ $\binom{4}{1}$ $\binom{4}{2}$ $\binom{4}{3}$ $\binom{4}{4}$

Row 5: $\binom{5}{0}$ $\binom{5}{1}$ $\binom{5}{2}$ $\binom{5}{3}$ $\binom{5}{4}$ $\binom{5}{5}$

Row 6: $\binom{6}{0}$ $\binom{6}{1}$ $\binom{6}{2}$ $\binom{6}{3}$ $\binom{6}{4}$ $\binom{6}{5}$ $\binom{6}{6}$

Row 7: $\binom{7}{0}$ $\binom{7}{1}$ $\binom{7}{2}$ $\binom{7}{3}$ $\binom{7}{4}$ $\binom{7}{5}$ $\binom{7}{6}$ $\binom{7}{7}$

Row 8: $\binom{8}{0}$ $\binom{8}{1}$ $\binom{8}{2}$ $\binom{8}{3}$ $\binom{8}{4}$ $\binom{8}{5}$ $\binom{8}{6}$ $\binom{8}{7}$ $\binom{8}{8}$

Row 9: $\binom{9}{0}$ $\binom{9}{1}$ $\binom{9}{2}$ $\binom{9}{3}$ $\binom{9}{4}$ $\binom{9}{5}$ $\binom{9}{6}$ $\binom{9}{7}$ $\binom{9}{8}$ $\binom{9}{9}$

Row 10: $\binom{10}{0}$ $\binom{10}{1}$ $\binom{10}{2}$ $\binom{10}{3}$ $\binom{10}{4}$ $\binom{10}{5}$ $\binom{10}{6}$ $\binom{10}{7}$ $\binom{10}{8}$ $\binom{10}{9}$ $\binom{10}{10}$

Pascal's triangle is an important thing to know since most combinatorics is derived through that. Later on when you prepare for the AIME, you'll realize that some recursion problems involve Pascal's Triangle so definitely remember this concept.

Pascal's Identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

The image above is an important identity to know because it can help you in simplifying combinatorial expressions in a short amount of time.

Hockey Stick Identity

.

$$\binom{n+1}{k+1} = \binom{n}{k}+\binom{n-1}{k}+\binom{n-2}{k}+\ldots+\binom{k+1}{k}+\binom{k}{k}.$$

You should absolutely memorize this theorem because in some problems, you'll have a lot of combinatorial expressions. Computing them will take a long time. Knowing this identity will make it super fast.

Probability

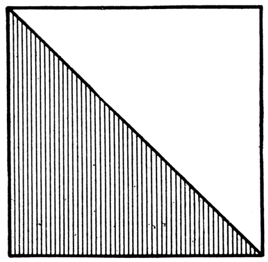
(Don't underestimate probability by thinking that it's just number of ways divided by the amount of ways)

$$\text{Probability(P)} = \frac{\text{Number of ways an outcome can occur}}{\text{Total number of outcomes possible}}$$

As you will see with some examples at the end of the handout, probability can get complicated.

Geometric Probability

If you've ever seen a problem in which the number of cases you want is infinite, then all you do is "graph" it or draw it in a shape. For example, pretend that I have two numbers x and y which are both in the range of 0 and 1. We want to know the probability of getting x+y < 1. There are an infinite amount of options for this. However, we can make a unit square for this problem and graph out x+y = 2. Then, we take the region that satisfies our statement.



This shows that whenever the desired outcomes can't be counted, you graph it out and find the area of the desired region divided by the area of the entire region.

Expected Value

Expected value is what the average of a certain outcomes will be. For example, pretend that I have a probability of ⅓ of winning one dollar. I also have a probability of ½ of losing 2 dollars. I also have a probability of ⅙ of not gaining or losing any money. Then, we find the expected value by multiplying the chance of the outcome to the outcome. After that, you add all of those numbers. In this case, the first outcome is winning a dollar and there is ⅓ chance of that. Furthermore, the second outcome states you have a probability of ½ when it comes to losing 2 dollars. You write the equation out for all 3 outcomes. $(\frac{1}{3} \times 1) + (\frac{1}{2} \times (-2)) + (\frac{1}{6} \times 0)$. After writing the expression for the expected value, now you compute it. Remember that I put negative 2 because you LOSE two dollars with a probability of ½. After multiplying the terms, you get ⅓ + -1 + 0. Adding them up gives you -⅔ which is the expected value.

Ball and Urn Counting

This is also known as sticks and stones along with stars and bars.

This is a good formula for counting problems and appears quite often.

If you have k **indistinguishable** items and n **distinguishable bins,** in which you want to see how many ways there are to place the items in the bins, then the formula for that is:

$$\binom{n+k-1}{k}$$

P.I.E.: Principle of Inclusion-Exclusion

This principle is all about overcounting and undercounting. It can be very helpful for hard combinatorial problems.

Let's pretend that we have 2 sets that are A and B. The numbers in A are 3, 4, 5, and 6 while the numbers in B are 5, 6, 8, 9. The way to find the union of these two sets is

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

The || represents the number of terms in those sets. The symbol on the very right means the number of terms that are common in both sets. For the example I showed, it means that the union of the two sets is the number of terms in set A added to the number of terms in set B. After that, you subtract the number of terms that are common between both of the sets to find the union.

This also works if there are more than two sets. Below is the way to represent it when there are 3 sets.

If A, B, C are finite sets, then

$$\#(A \cup B \cup C) = \#(A) + \#(B) + \#(C) - \#(A \cap B) - \#(A \cap C) - \#(B \cap C) + \#(A \cap B \cap C).$$

Vandermonde's Identity

$$\binom{n}{0}\binom{m}{m} + \binom{n}{1}\binom{m}{m-1} + \cdots + \binom{n}{m}\binom{m}{0} = \binom{m+n}{m}$$

Sum of a Row in Pascal's Triangle

This is something very important to remember since it can make your computation a lot easier.

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

Extra Things to know that involve combinatorics

$\binom{n}{4}$ is the maximum number of intersection points of diagonals inside a convex n-gon.

Binomial Theorem

We know that $(x + y)^2$ $is\ simply$ $x^2 + 2xy + y^2$. I'm sure that to find the answer, you would multiply each term with each other. That method is great, but what if you want to compute $(x + y)^6$. Calculating that can be painful by hand. That's when the binomial theorem comes in.

Binomial Theorem

$$(a+b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{n} a^0 b^n$$

$$\text{where } \binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$$

The terminology can be confusing in this theorem. If that's the case for you, then watch the recordings of our seminar since we explained it there.

Pigeonhole Principle

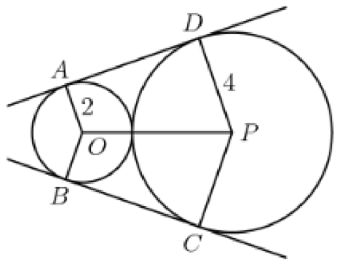
This principle involves more common sense than mathematical theory. It states that if we have n amount of holes and we have more than n+1 pigeons, then there will be multiple pigeons in some holes. In reality, this is just common sense. Pretend you have 3 waffle cones, and 5 scoops of ice cream. We already know that there is no possible way to distribute all 5 scoops of the ice cream to the cones so that each of the coins only have one scoop. This means that some scoops will have more than one. Using some common sense, we just found another example of this principle.

Partner Channel

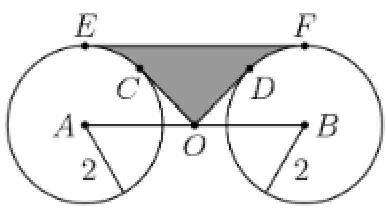
You can find great resources on this channel for ORIGINAL problems full with detailed solutions and videos on concepts: https://www.youtube.com/channel/UCbMEii6via2DLkF98xPNeVw

Practice Problems for Geometry

1. Circles with centers O and P have radii 2 and 4, respectively, and are externally tangent. Points A and B on the circle with center O and points C and D on the circle with center P are such that AD and BC are common external tangents to the circles. What is the area of the concave hexagon $AOBCPD$?



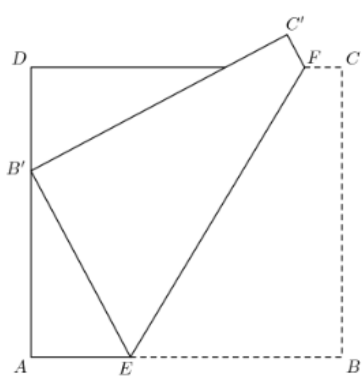
2. Circles centered at A and B each have radius 2, as shown. Point O is the midpoint of $\overline{AB}$, and $OA = 2\sqrt{2}$. Segments OC and OD are tangent to the circles centered at A and B, respectively, and EF is a common tangent. What is the area of the shaded region $ECODF$?



3. Points A, B, C, D and E are located in 3-dimensional space with $AB = BC = CD = DE = EA = 2$ and $\angle ABC = \angle CDE = \angle DEA = 90^o$. The plane of $\triangle ABC$ is parallel to $\overline{DE}$. What is the area of $\triangle BDE$?

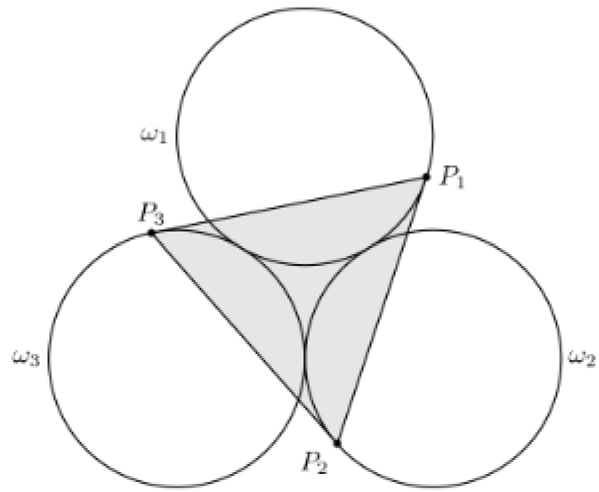
4. A flat board has a circular hole with radius 1 and a circular hole with radius 2 such that the distance between the centers of the two holes is 7. Two spheres with equal radii sit in the two holes such that the spheres are tangent to each other. The square of the radius of the spheres is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

5. A rectangular piece of paper whose length is $\sqrt{3}$ times the width has area A. The paper is divided into three equal sections along the opposite lengths, and then a dotted line is drawn from the first divider to the second divider on the opposite side as shown. The paper is then folded flat along this dotted line to create a new shape with area B. What is the ratio $\frac{B}{A}$?
6. Two skaters, Allie and Billie, are at points A and B, respectively, on a flat, frozen lake. The distance between A and B is 100 meters. Allie leaves A and skates at a speed of 8 meters per second on a straight line that makes a 60° angle with AB. At the same time Allie leaves A, Billie leaves B at a speed of 7 meters per second and follows the straight path that produces the earliest possible meeting of the two skaters, given their speeds. How many meters does Allie skate before meeting Billie?
7. Point B is on $\overline{AC}$ with $AB = 9$ and $BC = 21$. Point D is not on $\overline{AC}$ so that $AD = CD$, and AD and BD are integers. Let s be the sum of all possible perimeters of $\triangle ACD$. Find s.
8. $ABCD$ is a rectangular sheet of paper that has been folded so that corner B is matched with point B' on edge AD. The crease is EF, where E is on AB and F is on CD. The dimensions $AE = 8, BE = 17$, and $CF = 3$ are given. The perimeter of rectangle $ABCD$ is m/n, where m and n are relatively prime positive integers. Find $m + n$.



9. Circle C with radius 2 has diameter $\overline{AB}$. Circle D is internally tangent to circle C at A. Circle E is internally tangent to circle C, externally tangent to circle D, and tangent to $\overline{AB}$. The radius of circle D is three times the radius of circle E, and can be written in the form $\sqrt{m} - n$, where m and n are positive integers. Find $m + n$.
10. Circles ω_1, ω_2, and ω_3 each have radius 4 and are placed in the plane so that each circle is externally tangent to the other two. Points P_1, P_2, and P_3 lie on ω_1, ω_2, and ω_3 respectively such that $P_1P_2 = P_2P_3 = P_3P_1$ and line P_iP_{i+1} is tangent to ω_i

for each $i = 1, 2, 3$, where $P_4 = P_1$. See the figure below. The area of $\triangle P_1P_2P_3$ can be written in the form $\sqrt{a} + \sqrt{b}$ for positive integers a and b. What is $a + b$?



Solutions to Geometry Problems

1. https://artofproblemsolving.com/wiki/index.php/2006_AMC_10B_Problems/Problem_24
2. https://artofproblemsolving.com/wiki/index.php/2007_AMC_10A_Problems/Problem_24
3. https://artofproblemsolving.com/wiki/index.php/2007_AMC_12B_Problems/Problem_25
4. https://artofproblemsolving.com/wiki/index.php/2020_AIME_I_Problems/Problem_6
5. https://artofproblemsolving.com/wiki/index.php/2014_AMC_10A_Problems/Problem_23
6. https://artofproblemsolving.com/wiki/index.php/1989_AIME_Problems/Problem_6
7. https://artofproblemsolving.com/wiki/index.php/2003_AIME_I_Problems/Problem_7
8. https://artofproblemsolving.com/wiki/index.php/2004_AIME_II_Problems/Problem_7
9. https://artofproblemsolving.com/wiki/index.php/2014_AIME_II_Problems/Problem_8
10. https://artofproblemsolving.com/wiki/index.php/2018_AMC_12B_Problems/Problem_25

Practice Problems for Algebra

1. Find the eighth term of the sequence $1440,\ 1716,\ 1848, \ldots,$ whose terms are formed by multiplying the corresponding terms of two arithmetic sequences.
2. Determine $3x_4 + 2x_5$ if x_1, x_2, x_3, x_4, and x_5 satisfy the system of equations below.

$$2x_1 + x_2 + x_3 + x_4 + x_5 = 6$$

$$x_1 + 2x_2 + x_3 + x_4 + x_5 = 12$$

$$x_1 + x_2 + 2x_3 + x_4 + x_5 = 24$$

$x_1 + x_2 + x_3 + 2x_4 + x_5 = 48$

$x_1 + x_2 + x_3 + x_4 + 2x_5 = 96$

3. Two geometric sequences $a_1, a_2, a_3, \dots$ and $b_1, b_2, b_3, \dots$ have the same common ratio, with $a_1 = 27$, $b_1 = 99$, and $a_{15} = b_{11}$. Find a_9.
4. For how many ordered pairs (x, y) of integers is it true that $0 < x < y < 10^6$ and that the arithmetic mean of x and y is exactly 2 more than the geometric mean of x and y?
5. In order to complete a large job, 1000 workers were hired, just enough to complete the job on schedule. All the workers stayed on the job while the first quarter of the work was done, so the first quarter of the work was completed on schedule. Then 100 workers were laid off, so the second quarter of the work was completed behind schedule. Then an additional 100 workers were laid off, so the third quarter of the work was completed still further behind schedule. Given that all workers work at the same rate, what is the minimum number of additional workers, beyond the 800 workers still on the job at the end of the third quarter, that must be hired after three-quarters of the work has been completed so that the entire project can be completed on schedule or before?
6. The polynomial $x^3 - 2004x^2 + mx + n$ has integer coefficients and three distinct positive zeros. Exactly one of these is an integer, and it is the sum of the other two. How many values of n are possible?
7. Real numbers r and s are roots of $p(x) = x^3 + ax + b$, and $r + 4$ and $s - 3$ are roots of $q(x) = x^3 + ax + b + 240$. Find the sum of all possible values of $|b|$.
8. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. How many real numbers x satisfy the equation $x^2 + 10,000\lfloor x \rfloor = 10,000x$?
9. Define a sequence recursively by $t_1 = 20$, $t_2 = 21$, and $t_n = \frac{5t_{n-1} + 1}{25t_{n-2}}$ for all $n \geq 3$. Then t_{2020} can be expressed as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.
10. Consider polynomials $P(x)$ of degree at most 3, each of whose coefficients is an element of $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. How many such polynomials satisfy $P(-1) = -9$?

Solutions to Algebra Practice Problems

1. https://artofproblemsolving.com/wiki/index.php/2003_AIME_II_Problems/Problem_8
2. https://artofproblemsolving.com/wiki/index.php/1986_AIME_Problems/Problem_4
3. https://artofproblemsolving.com/wiki/index.php/2012_AIME_II_Problems/Problem_2
4. https://artofproblemsolving.com/wiki/index.php/2000_AIME_I_Problems/Problem_6
5. https://artofproblemsolving.com/wiki/index.php/2004_AIME_II_Problems/Problem_5
6. https://artofproblemsolving.com/wiki/index.php/2004_AMC_12B_Problems/Problem_23
7. https://artofproblemsolving.com/wiki/index.php/2014_AIME_II_Problems/Problem_5
8. https://artofproblemsolving.com/wiki/index.php/2018_AMC_10B_Problems/Problem_25
9. https://artofproblemsolving.com/wiki/index.php/2020_AIME_II_Problems/Problem_6
10. https://artofproblemsolving.com/wiki/index.php/2018_AMC_12B_Problems/Problem_22

Practice Problems for Number Theory

1. It is known that, for all positive integers k,
   $$1^2+2^2+3^2+\ldots+k^2=\frac{k(k+1)(2k+1)}{6}.$$
   Find the smallest positive integer k such that $1^2+2^2+3^2+\ldots+k^2$ is a multiple of 200.
2. The number obtained from the last two nonzero digits of 90! is equal to n. What is n?
3. A positive integer n is *nice* if there is a positive integer m with exactly four positive divisors (including 1 and m) such that the sum of the four divisors is equal to n. How many numbers in the set $\{2010, 2011, 2012, \ldots, 2019\}$ are nice?
4. Let n be the least positive integer greater than 1000 for which
   $$\gcd(63, n+120) = 21 \quad \text{and} \quad \gcd(n+63, 120) = 60.$$
   What is the sum of the digits of n?
5. Let $S(n)$ equal the sum of the digits of positive integer n. For example, $S(1507)=13$. For a particular positive integer n, $S(n)=1274$. Which of the following could be the value of $S(n+1)$?
6. Let $a, b, c,$ and d be positive integers such that $\gcd(a,b)=24$, $\gcd(b,c)=36$, $\gcd(c,d)=54$, and $70<\gcd(d,a)<100$. Which of the following must be a divisor of a? (This problem can be solved using the answer choices, and they are needed)
   A. 5 **B**. 7 **C**. 11 **D**. 13 **E.** 17

7. What is the largest 2-digit prime factor of the integer $n = \binom{200}{100}$?
8. For a certain positive integer n less than 1000, the decimal equivalent of $\frac{1}{n}$ is $0.\overline{abcdef}$, a repeating decimal of period of 6, and the decimal equivalent of $\frac{1}{n+6}$ is $0.\overline{wxyz}$, a repeating decimal of period 4. In which interval does n lie?
9. A solid rectangular block is formed by gluing together N congruent 1-cm cubes face to face. When the block is viewed so that three of its faces are visible, exactly 231 of the 1-cm cubes cannot be seen. Find the smallest possible value of N.
10. Let n be a 5-digit number, and let q and r be the quotient and the remainder, respectively, when n is divided by 100. For how many values of n is $q+r$ divisible by 11?

Solutions to Number Theory Practice

1. https://artofproblemsolving.com/wiki/index.php/2002_AIME_II_Problems/Problem_7
2. https://artofproblemsolving.com/wiki/index.php/2010_AMC_12A_Problems/Problem_23
3. https://artofproblemsolving.com/wiki/index.php/2013_AMC_10B_Problems/Problem_24
4. https://artofproblemsolving.com/wiki/index.php/2020_AMC_10A_Problems/Problem_24
5. https://artofproblemsolving.com/wiki/index.php/2017_AMC_10A_Problems/Problem_20
6. https://artofproblemsolving.com/wiki/index.php/2018_AMC_10A_Problems/Problem_22
7. https://artofproblemsolving.com/wiki/index.php/1983_AIME_Problems/Problem_8
8. https://artofproblemsolving.com/wiki/index.php/2016_AMC_12B_Problems/Problem_22
9. https://artofproblemsolving.com/wiki/index.php/2004_AIME_II_Problems/Problem_3
10. https://artofproblemsolving.com/wiki/index.php/2003_AMC_10A_Problems/Problem_25

Practice Problems for Probability and Combinatorics:

1. The Daily Challenge with Po-Shen-Loh currently features 7 different modules. In how many different ways can PePe pick 3 different modules out of these 7?
2. There is an unlimited supply of congruent equilateral triangles made of colored paper. Each triangle is a solid color with the same color on both sides of the paper. A large equilateral triangle is constructed from four of these paper triangles. Two large triangles are considered distinguishable if it is not possible to place one on the other, using translations, rotations, and/or reflections, so that their corresponding small

triangles are of the same color. Given that there are six different colors of triangles from which to choose, how many distinguishable large equilateral triangles may be formed?

3. Two quadrilaterals are considered the same if one can be obtained from the other by a rotation and a translation. How many different convex cyclic quadrilaterals are there with integer sides and perimeter equal to 32?
4. Each vertex of convex pentagon $ABCDE$ is to be assigned a color. There are 6 colors to choose from, and the ends of each diagonal must have different colors. How many different colorings are possible?
5. The cards in a stack of $2n$ cards are numbered consecutively from 1 through $2n$ from top to bottom. The top n cards are removed, kept in order, and form pile $A.$ The remaining cards form pile $B.$ The cards are then restacked by taking cards alternately from the tops of pile B and $A,$ respectively. In this process, card number $(n+1)$ becomes the bottom card of the new stack, card number 1 is on top of this card, and so on, until piles A and B are exhausted. If, after the restacking process, at least one card from each pile occupies the same position that it occupied in the original stack, the stack is named magical. For example, eight cards form a magical stack because cards number 3 and number 6 retain their original positions. Find the number of cards in the magical stack in which card number 131 retains its original position.
6. Eight people are sitting around a circular table, each holding a fair coin. All eight people flip their coins and those who flip heads stand while those who flip tails remain seated. What is the probability that no two adjacent people will stand?
7. Jason rolls three fair standard six-sided dice. Then he looks at the rolls and chooses a subset of the dice (possibly empty, possibly all three dice) to reroll. After rerolling, he wins if and only if the sum of the numbers face up on the three dice is exactly $7.$ Jason always plays to optimize his chances of winning. What is the probability that he chooses to reroll exactly two of the dice?
8. For a particular peculiar pair of dice, the probabilities of rolling 1, 2, 3, 4, 5, and 6, on each die are in the ratio $1:2:3:4:5:6$. What is the probability of rolling a total of 7 on the two dice?
9. Rachel and Robert run on a circular track. Rachel runs counterclockwise and completes a lap every 90 seconds, and Robert runs clockwise and completes a lap every 80 seconds. Both start from the same line at the same time. At some random time between 10 minutes and 11 minutes after they begin to run, a photographer standing inside the track takes a picture that shows one-fourth of the track, centered on the starting line. What is the probability that both Rachel and Robert are in the picture?

10. Ten women sit in 10 seats in a line. All of the 10 get up and then reseat themselves using all 10 seats, each sitting in the seat she was in before or a seat next to the one she occupied before. In how many ways can the women be reseated?

Solutions to Probability and Combinatorics

1. (From ThePuzzlr Contest. The solution can be found here: https://youtu.be/hC31Xw5x_a0?t=744)
2. https://artofproblemsolving.com/wiki/index.php/2006_AIME_II_Problems/Problem_8
3. https://artofproblemsolving.com/wiki/index.php/2010_AMC_12A_Problems/Problem_25
4. https://artofproblemsolving.com/wiki/index.php/2011_AMC_10A_Problems/Problem_22
5. https://artofproblemsolving.com/wiki/index.php/2005_AIME_II_Problems/Problem_6
6. https://artofproblemsolving.com/wiki/index.php/2015_AMC_10A_Problems/Problem_22
7. https://artofproblemsolving.com/wiki/index.php/2020_AMC_10A_Problems/Problem_25
8. https://artofproblemsolving.com/wiki/index.php/2006_AMC_10B_Problems/Problem_21
9. https://artofproblemsolving.com/wiki/index.php/2009_AMC_10B_Problems/Problem_23
10. https://artofproblemsolving.com/wiki/index.php/2009_AMC_12B_Problems/Problem_21