Spring Mass Model and Predator Prey System

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1 A Spring-Mass model

Modeling situations involving a mass attached to a spring, such as in Figure 1, it is important to consider the second derivative.

We can assume that the force depends on the displacement of the spring and with the displacement being comparatively small, we can use Taylor's theorem to approximate F

$$F(X) = F(0) + F'(0)x$$

where $F'(0) = -k$, $F(0) = 0$ and x^n terms with $n \geq 2$ can be neglected for small displacement. Thus, we get to the equation known as the Hooke's Law

$$F = -kx$$

At the same time, according to Newton's Second Law,

$$F = mx$$

Setting the forces equal, we get a **second order differential equation** for an **undamped harmonic oscilator**

$$mx'' = -kx$$

1.1 Example 1

Suppose that we have a spring-mass system where $m = 1$ and $k = 1$. If the initial velocity of the spring is one unit per second and the initial position is at the equilibrium point, then we have the following initial value problem,

$$x'' + x = 0$$

$$x(0) = 0$$

$$x'(0) = 1$$

In this case, since $x''(t) = -x(t)$, we can make a guess that the solution is in the form of

$$Asin(t) + Bcos(t)$$

Then,

$$x'(t) = -Asin(t) + Bcos(t)$$

Furthermore, based of the initial condition,

$$x'(0) = 0$$

which means that $A = 0$ and $B = 1$. Thus,

$$x(t) = sin(t)$$

which is the solution for the differential equation in the example. Figure 2 shows the graph of this solution.

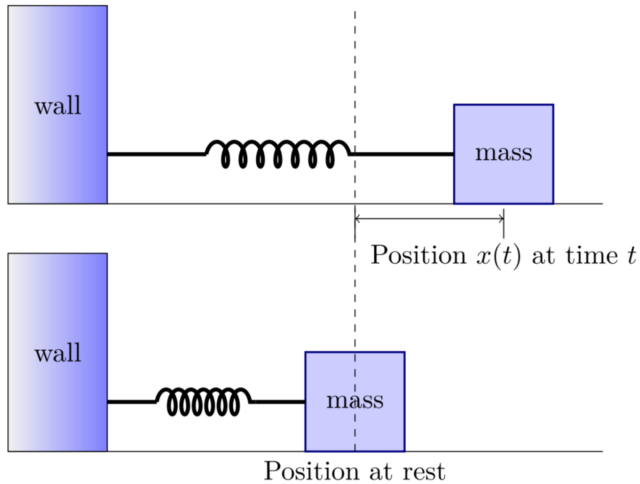

Figure 1: A spring-mass system

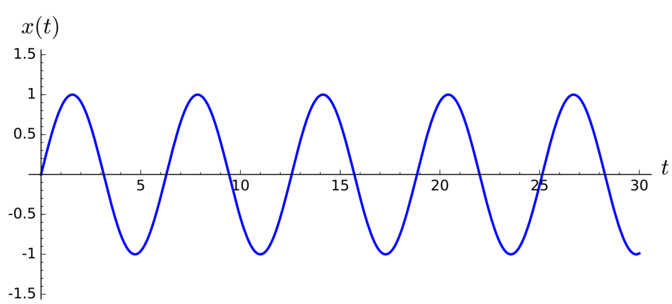

Figure 2: Undamped spring-mass system

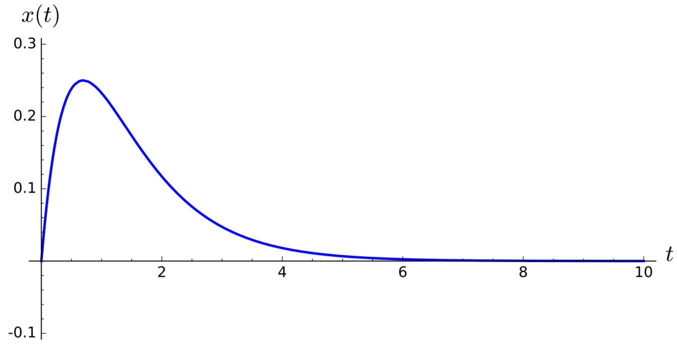

Figure 3: An over-damped spring-mass system

Now let us add a damping force to our system. For example, we might add a dashpot, a mechanical device that resists motion, to our system.

We can assume that the force excerted by the dashpot depends on velocity and acts opposite to the direction of the motion:

$$F = -bx'$$

Thus, the differtnial equaton for **damped harmonic ascilator** is

$$mx'' + bx' + kx = 0$$

1.2 Example 2

Suppose that we have a spring-mass system governed by the equation

$$x'' + 3x' + 2x = 0$$

We can assume that the solution is of a form $x(t) = e^{rt}$. Then,

$$x'' + 3x' + 2x = r^2e^{rt} + 3re^{rt} + 2e^{rt} = e^{rt}(r + 2)(r + 1) = 0$$

Since e^{rt} is never zero, it must be the case that $r = -2$ or $r = -1$. Therefore, the solution is a linear combination of e^{-2t} and e^{-t}. Using the initial condition, we obtain

$$x(t) = e^{-t} - e^{-2t}$$

Notice that the additional damping negates any oscillation in the system. In this case, we say that the harmonic oscillator is **overdamped**. Figure 3 shows graph of the solution for the overdamped system.

If we have a very strong srping and small amount of damping, the system will still oscilate. Such system is called **underdamped**. For example, our spring-mass system might be described by the initial value problem

$$x'' + 2x' + 50x = 0$$

$$x(0) = 0$$

$$x'(0) = 1$$

The solution for this equation is

$$1/7e^{-t}sin(7t)$$

and Figure 4 shows the graph of this solution.

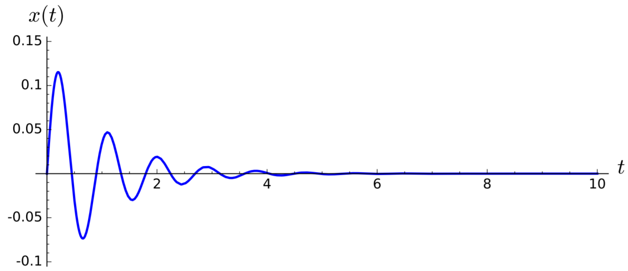

Figure 4: A spring-mass system

2 A Predator-Prey System

Some situations require more than one differential equation to model a particular phenomenon. We might use a system of differential equations to model two interacting species, say where one species preys on the other. For example, we can model how the population of Canadian lynx (lynx canadenis) interacts with a the population of snowshoe hare (lepus americanis).

If no lynx are present, we will assume that the hares reproduce at a rate proportional to their population and are not affected by overcrowding. That is, the hare population will grow exponentially,

$$\frac{dH}{dt} = aH$$

Since the lynx prey on the hares, we can argue that the rate at which the hares are consumed by the lynx is proportional to the rate at which the hares and lynx interact. Thus, the equation that predicts the rate of change of the hare population becomes

$$\frac{dH}{dt} = aH - bHL$$

If there is no food, the lynx population will decline at a rate proportional to itself,

$$\frac{dL}{dt} = -cL$$

The lynx receive benefit from the hare population. The rate at which lynx are born is proportional to the number of hares that are eaten, and this is proportional to the rate at which the hares and lynx interact. Consequently, the growth rate of the lynx population can be described by

$$\frac{dL}{dt} = -cL + dHL$$

Thus, we have a **system** of differntial equations that describes how the two populations interact:

$$\frac{dH}{dt} = aH - bHL$$

$$\frac{dL}{dt} = -cL + dHL$$

Figure 5 shows the graphical solution for this system.

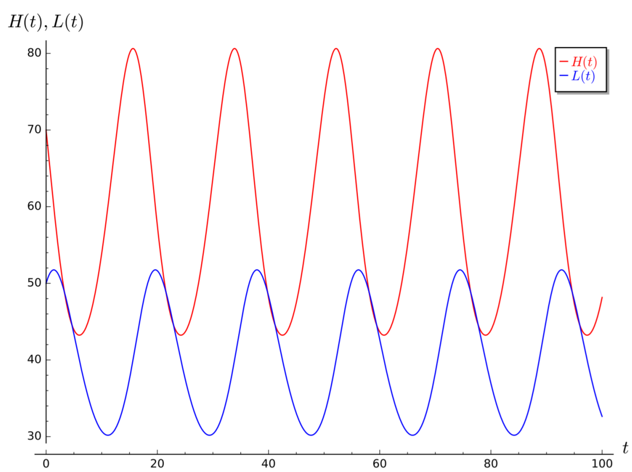

Figure 5: The predator-prey relationship