Chapter 3

Exercise 1.

(a) $x = 7k + 3$ for some $k \in \mathbb{Z}$.

(b) $x = 23k + 7$ for some $k \in \mathbb{Z}$.

(c) $x = 26k + 18$ for some $k \in \mathbb{Z}$.

(d) $x = 5k + 2$ for some $k \in \mathbb{Z}$.

(e) $x = 6k + 5$ for some $k \in \mathbb{Z}$.

(f) There are no $x \in \mathbb{Z}$ satisfying this equivalence.

Exercise 2.

(a) Not a group, there is no identity element.

(b) Is a group, identity is a, every element is its own inverse, table is associative and closed.

(c) Is a group, identity is a, $a^{-1} = a$, $b^{-1} = d$, $c^{-1} = c$, $d^{-1} = b$, associative and commutative.

(d) Not a group, identity is a but d has no inverse.

Exercise 3.

Symmetries of a rectangle:

- e: do nothing
- ρ: rotate 180°
- μ_1: flip horizontally
- μ_2: flip vertically

Cayley table for symmetries of a rectangle:

$\circ$	e	ρ	μ_1	μ_2
e	e	ρ	μ_1	μ_2
ρ	ρ	e	μ_2	μ_1
μ_1	μ_1	μ_2	e	ρ
μ_2	μ_2	μ_1	ρ	e

Cayley table for $(\mathbb{Z}_4, +)$:

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

These two groups are not the same. There is only 1 nontrivial proper subgroup of $(\mathbb{Z}_4, +)$, consisting of the elements 0 and 2, while there are 3 nontrivial proper subgroups of the symmetries of a rectangle, $H_1 = \{e, \rho\}$, $H_2 = \{e, \mu_1\}$, and $H_3 = \{e, \mu_3\}$.

Exercise 4.
Symmetries of a rhombus:

- e: do nothing
- ρ: rotate 180°
- μ_1: flip about the long diagonal
- μ_2: flip about the short diagonal

Cayley table for symmetries of a rhombus:

$\circ$	e	ρ	μ_1	μ_2
e	e	ρ	μ_1	μ_2
ρ	ρ	e	μ_2	μ_1
μ_1	μ_1	μ_2	e	ρ
μ_2	μ_2	μ_1	ρ	e

Comparing with the Cayley table for symmetries of a rectangle from above, we can see that the two groups are the same.

Exercise 5.
Symmetries of a square:

- e: do nothing
- ρ_1: rotate clockwise 90°
- ρ_2: rotate clockwise 180°
- ρ_3: rotate clockwise 270°
- μ_1: flip horizontally
- μ_2: flip vertically
- δ_1: flip along diagonal $y = x$
- δ_2: flip along diagonal $y = -x$

Cayley table for symmetries of a square:

$\circ$	e	ρ_1	ρ_2	ρ_3	μ_1	δ_1	μ_2	δ_2
e	e	ρ_1	ρ_2	ρ_3	μ_1	δ_1	μ_2	δ_2
ρ_1	ρ_1	ρ_2	ρ_3	e	δ_1	μ_2	δ_2	μ_1
ρ_2	ρ_2	ρ_3	e	ρ_1	μ_2	δ_2	μ_1	δ_1
ρ_3	ρ_3	e	ρ_1	ρ_2	δ_2	μ_1	δ_1	μ_2
μ_1	μ_1	δ_2	μ_2	δ_1	e	ρ_3	ρ_2	ρ_1
δ_1	δ_1	μ_1	δ_2	μ_2	ρ_1	e	ρ_3	ρ_2
μ_2	μ_2	δ_1	μ_1	δ_2	ρ_2	ρ_1	e	ρ_3
δ_2	δ_2	μ_2	δ_1	μ_1	ρ_3	ρ_2	ρ_1	e

There are 24 ways to permute 4 objects. However, not each permutation is a valid symmetry of the square, e.g. (A, C, B, D).

Exercise 6.

Multiplication table for $U(12)$:

$\cdot$	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

Exercise 7.

Let $S = \mathbb{R} \setminus \{-1\}$ and define $*$ on S by $a * b = a + b + ab$.

Proof. $(S, *)$ is an abelian group.
(Closed) Addition and multiplication are closed under the reals. We will show that there are no elements $a, b \in S$ such that $a * b = -1$.
Suppose for the sake of contradiction that $a, b \in S$ with $a * b = -1$. Then $a * b = a + b + ab = -1$, and rearranging and factoring gives $(a + 1)(b + 1) = 0$. This implies that either a or b is -1, which is a contradiction, since a and b are in S. Thus S is closed under $*$.

(Associative) Let $a, b, c \in S$. Then

$$
\begin{aligned}
(a * b) * c &= (a + b + ab) * c \\
&= (a + b + ab) + c + (ac + bc + abc) \\
&= a + (b + c + bc) + (ab + ac + abc) \\
&= a * (b + c + ab) \\
&= a * (b * c)
\end{aligned}
$$

(Identity) The identity element is 0. Let $a \in S$. Then $0 * a = 0 + a + 0 = a$, and $a * 0 = a + 0 + 0 = a$.

(Inverse) Let $a \in S$. Then the inverse a^{-1} is given by $a^{-1} = -\frac{a}{a+1}$. We can see this because $a * a^{-1} = a - \frac{a}{a+1} - \frac{a^2}{a+1} = 0$.

(Commutative) Let $a, b \in S$. Then $a * b = a + b + ab = b + a + ba = b * a$.

Since $(S, *)$ is closed, associative, has an identity and inverses, and is commutative, it is an abelian group. □

Exercise 8.

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$. Then $AB = \begin{bmatrix} 8 & 3 \\ 13 & 5 \end{bmatrix}$ but $BA = \begin{bmatrix} 4 & 7 \\ 5 & 9 \end{bmatrix}$.

Exercise 9.

Proof. The product of two matrices in $SL_2(\mathbb{R})$ has determinant one. Let $A, B \in SL_2(\mathbb{R})$ with $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$. Then $AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$.

Then

$$\begin{aligned}
\det(AB) &= (a_{11}b_{11} + a_{12}b_{21})(a_{21}b_{12} + a_{22}b_{22}) - (a_{11}b_{12} + a_{12}b_{22})(a_{21}b_{11} + a_{22}b_{21}) \\
&= (a_{11}a_{21}b_{11}b_{12} + a_{11}a_{22}b_{11}b_{22} + a_{12}a_{21}b_{12}b_{21} + a_{12}a_{22}b_{21}b_{22}) \\
&- (a_{11}a_{21}b_{11}b_{12} + a_{11}a_{22}b_{12}b_{21} + a_{12}a_{21}b_{11}b_{22} + a_{12}a_{22}b_{21}b_{22}) \\
&= a_{11}a_{22}b_{11}b_{22} + a_{12}a_{21}b_{12}b_{21} - a_{11}a_{22}b_{12}b_{21} - a_{12}a_{21}b_{11}b_{22} \\
&= (a_{11}a_{22} - a_{12}a_{21})(b_{11}b_{22} - b_{12}b_{21}) \\
&= \det(A)\det(B) \\
&= 1
\end{aligned}$$

Since A and B were arbitrary, the product of two matrices in $SL_2(\mathbb{R})$ has determinant one. □

Exercise 10.

Let H be the set of matrices of the form $\begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}$.

Proof. H is a group under matrix multiplication.

(Closed) Let $A, B \in H$ given by $A = \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & x' & y' \\ 0 & 1 & z' \\ 0 & 0 & 1 \end{bmatrix}$.

Then $AB = \begin{bmatrix} 1 & x + x' & y + y' + xz' \\ 0 & 1 & z + z' \\ 0 & 0 & 1 \end{bmatrix} \in H$.

(Associative) Matrix multiplication is associative.

(Identity) The matrix $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in H$ is the identity element. From matrix multiplication, we know that $AI = IA = A$ for any $A \in H$.

(Inverse) Let $A = \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} \in H$. The inverse A^{-1} is given by the matrix $\begin{bmatrix} 1 & -x & xz - y \\ 0 & 1 & -z \\ 0 & 0 & 1 \end{bmatrix}$.

We can see that $AA^{-1} = \begin{bmatrix} 1 & x - x & y + xz - y - xz \\ 0 & 1 & z - z \\ 0 & 0 & 1 \end{bmatrix} = I$.

Since $(H, *)$ is closed, associative, has an identity and inverses, it is a group. □

Exercise 11.
The proof that $\det(AB) = \det(A)\det(B)$ for $A, B \in GL_2(\mathbb{R})$ is nearly identical to the proof in exercise 9, except that $\det(A), \det(B) \neq 1$.

Proof. $GL_2(\mathbb{R})$ is closed.
Let $A, B \in GL_2(\mathbb{R})$. Since $\det(AB) = \det(A)\det(B)$, and $\det(A), \det(B) \neq 0$, then $\det(AB) \neq 0$, and $AB \in GL_2(\mathbb{R})$. □

Exercise 12.
Let $\mathbb{Z}_2^n = \{(a_1, a_2, \ldots, a_n) : a_i \in \mathbb{Z}_2\}$, and a binary operation on $\mathbb{Z}_2^n$ by

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$

Proof. $(\mathbb{Z}_2^n, +)$ is a group.
(Closed) Let $A, B \in \mathbb{Z}_2^n$. Then for each $i \in [n]$, $a_i + b_i \in \mathbb{Z}_2$, so $A + B \in \mathbb{Z}_2^n$.

(Associative) Let $A, B, C \in \mathbb{Z}_2^n$. Then

$$\begin{aligned}(A + B) + C &= (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) + C \\ &= (a_1 + b_1 + c_1, a_2 + b_2 + c_2, \ldots, a_n + b_n + c_n) \\ &= A + (b_1 + c_1, b_2 + c_2, \ldots, b_n + c_n) \\ &= A + (B + C)\end{aligned}$$

(Identity) The identity $\mathbf{0}$ is given by $(0, 0, \ldots, 0)$. We can see that for $A \in \mathbb{Z}_2^n$, $A + \mathbf{0} = \mathbf{0} + A = A$.

(Inverse) Let $A \in \mathbb{Z}_2^n$. Then $A^{-1} = (-a_1, -a_2, \ldots, -a_n)$. It is straightforward to compute that $A + A^{-1} = \mathbf{0}$.

Since $(\mathbb{Z}_2^n, +)$ is closed, associative, has an identity and inverses, it is a group. □

Exercise 13.

Proof. $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ is a group under multiplication.
The reals are closed under multiplication, and no two nonzero reals multiply to get zero. Multiplication over the reals is associative. 1 is the identity element, since $1 \cdot x = x \cdot 1 = 1$ for all $x \in \mathbb{R}^*$. The inverse of an element $x \in \mathbb{R}^*$ is given by $1/x$, since $x \cdot 1/x = 1$. □

Exercise 14.
Given the groups $\mathbb{R}^*$ and $\mathbb{Z}$, let $G = \mathbb{R}^* \times \mathbb{Z}$. Define a binary operation $\circ$ on G by $(a, m) \circ (b, n) = (ab, m + n)$.

Proof. $(G, \circ)$ is a group.
(Closed) Let $A, B \in G$ with $A = (a, m)$ and $B = (b, n)$. Then $A \circ B = (ab, m + n)$. Since $\mathbb{R}^*$ is closed under multiplication and $\mathbb{Z}$ is closed under addition, G is closed.

(Associatve) Let $A, B, C \in G$ with $A = (a, m)$, $B = (b, n)$, and $C = (c, p)$. Then

$$\begin{aligned}(A \circ B) \circ C &= (ab, m + n) \circ C \\ &= (abc, m + n + p) \\ &= A \circ (bc, n + p) \\ &= A \circ (B \circ C)\end{aligned}$$

(Identity) The identity is given by $(1, 0)$. We can see that for $A = (a, m) \in G$, $(1, 0) \circ A = (1 * a, 0 + m) = (a, m) = (a * 1, m + 0) = A \circ (1, 0)$.

(Inverse) Let $A = (a, m) \in G$. The inverse A^{-1} is given by $(1/a, -m)$. We can see that $A \circ A^{-1} = (a * 1/a, m - m) = (1, 0)$, and $A^{-1} \circ A = (1/a * a, -m + m) = (1, 0)$. Since $(G, \circ)$ is closed, associative, has an identity and inverses, it is a group. □

Exercise 15.
This is false; the symmetries of a triangle are nonabelian.

Exercise 16.
Consider the group of the symmetries of a triangle, and elements ρ_1 and μ_1. Then $(\rho_1\mu_1)^2 = \mu_3^2 = id$, but $\rho_1^2\mu_1^2 = \rho_2 id = \rho_2$.

Exercise 17.
Three examples of groups with eight elements are: $(\mathbb{Z}_8, +)$, D_4, and Q_8. Firstly $(\mathbb{Z}_8, +)$ is abelian, while D_4 and Q_8 are not. To compare D_4 and Q_8, we can look at the nontrivial proper subgroups. For Q_8, we have $\{1, -1\}, \{1, I, -1, -I\}, \{1, J, -1, -J\}$, and $\{1, K, -1, -K\}$. For D_4, we have $\{1, \rho_2\}, \{1, \mu_1\}, \{1, \delta_1\}, \{1, \mu_2\}, \{1, \delta_2\}, \{1, \rho_1, \rho_2, \rho_3\}, \{1, \rho_2, \mu_1, \mu_2\}$, and $\{1, \rho_2, \delta_1, \delta_2\}$. These subgroups are different, so the groups are different.

Exercise 18.

Proof. There are $n!$ permutations of a set containing n items.
Let $\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}$. Then we have n ways to choose a_1, $n-1$ ways to choose a_2, $\ldots$, 2 ways to choose a_{n-1}, and 1 way to choose a_n. Thus we have $(n)(n-1)\ldots(2)(1) = n!$ ways to permute n items. □

Exercise 19.
Let $n \in \mathbb{Z}^+$ and $a \in \mathbb{Z}_n$. By definition of modular congruence, $x \equiv y \pmod{n} \iff n \mid (x - y)$. Since $n \mid 0$, $n \mid (0 + a - a)$, so $0 + a \equiv a \pmod{n}$. Similarly, $n \mid (a + 0 - a)$, so $a + 0 \equiv a \pmod{n}$.

Exercise 20.
Similar to above, $n \mid 0$ so $n \mid (a \cdot 1 - a) \iff a \cdot 1 \equiv a \pmod{n}$.

Exercise 21.
Let $b = n - a$. Then $n \mid n \iff n \mid (a + n - a - 0) \iff n \mid (a + b - 0) \iff a + b \equiv 0 \pmod{n}$. A similar argument shows $b + a \equiv 0 \pmod{n}$.

Exercise 22.
Let $a, b, c \in \mathbb{Z}_n$. Then $(a+b)+c \equiv x \pmod{n} \iff n \mid (a+b)+c-x \iff n \mid a+(b+c)-x \iff a+(b+c) \equiv x \pmod{n}$. We also have $(ab)c \equiv x \pmod{n} \iff n \mid (ab)c-x \iff n \mid a(bc)-x \iff a(bc) \equiv x \pmod{n}$.

Exercise 23.
We have $n \mid 0 \iff n \mid (ab + ac - (ab + ac)) \iff n \mid (a(b + c) - (ab + ac)) \iff a(b + c) \equiv ab + ac \pmod{n}$.

Exercise 24. *Note:* This proof uses the identity in exercise 26.

Proof. $ab^na^{-1} = (aba^{-1})^n$ for $n \in \mathbb{Z}$, where a and b are elements in a group G.
Let $a, b \in G$.

We will first show that the identity holds for $n \in \mathbb{Z}^+ \cup \{0\}$ by induction on $\mathbb{Z}^+ \cup \{0\}$.
(Base Case): $n = 0$. Then $ab^0a^{-1} = aa^{-1} = e = (aba^{-1})^0$.
(Inductive Step): Assume $ab^na^{-1} = (aba^{-1})^n$ for some $n \in \mathbb{Z}^+ \cup \{0\}$. We want to show that $ab^{n+1}a^{-1} = (aba^{-1})^{n+1}$.
We have $(aba^{-1})^{n+1} = (aba^{-1})^n(aba^{-1})$. Applying the inductive hypothesis, we have $(aba^{-1})^n(aba^{-1}) = ab^na^{-1}aba^{-1} = ab^nba^{-1} = ab^{n+1}a^{-1}$, as desired.
By the principle of mathematical induction, $ab^na^{-1} = (aba^{-1})^n$ for all $n \in \mathbb{Z}^+ \cup \{0\}$.

Next, we will show that if $ab^na^{-1} = (aba^{-1})^n$ for some $n \in \mathbb{Z}^+ \cup \{0\}$, then $ab^{-n}a^{-1} = (aba^{-1})^{-n}$.
We have $(aba^{-1})^{-n} = ((aba^{-1})^{-1})^n = ((a^{-1})^{-1}b^{-1}a^{-1})^n = (ab^{-1}a^{-1})^n = ab^{-n}a^{-1}$, as desired.

Thus $ab^na^{-1} = (aba^{-1})^n$ for $n \in \mathbb{Z}$. □

Exercise 25.

Proof. For any $n > 2$, there exists $k \in U(n)$ such that $k^2 = 1$ and $k \neq 1$.
Consider $k = n-1$. Since $\gcd(n, n-1) \mid (n-(n-1)) \iff \gcd(n, n-1) \mid 1$, we can conclude that $\gcd(n, n-1) = 1$. Since k is relatively prime to n, $k \in U(n)$. Then $k^2 = n^2 - 2n + 1$, so $k^2 \equiv 1 \pmod{n}$. If $k = 1$, then $n \mid (n-2)$, forcing $n = 2$, contradicting $n > 2$. Therefore $k \neq 1$. □

Exercise 26.

Proof. $(g_1g_2 \dots g_{n-1}g_n)^{-1} = g_n^{-1}g_{n-1}^{-1} \dots g_2^{-1}g_1^{-1}$ for all $n \in \mathbb{Z}^+$.
We prove this by induction on $\mathbb{Z}^+$.
(Base Case): $n = 1$. $g_1^{-1} = g_1^{-1}$.
(Inductive Step): Assume $(g_1g_2 \dots g_{n-1}g_n)^{-1} = g_n^{-1}g_{n-1}^{-1} \dots g_2^{-1}g_1^{-1}$ for some $n \in \mathbb{Z}^+$. We want to show that $(g_1g_2 \dots g_{n-1}g_ng_{n+1})^{-1} = g_{n+1}^{-1}g_n^{-1}g_{n-1}^{-1} \dots g_2^{-1}g_1^{-1}$. We have $g_{n+1}^{-1}g_n^{-1}g_{n-1}^{-1} \dots g_2^{-1}g_1^{-1} = g_{n+1}^{-1}(g_1g_2 \dots g_{n-1}g_n)^{-1} = (g_1g_2 \dots g_{n-1}g_ng_{n+1})^{-1}$, from the identity that $(ab)^{-1} = b^{-1}a^{-1}$. □

Exercise 27.

Proof. If G is a group and $a, b \in G$, $xa = b$ has unique solutions in G.
Suppose that $xa = b$. We must show that such an x exists. Multiplying both sides of $xa = b$ by a^{-1}, we have $x = xe = xaa^{-1} = ba^{-1}$. To show uniqueness, suppose that x_1 and x_2 are both solutions of $xa = b$; then $x_1a = b = x_2a$. So $x_1 = x_1aa^{-1} = x_2aa^{-1} = x_2$. □

Exercise 28.
Note: I really don't want to do this proof because it's straightforward and boring, sorry :/

Exercise 29.

Proof. $ab = ac \implies b = c$ and $ba = ca \implies b = c$ for $a, b, c \in G$ in a group G.
Start with $ab = ac$. Multiply the left sides by a^{-1}, giving $a^{-1}ab = a^{-1}ac \implies b = c$. A similar argument follows for the other case. □

Exercise 30.

Proof. For a group G, if $a^2 = e$ for all $a \in G$, then G is abelian.
Assume $a^2 = e$ for all $a \in G$. Let $a, b \in G$. Since $(ab)^2 = e$, it follows that $(ab)(ab) = e$. Then $abab = e$. Multiplying by a on the left and b on the right, we get that $ba = ab$. Since a and b commute, then G is abelian. □

Exercise 31.

Proof. If G is a finite group of even order, then there exists an $a \in G$ such that $a \neq e$ and $a^2 = e$.
Assume for the sake of contradiction that no $a \in G$ satisfies $a \neq e$ and $a^2 = e$. Partition $G \setminus \{e\}$ into sets of a and its inverse. Since $a \neq a^{-1}$, these sets all have two elements. However, the number of elements in $G \setminus \{e\}$ is odd, so there is no way we were able to partition all elements into sets. Therefore there must exist an element $a \in G$ such that $a^2 = e$. □

Exercise 32.

Proof. If G is a group, and $(ab)^2 = a^2b^2$ for all $a, b \in G$, then G is abelian.
Let $a, b \in G$. Since $(ab)^2 = abab = aabb$, we can multiply the left side by a^{-1} and the right side by b^{-1} to get that $ba = ab$, so G is abelian. □

Exercise 33.
The subgroups of $\mathbb{Z}_3 \times \mathbb{Z}_3$ are $\{(0,0)\}$, $\{(0,0),(0,1),(0,2)\}$, $\{(0,0),(1,0),(2,0)\}$, $\{(0,0),(1,1),(2,2)\}$, $\{(0,0),(1,2),(2,1)\}$, and $\mathbb{Z}_3 \times \mathbb{Z}_3$.
The subgroups of $\mathbb{Z}_9$ are $\{0\}$, $\{0,3,6\}$, and $\mathbb{Z}_9$.

Exercise 34.
The subgroups of the symmetry group of an equilateral triangle are $\{id\}$, $\{id, \mu_1\}$, $\{id, \mu_2\}$, $\{id, \mu_3\}$, $\{id, \rho_1, \rho_2\}$, and the whole group.

Exercise 35.
Just like in exercise 17, we have $\{1\}$, $\{1, \rho_2\}$, $\{1, \mu_1\}$, $\{1, \delta_1\}$, $\{1, \mu_2\}$, $\{1, \delta_2\}$, $\{1, \rho_1, \rho_2, \rho_3\}$, $\{1, \rho_2, \mu_1, \mu_2\}$, $\{1, \rho_2, \delta_1, \delta_2\}$, and the whole group.

Exercise 36.

Proof. $H = \{2^k : k \in \mathbb{Z}\}$ is a subgroup of $\mathbb{Q}^*$.
We have $k = 0$ gives $2^0 = 1 \in H$. Let $a, b \in H$, with $a = 2^m$ and $b = 2^n$ for some $m, n \in \mathbb{Z}$. Then $ab^{-1} = 2^m 2^{-n} = 2^{m-n} \in H$. Thus H is a subgroup of $\mathbb{Q}^*$. □

Exercise 37.
Let $n \geq 0$ and $n\mathbb{Z} = \{nk : k \in \mathbb{Z}\}$.

Proof. $n\mathbb{Z}$ is a subgroup of $\mathbb{Z}$, and these subgroups are the only subgroups.
We have $k = 0$ gives $n \cdot 0 = 0 \in n\mathbb{Z}$. Let $a, b \in n\mathbb{Z}$ with $a = nk$ and $b = nl$ for some $k, l \in \mathbb{Z}$. Then $a - b = nk - nl = n(k - l) \in n\mathbb{Z}$.

Let H be a subgroup of $\mathbb{Z}$. We will show that $H = n\mathbb{Z}$ for some $n \geq 0$. If $H = \{0\}$ then $H = 0\mathbb{Z}$. Otherwise, H contains some positive integer. Since H is a subset of $\mathbb{Z}$, by the well-ordering principle, H contains a smallest positive integer. Call this integer n.
We will show that $H = n\mathbb{Z}$.

($\supseteq$) Since H is a subgroup of $\mathbb{Z}$, it is closed under addition. Since $n \in H$, $nk \in H$ for all $k \in \mathbb{Z}$.
($\subseteq$) Let $m \in H$. Then we can use the division algorithm to write $m = qn + r$ with $0 \leq r < n$. Since H is closed under subtraction, $r = m - qn \in H$. Since n was the smallest positive integer of H, r must be 0. Therefore $m \in n\mathbb{Z}$.
Therefore, $H = n\mathbb{Z}$, and since H was an arbitrary subgroups of $\mathbb{Z}$, all subgroups are in the form $n\mathbb{Z}$. $\square$

Exercise 38.

Proof. $\mathbb{T} = \{z \in \mathbb{C}^* : |z| = 1\}$ is a subgroup of $\mathbb{C}^*$.
The identity $z = 1 + 0i \in \mathbb{T}$. Let $z, w \in \mathbb{T}$ with $z = a + bi$ and $w = c + di$. Then $zw^{-1} = \dfrac{(a+bi)(c-di)}{c^2+d^2} = (a+bi)(c-di) = (ac+bd) + (bc-ad)i$.
We can see that $|zw^{-1}| = \sqrt{(ac+bd)^2 + (bc-ad)^2} = \sqrt{(a^2+b^2)(c^2+d^2)} = 1$, so $zw^{-1} \in \mathbb{T}$. $\square$

Exercise 39.
Let G consist of the 2×2 matrices of the form $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, where $\theta \in \mathbb{R}$.

Proof. G is a subgroup of $SL_2(\mathbb{R})$.
We have $\theta = 0$ gives $I \in G$. Let $A, B \in G$ with A given by θ and B given by ϕ. Then

$$\begin{aligned} AB^{-1} &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta\cos\phi + \sin\theta\sin\phi & \cos\theta\sin\phi - \sin\theta\cos\phi \\ \sin\theta\cos\phi - \cos\theta\sin\phi & \sin\theta\sin\phi + \cos\theta\cos\phi \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta-\phi) & -\sin(\theta-\phi) \\ \sin(\theta-\phi) & \cos(\theta-\phi) \end{bmatrix} \in G \end{aligned}$$

$\square$

Exercise 40.

Proof. $G = \{a + b\sqrt{2} : a, b \in \mathbb{Q} \text{ and } a, b \neq 0\}$ is a subgroup of $\mathbb{R}^*$ under multiplication.
We have $a = 1, b = 0$ is the identity and is in G. Let $A, B \in G$ with $A = a + q\sqrt{2}$ and $B = b + r\sqrt{2}$. Then $AB^{-1} = \dfrac{(a+q\sqrt{2})(b-r\sqrt{2})}{b^2-2r^2} = \dfrac{ab-2rq}{b^2-2r^2} + \dfrac{bq-ar}{b^2-2r^2}\sqrt{2} \in G$. $\square$

Exercise 41. Let G be the group of 2×2 matrices under addition.

Proof. $H = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + d = 0 \right\}$ is a subgroup of G.
The matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in H$ is the identity element. If you have two matrices $A = \begin{bmatrix} a & x \\ y & -a \end{bmatrix}$ and $B = \begin{bmatrix} b & z \\ w & -b \end{bmatrix}$, then $A - B = \begin{bmatrix} a-b & x-z \\ y-w & b-a \end{bmatrix}$ and we can see that $a - b + b - a = 0$ so $A - B \in H$. $\square$

Exercise 42.

Proof. $SL_2(\mathbb{Z})$ is a subgroup of $SL_2(\mathbb{R})$.
The matrix I is in $SL_2(\mathbb{Z})$ and serves as the identity. To show that for any $A, B \in SL_2(\mathbb{Z})$ that $AB^{-1} \in SL_2(\mathbb{Z})$, we can observe that the inverse of B is given by $B^{-1} = \dfrac{1}{\det B}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, but since $\det B = 1$, B^{-1} only has integer coefficients. We can then convince ourselves that $AB^{-1} \in SL_2(\mathbb{Z})$. □

Exercise 43.
Like in exercise 17, the subgroups of Q_8 are $\{1\}$, $\{1, -1\}$, $\{1, I, -1, -I\}$, $\{1, J, -1, -J\}$, $\{1, K, -1, -K\}$, and Q_8.

Exercise 44.
Let H_1 and H_2 be subgroup of G.

Proof. $H_1 \cap H_2$ is a subgroup of G.
Since H_1 and H_2 are both subgroups of G, we know that the identity e is in both subgroups, so $e \in H_1 \cap H_2$.
Let $g, h \in H_1 \cap H_2$. We want to show that $gh^{-1} \in H_1$ and $gh^{-1} \in H_2$. Since $g, h \in H_1 \cap H_2$, then $g, h \in H_1$. Since H_1 is a group, $gh^{-1} \in H_1$ under closure. A similar argument follows for H_2. □

Exercise 45.
Claim: If H and K are subgroups of a group G, then $H \cup K$ is a subgroup of G.

Proof. This is false.
Consider the subgroups of the symmetries of a triangle. Two such subgroups are $\{id, \mu_1\}$ and $\{id, \mu_2\}$. We can see that the union $\{id, \mu_1, \mu_2\}$ is not a subgroup since $\mu_1\mu_2^{-1} = \rho_2 \notin \{id, \mu_1, \mu_2\}$. □

Exercise 46.
Note: I'm lazy and I don't want to do the proof that it is false if G is not abelian. Consider S_3.
Claim: If H and K are subgroups of a group G, then $HK = \{hk : h \in H \text{ and } k \in K\}$ is a subgroup of G.

Proof. If G is abelian, the claim is true.
Assume G is abelian. Consider arbitrary H and K that are subgroups of G. Since $e \in H$ and $e \in K$, then $ee = e \in HK$. Then consider $g = ab, h = cd \in HK$. We see that $gh^{-1} = abd^{-1}c^{-1}$. Since G is abelian, this is equal to $ac^{-1}bd^{-1} \in HK$, as desired. □

Exercise 47.
Let G be a group and $Z(G) = \{x \in G : gx = xg \text{ for all } g \in G\}$.

Proof. $Z(G)$ is a subgroup of G.
We can see that $id \in Z(G)$ because $id \circ g = g \circ id = g$ for all $g \in G$. Suppose $a, b \in Z(G)$. Let $g \in G$. Then $ab^{-1}g = agb^{-1}$ because $b \in \mathbb{Z}(G)$ and $agb^{-1} = gab^{-1}$ because $a \in Z(G)$. □

Exercise 48.

Proof. Let $a, b \in G$. If $a^4b = ba$ and $a^3 = e$, then $ab = ba$.
We can see that $a^4b = ba \iff a^3ab = ba \iff eab = ba \iff ab = ba$. □

Exercise 49.
$(\mathbb{Z}, +)$ is an infinite group and $n\mathbb{Z}$ are the only subgroups of $\mathbb{Z}$, and are infinite (besides $\{0\}$, which is trivial) by exercise 37.

Exercise 50.
Note: I don't know how to do this problem :D

Exercise 51.

Proof. If $xy = x^{-1}y^{-1}$ for all $x, y \in G$, then G must be abelian.
Let $a, b \in G$. We can see that $ab = (b^{-1}a^{-1})^{-1} = (ab)^{-1} = b^{-1}a^{-1}$. Finally, we apply the assumption, so $ab = ba$. □

Exercise 52.

Proof. If $(xy)^2 = xy$ for all $x, y \in G$, then G must be abelian.
We can see that $xy = e$ for all $x, y \in G$. Taking $x = e$, we can see that $y = e$ for all $y \in G$. Therefore $G = \{e\}$, and since the trivial group is abelian, G is abelian. □

Exercise 53.
Claim: If G is a nonabelian group, then every nontrivial subgroup of G is nonabelian.

Proof. This is false.
Consider the symmetries of a triangle. We know that it is nonabelian. However, consider the rotation subgroup $\{e, \rho_1, \rho_2\}$. We can verify this subgroup is abelian, since $\rho_1\rho_2 = \rho_2\rho_1$. □

Exercise 54.
Let H be a subgroup of G and

$$C(H) = \{g \in G : gh = hg \text{ for all } h \in H\}$$

Proof. $C(H)$ is a subgroup of G.
We can see that $eh = he = h$ for all $h \in H$, so $e \in C(H)$. Let $a, b \in C(H)$, and let $h \in H$. Then $ab^{-1}h = ahb^{-1}$ because $h \in H$, and likewise $ahb^{-1} = hab^{-1}$. Since $ab^{-1}h = hab^{-1}$, $ab^{-1} \in C(H)$. □

Exercise 55.

Proof. If H is a subgroup of G, and $g \in G$, then gHg^{-1} is also a subgroup of G.
We know that $e \in H$, so $geg^{-1} = e \in gHg^{-1}$. Let $a, b \in gHg^{-1}$ with $a = gxg^{-1}$ and $b = gyg^{-1}$. Then $ab^{-1} = gxg^{-1}gy^{-1}g^{-1} = gxy^{-1}g^{-1}$, and since $xy^{-1} \in H$ because H is a group, then $ab^{-1} \in gHg^{-1}$. □