Chapter 3

Exercise 1.

(a) False. $U(8)$ is not cyclic. We can verify this manually by noticing that the order of every element is 2, and so no individual element can generate the group $U(8)$ of order 4.

(b) False. The generators of Z_{60} are all of the integers $1 \leq r < 60$ such that $\gcd(r, n) = 1$, and we can notice that $\gcd(49, 60) = 1$ but 49 is composite.

(c) False. Any element a that could be a generator for $\mathbb{Q}$, the element $a/2$ is not generated by a.

(d) False. Consider S_3. Every subgroup is cyclic, but the group is not.

(e) True. Suppose G is infinite. Consider an element $g \in G$. $\langle g \rangle$ cannot be finite, since that would imply there are infinitely many subgroups. But $\langle g \rangle$ cannot be infinite, since then $\langle g^n \rangle$ would provide infinite subgroups over all $n \in \mathbb{Z}^+$. Therefore if there are finitely many subgroups the group must be finite.

Exercise 2.

(a) $|5| = 12$

(b) $|\sqrt{3}| = \infty$

(c) $|\sqrt{3}| = \infty$

(d) $|-i| = 4$

(e) $|72| = 10$

(f) $|312| = 157$

Exercise 3.

(a) $\{\ldots, -14, -7, 0, 7, 14, \ldots\}$

(b) $\{0, 3, 6, 9, 12, 15, 18, 21\}$

(c) $\{0\}$, $\{0, 6\}$, $\{0, 4, 8\}$, $\{0, 3, 6, 9\}$, $\{0, 2, 4, 6, 8, 10\}$, $\mathbb{Z}_{12}$

(d) $\{0\}$, $\{0, 30\}$, $\{0, 20, 40\}$, $\{0, 15, 30, 45\}$, $\{0, 12, 24, 36, 48\}$, $\{0, 10, 20, 30, 40, 50\}$, $\{0, 6, 12, \ldots, 48, 54\}$, $\{0, 5, 10, \ldots, 50, 55\}$, $\{0, 4, 8, \ldots, 52, 56\}$, $\{0, 3, 6, \ldots, 54, 57\}$, $\{0, 2, 4, \ldots, 56, 58\}$, $\mathbb{Z}_{60}$

(e) $\{0\}$, $\mathbb{Z}_{13}$

(f) $\{0\}$, $\{0, 24\}$, $\{0, 16, 32\}$, $\{0, 12, 24, 36\}$, $\{0, 8, 16, 24, 32, 40\}$, $\{0, 6, 12, \ldots, 36, 42\}$, $\{0, 4, 8, \ldots, 40, 44\}$, $\{0, 3, 6, \ldots, 42, 45\}$, $\{0, 2, 4, \ldots, 44, 46\}$, $\mathbb{Z}_{48}$

(g) $\{1, 3, 7, 9\}$

(h) $\{1, 5, 7, 11, 13, 17\}$

(i) $\{\ldots, 1/49, 1/7, 1, 7, 49, \ldots\}$

(j) $\{1, i, -1, -i\}$

(k) $\{\ldots - 1/4, -i/2, 1, 2i, -4\}$

(l) $\{1, e^{i\pi/4}, e^{i\pi/2}, \ldots, e^{3i\pi/2}, e^{7i\pi/8}\}$

(m) $\{1, e^{i\pi/3}, e^{2i\pi/3}, e^{i\pi}, e^{4i\pi/3}, e^{5i\pi/3}\}$

Exercise 4.

(a) $\left\{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}, \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\right\}$

(b) $\{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1/3\\ 3 & 0\end{bmatrix}\}$

(c) $\left\{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1 & -1\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & -1\\ 1 & -1\end{bmatrix}, \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}, \begin{bmatrix}-1 & 1\\ -1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ -1 & 1\end{bmatrix}\right\}$

(d) $\left\{\ldots, \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}, \ldots, \begin{bmatrix}1 & -n\\ 0 & 1\end{bmatrix}\right\}$

(e) $\left\{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1 & -1\\ -1 & 0\end{bmatrix}, \begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}, \begin{bmatrix}3 & -2\\ -2 & 1\end{bmatrix}, \begin{bmatrix}5 & -3\\ -3 & 2\end{bmatrix}, \ldots, \begin{bmatrix}\phi_{n+2} & -\phi_{n+1}\\ -\phi_{n+1} & \phi_n\end{bmatrix}, \ldots\right\}$ where ϕ_n is the nth Fibonacci number

(f) $\left\{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}\sqrt{3}/2 & 1/2\\ -1/2 & \sqrt{3}/2\end{bmatrix}, \begin{bmatrix}1/2 & \sqrt{3}/2\\ -\sqrt{3}/2 & 1/2\end{bmatrix}, \ldots, \begin{bmatrix}\sqrt{3}/2 & -1/2\\ 1/2 & \sqrt{3}/2\end{bmatrix}\right\}$

Exercise 5.
0: 1; 1: 18; 2: 9; 3: 6; 4: 9; 5: 18; 6: 3; 7: 18; 8: 9; 9: 2; 10: 9; 11: 18; 12: 3; 13: 18; 14: 9; 15: 6; 16: 9; 17: 18

Exercise 6.
$id : 1$; $\rho_1 : 4$; $\rho_2 : 2$; $\rho_3 : 4$; $\mu_1 : 2$; $\mu_2 : 2$; $\delta_1 : 2$; $\delta_2 : 2$

Exercise 7.
$\{1\}, \{1, -1\}, \{1, I, -1, -I\}, \{1, J, -1, -J\}, \{1, K, -1, -K\}$

Exercise 8.
$\{1\}, \{1, 7, 13, 19\}, \{1, 11\}, \{1, 17, 19, 23\}, \{1, 19\}, \{1, 29\}$

Exercise 9.
4, 12, 20, 28

Exercise 10.

(a) 0

(b) 1, -1

(c) 1, -1

Exercise 11.
$|a| = 1, 2, 3, 4, 6, 8, 12, 24$

Exercise 12.
Cyclic group with 1 generator: $\mathbb{Z}_2$
Cyclic group with 2 generators: $\mathbb{Z}_3$
Cyclic group with 4 generators: $\mathbb{Z}_5$
It is impossible to find a cyclic group for any n generators, since the number of generators of $\mathbb{Z}_m$ is given by $\phi(m)$, Euler's totient function. There are some n for which $\phi(m) = n$ does not exist (the first being 14).

Exercise 13.
The only groups $U(n)$ for $n \leq 20$ that are not cyclic are $n = 8, 12, 15, 16, 20$.

Proof. $U(n)$ will be cyclic if and only if $n = 1, 2, 4, p^k$, or $2p^k$, where p is an odd prime number.
$(\Longleftarrow)$ Cases for $n = 1, 2, 4$ are trivial.

□

Exercise 14.
See from exercise 4 that A and B have finite orders from parts (a) and (c) respectively. We can notice that AB gives us the matrix in part (d), which did not have finite order.