Proposition 1. *For all types A, the type $S^2 \to A$ is equal to*

$$\sum_{x:A} \mathrm{refl}_x = \mathrm{refl}_x.$$

Proposition 2. *Suppose that A is a type, B a family of types over A and P a family of mere propositions over A, such that we have a family of functions $B(x) \to P(x)$ for all $x : A$. We then have an equality of types*

$$\sum_{x:A} B(x) = \sum \Big(t : \sum_{x:A} P(x)\Big) B(\mathrm{pr}_1 t).$$

A function f is said to be *invertible* if it admits a two-sided inverse. We define the type of invertibility proofs of f to be the type

$$\mathrm{inverse}(f) :\equiv \sum_{g:B\to A} (f \circ g = \mathrm{id}) \times (g \circ f = \mathrm{id})$$

and denote the type of all invertible maps between two types A and B by $\mathrm{Inv}(A, B) :\equiv \sum_{f:A\to B} \mathrm{inverse}(f)$.

A function f is said to be an *equivalence* if it admits both a left and a right sided inverse. We define the type of equivalence proofs of f to be the type

$$\mathrm{equiv}(f) :\equiv \sum_{g:B\to A} (f \circ g = \mathrm{id}) \times \sum_{h:B\to A} (h \circ f = \mathrm{id})$$

and similarly denote the type of all equivalences between two types A and B by $A \simeq B$.

We recall the following two important facts:

Proposition 3. *For all functions f, the type of equivalence proofs of f is a mere proposition.*

Proposition 4. *For all functions f, the types of invertibility and equivalence proofs of f are logically equivalent. In other words, a function is invertible if and only if it is an equivalence.*

Lemma 5. *Suppose A is a type. The type of invertibility proofs of id_A is equal to $\mathrm{id}_A = \mathrm{id}_A$.*

Proof. By reassociating the dependent sum type, we see that the type of invertibility proofs of id_A is equal to

$$\sum \Big(G : \sum_{g:A\to A} (g = \mathrm{id}_A)\Big) \left(\mathrm{pr}_1 G = \mathrm{id}_A\right).$$

Since the type $\sum_{g:A\to A} (g = \mathrm{id}_A)$ is contractible onto $(\mathrm{id}_A, \mathrm{refl})$, we see the whole type is equal to $\mathrm{pr}_1\,(\mathrm{id}_A, \mathrm{refl}) = \mathrm{id}_A$, which is itself equal to $\mathrm{id}_A = \mathrm{id}_A$. □

Proposition 6. *The type* $\mathrm{Inv}(A, B)$ *is equal to the type* $\sum_{e:A\simeq B} \mathrm{pr}_1 e = \mathrm{pr}_1 e$ *for all types* A *and* B*.*

Proof. We first view inverse and equiv as two families of types over $A \to B$. Proposition 3 then tells us that equiv is a family of mere propositions over $A \to B$ whereas proposition 4 in particular tells us that we have a family of functions $\mathrm{inverse}(f) \to \mathrm{equiv}(f)$ for all functions $f : A \to B$. By proposition 2, it then follows that the type $\mathrm{Inv}(A, B)$ is equal to the type $\sum_{e:A\simeq B} \mathrm{inverse}(\mathrm{pr}_1 e)$.

To construct the desired equality, it thus suffices to construct an equality between the types $\sum_{e:A\simeq B} \mathrm{inverse}(\mathrm{pr}_1 e)$ and $\sum_{e:A\simeq B} \mathrm{pr}_1 e = \mathrm{pr}_1 e$. By the fiberwise equivalence construction, it further suffices to construct an equality between $\mathrm{inverse}(\mathrm{pr}_1 e)$ and $\mathrm{pr}_1 e = \mathrm{pr}_1 e$ for all equivalences $e : A \simeq B$. The result then follows by equivalence induction and Lemma 5. □

Theorem 7. *The type* $\sum_{A,B:\mathcal{U}} \mathrm{Inv}(A, B)$ *is equal to the type* $S^2 \to \mathcal{U}$*.*

Proof. We first quantify the equality in Proposition 6 over $B : \mathcal{U}$, obtaining an equality between $\sum_{B:\mathcal{U}} \mathrm{Inv}(A, B)$ and $\sum_{B:\mathcal{U}} \sum_{e:A\simeq B} \mathrm{pr}_1 e = \mathrm{pr}_1 e$. Now, since the type $\sum_{B:\mathcal{U}} A \simeq B$ is contractible onto (A, Id_A), the second type is equal to $\mathrm{id}_A = \mathrm{id}_A$, which is itself equal to $\mathrm{refl}_A = \mathrm{refl}_A$ by univalence. Now quantifying this equality over $A : \mathcal{U}$, we obtain an equality between $\sum_{A,B:\mathcal{U}} \mathrm{Inv}(A, B)$ and $\sum_{A:\mathcal{U}} \mathrm{refl}_A = \mathrm{refl}_A$, which is equal to $S^2 \to \mathcal{U}$ by the universal property of the sphere. □