

1 Basic construction

Proposition 1. *For all types A , the type $\mathbb{S}^2 \rightarrow A$ is equal to*

$$\sum_{x:A} \text{refl}_x = \text{refl}_x.$$

Proposition 2. *Suppose that A is a type, B a family of types over A and P a family of mere propositions over A , such that we have a family of functions $B(x) \rightarrow P(x)$ for all $x : A$. We then have an equality of types*

$$\sum_{x:A} B(x) = \sum_{x:A} \left(t : \sum_{x:A} P(x) \right) B(\text{pr}_1 t).$$

A function f is said to be *invertible* if it admits a two-sided inverse. We define the type of invertibility proofs of f to be the type

$$\text{inverse}(f) := \sum_{g:B \rightarrow A} (f \circ g = \text{id}) \times (g \circ f = \text{id})$$

and denote the type of all invertible maps between two types A and B by $\text{Inv}(A, B) := \sum_{f:A \rightarrow B} \text{inverse}(f)$.

A function f is said to be an *equivalence* if it admits both a left and a right sided inverse. We define the type of equivalence proofs of f to be the type

$$\text{isequiv}(f) := \sum_{g:B \rightarrow A} (f \circ g = \text{id}) \times \sum_{h:B \rightarrow A} (h \circ f = \text{id})$$

and similarly denote the type of all equivalences between two types A and B by $A \simeq B$.

We recall the following two important facts:

Proposition 3. *For all functions f , the type of equivalence proofs of f is a mere proposition.*

Proposition 4. *For all functions f , the types of invertibility and equivalence proofs of f are logically equivalent. In other words, a function is invertible if and only if it is an equivalence.*

Lemma 5. *Suppose A is a type. The type of invertibility proofs of id_A is equal to $\text{id}_A = \text{id}_A$.*

Proof. By reassociating the dependent sum type, we see that the type of invertibility proofs of id_A is equal to

$$\sum_{g:A \rightarrow A} \left(G : \sum_{g:A \rightarrow A} (g = \text{id}_A) \right) (\text{pr}_1 G = \text{id}_A).$$

Since the type $\sum_{g:A \rightarrow A} (g = \text{id}_A)$ is contractible onto $(\text{id}_A, \text{refl})$, we see the whole type is equal to $\text{pr}_1(\text{id}_A, \text{refl}) = \text{id}_A$, which is itself equal to $\text{id}_A = \text{id}_A$. \square

Proposition 6. *The type $\text{Inv}(A, B)$ is equal to the type $\sum_{e:A=B} e = e$ for all types A and B .*

Proof. We view inverse and isequiv as two families of types over $A \rightarrow B$. Proposition 3 then tells us that isequiv is a family of mere propositions over $A \rightarrow B$, whereas proposition 4 in particular tells us that we have a family of functions $\text{inverse}(f) \rightarrow \text{isequiv}(f)$ for all functions $f : A \rightarrow B$. By proposition 2, it then follows that the type $\text{Inv}(A, B)$ is equal to the type $\sum_{e:A \simeq B} \text{inverse}(\text{pr}_1 e)$.

On the other hand, we first observe that the type $\sum_{e:A=B} e = e$ is equal to the type $\sum_{e:A \simeq B} e = e$ by univalence. Now, since isequiv is a family of mere propositions, $A \simeq B$ is a subtype of $A \rightarrow B$. By the subtype identity principle, it follows that the type $e = e$ is equal to the type $\text{pr}_1 e = \text{pr}_1 e$ for all equivalences $e : A \simeq B$.

To construct the desired equality, it thus suffices to construct an equality between the types $\sum_{e:A \simeq B} \text{inverse}(\text{pr}_1 e)$ and $\sum_{e:A \simeq B} \text{pr}_1 e = \text{pr}_1 e$. By the fiberwise equivalence construction, it further suffices to construct an equality between $\text{inverse}(\text{pr}_1 e)$ and $\text{pr}_1 e = \text{pr}_1 e$ for all equivalences $e : A \simeq B$. The result then follows by equivalence induction and Lemma 5. \square

Theorem 7. *The type $\sum_{A,B:\mathcal{U}} \text{Inv}(A, B)$ is equal to the type $\mathbb{S}^2 \rightarrow \mathcal{U}$.*

Proof. We first quantify the equality in Proposition 6 over $B : \mathcal{U}$, obtaining an equality between $\sum_{B:\mathcal{U}} \text{Inv}(A, B)$ and $\sum_{B:\mathcal{U}} \sum_{e:A=B} e = e$. Since the type $\sum_{B:\mathcal{U}} A = B$ is contractible onto (A, refl_A) , the second type is equal to $\text{refl}_A = \text{refl}_A$. Now quantifying over $A : \mathcal{U}$, we obtain an equality between $\sum_{A,B:\mathcal{U}} \text{Inv}(A, B)$ and $\sum_{A:\mathcal{U}} \text{refl}_A = \text{refl}_A$, which is equal to $\mathbb{S}^2 \rightarrow \mathcal{U}$ by the universal property of the sphere. \square

TODO: should this section be cut or be polished up to serve as motivation, introduction?

2 n -invertible maps

Definition 8. We define a notion of n -invertibility on $A \rightarrow B$ by induction on n . Let $f : A \rightarrow B$ be a function. We say that f is 0-invertible if there exists a function $g : B \rightarrow A$. We say that f is $(n+1)$ -invertible if there exist functions $g : B \rightarrow A$ and

$$r : \prod_{x:A} \prod_{y:B} (f(x) = y) \rightarrow (x = g(y)),$$

such that $r(x, y)$ is n -invertible for all x and y .

We thus define the type of $(n+1)$ -inverses to be

$$\text{inverse}_{n+1}(f) = \sum_{g:B \rightarrow A} \sum_{r:\text{Adj}(f,g)} \prod_{x:A} \prod_{y:B} \text{inverse}_n(r(x, y)),$$

where $\text{Adj}(f, g) = \prod_{x:A} \prod_{y:B} (f(x) = y) \rightarrow (x = g(y))$.

Remark 9. We note that 1-invertibility coincides with the ordinary notion of invertibility. By path induction, the data of a 1-invertible function $f : A \rightarrow B$ consists exactly of a function $g : B \rightarrow A$, together with homotopies of type $\prod_{x:A} x = g(f(x))$ and $\prod_{y:B} f(g(y)) = y$, witnessing that f is invertible.

Lemma 10. *The type $\text{inverse}_{n+1}(\text{id}_A)$ is equal to*

$$\prod_{x,y:A} \text{inverse}_n(\text{id}_{x=y})$$

for all $n : \mathbb{N}$.

Proof. TODO □

Proposition 11. *The type $\text{inverse}_n(\text{id}_A)$ is equal to $\prod_{x:A} \Omega^n(A, x)$ for all $n : \mathbb{N}$.*

Proof. We prove the result by induction on n . For the base case, we have

$$\text{inverse}_0(\text{id}_A) = (A \rightarrow A) = \prod_{x:A} (A, x) = \prod_{x:A} \Omega^0(A, x).$$

For the inductive step, we have

$$\begin{aligned} \text{inverse}_{n+1}(\text{id}_A) &= \prod_{x,y:A} \text{inverse}_n(\text{id}_{x=y}) = \prod_{x,y:A} \prod_{p:x=y} \Omega^n(x = y, p) \\ &= \prod_{x:A} \Omega^n(x = x, \text{refl}_x) = \prod_{x:A} \Omega^{n+1}(A, x). \end{aligned}$$

□

Proposition 12. *Let $n : \mathbb{N}$ and suppose $f : A \rightarrow B$ is $(n+1)$ -invertible. Then f is n -invertible.*

Proof. We again prove the result by induction on n . For the base case, suppose $f : A \rightarrow B$ is 1-invertible. We then have a function $g : B \rightarrow A$ together with some data, but g itself is enough to show that f is 0-invertible.

For the inductive step, suppose $f : A \rightarrow B$ is $(n+2)$ -invertible. We then have a function $g : B \rightarrow A$ and a certain dependent function r , such that $r(x, y)$ is $(n+1)$ -invertible for all $x, y : A$. By the inductive hypothesis, $r(x, y)$ is n -invertible for all $x, y : A$, showing that f is $(n+1)$ -invertible. □

Corollary 13. *Let $n : \mathbb{N}$ and suppose $f : A \rightarrow B$ is $(n+1)$ -invertible. Then f is an equivalence.*

Proof. Using the previous proposition, we can show that every $(n+1)$ -invertible function is 1-invertible. Since 1-invertibility coincides with invertibility, this shows that it is also an equivalence. □

Theorem 14. *Let \mathcal{U} be a universe and $A : \mathcal{U}$ a type. The type*

$$\sum_{B:\mathcal{U}} \sum_{f:A \rightarrow B} \text{inverse}_{n+1}(f)$$

of all $(n+1)$ -invertible maps with domain A is equal to $\Omega^{n+2}(\mathcal{U}, A)$.

Proof. Let $B : \mathcal{U}$ first be a type. Since $(n+1)$ -invertibility implies equivalence and equivalence is a mere proposition, we have

$$\sum_{f:A \rightarrow B} \text{inverse}_{n+1}(f) = \sum_{f:A \simeq B} \text{inverse}_{n+1}(\text{pr}_1 f).$$

Then, since the type $\sum_{B:\mathcal{U}} A \simeq B$ is contractible, it follows that

$$\sum_{B:\mathcal{U}} \sum_{f:A \simeq B} \text{inverse}_{n+1}(\text{pr}_1 f) = \text{inverse}_{n+1}(\text{id}_A).$$

By proposition 11, the second type is equal to $\prod_{x:A} \Omega^{n+1}(A, x)$. Finally, since dependent products commute with loop spaces, we have

$$\prod_{x:A} \Omega^{n+1}(A, x) = \Omega^{n+2}(\mathcal{U}, A).$$

TODO : why exactly is the final claim true? □

Corollary 15. *Let \mathcal{U} be a universe. The type $\sum_{A,B:\mathcal{U}} \sum_{f:A \rightarrow B} \text{inverse}_{n+1}(f)$ is equal to $\mathbb{S}^{n+2} \rightarrow \mathcal{U}$.*

Proof. Obtained by quantifying the equality in Theorem 14 over A and noting that $\sum_{A:\mathcal{U}} \Omega^{n+2}(\mathcal{U}, A) = \mathbb{S}^{n+2} \rightarrow \mathcal{U}$ by the universal property of the sphere. □

3 ∞ -invertible maps

Consider the diagram $\phi_n : \text{inverse}_{n+1}(f) \rightarrow \text{inverse}_n(f)$ defined in Proposition 12. We may explicitly define the ϕ_n to be

$$\begin{aligned} \phi_0(g, r, H) &= g \\ \phi_{n+1}(g, r, H) &= (g, r, \lambda x. \lambda y. \phi_n(H(x, y))). \end{aligned}$$

Taking the diagram's limit, we obtain a type $\text{inverse}_\infty(f)$, together with data

$$\begin{aligned} \psi_n &: \text{inverse}_\infty(f) \rightarrow \text{inverse}_n(f) \\ \alpha_n &: \phi_{n+1} \circ \psi_{n+1} = \psi_n. \end{aligned}$$

A function f , equipped with data of type $\text{inverse}_\infty(f)$, is said to be ∞ -invertible. Note that every ∞ -invertible function is an equivalence, since it is in particular 1-invertible by the data of the limit.

Proposition 16. *The type $\text{inverse}_\infty(f)$ is a mere proposition for all functions $f : A \rightarrow B$.*

Proof. We show that assuming f is ∞ -invertible, the type $\text{inverse}_\infty(f)$ is contractible. That is to say, we wish to show

$$\prod_{A,B:\mathcal{U}} \prod_{f:A \rightarrow B} \text{inverse}_\infty(f) \rightarrow \text{isContr}(\text{inverse}_\infty(f)).$$

Now, since ∞ -invertibility implies equivalence, it suffices to show

$$\prod_{A,B:\mathcal{U}} \prod_{f:A \rightarrow B} \text{isequiv}(f) \rightarrow \text{isContr}(\text{inverse}_\infty(\text{pr}_1 f)).$$

By equivalence induction, it then suffices to show only

$$\prod_{A:\mathcal{U}} \text{isContr}(\text{inverse}_\infty(\text{id}_A)).$$

TODO: reword the above

By definition, $\text{inverse}_\infty(\text{id}_A)$ is the limit of the diagram

$$\phi_n : \text{inverse}_{n+1}(\text{id}_A) \rightarrow \text{inverse}_n(\text{id}_A).$$

TODO: we could now try to use Proposition 11 somehow, this seems related to the contractibility of \mathbb{S}^∞ . \square

Proposition 17. *The type of ∞ -invertibility proofs satisfies the recursive equation*

$$\text{inverse}_\infty(f) = \sum_{g:B \rightarrow A} \sum_{r:\text{Adj}(f,g)} \prod_{x:A} \prod_{y:B} \text{inverse}_\infty(r(x,y))$$

for all $f : A \rightarrow B$.

Proof. We obtain the desired equality by showing the second type satisfies the universal property of the limit. Suppose then that P is a type, equipped with a cocone

$$\begin{aligned} \rho_n &: P \rightarrow \text{inverse}_n(f) \\ \beta_n &: \phi_{n+1} \circ \rho_{n+1} = \rho_n. \end{aligned}$$

TODO: not sure if this is a right approach. \square