

Define

$$\text{Adj}(f, g) := \prod_{x:A} \prod_{y:B} (f(x) = y \rightarrow (x = g(y))).$$

Suppose that  $g$  is a left inverse of  $f$ , witnessed by  $G : g \circ f \sim \text{id}_A$ . We then construct  $r_G : \text{Adj}(f, g)$ , given by induction by

$$r_G(x, f(x), \text{refl}_{f(x)}) = H(x).$$

This assignment induces an equivalence between  $g \circ f \sim \text{id}_A$  and  $\text{Adj}(f, g)$

Now, suppose that the type of left inverses of a function  $f$

$$\sum_{g:B \rightarrow A} g \circ f \sim \text{id}_A$$

is contractible onto  $(h, H)$ . By applying the equivalence between  $g \circ f \sim \text{id}_A$  and  $\text{Adj}(f, g)$ , we then obtain that the type

$$\sum_{g:B \rightarrow A} \text{Adj}(f, g)$$

is contractible onto  $(h, r_H)$ .

Consider the identity function  $\text{id}_A$ . Its type of left inverses is contractible onto  $(\text{id}_A, \lambda x. \text{refl}_x)$ . It then follows that the type

$$\sum_{g:A \rightarrow A} \text{Adj}(\text{id}_A, g)$$

is contractible onto  $(\text{id}_A, r_{\lambda x. \text{refl}_x})$ . We now claim that  $r_{\lambda x. \text{refl}_x}$  is equal to  $\lambda xy. \text{id}_{x=y}$ . By function extensionality, it suffices to show that

$$r_{\lambda x. \text{refl}_x}(x, y, p) = p$$

for all  $x, y : A$  and  $p : x = y$ . By path induction, it then suffices to show that

$$r_{\lambda x. \text{refl}_x}(x, x, \text{refl}_x) = \text{refl}_x$$

for all  $x : A$ , but this then follows by definition of  $r$ . We conclude that  $\sum_{g:A \rightarrow A} \text{Adj}(\text{id}_A, g)$  is contractible onto  $(\text{id}_A, \lambda xy. \text{id}_{x=y})$ .

With this, we can now show that  $\text{inverse}_{n+1}(\text{id}_A)$  is equal to  $\prod_{x,y:A} \text{inverse}_n(\text{id}_{x=y})$ . Unfolding  $\text{inverse}_{n+1}$ , we see that

$$\text{inverse}_{n+1}(\text{id}_A) = \sum_{g:A \rightarrow A} \sum_{r:\text{Adj}(\text{id}_A, g)} \prod_{x:A} \prod_{y:B} \text{inverse}_n(r(x, y)).$$

The result then follows by reassociating the sigma types and applying the above contractibility result.

In the formalisation I actually obtain the contractibility result by a more direct computational argument, but I thought the above may be nicer for a paper proof. As in the formalisation though, we can directly show that the type

$$\sum_{g:A \rightarrow A} \text{Adj}(\text{id}_A, g)$$

is contractible onto  $(\text{id}_A, \lambda xy. \text{id}_{x=y})$  in the following way:

We first characterise identifications in the above sigma type. To give an identification between  $(g, r)$  and  $(h, s)$ , we claim that it is sufficient to give a homotopy  $H : g \sim h$  and a witness to the commutative triangle

$$r(x, y, p) \cdot H(y) = s(x, y, p)$$

for all  $x, y : A$  and  $p : x = y$ . The claim follows by first applying homotopy induction on  $H$  and then noting that the commutativity data turns into just a homotopy between  $r$  and  $s$ , on which we may again apply homotopy induction.

Now let  $(g, r)$  be an arbitrary element of the above sigma type. Then

$$r : \prod_{x, y : A} (x = y) \rightarrow (x = g(y)),$$

so  $\lambda x. r(x, x, \text{refl}_x)$  gives a homotopy between  $\text{id}_A$  and  $g$ . To give an identification between  $(\text{id}_A, \lambda xy. \text{id}_{x=y})$  and  $(g, r)$ , the characterisation now tells us that it is sufficient to give an identification of type

$$p \cdot r(y, y, \text{refl}_y) = r(x, y, p)$$

for all  $x, y : A$  and  $p : x = y$ . This is then discharged by path induction on  $p$ .

Let me now turn to a potentially interesting thread, related to  $\infty$ -invertibility. Taking the coinductive definition of  $\infty$ -invertibility, we can unfold  $\text{inverse}_\infty(\text{id}_A)$  once to obtain

$$\text{inverse}_\infty(\text{id}_A) = \sum_{g:A \rightarrow A} \sum_{r:\text{Adj}(\text{id}_A, g)} \prod_{x:A} \prod_{y:B} \text{inverse}_\infty(r(x, y)).$$

As in the proof that  $\text{inverse}_{n+1}(\text{id}_A) = \prod_{x, y : A} \text{inverse}_n(\text{id}_{x=y})$ , we now obtain that  $\text{inverse}_\infty(\text{id}_A) = \prod_{x, y : A} \text{inverse}_\infty(\text{id}_{x=y})$ .

Now, the start of the proof of proposition 17 from the paper tells us that to show that  $\text{inverse}_\infty(f)$  is a proposition for all functions  $f$ , it suffices to show that  $\text{inverse}_\infty(\text{id}_A)$  is contractible. To understand  $\infty$ -invertibility, it thus seems crucial to understand the  $\infty$ -invertible structure of the identity function and by the above, this obeys the following curious recursive equation:

Define  $F(A) = \text{inverse}_\infty(\text{id}_A)$ . Then  $F(A) = \prod_{x, y : A} F(x = y)$ . Is there anything we can extract from this fact by itself?