1 Basic construction

Proposition 1. *For all types A, the type $\mathbb{S}^2 \to A$ is equal to*

$$\sum_{x:A} \mathrm{refl}_x = \mathrm{refl}_x.$$

Proposition 2. *Suppose that A is a type, B a family of types over A and P a family of mere propositions over A, such that we have a family of functions $B(x) \to P(x)$ for all $x : A$. We then have an equality of types*

$$\sum_{x:A} B(x) = \sum \Big(t : \sum_{x:A} P(x)\Big) B(\mathrm{pr}_1 t).$$

A function f is said to be *invertible* if it admits a two-sided inverse. We define the type of invertibility proofs of f to be the type

$$\mathrm{inverse}(f) :\equiv \sum_{g:B\to A} (f \circ g = \mathrm{id}) \times (g \circ f = \mathrm{id})$$

and denote the type of all invertible maps between two types A and B by $\mathrm{Inv}(A, B) :\equiv \sum_{f:A\to B} \mathrm{inverse}(f)$.

A function f is said to be an *equivalence* if it admits both a left and a right sided inverse. We define the type of equivalence proofs of f to be the type

$$\mathrm{isequiv}(f) :\equiv \sum_{g:B\to A} (f \circ g = \mathrm{id}) \times \sum_{h:B\to A} (h \circ f = \mathrm{id})$$

and similarly denote the type of all equivalences between two types A and B by $A \simeq B$.

We recall the following two important facts:

Proposition 3. *For all functions f, the type of equivalence proofs of f is a mere proposition.*

Proposition 4. *For all functions f, the types of invertibility and equivalence proofs of f are logically equivalent. In other words, a function is invertible if and only if it is an equivalence.*

Lemma 5. *Suppose A is a type. The type of invertibility proofs of id_A is equal to $\mathrm{id}_A = \mathrm{id}_A$.*

Proof. By reassociating the dependent sum type, we see that the type of invertibility proofs of id_A is equal to

$$\sum \Big(G : \sum_{g:A\to A} (g = \mathrm{id}_A)\Big) (\mathrm{pr}_1 G = \mathrm{id}_A).$$

Since the type $\sum_{g:A\to A} (g = \mathrm{id}_A)$ is contractible onto $(\mathrm{id}_A, \mathrm{refl})$, we see the whole type is equal to $\mathrm{pr}_1 (\mathrm{id}_A, \mathrm{refl}) = \mathrm{id}_A$, which is itself equal to $\mathrm{id}_A = \mathrm{id}_A$. $\square$

Proposition 6. *The type* $\mathrm{Inv}(A, B)$ *is equal to the type* $\sum_{e:A=B} e = e$ *for all types* A *and* B*.*

Proof. We view inverse and isequiv as two families of types over $A \to B$. Proposition 3 then tells us that isequiv is a family of mere propositions over $A \to B$, whereas proposition 4 in particular tells us that we have a family of functions $\mathrm{inverse}(f) \to \mathrm{isequiv}(f)$ for all functions $f : A \to B$. By proposition 2, it then follows that the type $\mathrm{Inv}(A, B)$ is equal to the type $\sum_{e:A\simeq B} \mathrm{inverse}(\mathrm{pr}_1 e)$.

On the other hand, we first observe that the type $\sum_{e:A=B} e = e$ is equal to the type $\sum_{e:A\simeq B} e = e$ by univalence. Now, since isequiv is a family of mere propositions, $A \simeq B$ is a subtype of $A \to B$. By the subtype identity principle, it follows that the type $e = e$ is equal to the type $\mathrm{pr}_1 e = \mathrm{pr}_1 e$ for all equivalences $e : A \simeq B$.

To construct the desired equality, it thus suffices to construct an equality between the types $\sum_{e:A\simeq B} \mathrm{inverse}(\mathrm{pr}_1 e)$ and $\sum_{e:A\simeq B} \mathrm{pr}_1 e = \mathrm{pr}_1 e$. By the fiberwise equivalence construction, it further suffices to construct an equality between $\mathrm{inverse}(\mathrm{pr}_1 e)$ and $\mathrm{pr}_1 e = \mathrm{pr}_1 e$ for all equivalences $e : A \simeq B$. The result then follows by equivalence induction and Lemma 5. $\square$

Theorem 7. *The type* $\sum_{A,B:\mathcal{U}} \mathrm{Inv}(A, B)$ *is equal to the type* $\mathbb{S}^2 \to \mathcal{U}$*.*

Proof. We first quantify the equality in Proposition 6 over $B : \mathcal{U}$, obtaining an equality between $\sum_{B:\mathcal{U}} \mathrm{Inv}(A, B)$ and $\sum_{B:\mathcal{U}} \sum_{e:A=B} e = e$. Since the type $\sum_{B:\mathcal{U}} A = B$ is contractible onto (A, refl_A), the second type is equal to $\mathrm{refl}_A = \mathrm{refl}_A$. Now quantifying over $A : \mathcal{U}$, we obtain an equality between $\sum_{A,B:\mathcal{U}} \mathrm{Inv}(A, B)$ and $\sum_{A:\mathcal{U}} \mathrm{refl}_A = \mathrm{refl}_A$, which is equal to $\mathbb{S}^2 \to \mathcal{U}$ by the universal property of the sphere. $\square$

TODO: should this section be cut or be polished up to serve as motivation, introduction?

2 n-invertible maps

Definition 8. We define a notion of n-invertibility on $A \to B$ by induction on n. Let $f : A \to B$ be a function. We say that f is 0-invertible if there exists a function $g : B \to A$. We say that f is $(n+1)$-invertible if there exist functions $g : B \to A$ and

$$r : \mathrm{Adj}(f, g) :\equiv \prod_{x:A} \prod_{y:B} (f(x) = y) \to (x = g(y)),$$

such that $r(x, y)$ is n-invertible for all x and y. We thus define a family of types $\mathrm{inverse}_n(f)$, such that $\mathrm{inverse}_0(f) = B \to A$ and

$$\mathrm{inverse}_{n+1}(f) = \sum_{g:B\to A} \sum_{r:\mathrm{Adj}(f,g)} \prod_{x:A} \prod_{y:B} \mathrm{inverse}_n(r(x, y)).$$

Remark 9. We note that the type $\mathrm{Adj}(f,g)$ is equal to $\prod_{x:A} x = g(f(x))$ by path induction, stating that g is a left inverse of f. However, we want to keep the type $\mathrm{Adj}(f,g)$ in its expanded form, so that we may speak about the invertibility of the function $f(x) = y \to x = g(y)$.

Lemma 10. *The notion of* 1*-invertibility coincides with the ordinary notion of invertibility. More precisely, the types* $\mathrm{inverse}_1(f)$ *and* $\mathrm{inverse}(f)$ *are equal for all functions* f*.*

Proof. Let $f : A \to B$ be a function. By definition, $\mathrm{inverse}_1(f)$ is equal to

$$\sum_{g:B\to A} \sum_{r:\mathrm{Adj}(f,g)} \prod_{x:A} \prod_{y:B} \mathrm{inverse}_0(r(x,y))$$

and $\mathrm{inverse}_0(r(x,y))$ is equal to

$$(x = g(y)) \to (f(x) = y).$$

We thus obtain that $\mathrm{inverse}_1(f)$ is equal to

$$\sum_{g:B\to A} \mathrm{Adj}(f,g) \times \mathrm{Adj}'(f,g),$$

where $\mathrm{Adj}'(f,g) = \prod_{x:A} \prod_{y:B} (x = g(y)) \to (f(x) = y)$. By applying path induction to both Adj and Adj' as in the remark above, we se that gives exactly the data of an invertible function. $\square$

Lemma 11. *The type* $\mathrm{inverse}_{n+1}(\mathrm{id}_A)$ *is equal to*

$$\prod_{x,y:A} \mathrm{inverse}_n(\mathrm{id}_{x=y})$$

for all $n : \mathbb{N}$*.*

Proof. TODO $\square$

Proposition 12. *The type* $\mathrm{inverse}_n(\mathrm{id}_A)$ *is equal to* $\prod_{x:A} \Omega^n(A,x)$ *for all* $n : \mathbb{N}$*.*

Proof. We prove the result by induction on n. For the base case, we have

$$\mathrm{inverse}_0(\mathrm{id}_A) = (A \to A) = \prod_{x:A} (A,x) = \prod_{x:A} \Omega^0(A,x).$$

For the inductive step, we have

$$\begin{aligned} \mathrm{inverse}_{n+1}(\mathrm{id}_A) &= \prod_{x,y:A} \mathrm{inverse}_n(\mathrm{id}_{x=y}) = \prod_{x,y:A} \prod_{p:x=y} \Omega^n(x=y,p) \\ &= \prod_{x:A} \Omega^n(x=x, \mathrm{refl}_x) = \prod_{x:A} \Omega^{n+1}(A,x). \end{aligned}$$

$\square$

Proposition 13. *Let $n : \mathbb{N}$ and suppose $f : A \to B$ is $(n+1)$-invertible. Then f is n-invertible.*

Proof. We again prove the result by induction on n. For the base case, suppose $f : A \to B$ is 1-invertible. We then have a function $g : B \to A$ together with some data, but g itself is enough to show that f is 0-invertible.

For the inductive step, suppose $f : A \to B$ is $(n+2)$-invertible. We then have a function $g : B \to A$ and a certain dependent function r, such that $r(x, y)$ is $(n+1)$-invertible for all $x, y : A$. By the inductive hypothesis, $r(x, y)$ is n-invertible for all $x, y : A$, showing that f is $(n+1)$-invertible. □

Corollary 14. *Let $n : \mathbb{N}$ and suppose $f : A \to B$ is $(n+1)$-invertible. Then f is an equivalence.*

Proof. Using the previous proposition, we can show that every $(n+1)$-invertible function is 1-invertible. Since 1-invertibility coincides with invertibility, this shows that it is also an equivalence. □

Theorem 15. *Let $\mathcal{U}$ be a universe and $A : \mathcal{U}$ a type. The type*

$$\sum_{B:\mathcal{U}} \sum_{f:A\to B} \mathrm{inverse}_{n+1}(f)$$

of all $(n+1)$-invertible maps with domain A is equal to $\Omega^{n+2}(\mathcal{U}, A)$.

Proof. Let $B : \mathcal{U}$ first be a type. Since $(n+1)$-invertibility implies equivalence and equivalence is a mere proposition, we have

$$\sum_{f:A\to B} \mathrm{inverse}_{n+1}(f) = \sum_{f:A\simeq B} \mathrm{inverse}_{n+1}(\mathrm{pr}_1 f).$$

Then, since the type $\sum_{B:\mathcal{U}} A \simeq B$ is contractible onto (A, id_A), we also have

$$\sum_{B:\mathcal{U}} \sum_{f:A\simeq B} \mathrm{inverse}_{n+1}(\mathrm{pr}_1 f) = \mathrm{inverse}_{n+1}\,(\mathrm{id}_A),$$

so we conclude that

$$\sum_{B:\mathcal{U}} \sum_{f:A\to B} \mathrm{inverse}_{n+1}(f) = \mathrm{inverse}_{n+1(\mathrm{id}_A)}$$

by transitivity. By proposition 12, the second type is equal to $\prod_{x:A} \Omega^{n+1}(A, x)$. Finally, since dependent products commute with loop spaces, we have

$$\prod_{x:A} \Omega^{n+1}(A, x) = \Omega^{n+2}(\mathcal{U}, A).$$

TODO : why exactly is the final claim true? □

Corollary 16. *Let $\mathcal{U}$ be a universe. The type $\sum_{A,B:\mathcal{U}} \sum_{f:A\to B} \mathrm{inverse}_{n+1}(f)$ is equal to $\mathbb{S}^{n+2} \to \mathcal{U}$.*

Proof. Obtained by quantifying the equality in Theorem 15 over A and noting that $\sum_{A:\mathcal{U}} \Omega^{n+2}(\mathcal{U}, A) = \mathbb{S}^{n+2} \to \mathcal{U}$ by the universal property of the sphere. □

3 ∞-invertible maps

Consider the family of functions $\phi_n : \mathrm{inverse}_{n+1}(f) \to \mathrm{inverse}_n(f)$ defined in Proposition 13. These may explicitly be written out to be

$$\begin{aligned} \phi_0(g, r, H) &= g \\ \phi_{n+1}(g, r, H) &= (g, r, \lambda x.\lambda y.\phi_n(H(x, y))). \end{aligned}$$

Considering this family of function as a diagram and taking its limit, we obtain a type $\mathrm{inverse}_\infty(f)$, together with data

$$\begin{aligned} \psi_n &: \mathrm{inverse}_\infty(f) \to \mathrm{inverse}_n(f) \\ \alpha_n &: \phi_{n+1} \circ \psi_{n+1} = \psi_n. \end{aligned}$$

A function f, equipped with data of type $\mathrm{inverse}_\infty(f)$, is said to be ∞-*invertible*. Note that every ∞-invertible function is an equivalence, since it is in particular 1-invertible by the data of the limit.

Proposition 17. *The type* $\mathrm{inverse}_\infty(f)$ *is a mere proposition for all functions* $f : A \to B$.

Proof. We show that assuming f is ∞-invertible, the type $\mathrm{inverse}_\infty(f)$ is contractible. That is to say, we wish to show

$$\prod_{A,B:\mathcal{U}} \prod_{f:A\to B} \mathrm{inverse}_\infty(f) \to \mathrm{isContr}(\mathrm{inverse}_\infty(f)).$$

Now, since ∞-invertibility implies equivalence, it suffices to show

$$\prod_{A,B:\mathcal{U}} \prod_{f:A\to B} \mathrm{isequiv}(f) \to \mathrm{isContr}(\mathrm{inverse}_\infty(\mathrm{pr}_1 f)).$$

By equivalence induction, it then suffices to show only

$$\prod_{A:\mathcal{U}} \mathrm{isContr}(\mathrm{inverse}_\infty(\mathrm{id}_A)).$$

TODO: reword the above

By definition, $\mathrm{inverse}_\infty(\mathrm{id}_A)$ is the limit of the diagram

$$\phi_n : \mathrm{inverse}_{n+1}(\mathrm{id}_A) \to \mathrm{inverse}_n(\mathrm{id}_A).$$

TODO: we could now try to use Proposition 12 somehow, this seems related to the contractibility of $\mathbb{S}^\infty$. □

Proposition 18. *The type of* ∞*-invertibility proofs satisfies the recursive equation*

$$\mathrm{inverse}_\infty(f) = \sum_{g:B\to A} \sum_{r:\mathrm{Adj}(f,g)} \prod_{x:A} \prod_{y:B} \mathrm{inverse}_\infty(r(x, y))$$

for all $f : A \to B$.

Proof. We obtain the desired equality by showing the second type satisfies the universal property of the limit. Suppose then that P is a type, equipped with a cocone

$$\rho_n : P \to \mathrm{inverse}_n(f)$$
$$\beta_n : \phi_{n+1} \circ \rho_{n+1} = \rho_n.$$

TODO: not sure if this is a right approach. $\square$