Define

$$\mathrm{Adj}(f,g) := \prod_{x:A}\prod_{y:B}(f(x)=y) \to (x = g(y)).$$

Suppose that g is a left inverse of f, witnessed by $G : g \circ f \sim \mathsf{id}_A$. We then construct $r_G : \mathrm{Adj}(f,g)$, given by induction by

$$r_G(x, f(x), \mathsf{refl}_{f(x)}) = H(x).$$

This map induces an equivalence between $g \circ f \sim \mathsf{id}_A$ and $\mathrm{Adj}(f,g)$

Now, suppose that the type of left inverses of a function f

$$\sum_{g:B\to A} g \circ f \sim \mathsf{id}_A$$

is contractible onto (h, H). By applying the equivalence between $g \circ f \sim \mathsf{id}_A$ and $\mathrm{Adj}(f,g)$, we then obtain that the type

$$\sum_{g:B\to A} \mathrm{Adj}(f,g)$$

is contractible onto (h, r_H).

Consider the identity function id_A. Its type of left inverses is contractible onto $(\mathsf{id}_A, \lambda x.\mathsf{refl}_x)$. It then follows that the type

$$\sum_{g:A\to A} \mathrm{Adj}(\mathsf{id}_A, g)$$

is contractible onto $(\mathsf{id}_A, r_{\lambda x.\mathsf{refl}_x})$. We now claim that $r_{\lambda x.\mathsf{refl}_x}$ is equal to $\lambda xy.\mathsf{id}_{x=y}$. By function extensionality, it suffices to show that

$$r_{\lambda x.\mathsf{refl}_x}(x,y,p) = p$$

for all $x, y : A$ and $p : x = y$. By path induction, it then suffices to show that

$$r_{\lambda x.\mathsf{refl}_x}(x,x,\mathsf{refl}_x) = \mathsf{refl}_x$$

for all $x : A$, but this then follows by definiton of r. We conclude that $\sum_{g:A\to A} \mathrm{Adj}(\mathsf{id}_A, g)$ is contractible onto $(\mathsf{id}_A, \lambda xy.\mathsf{id}_{x=y})$.

With this, we can now show that $\mathsf{inverse}_{n+1}(\mathsf{id}_A)$ is equal to $\prod_{xy:A}\mathsf{inverse}_n(\mathsf{id}_{x=y})$. Unfolding $\mathsf{inverse}_{n+1}$, we see that

$$\mathsf{inverse}_{n+1}(\mathsf{id}_A) = \sum_{g:A\to A}\ \sum_{r:\mathrm{Adj}(\mathsf{id}_A,g)}\ \prod_{x:A}\prod_{y:B}\mathsf{inverse}_n(r(x,y)).$$

The result then follows by reassociating the sigma types and applying the above contractibility result.