# 1 Basic construction

**Proposition 1.** *For all types $A$, the type $\mathbb{S}^2 \to A$ is equal to*

$$\sum_{x:A} \mathrm{refl}_x = \mathrm{refl}_x.$$

**Proposition 2.** *Suppose that $A$ is a type, $B$ a family of types over $A$ and $P$ a family of mere propositions over $A$, such that we have a family of functions $B(x) \to P(x)$ for all $x : A$. We then have an equality of types*

$$\sum_{x:A} B(x) = \sum \Big(t : \sum_{x:A} P(x)\Big) B(\mathrm{pr}_1 t).$$

A function $f$ is said to be *invertible* if it admits a two-sided inverse. We define the type of invertibility proofs of $f$ to be the type

$$\mathrm{inverse}(f) := \sum_{g:B\to A} (f \circ g = \mathrm{id}) \times (g \circ f = \mathrm{id})$$

and denote the type of all invertible maps between two types $A$ and $B$ by $\mathrm{Inv}(A,B) := \sum_{f:A\to B} \mathrm{inverse}(f)$.

A function $f$ is said to be an *equivalence* if it admits both a left and a right sided inverse. We define the type of equivalence proofs of $f$ to be the type

$$\mathrm{isequiv}(f) := \sum_{g:B\to A} (f \circ g = \mathrm{id}) \times \sum_{h:B\to A} (h \circ f = \mathrm{id})$$

and similarly denote the type of all equivalences between two types $A$ and $B$ by $A \simeq B$.

We recall the following two important facts:

**Proposition 3.** *For all functions $f$, the type of equivalence proofs of $f$ is a mere proposition.*

**Proposition 4.** *For all functions $f$, the types of invertibility and equivalence proofs of $f$ are logically equivalent. In other words, a function is invertible if and only if it is an equivalence.*

**Lemma 5.** *Suppose $A$ is a type. The type of invertibility proofs of $\mathrm{id}_A$ is equal to $\mathrm{id}_A = \mathrm{id}_A$.*

*Proof.* By reassociating the dependent sum type, we see that the type of invertibility proofs of $\mathrm{id}_A$ is equal to

$$\sum \Big(G : \sum_{g:A\to A} (g = \mathrm{id}_A)\Big) (\mathrm{pr}_1 G = \mathrm{id}_A).$$

Since the type $\sum_{g:A\to A} (g = \mathrm{id}_A)$ is contractible onto $(\mathrm{id}_A, \mathrm{refl})$, we see the whole type is equal to $\mathrm{pr}_1 (\mathrm{id}_A, \mathrm{refl}) = \mathrm{id}_A$, which is itself equal to $\mathrm{id}_A = \mathrm{id}_A$. $\qquad\square$

**Proposition 6.** *The type* $\mathrm{Inv}(A, B)$ *is equal to the type* $\sum_{e:A=B} e = e$ *for all types $A$ and $B$.*

*Proof.* We view inverse and isequiv as two families of types over $A \to B$. Proposition 3 then tells us that isequiv is a family of mere propositions over $A \to B$, whereas proposition 4 in particular tells us that we have a family of functions $\mathrm{inverse}(f) \to \mathrm{isequiv}(f)$ for all functions $f : A \to B$. By proposition 2, it then follows that the type $\mathrm{Inv}(A, B)$ is equal to the type $\sum_{e:A\simeq B} \mathrm{inverse}(\mathrm{pr}_1 e)$.

On the other hand, we first observe that the type $\sum_{e:A=B} e = e$ is equal to the type $\sum_{e:A\simeq B} e = e$ by univalence. Now, since isequiv is a family of mere propositions, $A \simeq B$ is a subtype of $A \to B$. By the subtype identity principle, it follows that the type $e = e$ is equal to the type $\mathrm{pr}_1 e = \mathrm{pr}_1 e$ for all equivalences $e : A \simeq B$.

To construct the desired equality, it thus suffices to construct an equality between the types $\sum_{e:A\simeq B} \mathrm{inverse}(\mathrm{pr}_1 e)$ and $\sum_{e:A\simeq B} \mathrm{pr}_1 e = \mathrm{pr}_1 e$. By the fiberwise equivalence construction, it further suffices to construct an equality between $\mathrm{inverse}(\mathrm{pr}_1 e)$ and $\mathrm{pr}_1 e = \mathrm{pr}_1 e$ for all equivalences $e : A \simeq B$. The result then follows by equivalence induction and Lemma 5. $\square$

**Theorem 7.** *The type* $\sum_{A,B:\mathcal{U}} \mathrm{Inv}(A, B)$ *is equal to the type* $\mathbb{S}^2 \to \mathcal{U}$.

*Proof.* We first quantify the equality in Proposition 6 over $B : \mathcal{U}$, obtaining an equality between $\sum_{B:\mathcal{U}} \mathrm{Inv}(A, B)$ and $\sum_{B:\mathcal{U}} \sum_{e:A=B} e = e$. Since the type $\sum_{B:\mathcal{U}} A = B$ is contractible onto $(A, \mathrm{refl}_A)$, the second type is equal to $\mathrm{refl}_A = \mathrm{refl}_A$. Now quantifying over $A : \mathcal{U}$, we obtain an equality between $\sum_{A,B:\mathcal{U}} \mathrm{Inv}(A, B)$ and $\sum_{A:\mathcal{U}} \mathrm{refl}_A = \mathrm{refl}_A$, which is equal to $\mathbb{S}^2 \to \mathcal{U}$ by the universal property of the sphere. $\square$

TODO: should this section be cut or be polished up to serve as motivation, introduction?

## 2   $n$-invertible maps

**Definition 8.** We define a notion of $n$-invertibility on $A \to B$ by induction on $n$. Let $f : A \to B$ be a function. We say that $f$ is 0-invertible if there exists a function $g : B \to A$. We say that $f$ is $(n+1)$-invertible if there exist functions $g : B \to A$ and

$$r : \mathrm{Adj}(f, g) := \prod_{x:A} \prod_{y:B} (f(x) = y) \to (x = g(y)),$$

such that $r(x, y)$ is $n$-invertible for all $x$ and $y$. We thus define a family of types $\mathrm{inverse}_n(f)$, such that $\mathrm{inverse}_0(f) = B \to A$ and

$$\mathrm{inverse}_{n+1}(f) = \sum_{g:B\to A} \sum_{r:\mathrm{Adj}(f,g)} \prod_{x:A} \prod_{y:B} \mathrm{inverse}_n(r(x, y)).$$

*Remark* 9. We note that the type $\text{Adj}(f, g)$ is equal to $\prod_{x:A} x = g(f(x))$ by path induction, stating that $g$ is a left inverse of $f$. However, we want to keep the type $\text{Adj}(f, g)$ in its expanded form, so that we may speak about the invertibility of the function $f(x) = y \to x = g(y)$.

**Lemma 10.** *The notion of 1-invertibility coincides with the ordinary notion of invertibility. More precisely, the types $\text{inverse}_1(f)$ and $\text{inverse}(f)$ are equal for all functions $f$.*

*Proof.* Let $f : A \to B$ be a function. By definition, $\text{inverse}_1(f)$ is equal to

$$\sum_{g:B\to A} \sum_{r:\text{Adj}(f,g)} \prod_{x:A} \prod_{y:B} \text{inverse}_0(r(x, y))$$

and $\text{inverse}_0(r(x, y))$ is equal to

$$(x = g(y)) \to (f(x) = y).$$

We thus obtain that $\text{inverse}_1(f)$ is equal to

$$\sum_{g:B\to A} \text{Adj}(f, g) \times \text{Adj}'(f, g),$$

where $\text{Adj}'(f, g) = \prod_{x:A} \prod_{y:B} (x = g(y)) \to (f(x) = y)$. By applying path induction to both $\text{Adj}$ and $\text{Adj}'$ as in the remark above, we se that gives exactly the data of an invertible function. $\square$

**Lemma 11.** *The type $\text{inverse}_{n+1}(\text{id}_A)$ is equal to*

$$\prod_{x,y:A} \text{inverse}_n(\text{id}_{x=y})$$

*for all $n : \mathbb{N}$.*

*Proof.* TODO $\square$

**Proposition 12.** *The type $\text{inverse}_n(\text{id}_A)$ is equal to $\prod_{x:A} \Omega^n(A, x)$ for all $n : \mathbb{N}$.*

*Proof.* We prove the result by induction on $n$. For the base case, we have

$$\text{inverse}_0(\text{id}_A) = (A \to A) = \prod_{x:A}(A, x) = \prod_{x:A} \Omega^0(A, x).$$

For the inductive step, we have

$$\text{inverse}_{n+1}(\text{id}_A) = \prod_{x,y:A} \text{inverse}_n(\text{id}_{x=y}) = \prod_{x,y:A} \prod_{p:x=y} \Omega^n(x = y, p)$$
$$= \prod_{x:A} \Omega^n(x = x, \text{refl}_x) = \prod_{x:A} \Omega^{n+1}(A, x).$$

$\square$

**Proposition 13.** *Let* $n : \mathbb{N}$ *and suppose* $f : A \to B$ *is* $(n + 1)$-*invertible. Then* $f$ *is* $n$-*invertible.*

*Proof.* We again prove the result by induction on $n$. For the base case, suppose $f : A \to B$ is 1-invertible. We then have a function $g : B \to A$ together with some data, but $g$ itself is enough to show that $f$ is 0-invertible.

For the inductive step, suppose $f : A \to B$ is $(n + 2)$-invertible. We then have a function $g : B \to A$ and a certain dependent function $r$, such that $r(x, y)$ is $(n + 1)$-invertible for all $x, y : A$. By the inductive hypothesis, $r(x, y)$ is $n$-invertible for all $x, y : A$, showing that $f$ is $(n + 1)$-invertible. $\square$

**Corollary 14.** *Let* $n : \mathbb{N}$ *and suppose* $f : A \to B$ *is* $(n + 1)$-*invertible. Then* $f$ *is an equivalence.*

*Proof.* Using the previous proposition, we can show that every $(n+1)$-invertible function is 1-invertible. Since 1-invertibility coincides with invertibility, this shows that it is also an equivalence. $\square$

**Theorem 15.** *Let* $\mathcal{U}$ *be a universe and* $A : \mathcal{U}$ *a type. The type*

$$\sum_{B:\mathcal{U}} \sum_{f:A\to B} \text{inverse}_{n+1}(f)$$

*of all* $(n + 1)$-*invertible maps with domain* $A$ *is equal to* $\Omega^{n+2}(\mathcal{U}, A)$.

*Proof.* Let $B : \mathcal{U}$ first be a type. Since $(n + 1)$-invertibility implies equivalence and equivalence is a mere proposition, we have

$$\sum_{f:A\to B} \text{inverse}_{n+1}(f) = \sum_{f:A\simeq B} \text{inverse}_{n+1}(\text{pr}_1 f).$$

Then, since the type $\sum_{B:\mathcal{U}} A \simeq B$ is contractible onto $(A, \text{id}_A)$, we also have

$$\sum_{B:\mathcal{U}} \sum_{f:A\simeq B} \text{inverse}_{n+1}(\text{pr}_1 f) = \text{inverse}_{n+1}(\text{id}_A),$$

so we conclude that

$$\sum_{B:\mathcal{U}} \sum_{f:A\to B} \text{inverse}_{n+1}(f) = \text{inverse}_{n+1(\text{id}_A)}$$

by transitivity. By proposition 12, the second type is equal to $\prod_{x:A} \Omega^{n+1}(A, x)$. Finally, since dependent products commute with loop spaces, we have

$$\prod_{x:A} \Omega^{n+1}(A, x) = \Omega^{n+2}(\mathcal{U}, A).$$

TODO : why exactly is the final claim true? $\square$

**Corollary 16.** *Let* $\mathcal{U}$ *be a universe. The type* $\sum_{A,B:\mathcal{U}} \sum_{f:A\to B} \text{inverse}_{n+1}(f)$ *is equal to* $\mathbb{S}^{n+2} \to \mathcal{U}$.

*Proof.* Obtained by quantifying the equality in Theorem 15 over $A$ and noting that $\sum_{A:\mathcal{U}} \Omega^{n+2}(\mathcal{U}, A) = \mathbb{S}^{n+2} \to \mathcal{U}$ by the universal property of the sphere. $\square$

# 3 ∞-invertible maps

Consider the family of functions $\phi_n : \mathrm{inverse}_{n+1}(f) \to \mathrm{inverse}_n(f)$ defined in Proposition 13. These may explicitly be written out to be

$$\phi_0(g, r, H) = g$$
$$\phi_{n+1}(g, r, H) = (g, r, \lambda x.\lambda y.\phi_n(H(x, y))).$$

Considering this family of function as a diagram and taking its limit, we obtain a type $\mathrm{inverse}_\infty(f)$, together with data

$$\psi_n : \mathrm{inverse}_\infty(f) \to \mathrm{inverse}_n(f)$$
$$\alpha_n : \phi_{n+1} \circ \psi_{n+1} = \psi_n.$$

A function $f$, equipped with data of type $\mathrm{inverse}_\infty(f)$, is said to be *∞-invertible*. Note that every ∞-invertible function is an equivalence, since it is in particular 1-invertible by the data of the limit.

**Proposition 17.** *The type $\mathrm{inverse}_\infty(f)$ is a mere proposition for all functions $f : A \to B$.*

*Proof.* We show that assuming $f$ is ∞-invertible, the type $\mathrm{inverse}_\infty(f)$ is contractible. That is to say, we wish to show

$$\prod_{A,B:\mathcal{U}} \prod_{f:A \to B} \mathrm{inverse}_\infty(f) \to \mathrm{isContr}(\mathrm{inverse}_\infty(f)).$$

Now, since ∞-invertibility implies equivalence, it suffices to show

$$\prod_{A,B:\mathcal{U}} \prod_{f:A \to B} \mathrm{isequiv}(f) \to \mathrm{isContr}(\mathrm{inverse}_\infty(\mathrm{pr}_1 f)).$$

By equivalence induction, it then suffices to show only

$$\prod_{A:\mathcal{U}} \mathrm{isContr}(\mathrm{inverse}_\infty(\mathrm{id}_A)).$$

TODO: reword the above

By definition, $\mathrm{inverse}_\infty(\mathrm{id}_A)$ is the limit of the diagram

$$\phi_n : \mathrm{inverse}_{n+1}(\mathrm{id}_A) \to \mathrm{inverse}_n(\mathrm{id}_A).$$

TODO: we could now try to use Proposition 12 somehow, this seems related to the contractibility of $\mathbb{S}^\infty$. $\qquad\square$

**Proposition 18.** *The type of ∞-invertibility proofs satisfies the recursive equation*

$$\mathrm{inverse}_\infty(f) = \sum_{g:B \to A} \sum_{r:\mathrm{Adj}(f,g)} \prod_{x:A}\prod_{y:B} \mathrm{inverse}_\infty(r(x, y))$$

*for all $f : A \to B$.*

*Proof.* We obtain the desired equality by showing the second type satisfies the universal property of the limit. Suppose then that $P$ is a type, equipped with a cocone

$$\rho_n : P \to \text{inverse}_n(f)$$
$$\beta_n : \phi_{n+1} \circ \rho_{n+1} = \rho_n.$$

TODO: not sure if this is a right approach. $\square$