

Define

$$\text{Adj}(f, g) := \prod_{x:A} \prod_{y:B} (f(x) = y) \rightarrow (x = g(y)).$$

Suppose that g is a left inverse of f , witnessed by $G : g \circ f \sim \text{id}_A$. We then construct $r_G : \text{Adj}(f, g)$, given by induction by

$$r_G(x, f(x), \text{refl}_{f(x)}) = H(x).$$

This assignment induces an equivalence between $g \circ f \sim \text{id}_A$ and $\text{Adj}(f, g)$

Now, suppose that the type of left inverses of a function f

$$\sum_{g:B \rightarrow A} g \circ f \sim \text{id}_A$$

is contractible onto (h, H) . By applying the equivalence between $g \circ f \sim \text{id}_A$ and $\text{Adj}(f, g)$, we then obtain that the type

$$\sum_{g:B \rightarrow A} \text{Adj}(f, g)$$

is contractible onto (h, r_H) .

Consider the identity function id_A . Its type of left inverses is contractible onto $(\text{id}_A, \lambda x.\text{refl}_x)$. It then follows that the type

$$\sum_{g:A \rightarrow A} \text{Adj}(\text{id}_A, g)$$

is contractible onto $(\text{id}_A, r_{\lambda x.\text{refl}_x})$. We now claim that $r_{\lambda x.\text{refl}_x}$ is equal to $\lambda xy.\text{id}_{x=y}$. By function extensionality, it suffices to show that

$$r_{\lambda x.\text{refl}_x}(x, y, p) = p$$

for all $x, y : A$ and $p : x = y$. By path induction, it then suffices to show that

$$r_{\lambda x.\text{refl}_x}(x, x, \text{refl}_x) = \text{refl}_x$$

for all $x : A$, but this then follows by definition of r . We conclude that $\sum_{g:A \rightarrow A} \text{Adj}(\text{id}_A, g)$ is contractible onto $(\text{id}_A, \lambda xy.\text{id}_{x=y})$.

With this, we can now show that $\text{inverse}_{n+1}(\text{id}_A)$ is equal to $\prod_{xy:A} \text{inverse}_n(\text{id}_{x=y})$. Unfolding inverse_{n+1} , we see that

$$\text{inverse}_{n+1}(\text{id}_A) = \sum_{g:A \rightarrow A} \sum_{r:\text{Adj}(\text{id}_A, g)} \prod_{x:A} \prod_{y:B} \text{inverse}_n(r(x, y)).$$

The result then follows by reassociating the sigma types and applying the above contractibility result.