

Define

$$\text{Adj}(f, g) := \prod_{x:A} \prod_{y:B} (f(x) = y) \rightarrow (x = g(y)).$$

Suppose that g is a left inverse of f , witnessed by $G : g \circ f \sim \text{id}_A$. We then construct $r_G : \text{Adj}(f, g)$, given by induction by

$$r_G(x, f(x), \text{refl}_{f(x)}) = H(x).$$

This assignment induces an equivalence between $g \circ f \sim \text{id}_A$ and $\text{Adj}(f, g)$

Now, suppose that the type of left inverses of a function f

$$\sum_{g:B \rightarrow A} g \circ f \sim \text{id}_A$$

is contractible onto (h, H) . By applying the equivalence between $g \circ f \sim \text{id}_A$ and $\text{Adj}(f, g)$, we then obtain that the type

$$\sum_{g:B \rightarrow A} \text{Adj}(f, g)$$

is contractible onto (h, r_H) .

Consider the identity function id_A . Its type of left inverses is contractible onto $(\text{id}_A, \lambda x.\text{refl}_x)$. It then follows that the type

$$\sum_{g:A \rightarrow A} \text{Adj}(\text{id}_A, g)$$

is contractible onto $(\text{id}_A, r_{\lambda x.\text{refl}_x})$. We now claim that $r_{\lambda x.\text{refl}_x}$ is equal to $\lambda xy.\text{id}_{x=y}$. By function extensionality, it suffices to show that

$$r_{\lambda x.\text{refl}_x}(x, y, p) = p$$

for all $x, y : A$ and $p : x = y$. By path induction, it then suffices to show that

$$r_{\lambda x.\text{refl}_x}(x, x, \text{refl}_x) = \text{refl}_x$$

for all $x : A$, but this then follows by definition of r . We conclude that $\sum_{g:A \rightarrow A} \text{Adj}(\text{id}_A, g)$ is contractible onto $(\text{id}_A, \lambda xy.\text{id}_{x=y})$.

With this, we can now show that $\text{inverse}_{n+1}(\text{id}_A)$ is equal to $\prod_{x,y:A} \text{inverse}_n(\text{id}_{x=y})$. Unfolding inverse_{n+1} , we see that

$$\text{inverse}_{n+1}(\text{id}_A) = \sum_{g:A \rightarrow A} \sum_{r:\text{Adj}(\text{id}_A, g)} \prod_{x:A} \prod_{y:B} \text{inverse}_n(r(x, y)).$$

The result then follows by reassociating the sigma types and applying the above contractibility result.

In the formalisation I actually obtain the contractibility result by a more direct computational argument, but I thought the above may be nicer for a paper proof. As in the formalisation though, we can directly show that the type

$$\sum_{g:A \rightarrow A} \text{Adj}(\text{id}_A, g)$$

is contractible onto $(\text{id}_A, \lambda xy.\text{id}_{x=y})$ in the following way:

We first characterise identifications in the above sigma type. To give an identification between (g, r) and (h, s) , we claim that it is sufficient to give a homotopy $H : g \sim h$ and a witness to the commutative triangle

$$r(x, y, p) \cdot H(y) = s(x, y, p)$$

for all $x, y : A$ and $p : x = y$. The claim follows by first applying homotopy induction on H and then noting that the commutativity data turns into just a homotopy between r and s , on which we may again apply homotopy induction.

Now let (g, r) be an arbitrary element of the above sigma type. Then

$$r : \prod_{x, y : A} (x = y) \rightarrow (x = g(y)),$$

so $\lambda x.r(x, x, \text{refl}_x)$ gives a homotopy between id_A and g . To give an identification between $(\text{id}_A, \lambda xy.\text{id}_{x=y})$ and (g, r) , the characterisation now tells us that it is sufficient to give an identification of type

$$p \cdot r(y, y, \text{refl}_y) = r(x, y, p)$$

for all $x, y : A$ and $p : x = y$. This is then discharged by path induction on p .

Let me now turn to a potentially interesting thread, related to ∞ -invertibility. Taking the coinductive definition of ∞ -invertibility, we can unfold $\text{inverse}_\infty(\text{id}_A)$ once to obtain

$$\text{inverse}_\infty(\text{id}_A) = \sum_{g : A \rightarrow A} \sum_{r : \text{Adj}(\text{id}_A, g)} \prod_{x : A} \prod_{y : B} \text{inverse}_\infty(r(x, y)).$$

As in the proof that $\text{inverse}_{n+1}(\text{id}_A) = \prod_{x, y : A} \text{inverse}_n(\text{id}_{x=y})$, we now obtain that $\text{inverse}_\infty(\text{id}_A) = \prod_{x, y : A} \text{inverse}_\infty(\text{id}_{x=y})$.

Now, the start of the proof of proposition 17 from the paper tells us that to show that $\text{inverse}_\infty(f)$ is a proposition for all functions f , it suffices to show that $\text{inverse}_\infty(\text{id}_A)$ is contractible. To understand ∞ -invertibility, it thus seems crucial to understand the ∞ -invertible structure of the identity function and by the above, this obeys the following curious recursive equation:

Define $F(A) = \text{inverse}_\infty(\text{id}_A)$. Then $F(A) = \prod_{x, y : A} F(x = y)$. Is there anything we can extract from this fact by itself?