

Proposition 1. *For all types A , the type $S^2 \rightarrow A$ is equal to*

$$\sum_{x:A} \text{refl}_x = \text{refl}_x.$$

Proposition 2. *Suppose that A is a type, B a family of types over A and P a family of mere propositions over A , such that we have a family of functions $B(x) \rightarrow P(x)$ for all $x : A$. We then have an equality of types*

$$\sum_{x:A} B(x) = \sum_{x:A} \left(t : \sum_{x:A} P(x) \right) B(\text{pr}_1 t).$$

A function f is said to be *invertible* if it admits a two-sided inverse. We define the type of invertibility proofs of f to be the type

$$\text{inverse}(f) := \sum_{g:B \rightarrow A} (f \circ g = \text{id}) \times (g \circ f = \text{id})$$

and denote the type of all invertible maps between two types A and B by $\text{Inv}(A, B) := \sum_{f:A \rightarrow B} \text{inverse}(f)$.

A function f is said to be an *equivalence* if it admits both a left and a right sided inverse. We define the type of equivalence proofs of f to be the type

$$\text{equiv}(f) := \sum_{g:B \rightarrow A} (f \circ g = \text{id}) \times \sum_{h:B \rightarrow A} (h \circ f = \text{id})$$

and similarly denote the type of all equivalences between two types A and B by $A \simeq B$.

We recall the following two important facts:

Proposition 3. *For all functions f , the type of equivalence proofs of f is a mere proposition.*

Proposition 4. *For all functions f , the types of invertibility and equivalence proofs of f are logically equivalent. In other words, a function is invertible if and only if it is an equivalence.*

Lemma 5. *Suppose A is a type. The type of invertibility proofs of id_A is equal to $\text{id}_A = \text{id}_A$.*

Proof. By reassociating the dependent sum type, we see that the type of invertibility proofs of id_A is equal to

$$\sum \left(G : \sum_{g:A \rightarrow A} (g = \text{id}_A) \right) (\text{pr}_1 G = \text{id}_A).$$

Since the type $\sum_{g:A \rightarrow A} (g = \text{id}_A)$ is contractible onto $(\text{id}_A, \text{refl})$, we see the whole type is equal to $\text{pr}_1(\text{id}_A, \text{refl}) = \text{id}_A$, which is itself equal to $\text{id}_A = \text{id}_A$. \square

Proposition 6. *The type $\text{Inv}(A, B)$ is equal to the type $\sum_{e:A=B} e = e$ for all types A and B .*

Proof. We view inverse and equiv as two families of types over $A \rightarrow B$. Proposition 3 then tells us that equiv is a family of mere propositions over $A \rightarrow B$, whereas proposition 4 in particular tells us that we have a family of functions $\text{inverse}(f) \rightarrow \text{equiv}(f)$ for all functions $f : A \rightarrow B$. By proposition 2, it then follows that the type $\text{Inv}(A, B)$ is equal to the type $\sum_{e:A \simeq B} \text{inverse}(\text{pr}_1 e)$.

On the other hand, we first observe that the type $\sum_{e:A=B} e = e$ is equal to the type $\sum_{e:A \simeq B} e = e$ by univalence. Now, since equiv is a family of mere propositions, $A \simeq B$ is a subtype of $A \rightarrow B$. By the subtype identity principle, it follows that the type $e = e$ is equal to the type $\text{pr}_1 e = \text{pr}_1 e$ for all equivalences $e : A \simeq B$.

To construct the desired equality, it thus suffices to construct an equality between the types $\sum_{e:A \simeq B} \text{inverse}(\text{pr}_1 e)$ and $\sum_{e:A \simeq B} \text{pr}_1 e = \text{pr}_1 e$. By the fiberwise equivalence construction, it further suffices to construct an equality between $\text{inverse}(\text{pr}_1 e)$ and $\text{pr}_1 e = \text{pr}_1 e$ for all equivalences $e : A \simeq B$. The result then follows by equivalence induction and Lemma 5. \square

Theorem 7. *The type $\sum_{A,B:\mathcal{U}} \text{Inv}(A, B)$ is equal to the type $S^2 \rightarrow \mathcal{U}$.*

Proof. We first quantify the equality in Proposition 6 over $B : \mathcal{U}$, obtaining an equality between $\sum_{B:\mathcal{U}} \text{Inv}(A, B)$ and $\sum_{B:\mathcal{U}} \sum_{e:A=B} e = e$. Since the type $\sum_{B:\mathcal{U}} A = B$ is contractible onto (A, refl_A) , the second type is equal to $\text{refl}_A = \text{refl}_A$. Now quantifying over $A : \mathcal{U}$, we obtain an equality between $\sum_{A,B:\mathcal{U}} \text{Inv}(A, B)$ and $\sum_{A:\mathcal{U}} \text{refl}_A = \text{refl}_A$, which is equal to $S^2 \rightarrow \mathcal{U}$ by the universal property of the sphere. \square