

# Localic Stone Duality

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## 0.1 Preliminaries

Suppose throughout that  $P$  is a poset with order relation  $\leq$ .

**Definition 1.** We say that a subset  $S \subseteq P$  is *directed* if for every  $x, y \in S$  there exists  $z \in S$ , such that  $x \leq z$  and  $y \leq z$ .

We say that  $S$  is *downwards closed* if whenever  $x \in S$  and  $y \leq x$ , we also have  $y \in S$ .

If  $S$  is both directed and downwards closed, we call  $S$  an *ideal* of  $P$ .

**Lemma 2.** For all  $x \in P$ , the set  $\downarrow x = \{y \in P \mid y \leq x\}$  is an ideal of  $P$ .

**Lemma 3.** Suppose the poset  $P$  has binary joins  $\vee$  and  $S \subseteq P$  is downwards closed. Then  $P$  is directed (and hence an ideal) if and only if  $P$  is closed under binary joins.

**Definition 4.** Whenever a directed subset  $S \subseteq P$  has a join, we call it a *directed join* and denote it by  $\bigvee^\uparrow S$ . If  $P$  has all directed joins, we call  $P$  a *directed complete partial order*, or *dcpo* for short.

**Lemma 5.** To construct arbitrary joins in a dcpo  $P$ , it suffices to construct all finite joins. Furthermore, to show meets distribute over arbitrary joins, it suffices to show they distribute over every directed and finite joins.

## 0.2 Ideal completions

**Definition 6.** We define  $\text{Idl}(P)$  to be the set of all ideals of  $P$  and call it *the ideal completion of  $P$* . It is easy to see that  $\text{Idl}(P)$  forms a poset under subset inclusion and that there is a monotone function  $\eta : P \rightarrow \text{Idl}(P)$ , given by  $\eta(x) = \downarrow x$ .

**Proposition 7.** The set  $\text{Idl}(P)$  is a dcpo, with directed joins given by unions.

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For my project, I would like to formalize the stone duality and some related facts. We talked about the possibility of contributing to an existing formalization, but I think it would be good if I tried to plan out the project myself. I am also interested in isolating the constructive part of the proof, and possibly doing the classical part as a stretch goal. What I am planning to formalize is 1. Given a poset  $P$ , construct its ideal completion  $\text{Idl}(P)$  and show that it forms the free dcpo over  $P$ . If  $P$  is a distributive lattice, show that it forms the free frame over  $P$ , call this the coherent frame over  $P$ . 2. Introduce compact elements, show some facts about the subposet of compact elements  $KP$ . If  $P$  is a boolean algebra, show that  $K$  is functorial. 3. Show that  $K\text{Idl}(P)$  is isomorphic to  $P$  for every poset  $P$ . If  $P$  is a boolean algebra, show that this isomorphism is natural. 4. Introduce algebraic dcpos. Show that a dcpo  $A$  is algebraic if and only if it is isomorphic to  $\text{Idl}(KA)$ .