Gauging Topological order : Towards Fracton phases

Tanaka Shuma, Nishida laboratory

November 27, 2025

Our goal : Property of Fracton phase

Many gapped fracton phases have non-topological property.

Ex

The ground state degeneracy (GSD) of the X-cube : GSD=2^{6L-3} (The size of torus is $L \times L \times L$)

Such a system cannot be described conventional TQFT because it also depends on the geometry of the system(not only the topology of the system!)

Our goal

Connect such an exotic (Geometry-dependent) system to TQFT through Gauging symmetry!

Outline of the discussion

Pranay Gorantla et al. arXiv:2509.19440(2025).

- We introduce the Toric code in 2+1D, This is one of the most basic system written in TQFT.
- We stack the defined two-dimensional Toric codes along the xy, yz and zx planes,respectively, to construct a new three-dimensional Hamiltonian. We call this Hamiltonian 3-foliated Toric code.
- We gauge the symmetry which 3-foliated Toric code has. We get X-cube model and this is a typical model of emergence of fracton written in exotic theory.

Toric code in 2+1D : Hamiltonian

Hamiltonian

$$H_{TC} = -\sum_{x,y}[V_{(x,y)} + P_{(x,y)}] + (\mathrm{h.c.}).$$

Stabilizar

$$V_{(x,y)} = X^{\dagger}_{(x-\frac{1}{2},y)} X_{(x+\frac{1}{2},y)} X^{\dagger}_{(x,y-\frac{1}{2})} X_{(x,y-\frac{1}{2})}$$

$$P_{(x,y)} = Z^{\dagger}_{(x,y+\frac{1}{2})} Z_{(x+1,y+\frac{1}{2})} Z^{\dagger}_{(x+\frac{1}{2},y)} Z_{(x+\frac{1}{2},y+1)}$$

A qubit is placed on every edge.

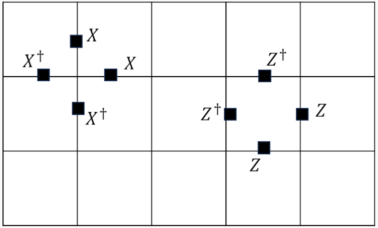


Figure 1: The product of the four X operators on the left star represents $V_{(x,y)}$, and the product of the four Z operators on the right plaquette represents $P_{(x,y)}$

X and Z are Pauli operators.($X^2 = Z^2 = 1, XZ = -ZX$)

Toric code in 2+1D : TQFT

In low energy limit, Toric code can be written in BF theory.

Lagrangian

$$\mathcal{L}_{TC} = \frac{iN}{2\pi} b_m \wedge da_e$$

Canonical quantization

$$[\hat{a}_{e,k}(x), \hat{b}_{m,i}(y)] = -\frac{2\pi i}{N}\epsilon^{ki}\delta(x-y)$$

Gauge symmetry

$$a_e \sim a_e + d\alpha_e$$

$$b_m \sim b_m + d\beta_m$$

A.Kitaev, Annals of Physics 303, 2 (2003).

1-form symmetry : symmetry that acts along a line

$[W(C), H_{TC}] = [V(C'), H_{TC}] = 0$

Symmetries(Wilson lines)

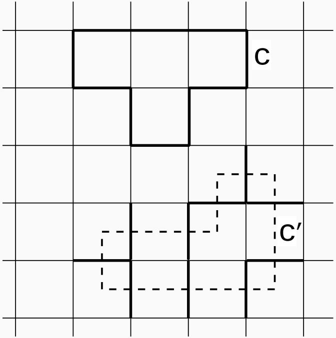
$$W(C) = \prod_{e \subset C} Z_e \quad W(C) = \exp\left(i \oint_C a_e\right)$$

$$V(C') = \prod_{e \subset C'} X_e \quad V(C') = \exp\left(i \oint_{C'} b_m\right)$$

Commutationrelation

$W(C)^2 = V(C')^2 = 1$

$W(C)V(C') = e^{-\pi i Link(C,C')} V(C')W(C)$



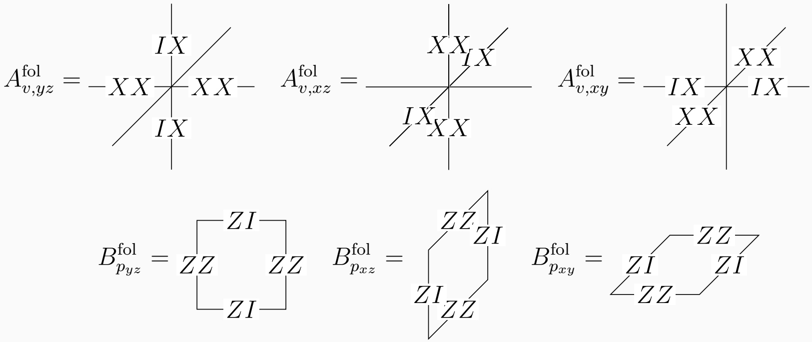
Hamiltonian

$$H_{\mathrm{fol}} = -\sum_{v}(A^{\mathrm{fol}}_{v,yz} + A^{\mathrm{fol}}_{v,xz} + A^{\mathrm{fol}}_{v,xy}) - \sum_{p} B^{\mathrm{fol}}_{p}$$

$AB \equiv A \otimes B$

v : vertex, p : plaquette

Structure of stabilizars



3-foliated stacks of Toric code : Symmetries

- 3-foliated Toric code has some symmetry. We focus on the $G_m^{(1),fol}$ symmetry. $G_m^{(1),fol}$:1-form symmetry generated by type-m particles.(Higher form symmetry)

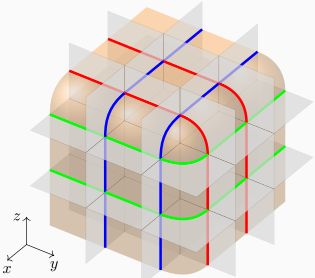
  Wilson lines :$U_m(\Sigma_i) = \prod_{e_i \subset \Sigma_i} XX_{e_i}$ $U_m(\Sigma_i) = \prod_{e_i \subset \Sigma_i} IX_{e_i}$ $\quad \Sigma_i = \hat{\Gamma}_x^{(i_0,l)}, \hat{\Gamma}_y^{(j_0,m)}, \hat{\Gamma}_z^{(k_0,n)}$
- $\hat{\Gamma}_x^{(i_0,m)}$ consists of e_z edges along the x-direction with (y,z) coordinate $(i_0, m+\frac{1}{2})$. (e_z is a edge oriented in the z direction.)$3 \times 2 \times L = 6L$ independent global symmetry generated by Wilson lines.
- $G_m^{(1),fol}$ has subgroup. We call this $G_m^{(1)}$.

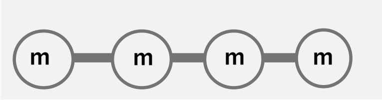
Subgroup

$$\prod_{j_0} U_m(\hat{\Gamma}_z^{(j_0,i)}) \prod_{k_0} U_m(\hat{\Gamma}_y^{(k_0,i)}) = \prod_{e_x \in \hat{\Sigma}_{yz}^{(i)}} XI_{e_x} = U_m^{(1)}(\hat{\Sigma}_{yz}^{(i)}) \otimes I$$

3-foliated Toric code : physical meaning of the gauging

- Now, we want to gauge the $G_m^{(1)}$ symmetry. What does this mean physically?
- Grey layer represents Toric code in 2+1D.
- Red, blue, green lines represent Wilson line of the Toric code in 2+1D. These lines form $G_m^{(1),fol}$.
- Orange surface represents $G_m^{(1)}$.
- $G_m^{(1)}$ is the 1-form symmetry generated precisely by the Wilson surfaces of the p-strings. And when the p-strings condense, this $G_m^{(1),fol}$ is gauged (i.e. becomes trivial.)
- p-string is a string-like excitation formed by connecting the m anyons residing each layer.





Gauging symmetry(1/3)

The 3+1D Toric code has a $G_m^{(1)}$ symmetry, so it can be viewed as the gauged toric code.

Gauging of Toric code in TQFT

$$\mathcal{L}_{TC} = \frac{iN}{2\pi} b_m \wedge da_e \xmapsto{\text{gauging } G_m^{(1)}} \mathcal{L}_{TC,\text{gauged}} = \frac{iN}{2\pi} b_m \wedge (da_e - b_e) + a_m \wedge db_e$$

$$\text{Faraday's law} \ : db_m = 0 \qquad \text{Gauss law} \ : da_e - b_e = 0$$

Correspondense between lattice model and TQFT

$$Z_e \sim \exp\Big(i \int_e a_e\Big) \quad Z_p^g \sim \exp\Big(i \int_p b_e\Big) \quad X_e \sim \exp\Big(i \int_{\hat{\Sigma}_e} b_m\Big) \quad X_p^g \sim \exp\Big(i \int_{\hat{\Gamma}_p} a_m\Big)$$

Gauging symmetry(2/3)

- Integrating Gauss law $da_e - b_e = 0$ on p plaquette. $\int_p da_e = \oint_{\partial p} a_e = \sum_{e\subset p} \int_e a_e$

$$\boxed{Z_p^g \sim \exp\left(i \int_p b_e\right) = \exp\left(i \sum_{e\subset p} \int_e a_e\right) = \prod_{e\subset p} \exp\left(i \int_e a_e\right) \sim \prod_{e\subset p} Z_e}$$

- From Faraday's law, b_m is written locally as the exterior derivative of a one-form.

$$b_m = da_m$$

Integrating Faraday's law on dual surface of e, $\hat{\Sigma}_e$. $\int_{\hat{\Sigma}_e} da_m = \int_{\partial\hat{\Sigma}_e} a_m = \sum_{p\supset e} \int_{\hat{\Gamma}_p} a_m$.

$$\boxed{X_e \sim \exp\left(i \int_{\hat{\Sigma}_e} b_m\right) = \exp\left(i \sum_{p\supset e} \int_{\hat{\Gamma}_p} a_m\right) = \prod_{p\supset e} \exp\left(i \int_{\hat{\Gamma}_p} a_m\right) \sim \prod_{p\supset e} X_p^g}$$

Gauging symmetry(3/3)

gauge map of $G_m^{(1)}$: $D^{G_m^{(1)}}$

$$\prod_{e\subset p} Z_e \xmapsto{\text{gauge}} Z_p^g \qquad X_e \xmapsto{\text{gauge}} \prod_{p\supset e} X_p^g$$

(c.f. $\partial_\mu\phi\partial^\mu\phi \mapsto D_\mu\phi D^\mu\phi$)

$$H_{\mathrm{fol}} = -\sum_v (A^{\mathrm{fol}}_{v,yz} + A^{\mathrm{fol}}_{v,xz} + A^{\mathrm{fol}}_{v,xy}) - \sum_p B_p^{\mathrm{fol}}$$

$$\downarrow \quad \text{gauging map } D^{G_m^{(1)}}$$

$$H_{XC,e} = -\sum_v (A^{XC,e}_{v,yz} + A^{XC,e}_{v,xz} + A^{XC,e}_{v,xy}) - \sum_p B_p^{XC,e} - \sum_c V_c^{XC,e}$$

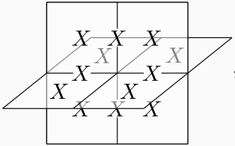
$V_c^{XC,e} = \prod_{p\supset c} Z_p$ is a Lagrange multiplier that enforces the local generator of the dual 1-form symmetry on each cube. We act $D^{G_m^{(1)}} \otimes I$ on 3-foliated Toric code.

Gauged Hamiltonian

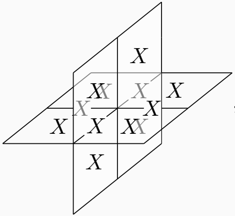
Hamiltonian

$$H_{XC,e} = -\sum_v (A^{XC,e}_{v,yz} + A^{XC,e}_{v,xz} + A^{XC,e}_{v,xy}) - \sum_p B^{XC,e}_p - \sum_c V^{XC,e}_c$$

$$A^{\mathrm{XC,e}}_{v,yz} = \prod_{e_y,e_z \supset v} X_e \prod_{p \supset e_y} X_p \quad =$$

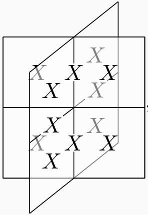


$$, \quad A^{\mathrm{XC,e}}_{v,xy} = \prod_{e_x,e_y \supset v} X_e \prod_{p \supset e_x} X_p \quad =$$



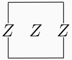
,

$$A^{\mathrm{XC,e}}_{v,xz} = \prod_{e_x,e_z \supset v} X_e \prod_{p \supset e_z} X_p \quad =$$



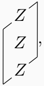
,

$$B^{\mathrm{XC,e}}_{p_{yz}} = Z_p \prod_{e_z \supset p} Z_e \quad =$$



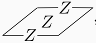
,

$$B^{\mathrm{XC,e}}_{p_{xz}} = Z_p \prod_{e_x \supset p} Z_e \quad =$$



,

$$B^{\mathrm{XC,e}}_{p_{xy}} = Z_p \prod_{e_y \supset p} Z_e \quad =$$



,

Unitary transformation

$$U_{SEF} = \prod_{p_{xz}} \prod_{e_z \supset p_{xz}} CX_{e_z, p_{xz}} \times \prod_{p_{xy}} \prod_{e_x \supset p_{xy}} CX_{e_x, p_{xy}} \times \prod_{p_{yz}} \prod_{e_y \supset p_{yz}} CX_{e_y, p_{yz}}$$

where CX acts as

$$CX(X \otimes I)CX = X \otimes X \qquad CX(I \otimes X)CX = I \otimes X$$

$$CX(Z \otimes I)CX = Z \otimes I \qquad CX(I \otimes Z)CX = Z \otimes Z$$

By transformation,

$$A^{XC,e}_{v,yz} \xmapsto{U_{SEF}} \prod_{e_y, e_z \supset v} X_e = A_{v,yz} \qquad A^{XC,e}_{v,xz} \xmapsto{U_{SEF}} \prod_{e_x, e_z \supset v} X_e = A_{v,xz}$$

$$A^{XC,e}_{v,xy} \xmapsto{U_{SEF}} \prod_{e_x, e_y \supset v} X_e = A_{v,xy} \qquad B^{XC,e}_{p} \xmapsto{U_{SEF}} Z_p \qquad V_c \xmapsto{U_{SEF}} \prod_{e \subset c} Z_e \prod_{p \subset c} Z_p$$

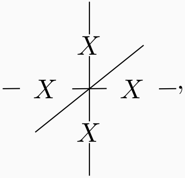
Result : X-cube

We restrict to the subspace where $B_p^{XC,e} = Z_p = I$.

Hamiltonian of X-cube

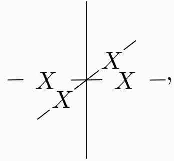
$$H_{XC} = -\sum_v (A_{v,yz} + A_{v,xz} + A_{v,xy}) - \sum_c W_c$$

$$A_{v,yz} = \prod_{e_y, e_z \supset v} X_e \quad =$$



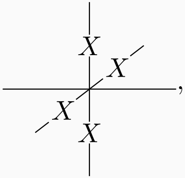
,

$$A_{v,xy} = \prod_{e_x, e_y \supset v} X_e \quad =$$



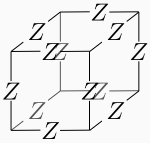
,

$$A_{v,xz} = \prod_{e_x, e_z \supset v} X_e \quad =$$



,

$$W_c = \prod_{e \subset c} Z_e \quad =$$



.

Future work

T. Devakul et al. arXiv:2009.01252(2021).

Meng-Yuan Li, Peng Ye, arXiv:2510.15766(2025).

- Topology vs Geometry
  - Exotic model like X-cube exhibit both topological and geometric features. To cleary disentangle these two contributions, one can make use of subsystem entanglement entropy(SEE).
- Fractal symmetry
  - The gauging web developed in this work systematizes X-cube type fracton phases by gauging planar subsystem symmetries.By applying a similar construction to fractal subsystem symmetries, it may be possible to build a more general gauging web that includes type-II fracton phases and fractal SPT phases.

Thank you for listening.