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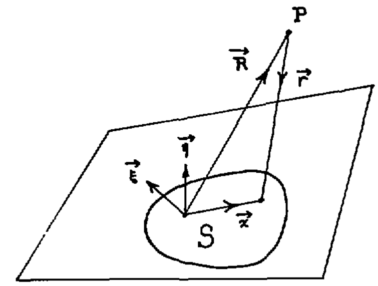

Fig. 1.

Far-Field Approximations to the Kirchhoff–Helmholtz Representations of Scattered Fields

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Abstract—**The standard far-field approximation to the Kirchhoff formula for the field scattered by a flat metallic plate S of arbitrary shape is given by a certain surface (double) integral. This double integral can be reduced to a line integral evaluated around the boundary of S. Moreover, if S is a polygon, this line integral can be reduced to a closed form expression involving no integrations at all. The use of such line integral representations can easily reduce the costs of numerical calculation by orders of magnitude. If the integrands are to be sampled p times per wavelength to achieve an acceptable degree of precision, and if A is the area of S, then the numerical evaluation of the double integral requires p^2A/λ^2 functional evaluations whereas the line integral only requires $p\sqrt{A/\lambda^2}$. If S is a polygon with N vertices, then only $2N$ functional evaluations are required to evaluate the closed form expression with no quadrature error at all.**

I. Introduction

In the Kirchhoff theory of physical optics the field diffracted through a plane aperture S (or scattered by a flat metallic reflector S) is represented by a Helmholtz integral, which is a certain (double) integral evaluated over S. The standard far-field approximation to the Helmholtz integral can be reduced to a line integral evaluated over the boundary of S by an easy application of Green's theorem (or Stokes' theorem). Moreover, as we shall show, if S is a polygon with N sides, then this line integral can be further reduced to an expression of the form $t_1 + t_2 + \cdots + t_N$, where each t_n is a certain complex quantity evaluated at the vertices of S. In other words, if S is a polygon, then the standard far-field approximation to the Helmholtz integral can be reduced to a closed form expression which involves no integrations at all.

These results (for apertures) will be given in Section II and derived in Section III. The corresponding results for the case when S is a flat reflector will be written out in Section IV.

Applications will be discussed in Section V. The reduction of the Helmholtz integral to a line integral is very important, since the use of such line integral representations in machine calculations can easily reduce costs by orders of magnitude. From the standpoint of theory, they are interesting because they show very clearly how the Kirchhoff theory predicts certain geometrical optics effects and also those edge effects which are implied by the Fermat principle.

The reduction of the far-field approximation of the Helmholtz integral to a line integral is in no way remarkable. In fact, if the incident radiation is a plane or spherical wave, then according to some old results of Maggi and Rubinowicz the Helmholtz integral itself can be represented as a line integral [1], [5]. The Maggi–Rubinowicz representations do not involve any far-field approximations and are *mathematically* exact, although the Kirchhoff theory itself is a *physical* approximation which is only valid when the dimensions of S are not too small in relation to the wavelength of the incident field. Generalizations of the Maggi–Rubinowicz results for other classes of incident radiation have been discussed by Miyamoto and Wolf [3] and Rubinowicz [4], [5]. (See also [2].)

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II. The Results for Apertures

A. Notation

We suppose that a plane monochromatic wave with wavelength λ (and wave number $k = 2\pi/\lambda$) is incident on an opaque screen with aperture S. The unit vector ξ normal to the incident wave front will be made to point from the illuminated to the shadowed side of the screen. The "field point" P will be the point on the shadowed side of the screen at which the diffracted field is measured. The geometry is shown in Fig. 1.

- ξ Unit normal to the incident wave front.
- η Unit normal to the plane of S, pointing into the half-space containing P.
- r Vector drawn *from* P to a general point on plane of S.
- R Vector drawn from a fixed point in S (e.g., the centroid) to P, so that R represents a "mean" distance from S to P.
- R_1 $= R/R$.
- x $= r + R$, (so that x lies in the plane of S).
- w Projection of $[\xi - R_1]$ onto plane of S.

Since the projection of any vector v onto the plane of S is given by

$$\mathrm{Proj}\,(v) = v - (v \cdot \eta)\eta$$

we have

$$w = (\xi - R_1) - [(\xi - R_1) \cdot \eta]\eta. \tag{2.1}$$

Let u_P denote the space dependent part of the diffracted field at P. Then for any class of incident radiation $u = u(r)$, the Kirchhoff–Helmholtz representation of u_P is given by

$$u_P = \frac{1}{4\pi}\iint_S \left\{u(r)\nabla\left(\frac{e^{ikr}}{r}\right) - \left(\frac{e^{ikr}}{r}\right)\nabla u(r)\right\}\cdot \eta \, dA.$$

In our particular case, the incident field is a plane wave which will be normalized so that it has unit field strength, and zero phase at the point of S given by $x = 0$. Hence, writing u as a function of r (as is required in the Kirchhoff–Helmholtz representation)

$$u(r) = e^{ik\xi\cdot(r+R)}, \qquad (= e^{ik\xi\cdot x} \text{ on } S).$$

Hence

$$u_P = \frac{1}{4\pi} \iint_S e^{ik(r+\xi\cdot x)} \left\{ \left(\frac{ik}{r} - \frac{1}{r^2} \right) \frac{r}{r} - \frac{ik}{r}\, \xi \right\} \cdot \eta \, dA. \quad (2.2)$$

Taking $x = [x_1, x_2]$ as Euclidean coordinates in the plane of S, and expanding $r + \xi \cdot x$ as a function of x around $x = 0$, we have

$$r + \xi \cdot x = R + (\xi \cdot x) - (R_1 \cdot x) + \text{higher order terms in } x$$
$$= R + (w \cdot x) + \text{higher order terms in } x. \quad (2.3)$$

In the far-field approximation, we have $R \gg D^2/\lambda$, where D is the diameter of S (= supremum of all lengths $\overline{Q_1Q_2}$, where Q_1 and Q_2 are general points lying in S). Neglecting the $(1/r^2)$ term in the integrand in (2.2) and the "higher order terms" in (2.3), the far-field approximation to (2.2) is

$$u_P = -ik(R_1 \cdot \eta + \xi \cdot \eta) \frac{e^{ikR}}{4\pi R} \iint_S e^{ikw\cdot x} \, dx_1 \, dx_2. \quad (2.4)$$

Remark: These results are standard, but it is common for some authors to write $-i$ where we have written i. (Cf. [6].)

B. Statement of Results for Apertures

Let ∂S denote the boundary of S, and let $x = x(t)$ be a parametric representation of ∂S. For any vector $a = [a_1, a_2]$ in the plane of S let $a^* = [a_2, -a_1]$ be the vector which is obtained by rotating a through an angle of 90° *clockwise*, as seen from an observer at P or any other point in the shadowed half-space. Note that

$$a^* = a \times \eta \qquad a^{**} = -a \qquad a^* \cdot b = -a \cdot b^*. \quad (2.5)$$

i) The reduction of the double integral in (2.4) to a line integral is given by

$$u_P = \frac{(R_1 \cdot \eta + \xi \cdot \eta)}{w^2} \frac{e^{ikR}}{4\pi R} \int_{\partial S} e^{ikw\cdot x} (w_2 \, dx_1 - w_1 \, dx_2) \quad (2.6a)$$

equivalently,

$$u_P = \frac{(R_1 \cdot \eta + \xi \cdot \eta)}{w^2} \frac{e^{ikR}}{4\pi R} \int_{\partial S} e^{ik[w\cdot x(t)]} [w^* \cdot \dot{x}(t)] \, dt. \quad (2.6b)$$

Let S be an N-gon whose vertices are $a_1, \cdots, a_N$. Set $a_{N+1} = a_1$, and for $1 \le n \le N$ set $\Delta a_n = a_{n+1} - a_n$.

ii) The reduction of (2.4) to a form involving no integrations in the case when S is an N-gon is given by

$$u_P = \frac{(R_1 \cdot \eta + \xi \cdot \eta)}{w^2} \frac{e^{ikR}}{4\pi R} (T_1 + \cdots + T_N)$$

where, for $1 \le n \le N$,

$$T_n = (w^* \cdot \Delta a_n) \frac{\sin \left[\frac{k}{2} \, w \cdot \Delta a_n \right]}{\left[\frac{k}{2} \, w \cdot \Delta a_n \right]} \exp \left[\frac{ik}{2} \, w \cdot (a_n + a_{n+1}) \right]. \quad (2.7)$$

When $w = 0$, a direct substitution into (2.4) yields

$$u_P = -ik(R_1 \cdot \eta + \xi \cdot \eta) \frac{e^{ikR}}{4\pi R} A$$

where A = area of S. Geometrically, $w = 0$ means that $\xi = R_1$. Hence

$$w = 0 \text{ implies } u_P = -ik(R_1 \cdot \eta) \frac{e^{ikR}}{2\pi R} A. \quad (2.8)$$

III. The Derivations

i) The derivations depend on the following simple lemma.

Lemma: Let $w = [w_1, w_2]$ be a constant vector. Then

$$\iint_S e^{ikw\cdot x} \, dx_1 \, dx_2 = \frac{i}{kw^2} \int_{\partial S} e^{ikw\cdot x} (w_2 \, dx_1 - w_1 \, dx_2). \quad (3.1)$$

Proof: Apply Green's theorem (or Stokes' theorem) to the right side of (3.1):

$$\int_{\partial S} e^{ikw\cdot x} (w_2 \, dx_1 - w_1 \, dx_2)$$
$$= \iint_S \left\{ -w_1 \frac{\partial}{\partial x_1} (e^{ikw\cdot x}) - w_2 \frac{\partial}{\partial x_2} (e^{ikw\cdot x}) \right\} dx_1 \, dx_2$$
$$= -ikw^2 \iint_S e^{ikw\cdot x} \, dx_1 \, dx_2.$$

ii) To obtain (2.6), apply (3.1) to the right side of (2.4).

iii) Suppose now that S is an N-gon. To obtain (2.7) we use (2.6b). Let $x(t) = (1 - t)a_n + ta_{n+1}$ be the parametric representation of the nth side of S (with $a_{N+1} = a_1$ as before). The contribution of the nth side of S to the integral in (2.6b) is

$$\int_0^1 (w^* \cdot \Delta a_n) \exp \left[ikw \cdot \{(1 - t)a_n + ta_{n+1}\} \right] dt$$
$$= (w^* \cdot \Delta a_n) e^{ikw\cdot a_n} \int_0^1 e^{iktw\cdot \Delta a_n} \, dt$$
$$= (w^* \cdot \Delta a_n) \frac{e^{ikw\cdot a_n}}{ikw \cdot \Delta a_n} (e^{ikw\cdot \Delta a_n} - 1)$$
$$= (w^* \cdot \Delta a_n) \frac{e^{ikw\cdot a_n}}{ikw \cdot \Delta a_n} e^{ikw\cdot \Delta a_n/2} (2i) \sin \left[\frac{k}{2} \, w \cdot \Delta a_n \right]$$
$$= (w^* \cdot \Delta a_n) \frac{\sin \left[\frac{k}{2} \, w \cdot \Delta a_n \right]}{\left[\frac{k}{2} \, w \cdot \Delta a_n \right]} \exp \left[\frac{ik}{2} \, w \cdot (a_n + a_{n+1}) \right]$$
$$= T_n.$$

IV. Application to Flat Reflectors

Suppose a flat metallic reflector S is illuminated by a spherical wave. Then one can apply the Kirchhoff–Helmholtz theory to compute the field scattered by S by treating S as an aperture and replacing the true source with its "image" source (located symmetrically on the opposite side of the plane of S). Similarly, when the incident radiation is a plane wave whose wave unit normal vector is the "ray" ξ_i, the Kirchhoff–Helmholtz theory can be applied by replacing the incident ray ξ_i with the "reflected" ray ξ_r, given by

$$\xi_r = \xi_i - 2(\xi_i \cdot \eta)\eta. \quad (4.1)$$

Hence, in order to apply the results (2.6), (2.7) to the case when S is a reflector, we replace each occurrence of the vector ξ in (2.1), (2.6), and (2.7) with the vector ξ_r given by (4.1).

Since the projections of ξ_r and ξ_i onto the plane of S are equal, the new expression for w becomes

$$w = (\xi_i - R_1) - [(\xi_i - R_1) \cdot \eta]\eta. \tag{4.2}$$

The Kirchhoff representation (2.4) becomes

$$u_P = -ik(R_1 \cdot \eta - \xi_i \cdot \eta)\frac{e^{ikR}}{4\pi R}\iint_S e^{ikw\cdot x}\, dx_1\, dx_2. \tag{4.3}$$

Geometrically, $w = 0$ means that $\xi_r = R_1$. Hence [cf. (2.8)]

$$w = 0 \text{ implies } u(P) = -ik(R_1 \cdot \eta)\frac{e^{ikR}}{2\pi R}A. \tag{4.4}$$

For the case $w \neq 0$, (2.6b) becomes

$$u_P = \frac{(R_1 \cdot \eta - \xi_i \cdot \eta)}{w^2}\frac{e^{ikR}}{4\pi R}\int_{\partial S} e^{ik[w\cdot x(t)]}[w^* \cdot \dot{x}(t)]\, dt. \tag{4.5}$$

For the case when $w \neq 0$ and S is an N-gon, (2.7) becomes

$$u_P = \frac{(R_1 \cdot \eta - \xi_i \cdot \eta)}{w^2}\frac{e^{ikR}}{4\pi R}(T_1 + \cdots + T_N)$$

where for $1 \leq n \leq N$,

$$T_n = (w^* \cdot \Delta a_n)\frac{\sin\left[\frac{k}{2}\, w \cdot \Delta a_n\right]}{\left[\frac{k}{2}\, w \cdot \Delta a_n\right]}\exp\left[\frac{ik}{2}\, w \cdot (a_n + a_{n+1})\right]. \tag{4.6}$$

V. Applications

The use of the line integral representations (2.6) can easily reduce the costs of numerical calculations by orders of magnitude. The reason for this is that if N sample points are required to evaluate the line integral of a function over the boundary of a domain, then N^2 points would be required to evaluate the corresponding surface (double) integral of the function over the domain with the same degree of precision. (More generally, the error in numerically integrating a function over a d-dimensional hypercube varies as $N^{-C/d}$, where C is a constant involving bounds or the derivatives of the integrand.)

For example, suppose that S has a circumference of 20λ, and suppose that it is determined that the integrand in (2.6) must be sampled twice per wavelength to obtain an exceptable degree of precision. Then $N = 40$ sample points would be required to evaluate the line integral (2.6), and $N^2 = 1600$ sample points would be required to evaluate the corresponding double integral (2.4). Moreover, if S were, say, an irregular quadrilateral, then one could achieve an even greater economy in calculation by the use of (2.7), which requires only 8 sample points, viz., the points $a_{n+1} \pm a_n$, $1 \leq n \leq 4$. Formula (2.7) also has the advantage that it is "exact," so that the accuracy attained by its use does not depend on the wavelength λ.

Remark: The same considerations apply to the Maggi–Rubinowicz line integral representations of the "exact" Kirchhoff–Helmholtz integral (involving *no* far-field assumptions). If this latter integral can be expressed *in closed form* as a line integral (which is always possible when the incident field is a plane wave of unit strength, a spherical wave, or dipole radiation [5]) then the use of the line integral representation will reduce the number of sample points by a *factor* proportional to $p\sqrt{A/\lambda^2}$, where A is the area of S and p is the number of sample points per wavelength required to achieve an acceptable degree of precision.

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Angular Phase Distribution of the Field of an Electric Dipole in a Hot Magnetoplasma

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Abstract—**The angular distribution of the phase of the fields of an electric dipole with and without a sheath in a hot magnetoplasma is calculated. Like the amplitude distribution of the field, the phase distribution can also be used for estimating the electron temperature.**

Introduction

A new method for plasma diagnostics through the measurements on the radiation pattern of a short electric dipole has been reported in a series of recent papers [1]–[5]. In the theoretical work referred to, only the angular distribution of the amplitude of the field of the electric dipole has been reported. In this communication we report the theoretical calculations of the phase. The sheath around the dipole, which has a profound effect on the radiation field, is included in a simple way [6].

The potential of the electric dipole in a warm uniaxial plasma is given by

$$V(r,\theta) = \frac{1}{2\pi^2\varepsilon_0}\int_0^\infty F(k_z)K_0(r\sin\theta\varepsilon)I_0(a\varepsilon) \cdot \cos(r\cos\theta k_z)e^{-a_{sh}k_z} \tag{1}$$

$$\varepsilon = k_z K_{11}{}^{1/2} \tag{2}$$

$$K_{11} = 1 - \left(\frac{\omega_p{}^2}{k_z{}^2 V_t{}^2}\right)Z'\left(\frac{\omega}{k_z V_t}\right) \tag{3}$$

$$F(k_z) = \frac{-4i\sin^2(k_z l/2)}{k_z} \tag{4}$$

where K_0 and I_0 are the modified Bessel functions, ω_p is the plasma frequency, V_t is the rms electron thermal velocity, l and a are the dipole halflength and radius, respectively, and a_{sh} is the effective sheath radius [6], Z' is the derivative of the plasma dispersion function [7], r is the distance from the dipole, and θ is the angle measured from the static magnetic field B_0.

The uniaxial assumption for the magnetoplasma yields good results if the theory and the experiments are compared in terms of the resonance cone angle rather than the frequencies. Since the amplitude and the phase of the field are useful for plasma diagnostics only near the resonance cone, we have represented the field by a potential, which is a valid approximation near the resonance cone [1]. $F(k_z)$ in (4) is the Fourier transform of the charge on the dipole assuming a triangular current distribution.

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