Knapsack Problem

Sakeena | 03-02-2022

Knapsack Problem

→ more speific

- **Problem:** Given n items {1, 2, … , n}, and each has a given non-negative weight w_i **and** an associated value v_i (for i={1, 2, … , n}) . We also have a bound W. We would like to select a subset S if the items so that $\sum_{i\in S} w_i \leq W$, and subject to the restriction that $\sum_{i\in S} v_i$ is as large as possible.
- Informally: Consider, we have a knapsack that can hold upto W kgs. If we want to choose from n items to add to the bag where each item is w_i kgs and costs v_i INR, to keep the knapsack as valueable as possible, which items will you choose?
- **More general case of Subset Sum Problem**

0-1 knapsack { either pick an item or leave it.
eg: microwave

v/s

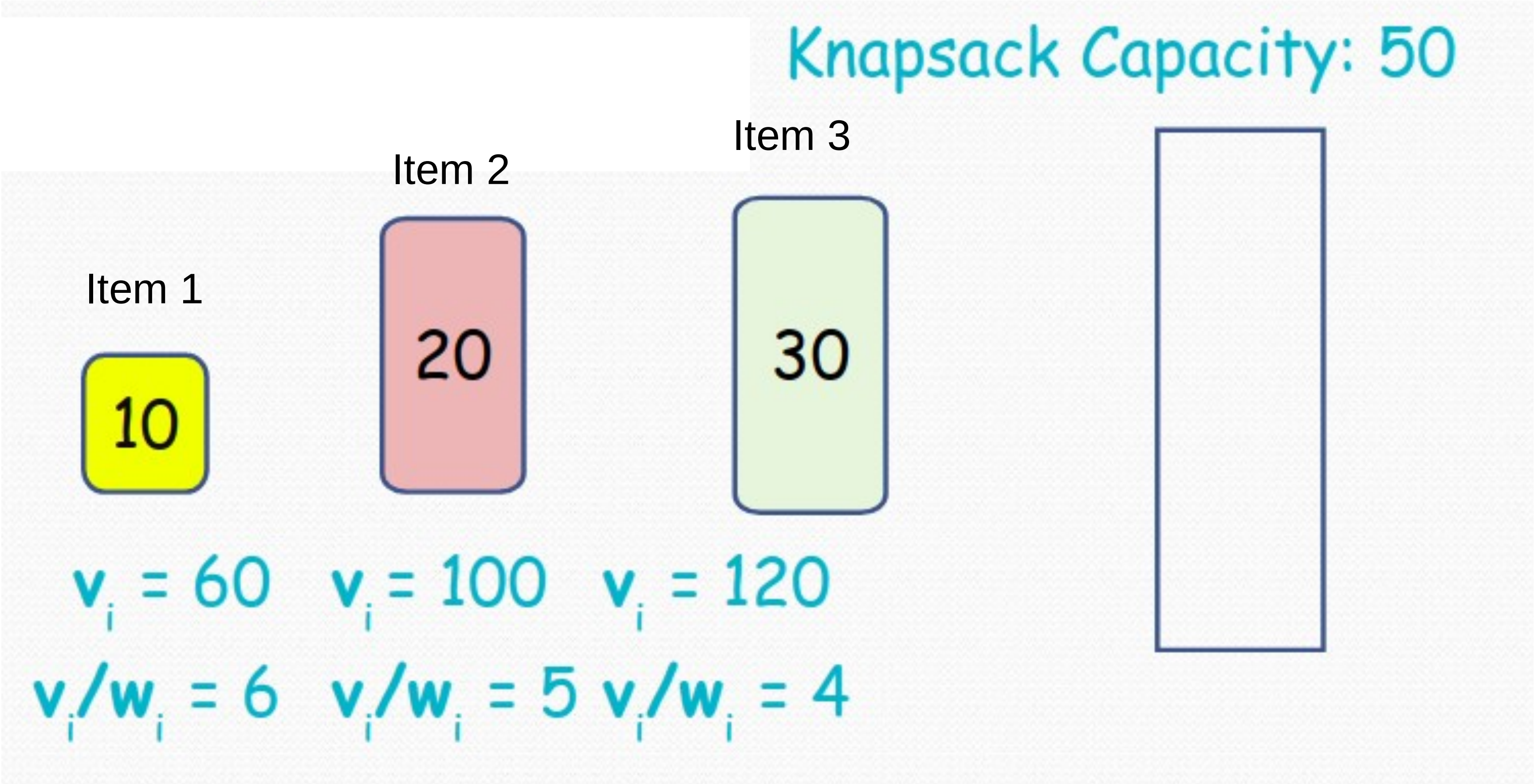
Fractional knapsack { can take a fraction of an item.
eg:- sugar

Greedy approach

- Which item is the most valuable?
- The one which has the greatest value to weight ratio.
- That is, value as high as possible along with weight as low as possible.
- Approach: Pick the items in the decreasing order of value per unit weight i.e. highest first.

Example

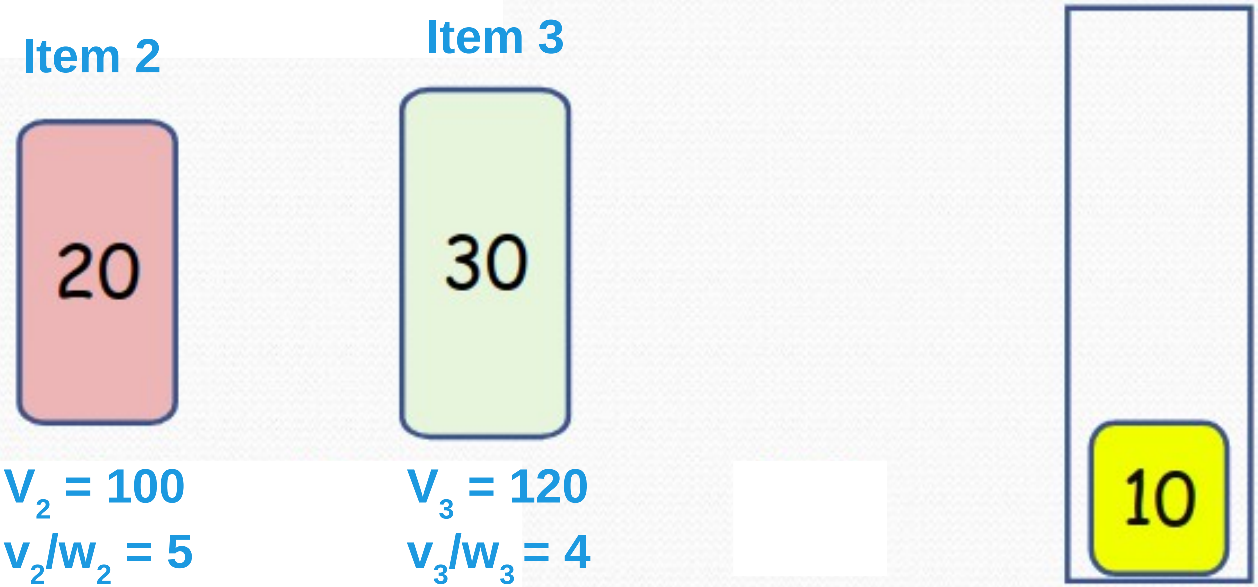
Greedy approach: Pick the items in the decreasing order of value per unit weight i.e. highest first.



Remaining capacity of knapsack = 40

Item 2

Item 3



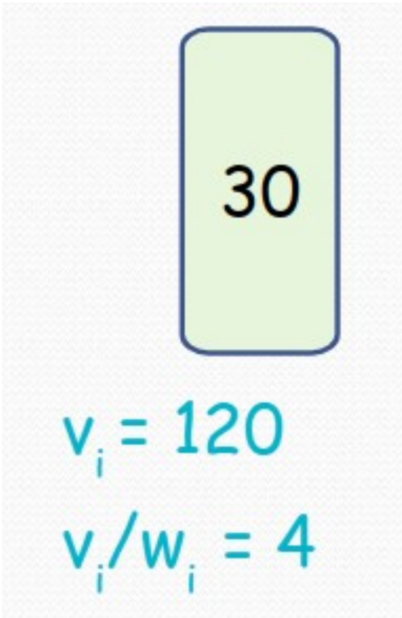
$V_2 = 100$
$v_2/w_2 = 5$

$V_3 = 120$
$v_3/w_3 = 4$

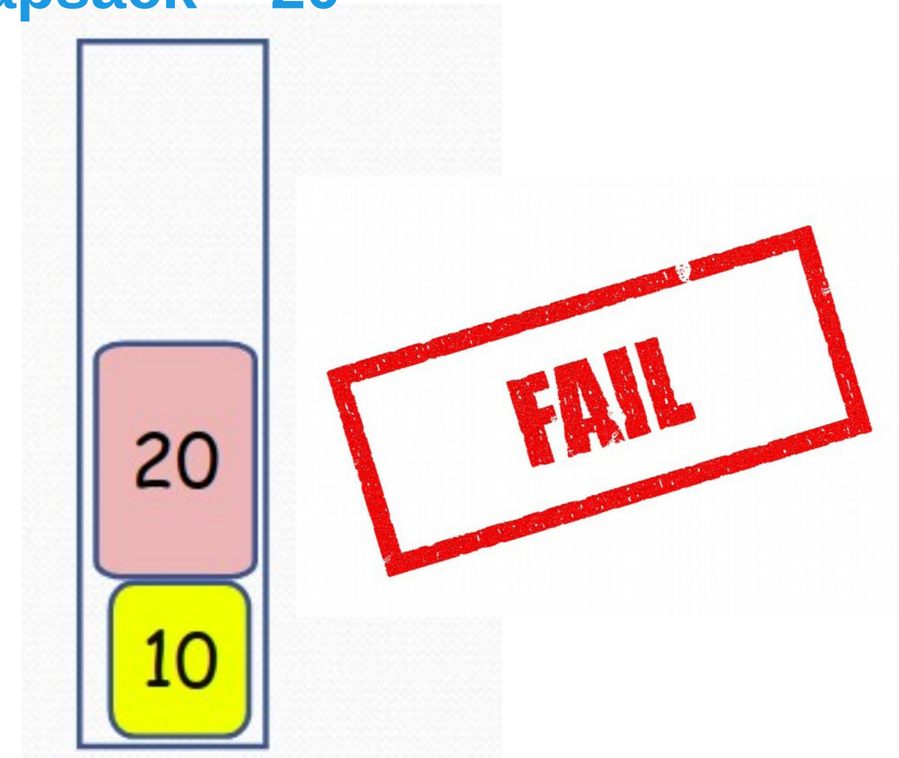
Current value of knapsack: 60

Remaining capacity of knapsack = 20

Item 3



Item 3 cannot be added to knapsack because capacity does not allow.



Current value of knapsack = 60 + 100 = 160

SUBOPTIMAL

Optimal value?

DP solution to 0-1 knapsack

- Let OPT(n, W) denote the value of optimal solution with n objects and capacity W.
- Working on similar lines as in WIS and Subset sums,
- If n does not belong to OPT, then OPT(n, W) = OPT(n-1, W)
- If n belongs to OPT then ? Which sub-problem to consider? OPT(n-1, W)?
- But OPT(n-1, W) denote the optimal solution with knapsack capacity W. If n belongs to OPT then we have reduced capacity in our knapsack for the smaller sub-problems. i.e. we need to consider OPT(n – 1, W – w_n). Then,
- **OPT(n, W) = max{ OPT(n-1, W) , $\cancel{w_n}$ + OPT(n-1, W-w_n)**

v_n

DP solution to 0-1 knapsack

- If we knew the exact value (say K) of OPT knapsack, then to compute OPT(n, K) we know that for n-1 objects, we have to solve exactly two problems : OPT(n-1, K) and OPT(n -1, K – w_n).
- But obviously, we don’t know that so we make a guess ***w*** for K (i.e. try out all possible values for K) and solve the problem for w. So, we add a dimension(/variable) to our problem.
- Similarly while dealing with objects {1 … i} we need to solve OPT(i-1, K) and OPT(i -1, K – w_i).
- Thus in general we need to define OPT(i, w) for every i < n and every w < W

DP solution for knapsack

- Let Opt[n, W] be the optimal value obtained when considering objects {1 … n} and filling a knapsack of capacity W
  - m[0,w] = 0 (any capacity, but 0 items=> optimal =0)
  - m[i,0] = 0 (any no. of items. But 0 capacity=> optimal =0)
  - m[i,w] = m[i-1,w] if $w_i > w$ – cannot include
  - $m[i,w] = \max\{\underbrace{m[i-1, w-w_i] + vi}_{\text{can include + will include}} , \underbrace{m[i-1, w]}_{\text{can include + will not include}}\}$ if $w_i <= w$

Example

- n = 4
- W = 5
- Elements (weight, value): (2,3), (3,4), (4,5), (5,6)

Find optimal value and optimal solution.

- Create M. Dimensions?
- What cell will denote the optimal value?

Time Complexity?

Weight	Value
1	3
2	4
3	5
4	6

$W = 5$ kg is knapsack capacity.

4	0	3	4	7	8	9
3	0	3	4	7	8	9
2	0	3	4	7	7	7
1	0	3	3	3	3	3
0	0	0	0	0	0	0
	0	1	2	3	4	5

Ans = 9

Please read "Sequence Alignment" posted on portal.