CS 214 – Algorithms and Complexity

Shanghai Jiaotong University, Fall 2016

- Handed out: Friday, 2016-10-28
- Due: Wednesday, 2016-11-02
- Feedback: Friday, 2016-11-04
- Revision due: Monday, 2016-11-07

3 Minimum Spanning Trees

Throughout this assignment, let $G = (V, E)$ be a connected graph and $w : E \to \mathbb{R}^+$ be a weight function.

Exercise 1. [Good sets and the Cut Lemma] A set $X \subseteq E$ is called *good* if there exists a minimum spanning tree T of G such that $X \subseteq E(T)$.

1. State the Cut Lemma from last class and sketch its proof. Draw a picture!
2. Prove the inverse of the cut lemma: If X is good, $e \notin X$, and $X \cup e$ is good, then there is a cut $S, V \setminus S$ such that (i) no edge from X crosses this cut and (ii) e is a minimum weight edge of G crossing this cut.

Definition 2. *For $c \in \mathbb{R}$ and a weighted graph $G = (V, E)$, let $G_c := (V, \{e \in E \mid w(e) \le c\})$. That is, G_c is the subgraph of G consisting of all edges of weight at most c.*

Lemma 3. *Let T be a minimum spanning tree of G, and let $c \in \mathbb{R}$. Then T_c and G_c have exactly the same connected components. (That is, two vertices $u, v \in V$ are connected in T_c if and only if they are connected in G_c).*

Exercise 4.
1. Illustrate Lemma 3 with an example!
2. Prove the lemma.

Definition 5. *For a weighted graph G, let $m_c(G) := |\{e \in E(G) \mid w(e) \leq c\}|$, i.e., the number of edges of weight at most c (so G_c has $m_c(G)$ edges).*

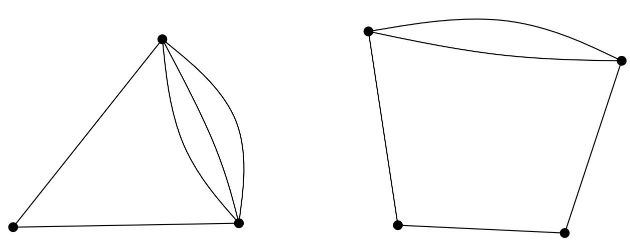
Lemma 6. *Let T, T' be two minimum spanning trees of G. Then $m_c(T) = m_c(T')$.*

Exercise 7.
1. Illustrate Lemma 6 with an example!
2. Prove the lemma.

Exercise 8. Suppose no two edges of G have the same weight. Show that G has exactly one minimum spanning tree!

Definition 9. *Suppose H is a graph, not necessarily connected. A* spanning forest *is an acyclic spanning graph with a maximum number of edges. Equivalently, it contains a spanning tree for each connected component of H.*

A *multigraph* is a graph that can have multiple edges, called "parallel edges". Without defining it formally, we illustrate it:



A multigraph.

All other definitions, like connected components, spanning trees, spanning forests, are the same as for normal (non-multi) graphs. However, when two spanning forests use different parallel edges, we consider them different:

The same multigraph with two different spanning forests.

Exercise 10. How many spanning forests does the above multigraph on 7 vertices have? Justify your answer!

Exercise 11. Suppose you have a polynomial-time algorithm that, given a multigraph H, computes the number of spanning forests of H. Using this algorithm as a subroutine, design a polynomial-time algorithm that, given a weighted graph G, computes the number of minimum spanning trees of G.