Homework 3

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Exercise 1

1. **Definition 1** (Cut Lemma)**.** *Suppose edge set X is good, pick any vertex set $S \subseteq V$ s.t. there is no edge in X that goes from S to $V \backslash S$. Let $e \in E$ be the edge going from S to $V \backslash X$ with the cheapest weight, then $X \cup \{e\}$ is also good.*

   Proof.

   (1) If the cheapest edge e happens to be in the tree T, then the case is trivial.

   (2) If the cheapest edge e is not in the tree T, since T is already a tree, adding any edge to it will result in a circle and there must exist another edge e' which also goes from S to $V \backslash X$. If we remove this edge e', we will get another graph $T' = T \cup \{e\} - \{e'\}$. Next, we are going to prove that it is also a minimum spanning tree.

      (a) First, we prove that T' is a tree. Since T is a tree, adding a edge to it will form a circle. Then we remove the edge e' from $T \cup \{e\}$ where e' is part of a circle and removing it will not disconnect the graph, hence $T' = T \cup \{e\} - e'$ is also connected. On the other hand, in the connected graph T', $|E| - |V| = 1$, therefore T' is a tree.

      (b) Next, we prove that T' is a minimum spanning tree. Since substitute e' for e will not affect spanning property of minimum spanning tree, all we need to prove is it takes minimum weight. From the equation $weight(T') = weight(T) - w(e) + w(e')$, since e' is chosen to be the edge with minimum weight, thus $weight(T') < weight(T)$. Therefore T' is a minimum spanning tree.

   Combine (1) and (2), cut lemma is proved. □

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