$$\Gamma: x = s,\, y = 0,\, z = h(s),\ s \in \mathbb{R}. \tag{2.47}$$

**Characteristic equations and the initial conditions for solving them**

$$\frac{dx}{dt} = z,\ \frac{dy}{dt} = 1,\ \frac{dz}{dt} = 0, \tag{2.48}$$

with initial conditions

$$x(0) = s,\ y(0) = 0,\ z(0) = h(s). \tag{2.49}$$

The characteristic equations is a system of linear ODE and hence the IVP has a unique solution defined for all $t \in \mathbb{R}$.

**Solutions of the IVP for Characteristic equations**

The unique solution to (2.48)-(2.49) is given by

$$x = X(s,t) = h(s)t + s,\ y = Y(s,t) = t,\ z = Z(s,t) = h(s) \text{ defined for } (s,t) \in \mathbb{R}\times\mathbb{R}. \tag{2.50}$$

**Comment on the Base characteristics**

Projection of a characteristic curve (corresponds to a fixed $s$) to the $xy$-plane are given by parametric equations

$$\{(h(s)t + s, t) : t \in \mathbb{R}\} \tag{2.51}$$

In cartesian coordinates, the equation is given by

$$X = h(s)Y + s. \tag{2.52}$$

This is a straight line with slope $\dfrac{1}{h(s)}$. Base characteristics correspond to different $s_1$ and $s_2$ meet if and only if the following system has a solution

$$\begin{pmatrix} 1 & -h(s_1) \\ 1 & -h(s_2) \end{pmatrix} \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \tag{2.53}$$

Since $s_1 \neq s_2$, even if $h(s_1) = h(s_2)$ the corresponding base characteristics are two parallel lines and hence do not intersect. If $h(s_1) - h(s_2) \neq 0$, then both straight lines meet at a unique point given by

$$\begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} = \frac{1}{h(s_1) - h(s_2)} \begin{pmatrix} s_2 h(s_1) - s_1 h(s_2) \\ s_2 - s_1 \end{pmatrix} \tag{2.54}$$

**Comments on Jacobian and explain if the local existence and uniqueness theorem can be applied or if it is a characteristic Cauchy problem**

$$J = \frac{\partial(X,Y)}{\partial(s,t)} = \begin{vmatrix} h'(s)t + 1 & h(s) \\ 0 & 1 \end{vmatrix} = 1 + h'(s)t. \tag{2.55}$$

We expect something wrong at $(s,t)$ where $1+h'(s)t=0$, *i.e.,* at $t=-\frac{1}{h'(s)}$. We will come back to this later on. Nevertheless,

$$J=\frac{\partial(X,Y)}{\partial(s,t)}|_{(s,t)=(s_0,0)}=\begin{vmatrix} 1 & h(s_0) \\ 0 & 1 \end{vmatrix}\neq 0. \tag{2.56}$$

Thus we can apply the local existence theorem (Theorem 2.17) and we get a unique solution to the Cauchy problem near $(s_0,0,h(s_0))\in\Gamma$.

**The Final solution(s)**

Eliminating $s,t$, we get

$$z=Z(S(x,y),T(x,y))=h(x-yz) \tag{2.57}$$

Thus the solution to Cauchy problem is given implicitly by

$$u=h(x-yu). \tag{2.58}$$

∎

**Special case:** Let us now analyze what happens when $h$ is a monotonic function.

(i) Consider the following non-increasing function

$$h(x)=\begin{cases} 1 & \text{if } x\leq 0, \\ 1-x & \text{if } 0\leq x\leq 1, \\ 0 & \text{if } x\geq 1. \end{cases} \tag{2.59}$$

Since $h$ is non-increasing, the characteristic slope is non-decreasing and hence characteristics collide.

(ii) If $h$ is non-decreasing, the characteristic slope is non-increasing and hence characteristics never collide, and thus giving rise to smooth solutions.

Note that if more than one base characteristic passes through a point $(x,y)$ where $y>0$, then multiplicity of information reaches the point $(x,y)$ from the initial curve, and these information could be conflicting. Thus the notion of solution at $(x,y)$ breaks down. In the present example, let $\gamma_{s_1}$ and $\gamma_{s_2}$ be two base characteristics that pass through a point $(x,y)$. Since $x=h(s_1)y+s_1$ and $x=h(s_2)y+s_2$, we get

$$y=\frac{s_2-s_1}{h(s_1)-h(s_2)}. \tag{2.60}$$

If the function $h$ is increasing *i.e.,* $h(s_1)<h(s_2)$ whenever $s_1<s_2$, then $\frac{s_2-s_1}{h(s_1)-h(s_2)}<0$ which rules out the possibility of intersecting base characteristics. On the other hand, if the function $h$ is decreasing then $\frac{s_2-s_1}{h(s_1)-h(s_2)}>0$ and thus base characteristics do intersect.

If the function $h$ is decreasing, then we expect an infinite slope of a tangent to the wave profile at some $y>0$. Indeed differentiating both sides of $u(x,y)=h(x-yu(x,y))$ w.r.t. $x$ to obtain

$$u_x(x,y)=h'(x-yu(x,y))(1-yu_x(x,y)).$$

Re-arranging terms in the last equation yields

$$u_x(x,y) = \frac{h'(x - y u(x,y))}{1 + y h'(x - y u(x,y))} \tag{2.61}$$

Note that $u_x(x,y)$ will become infinite at any point for which $1 + y h'(x - y u(x,y)) = 0$. This phenomenon is referred to as *gradient catastrophe*.

Using the implicit formula (2.58) for the solution to Cauchy problem for Burgers equation, we will obtain solutions explicitly for three different Cauchy data in the following examples.

**Example 2.24 (Characteristics do not reach some points).** Consider Burgers equation with Cauchy data given by

$$h(x) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases} \tag{2.62}$$

Equation for the family of base characteristics (indexed by $s \in \mathbb{R}$) is given by

$$y = \frac{1}{h(s)} x - \frac{s}{h(s)}.$$

Thus a base characteristic passing through a point $(s,0)$ with $s < 0$ is given by the line $y = -x + s$, which is a family of lines parallel to $y = -x$. and when $s \geq 0$ the base characteristic passing through a point $(s,0)$ is given by the line $y = x - s$, which is a family of lines parallel to $y = x$. Note that no characteristic passes through the region in the upper half-plane which is bounded by the lines $y = x$ and $y = -x$. In other words, information from the initial data does not reach this region. The question is how to make sense of solution at those points? This will be discussed in the Section **??**. ∎

**Example 2.25 (Intersecting base characteristics).** Consider Burgers equation with Cauchy data given by

$$h(x) = \begin{cases} 1 & \text{if } x < 0, \\ -1 & \text{if } x \geq 0. \end{cases} \tag{2.63}$$

Equation for the family of base characteristics (indexed by $s \in \mathbb{R}$) is given by

$$y = \frac{1}{h(s)} x - \frac{s}{h(s)}.$$

Thus a base characteristic passing through a point $(s,0)$ with $s < 0$ is given by the line $y = x - s$, which is a family of lines parallel to $y = x$. and when $s \geq 0$ the base characteristic passing through a point $(s,0)$ is given by the line $y = -x + s$, which is a family of lines parallel to $y = -x$. Note that the base characteristics emanating from the negative $x$-axis intersect those emanating from the positive $x$-axis. Thus there is more than one characteristic passing through the region above $x$-axis that is bounded by the lines $y = x$ and $y = -x$. At other parts of the upper half-plane, solution is uniquely determined, namely the region which is left to the line $y = -x$, and the region which is right to the line $y = x$. Multiplicity of information is reaching rest of the upper half-plane. The question is how to make sense of solution at those points? This will be discussed in the Section **??**. ∎

**Example 2.26.** Consider Burgers equation with Cauchy data given by (2.59). ∎

### 2.2.5 ▪ Domain of dependence, Domain of influence

In this subsection we discuss the concepts of domains of dependence and influence in the context of two examples. These concepts are characteristic of hyperbolic partial differential equations. We revisit these concepts in the context of wave equations in Section 5.2.

**Example 2.27.** Consider the following Cauchy problem

$$u_x + u_y = 0, \tag{2.64a}$$
$$u(x,0) = \sin x, \quad \text{for } x \geq 0. \tag{2.64b}$$

The solution of the given Cauchy problem is given by

$$u(x,y) = \sin(x-y) \text{ for } y \leq x.$$

Note that no base characteristic passes through the points in the first quadrant of $xy$-plane which lie above the line $y = x$. Thus the information from the initial data does not reach those points. In other words, domain of influence of the initial data is the points in the first quadrant of $xy$-plane that lie on or below the line $y = x$. Note that the value of the solution at a point $(x_0, y_0)$ with $y_0 \leq x_0$, which is given by

$$u(x_0,y_0) = \sin(y_0 - x_0),$$

depends only on the value of the initial condition at the point $(\xi_0, 0) := (y_0 - x_0, 0)$ on the $x$-axis. In other words, domain of dependence for a solution at the point $(x_0, y_0)$ is the point $(\xi_0, 0)$.

If we consider an interval $[\alpha, \beta]$ on the $x$-axis, then the domain of influence of this interval is the region in the first quadrant of $xy$-plane which is enclosed by the two base characteristics originating from $(\alpha, 0)$ and $(\beta, 0)$ respectively. This region turns out to be

$$\{(x,y) \in \mathbb{R}^2 : \alpha \leq x \leq \beta,\ y \geq 0,\ x-\beta \leq y \leq x-\alpha\}.$$

∎

**Example 2.28.** This is a continuation of Example 2.23. As a consequence of the third equation in (2.48), the solution $u$ is constant along each of the base characteristic curves. Thus each base characteristic is a straight line. Let $\gamma_s$ denote the base characteristic passing through $(s, 0)$. The equation for $\gamma_s$ is given by the equation

$$x = h(s)y + s.$$

Let $\gamma_s$ denote a base characteristic and let $(x,y)$ be an arbitrary point on $\gamma_s$. Let this base characteristic intersect the $x$-axis at the point $(s,0)$ where the initial values are prescribed. Then the value of $u(x,y)$ equals $h(s)$. Thus the solution at any point $(x,y)$ on $\gamma_s$ depends on the initial data at only one point, namely, $(s,0)$. Thus we may say that domain of dependence for solution at any point $(x,y)$ on a base characteristic is the point $(s,0)$. We may also interpret this by saying that the information (from initial conditions) from a point $(s,0)$ is transported along the base characteristic passing through it. Thus the base characteristic $\gamma_s$ is the domain of influence of the point $(s,0)$. ∎

### 2.2.6 ▪ Reasons behind the local nature of the existence theorem: A discussion

1. Even when the PDE is linear, the characteristic equations (2.25) are nonlinear ODEs in $x$ and $y$. From the theory of Ordinary differential equations we know that, in general, we can assert only the local existence of solutions to IVPs for nonlinear ODEs even when the vector fields are smooth. This immediately suggests that we can expect at most a local existence theorem even for a linear first-order PDE, and hence for the quasilinear first order PDE. Thus we understand that in order to have a 'global existence theorem' for quasilinear PDE, the coefficients should satisfy extra hypothesis so that at least the corresponding characteristic system of ODEs have global solutions.
2. The parametric representation of an integral surface might be misleading. We have seen many examples where the integral surface is described by smooth functions in $(s,t)$-variables (a consequence of differentiable dependence of solutions to IVPs for ODEs on initial conditions: more precisely, dependence of solutions on $t$ is due to the characteristic vector field *i.e.,* the PDE; and the dependence on $s$ is coming through IVPs for characteristic system of ODEs and also on the description of the initial curve). But in terms of $(x,y)$-variables it is not smooth. Inverse function theorem plays its role here. Now either we will be able to apply Inverse function theorem or we can not apply. Even when we can apply Inverse function theorem, the conclusions are only local. A good question to think about is "When can we have global conclusions from Inverse function theorem? When would such a situation arise for a quasilinear PDE?"
3. A characteristic curve might intersect the initial curve more than once. Since a characteristic curve carries information from the initial curve, when a characteristic curve intersects the initial curve more than once, the information carried from different intersection points should not be in conflict if we expect to have global solutions for the Cauchy problem. Also base characteristics corresponding to distinct characteristics should not intersect for a similar reason if we are hoping for a global solution for the Cauchy problem.
4. For ODEs when the vector field is defined in the whole space, through every point there passes a solution curve. For quasilinear first order PDE, characteristic curves may not pass through some parts of $\mathbb{R}^3$ and hence information from the initial curves does not propagate in those areas. This suggests the concepts of domain of influence (of initial curve), domain of dependence (of a solution at a point on the initial curve), finite speed of propagation (of information from the initial curve when one of the independent variables has the interpretation of time).

### 2.2.7 ▪ Special cases of linear and semilinear equations

#### Linear equation

Let us consider the linear PDE given by

$$a(x,y)\,u_x + b(x,y)\,u_y = c(x,y)u + d(x,y), \tag{2.65}$$

where $a,b,c,d$ are continuously differentiable functions defined on a domain $\Omega_0 \subseteq \mathbb{R}^2$. The three dimensional system of characteristic ODEs corresponding to the equation (2.65) are given by

$$\frac{dx}{dt} = a(x,y) \tag{2.66a}$$
$$\frac{dy}{dt} = b(x,y) \tag{2.66b}$$
$$\frac{dz}{dt} = c(x,y)z + d(x,y). \tag{2.66c}$$

The equations (2.66a) and (2.66b) determine the base characteristics which is the projection of characteristic curves. The characteristic curves are obtained by solving for $z = z(t)$ the linear non-homogeneous equation

$$\frac{dz}{dt} = c\,(x(t), y(t))\,z + d\,(x(t), y(t)).$$

That is, the characteristic curves are given by $(x(t), y(t), z(t))$.

#### Semilinear equation

We are considering the Semilinear PDE given by

$$a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u) \tag{2.67}$$

The three dimensional system of characteristic ODEs corresponding to the equation (2.67) are given by

$$\frac{dx}{dt} \quad = a(x,y) \tag{2.68a}$$
$$\frac{dy}{dt} \quad = b(x,y) \tag{2.68b}$$
$$\frac{dz}{dt} = c(x,y,z). \tag{2.68c}$$

The equations (2.68a) and (2.68b) determine the base characteristics which is the projection of characteristic curves. The characteristic curves are obtained by solving for $z = z(t)$ the nonlinear equation

$$\frac{dz}{dt} = c\,(x(t), y(t), z).$$

That is, the characteristic curves are given by $(x(t), y(t), z(t))$.

**Remark 2.29 (Comparison with quasilinear case).** Note that unlike for a general quasilinear PDE, the characteristic equations decouple when the PDE is either linear or semilinear. The base characteristics can be determined first, and then the characteristics. ■

### 2.2.8 ▪ Quasilinear equation in $n$-independent variables

Quasilinear equation in $n$-independent variables is of the form

$$\sum_{i=1}^{n} a_i(x_1, x_2, \cdots, x_n, u)u_{x_i} = c(x_1, x_2, \cdots, x_n, u), \tag{2.69}$$

where $a_1, a_2, \cdots, a_n, c$ are $C^1$ functions defined on a domain $\Omega$ in $\mathbb{R}^{n+1}$.

Cauchy data is prescribed on an $(n-1)$-dimensional manifold $\Gamma$ in $\mathbb{R}^{n+1}$, and the integral surface represents an $n$-dimensional manifold $S$ in $\mathbb{R}^{n+1}$.

We require $S$ to pass through $\Gamma$. Let $\Gamma$ be represented by the equations

$$x_i = f_i(s_1, s_2, \cdots, s_{n-1}) \text{ for } i = 1, 2, \cdots, n \tag{2.70}$$

$$z = h(s_1, s_2, \cdots, s_{n-1}) \tag{2.71}$$

Alternatively, the Cauchy data can be stated as

$$u\left(f_1(s_1, s_2, \cdots, s_{n-1}), \cdots, f_n(s_1, s_2, \cdots, s_{n-1})\right) = h(s_1, s_2, \cdots, s_{n-1})$$

The $(n+1)$-dimensional system of characteristic ODEs corresponding to the equation (2.69) are given by

$$\frac{dx_i}{dt} = a_i(x_1, x_2, \cdots, x_n, z) \text{ for } i = 1, 2, \cdots, n \tag{2.72a}$$

$$\frac{dz}{dt} = c(x_1, x_2, \cdots, x_n, z) \tag{2.72b}$$

The equations (2.72) represent the integral surface $z = u(x_1, x_2, \cdots, x_n)$ in parametric form provided

$$J = \left| \begin{array}{ccc} f_{1,s_1}(s_1, s_2, \cdots, s_{n-1}) & \cdots & f_{n,s_1}(s_1, s_2, \cdots, s_{n-1}) \\ f_{1,s_2}(s_1, s_2, \cdots, s_{n-1}) & \cdots & f_{n,s_2}(s_1, s_2, \cdots, s_{n-1}) \\ \cdots & \cdots & \cdots \\ f_{1,s_{n-1}}(s_1, s_2, \cdots, s_{n-1}) & \cdots & f_{n,s_{n-1}}(s_1, s_2, \cdots, s_{n-1}) \\ a_1(x_1, x_2, \cdots, x_n, z) & \cdots & a_n(x_1, x_2, \cdots, x_n, z) \end{array} \right|_{(s_1, s_2, \cdots, s_{n-1}) = (s_1^0, s_2^0, \cdots, s_{n-1}^0)} \neq 0.$$

Denote the solution of the $(n+1)$-dimensional system of ODEs (2.72) satisfying the initial conditions (2.70) by

$$x_i = X_i(s_1, s_2, \cdots, s_{n-1}, t) \text{ for } i = 1, 2, \cdots, n \tag{2.73}$$

$$z = Z(s_1, s_2, \cdots, s_{n-1}, t). \tag{2.74}$$

If $J \neq 0$, then we can solve for $s_1, s_2, \cdots, s_{n-1}, t$ in terms of $x_1, x_2, \cdots, x_n$ from the first $n$ equations of (2.73), say

$$s_i = S_i(x_1, x_2, \cdots, x_n) \text{ (for } i = 1, \cdots, n), \quad t = T(x_1, x_2, \cdots, x_n).$$

Substituting for $s_1, s_2, \cdots, s_{n-1}, t$ in the last equation of (2.73), we obtain the integral surface corresponding to the given Cauchy problem in the form

$$z = u(x_1, x_2, \cdots, x_n) = Z\left(S_1(x_1, x_2, \cdots, x_n), \cdots, S_{n-1}(x_1, x_2, \cdots, x_n), T(x_1, x_2, \cdots, x_n)\right).$$

## 2.3 ▪ Cauchy problem for fully nonlinear equation

We learnt the method of characteristics to solve Cauchy problem for Quasilinear PDE. We wish to extend it to construct solutions of Cauchy problems for fully nonlinear equations

$$F(x, y, u, u_x, u_y) = 0.$$

We use $p$ to denote $u_x$, and $q$ to denote $u_y$, which is a standard notation in the context of first order PDE.

**Cauchy Problem**

The Cauchy problem for the fully nonlinear equation

$$F(x,y,u,p,q)=0 \tag{2.75}$$

consists of finding a solution $u$ of (2.75) such that the integral surface $z=u(x,y)$ contains the initial curve $\Gamma$ whose parametric equation is given by

$$\Gamma: \ x=f(s), y=g(s), z=h(s) \quad s\in I, \tag{2.76}$$

where $I\subseteq\mathbb{R}$ is an interval. In other words, $u$ should not only solve (2.75) but also satisfy the following equation:

$$h(s)=u(f(s),g(s)).$$

Observe that when $F$ has the form

$$F(x,y,u,p,q)=a(x,y,u)p+b(x,y,u)q-c(x,y,u), \tag{2.77}$$

the fully nonlinear equation (2.75) reduces to a quasilinear equation. Thus (2.75) is a general case of quasilinear equations.

Since we want the equation (2.75) to repesent a first order PDE, we assume

**Hypothesis 1**

(i) $F:D\subseteq\mathbb{R}^5\to\mathbb{R}$ is a continuously differentiable function on $D$. Equivalently, the partial derivatives $F_x,F_y,F_z,F_p,F_q$ are continuous on $D$.
(ii) The partial derivatives $F_p$ and $F_q$ satisfy

$$F_p^2(x,y,u,p,q)+F_q^2(x,y,u,p,q)\neq 0 \quad \forall (x,y,u,p,q)\in D. \tag{2.78}$$

### 2.3.1 ▪ Solving Cauchy problem

In this subsection we discuss existence of solutions to the Cauchy problem (2.75)-(2.76), by closely following the proof of existence of solutions to Cauchy problem for quasilinear PDEs. We divide the discussion into several steps. At the end of each of these steps, we give some sufficient conditions on $F$ and $\Gamma$ that mathematically justifies each step.
**Step 1: Characteristic system of ODEs**
**Step 2: Finding initial conditions for Characteristic system of ODEs**
**Step 3: Application of Inverse Function Theorem**
**Step 4: Final solution**

#### Step 1: Finding characteristic system of ODEs using geometry

For quasilinear PDE defined by (2.75)-(2.77), $(a,b,c)$ describes a tangential direction at every point on an integral surface and this direction was called a characteristic direction.

Using characteristic direction, we constructed characteristic curves and the integral surface was found using them.

For the fully nonlinear case (2.75), there is no such automatic choice of a characteristic direction. We follow a geometric argument to choose one such characteristic direction.

Let $P_0(x_0, y_0, z_0)$ be a point on a given integral surface $z = u(x, y)$, where $u$ is twice continuously differentiable. The tangent plane at $P_0$ to the integral surface $z = u(x, y)$ is given by

$$z - z_0 = p_0(x - x_0) + q_0(y - y_0), \tag{2.79}$$

where $p_0 = u_x(x_0, y_0)$ and $q_0 = u_y(x_0, y_0)$.

However, if $u$ is unknown, we do not know $p_0$ and $q_0$ as well, but we know that $p_0, q_0$ are constrained to satisfy the equation

$$F(x_0, y_0, z_0, p_0, q_0) = 0.$$

Thus the system of equations

$$z - z_0 = l(x - x_0) + m(y - y_0) \tag{2.80a}$$

$$F(x_0, y_0, z_0, l, m) = 0 \tag{2.80b}$$

determine a one-parameter family of planes passing through the point $P_0$ and one of them is the tangent plane to the given integral surface $z = u(x, y)$ at $P_0$.

The above discussion shows us the way to find a characteristic direction. The geometric idea is now to consider the envelope of the family of planes described by (2.80). Recall that the envelope has two properties: Envelope touches each member of the family of planes, and at the points of intersection where envelope touches a member of the family they have a common tangent plane. The point $P_0$ will be there on the envelope and we consider the tangent plane at $P_0$ to the envelope and choose a direction in it. That direction will be chosen as a characteristic direction.

#### Computation of the envelope of the family of planes (2.80a) and Characteristic directions

Let us assume that there exists a differentiable function $m = m(l)$ such that

$$F(x_0, y_0, z_0, l, m(l)) = 0. \tag{2.81}$$

Now the family of planes in (2.80a) is given by

$$z - z_0 = l(x - x_0) + m(l)(y - y_0) \tag{2.82}$$

Differentiating the equations (2.82) and (2.81) w.r.t. $l$ yield

$$0 = (x - x_0) + m'(l)(y - y_0) \tag{2.83a}$$

$$F_p(x_0, y_0, z_0, l, m(l)) + F_q(x_0, y_0, z_0, l, m(l))m'(l) = 0 \tag{2.83b}$$

Eliminating $m'(l)$ from (2.83) and using (2.82) we get

$$\frac{x - x_0}{F_p(x_0, y_0, z_0, l, m(l))} = \frac{y - y_0}{F_q(x_0, y_0, z_0, l, m(l))} = \frac{z - z_0}{lF_p(x_0, y_0, z_0, l, m(l)) + m(l)F_q(x_0, y_0, z_0, l, m(l))} \tag{2.84}$$

This is the equation of a collection of lines passing through $P_0(x_0,y_0,z_0)$ with directions[8]

$$\Big(F_p(x_0,y_0,z_0,l,m(l)),F_q(x_0,y_0,z_0,l,m(l)),lF_p(x_0,y_0,z_0,l,m(l))+m(l)F_q(x_0,y_0,z_0,l,m(l))\Big). \tag{2.85}$$

corresponding to each solution $(l,m(l))$ of (2.81). Union of these lines as $l$ varies defines a 'cone' in general, and is called **Monge Cone**, and each of these lines (corresponding to each $l$) is a generator of the cone, and any of them may be taken as a **characteristic direction**. Equivalently, different generators are obtained by different choices of $p_0$ and $q_0$ satisfying the equation

$$F(x_0,y_0,z_0,p_0,q_0)=0, \tag{2.86}$$

and the characteristic direction(s) is given by

$$\Big(F_p(x_0,y_0,z_0,p_0,q_0),F_q(x_0,y_0,z_0,p_0,q_0),lF_p(x_0,y_0,z_0,p_0,q_0)+m(l)F_q(x_0,y_0,z_0,p_0,q_0)\Big). \tag{2.87}$$

We follow the ideas from quasilinear case and hope to find an integral surface using "characteristic curves". Thus we are led to looking at curves whose tangential direction at each of its points is given by (2.87). But then this direction also depends on $p_0$ and $q_0$. How to find them? Instead let us also find equations for $l(t),m(t)$. So add more equations.

Let $(x(t),y(t),z(t))(t\in J)$ ($J\subseteq\mathbb{R}$ is an interval) be a curve in parametric form that passes through a point $P_0(x_0,y_0,z_0)$ on an integral surface $z=u(x,y)$ at $t=0$, and lies on the integral surface $z=u(x,y)$. Then necessaril the tangential direction $(x'(t),y'(t),z'(t))$ at an arbitrary point on the curve should belong to the tangent plane at $P(x(t),y(t),z(t))$ to the integral surface. Therefore we get that the following system of characteristic ODEs are satisfied along the curve:

$$\frac{dx}{dt}=F_p(x,y,z,p,q) \tag{2.88a}$$

$$\frac{dy}{dt}=F_q(x,y,z,p,q) \tag{2.88b}$$

$$\frac{dz}{dt}=pF_p(x,y,z,p,q)+qF_q(x,y,z,p,q). \tag{2.88c}$$

along with $(x(t_0),y(t_0),z(t_0))=(x_0,y_0,z_0)$.

Notice that above system does not determine characteristic curves because the unknowns $p,q$ appear explicitly in the equations, unlike quasilinear, semilinear cases. It means we have to complete the system (2.88) by adding equations for $p,q$ such that $(p,q,-1)$ is the normal direction to the integral surface $z=u(x,y)$. Thus we have to find five quantities $(x(t),y(t),z(t),p(t),q(t))$. Geometrically we want to determine not only the points through which the curve passes but also a tangent plane to this curve which is also tangent to a solution surface. Such a set of small planes is called a **strip** and the five quantities are called its elements. Strip can be though of as a point $(x(t),y(t),z(t))$ along with direction of normal to the integral surface $(p(t),q(t),-1)$.

### Derivation of equations for Characteristic strips

Differentiating the equation (2.75) w.r.t. $x$ gives

$$F_x+pF_z+F_pp_x+F_qp_y=0, \tag{2.89}$$

[8] Note we used Hypothesis 1 here.

since $u$ is twice continuously differentiable function, and as a consequence $p_y = q_x$.

Let $(x(t), y(t), z(t))$ lie on the integral surface $z = u(x, y)$.

(i) We want $p(t)$ also to satisfy $p(t) = p(x(t), y(t))$. On differentiating w.r.t. $t$, we get

$$p'(t) = x'(t)p_x(x(t), y(t)) + y'(t)p_y(x(t), y(t)) \tag{2.90}$$

Since (2.88) holds, we must have

$$x'(t)p_x + y'(t)p_y = -(F_x + p(t)F_z) \tag{2.91}$$

Thus we get

$$p'(t) = -(F_x + p(t)F_z)) \tag{2.92}$$

(ii) We want $q(t)$ also to satisfy $q(t) = q(x(t), y(t))$. On differentiating w.r.t. $t$, we get

$$q'(t) = x'(t)q_x(x(t), y(t)) + x'(t)q_y(x(t), y(t)). \tag{2.93}$$

Since (2.88) holds, we must have

$$x'(t)q_x + y'(t)q_y = -\left(F_y + q(t)F_z\right) \tag{2.94}$$

Thus we get

$$q'(t) = -\left(F_y + q(t)F_z\right)). \tag{2.95}$$

Thus we get the following system of ODEs satisfied by the characteristic strip.

$$\frac{dx}{dt} = F_p(x, y, z, p, q) \tag{2.96a}$$
$$\frac{dy}{dt} = F_q(x, y, z, p, q) \tag{2.96b}$$
$$\frac{dz}{dt} = pF_p(x, y, z, p, q) + qF_q(x, y, z, p, q) \tag{2.96c}$$
$$\frac{dp}{dt} = -F_x(x, y, z, p, q) - pF_z(x, y, z, p, q) \tag{2.96d}$$
$$\frac{dq}{dt} = -F_y(x, y, z, p, q) - qF_z(x, y, z, p, q) \tag{2.96e}$$

Note that along a characteristic strip we have

$$\frac{d}{dt}F\left(x(t), y(t), z(t), p(t), q(t)\right) = 0. \tag{2.97}$$

Therefore if we choose an initial strip $(x_0, y_0, z_0, p_0, q_0)$ satisfying $F(x_0, y_0, z_0, p_0, q_0) = 0$, then it follows that

$$F\left(x(t), y(t), z(t), p(t), q(t)\right) = 0 \quad \text{for all} \quad t \in J. \tag{2.98}$$

where $(x(t), y(t), z(t), p(t), q(t))$ is the characteristic strip satisfying

$$(x(t), y(t), z(t), p(t), q(t))|_{t=0} = (x_0, y_0, z_0, p_0, q_0). \tag{2.99}$$

### Determination of an initial strip

Recall that the given Cauchy data is

$$\Gamma: \; x = f(s), y = g(s), z = h(s) \quad s \in I. \tag{2.100}$$

Since want to find an integral surface $z = u(x,y)$ that contains $\Gamma$ in it, we must have

$$h(s) = u(f(s), g(s)) \tag{2.101}$$

Differentiating the last equation equation gives

$$h'(s) = u_x(f(s), g(s))f'(s) + u_y(f(s), g(s))g'(s) \tag{2.102}$$

Note that $u_x(f(s), g(s)) = p(s)$ and $u_y(f(s), g(s)) = q(s)$ determine the normal to the integral surface and must satisfy

$$\begin{aligned} F(f(s), g(s), h(s), p(s), q(s)) &= 0 && (2.103\text{a}) \\ p(s)f'(s) + q(s)g'(s) = h'(s). & && (2.103\text{b}) \end{aligned}$$

Thus we would like to complete $\Gamma$ into a strip by finding two functions

$$p = p(s), \quad q = q(s)$$

which satisfy the system of equations (2.103).

Assume a special solution $p_0, q_0$ of the system

$$\begin{aligned} F(f(s_0), g(s_0), h(s_0), p_0, q_0) &= 0 && (2.104\text{a}) \\ p_0 f'(s_0) + q_0 g'(s_0) = h'(s_0) & && (2.104\text{b}) \end{aligned}$$

satisfying

$$\Delta = \begin{vmatrix} f'(s_0) & F_p(f(s_0), g(s_0), h(s_0), p_0, q_0) \\ g'(s_0) & F_q(f(s_0), g(s_0), h(s_0), p_0, q_0) \end{vmatrix} \neq 0. \tag{2.105}$$

Applying implicit function theorem, we can solve for $p, q$ in terms of $x, y, z$ near $(p_0, q_0)$. As a consequence, there exist unique functions $p = p(s)$ and $q = q(s)$ which are continuously differentiable for $s$ near $s_0$, say $s \in I$ once again, and such that $p(s_0) = p_0$ and $q(s_0) = q_0$.

### Step 2: Finding characteristic strips

Through each element $(f(s), g(s), h(s), p(s), q(s))$ $(s \in I)$, pass the characteristic strip with parameter $t$ that reduces at $t = 0$ to $(f(s), g(s), h(s), p(s), q(s))$. That is, solve the system of ODEs (2.96) such that

$$(x(0), y(0), z(0), p(0), q(0)) = (f(s), g(s), h(s), p(s), q(s)).$$

Let this characteristic strip be defined by the functions

$$x = X(s,t), y = Y(s,t), z = Z(s,t), p = P(s,t), q = Q(s,t). \tag{2.106}$$

As in the analysis for quasilinear equation, by restricting $s$ to a small interval that contains $s_0$, we can get an interval $J$ on which the functions defining the characteristic strip are defined. Thus the domain of functions in (2.106) may be taken to be $I \times J$.

The functions in (2.106) have the following properties:

(i)

$$X(s,0)=f(s),\ Y(s,0)=g(s),\ Z(s,0)=h(s),\ P(s,0)=p(s),\ Q(s,0)=q(s).$$

(ii) Further, in view of equations (2.97)-(2.98), we have

$$F\left(X(s,t),Y(s,t),Z(s,t),P(s,t),Q(s,t)\right)=0 \quad \text{for} \quad (s,t)\in I\times J.$$

At the end of Steps 1 and 2, we have the following proposition

**Proposition 2.30.** *If there exists an integral surface $z=u(x,y)$, where $u$ is a twice continuously differentiable function, of the fully nonlinear equation (2.75) passing through $\Gamma$ (or a part thereof) containing the element $(x_0,y_0,z_0,p_0,q_0)$, then that surface must be the union of the supports of the characteristic strips through the points of $\Gamma$ (respectively, a part thereof), provided that the Cauchy data is satisfied and equations (2.104) and (2.105) are satisfied.*

**Remark 2.31.** The above proposition establishes uniqueness part of the existence-uniqueness theorem that will be stated at the end of Step 4. ∎

### Step 3: Applicaton of inverse function theorem

In view of the assumption (2.105), we can apply Inverse function theorem to the function

$$(s,t)\in I\times J\longmapsto (X(s,t),Y(s,t))\in\mathbb{R}^2 \tag{2.107}$$

at $(s,t)=(s_0,0)$, and conclude that there exists a neighbourhood of $(s_0,0)$ and a neighbourhood of $(f(s_0),g(s_0))$ such that the map (2.107) is invertible. That is, we get two functions $S,T$ defined on a neighbourhood of the point $(x_0,y_0)$ such that

$$s=S(x,y),\ t=T(x,y). \tag{2.108}$$

Let us define

$$u(x,y):=Z(S(x,y),T(x,y)),\ p(x,y):=P(S(x,y),T(x,y)),\ q(x,y):=Q(S(x,y),T(x,y)). \tag{2.109}$$

To show that $u$ defined in (2.109) solves the Cauchy problem for PDE (2.75), we have to show that

$$p(x,y)=u_x(x,y),\ q(x,y)=u_y(x,y). \tag{2.110}$$

This is proved by showing that both $p(x,y)$ and $u_x(x,y)$ satisfy the same system of linear non-homogeneous system with a unique solution. Also $q(x,y)$ and $u_y(x,y)$ satisfy the same system of linear non-homogeneous system with a unique solution.

Since $z=u(x,y)$, $x=X(s,t)$, and $y=Y(s,t)$, we get by differentiating the equation $z=u(x,y)$ w.r.t. $s$ and $t$ the system of equations

$$z_s=u_xX_s+u_yY_s \tag{2.111a}$$
$$z_t=u_xX_t+u_yY_t. \tag{2.111b}$$

To prove that $u_x=P(s,t)$ and $u_y=Q(s,t)$, it suffices to show that

$$z_s = PX_s + QY_s \tag{2.112a}$$
$$z_t = PX_t + QY_t \tag{2.112b}$$

since $\begin{vmatrix} X_s & Y_s \\ X_t & Y_t \end{vmatrix} = \Delta \neq 0$ and hence solution of (2.111) is unique.

The second equation of (2.112) is obviously true since $z_t = pF_p + qF_q$. To obtaine the first equation of (2.112), assume

$$A(s,t) = z_s - PX_s - QY_s.$$

Observe that $A(s,0) = h'(s) - pf'(s) - qg'(s) = 0$ for all $s$. Differentiating w.r.t. $t$, we have

$$A_t = z_{st} - P_tX_s - Q_tY_s - PX_{st} - QY_{st} \tag{2.113}$$
$$= \frac{\partial}{\partial s}(z_t - PX_t - QY_t) + P_sX_t + Q_sY_t - P_tX_s - Q_tY_s \tag{2.114}$$
$$= P_sF_p + Q_sF_q + X_s(F_x + PF_z) + Y_s(F_y + QF_z) \tag{2.115}$$
$$= \frac{\partial F}{\partial s} - F_z(z_s - PX_s - QY_s) \tag{2.116}$$
$$= -F_zA \tag{2.117}$$

That is, $A_t + F_zA = 0$. This is a linear first order ordinary differential equation, $F_z$ is continuous, and $A(s,0) = 0$. Hence

$$A(s,t) = 0 \text{ for all } t.$$

This is true for all $s$. Therefore

$$z_s = PX_s + QY_s.$$

Thus

$$F(X(s,t), Y(s,t), Z(s,t), P(s,t), Q(s,t)) = F\Big(X(s,t), Y(s,t), Z(s,t), Z_x(s,t), Z_y(s,t)\Big) = 0.$$

$$F(x,y,u(x,y),p(x,y),q(x,y)) = 0 \tag{2.118}$$

and Cauchy condition is satisfied.

Thus we have proved the following local existence and uniqueness theorem:

**Theorem 2.32.**

1. *Let $F : D \subseteq \mathbb{R}^5 \to \mathbb{R}$ be a twice continuously differentiable function on $D$. Equivalently, the function $F$ and all the partial derivatives upto second order are continuous on $D$.*
2. *The partial derivatives $F_p$ and $F_q$ satisfy*

$$F_p^2(x,y,u,p,q) + F_q^2(x,y,u,p,q) \neq 0 \quad \forall (x,y,u,p,q) \in D. \tag{2.119}$$

3. *Let $\Gamma$ be a twice continuously differentiable parametrized curve: that is,*

$$\Gamma: \ x = f(s), y = g(s), z = h(s) \quad s \in I, \tag{2.120}$$

*where $I \subseteq \mathbb{R}$ is an interval, and the functions $f, g, h$ are twice continuously differentiable on the interval $I$.*

4. *Assume that the system*

$$F(f(s), g(s), h(s), p(s), q(s)) \quad = 0 \tag{2.121a}$$
$$p(s)f'(s) + q(s)g'(s) = h'(s). \tag{2.121b}$$

*admits a solution $(p(s), q(s))$ where $p(s)$ and $q(s)$ are twice continuously differentiable functions on the interval $I$ such that the strip $(f(s), g(s), h(s), p(s), q(s)) \in D$ for all $s \in I$, and such that the transversality condition holds:*

$$\Delta = \begin{vmatrix} f'(s_0) & F_p(f(s_0), g(s_0), h(s_0), p_0, q_0) \\ g'(s_0) & F_q(f(s_0), g(s_0), h(s_0), p_0, q_0) \end{vmatrix} \neq 0. \tag{2.122}$$

*Then the fully nonlinear PDE*

$$F(x, y, u, p, q) = 0, \tag{2.123}$$

*where $p$ and $q$ denote $u_x$ and $u_y$ respectively admits a solution $u = u(x, y)$ on an open set $U$ containing the point $(f(s_0), g(s_0))$ that satisfies $u(f(s), g(s)) = h(s)$ for $s \in I$ such that $(f(s), g(s)) \in U$. Moreover the function $u$ is twice continuously differentiable on $U$.*

*Further,*

(i) *If the system (2.121) has a unique solution for $(p(s), q(s))$, and the transversality condition (2.122) holds along the strip $(f(s), g(s), h(s), p(s), q(s))$, then the local solution is unique.*
(ii) *If the system (2.121) has many solutions for $(p(s), q(s))$, and for each such solution if the transversality condition holds along the resultant strip, then the Cauchy problems possesses a unique local solution.*
(iii) *If the system (2.121) has no solutions for $(p(s), q(s))$, the Cauchy problem for (2.123) does not have a solution.*

### 2.3.2 ▪ Fully nonlinear equation in $n$-independent variables

Consider

$$F(x_1, x_2, \cdots, x_n, z, p_1, p_2, \cdots, p_n) = 0, \tag{2.124}$$

where $F$ is a $C^2$ function, $p_i = z_{x_i}$, $z = u(x_1, x_2, \cdots, x_n)$ is an unknown function.

The Cauchy problem is to find an integral surface in the $x_1, x_2, \cdots, x_n, z$ space which passes through a given $(n-1)$-dimensional manifold $\Gamma$, represented parametrically by

$$x_i = f_i(s_1, s_2, \cdots, s_{n-1}) \text{ for } i = 1, 2, \cdots, n \tag{2.125}$$
$$z = h(s_1, s_2, \cdots, s_{n-1}) \tag{2.126}$$

**Step 1:** Complete $\Gamma$ into a strip of characteristic elements, by finding functions

$$p_i = \phi_i(s_1, s_2, \cdots, s_{n-1}) \text{ for } i = 1, 2, \cdots, n$$

such that

$$\frac{\partial h}{\partial x_i} = \sum_{k=1}^{n} \phi_k \frac{\partial f_k}{\partial x_i} \text{ for } i = 1, 2, \cdots, n, \tag{2.127a}$$

$$F(f_1, f_2, \cdots, f_n, h, \phi_1, \phi_2, \cdots, \phi_n) = 0. \tag{2.127b}$$

**Step 2:** Pass through point $P$ of $\Gamma$ a characteristic strip which reduces at $t = 0$ to the plane element $(f_1(s), f_2(s), \cdots, f_n(s), h(s), \phi_1(s), \phi_2(s), \cdots, \phi_n(s))$, where $s = (s_1, s_2, \cdots, s_{n-1})$.

The above characteristic strip is a set of ‘elements’ $(x_1, x_2, \cdots, x_n, z, p_1, p_2, \cdots, p_n)$ depending on a parameter $t$ and satisfying the system of ODEs

$$\frac{dx_i}{dt} = p_i(x_1, x_2, \cdots, x_n, z) \text{ for } i = 1, 2, \cdots, n \tag{2.128a}$$

$$\frac{dz}{dt} = \sum_{i=1}^{n} p_i F_{p_i} \tag{2.128b}$$

$$\frac{dp_i}{dt} = -F_{x_i} - p_i F_z \text{ for } i = 1, 2, \cdots, n \tag{2.128c}$$

The completion of $\Gamma$ into a strip of characteristic elements can be achieved if

$$\Delta = \begin{vmatrix} \frac{\partial f_1}{\partial s_1} & \cdots & \frac{\partial f_n}{\partial s_1} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_1}{\partial s_{n-1}} & \cdots & \frac{\partial f_n}{\partial s_{n-1}} \\ F_{p_1} & \cdots & F_{p_n} \end{vmatrix} \neq 0 \tag{2.129}$$

at some point $P_0 = (x_{10}, \cdots, x_{n0})$ of $\Gamma$.

## Exercises

### Origin of PDEs

2.1. Let $f(x,y,a,b)=(x-a)^2+(y-b)^2$. Get a PDE by eliminating the parameters $a$ and $b$. (Answer: $u_x^2+u_y^2=4u$.)

### Linear equations

2.2. Check if local existence theorem is applicable for each of the following Cauchy problems. Find solutions whenever they exist, and mention domains of these solutions.

(i) $u_t+cu_x=0, \quad u(x,0)=f(x)$.

(ii) $x\,u_x+u_y=1, \quad u(x,0)=e^x$.

(iii) $x\,u_x+u_y=1, \quad u(0,y)=e^y$.

(iv) $x\,u_x+(y^2+1)\,u_y=u, \quad u(x,0)=e^x$.

(v) $x\,u_x+(x+y)\,u_y=u+1 \quad u(x,0)=x^2$.

(vi) $x\,u_x+y\,u_y=u+1, \quad u(x,x)=x^2$.

(vii) $x\,u_y-y\,u_x=u, \quad u(x,0)=h(x),\ x\geq 0$. (Ans. $u=h(\sqrt{x^2+y^2})e^{\arctan\frac{y}{x}}$)

(viii) $u_x+2xu_y=2xu, \quad u(x,0)=x^2$ for all $x\in\mathbb{R}$. (Ans. $u=(x^2-y)e^y$)

(ix) $u_x+2xu_y=2xu, \quad u(0,y)=y^2$ for all $y\in\mathbb{R}$. (Ans. $u=(y-x^2)^2e^{x^2}$)

(x) $u_x+2xu_y=2xu, \quad u(x,0)=x^2$ for $x\geq 0$ and $u(0,y)=y^2$ for $y\geq 0$. (Ans. $u=\begin{cases}(x^2-y)e^y & \text{for } x\geq 0,\ y\leq x^2,\\ (y-x^2)^2e^{x^2} & \text{for } x\geq 0,\ y\geq x^2\end{cases}$ )

(xi) $u_x+2u_y=1+u$ such that $u=\sin x$ on the line $y=3x+1$. (Answer: $u(x,y)=\exp(y-3x-1)[1+\sin(y-2x-1)]-1$)

(xii) $xu_x-yu_y=y$ such that $u(1,y)=y,\ y\in\mathbb{R}$.

(xiii) $u_x=1$ such that $u(0,y)=g(y)$.

(xiv) $u_x+u_y=2$ such that $u(x,0)=x^2$.

(xv) $xu_x+yu_y=2u$ such that $u(x,0)=x^2,\ x>0$.

(xvi) $xu_x+(x^2+y)u_y=1-\left(\frac{y}{x}-x\right)u$ such that $u(1,y)=0$.

2.3. Solve the equation $u_x+3y^{\frac{2}{3}}u_y=2$ subject to the condition $u(x,1)=1+x$. Discuss how the non-smoothness of characteristic vector field affects the solution. (Answer: $u(x,y)=x+y^{\frac{1}{3}}$)

2.4. Solve the PDE $u_x+3u_y=u$ subject to Cauchy condition $u=\cos x$ on the line $y=\alpha x$. Find the value of $\alpha$ for which the method fails and interpret the result. (Answer: $u(x,y)=\exp\left(\frac{y-\alpha x}{3-\alpha}\right)\cos\left(\frac{3x-y}{3-\alpha}\right)$)

2.5. Let $u$ be a continuously differentiable function in the closed unit disk $\overline{B}(0,1)$ and is a solution of

$$a(x,y)u_x+b(x,y)u_y=-u \text{ in } B(0,1).$$

Let $a(x,y)x+b(x,y)y>0$ on the unit circle. Prove that $u$ vanishes identically.

(Hint: Show that $\max\limits_{\overline{B}(0,1)} u \leq 0$ and $\min\limits_{\overline{B}(0,1)} u \geq 0$.)

2.6. Let $u$ be a continuously differentiable function on the closure of $D$ where $D$ is given by

$$D = \left\{(x, y) \in \mathbb{R}^2 : |x| < 1, |y| < 1\right\}.$$

Let $u$ be a solution of

$$a(x, y)u_x + b(x, y)u_y = -u,$$

where $a, b$ are positive differentiable functions in the entire plane. Then

(a) Prove that the projection on the $xy$-plane of each characteristic curve passing through a point in $D$ intersects the boundary of $D$ at exactly two points. (Hint: The projection on the $xy$-plane of each characteristic curve has a positive direction and it propagates with a strictly positive speed in the square. Therefore, it intersects the boundary of $D$ exactly twice.)

(b) Show that if $u$ is positive on the boundary of $D$, then it is positive at every point in $D$. (Hint: Suppose that $u$ is positive on the boundary of $D$ and $u \leq 0$ at some point in $D$. Consider the base characteristic passing through this point. Consider what happens to solution along this characteristic and get a contradiction.)

(c) Suppose that $u$ attains a local minimum (maximum) at a point $(x_0, y_0) \in D$. Evaluate $u(x_0, y_0)$. (Hint: use your calculus)

(d) Denote by $m$ the minimal value of $u$ on the boundary of $D$. Assume $m > 0$. Show that $u(x, y) \geq m$ for all $(x, y) \in D$. (Hint: If this were not true, then $u$ has a global minimum in $D$ and hence it is a local minimum. Use (c) and (b), and conclude.)

### Semilinear equations

2.7. Solve $u_x + u_y = u^2, \quad u(x, 0) = h(x)$.

2.8. $u_x + u_y = u^2, \quad u(x, -x) = x$.

### Quasilinear equations

2.9. Solve $u_y = x u u_x, \quad u(x, 0) = x$. (Ans. $x = u e^{-yu}$)

2.10. Consider the first order quasilinear PDE given by

$$u u_x + u_y = 1. \tag{2.130}$$

Solve the Cauchy problems associated with (2.130) and each of the following Cauchy data.

(i) (Jacobian is non-vanishing)

$$x = s, \ y = s, \ z = \frac{s}{2}, \ 0 \leq s \leq 1. \tag{2.131}$$

(ii) (Jacobian is identically equal to zero, infinite number of solutions to Cauchy problem)

$$x = \frac{s^2}{2}, \ y = s, \ z = s, \ 0 \leq s \leq 1. \tag{2.132}$$

Find all solutions and sketch some of them.

(iii) (Jacobian is identically equal to zero, no solutions to Cauchy problem)

$$x = s^2,\ y = 2s,\ z = s,\ 0 \le s \le 1. \tag{2.133}$$

Try solving this problem and see what happens.

(iv) (Jacobian is identically equal to zero on a subset)

$$x = s^2,\ y = 2s,\ z = s,\ 0 \le s \le 3. \tag{2.134}$$

Try solving this problem and see what happens.

2.11. Suppose $\gamma$ is a curve that lies on two integral surfaces corresponding to the quasilinear PDE (2.6). Can we conclude that $\gamma$ is a characteristic curve? If yes, give a proof. If not, explain where the proof of Corollary 2.11 fails.

2.12. Solve the PDE $u_t + f(u)u_x = 0$ subject to Cauchy condition $u(x,0) = g(x)$, where $f$ is a continuously differentiable function and $g$ is a continuous function. Also find an expression for $u_x(x,t)$ and hence find a condition that must be satisfied that ensures the existence of $u_x(x,t)$ for all $t > 0$.

2.13. Solve the PDE $uu_t + u_x = 0$ subject to Cauchy condition $u(x,1) = \frac{1}{x}$ for $x \ge 1$.

2.14. Solve the PDE $u_t + uu_x = t$ subject to Cauchy condition $u(x,0) = -2x$. Find the domain in the upper half of the $(x,t)$-plane $t \ge 0$ where the solution is valid.

2.15. Solve the PDE $u_t + uu_x = 0$ subject to Cauchy condition $u(x,0) = x$ for $1 \le x \le 2$. Find the domain in the upper half-plane $t \ge 0$ where the solution is valid.

2.16. Solve the PDE $u_t + uu_x = 0$ subject to Cauchy condition $u(x,0) = -x$ for $1 \le x \le 2$. Find the domain in the upper half-plane $t \ge 0$ where the solution is valid.

2.17. Explore the reasons behind the assertion of only local existence of solutions to Cauchy problem in Theorem 2.17. Illustrate with an example each for each of your reasons.

2.18. Explain why the procedure given in the proof of Lemma 2.20 to construct a solution when $J \equiv 0$ fails in the case where there is no solution for the Cauchy problem.

**Nonlinear equations**

In the following exercises, we use the notations $p = u_x$ and $q = u_y$. Find if the following Cauchy problems have solutions, and find them whenever they exist. In each case determine the domains on which solutions exist and also comment on uniqueness of solutions.

2.19. $pq = 1, \quad u(x,0) = \log x.$

2.20. $pq = u, \quad u(x,1) = x,\ 0 \le x \le 1.$(Ans. $u = xy$)

2.21. $pq = 2, \quad u(x,x) = 3x,\ 0 \le x \le 1.$ (Ans. $u = y + 2x$)

2.22. $p^2 + q^2 + 2(p-x)(q-y) - 2u = 0, \quad u(x,0) = 0,\ 0 \le x \le 1.$ (Ans. $u = 2xy - \frac{3}{2}y^2$, $u = \frac{1}{2}y^2$)

2.23. $p^3 - q = 0, \quad u(x,0) = 2x\sqrt{x},\ 0 \le x \le 1.$ (Ans. $u = \frac{2x\sqrt{x}}{\sqrt{1-28y}}$)

2.24. $p + \frac{1}{2}q^2 = 1, \quad u(0,y) = y^2,\ 0 \le y \le 1.$ (Ans. $u = x + \frac{y^2}{1+2x}$)

2.25. $u = xp + yq + \frac{p^2+q^2}{2}, \quad u(x,0) = \frac{1-x^2}{2}.$ (Ans. $u = \pm y + \frac{1-x^2}{2}$)

2.26. $2p^2x + qy = u, \quad u(x,1) = \frac{x}{2}.$ (Ans. $u = \frac{x}{2}$)

2.27. $p^2 + q^2 = 1, \quad u(x,0) = 0.$

2.28. $p^2+q^2=n_0^2, \quad u(x,2x)=1.$

2.29. $p^2+q^2=u, \quad u(x,0)=x^2+1.$

2.30. $p^2+q^2=u^2, \quad u(x,0)=1$. (Ans. $u=\exp(\pm y)$)

2.31. $p^2+q^2=u^2$ with the initial curve $\Gamma$ given by the parametrization $(\cos s,\sin s,1)$.(Ans. $u=\exp(\pm(1-\sqrt{x^2+y^2}))$)

2.32. $p^2+q^2=u$ with the initial curve $\Gamma$ given by the parametrization $(\cos s,\sin s,1)$.

2.33. $p^2+q=0, \quad u(x,0)=x.$

2.34. $q^2-p=0, \quad u(x,0)=f(x).$

2.35. $p^2+q^2=1, \quad u(\cos s,\sin s)=0.$

2.36. $p^2-3q^2=u, \quad u(x,0)=x^2$. (Ans. $u=(x+\frac{y}{2})^2$ if $q_0(s)=s$, and $u=(x-\frac{y}{2})^2$ if $q_0(s)=-s$)