Partial Differential Equations

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Homework

1. From the lecture notes: Chapter 3, problems 2 and 3; Chapter 5, problem 1.
2. Find the dispersion relation for the linear Benjamin-Bona-Mahoney equation

$$u_t + u_x - u_{xxt} = 0.$$

and write down the soluiton to the equation with initial conditions (integrable and square integrable) $u(x,t) = u_0(x), x \in \mathbb{R}$.
3. One-dimensional waves on the surface of the ocean can be modeled by the Euler equations given below.

$$\begin{aligned} \nabla^2\phi &= 0, && -h < z < \zeta(x,t), \\ \phi_z &= 0, && z = -h, \\ \zeta_t + \phi_x\zeta_x &= \phi_z, && z = \zeta(x,t), \\ \phi_t + g\zeta + \frac{1}{2}\left(\phi_x^2 + \phi_z^2\right) &= T\frac{\zeta_{xx}}{(1+\zeta_x^2)^{3/2}}, && z = \zeta(x,t). \end{aligned}$$

Here $z = \zeta(x,t)$ is the surface of the water, $\phi(x,z,t)$ is the velocity potential so that $v = \nabla\phi$ is the velocity of the water, g is the acceleration of gravity, and $T > 0$ is the coefficient of surface tension. This is a nonlinear free-surface problem (i.e., we have to solve an equation inside a domain whose boundaries we do not know), which has been the topic of much research since 1840s.

(a) Check that $\zeta = 0$ (flat water) with $\phi = 0$ provides a solution to this problem.

(b) By linearizing around this trivial solution, we find the following system of linear equations

$$\begin{aligned} \nabla^2\phi &= 0, && -h < z < \zeta(x,t), \\ \phi_z &= 0, && z = -h, \\ \zeta_t &= \phi_z, && z = 0, \\ \phi_t + g\zeta &= T\zeta_{xx}, && z = 0. \end{aligned}$$

Find the dispersion relation for this problem.

(c) Having found that without surface tension ($T = 0$) the dispersion relation is $\omega^2 = gk\tanh(kh)$, find the group velocities for the case of long waves in shallow water (kh small), and for the case of deep water (kh big).

4. Everything that we have done for continuous space equations also works for equations with a discrete space variable. Consider the discrete linear Schröodinger equation:

$$i\frac{d\psi_n}{dt} + \frac{1}{h^2}(\psi_{n+1} - 2\psi_n + \psi_{n-1}) = 0,$$

where h is a real constant, n is any integer, $t > 0$, $\psi_n \to 0$ as $|n| \to \infty$, and $\psi_n(0) = \psi_{n,0}$ is given.

(a) The discrete analogue of the Fourier transform is given by

$$\psi_n(t) = \frac{1}{2\pi} \oint_{|z|=1} \hat{\phi}(z,t) z^{n-1} dz,$$

and its inverse

$$\hat{\psi}(z,t) = \sum_{n=-\infty}^{\infty} \psi_m(t) z^{-m}.$$

Show that these two transformations are indeed inverse each other.

(b) The disperation relation of a semi-discrete problem is obtained by looking for solutions of the form $\psi_n = z^n e^{-i\omega t}$. Show that for the semi-discrete Schrödinger equation

$$\omega(z) = -\frac{(z-1)^2}{zh^2}.$$

How does this compare to the dispersion relation of the continuous space problem? Specifically, demonstrate that you recover the dispersion relationship for the continuous problem as $h \to 0$.

5. A function is said to be two-frequency quasiperiodic if it has an expansion of the form

$$f(x) = \sum_{n_1,n_2=-\infty}^{\infty} a_{n_1,n_2} e^{i(n_1\omega_1+n_2\omega_2)x},$$

where the frequencies ω_1, ω_2 are incommensurable, i.e., their ratio is not rational.

(a) Argue formally that the coefficients a_{n_1,n_2} are determined by

$$a_{n_1,n_2} = \lim_{L\to+\infty} \frac{1}{2L} \int_{-L}^{L} e^{-i(n_1\omega_1+n_2\omega_2)x} f(x) dx.$$

(b) Using this results, genearalize Bessel's inequality to quasiperiodic functions of two frequencies.

(c) Can you generalize this to quasiperiodic fucntions of $N-$frequencies? How would you define those? How does the formula for "Fourier coefficients" change?

6. The following system of partial differential equations was recently proposed by Bona, Chen and Saut (J. Nonlinear Science, Vol. 12, 283-318, 2002) to describe small (but finite) amplitude waves in water

$$\begin{cases} \eta_t + \omega_x + (\omega\eta)_x + a\omega_{xxx} - b\eta_{xxt} &= 0, \\ \omega_t + \eta_x + \omega\omega_x + c\eta_{xxx} - d\omega_{xxt} &= 0, \end{cases}$$

where η represents a nondimensional elevation of the water surface, and ω is related to the velocity potential evaluated at a certain height dependent on the real parameters a, b, c and d.

(a) By linearizing around the zero solution, find the linear dispersion relation for this system.

(b) For the special case $b = -c > 0$, find conditions on a, b and d so that the equation is dispersive.