this proof not provide a method: although this celebrated proof, due to Erdős, was discovered in 1947, no construction of such graphs has been found up until now, and this lower bound for Ramsey numbers remains one of the most convincing pieces of evidence for the power of the probabilistic method.

Exercises

1. Prove that $\sqrt{2} \leq \sqrt[k]{r(k)} \leq 4$ (for all $k \geq 2$).
2. Construct a graph G showing that $r(k) \geq (k-1)^2$.
3. The graph witnessing $r(4) > 17$ may look complicated, but actually, it is easy to remember. For example, it is enough to remember this: 17; $1, 2, 4, 8$. Or this: quadratic residues modulo 17. Can you explain these two somewhat cryptic memory aids?
4. Prove $r(4) = 18$ by considering the example given in the text and by "improving" the proof of Theorem 11.2.1.

12
Generating functions

In this chapter, we are going to present a useful calculation technique. The basic idea, quite a surprising one, is to consider an infinite sequence of real numbers and associate a certain continuous function with it, the so-called generating function of the sequence. Problems about sequences can then be solved by calculations with functions.

In this introductory text, we will only get to simpler examples, whose solution can mostly be found also without generating functions. In some problems, various tricks can reach the goal even quicker than the method of generating functions (but it may not be easy to discover such tricks). This should not discourage the reader from learning about generating functions, since these are a powerful tool also for more advanced problems where other methods fail or become too complicated.

In some solutions, alternative methods are mentioned, for others they are indicated in the exercises. Sometimes the reader will perhaps find even simpler solutions.

12.1 Combinatorial applications of polynomials

How do we multiply the polynomials $p(x) = x + x^2 + x^3 + x^4$ and $q(x) = x + x^3 + x^4$? Here is a simple rule: multiply each term of $p(x)$ by each term of $q(x)$ and add all these products together. Adding the products is simple, since all the products have coefficient 1. In this way, we calculate that $p(x)q(x) = x^8 + 2x^7 + 2x^6 + 3x^5 + 2x^4 + x^3 + x^2$.

Let us now ask a different question. We pick some power of x, say x^5, and we want to know its coefficient in $p(x)q(x)$, without calculating the whole product. In our case, x^5 appears by multiplying the term x in $p(x)$ by x^4 in $q(x)$, and also by multiplying x^2 in $p(x)$ by x^3 in $q(x)$, and finally by multiplying x^4 in $p(x)$ by x in $q(x)$. Each of these possibilities adds 1 to the resulting coefficient, and hence the coefficient of x^5 in the product $p(x)q(x)$ is 3.

This is as if we had 4 silver coins with values 1, 2, 3, and 4 doublezons (these are the exponents of x in the polynomial $p(x)$) and

3 golden coins with values 1, 3, and 4 doublezons (corresponding to the exponents of x in $q(x)$), and asked how may ways there are to pay 5 doublezons by one silver and one golden coin. Mathematically speaking, the coefficient of x^5 is the number of ordered pairs (i,j), where $i+j=5$, $i \in \{1,2,3,4\}$, and $j \in \{1,3,4\}$.

Let us express the consideration just made for the two particular polynomials somewhat more generally. Let I and J be finite sets of natural numbers. Let us form the polynomials $p(x) = \sum_{i\in I} x^i$ and $q(x) = \sum_{j\in J} x^j$ (note that the coefficients in such polynomials are 0s and 1s). Then, for any natural number r, the number of solutions (i,j) of the equation

$$i+j=r$$

with $i \in I$ and $j \in J$ equals the coefficient of x^r in the product $p(x)q(x)$.

A further, more interesting generalization of this observation deals with a product of 3 or more polynomials. Let us illustrate it on a particular example first.

Problem. How many ways are there to pay the amount of 21 doublezons if we have 6 one-doublezon coins, 5 two-doublezon coins, and 4 five-doublezon coins?

Solution. The required number equals the number of solutions of the equation

$$i_1+i_2+i_3=21$$

with

$$i_1 \in \{0,1,2,3,4,5,6\},\ i_2 \in \{0,2,4,6,8,10\},\ i_3 \in \{0,5,10,15,20\}.$$

Here i_1 is the amount paid by coins of value 1 doublezon, i_2 the amount paid by 2-doublezon coins, and i_3 the amount paid by 5-doublezon coins.

This time we claim that the number of solutions of this equation equals the coefficient of x^{21} in the product

$$\left(1+x+x^2+x^3+\cdots+x^6\right)\left(1+x^2+x^4+x^6+x^8+x^{10}\right)$$
$$\times\left(1+x^5+x^{10}+x^{15}+x^{20}\right)$$

(after multiplying out the parentheses and combining the terms with the same power of x, of course). Indeed, a term with x^{21} is obtained by taking some term x^{i_1} from the first parentheses, some term x^{i_2} from the second, and x^{i_3} from the third, in such a way that $i_1+i_2+i_3=21$. Each such possible selection of i_1, i_2, and i_3 contributes 1 to the considered coefficient of x^{21} in the product.

How does this help us in solving the problem? From a purely practical point of view, this allows us to get the answer easily by computer, provided that we have a program for polynomial multiplication at our disposal. In this way, the authors have also found the result: 9. Since we only deal with relatively few coins, the solution can also be obtained by listing all possibilities, but it could easily happen that we forget some. However, the method explained above is most significant as a prelude to handling more complicated situations.

A combinatorial meaning of the binomial theorem. The binomial theorem asserts that

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n. \tag{12.1}$$

On the left-hand side, we have a product of n polynomials, each of them being $1+x$. Analogously to the above considerations with coins, the coefficient of x^r after multiplying out the parentheses equals the number of solutions of the equation

$$i_1 + i_2 + \cdots + i_n = r,$$

with $i_1, i_2, \ldots, i_n \in \{0,1\}$. But each such solution to this equation means selecting r variables among $i_1, i_2, \ldots, i_n$ that equal 1—the $n-r$ remaining ones must be 0. The number of such selections is the same as the number of r-element subsets of an n-element set, i.e. $\binom{n}{r}$. This means that the coefficient of x^r in the product $(1+x)^n$ is $\binom{n}{r}$. We have just proved the binomial theorem combinatorially!

If we play with the polynomial $(1+x)^n$ and similar ones skillfully, we can derive various identities and formulas with binomial coefficients. We have already seen simple examples in Section 3.3, namely the formulas $\sum_{k=0}^{n}\binom{n}{k} = 2^n$ and $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$ obtained by substituting $x=1$ and $x=-1$ into (12.1), respectively.

For the next example, the reader should be familiar with the notion of a derivative (of a polynomial).

12.1.1 Example. For all $n \geq 1$, we have

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.$$

Proof. This equality can be proved by differentiating both sides of the formula (12.1) as a function of the variable x. On both sides, we must obtain the same polynomial. By differentiating the left-hand side we get $n(1+x)^{n-1}$, and differentiating the right-hand side yields $\sum_{k=0}^{n} k\binom{n}{k}x^{k-1}$. By setting $x = 1$ we get the desired identity. □

An example of a different type is based on the equality of coefficients in two different expressions for the same polynomial.

Another proof of Proposition 3.3.4. We want to prove

$$\sum_{i=0}^{n}\binom{n}{i}^2 = \binom{2n}{n}.$$

Consider the identity

$$(1+x)^n(1+x)^n = (1+x)^{2n}.$$

The coefficient of x^n on the right-hand side is $\binom{2n}{n}$ by the binomial theorem. On the left-hand side, we can expand both the powers $(1+x)^n$ according to the binomial theorem, and then we multiply the two resulting polynomials. The coefficient of x^n in their product can be expressed as $\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \cdots + \binom{n}{n}\binom{n}{0}$, and this must be the same number as the coefficient of x^n on the right-hand side. This leads to

$$\sum_{i=0}^{n}\binom{n}{i}\binom{n}{n-i} = \binom{2n}{n}.$$

So we have proved Proposition 3.3.4 in a different way. □

A number of other sums and formulas can be handled similarly. But if we try some more complicated calculations, we soon encounter, as an unpleasant restriction, the fact that we work with polynomials having finitely many terms only. It turns out that the "right" tool for such calculations is an object analogous to a polynomial but with possibly infinitely many powers of x, the so-called power series. This is the subject of the next section.

Exercises

1. Let $a(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ and $b(x) = b_0 + b_1x + b_2x^2 + \cdots + b_mx^m$ be two polynomials. Write down a formula for the coefficient of x^k in the product $a(x)b(x)$, where $0 \le k \le n+m$.

2. A coffee shop sells three kinds of cakes—Danish cheese cakes, German chocolate cakes, and brownies. How many ways are there to buy 12 cakes in such a way that at least 2 cakes of each kind are included, but no more than 3 German chocolate cakes? Express the required number as a coefficient of a suitable power of x in a suitable product of polynomials.

3. How many ways are there to distribute 10 identical balls among 2 boys and 2 girls, if each boy should get at least one ball and each girl should get at least 2 balls? Express the answer as a coefficient of a suitable power of x in a suitable product of polynomials.

4. Prove the multinomial theorem 3.3.5 in a similar way as the binomial theorem was proved in the text.

5. Calculate the sum in Example 12.1.1 by a suitable manipulation of the expression $k\binom{n}{k}$ and by using the binomial theorem.

6. Compute the sum $\sum_{i=0}^{n}(-1)^i\binom{n}{i}\binom{n}{n-i}$.

12.2 Calculation with power series

Properties of power series. A *power series* is an infinite series of the form $a_0 + a_1x + a_2x^2 + \cdots$, where $a_0, a_1, a_2, \ldots$ are real numbers and x is a variable attaining real values.[1] This power series will usually be denoted by $a(x)$.

A simple example of a power series is

$$1 + x + x^2 + x^3 + \cdots \qquad (12.2)$$

(all the a_i are 1). If x is a real number in the interval $(-1, 1)$, then this series converges and its sum equals $\frac{1}{1-x}$ (this is the well-known formula for the sum of an infinite geometric series; if you're not familiar with it you should certainly consider doing Exercise 1). In this sense, the series (12.2) determines the function $\frac{1}{1-x}$. Conversely, this function contains all the information about the series (12.2). Indeed, if we differentiate the function k times and then substitute $x = 0$ into the result, we obtain exactly $k!$ times the coefficient of x^k. In other words, the series (12.2) is the Taylor series of the function $\frac{1}{1-x}$ at $x = 0$. Hence the function $\frac{1}{1-x}$ can be understood as an incarnation of the infinite sequence $(1, 1, 1, \ldots)$ and vice versa. Such

[1] It is extremely useful to consider also complex values of x and to apply methods from the theory of functions of a complex variable. But we are not going to get this far in our introductory treatment.

a transmutation of infinite sequences into functions and back is a key step in the technique of generating functions.

In order to explain what generating functions are, we have to use some notions of elementary calculus (convergence of infinite series, derivative, Taylor series), as we have already done in the example just given. If you're not familiar with enough calculus to understand this introduction, you still need not give up reading, because almost nothing from calculus is usually needed to solve problems using generating functions. For example, you can accept as a matter of faith that the infinite series $1 + x + x^2 + \cdots$ and the function $\frac{1}{1-x}$ mean the same thing, and use this and similar facts listed below for calculations.

The following proposition says that if the terms of a sequence $(a_0, a_1, a_2, \ldots)$ do not grow too fast, then the corresponding power series $a(x) = a_0 + a_1x + a_2x^2 + \cdots$ indeed defines a function of the real variable x, at least in some small neighborhood of 0. Further, from the knowledge of the values of this function we can reconstruct the sequence $(a_0, a_1, a_2, \ldots)$ uniquely.

12.2.1 Proposition. *Let $(a_0, a_1, a_2, \ldots)$ be a sequence of real numbers, and let us suppose that for some real number K, we have $|a_n| \leq K^n$ for all $n \geq 1$. Then for any number $x \in (-\frac{1}{K}, \frac{1}{K})$, the series $a(x) = \sum_{i=0}^{\infty} a_i x^i$ converges (even absolutely), and hence the value of its sum defines a function of the real variable x on this interval. This function will also be denoted by $a(x)$. The values of the function $a(x)$ on an arbitrarily small neighborhood of 0 determine all the terms of the sequence $(a_0, a_1, a_2, \ldots)$ uniquely. That is, the function $a(x)$ has derivatives of all orders at 0, and for all $n = 0, 1, 2, \ldots$ we have*

$$a_n = \frac{a^{(n)}(0)}{n!}$$

($a^{(n)}(0)$ stands for the nth derivative of the function $a(x)$ at the point 0).

A proof follows from basic results of mathematical analysis, and we omit it here (as we do for proofs of a few other results in this section). Most natural proofs are obtained from the theory of functions of a complex variable, which is usually covered in more advanced courses only. But we need very little for our purposes, and this can be proved, somewhat more laboriously, also from basic theorems on limits and derivatives of functions of a real variable. In the subsequent examples, we will not explicitly check that the power series in question converge in some neighborhood of 0 (i.e. the assumptions of Proposition 12.2.1). Usually it is easy. Moreover, in many cases, this can be avoided in the following way: once we find a correct solution of a problem using generating functions, in a possibly very suspicious manner, we can verify the solution by some other method, say by induction. And, finally, we should remark that there is also a theory of the so-called *formal power series*, which allows one to work even with power series that never converge (except at 0) in a meaningful way. So convergence is almost never a real issue in applications of generating functions.

Now, finally, we can say what a generating function is:

12.2.2 Definition. *Let* $(a_0, a_1, a_2, \ldots)$ *be a sequence of real numbers. By the* generating function *of this sequence*[2] *we understand the power series* $a(x) = a_0 + a_1x + a_2x^2 + \cdots$.

If the sequence $(a_0, a_1, a_2, \ldots)$ has only finitely many nonzero terms, then its generating function is a polynomial. Thus, in the preceding section, we have used generating functions of finite sequences without calling them so.

Manufacturing generating functions. In applications of generating functions, we often encounter questions like "What is the generating function of the sequence $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$?" Of course, the generating function is $1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + \cdots$ by definition, but is there a nice closed formula? In other words, doesn't this power series define some function well known to us from calculus, say? (This one does, namely $-\frac{\ln(1-x)}{x}$; see below.) The answer can often be found by an "assembling" process, like in a home workshop. We have a supply of parts, in our case sequences like $(1, 1, 1, \ldots)$, for which we know the generating function right away. We also have a repertoire of simple operations on sequences and their corresponding operations on the generating functions. For example, knowing the generating function $a(x)$ of some sequence $(a_0, a_1, a_2, \ldots)$, the generating function of the sequence $(0, a_0, a_1, a_2, \ldots)$ equals $x\,a(x)$. With some skill, we can usually "assemble" the sequence we want.

Let us begin with a supply of the "parts". Various examples of Taylor (or Maclaurin) series mentioned in calculus courses belong to

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[2] A more detailed name often used in the literature is the *ordinary generating function*. This suggests that other types of generating functions are also used in mathematics. We briefly mention the so-called *exponential generating functions*, which are particularly significant for combinatorial applications. The exponential generating function of a sequence $(a_0, a_1, a_2, \ldots)$ is the power series $\sum_{i=0}^{\infty}(a_i/i!)x^i$. For example, the sequence $(1, 1, 1, \ldots)$ has the exponential generating function e^x. In the sequel, except for a few exercises, we restrict ourselves to ordinary generating functions.

such a supply. For instance, we have

$$\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+\cdots=-\ln(1-x) \tag{12.3}$$

(valid for all $x\in(-1,1)$) and

$$1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots=\mathrm{e}^x$$

(holds for all real x). Lots of other examples can be found in calculus textbooks. Here is another easy calculus result, which is used particularly often:

12.2.3 Proposition (Generalized binomial theorem). *For an arbitrary real number r and for any nonnegative integer k, we define the binomial coefficient $\binom{r}{k}$ by the formula*

$$\binom{r}{k}=\frac{r(r-1)(r-2)\dots(r-k+1)}{k!}$$

(in particular, we set $\binom{r}{0}=1$). Then the function $(1+x)^r$ is the generating function of the sequence $\left(\binom{r}{0},\binom{r}{1},\binom{r}{2},\binom{r}{3},\dots\right)$. The power series $\binom{r}{0}+\binom{r}{1}x+\binom{r}{2}x^2+\cdots$ always converges for all $|x|<1$.

A proof again belongs to the realm of calculus, and it is easily done via the Taylor series.

For combinatorial applications, it is important to note that for r being a negative integer, the binomial coefficient $\binom{r}{k}$ can be expressed using the "usual" binomial coefficient (involving nonnegative integers only): $\binom{r}{k}=(-1)^k\binom{-r+k-1}{k}=(-1)^k\binom{-r+k-1}{-r-1}$. Hence for negative integer powers of $1-x$ we obtain

$$\frac{1}{(1-x)^n}=\binom{n-1}{n-1}+\binom{n}{n-1}x+\binom{n+1}{n-1}x^2+\cdots+\binom{n+k-1}{n-1}x^k+\cdots.$$

Note that the equality $\frac{1}{1-x}=1+x+x^2+\cdots$ is a particular case for $n=1$.

Operations with sequences and with their generating functions. To facilitate the above-mentioned process of "assembling" generating functions for given sequences, we list some important operations. In the sequel, let $(a_0,a_1,a_2,\dots)$ and $(b_0,b_1,b_2,\dots)$ be sequences and let $a(x)$ and $b(x)$ be their respective generating functions.

A. If we add the sequences term by term, then the corresponding operation with generating functions is simply their *addition*. That is, the sequence $(a_0+b_0, a_1+b_1, a_2+b_2, \ldots)$ has generating function $a(x)+b(x)$.

B. Another simple operation is *multiplication by a fixed real number α*. The sequence $(\alpha a_0, \alpha a_1, \alpha a_2, \ldots)$ has generating function $\alpha \cdot a(x)$.

C. If n is a natural number, then the generating function $x^n a(x)$ corresponds to the sequence

$$(\underbrace{0,0,\ldots,0}_{n\times}, a_0, a_1, a_2, \ldots).$$

This is very useful for *shifting the sequence to the right* by a required number of positions.

D. What do we do if we want to *shift the sequence to the left*, i.e. to gain the generating function for the sequence $(a_3, a_4, a_5, \ldots)$, say? Obviously, we have to divide by x^3, but we must not forget to subtract the 3 initial terms first. The correct generating function of the above sequence is

$$\frac{a(x) - a_0 - a_1 x - a_2 x^2}{x^3}.$$

E. *Substituting αx for x.* Let α be a fixed real number, and let us consider the function $c(x) = a(\alpha x)$. Then $c(x)$ is the generating function of the sequence $(a_0, \alpha a_1, \alpha^2 a_2, \ldots)$. For example, we know that $\frac{1}{1-x}$ is the generating function of the sequence of all 1s, and so by the rule just given, $\frac{1}{1-2x}$ is the generating function of the sequence of powers of 2: $(1,2,4,8,\ldots)$. This operation is also used in the following trick for replacing all terms of the considered sequence with an odd index by 0: as the reader can easily check, the function $\frac{1}{2}(a(x)+a(-x))$ corresponds to the sequence $(a_0, 0, a_2, 0, a_4, 0, \ldots)$.

F. Another possibility is a *substitution of x^n for x*. This gives rise to the generating function of the sequence whose term number nk equals the kth term of the original sequence, and all of whose other terms are 0s. For instance, the function $a(x^3)$ produces the sequence $(a_0, 0, 0, a_1, 0, 0, a_2, 0, 0, \ldots)$. A more complicated operation generalizing both E and F is the substitution of one power series for x into another power series. We will meet only a few particular examples in the exercises.

Let us see some of the operations listed so far in action.

Problem. What is the generating function of the sequence

$$(1,1,2,2,4,4,8,8,\ldots),$$

i.e. $a_n = 2^{\lfloor n/2 \rfloor}$?

Solution. As was mentioned in E, the sequence $(1,2,4,8,\ldots)$ has generating function $1/(1-2x)$. By F we get the generating function $1/(1-2x^2)$ for the sequence $(1,0,2,0,4,0,\ldots)$, and by C, the sequence $(0,1,0,2,0,\ldots)$ has generating function $x/(1-2x^2)$. By addition we finally get the generating function for the given sequence; that is, the answer is $(1+x)/(1-2x^2)$. □

G. Popular operations from calculus, *differentiation* and *integration* of generating functions, mean the following in the language of sequences. The derivative of the function $a(x)$, i.e. $a'(x)$, corresponds to the sequence

$$(a_1, 2a_2, 3a_3, \ldots).$$

In other words, the term with index k is $(k+1)a_{k+1}$ (a power series is differentiated term by term in the same way as a polynomial). The generating function $\int_0^x a(t)\mathrm{d}t$ gives the sequence $(0, a_0, \frac{1}{2}a_1, \frac{1}{3}a_2, \frac{1}{4}a_3, \ldots)$; that is, for all $k \geq 1$, the term with index k equals $\frac{1}{k}a_{k-1}$. For instance, we can derive the power series (12.3) for $\ln(1-x)$ by integrating the function $\frac{1}{1-x}$.

Here is an example where differentiation helps:

12.2.4 Problem. What is the generating function for the sequence $(1^2, 2^2, 3^2, \ldots)$ of squares, i.e. for the sequence $(a_0, a_1, a_2, \ldots)$ with $a_k = (k+1)^2$?

Solution. We begin with the sequence of all 1s with the generating function $\frac{1}{1-x}$. The first derivative of this function, $1/(1-x)^2$, gives the sequence $(1,2,3,4,\ldots)$ by G. The second derivative is $2/(1-x)^3$, and its sequence is $(2\cdot 1, 3\cdot 2, 4\cdot 3, \ldots)$, again according to G; the term with index k is $(k+2)(k+1) = (k+1)^2 + k + 1$. But we want $a_k = (k+1)^2$, and so we subtract the generating function of the sequence $(1,2,3,\ldots)$. We thus get

$$a(x) = \frac{2}{(1-x)^3} - \frac{1}{(1-x)^2}\,.$$

□

H. The last operation in our list is perhaps the most interesting one: *multiplication of generating functions*. The product $a(x)b(x)$ is the generating function of the sequence $(c_0, c_1, c_2, \ldots)$, where the numbers c_k are given by the equations

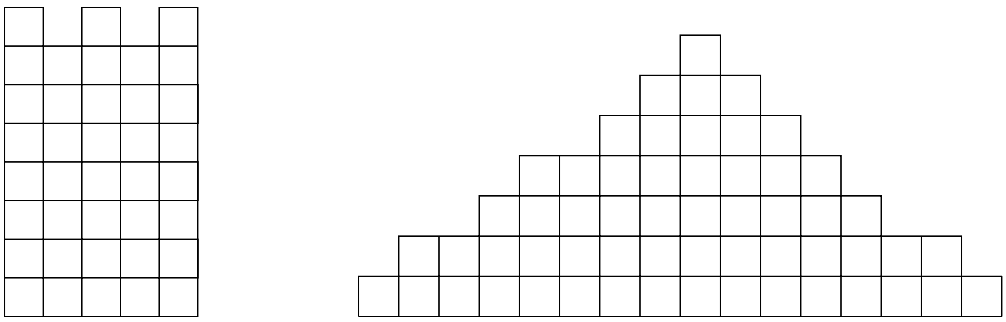
$$\begin{array}{rcl} c_0 & = & a_0 b_0 \\ c_1 & = & a_0 b_1 + a_1 b_0 \\ c_2 & = & a_0 b_2 + a_1 b_1 + a_2 b_0 \\ & & \vdots \end{array}$$

and in general we can write

$$c_k = \sum_{i,j \geq 0:\ i+j=k} a_i b_j. \tag{12.4}$$

This is easy to remember—the terms in the product $a(x)b(x)$ up to the kth one are the same as in the product of the polynomials $(a_0 + a_1 x + \cdots + a_k x^k)$ and $(b_0 + b_1 x + \cdots + b_k x^k)$.

Multiplication of generating functions has a combinatorial interpretation which we now explain in a somewhat childish example. A natural example comes in Section 12.4. Suppose that we have a supply of indistinguishable wooden cubes, and we know that we can build a tower in a_i different ways from i cubes, $i = 0, 1, 2, \ldots$, and also that we can build a pyramid in b_j different ways from j cubes, $j = 0, 1, 2, \ldots$.



If we now have k cubes altogether, then $c_k = a_0 b_k + a_1 b_{k-1} + \cdots + a_k b_0$ is the number of ways to build one tower and one pyramid at the same time (assuming that a tower and a pyramid never have any cubes in common). In short, the generating function for the number of ordered pairs (tower, pyramid) is obtained as the product of the generating functions for the number of towers and for the number of pyramids.

Remark. The operations listed above are useful not only for finding the generating function for a given sequence, but often also for the reverse task, namely finding the sequence corresponding to a given generating function. In principle, the sequence can be found by computing the Taylor series (according to Proposition 12.2.1), i.e. by a repeated differentiation, but in practice this is seldom a good method.

We conclude this section with an example of an application of generating functions. More examples are supplied in the exercises, and several more sophisticated problems follow in subsequent sections.

Problem. A box contains 30 red, 40 blue, and 50 white balls; balls of the same color are indistinguishable. How many ways are there of selecting a collection of 70 balls from the box?

Solution. Armed with the results of Section 12.1, we find that the number we seek equals the coefficient of x^{70} in the product

$$(1+x+x^2+\cdots+x^{30})(1+x+x^2+\cdots+x^{40})(1+x+x^2+\cdots+x^{50}).$$

We need not multiply this out. Instead, we can rewrite

$$1+x+x^2+\cdots+x^{30} = \frac{1-x^{31}}{1-x}\,. \tag{12.5}$$

To see this equality, we can recall a formula for the sum of the first n terms of a geometric series. In case we cannot remember it, we can help ourselves as follows. We begin with the generating function of the sequence $(1,1,1,\ldots)$, which is $\frac{1}{1-x}$, and we subtract from it the generating function of the sequence

$$(\underbrace{0,0,\ldots,0}_{31\times},1,1,\ldots),$$

which is $x^{31}/(1-x)$ by item C above. The result is $(1-x^{31})/(1-x)$, which is the generating function for the sequence

$$(\underbrace{1,1,\ldots,1}_{31\times},0,0,\ldots).$$

This shows that (12.5) holds.

Hence we can rewrite the whole product as

$$\frac{1-x^{31}}{1-x}\cdot\frac{1-x^{41}}{1-x}\cdot\frac{1-x^{51}}{1-x}=\frac{1}{(1-x)^3}\,(1-x^{31})(1-x^{41})(1-x^{51}).$$

The factor $(1-x)^{-3}$ can be expanded according to the generalized binomial theorem 12.2.3. In the product of the remaining factors $(1-x^{31})(1-x^{41})(1-x^{51})$, it suffices to find the coefficients of powers up to x^{70}, which is quite easy. We get

$$\left(\binom{2}{2}+\binom{3}{2}x+\binom{4}{2}x^2+\cdots\right)(1-x^{31}-x^{41}-x^{51}+\cdots),$$

where the dots $\cdots$ in the second parentheses stand for powers higher than x^{70}. The coefficient of x^{70} in this product is $\binom{70+2}{2}-\binom{70+2-31}{2}-\binom{70+2-41}{2}-\binom{70+2-51}{2}=1061$.

Exercises

1. (a) Show that $(1-x)(1+x+x^2+\cdots+x^n)$ and $1-x^{n+1}$ are the same polynomials, and deduce the formula $1+x+x^2+\cdots+x^n=(1-x^{n+1})/(1-x)$. For which values of x is it correct?

   (b) Using the formula in (a), show that the infinite series $1+x+x^2+\cdots$ converges to $\frac{1}{1-x}$ for all $x\in(-1,1)$, and diverges for all x outside this interval. (This requires basic notions from calculus.)

2. (a) Determine the coefficient of x^{15} in $(x^2+x^3+x^4+\cdots)^4$.

   (b) Determine the coefficient of x^{50} in $(x^7+x^8+x^9+x^{10}+\cdots)^6$.

   (c) Determine the coefficient of x^5 in $(1-2x)^{-2}$.

   (d) Determine the coefficient of x^4 in $\sqrt[3]{1+x}$.

   (e) Determine the coefficient of x^3 in $(2+x)^{3/2}/(1-x)$.

   (f) Determine the coefficient of x^4 in $(2+3x)^5\sqrt{1-x}$.

   (g) Determine the coefficient of x^3 in $(1-x+2x^2)^9$.

3. Find generating functions for the following sequences (express them in a closed form, without infinite series!):

   (a) $0,0,0,0,-6,6,-6,6,-6,\ldots$

   (b) $1,0,1,0,1,0,\ldots$

   (c) $1,2,1,4,1,8,\ldots$

   (d) $1,1,0,1,1,0,1,1,0,\ldots$

4. Find the probability that we get exactly 12 points when rolling 3 dice.

5. Let a_n be the number of ordered triples (i, j, k) of integer numbers such that $i \geq 0$, $j \geq 1$, $k \geq 1$, and $i+3j+3k = n$. Find the generating function of the sequence $(a_0, a_1, a_2, \ldots)$ and calculate a formula for a_n.

6. Let a_n be the number of ordered r-tuples $(i_1, \ldots, i_r)$ of nonnegative integers with $i_1+i_2+\cdots+i_r = n$; here r is some fixed natural number.

   (a) Find the generating function of the sequence $(a_0, a_1, a_2, \ldots)$.

   (b) Find a formula for a_n. (This has been solved by a different method in Section 3.3.)

7. Solve Problem 12.2.4 without using differentiation—apply the generalized binomial theorem instead.

8. Let a_n be the number of ways of paying the sum of n doublezons using coins of values 1, 2, and 5 doublezons.

   (a) Write down the generating function for the sequence $(a_0, a_1, \ldots)$.

   (b) *Using (a), find a formula for a_n (reading the next section can help).

9. (a) Check that if $a(x)$ is the generating function of a sequence $(a_0, a_1, a_2, \ldots)$ then $\frac{1}{1-x}a(x)$ is the generating function of the sequence of partial sums $(a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots)$.

   (b) Using (a) and the solution to Problem 12.2.4, calculate the sum $\sum_{k=1}^{n} k^2$.

   (c) By a similar method, calculate the sum $\sum_{k=1}^{n} k^3$.

   (d) For natural numbers n and m, compute the sum $\sum_{k=0}^{m}(-1)^k \binom{n}{k}$.

   (e) Now it might seem that by this method we can calculate almost any sum we can think of, but it isn't so simple. What happens if we try computing the sum $\sum_{k=1}^{n} \frac{1}{k}$ in this manner?

10. *Let n, r be integers with $n \geq r \geq 1$. Pick a random r-element subset of the set $\{1, 2, \ldots, n\}$ and call it R (all the $\binom{n}{r}$ possible subsets have the same probability of being picked for R). Show that the expected value of the smallest number in R is $\frac{n+1}{r+1}$.

11. (Calculus required) *Prove the formula for the product of power series. That is, show that if $a(x)$ and $b(x)$ are power series satisfying the assumptions of Proposition 12.2.1, then on some neighborhood of 0, the power series $c(x)$ with coefficients given by the formula (12.4) converges to the value $a(x)b(x)$.

12. (Calculus required) Let $a(x) = a_0 + a_1x + a_2x^2 + \cdots$ be a power series with nonnegative coefficients, i.e. suppose that $a_i \geq 0$ for all i. We define its *radius of convergence* ρ by setting

$$\rho = \sup\{x \geq 0\colon\ a(x) \text{ converges}\}.$$

   (a) *Prove that $a(x)$ converges for each real number $x \in [0, \rho)$, and that the function $a(x)$ is continuous in the interval $[0, \rho)$.

(b) Find an example of a sequence $(a_0, a_1, a_2, \ldots)$ with $\rho = 1$ such that the series $a(\rho)$ diverges.

(c) Find an example of a sequence $(a_0, a_1, a_2, \ldots)$ with $\rho = 1$ such that the series $a(\rho)$ converges.

13. (A warning example; calculus required) Define a function f by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

(a) *Show that all derivatives of f at 0 exist and equal 0.

(b) Prove that f is not given by a power series at any neighborhood of 0.

14. (Exponential generating functions) In a footnote, we have defined the *exponential generating function* of a sequence $(a_0, a_1, a_2, \ldots)$ as the power series $A(x) = \sum_{i=0}^{\infty}(a_i/i!)x^i$. Here we are going to sketch some combinatorial applications.

For a group of n people, one can consider various types of arrangements. For instance, one type of arrangement of n people would be to select one of the people as a leader, and the others as a crowd. For each group of n people, there are n arrangements of this type. Other examples of types of arrangements are: arranging the n people in a row from left to right ($n!$ possibilities), arranging the n people in a ring for a folk dance (there are $(n-1)!$ possibilities), letting the people be just an unorganized crowd (1 possibility), arranging the people into a jury (12 possibilities for $n = 12$, with electing one chairman, and 0 possibilities otherwise).

(a) For each type of arrangement mentioned above, write down the exponential generating function of the corresponding sequence (i.e. for the sequence $(a_0, a_1, a_2, \ldots)$, where a_i is the number of arrangements of the considered type of i people).

(b) Let a_n be the number of possible arrangements of some type A for a group of n people, and let b_n be the number of possible arrangements of some type B for a group of n people. Let an arrangement of type C mean the following: divide the given group of n people into two groups, the First Group and the Second Group, and arrange the First Group by an arrangement of type A and the Second Group by an arrangement of type B. Let c_n be the number of arrangements of type C for a group of n people. Check that if $A(x)$, $B(x)$, and $C(x)$ are the corresponding exponential generating functions, then $C(x) = A(x)B(x)$. Discuss specific examples with the arrangement types mentioned in the introduction to this exercise.

(c) *Let $A(x)$ be the exponential generating function for arrangements of type A as in (b), and, moreover, suppose that $a_0 = 0$ (no empty group is allowed). Let an arrangement of type D mean dividing the given n people into k groups, the First Group, Second Group, ..., kth Group ($k = 0, 1, 2, \ldots$), and arranging each group by an arrangement of type A. Express the exponential generating function $D(x)$ using $A(x)$.

(d) *Let $A(x)$ be as in (c), and let an arrangement of type E mean dividing the given n people into some number of groups and organizing each group by an arrangement of type A (but this time it only matters who is with whom, the groups are not numbered). Express the exponential generating function $E(x)$ using $A(x)$.

(e) How many ways are there of arranging n people into pairs (it only matters who is with whom, the pairs are not numbered)? Solve using (d).

15. *Using part (d) of Exercise 14, find the exponential generating function for the *Bell numbers*, i.e. the number of equivalences on n given points. Calculate the first few terms of the Taylor series to check the result numerically (see also Exercise 3.8.8).

16. Twelve students should be assigned work on 5 different projects. Each student should work on exactly one project, and for every project at least 2 and at most 4 students should be assigned. In how many ways can this be done? Use the idea of part (b) in Exercise 14.

17. (Hatcheck lady strikes again)

    (a) *Using parts (a) and (d) of Exercise 14, write down the exponential generating function for arranging n people into one or several rings for a dance (a ring can have any number of persons including 1). Can you derive the number of such arrangements in a different way?

    (b) Consider arrangements as in (a), but with each ring having at least 2 people. Write down the exponential generating function.

    (c) *Using the result of (b), find the number of arrangements of n people into rings with at least 2 people each. If your calculations are correct, the result should equal the number of permutations without a fixed point (see Section 3.8). Explain why. (This actually gives an alternative solution to the hatcheck lady problem, without inclusion–exclusion!)

12.3 Fibonacci numbers and the golden section

We will investigate the sequence $(F_0, F_1, F_2, \ldots)$ given by the following rules:

$$F_0 = 0,\ F_1 = 1,\ F_{n+2} = F_{n+1} + F_n \quad \text{for } n = 0, 1, 2, \ldots$$

This sequence has been studied by Leonardo of Pisa, called Fibonacci, in the 13th century, and it is known as the Fibonacci numbers. Its first few terms are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

Fibonacci illustrated this sequence with a not too realistic example concerning the reproduction of rabbits in his treatise, but the sequence does have a certain relevance in biology (for instance, the number of petals of various flowers is very often a Fibonacci number; see Stewart [46] for a possible explanation). In mathematics and computer science it surfaces in various connections and quite frequently.

We show how to find a formula for the nth Fibonacci number using generating functions. So let $F(x)$ denote the generating function of this sequence. The main idea is to express the generating function of a sequence whose kth term equals $F_k - F_{k-1} - F_{k-2}$ for all $k \geq 2$. By the relation defining the Fibonacci numbers, this sequence must have all terms from the second one on equal to 0. On the other hand, such a generating function can be constructed from $F(x)$ using the operations discussed in Section 12.2. In this way, we get an equation from which we can determine $F(x)$.

Concretely, let us take the function $F(x) - xF(x) - x^2F(x)$ corresponding to the sequence

$$(F_0, F_1 - F_0, F_2 - F_1 - F_0, F_3 - F_2 - F_1, F_4 - F_3 - F_2, \ldots)$$
$$= (0, 1, 0, 0, 0, \ldots).$$

In the language of generating functions, this means $(1-x-x^2)F(x) = x$, and hence

$$F(x) = \frac{x}{1 - x - x^2}\,. \tag{12.6}$$

If we now tried to compute the Taylor series of this function by differentiation, we wouldn't succeed (we would only recover the definition of the Fibonacci numbers). We must bring in one more trick, which is well known in calculations of integrals by the name *decomposition into partial fractions.* In our case, this method guarantees that the rational function on the right-hand side of (12.6) can be rewritten in the form

$$\frac{x}{1 - x - x^2} = \frac{A}{x - x_1} + \frac{B}{x - x_2}\,,$$

where x_1, x_2 are the roots of the quadratic polynomial $1-x-x^2$ and A and B are some suitable constants. For our purposes, a slightly different form will be convenient, namely

$$\frac{x}{1-x-x^2} = \frac{a}{1-\lambda_1 x} + \frac{b}{1-\lambda_2 x}, \tag{12.7}$$

where $\lambda_1 = \frac{1}{x_1}$, $\lambda_2 = \frac{1}{x_2}$, $a = -\frac{A}{x_1}$, and $b = -\frac{B}{x_2}$. From (12.7), it is easy to write down a formula for F_n. As the reader is invited to check, we get $F_n = a\lambda_1^n + b\lambda_2^n$.

We omit the calculation of the roots x_1, x_2 of the quadratic equation here, as well as the calculation of the constants a and b, and we only present the result:

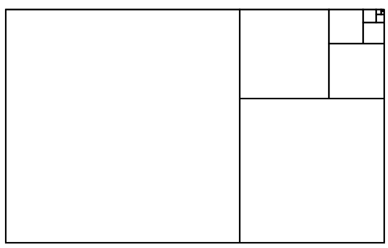
$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

It is remarkable that this expression full of irrational numbers should give an integer for each natural number n—but it does.

With approximate numerical values of the constants, the formula looks as follows:

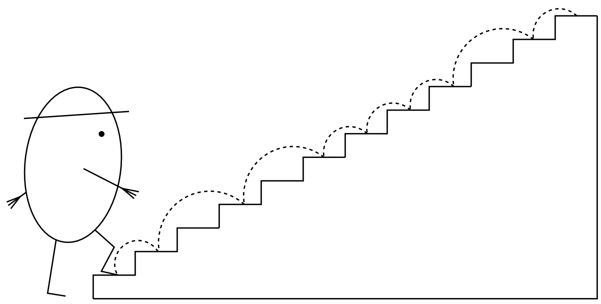
$$F_n = (0.4472135\ldots)\left[(1.6180339\ldots)^n - (-0.6180339\ldots)^n\right].$$

We see that for large n, the numbers F_n behave roughly as $\frac{1}{\sqrt{5}}\lambda_1^n$, and that the ratio F_n/F_{n+1} converges to the limit $\frac{1}{\lambda_1} = 0.6180339\ldots$. This ratio was highly valued in ancient Greece; it is called the *golden section.* A rectangle with side ratio equal to the golden section was considered the most beautiful and proportional one among all rectangles. (Using the picture below, you can check whether your aesthetic criteria for rectangles agree with those of the ancient Greeks.) If we cut off a square from such a golden-section rectangle, the sides of the remaining rectangle again determine the golden section:



Similar to the Fibonacci numbers, the golden section is encountered surprisingly often in mathematics.

Another derivation: a staircase. Consider a staircase with n stairs. How many ways are there to ascend the staircase, if we can climb by 1 stair or by 2 stairs in each step?



In other words, how many ways are there to write the number n as a sum of 1s and 2s, or how many solutions are there to the equation

$$s_1 + s_2 + \cdots + s_k = n,$$

with $s_i \in \{1, 2\}$, $i = 1, 2, \ldots, k$, $k = 0, 1, 2, \ldots$? If we denote the number of such solutions by S_n, we get $S_1 = 1$, $S_2 = 2$, and it is not difficult to see that for $n \geq 1$, we have $S_{n+2} = S_{n+1} + S_n$ (think it over!). This implies that S_n is exactly the Fibonacci number F_{n-1}.

We derive the generating function of the sequence $(S_0, S_1, S_2, \ldots)$ in a different way. According to the recipe from Section 12.1, we get that for a given k, the number of solutions to the equation $s_1+s_2+\cdots+s_k = n$ with $s_i \in \{1, 2\}$ equals the coefficient of x^n in the product $(x+x^2)^k$. We can choose k arbitrarily, however (it is not prescribed by how many steps we should ascend the staircase), and therefore S_n is the coefficient of x^n in the sum $\sum_{k=0}^{\infty}(x + x^2)^k$. Thus, this sum is the generating function of the sequence $(S_0, S_1, S_2, \ldots)$. The generating function can be further rewritten: the sum is a geometric series with quotient $x + x^2$ and so it equals $1/(1 - x - x^2)$. Hence the generating function for the Fibonacci numbers is $x/(1 - x - x^2)$, as we have already derived by a different approach.

Recipes for some recurrences. By the above method, one can find a general form of a sequence $(y_0, y_1, y_2, \ldots)$ satisfying the equation

$$y_{n+k} = a_{k-1}y_{n+k-1} + a_{k-2}y_{n+k-2} + \cdots + a_1y_{n+1} + a_0y_n \qquad (12.8)$$

for all $n = 0, 1, 2, \ldots$, where k is a natural number and $a_0, a_1, \ldots, a_{k-1}$ are real (or complex) numbers. For instance, for the Fibonacci numbers we would set $k = 2$, $a_0 = a_1 = 1$. Let us denote the set of all sequences $(y_0, y_1, y_2, \ldots)$ satisfying (12.8) by the symbol $\mathcal{Y}$ (so $\mathcal{Y}$ depends on k and on $a_0, a_1, \ldots, a_{k-1}$). This set contains many sequences in general, because the first k terms of the sequence $(y_0, y_1, y_2, \ldots)$ can be chosen at will (while all the remaining terms are determined by the relation (12.8)). In the sequel, we will describe what the sequences of $\mathcal{Y}$ look like, but first we take a detour into terminology.

A learned name for Eq. (12.8) is a *homogeneous linear recurrence of kth degree with constant coefficients*. Let us try to explain the parts of this complicated name.

- A *recurrence* or *recurrent relation* is used as a general notion denoting a relation (usually a formula) expressing the nth term of a sequence via several previous terms of the sequence.[3]
- *Homogeneous* appears in the name since whenever $(y_0, y_1, y_2, \ldots) \in \mathcal{Y}$ then also $(\alpha y_0, \alpha y_1, \alpha y_2, \ldots) \in \mathcal{Y}$ for any real number α. (In mathematics, "homogeneity" usually means "invariance to scaling".) On the other hand, an example of an inhomogeneous recurrence is $y_{n+1} = y_n + 1$.
- The word *linear* here means that the values of y_j always appear in the first power in the recurrence and are not multiplied together. A nonlinear recurrence is, for example, $y_{n+2} = y_{n+1}y_n$.
- Finally, the phrase *with constant coefficients* expresses that a_0, $a_1, \ldots, a_{k-1}$ are fixed numbers independent of n. One could also consider a recurrence like $y_{n+1} = (n-1)y_n$, where the coefficient on the right-hand side is a function of n.

So much for the long name. Now we are going to formulate a general result about solutions of the recurrent relation of the considered type. We define the *characteristic polynomial* of the recurrence (12.8) as the polynomial

$$p(x) = x^k - a_{k-1}x^{k-1} - a_{k-2}x^{k-2} - \cdots - a_1 x - a_0.$$

For example, the characteristic polynomial of the recurrence relation for the Fibonacci numbers is $x^2 - x - 1$. Let us recall that any polynomial of degree k with coefficient 1 at x^k can be written in the form

$$(x - \lambda_1)(x - \lambda_2) \ldots (x - \lambda_k),$$

where $\lambda_1, \ldots, \lambda_k$ are (generally complex) numbers called the *roots* of the given polynomial.

12.3.1 Proposition. *Let $p(x)$ be the characteristic polynomial of the homogeneous linear recurrence (12.8).*

(i) (Simple roots) *Suppose that $p(x)$ has k* pairwise distinct *roots $\lambda_1, \ldots, \lambda_k$. Then for any sequence $y = (y_0, y_1, \ldots) \in \mathcal{Y}$ satisfying (12.8), complex constants $C_1, C_2, \ldots, C_k$ exist such that for all n, we have*

$$y_n = C_1\lambda_1^n + C_2\lambda_2^n + \cdots + C_k\lambda_k^n.$$

(ii) (The general case) *Let $\lambda_1, \ldots, \lambda_q$ be pairwise different complex numbers, and let $k_1, \ldots, k_q$ be natural numbers with $k_1 + k_2 + \cdots +$*

[3] Another name sometimes used with this meaning is *difference equation*. This is related to the so-called difference of a function, which is a notion somewhat similar to derivative. The theory of difference equations is often analogous to the theory of differential equations.

$k_q = k$ such that

$$p(x) = (x-\lambda_1)^{k_1}(x-\lambda_2)^{k_2}\ldots(x-\lambda_q)^{k_q}.$$

Then for any sequence $y = (y_0, y_1, \ldots) \in \mathcal{Y}$ satisfying (12.8), complex constants C_{ij} exist ($i = 1, 2, \ldots, q$, $j = 0, 1, \ldots, k_i - 1$) such that for all n, we have

$$y_n = \sum_{i=1}^{q} \sum_{j=0}^{k_i-1} C_{ij} \binom{n}{j} \lambda_i^n.$$

How do we solve a recurrence of the form (12.8) using this proposition? Let us give two brief examples. For the recurrence relation $y_{n+2} = 5y_{n+1} - 6y_n$, the characteristic polynomial is $p(x) = x^2 - 5x + 6 = (x-2)(x-3)$. Its roots are $\lambda_1 = 2$ and $\lambda_2 = 3$, and Proposition 12.3.1 tells us that we should look for a solution of the form $C_1 2^n + C_2 3^n$. Being given some initial conditions, say $y_0 = 2$ and $y_1 = 5$, we must determine the constants C_1, C_2 in such a way that the formula gives these required values for $n = 0, 1$. In our case, we would set $C_1 = C_2 = 1$.

And one more example, a quite artificial one, with multiple roots: the equation $y_{n+5} = 8y_{n+4} + 25y_{n+3} - 38y_{n+2} + 28y_{n+1} - 8y_n$ has characteristic polynomial[4] $p(x) = (x-1)^2(x-2)^3$, and so Proposition 12.3.1 says that the solution should be sought in the form $y_n = C_{10} + C_{11}n + C_{20}2^n + C_{21}n2^n + C_{22}\binom{n}{2}2^n$. The values of the constants should again be calculated according to the values of the first 5 terms of the sequence $(y_0, y_1, y_2, \ldots)$.

The procedure just demonstrated for solving the recurrence (12.8) can be found in all kinds of textbooks, and the generating functions are seldom mentioned in this connection. Indeed, Proposition 12.3.1 can be proved in a quite elegant manner using linear algebra (as indicated in Exercise 16), and once we know from somewhere that the solution should be of the indicated form, generating functions become unnecessary. However, the method explained above for Fibonacci numbers really finds the correct form of the solution. Moreover, it is a nice example of an application of generating functions, and a similar approach can sometimes be applied successfully for recurrences of other types, where no general solution method is known (or where it is difficult to find such a method in the literature, which is about the same in practice).

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[4]Of course, the authors have selected the coefficients so that the characteristic polynomial comes out very nicely. The instructions given above for solving homogeneous linear recurrence relations with constant coefficients (as well as the method with generating functions) leave aside the question of finding the roots of the characteristic polynomial. In examples in various textbooks, the recurrences mostly have degree 1 or 2, or the coefficients are chosen in such a way that the roots are small integers.

Exercises

1. *Determine the number of n-term sequences of 0s and 1s containing no two consecutive 0s.

2. Prove that any natural number $n \in \mathbf{N}$ can be written as a sum of mutually distinct Fibonacci numbers.

3. Express the nth term of the sequences given by the following recurrence relations (generalize the method used for the Fibonacci numbers, or use Proposition 12.3.1).

   (a) $a_0 = 2,\ a_1 = 3,\ a_{n+2} = 3a_n - 2a_{n+1}\ \ (n = 0, 1, 2, \ldots)$,

   (b) $a_0 = 0,\ a_1 = 1,\ a_{n+2} = 4a_{n+1} - 4a_n\ \ (n = 0, 1, 2, \ldots)$,

   (c) $a_0 = 1, a_{n+1} = 2a_n + 3\ \ (n = 0, 1, 2, \ldots)$.

4. In a sequence $(a_0, a_1, a_2, \ldots)$, each term except for the first two is the arithmetic mean of the preceding two terms, i.e. $a_{n+2} = (a_{n+1} + a_n)/2$. Determine the limit $\lim_{n\to\infty} a_n$ (as a function of a_0, a_1).

5. Solve the recurrence relation $a_{n+2} = \sqrt{a_{n+1}a_n}$ with initial conditions $a_0 = 2$, $a_1 = 8$, and find $\lim_{n\to\infty} a_n$.

6. (a) Solve the recurrence $a_n = a_{n-1} + a_{n-2} + \cdots + a_1 + a_0$ with the initial condition $a_0 = 1$.

   (b) *Solve the recurrence $a_n = a_{n-1} + a_{n-3} + a_{n-4} + a_{n-5} + \cdots + a_1 + a_0$ $(n \geq 3)$ with $a_0 = a_1 = a_2 = 1$.

7. *Express the sum

$$S_n = \binom{2n}{0} + 2\binom{2n-1}{1} + 2^2\binom{2n-2}{2} + \cdots + 2^n\binom{n}{n}$$

   as a coefficient of x^{2n} in a suitable power series, and find a simple formula for S_n.

8. Calculate $\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}(-4)^{-k}$.

9. *Show that the number $\frac{1}{2}\left[(1+\sqrt{2})^n + (1-\sqrt{2})^n\right]$ is an integer for all $n \geq 1$.

10. *Show that the number $(6+\sqrt{37})^{999}$ has at least 999 zeros following the decimal point.

11. *Show that for any $n \geq 1$, the number $(\sqrt{2}-1)^n$ can be written as the difference of the square roots of two consecutive integers.

12. *Find the number of n-term sequences consisting of letters a, b, c, d such that a is never adjacent to b.