that the transversality condition guarantees that the side condition curve is not a characteristic line (Why ?). Examples 4 and 5 show that the transversality condition is necessary (cf. also Problem 9). In the next example, the side condition curve is $y = x^3$, instead of a line.

Example 7. Solve the PDE $u_x - u_y + u = 0$, subject to the condition $u(x,x^3) = e^{-x}(x + x^3)$.

Solution. Here u is specified on the curve $y = x^3$. This curve intersects each characteristic line $x+y = d$ exactly once, and transversely (since the slope of the curve x^3 is $3x^2$ which is never the same as the slope (-1) of the characteristic curves). With $v(w,z) \equiv u(x,y)$, where

$$\begin{cases} w = x + y \\ z = y \end{cases} ; \qquad \begin{cases} x = w - z \\ y = z\,. \end{cases}$$

the PDE becomes $-v_z + v = 0$. The general solution is $v(w,z) = C(w)e^z$, whence $u(x,y) = C(x+y)e^y = D(x+y)e^{-x}$. In order to meet the side condition, we need to choose D, so that $e^{-x}(x+x^3) = u(x,x^3) = D(x+x^3)e^{-x}$. The choice that works is the function $D(r) = r$, and the solution of the problem is then $u(x,y) = (x+y)e^{-x}$. □

An Application to Population or Inventory Analysis

Under certain natural assumptions, here we derive and solve a first–order PDE which governs the way in which the composition, with respect to age, of a population of individuals, changes with time. The individuals need not be biological organisms, but they could be manufactured items (e.g., light bulbs, transistors, food products) or more generally any collection of similar objects which become defective with age according to a statistical pattern. Thus, perhaps this first–order PDE has a greater variety of applications than the heat, wave and Laplace equations.

Suppose that at time t a certain population has approximately $P(y,t)\cdot\Delta y$ individuals between the ages of y and $y + \Delta y$. In other words, at a fixed time t, $P(y,t)$ is **the population density** with respect to the age variable y. At time t, the number of individuals between the ages of a and b is then $\int_a^b P(y,t)\,dy$. We suppose that the number of individuals of age between y and $y + \Delta y$, which die in the time interval from t to $t + \Delta t$ is approximately $D(y,t)\cdot P(y,t)\cdot\Delta y\cdot\Delta t$, for some function $D(y,t)$ which has been statistically determined, say by observation. One usually expects that the "death rate density" $D(y,t)$ increases as y increases (i.e., older individuals may be more likely to expire), and $D(y,t)$ could very well depend on t because of seasonal variations (e.g., air conditioners are more likely to die in the summer). If individuals never expire (i.e., $D(y,t) = 0$), then $P(y,t+\Delta t) = P(y-\Delta t,t)$ for any y and time interval Δt with $0 \le \Delta t \le y$, since the population density at time $t+\Delta t$ is just a translate, by the age difference Δt, of what it was at time t. However, if $D \neq 0$, then we must take into account that a number of individuals will die, as time advances from t to $t+\Delta t$. Indeed,

$$P(y,t+\Delta t) = P(y-\Delta t,t) - \int_0^{\Delta t} D(y-\Delta t+s,t+s)P(y-\Delta t+s,t+s)\ ds. \tag{21}$$

Differentiating both sides of (21) with respect to Δt, and then setting $\Delta t = 0$, we obtain

$$P_t(y,t) = -P_y(y,t) - D(y,t)P(y,t) \quad \text{or} \quad P_y + P_t + D(y,t)P = 0\ . \tag{22}$$

The coefficients for P_y and P_t are constant (both are 1), but the coefficient $D(y,t)$ of P is not necessarily constant. Nevertheless, all of the theory of this section still applies to PDEs of the form $au_x + bu_y + c(x,y)u = f(x,y)$ (i.e., only the constancy of the coefficients of u_x and u_y is needed to reduce this PDE to an ODE, by a linear change of variables). For equation (22), the family of characteristic lines is $y - t = d$. Hence, we make the change of variables

$$\begin{cases} w = y - t \\ z = y \end{cases} \quad ; \quad \begin{cases} t = z - w \\ y = z\ . \end{cases} \tag{23}$$

With $Q(w,z) \equiv P(y,t)$, we have $P_y + P_t = Q_w w_y + Q_z z_y + Q_w w_t + Q_z z_t = Q_z$, and (22) becomes $Q_z + D(z,z-w)Q = 0$. The integrating factor is $\exp[\int D(z,z-w)\ dz]$, and we obtain

$$Q(w,z) = C(w) \exp\left[-\int_w^z D(\zeta,\zeta-w)\ d\zeta \right], \tag{24}$$

where $C(w)$ is an arbitrary C^1 function, and the lower limit w in the integral is introduced for future convenience, but it can be replaced by any C^1 function of w (Why ?). Hence,

$$P(y,t) = C(y-t) \exp\left[-\int_{y-t}^{y} D(\zeta,\zeta-y+t)\ d\zeta \right].$$

If we set $t = 0$, then we obtain $P(y,0) = C(y)$. Thus, $C(y)$ is just the initial population density This is why we chose w for the lower limit in (24). We have (with $\tau \equiv \zeta-y+t$)

$$\boxed{P(y,t) = P(y-t,0) \exp\left[-\int_{y-t}^{y} D(\zeta,\zeta-y+t)\ d\zeta \right] = P(y-t,0) \exp\left[-\int_0^t D(y-t+\tau,\tau)\ d\tau \right].} \tag{25}$$

Note that since $P(y,0)$ has not yet been defined for $y < 0$ (i.e. for negative ages), the solution (25) is undefined for $t > y$. For $y < 0$, it is convenient to define $P(y,0)$, so that $P(y,0)\cdot\Delta y$ is approximately the number of individuals to be produced between $|y|$ and $|y| + \Delta y$ time units into the future. In other words, for $y < 0$, $P(y,0)$ is the production rate at $-y$ time units into the future. Naturally, we take $D(y,t) = 0$ for $y < 0$. Then (25) defines $P(y,t)$ for all (y,t). In the case of a constant rate of production (say C) and when $D(y,t) = D(y)$ is time–independent (for $y > 0$), we can determine (using the middle expression in (25)) the