Journal of Nonlinear and Convex Analysis
Volume 19, Number 3, 2018, 433–460

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

CUBIC CONVERGENCE OF NEWTON-STEFFENSEN'S METHOD FOR OPERATORS WITH LIPSCHITZ CONTINUOUS DERIVATIVE

YAN ZHANG, YAOHUA HU*, CARISA KWOK WAI YU†, AND JINHUA WANG‡

Abstract. The present paper is concerned with the semilocal as well as the local convergence issues of Newton-Steffensen's method for solving nonlinear operator equations on Banach spaces. The cubical convergence results are established under the assumption that the first derivative of the involved operator is Lipschitz continuous. More precisely, the convergence criteria are established, and estimates of radii of convergence balls for Newton-Steffensen's method are also provided under some mild conditions. The obtained results improve the corresponding ones in [Appl. Math. Comput., 169, 242–246 (2005)]. Finally, preliminary numerical results illustrate that the generated sequences converge cubically under the assumption that the first derivative of the operator is Lipschitz continuous. It should be remarked that in these numerical experiments the second derivatives of the operators do not satisfy Lipschitz continuous and so the corresponding results in [Appl. Math. Comput., 169, 242–246 (2005)] are not applicable.

1. Introduction

Let $F : D \subseteq X \to Y$ be a nonlinear operator with its Fréchet derivative being denoted by F'. Finding solutions of the nonlinear operator equation

$$F(x) = 0 \tag{1.1}$$

on Banach spaces is a very general problem which is widely studied in both theoretical and applied areas of mathematics.

When F is Fréchet differentiable, the most important method to find an approximation of a solution of (1.1) is Newton's method which takes the following form:

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \quad n = 0, 1, 2, \ldots, \tag{1.2}$$

where $x_0 \in D$ is an initial point. The sequence generated by Newton's method is *quadratically* convergent under the assumption:

- the first Fréchet derivative of F is Lipschitz continuous.

2010 *Mathematics Subject Classification.* 65K05, 93E24.

Key words and phrases. Newton-Steffensen's method, Lipschitz continuous, convergence criterion, convergence radius.

*Corresponding author. This author was partially supported by the National Natural Science Foundation of China (grant 11601343, 11601344) and Natural Science Foundation of Guangdong (grant 2016A030310038).

†This author was partially supported by grants from the Research Grants Council of the Hong Kong Special Administrative Region, China (UGC/FDS14/P03/14 and UGC/FDS14/P02/15).

‡This author was partially supported by the National Natural Science Foundation of China (grant 11371325, 11771397) and Zhejiang Provincial Natural Science Foundation of China (grant LY17A010021).

There are several kinds of cubic generalizations for Newton's method. The most important family is the Euler-Halley family, see for example, [1, 3–5, 8, 9, 12, 13, 25] and references therein. These cubic methods require the computation of the second derivatives of the operator F at each step, and thus the operation cost of which may be very large in practical calculation. In addition, the Lipschitz continuity assumption of second Fréchet derivative is required to ensure cubic convergence rate.

To reduce the operation cost but retain the cubical convergence, several variations of cubic methods are provided, such as convex acceleration of Newton's method, super-Halley method, deformed Euler-Halley method and so on (see [6, 7, 10, 11, 19, 27]). One of the advantages of these variations of cubic methods is that we do not need to calculate the second derivative operator at each step, where the second derivative operator is generally replaced by a finite difference between first derivatives. However, in order to obtain cubic convergence, the second derivative of F is still assumed to be Lipschitz continuous (see [6, 7, 10, 11, 19, 27]).

In particular, in [19], Sharma proposed the following Newton-Steffensen's method which avoids the computation of the second derivative but has *cubical* convergence under the assumption that the function is sufficiently smooth in a neighborhood. Let $f : \mathbb{R} \to \mathbb{R}$. The method is defined as follows:

$$(1.3)\qquad \begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{f(x_n)}{g(x_n)}, \end{cases} \qquad n = 0, 1, \dots,$$

where $g(x_n) = \dfrac{f(y_n) - f(x_n)}{y_n - x_n}$. This method has been extended to Banach spaces in [28], which is described as follows:

$$(1.4)\qquad \begin{cases} y_n = x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} = x_n - [y_n, x_n; F]^{-1}F(x_n), \end{cases} \qquad n = 0, 1, \dots,$$

where the divided difference operator is defined by

$$(1.5)\qquad [y_n, x_n; F] = \int_0^1 F'(x_n + t(y_n - x_n))\, \mathrm{d}t.$$

In [28], in order to obtain cubical convergence, the second Fréchet derivative of F is still assumed to be Lipschitz continuous.

To the best of our knowledge, in order to get cubical convergence, all the known results (see [6,7,10,11,19,27,28]) need that the operator should to be second Fréchet differentiable or even sufficiently smooth. In fact, sometimes, the operator dose not have second Fréchet derivative. Hence, it's interesting and important if we can still have cubical convergence but only under the assumption that the operator just have first Fréchet derivative. Indeed, we establish this result in our present paper. More precisely, in the present paper, we continue to study the cubical convergence of Newton-Steffensen's method given by (1.4). In order to get cubical convergence of Newton-Steffensen's method, we just only assume that the first Fréchet derivative of F is Lipschitz continuous, which is different from the assumptions made for all the results mentioned above. Our main contribute is as follows:

- the *cubical* convergence results are established under the assumption that the *first* Fréchet derivative of F is Lipschitz continuous which is the same condition for Newton's method.

Hence, our results significantly improve the corresponding results in [19].

Recall that, in general, the study about convergence issue of Newton's method is mainly centered on two types: local and semi-local convergence analysis. The local convergence issue is, based on the information around a solution, to seek estimates of the radii of convergence balls; while the semi-local one is, based on the information around an initial point, to give criteria ensuring the convergence of Newton's method. We refer the reader to [14–16, 20–24, 26] and the bibliographies therein for various results, examples, discussions, and applications.

In the present paper, we study the local and semi-local convergence analysis of Newton-Steffensen's method given by (1.4). More precisely, under the assumption that the first Fréchet derivative of F is Lipschitz continuous, convergence criteria are established, and convergence radii are also estimated. Finally, several numerical examples are provided to illustrate how fast the generated sequences converge. It should be noted that in these examples, first Fréchet derivative of F is Lipschitz continuous while the second Fréchet derivative of F not. Hence, our results are applicable while the corresponding results in [19] fail.

The remainder of the present paper is organized as follows. Some basic definitions, notations, and preliminary results are given in Section 2. In Section 3, convergence criteria are established. The estimates of convergence radii are obtained in Section 4. In the last section, numerical examples are presented.

2. Notations and preliminary results

Throughout this paper, we assume that X and Y are two Banach spaces. Let $D \subseteq X$ be an open subset and let $F : D \subseteq X \to Y$ be a nonlinear operator with continuous Fréchet derivative F'. For $x \in X$ and $r > 0$, we use $\mathbf{B}(x, r)$ and $\overline{\mathbf{B}(x, r)}$ to denote the open ball with radius r and center x and its closure, respectively.

Let $L > 0$ be a constant. Let $\beta > 0$, and define

$$h(t) = \beta - t + \frac{L}{2}t^2 \text{ for each } t \geq 0. \tag{2.1}$$

Then

$$h'(t) = -1 + Lt \text{ and } h''(t) = L. \tag{2.2}$$

The following lemma is clear.

Lemma 2.1. *Suppose that*

$$\beta L \leq \frac{1}{2}. \tag{2.3}$$

Then h is strictly decreasing on $[0, \frac{1}{L}]$ and strictly increasing on $[\frac{1}{L}, +\infty)$. Moreover, if $\beta L < \frac{1}{2}$, h has two zeros, denoted respectively by $t^* = \frac{1-\sqrt{1-2L\beta}}{L}$ and $t^{**} = \frac{1+\sqrt{1-2L\beta}}{L}$, such that

$$\beta < t^* < \frac{1}{L} < t^{**}; \tag{2.4}$$

if $\beta L = \frac{1}{2}$, *then* h *has a unique zero* t^* *in* $(\beta, +\infty)$ *(in fact,* $t^* = \frac{1}{L}$*).*

Let $\{s_n\}$ and $\{t_n\}$ denote the corresponding sequences generated by Newton-Steffensen's method for the majoring function h with the initial point $t_0 = 0$, that is,

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{f(x_n)}{g(x_n)}, \end{cases} \quad n = 0, 1, \ldots. \tag{2.5}$$

The following lemma describes the convergence properties of the sequences $\{s_n\}$ and $\{t_n\}$.

Lemma 2.2. *Suppose that (2.3) holds. Let* $\{s_n\}$ *and* $\{t_n\}$ *be the sequences generated by (2.5). Then*

$$0 \le t_n < s_n < t_{n+1} < t^* \quad \text{for all } n \ge 0. \tag{2.6}$$

Moreover, $\{s_n\}$ *and* $\{t_n\}$ *converge increasingly to the same point* t^**.*

Proof. Note that $0 = t_0 < s_0 = \beta$ and $t_1 = \frac{\beta}{1-\frac{L}{2}\beta}$. Clearly, $s_0 < t_1$. Furthermore, since $L\beta \le \frac{1}{2}$, it's easy to verify that $t_1 < t^* = \frac{1-\sqrt{1-2L\beta}}{L}$. Hence (2.6) holds for $n = 0$.

Now we assume that

$$0 \le t_{n-1} < s_{n-1} < t_n < t^* \quad \text{for some } n \ge 1.$$

By Lemma 2.1, we have $h(t) \ge 0$ for each $t \in [0, t^*]$ and $h(t_n)/h'(t_n) < 0$. The later implies that $s_n > t_n$. Define function

$$N(t) = t - \frac{h(t)}{h'(t)}, \quad t \in [0, t^*].$$

Then, $N'(t) = h(t)h''(t)/h'(t)^2 > 0$, which implies that $N(t)$ is increasing monotonically in $[0, t^*]$. Therefore we have

$$s_n = t_n - \frac{h(t_n)}{h'(t_n)} < t^* - \frac{h(t^*)}{h'(t^*)} = t^*.$$

Since h is convex in $[0, t^*]$, we get $h'(t_n) < (h(s_n) - h(t_n))/(s_n - t_n)$ and so $s_n < t_{n+1}$.

It remains to show $t_{n+1} < t^*$. To this end, we first verify the following inequality:

$$t' - \left(\frac{h(t) - h(t')}{t - t'}\right)^{-1} h(t') < t'' - \left(\frac{h(t'') - h(t)}{t'' - t}\right)^{-1} h(t'') \tag{2.7}$$

for all $t', t, t'' \in [0, t^*]$ and $t' < t < t''$. To do this, for a fixed $t \in (0, t^*)$, let g denote the function defined by

$$g(s) := s - \left(\frac{h(s) - h(t)}{s - t}\right)^{-1} h(s) \quad \textit{for any } s \in [0, t^*].$$

Then

$$g'(s) = \left(\frac{h(t)}{(h(s) - h(t))^2}\right) [(h(t) - h(s)) - h'(s)(t - s)].$$

Since h is convex in $[0,t^*]$, we have $(h(t)-h(s))-h'(s)(t-s)>0$. Observe further that $h(t)>0$. Hence $g'(s)>0$ and so g is strict monotonically increasing on $[0,t^*]$. Thus, (2.7) is seen to hold. By (2.7), we conclude that

$$t_{n+1}=t_n-\left(\frac{h(s_n)-h(t_n)}{s_n-t_n}\right)^{-1}h(t_n)<t^*-\left(\frac{h(t^*)-h(s_n)}{t^*-s_n}\right)^{-1}h(t^*)=t^*.$$

Therefore, (2.6) holds for all $n\geq 0$. The inequalities in (2.6) imply that $\{s_n\}$ and $\{t_n\}$ converge increasingly to some same point, say τ. Clearly $\tau\in[0,t^*]$ and τ is a zero of h in $[0,t^*]$. Noting that t^* is the unique zero of h in $[0,\frac{1}{L}]$, one has that $\tau=t^*$. The proof is complete. □

Lemma 2.3. *Let $\{s_k\}$ and $\{t_k\}$ be the sequences generated by (2.5). Then for all $k\geq 0$,*

$$-\frac{L}{2h'(t_k)}(t_{k+1}-t_k)(s_k-t_k)=t_{k+1}-s_k, \tag{2.8}$$

and

$$\frac{L}{2}(t_{k+1}-s_k)(t_{k+1}-t_k)=h(t_{k+1}) \tag{2.9}$$

Proof. Observe that

$$\begin{aligned}-\frac{L}{2h'(t_k)}(t_{k+1}-t_k)(s_k-t_k)&=-\frac{1}{h'(t_k)}(t_{k+1}-t_k)(s_k-t_k)\int_0^1 Lt\mathrm{d}t\\&=-\frac{1}{h'(t_k)}\frac{t_{k+1}-t_k}{s_k-t_k}\int_{t_k}^{s_k}L(s_k-u)\mathrm{d}u\\&=-\frac{1}{h'(t_k)}\frac{t_{k+1}-t_k}{s_k-t_k}\int_{t_k}^{s_k}(s_k-u)\mathrm{d}h'(u)\\&=t_{k+1}-t_k+\frac{h(t_k)}{h'(t_k)}\\&=t_{k+1}-s_k.\end{aligned}$$

Hence, (2.8) is seen to hold. Note further that

$$\begin{aligned}&\frac{L}{2}(t_{k+1}-s_k)(t_{k+1}-t_k)\\&\quad=(t_{k+1}-s_k)(t_{k+1}-t_k)\int_0^1 Lt\mathrm{d}t\\&\quad=(t_{k+1}-t_k)\int_0^1\int_{t_k+t(s_k-t_k)}^{t_k+t(s_k-t_k)+t(t_{k+1}-s_k)}L\mathrm{d}u\mathrm{d}t\\&\quad=(t_{k+1}-t_k)\int_0^1\int_{t_k+t(s_k-t_k)}^{t_k+t(s_k-t_k)+t(t_{k+1}-s_k)}h''(u)\mathrm{d}u\mathrm{d}t\\&\quad=(t_{k+1}-t_k)\int_0^1(h'(t_k+t(t_{k+1}-t_k))-h'(t_k+t(s_k-t_k)))\mathrm{d}t\\&\quad=h(t_{k+1}).\end{aligned}$$

Thus, (2.9) holds. □

The following L Lipschitz condition was introduced by Wang in [21].

Definition 2.4. Let $\bar{x}$ be a given point such that $F'(\bar{x})^{-1}$ exists, and let $\boldsymbol{B}(\bar{x}, r) \subseteq D$. Then $F'(\bar{x})^{-1}F'$ is said to satisfy the L Lipschitz condition on $\boldsymbol{B}(\bar{x}, r)$ if for all $x, x' \in \boldsymbol{B}(\bar{x}, r)$ with $\|x - x'\| \leq r$

$$\|F'(\bar{x})^{-1}(F'(x) - F'(x'))\| \leq L\|x - x'\|. \tag{2.10}$$

By the well known Banach Lemma, we have the following lemma which is useful in the proof of our convergence results.

Lemma 2.5. *Let $0 < r \leq \frac{1}{L}$. Let $\bar{x}$ be such that $F'(\bar{x})^{-1}$ exists, and let $\mathbf{B}(\bar{x}, r) \subseteq D$. Suppose that $F'(\bar{x})^{-1}F'$ satisfies the L Lipschitz condition on $\mathbf{B}(\bar{x}, r)$. Then, for each $x \in \boldsymbol{B}(\bar{x}, r)$, $F'(x)^{-1}$ exists and*

$$\|F'(x)^{-1}F'(\bar{x})\| \leq \frac{1}{1 - L\|x - \bar{x}\|} = -\frac{1}{h'(\|x - \bar{x}\|)}. \tag{2.11}$$

3. Convergence criterion

Throughout this section, let $x_0 \in D$ be the initial point such that the inverse $F'(x_0)^{-1}$ exists and let $\mathbf{B}(x_0, \frac{1}{L}) \subseteq D$. Below, we list a series of useful lemmas. Recall that the divided difference operator $[y, x; F]$ is defined by (1.5). The following lemma gives the expressions of some desired estimates in the proof of Lemma 3.2.

Lemma 3.1. *Let $x \in \mathbf{B}(x_0, \frac{1}{L})$ be such that $F'(x)^{-1}$ exists. Define*

$$y := x - F'(x)^{-1}F(x) \quad \text{and} \quad \bar{x} := x - [y, x; F]^{-1}F(x).$$

Then the following formulae hold:

(i) $\bar{x} - y = F'(x)^{-1} \displaystyle\int_0^1 [F'(x) - F'(x + t(y - x))]\,\mathrm{d}t(\bar{x} - x)$.

(ii) $F(\bar{x}) = \displaystyle\int_0^1 [F'(x + t(\bar{x} - x)) - F'(x + t(y - x))]\,\mathrm{d}t(\bar{x} - x)$.

Proof. For (i), we notice that

$$\begin{aligned}
\bar{x} - y &= \bar{x} - x + F'(x)^{-1}F(x) \\
&= \bar{x} - x - F'(x)^{-1}[y, x; F](\bar{x} - x) \\
&= F'(x)^{-1} \int_0^1 [F'(x) - F'(x + t(y - x))]\,\mathrm{d}t(\bar{x} - x).
\end{aligned}$$

As for (ii), one has that

$$\begin{aligned}
F(\bar{x}) &= F(\bar{x}) - F(x) - [y, x; F](\bar{x} - x) \\
&= \int_0^1 F'(x + t(\bar{x} - x))(\bar{x} - x)\,\mathrm{d}t - \int_0^1 F'(x + t(y - x))(\bar{x} - x)\,\mathrm{d}t \\
&= \int_0^1 [F'(x + t(\bar{x} - x)) - F'(x + t(y - x))]\,\mathrm{d}t(\bar{x} - x).
\end{aligned}$$

The proof is complete. □

Recall that $t^* = \frac{1-\sqrt{1-2L\beta}}{L}$, $\{s_n\}$ and $\{t_n\}$ are the corresponding sequences generated by Newton-Steffensen's method (2.5) for the majoring function h with the initial point $t_0 = 0$.

Lemma 3.2. *Suppose that $\beta L \le \frac{1}{2}$. Suppose further that $\|F'(x_0)^{-1}F(x_0)\| \le \beta$ and $F'(x_0)^{-1}F'$ satisfies the L Lipschitz condition on $\mathbf{B}(x_0, t^*)$. Let $\{x_n\}$ be a sequence generated by (1.4) with initial point x_0. Then, $\{x_n\}$ is well defined and the following estimates hold for any $n \ge 1$:*

(i) $\|y_{n-1} - x_{n-1}\| \le s_{n-1} - t_{n-1}$, $\|x_n - x_{n-1}\| \le t_n - t_{n-1}$, $\|x_n - y_{n-1}\| \le t_n - s_{n-1}$.

(ii) $\left\|[y_{n-1}, x_{n-1}; F]^{-1}F'(x_0)\right\| \le -\dfrac{s_{n-1} - t_{n-1}}{h(s_{n-1}) - h(t_{n-1})}$.

(iii) $\left\|F'(x_0)^{-1}F(x_n)\right\| \le h(t_n)\left(\dfrac{\|x_n - x_{n-1}\|}{t_n - t_{n-1}}\right)^2\left(\dfrac{\|y_{n-1} - x_{n-1}\|}{s_{n-1} - t_{n-1}}\right)$.

Proof. Note that $\|y_0 - x_0\| = \|F'(x_0)^{-1}F(x_0)\| \le \beta = s_0 - t_0$. Hence, the first inequality of (i) holds for $n = 1$. This together with (2.2) gives that

$$\begin{aligned}\left\|F'(x_0)^{-1}([y_0, x_0; F] - F'(x_0))\right\| &\le \int_0^1 \left\|F'(x_0)^{-1}(F'(x_0 + t(y_0 - x_0) - F'(x_0))\right\| \,\mathrm{d}t \\ &\le \int_0^1 Lt\|y_0 - x_0\|\mathrm{d}t \\ &\le \int_0^1\int_0^{t(s_0 - t_0)} h''(u)\,\mathrm{d}u\mathrm{d}t \\ &= \frac{h(s_0) - h(t_0)}{s_0 - t_0} + 1.\end{aligned}$$

Since $h'(t) < 0$ in $(0, \frac{1}{L})$, we have $(h(s_0) - h(t_0))/(s_0 - t_0) < 0$. Thus, it follows from Banach lemma that $[y_0, x_0; F]^{-1}$ exists and satisfies

$$\left\|[y_0, x_0; F]^{-1}F'(x_0)\right\| \le \frac{1}{1 - \left(\dfrac{h(s_0) - h(t_0)}{s_0 - t_0} + 1\right)} = -\frac{s_0 - t_0}{h(s_0) - h(t_0)}. \tag{3.1}$$

Hence

$$\|x_1 - x_0\| \le \left\|[y_0, x_0; F]^{-1}F'(x_0)\right\| \left\|F'(x_0)^{-1}F(x_0)\right\| \le -\frac{s_0 - t_0}{h(s_0) - h(t_0)}h(t_0) = t_1 - t_0.$$

Thus, the second inequality of (i) holds for $n = 1$. By Lemma 3.1, we have

$$x_1 - y_0 = F'(x_0)^{-1}\int_0^1 [F'(x_0) - F'(x_0 + t(y_0 - x_0))]\,\mathrm{d}t(x_1 - x_0). \tag{3.2}$$

By (2.2) and L Lipschitz assumption, we have

$$\begin{aligned}\|x_1 - y_0\| &\le \int_0^1 \|F'(x_0)^{-1}[F'(x_0) - F'(x_0 + t(y_0 - x_0))]\,\mathrm{d}t\|x_1 - x_0\| \\ &\le \int_0^1 Lt\|y_0 - x_0\|\mathrm{d}t\|x_1 - x_0\|\end{aligned}$$

$$
\begin{aligned}
(3.3)\qquad &= -\frac{1}{h'(t_0)}(t_1-t_0)(s_0-t_0)\int_0^1 Lt\mathrm{d}t\frac{\|x_1-x_0\|}{t_1-t_0}\frac{\|y_0-x_0\|}{s_0-t_0}\\
&= (t_1-s_0)\frac{\|y_0-x_0\|}{s_0-t_0}\frac{\|x_1-x_0\|}{t_1-t_0},
\end{aligned}
$$

where the last equality holds because of (2.8). Consequently, statement (i) holds for $n = 1$.

Statement (ii) for the case $n = 1$ is verified by (3.1). Below, we consider the case $n = 1$ for (iii). First we have the following expression of $F(x_1)$ due to Lemma 3.1:

$$
F(x_1) = \int_0^1 [F'(x_0+t(x_1-x_0)) - F'(x_0+t(y_0-x_0))]\,\mathrm{d}t(x_1-x_0),
$$

Thus by (3.3), we get

$$
\begin{aligned}
&\|F'(x_0)^{-1}F(x_1)\|\\
&\quad\le \int_0^1 \left\|F'(x_0)^{-1}[F'(x_0+t(x_1-x_0)) - F'(x_0+t(y_0-x_0))]\right\|\,\mathrm{d}t\|x_1-x_0\|\\
&\quad\le \int_0^1 Lt\|x_1-y_0\|\mathrm{d}t\|x_1-x_0\|\\
&\quad\le (t_1-t_0)(t_1-s_0)\int_0^1 Lt\mathrm{d}t\left(\frac{\|x_1-x_0\|}{t_1-t_0}\right)^2\left(\frac{\|y_0-x_0\|}{s_0-t_0}\right)\\
&\quad= h(t_1)\left(\frac{\|x_1-x_0\|}{t_1-t_0}\right)^2\left(\frac{\|y_0-x_0\|}{s_0-t_0}\right),
\end{aligned}
$$

where the last equality holds because of (2.9). Therefore statement (iii) holds for $n = 1$.

Assume that statements (i)-(iii) are true for $n = k(\ge 1)$. Blow, we use mathematical induction to prove that they also hold for $n = k+1$. First, by statement (i), we have

$$
\|x_k-x_0\| \le \sum_{i=0}^{k-1}\|x_{i+1}-x_i\| \le \sum_{i=0}^{k-1}(t_{i+1}-t_i) = t_k < t^* < \frac{1}{L}. \tag{3.4}
$$

Therefore, it follows from Lemma 2.5 that $F'(x_k)^{-1}$ exists and

$$
\|F'(x_k)^{-1}F'(x_0)\| \le \frac{1}{1-L\|x_k-x_0\|} \le -\frac{1}{h'(t_k)}. \tag{3.5}
$$

Noting that

$$
\|F'(x_0)^{-1}F(x_k)\| \le h(t_k)
$$

by the inductive hypothesis of (i) and (iii). This, together with (3.5), implies that

$$
\|y_k-x_k\| \le \|F'(x_k)^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(x_k)\| \le -\frac{h(t_k)}{h'(t_k)} = s_k - t_k, \tag{3.6}
$$

where the last equality holds because of (2.5). Hence, the first inequality of (i) holds for $n = k+1$. Observe from L Lipschitz assumption, (3.4) and (3.6) that

$$
\|F'(x_0)^{-1}([y_k, x_k; F] - F'(x_0))\|
$$

$$
\begin{aligned}
&\le \int_0^1 \left\|F'(x_0)^{-1}(F'(x_k+t(y_k-x_k)-F'(x_0))\right\| \,\mathrm{d}t\\
&\le \int_0^1 L(\|x_k-x_0\|+t\|y_k-x_k\|)\mathrm{d}t\\
&\le \int_0^1\int_0^{t_k+t(s_k-t_k)} h''(u)\,\mathrm{d}u\mathrm{d}t\\
&= \frac{h(s_k)-h(t_k)}{s_k-t_k}+1.
\end{aligned}
$$

It follows from Lemma 2.2 and the monotonicity of h that $(h(s_k)-h(t_k))/(s_k-t_k) < 0$. Thus, we have $\|F'(x_0)^{-1}([y_k,x_k;F]-F'(x_0))\| < 1$ and by Banach lemma $[y_k,x_k;F]^{-1}F'(x_0)$ exists and satisfies

$$
\begin{aligned}
\|[y_k,x_k;F]^{-1}F'(x_0)\| &\le \frac{1}{1-\left(\dfrac{h(s_k)-h(t_k)}{s_k-t_k}+1\right)}\\
&= -\frac{s_k-t_k}{h(s_k)-h(t_k)}.
\end{aligned}
\tag{3.7}
$$

Combining this with the inductive hypothesis of (iii) yields that

$$
\begin{aligned}
&\|x_{k+1}-x_k\| \le \|[y_k,x_k;F]^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(x_k)\|\\
&\le -\left(\frac{h(s_k)-h(t_k)}{s_k-t_k}\right)^{-1} h(t_k)\left(\frac{\|x_k-x_{k-1}\|}{t_k-t_{k-1}}\right)^2\left(\frac{\|y_{k-1}-x_{k-1}\|}{s_{k-1}-t_{k-1}}\right)\\
&= (t_{k+1}-t_k)\left(\frac{\|x_k-x_{k-1}\|}{t_k-t_{k-1}}\right)^2\left(\frac{\|y_{k-1}-x_{k-1}\|}{s_{k-1}-t_{k-1}}\right).
\end{aligned}
\tag{3.8}
$$

This implies that $\|x_{k+1}-x_k\| \le t_{k+1}-t_k$ and so the second inequality of (i) holds.

On the other hand, by L Lipschitz assumption, (2.11) and Lemma 3.1 (i) with $(x_{n+1},x_n,y_n$ in place of $\bar{x},x,y)$, we have

$$
\begin{aligned}
\|x_{k+1}-y_k\| &= \left\|F'(x_k)^{-1}\int_0^1\left[F'(x_k)-F'(x_k+t(y_k-x_k))\right]\mathrm{d}t(x_{k+1}-x_k)\right\|\\
&\le \|F'(x_k)^{-1}F'(x_0)\|\cdot\\
&\quad \left\|\int_0^1 F'(x_0)^{-1}[F'(x_k)-F'(x_k+t(y_k-x_k))]\,\mathrm{d}t\right\|\|x_{k+1}-x_k\|\\
&\le -\frac{1}{h'(\|x_k-x_0\|)}\int_0^1 Lt\|y_k-x_k\|\mathrm{d}t\|x_{k+1}-x_k\|\\
&\le -\frac{1}{h'(t_k)}(t_{k+1}-t_k)(s_k-t_k)\int_0^1 Lt\mathrm{d}t\frac{\|x_{k+1}-x_k\|}{t_{k+1}-t_k}\frac{\|y_k-x_k\|}{s_k-t_k}\\
&= (t_{k+1}-s_k)\left(\frac{\|y_k-x_k\|\|x_{k+1}-x_k\|}{(s_k-t_k)(t_{k+1}-t_k)}\right),
\end{aligned}
\tag{3.9}
$$

where the last equality holds because of (2.8). This implies that $\|x_{k+1}-y_k\| \le t_{k+1}-s_k$. Hence, statement (i) holds for $n=k+1$.

Statement (ii) holds for $n = k+1$ by (3.7). Next, we will show that (iii) also holds for $n = k+1$. First, it follows from L Lipschitz assumption and (3.9) that

$$\begin{aligned}
&\|F'(x_0)^{-1}F(x_{k+1})\| \\
\leq\ & \int_0^1 \left\|F'(x_0)^{-1}[F'(x_k + t(x_{k+1} - x_k)) - F'(x_k + t(y_k - x_k))]\right\| \,\mathrm{d}t\|x_{k+1} - x_k\| \\
\leq\ & \int_0^1 Lt\|x_{k+1} - y_k\|\mathrm{d}t\|x_{k+1} - x_k\| \\
\leq\ & (t_{k+1} - s_k)(t_{k+1} - t_k)\int_0^1 Lt\mathrm{d}t\left(\frac{\|x_{k+1} - x_k\|}{t_{k+1} - t_k}\right)^2\left(\frac{\|y_k - x_k\|}{s_k - t_k}\right) \\
=\ & h(t_{k+1})\left(\frac{\|x_{k+1} - x_k\|}{t_{k+1} - t_k}\right)^2\left(\frac{\|y_k - x_k\|}{s_k - t_k}\right),
\end{aligned}$$

where the last equality holds because of (2.9). Therefore statement (iii) is confirmed for $n = k+1$. Hence (i)-(iii) hold for all $n \geq 0$.

Furthermore, by statement (i), one has, for any $n \geq 0$, $\|x_n - x_0\| \leq t_n < t^* < \frac{1}{L}$. Thus, by Lemma 2.5, we know that $F'(x_n)^{-1}$ exists for each $n \geq 1$. Moreover, by statement (ii), we know that for all $n \geq 1$, $[y_n, x_n, F]^{-1}$ exists. Thus $\{x_n\}$ is well defined. The proof is complete. □

Recall that $t^* = \frac{1-\sqrt{1-2L\beta}}{L}$, and the sequences $\{s_n\}$ and $\{t_n\}$ are defined by (2.5). We are now ready to prove a semilocal convergence theorem for Newton-Steffensen's method (1.4) under L Lipschitz condition.

Theorem 3.3. *Suppose that $\beta L \leq \frac{1}{2}$. Suppose further that $\|F'(x_0)^{-1}F(x_0)\| \leq \beta$ and $F'(x_0)^{-1}F'$ satisfies the L Lipschitz condition on $\mathbf{B}(x_0, t^*)$. Let $\{x_n\}$ be a sequence generated by (1.4) with initial point x_0. Then, $\{x_n\}$ is well defined and converges to a solution $x^* \in \overline{\mathbf{B}(x_0, t^*)}$ of equation (1.1) with Q-cubic rate and x^* is the unique solution in $\overline{\mathbf{B}(x_0, t^*)}$. Moreover, the following estimate holds:*

$$\|x^* - x_n\| \leq (t^* - t_n)\left(\frac{\|x^* - x_m\|}{t^* - t_m}\right)^{3^{n-m}} \quad \text{for all} \quad n \geq m \geq 0. \tag{3.10}$$

Proof. The uniqueness ball can be obtained by Theorem 1.5 in [21]. Moreover, it follows from Lemma 3.2 that $\{x_n\}$ is well defined. In addition, from Lemmas 2.2 and 3.2 (i), we can see that $\{x_n\}$ is a Cauchy sequence, so it converges to a limit, say x^*. Below, we show that x^* is a solution of equation (1.1). It follows from Lemma 3.2 that

$$\|F'(x_0)^{-1}F(x_n)\| \leq h(t_n) \quad \text{for all } n \geq 0.$$

Letting $n \to \infty$ in the preceding relation gives that the limit x^* is a solution of equation (1.1). Moreover, Lemma 3.2 (i) gives

$$\|x^* - x_n\| \leq t^* - t_n. \tag{3.11}$$

Next, we verify that estimate (3.10) is true. First we have

$$\begin{aligned}
\|x^* - y_n\| &= \|x^* - x_n + F'(x_n)F(x_n)\| \\
&= \|x^* - x_n - F'(x_n)(F(x^*) - F(x_n))\| \\
&= \left\| -F'(x_n)^{-1} \int_0^1 [F'(x_n + t(x^* - x_n)) - F'(x_n)](x^* - x_n)\, \mathrm{d}t \right\| \\
&\le \|F'(x_n)^{-1} F'(x_0)\| \cdot \\
&\quad \int_0^1 \left\| F'(x_0)^{-1}[F'(x_n) - F'(x_n + t(x^* - x_n))](x^* - x_n) \right\| \,\mathrm{d}t.
\end{aligned} \tag{3.12}$$

This, together with L Lipschitz assumption and (2.11), gives the following estimate:

$$\begin{aligned}
\|x^* - y_n\| &\le -\frac{1}{h'(t_n)} \int_0^1 Lt\|x^* - x_n\|\mathrm{d}t\|x^* - x_n\| \\
&= -\frac{1}{h'(t_n)}(t^* - t_n)^2 \int_0^1 Lt\mathrm{d}t \left(\frac{\|x^* - x_n\|}{t^* - t_n} \right)^2 \\
&= (t^* - s_n) \left(\frac{\|x^* - x_n\|}{t^* - t_n} \right)^2,
\end{aligned} \tag{3.13}$$

where the last equality holds because it's easy to verify that

$$\begin{aligned}
-\frac{1}{h'(t_n)}(t^* - t_n)^2 \int_0^1 Lt\mathrm{d}t &= -\frac{1}{h'(t_n)} \int_{t_n}^{t^*} L(t^* - u)\mathrm{d}u \\
&= -\frac{1}{h'(t_n)} \int_{t_n}^{t^*} (t^* - u)\mathrm{d}h'(u) \\
&= t^* - s_n.
\end{aligned}$$

Since $[y_n, x_n; F](x_{n+1} - x_n) + F(x_n) = 0$, we observe that

$$\begin{aligned}
&[y_n, x_n; F](x_{n+1} - x^*) \\
&= [y_n, x_n; F](x_{n+1} - x^*) - [y_n, x_n; F](x_{n+1} - x_n) - F(x_n) + F(x^*) \\
&= [y_n, x_n; F](x_n - x^*) - F(x_n) + F(x^*) \\
&= \int_0^1 \left[F'(x_n + t(x^* - x_n)) - F'(x_n + t(y_n - x_n)) \right] (x^* - x_n)\, \mathrm{d}t.
\end{aligned}$$

Thus by L Lipschitz assumption and (3.13), one has

$$\begin{aligned}
\|F'(x_0)^{-1}[y_n, x_n; F](x_{n+1} - x^*)\| &\le \int_0^1 Lt\|x^* - y_n\|\mathrm{d}t\|x^* - x_n\| \\
&\le (t^* - s_n) \int_0^1 Lt\mathrm{d}t \left(\frac{\|x^* - x_n\|}{t^* - t_n} \right)^2 \|x^* - x_n\| \\
&\le (t^* - s_n)(t^* - t_n) \int_0^1 Lt\mathrm{d}t \left(\frac{\|x^* - x_n\|}{t^* - t_n} \right)^3.
\end{aligned} \tag{3.14}$$

Note that

$$
\begin{aligned}
(t^*-s_n)(t^*-t_n)\int_0^1 Lt\mathrm{d}t &= \int_0^1\int_{t_n+t(s_n-t_n)}^{t_n+t(s_n-t_n)+t(t^*-s_n)} L\mathrm{d}u\mathrm{d}t(t^*-t_n)\\
&= \int_0^1\int_{t_n+t(s_n-t_n)}^{t_n+t(s_n-t_n)+t(t^*-s_n)} h''(u)\mathrm{d}u\mathrm{d}t(t^*-t_n)\\
&= (t_{n+1}-t^*)\frac{h(s_n)-h(t_n)}{s_n-t_n}.
\end{aligned}
$$

This, together with (3.14), yields that

$$
\|F'(x_0)^{-1}[y_n,x_n;F](x_{n+1}-x^*)\| \le (t_{n+1}-t^*)\frac{h(s_n)-h(t_n)}{s_n-t_n}\left(\frac{\|x^*-x_n\|}{t^*-t_n}\right)^3. \tag{3.15}
$$

Combining Lemma 3.2 (ii) with (3.15) gives that

$$
\begin{aligned}
\|x_{n+1}-x^*\| &\le \left\|[y_n,x_n;F]^{-1}F'(x_0)\right\|\left\|F'(x_0)^{-1}[y_n,x_n;F](x_{n+1}-x^*)\right\|\\
&\le (t^*-t_{n+1})\left(\frac{\|x^*-x_n\|}{t^*-t_n}\right)^3.
\end{aligned}
$$

Therefore the error estimate (3.10) follows. Also, from the previous inequality, we know that the convergence rate of $\{x_n\}$ to x^* is Q-cubic. The proof is complete. □

4. Convergence ball

Now we begin to study the local convergence properties for Newton-Steffensen's method (1.4). Throughout this section, we suppose $x^*\in D$ such that $F(x^*)=0$, $\mathbf{B}(x^*,\frac{1}{L})\subseteq D$ and the inverse $F'(x^*)^{-1}$ exists. First, we define an auxiliary function which is useful for the analysis of convergence ball:

$$
G(t) := \frac{Lt}{2(1-Lt)} \quad \text{for each } t\in\left(0,\frac{1}{L}\right). \tag{4.1}
$$

Note that G increases monotonically from 0 to $+\infty$ when t increases monotonically from 0 to $\frac{1}{L}$. Hence the following lemma holds.

Lemma 4.1. *There exists a unique point $r_1=\frac{2}{3L}\in(0,\frac{1}{L})$ such that $G(r_1)=1$. Moreover $G(t)$ and $G(t)/t$ increase monotonically on $(0,r_1)$.*

Now, we are ready to establish the theorem about estimate of radius of convergence ball.

Theorem 4.2. *Suppose that $F'(x^*)^{-1}F'$ satisfies the L Lipschitz condition on $\mathbf{B}(x^*,\frac{2}{3L})$. Let $x_0\in\mathbf{B}(x^*,\frac{2}{3L})$. Let $\{x_n\}$ be a sequence generated by (1.4) with initial point x_0. Then, $\{x_n\}$ converges to x^* and the following assertion holds for all $n=0,1,\ldots$:*

$$
\|x_n-x^*\|\le q^{3^n-1}\|x_0-x^*\|, \tag{4.2}
$$

where

$$
q=G(t_0)<1,\quad t_0=\|x_0-x^*\|. \tag{4.3}
$$

Proof. For each $n = 0, 1, \ldots$, we write $t_n := \|x_n - x^*\|$. It is sufficient to show that

$$t_{n+1} \le t_n \quad \text{and} \quad \|x_{n+1} - x^*\| \le G(t_n)^2 \|x_n - x^*\|, \quad n = 0, 1, \ldots. \tag{4.4}$$

In fact, by noticing the monotonicity of $G(t)/t$, we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &\le (G(t_n)/t_n)^2 \|x_n - x^*\|^3 \\
&\le (G(t_0)/t_0)^2 \|x_n - x^*\|^3 \\
&= \left(\frac{q}{t_0}\right)^2 \|x_n - x^*\|^3, \quad n = 0, 1, \ldots.
\end{aligned}$$

From this we can easily establish (4.2) by mathematical induction.

Now we prove (4.4) by mathematical induction. Note that for all n, as $[y_n, x_n; F](x_{n+1} - x_n) + F(x_n) = 0$, one has

$$\begin{aligned}
&[y_n, x_n; F](x_{n+1} - x^*) \\
&= [y_n, x_n; F](x_{n+1} - x^*) - [y_n, x_n; F](x_{n+1} - x_n) - F(x_n) + F(x^*) \\
&= [y_n, x_n; F](x_n - x^*) - F(x_n) + F(x^*) \\
&= \int_0^1 \left[F'(x_n + t(x^* - x_n)) - F'(x_n + t(y_n - x_n))\right](x^* - x_n)\, \mathrm{d}t.
\end{aligned}$$

This gives that

$$\begin{aligned}
x_{n+1} - x^* &= [y_n, x_n; F]^{-1} \cdot \\
&\quad \int_0^1 \left[F'(x_n + t(x^* - x_n)) - F'(x_n + t(y_n - x_n))\right](x^* - x_n) \mathrm{d}t
\end{aligned} \tag{4.5}$$

Similarly, we also have

$$\begin{aligned}
y_n - x^* &= x_n - F'(x_n)^{-1} F(x_n) - x^* \\
&= F'(x_n)^{-1} \left[F(x^*) - F(x_n) + F'(x_n)(x_n - x^*)\right] \\
&= F'(x_n)^{-1} \int_0^1 \left[F'(x_n + t(x^* - x_n)) - F'(x_n)\right](x^* - x_n) \mathrm{d}t.
\end{aligned} \tag{4.6}$$

For the case $n = 0$, by L Lipschitz assumption, (4.6) and (2.11), we get that

$$\begin{aligned}
\|y_0 - x^*\| &\le \|F'(x_0)^{-1} F'(x^*)\| \cdot \\
&\quad \left\| \int_0^1 F'(x^*)^{-1} \left[F'(x_0 + t(x^* - x_0)) - F'(x_0)\right] \mathrm{d}t \right\| \|x^* - x_0\| \\
&\le \frac{1}{1 - L\|x_0 - x^*\|} \int_0^1 Lt \|x_0 - x^*\| \mathrm{d}t \|x^* - x_0\| \\
&\le \frac{\frac{L}{2} t_0}{1 - L t_0} t_0 \\
&= G(t_0) t_0.
\end{aligned} \tag{4.7}$$

Hence by (4.3), we have

$$\|y_0 - x^*\| \le t_0. \tag{4.8}$$

This, together with L Lipschitz assumption, yields that

$$\begin{aligned}
&\|F'(x^*)^{-1}([y_0, x_0; F] - F'(x^*))\| \\
&= \left\| \int_0^1 F'(x^*)^{-1}[F'(x_0 + t(y_0 - x_0) - F'(x^*)]\,\mathrm{d}t \right\| \\
&\leq \int_0^1 L\|x_0 + t(y_0 - x_0) - x^*\|\mathrm{d}t \\
&\leq \int_0^1 L[(1-t)\|x_0 - x^*\| + t\|y_0 - x^*\|]\mathrm{d}t \\
&\leq \int_0^1 Lt_0\mathrm{d}t \\
&= Lt_0 \\
&< 1.
\end{aligned}$$

It follows from Banach lemma that

$$\|[y_0, x_0; F]^{-1}F'(x^*)\| \leq \frac{1}{1 - Lt_0}.$$

Combining this with L Lipschitz assumption, (4.5) and (4.7), we get

$$\begin{aligned}
&\|x_1 - x^*\| \\
&\leq \|[y_0, x_0; F]^{-1}F'(x^*)\| \cdot \\
&\quad \left\| F'(x^*)^{-1} \int_0^1 \left[F'(x_0 + t(x^* - x_0)) - F'(x_0 + t(y_0 - x_0))\right] \mathrm{d}t \right\| \|x_0 - x^*\| \\
&\leq \frac{1}{1 - Lt_0} \int_0^1 Lt\|y_0 - x^*\|\mathrm{d}t\|x_0 - x^*\| \\
&= \frac{1}{1 - Lt_0} \frac{L}{2} G(t_0)t_0\|x_0 - x^*\| \\
&= G(t_0)^2\|x_0 - x^*\|.
\end{aligned} \tag{4.9}$$

This with (4.3) implies that $t_1 < t_0$. Hence the inequalities in (4.4) hold for $n = 0$.

Now assume that the inequalities in (4.4) hold for up to some $n \geq 1$. Thus $t_{n+1} \leq t_n < \frac{2}{3L} < \frac{1}{L}$. It follows from L Lipschitz assumption, (4.6) and (2.11) that

$$\begin{aligned}
&\|y_{n+1} - x^*\| \\
&\leq \|F'(x_{n+1})^{-1}F'(x^*)\| \cdot \\
&\quad \left\| \int_0^1 F'(x^*)^{-1} \left[F'(x_{n+1} + t(x^* - x_{n+1})) - F'(x_{n+1})\right] \mathrm{d}t \right\| \|x^* - x_{n+1}\| \\
&\leq \frac{1}{1 - L\|x_{n+1} - x^*\|} \int_0^1 Lt\|x_{n+1} - x^*\|\mathrm{d}t\|x^* - x_{n+1}\| \\
&= \frac{\frac{L}{2}t_{n+1}}{1 - Lt_{n+1}} t_{n+1} \\
&= G(t_{n+1})t_{n+1}.
\end{aligned} \tag{4.10}$$

By the monotonicity of G, one has

$$\|y_{n+1} - x^*\| \leq G(t_{n+1})t_{n+1} \leq G(t_0)t_{n+1} \leq t_{n+1}.$$

Similarly, we have

$$\begin{aligned}
&\|F'(x^*)^{-1}([y_{n+1}, x_{n+1}; F] - F'(x^*))\| \\
&= \left\| \int_0^1 F'(x^*)^{-1}[F'(x_{n+1} + t(y_{n+1} - x_{n+1}) - F'(x^*)] \, \mathrm{d}t \right\| \\
&\leq \int_0^1 L\|x_{n+1} + t(y_{n+1} - x_{n+1}) - x^*\| \mathrm{d}t \\
&\leq \int_0^1 L[(1-t)\|x_{n+1} - x^*\| + t\|y_{n+1} - x^*\|] \mathrm{d}t \\
&\leq Lt_{n+1} \\
&< 1.
\end{aligned}$$

Using Banach lemma again, one has

$$\|[y_{n+1}, x_{n+1}; F]^{-1}F'(x^*)\| \leq \frac{1}{1 - Lt_{n+1}}.$$

This together with L Lipschitz assumption, (4.5) and (4.10) yields that

$$\begin{aligned}
\|x_{n+2} - x^*\| \leq {} & \|[y_{n+1}, x_{n+1}; F]^{-1}F'(x^*)\| \cdot \\
& \left\| F'(x^*)^{-1} \int_0^1 \big[F'(x_{n+1} + t(x^* - x_{n+1})) \right. \\
& \left. . - F'(x_{n+1} + t(y_{n+1} - x_{n+1}))\big] (x^* - x_{n+1}) \mathrm{d}t \right\| \\
\leq {} & \frac{1}{1 - Lt_{n+1}} \int_0^1 Lt\|y_{n+1} - x^*\| \mathrm{d}t \|x_{n+1} - x^*\| \\
\leq {} & \frac{\frac{L}{2}}{1 - Lt_{n+1}} G(t_{n+1})t_{n+1}\|x_{n+1} - x^*\| \\
= {} & G(t_{n+1})^2\|x_{n+1} - x^*\|.
\end{aligned}$$

This together with the monotonicity of G yields that

$$\|x_{n+2} - x^*\| \leq G(t_0)^2 t_{n+1} \leq t_{n+1}.$$

Thus the inequalities in (4.4) hold for $n + 1$ and hence they hold for each n. The proof is complete. □

5. Numerical experiments

This section is devoted to applications of convergence results obtained in previous sections. We consider the following nonlinear boundary values problem of second order, which is a modification of an example in [28] and also appears in [2, 18],

$$\begin{cases} z'' + z^{2+\lambda} + \gamma z^2 = 0, \quad \gamma \in \mathbb{R}, \ \lambda \in (0, 1), \\ z(0) = z(1) = 0. \end{cases} \tag{5.1}$$

We divide the interval $[0,1]$ into n subintervals and let $d = 1/n$. We denote the point of subdivision by $l_i = id$, $i = 1, 2, \ldots, n$, and the corresponding values of the function $z_i := z(l_i)$. Obviously, by the boundary condition, we have $z_0 = z_n = 0$. A simple approximation for second derivative is

$$z''(l) \approx \frac{z(l+d) - 2z(l) + z(l-d)}{d^2},$$

$$z''(l_i) \approx \frac{z_{i+1} - 2z_i + z_{i-1}}{d^2}, \quad i = 1, 2, \ldots, n-1.$$

Noting that $z_0 = z_n = 0$, we have the following system of nonlinear equations:

$$\left\{ \begin{array}{l} 2z_1 - d^2 z_1^{2+\lambda} - d^2\gamma z_1^2 - z_2 = 0, \\ -z_{i-1} + 2z_i - d^2 z_i^{2+\lambda} - d^2\gamma z_i^2 - z_{i+1} = 0, \quad i = 2, 3, \ldots, n-2, \\ -z_{n-2} + 2z_{n-1} - d^2 z_{n-1}^{2+\lambda} - d^2\gamma z_{n-1}^2 = 0. \end{array} \right. \tag{5.2}$$

By (5.2), we can obtain an operator $F : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ defined by

$$F(z) = Mz - d^2 f(z) - d^2\gamma g(z), \tag{5.3}$$

where

$$z = (z_1, z_2, \ldots, z_{n-1})^T, \; f(z) = (z_1^{2+\lambda}, z_2^{2+\lambda}, \ldots, z_{n-1}^{2+\lambda})^T, \; g(z) = (z_1^2, z_2^2, \ldots, z_{n-1}^2)^T$$

and

$$M = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix} \in \mathbb{R}^{(n-1)\times(n-1)}.$$

Then

$$F'(z) = M - (2+\lambda)d^2 \begin{pmatrix} z_1^{1+\lambda} & & & \\ & z_2^{1+\lambda} & & \\ & & \ddots & \\ & & & z_{n-1}^{1+\lambda} \end{pmatrix} - 2d^2\gamma \begin{pmatrix} z_1 & & & \\ & z_2 & & \\ & & \ddots & \\ & & & z_{n-1} \end{pmatrix}.$$

From [17], we know that for any $v, w \in \mathbb{R}^{n-1}$,

$$\begin{aligned} [v, w, F]_{ij} = \frac{1}{v_j - w_j} &(F_i(v_1, v_2, \ldots, v_j, w_{j+1}, \ldots, w_{n-1}) \\ &- F_i(v_1, v_2, \ldots, v_{j-1}, w_j, \ldots, w_{n-1})). \end{aligned}$$

Thus

$$[v, w, F] = M - d^2 \begin{pmatrix} \frac{v_1^{2+\lambda} - w_1^{2+\lambda} + \gamma(v_1^2 - w_1^2)}{v_1 - w_1} & & & \\ & \frac{v_2^{2+\lambda} - w_2^{2+\lambda} + \gamma(v_2^2 - w_2^2)}{v_2 - w_2} & & \\ & & \ddots & \\ & & & \frac{v_{n-1}^{2+\lambda} - w_{n-1}^{2+\lambda} + \gamma(v_{n-1}^2 - w_{n-1}^2)}{v_{n-1} - w_{n-1}} \end{pmatrix}.$$

For any $v \in \mathbb{R}^{n-1}$, let $\|v\| = \max_{1 \le i \le n-1} |v_i|$. Then the corresponding norm for $A \in \mathbb{R}^{(n-1)\times(n-1)}$ is

$$\|A\| = \max_{1 \le i \le n-1} \sum_{j=1}^{n-1} |a_{ij}|.$$

Let $x_0 \in \mathbb{R}^{n-1}$, $r > 0$, and

$$\eta = \sup_{\theta \in \overline{\mathbf{B}(x_0,r)}} (\lambda + 1)\|\theta^\lambda\|.$$

Note that for any $z,\ v \in \mathbf{B}(x_0, r)$

$$\begin{aligned}
\|F'(z) - F'(v)\| &= d^2 \left(\max_{1 \le i \le n-1} |(2+\lambda)(z_i^{1+\lambda} - v_i^{1+\lambda}) + 2\gamma(z_i - v_i)| \right) \\
&\le d^2 \left((2+\lambda) \max_{1 \le i \le n-1} |z_i^{1+\lambda} - v_i^{1+\lambda}| + 2\gamma \max_{1 \le i \le n-1} |z_i - v_i| \right) \\
&\le d^2 \left((2+\lambda)\eta + 2\gamma \right) \|z - v\|.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\|F'(x_0)^{-1}(F'(z) - F'(v))\| &\le \|F'(x_0)^{-1}\| \|F'(z) - F'(v)\| \\
&\le d^2 \left((2+\lambda)\eta + 2\gamma \right) \|F'(x_0)^{-1}\| \|z - v\|.
\end{aligned}$$

This means that $F'(x_0)^{-1}F'$ satisfies L Lipschitz condition with $L = d^2 \left((2+\lambda)\eta + 2\gamma \right) \|F'(x_0)^{-1}\|$. Let $u_i = (u_i^1, \ldots, u_i^{n-1})^T \in \mathbb{R}^{n-1}$, $\quad i = 1, 2$. It follows that

$$\begin{aligned}
&F''(z)u_1u_2 \\
&= -d^2 \left(((2+\lambda)(1+\lambda)z_1^\lambda + 2\gamma)u_1^1u_2^1, \ldots, ((2+\lambda)(1+\lambda)z_{n-1}^\lambda + 2\gamma)u_1^{n-1}u_2^{n-1} \right)^T
\end{aligned}$$

and

$$\begin{aligned}
&(F''(z) - F''(v))u_1u_2 \\
&\qquad = -(2+\lambda)(1+\lambda)d^2 \left((z_1^\lambda - v_1^\lambda)u_1^1u_2^1, \ldots, (z_{n-1}^\lambda - v_{n-1}^\lambda)u_1^{n-1}u_2^{n-1} \right)^T.
\end{aligned}$$

Hence

$$\|F''(z) - F''(v)\| = d^2(2+\lambda)(1+\lambda) \max_{1 \le i \le n-1} |z_i^\lambda - v_i^\lambda|.$$

Below, we show that

(5.4) $\qquad F''$ is not Lipschitz continuous at a neighborhood of 0.

Assume on the contrary that F'' is Lipschitz continuous at $\mathbf{B}(0, r)$ with modulus $L > 0$. Then, for any $x \in \mathbf{B}(0, r)$ with $x \neq 0$,

$$\begin{aligned}
\|F''(x) - F''(0)\| &= d^2(2+\lambda)(1+\lambda) \max_{1 \le i \le n-1} |x_i^\lambda - 0_i^\lambda| \\
&= d^2(2+\lambda)(1+\lambda)\|x^\lambda\| \le L\|x\|,
\end{aligned}$$

which implies that

$$\frac{\|x^\lambda\|}{\|x\|} \le \frac{L}{d^2(2+\lambda)(1+\lambda)}, \tag{5.5}$$

where $x^\lambda = (x_1^\lambda, \dots, x_{n-1}^\lambda)$. Since $x \neq 0$ and $x \in \mathbf{B}(0,r)$ is arbitrary, letting $x \to 0$ in (5.5), we get that

$$\lim_{x\to 0} \frac{\|x^\lambda\|}{\|x\|} = +\infty \leq \frac{L}{d^2(2+\lambda)(1+\lambda)},$$

where the first equality holds because of $0 < \lambda < 1$. This is a contradiction and so assertion (5.4) holds.

We apply Newton-Steffensen's method (1.4) to find a solution z^* of the equation

$$F(z) = 0 \tag{5.6}$$

and study this problem in two cases: $\gamma = 0$ and $\gamma \neq 0$, where F is given by (5.3). Set

$$\beta := \|F'(x_0)^{-1}F(x_0)\|.$$

Using MATLAB(version R2008a) and the stopping tolerance for iterations is $\|F(z_k)\| \leq eps$, where $eps \approx 2.2204e-16$.

5.1. **When $\gamma = 0$.** The problem (5.1) turns into

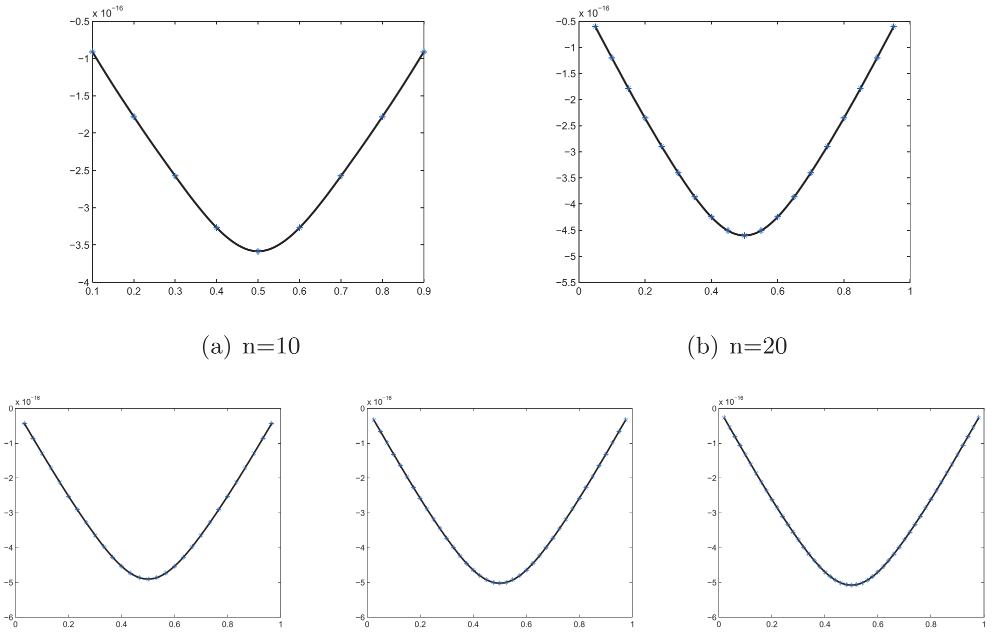
$$\begin{cases} z'' + z^{2+\lambda} = 0, \quad \lambda \in (0,1), \\ z(0) = z(1) = 0. \end{cases} \tag{5.7}$$

We apply Newton-Steffensen's method to approach the approximate solution z^* of $F(z) = 0$. Since a solution of $F(z) = 0$ would vanish at the endpoints and be positive in the interior of $[0,1]$, we can choose $x_0 = (\alpha \sin \frac{1}{n}\omega\pi, \dots, \alpha \sin \frac{n-1}{n}\omega\pi)^T$ as the initial approximation, where $\alpha,\ \omega \in \mathbb{R}$. Below, we consider four cases:

Case I. Let $\lambda = \frac{1}{6}$. Let $n = 10,\ 20,\ 30,\ 40,\ 50$. The selection of initial points and numerical results with the starting points are listed in Table 1 which shows the effectiveness of our method. By Newton-Steffensen's method, we get solutions z^* of (5.6). Using cubic spline interpolation, the approximate solutions $\hat{z}^*$ of (5.7) can be shown in Figure 1.

n	α	ω	iteration	error		
				$k=0$	$k=1$	$k=2$
10	1/2	5	2	5.0000e-1	1.1843e-4	0.0000
20	1/2	5	2	5.0000e-1	1.2395e-4	0.0000
30	1/2	5	2	5.0000e-1	1.2601e-4	0.0000
40	1/2	5	2	5.0000e-1	1.2683e-4	0.0000
50	1/2	5	2	5.0000e-1	1.2726e-4	0.0000

TABLE 1. Initial points and values of $\|x_k - z^*\|$.



(a) n=10 (b) n=20

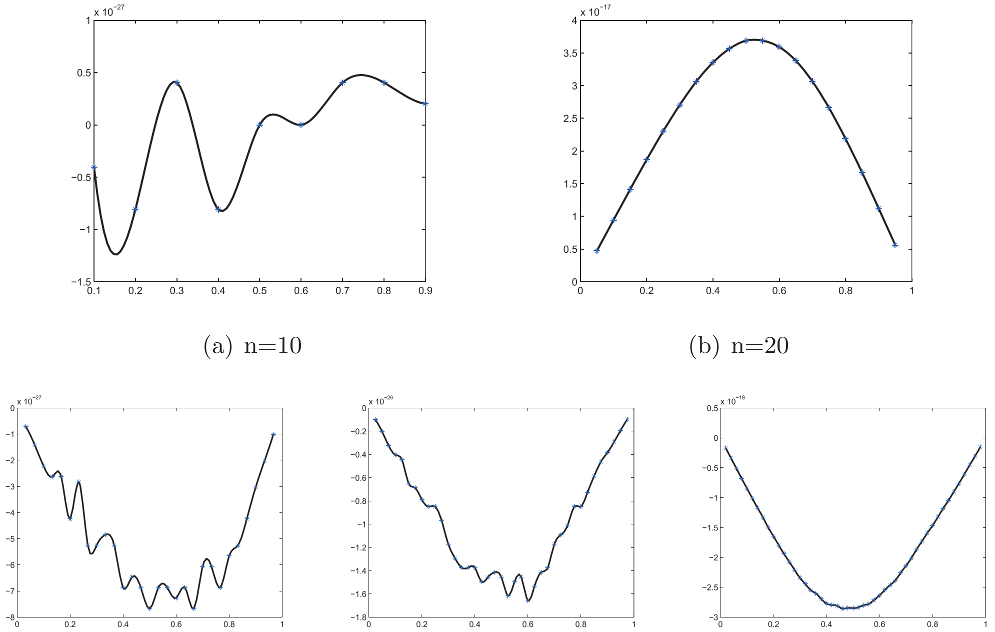
(c) n=30 (d) n=40 (e) n=50

FIGURE 1. z^* and the approximate solutions $\hat{z}^*$.

Case II. Let $\lambda = \frac{1}{4}$. For $n = 10,\ 20,\ 30,\ 40,\ 50$, the initial points and numerical results are listed in Table 2. By Newton-Steffensen's method, we get the solutions z^* of (5.6). Moreover, by cubic spline interpolation, the approximate solutions $\hat{z}^*$ of (5.7) can be shown in Figure 2.

n	α	ω	iteration	error			
				$k=0$	$k=1$	$k=2$	$k=3$
10	1	8	3	9.5105e-1	2.0535e-3	4.3562e-12	0.0000
20	1	1/8	2	3.6447e-1	7.9285e-5	0.0000	0.0000
30	1	8	3	9.9452e-1	1.9880e-3	3.1687e-12	0.0000
40	1	8	3	9.5106e-1	1.9285e-3	2.8207e-12	0.0000
50	1/3	8	2	3.3268e-1	8.4899e-5	0.0000	0.0000

TABLE 2. Initial points and values of $\|x_k - z^*\|$.



(a) n=10 (b) n=20

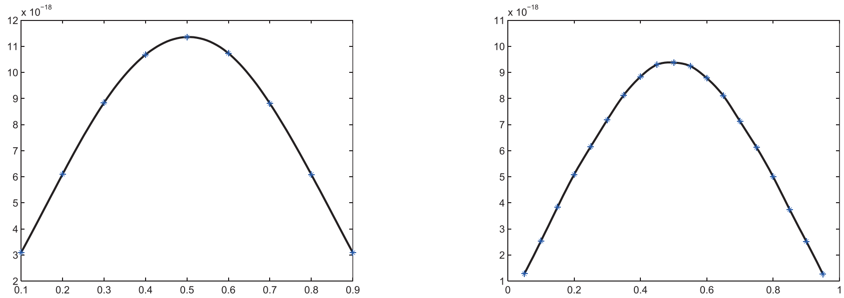
(c) n=30 (d) n=40 (e) n=50

FIGURE 2. z^* and the approximate solutions $\hat{z}^*$.

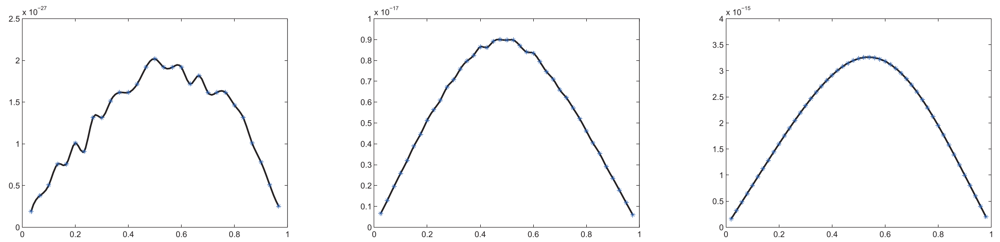
Case III. Let $\lambda = \frac{1}{2}$. When $n = 10,\ 20,\ 30,\ 40,\ 50$, Table 3 shows the initial points and numerical results, respectively. By Newton-Steffensen's method, we get the solutions z^* of (5.6). Furthermore by cubic spline interpolation, the approximate solutions $\hat{z}^*$ of (5.7) can be shown in Figure 3.

n	α	ω	iterations	errors		
				$k=0$	$k=1$	$k=2$
10	1/2	5	2	5.0015e-1	1.8853e-4	0.0000
20	1/2	5	2	5.0000e-1	1.7655e-4	0.0000
30	1	8	3	9.9452e-1	3.0359e-3	8.8226e-13
40	1/2	5	2	5.0014e-1	1.7691e-4	0.0000
50	2	1/8	2	7.5083e-1	7.7171e-4	0.0000

TABLE 3. Initial points and values of $\|x_k - z^*\|$.



(a) n=10 (b) n=20



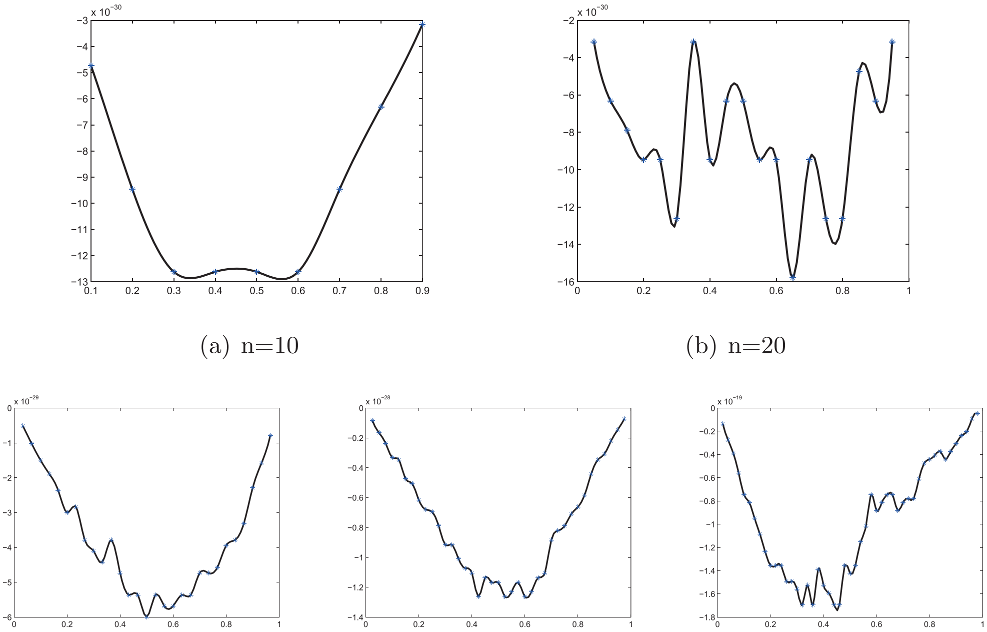
(c) n=30 (d) n=40 (e) n=50

FIGURE 3. z^* and the approximate solutions $\hat{z}^*$.

Case IV. Let $\lambda = \frac{2}{3}$. When $n = 10,\ 20,\ 30,\ 40,\ 50$, we give, respectively, the initial points and numerical results in Table 4. Using Newton-Steffensen's method, one has the solutions z^* of (5.6). Moreover, by cubic spline interpolation, the approximate solutions $\hat{z}^*$ of (5.7) can be shown in Figure 4.

n	α	ω	iterations	errors			
				$k=0$	$k=1$	$k=2$	$k=3$
10	1	8	3	9.5105e-1	2.9748e-3	1.6672e-13	0.0000
20	1	8	3	9.5106e-1	2.7745e-3	9.4290e-14	0.0000
30	1	8	3	9.9452e-1	2.7818e-3	9.4255e-14	0.0000
40	1	8	3	9.5106e-1	2.7672e-3	9.1616e-14	0.0000
50	1	1/8	2	5.0000e-1	1.7698e-4	0.0000	0.0000

TABLE 4. Initial points and values of $\|x_k - z^*\|$.



(a) n=10 (b) n=20

(c) n=30 (d) n=40 (e) n=50

FIGURE 4. z^* and the approximate solutions $\hat{z}^*$.

5.2. **When** $\gamma \neq 0$**.** In this subsection, we consider $\gamma = 1$. Then the problem (5.1) becomes

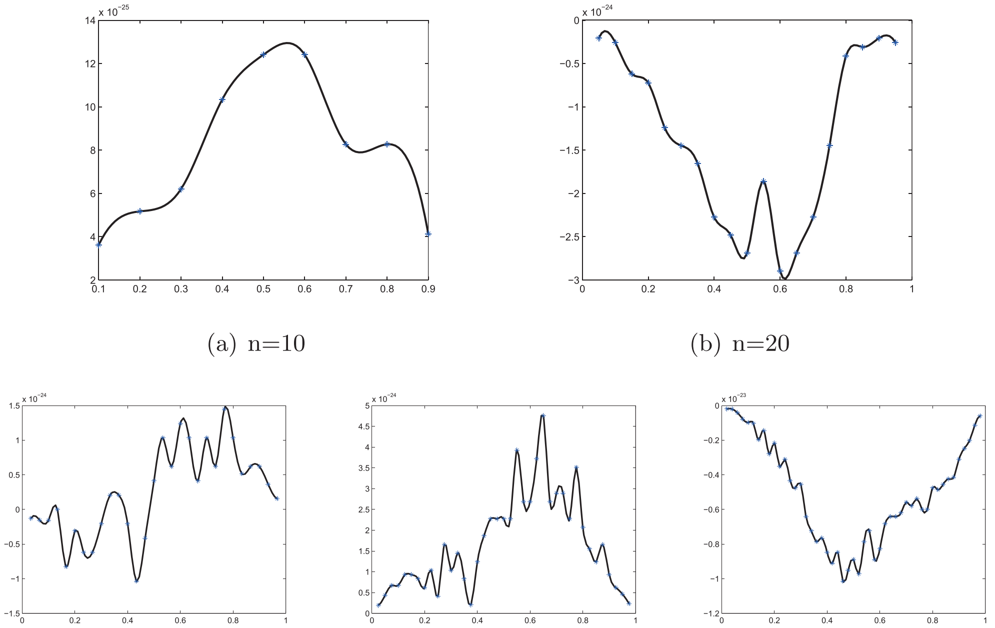
$$\begin{cases} z'' + z^{2+\lambda} + z^2 = 0, & \lambda \in (0,1), \\ z(0) = z(1) = 0. \end{cases} \tag{5.8}$$

As in the last subsection, we choose $x_0 = (\alpha \sin \frac{1}{n}\omega\pi, \ldots, \alpha \sin \frac{n-1}{n}\omega\pi)^T$ as an initial approximation, where $\alpha,\ \omega \in \mathbb{R}$. Also, we discuss the following cases.

Case V. Let $\lambda = \frac{1}{6}$. We consider five cases: $n = 10,\ 20,\ 30,\ 40,\ 50$. The initial points and numerical results are shown in Table 5. Also the solutions z^* of (5.6) can be got by Newton-Steffensen's method. By cubic spline interpolation, the approximate solutions $\hat{z}^*$ of (5.8) can be shown in Figure 5.

n	α	ω	iterations	errors			
				$k=0$	$k=1$	$k=2$	$k=3$
10	2	1/8	3	6.9223e-1	4.8319e-3	1.7056e-9	0.0000
20	2	1/8	3	7.2894e-1	4.8085e-3	1.6609e-9	0.0000
30	2	1/8	3	7.4111e-1	4.8041e-3	1.6575e-9	0.0000
40	2	1/8	3	7.4719e-1	4.8068e-3	1.6554e-9	0.0000
50	2	1/8	3	7.5083e-1	4.8079e-3	1.6539e-9	0.0000

TABLE 5. Initial points and values of $\|x_k - z^*\|$.



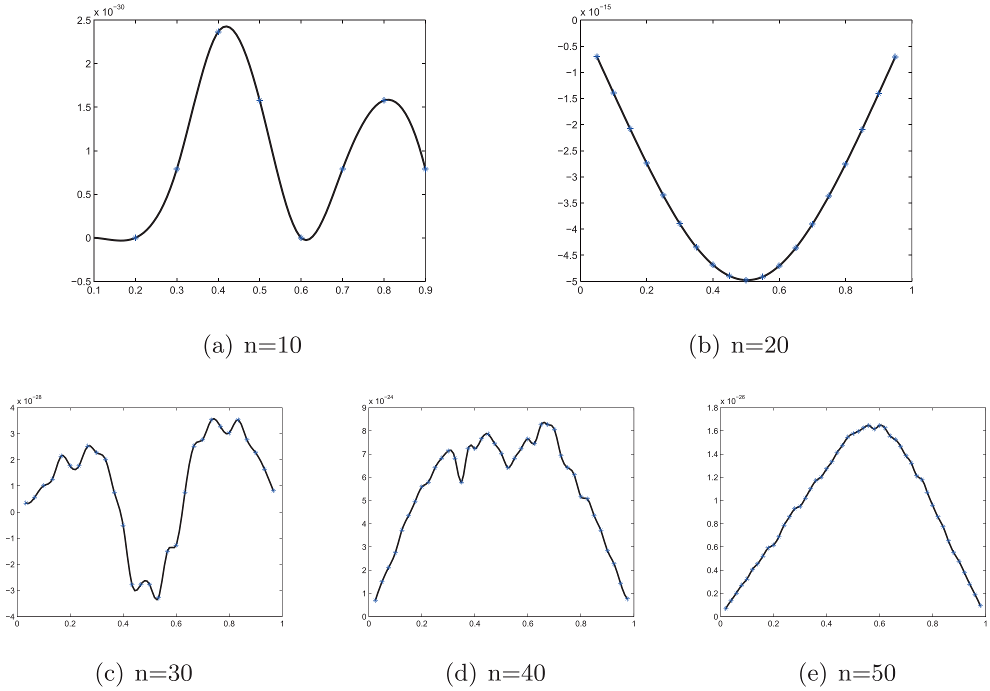
(a) n=10 (b) n=20

(c) n=30 (d) n=40 (e) n=50

FIGURE 5. z^* and the approximate solutions $\hat{z}^*$.

n	α	ω	iterations	errors			
				$k=0$	$k=1$	$k=2$	$k=3$
10	1/3	8	3	3.1702e-1	8.8360e-5	7.1089e-15	0.0000
20	1/3	8	2	3.1702e-1	8.4074e-5	0.0000	0.0000
30	1/2	15	3	5.0000e-1	2.7987e-4	2.1836e-13	0.0000
40	2	1/8	3	7.4719e-1	4.6377e-3	1.1880e-9	0.0000
50	1	1/8	3	3.7542e-1	1.9735e-2	1.1034e-7	0.0000

TABLE 6. Initial points and values of $\|x_k - z^*\|$.



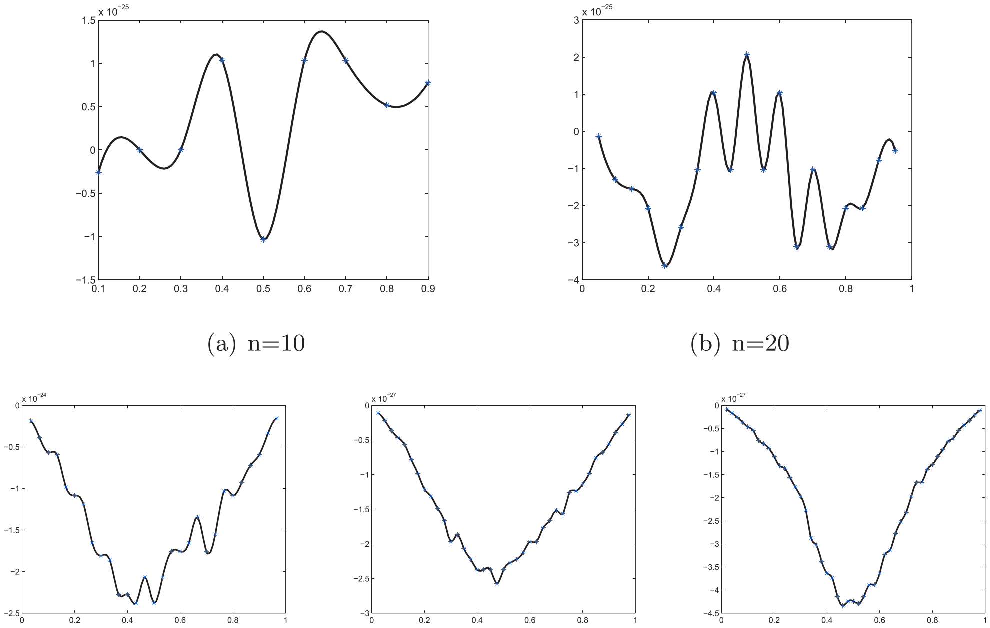
(a) n=10 (b) n=20

(c) n=30 (d) n=40 (e) n=50

FIGURE 6. z^* and the approximate solutions $\hat{z}^*$.

n	α	ω	iterations	errors			
				$k=0$	$k=1$	$k=2$	$k=3$
10	2	1/8	3	6.9223e-1	4.2252e-3	6.3449e-10	0.0000
20	2	1/8	3	7.2894e-1	4.2085e-3	6.1992e-10	0.0000
30	2	1/8	3	7.4111e-1	4.2053e-3	6.1899e-10	0.0000
40	1	1/8	3	3.7359e-1	3.5413e-4	3.3385e-13	0.0000
50	1	1/8	3	3.7542e-1	3.5411e-4	3.3359e-13	0.0000

TABLE 7. Initial points and values of $\|x_k - z^*\|$.



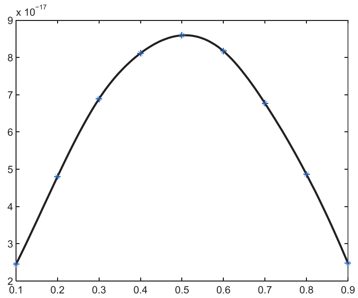
(a) n=10 (b) n=20

(c) n=30 (d) n=40 (e) n=50

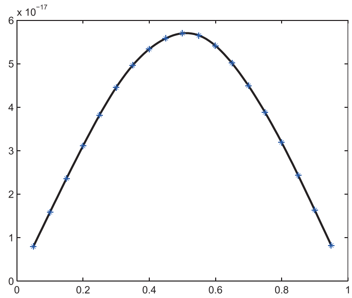
FIGURE 7. z^* and the approximate solutions $\hat{z}^*$.

n	α	ω	iterations	errors			
				$k=0$	$k=1$	$k=2$	$k=3$
10	1/4	8	2	2.3776e-1	2.6460e-5	0.0000	0.0000
20	1/4	8	2	2.3776e-1	2.5188e-5	0.0000	0.0000
30	1/2	5	3	5.0000e-1	3.8819e-4	3.1039e-13	0.0000
40	1	1/8	3	3.7359e-1	3.1264e-4	2.2217e-13	0.0000
50	1/4	8	2	2.4951e-1	2.4837e-5	0.0000	0.0000

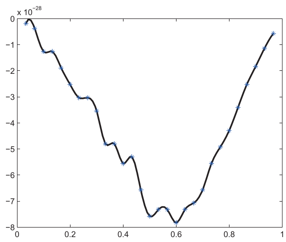
TABLE 8. Initial points and values of $\|x_k - z^*\|$.



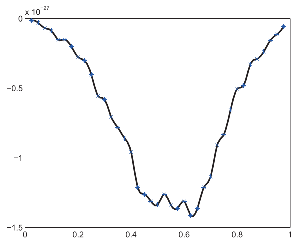
(a) n=10



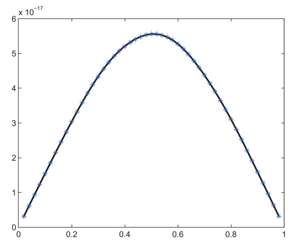
(b) n=20



(c) n=30



(d) n=40



(e) n=50

FIGURE 8. z^* and the approximate solutions $\hat{z}^*$.

Case VI. Let $\lambda = \frac{1}{4}$. For $n = 10,\ 20,\ 30,\ 40,\ 50$, the option of initial points and numerical results are shown in Table 6. By Newton-Steffensen's method, we get the solutions z^* of (5.6). Using cubic spline interpolation, we get the approximate solutions $\hat{z}^*$ of (5.8) which are shown in Figure 6.

Case VII. Let $\lambda = \frac{1}{2}$ and $n = 10,\ 20,\ 30,\ 40,\ 50$. The option of initial points, numerical results and errors are shown in Table 7. By Newton-Steffensen's method, we get the solutions z^* of (5.6). Using cubic spline interpolation, we get the approximate solutions $\hat{z}^*$ of (5.8) which are shown in Figure 7.

From Table 7, we know that errors are smaller and smaller as the increase of n under the condition of the same initial point.

Case VIII. Let $\lambda = \frac{2}{3}$. For $n = 10,\ 20,\ 30,\ 40,\ 50$, the option of initial points, numerical results and errors are shown in Table 8. One has the solutions z^* of (5.6) by Newton-Steffensen's method and using cubic spline interpolation, we get the approximate solutions $\hat{z}^*$ of (5.8) which are shown in Figure 8.

REFERENCES

[1] I. K. Argyros and D. Chen, *Results on the Chebyshev method in Banach spaces*, Proyecciones **12** (1993), 119–128.

[2] I. K. Argyros and S. K. Khattri, *On the secant method*, J. Complexity **29** (2013), 454–471.

[3] V. Candela and A. Marquina, *Recurrence relations for rational cubic methods I: the Halley method*, Computing **44** (1990), 169–184.

[4] V. Candela and A. Marquina, *Recurrence relations for rational cubic methods II: the Chebyshev method*, Computing **45** (1990), 355–367.

[5] D. Chen, I. K. Argyros and Q. S. Qian, *A note on the Halley method in Banach spaces*, Appl. Math. Comput. **58** (1993), 215–224.

[6] J. A. Ezquerro and M. A. Hernández, *Avoiding the computation of the second Frechet-derivative in the convex acceleration of Newton's method*, J. Comput. Appl. Math. **96** (1998), 1–12.

[7] J. A. Ezquerro and M. A. Hernández, *A modification of the super-Halley method under mild differentiability conditions*, J. Comput. Appl. Math. **114** (2000), 405–409.

[8] J. M. Gutiérrez and M. A. Hernández, *A family of Chebyshev-Halley type methods in Banach spaces*, Bull. Aust. Math. Soc. **55** (1997), 113–130.

[9] M. A. Hernández, *A note on Halley's method*, Numer. Math. **59** (1991), 273–276.

[10] M. A. Hernández, *Second-derivative-free variant of the Chebyshev method for nonlinear equations*, J. Optim. Theory Appl. **104**(3) (2000), 501-515.

[11] M. A. Hernández, *Chebyshevs approximation algorithms and applications*, Comput. Math. Appl. **41** (2001), 433–445.

[12] M. A. Hernández and M. A. Salanova, *A family of Chebyshev-Halley type methods*, Internat. J. Comput. Math. **47** (1993), 59–63.

[13] M. A. Hernández and M. A. Salanova, *Modification of the Kantorovich assumptions for semilocal convergence of the Chebyshev method*, J. Comput. Appl. Math. **126** (2000), 131–143.

[14] Y. H. Hu, C. Li and X. Q. Yang, *On convergence rates of linearized proximal algorithms for convex composite optimization with applications*, SIAM J. Optim. **26** (2016), 1207–1235.

[15] L. V. Kantorvich, *On Newton's method for functional equations*, Dokl. Akad. Nauk SSSR (N.S.) **59** (1948), 1237–1240.

[16] L. V. Kantorvich and G. P. Akilov, *Functional Analysis*, Pergamon Press, Oxford, 1982.

[17] F. A. Potra and V. Pták, *Nondiscrete Induction and Iterative Processes*, Research Notes in Mathematics, 103. Pitman (Advanced Publishing Program), Boston, MA, 1984.

[18] J. Rokne, *Newton's method under mild differentiability conditions with error analysis*, NzLmer. Math. **18** (1972), 401–412.

[19] J. R. Sharma, *A composite third order Newton-Steffensen method for solving nonlinear equations*, Appl. Math. Comput. **169** (2005), 242–246.

[20] S. Smale, *Newtons method estimates from data at one point*, In:Ewing, R., Gross, K., Martin, C.(eds.) The Merging of Disciplines NewDirections in Pure, Applied and Computational Mathematics, Springer, New York, 1986, pp. 185–196.

[21] X. H. Wang, *Convergence of Newton's method and inverse function theorem in Banach Space*, Math. Comp. **68** (1999), 169–186.

[22] X. H. Wang, *Convergence of Newton's method and uniqueness of the solution of equations in Banach space*, IMA J. Numer. Anal. **20** (2000), 123–134.

[23] X. H. Wang and C. Li, *Local and global behavior for algorithms of solving equations*, Chinese Sci. Bull. **46** (2001), 441–448.

[24] X. H. Wang and C. Li, *Convergence of Newton's method and uniqueness of the solution of equations in Banach Spaces II*, Acta Math. Sin. (engl. ser.) **19** (2003), 405–412.

[25] X. H. Wang and C. Li, *On the united theory of the family of Euller–Halley types with cubical convergence in Banach spaces*, J. Comput. Math. **21** (2003), 195–200.

[26] X. H. Wang, C. Li and M. J. Lai, *A unified convergence theory for Newton-type methods for zeros of nonlinear operators in Banach spaces*, BIT Numer. Math. **42** (2002), 206–213.

[27] X. T. Ye and C. Li, *Convergence of the family of the deformed Euler-Halley iterations under the Hölder condition of the second derivative*, J. Comput. Appl. Math. **194** (2006), 294–308.

[28] S. H. Yu, X. B. Xu, J. Q. Li, and Y. H. Ling, *Convergence behavior for Newton-Steffensen's method under Lipschitz condition of second derivative*, Taiwanese J. Math. **15** (2011), 2577–2600.

Manuscript received September 10, 2017
revised November 5, 2017

Y. Zhang
Zhejiang University of Water Resources and Electric Power, Hangzhou 310018, P. R. China
E-mail address: `zhangyan198421@163.com`

Y. Hu
College of Mathematics and Statistics, Shenzhen University, Shenzhen 518060, P. R. China
E-mail address: `mayhhu@szu.edu.cn`

C. K. W. Yu
Department of Mathematics and Statistics, Hang Seng Management College, Hong Kong
E-mail address: `carisayu@hsmc.edu.hk`

J. Wang
Department of Mathematics, Zhejiang University of Technology, Hangzhou 310032, P. R. China
E-mail address: `wjh@zjut.edu.cn`