Business Analytics

Regression Diagnostics

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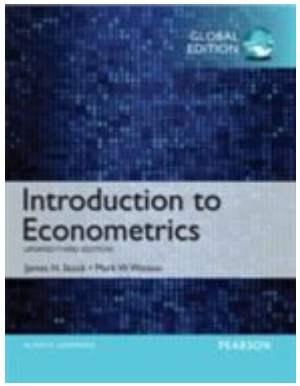
Course Content

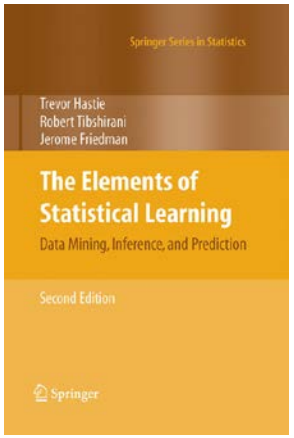
- Introduction
- Regression Analysis
- **Regression Diagnostics**
- Logistic and Poisson Regression
- Naive Bayes and Bayesian Networks
- Decision Tree Classifiers
- Data Preparation and Causal Inference
- Model Selection and Learning Theory
- Ensemble Methods and Clustering
- High-Dimensional Problems
- Association Rules and Recommenders
- Neural Networks

Recommended Literature

- **Introduction to Econometrics**
  - Stock, James H., and Mark W. Watson
  - Chapter 6, 7, 10, 17, 18



- **The Elements of Statistical Learning**
  (Trevor Hastie, Robert Tibshirani, Jerome Friedman)
  - http://web.stanford.edu/~hastie/Papers/ESLII.pdf
  - Section 3: Linear Methods for Regression



- **An Introduction to Statistical Learning: With Applications in R**
  (Gareth James, Trevor Hastie, Robert Tibshirani)
  - Section 3: Linear Regression

Multiple Linear Regression

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & x_{11} & x_{12} & \dots & x_{1p} \\ 1 & x_{21} & x_{22} & \dots & x_{2p} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{n1} & x_{n2} & \dots & x_{np} \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix}$$

y	$\boldsymbol{X}$	β	$+ \varepsilon$
$(n \times 1)$	$(n \times (p+1))$	$((p+1) \times 1)$	$(n \times 1)$

Reminder: Least Squares Estimation

$\mathbf{X}$ is $n \times (p+1)$, y is the vector of outputs

RSS(β) = $(y - \mathbf{X}\beta)^T (y - \mathbf{X}\beta)$

If $\mathbf{X}$ is full rank, then $\mathbf{X}^T\mathbf{X}$ is positive definite

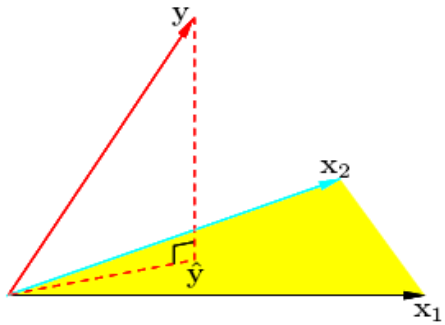
$$RSS = (y^T y - 2\beta^T \mathbf{X}^T \mathrm{y} + \beta^T \mathbf{X}^T \mathbf{X}\beta)$$

$$\frac{\partial RSS}{\partial \beta} = -2\mathbf{X}^T \mathrm{y} + 2\mathbf{X}^T \mathbf{X}\beta = 0$$ First-order condition

$$\beta = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T y$$

$$\hat{y} = \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T y$$

"Hat" or projection matrix H



Source: Hastie et al. 2016, p. 46

Goal for this Class

The linear regression model for **a random sample** (no bias in the sampling) is computationally simple and „best“ if certain assumptions are satisfied.

Today, we will discuss the **main assumptions** and introduce selected tests that can help you check whether you can assume that the assumptions are satisfied in your data set.

There are several alternative tests that have been developed for each assumption. We introduce one of these tests, which will enable you to use the multiple linear regression in applications.

Gauss-Markov Theorem

The Gauss-Markov theorem states that in a linear regression model in which the errors

- have expectation zero and
- are uncorrelated and
- have equal variances,

the *best linear unbiased estimator* (BLUE) of the coefficients is given by the ordinary least squares (OLS) estimator.

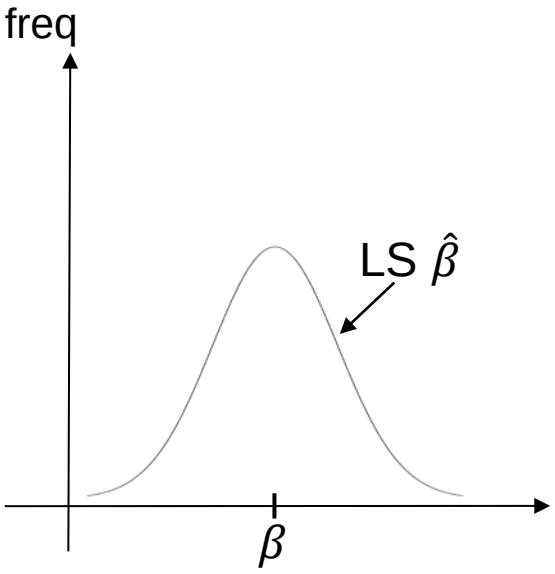
- „Unbiased“ means $E(\hat{\beta}_j) = \beta_j$
- „Best“ means giving the ***lowest variance*** of the estimate as compared to other linear unbiased estimators.
  - Restriction to unbiased estimation is not always the best
    (will be discussed in the context of the ridge regression later)

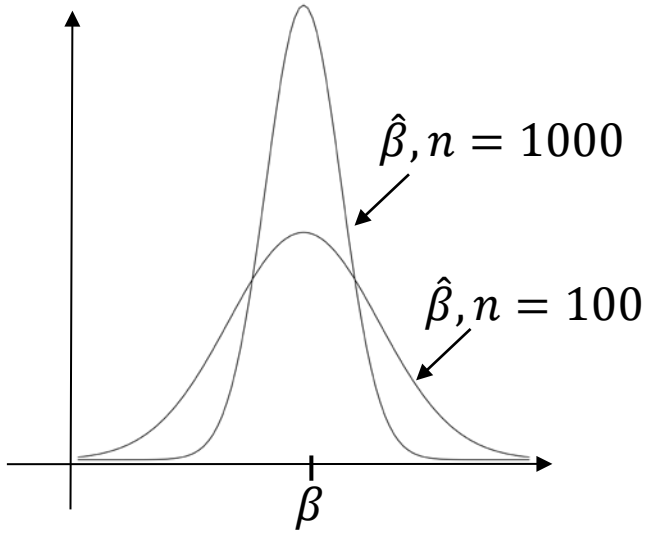
Bias, Consistency, and Efficiency

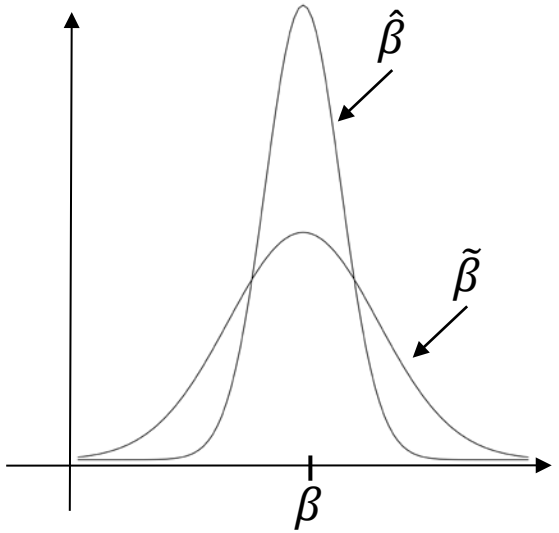
Unbiased
$E(\hat{\beta}) = \beta$
Expected value for estimator “is true”

Consistent
$var(\hat{\beta})$ decreases with increasing sample size n

Efficient
$var(\hat{\beta}) < var(\tilde{\beta})$
estimator $\hat{\beta}$ has lower variance than any other estimator, $\tilde{\beta}$







Gauss-Markov Assumptions in Detail

The OLS estimator is the best linear unbiased estimator (BLUE), iff

1) **Linearity**

   Linear relationship in parameters β

2) **No multicollinearity** of predictors

   No linear dependency between predictors

3) **Homoscedasticity**

   The residuals exhibit constant variance

4) **No autocorrelation**

   There is no correlation between the i^{th} and j^{th} residual terms

5) **Expected value** of the residual vector, given X, is 0 ($E(\varepsilon|X) = 0$)

   (i.e. exogeneity ($cov(\varepsilon, X) = 0$))

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When Linearity Does Not Hold: Try to Reformulate

For non-linear regressions the OLS estimator from the last class might not be appropriate. Often, you can adapt your data such that you can still use the multiple linear regression. The following reformulations lead to models that are again linear in β:

- Polynomial regression (still a linear model):

$$Y = \beta_0 + \beta_1 X + \beta_2 X^2 + \cdots + \varepsilon$$

- Transform either X or Y or both variables, e.g.:

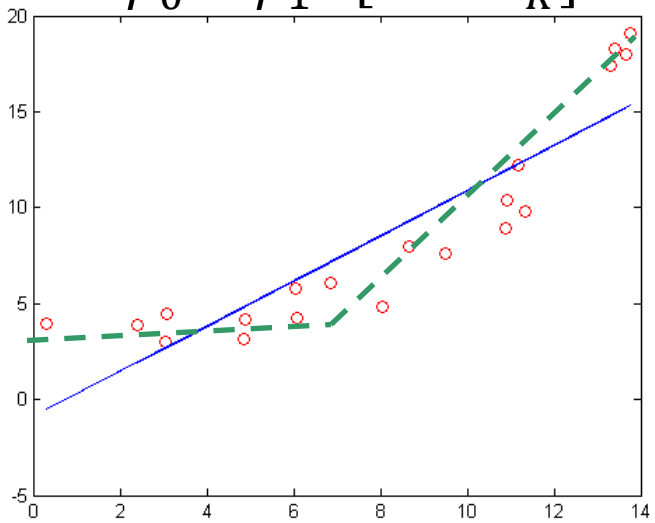
$$Log(Y) = \beta_0 + \beta_1 Log(X) + \varepsilon$$

- Piecewise linear regression:

$$Y = \beta_0 + \beta_1 X[X > X_K] + \varepsilon$$

where $[X > X_K] = 0$ if $X \leq X_K$ and $[X > X_K] = 1$ if $X > X_K$

$$Y = \beta_0 + \beta_1 X[X > X_K] + \varepsilon$$



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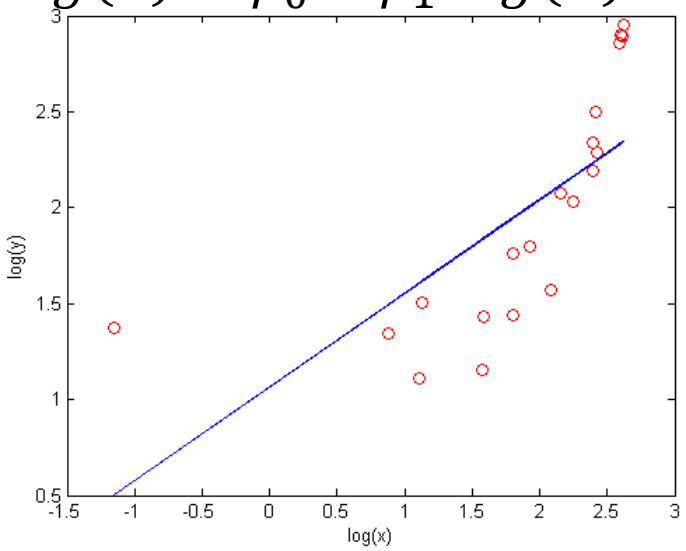
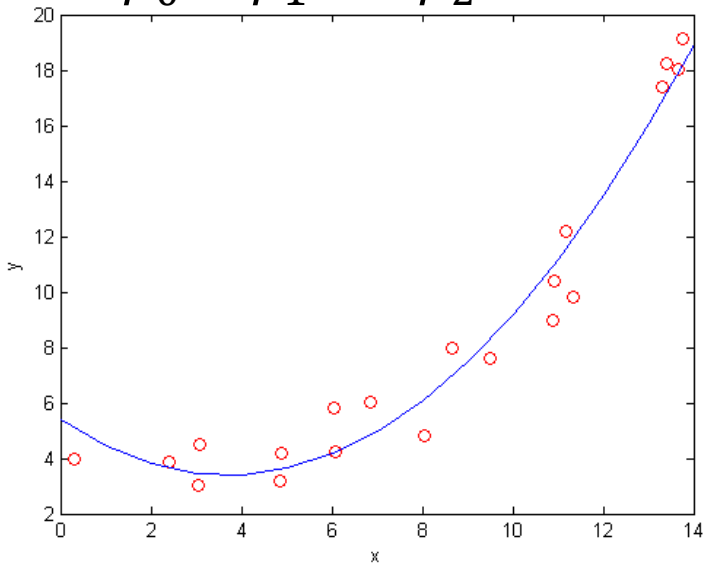


$$Y = \beta_0 + \beta_1 X + \beta_2 X^2 + \cdots + \varepsilon$$



Outliers

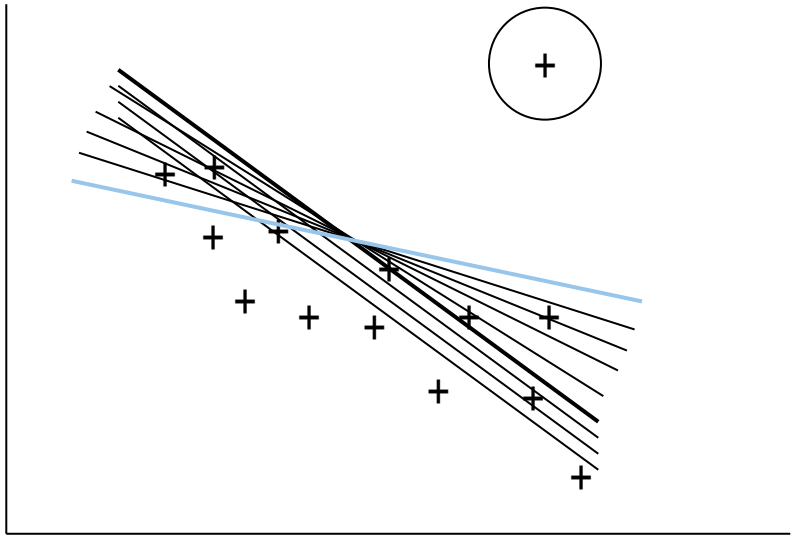
- An outlier is an observation that is unusually small or large

Several possibilities need to be investigated when an outlier is observed:

- There was an error in recording the value
- The point does not belong in the sample
- The observation is valid

- Identify outliers from the scatter diagram

- There are also methods for “robust” regression



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5) **Expected value** of the residual vector, given X, is 0 ($E(\varepsilon|X) = 0$)

   (i.e. exogeneity ($cov(\varepsilon, X) = 0$))

Multicollinearity

- Independent variables must not be linearly dependent. If two independent variables were dependent, one could easily omit one.
- To check for linear dependencies of two columns in a matrix, we can use the **rank**.
  - The **rank** of the data matrix $\mathbf{X}$ is p, the number of columns
  - $p < n$, the number of observations
  - If there is an exact linear relationships among independent variables $rank(\mathbf{X}) = p$, $\mathbf{X}$ has full column rank
  - If $rank(\mathbf{X}) < p$, $\mathbf{X}$ is singular -> impossible to calculate the inverse
    - Remember: $\hat{y} = \mathbf{X}\hat{\beta} = \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T y$
- Also, **high correlation** between independent variables leads to issues wrt. the significance of predictors

Check for Multicollinearity

- A basic check of multicollinearity is to calculate the **correlation coefficient** for each pair of predictor variables
  - Large correlations (both positive and negative) indicate problems
    - large means greater than the correlations between predictors and response
  - It is possible that the pairwise correlations are small, and yet a linear dependence exists among three or even more variables.

- Alternatively use the *variance inflation factor* (*VIF*)

Variance Inflation Factor

- $\text{VIF} = \frac{1}{1-R_k^2}$, where the R_k^2 value here is the value when the predictor in question (k) is set as the dependent variable

- For example, if the VIF = 10, then the respective R_k^2 would be 90%. This would mean that 90% of the variance in the predictor in question can be explained by the other independent variables

- Because so much of the variance is captured elsewhere, removing the predictor in question should not cause a substantive decrease in overall R^2
  - The rule of thumb is to remove variables with VIF scores greater than 10

Consequence - Non-Significance

- If a variable has a non-significant *t*-value, then either
  - The variable is not related to the response, or
    (-> Small t-value, small VIF, small correlation with response)
  - The variable is related to the response, but it is not required in the regression because it is strongly related to a third variable that is in the regression, so we don’t need both
    (-> Small t-value, big VIF, big correlation with response)
- The usual remedy is to drop one or more variables from the model

Example

Y	X1	X2	X3	X4
78.5	7	26	6	60
74.3	1	29	15	52
104.3	11	56	8	20
87.6	11	31	8	47
95.9	7	52	6	33
109.2	11	55	9	22
102.7	3	71	17	6
72.5	1	31	22	44
93.1	2	54	18	22
115.9	21	47	4	26
83.8	1	40	23	34
113.3	11	66	9	12
109.4	10	68	8	12

Example

```
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  62.4054    70.0710   0.891   0.3991
X1            1.5511     0.7448   2.083   0.0708
X2            0.5102     0.7238   0.705   0.5009
X3            0.1019     0.7547   0.135   0.8959
X4           -0.1441     0.7091  -0.203   0.8441
Residual standard error: 2.446 on 8 degrees of freedom
Multiple R-Squared: 0.9824,     Adjusted R-squared: 0.9736
F-statistic: 111.5 on 4 and 8 DF,  p-value: 4.756e-07
```

Large p-values

Big R-squared

```
> round(cor(cement.df),2)
      Y     X1    X2    X3    X4
Y     1.00  0.73  0.82 -0.53 -0.82
X1    0.73  1.00  0.23 -0.82 -0.25
X2    0.82  0.23  1.00 -0.14 -0.97
X3   -0.53 -0.82 -0.14  1.00  0.03
X4   -0.82 -0.25 -0.97  0.03  1.00
```

Big correlation

Data

```
> vif(fit)
       X1        X2        X3        X4
 38.49621 254.42317  46.86839 282.51286
```

Drop X4

```
> vif(fit)
      X1       X2       X3
3.251068 1.063575 3.142125
```

VIF's now small

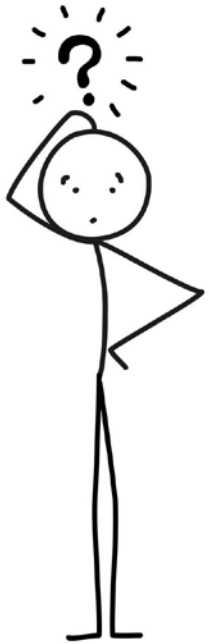
```
             Estimate Std. Error t value Pr(>|t|)
(Intercept) 48.19363    3.91330  12.315 6.17e-07 ***
X1           1.69589    0.20458   8.290 1.66e-05 ***
X2           0.65691    0.04423  14.851 1.23e-07 ***
X3           0.25002    0.18471   1.354    0.209

Residual standard error: 2.312 on 9 degrees of freedom
Multiple R-Squared: 0.9823,    Adjusted R-squared: 0.9764
F-statistic: 166.3 on 3 and 9 DF,  p-value: 3.367e-08
```

X1, X2 now signif

R-squared hardly decreased

Please explain the intuition behind the VIF.



Gauss-Markov Assumptions in Detail

The OLS estimator is the best linear unbiased estimator (BLUE), iff

1) **Linearity**

   Linear relationship in parameters β

2) **No multicollinearity** of predictors

   No linear dependency between predictors

3) Homoscedasticity

   The residuals exhibit constant variance and are **normally** distributed

4) **No autocorrelation**

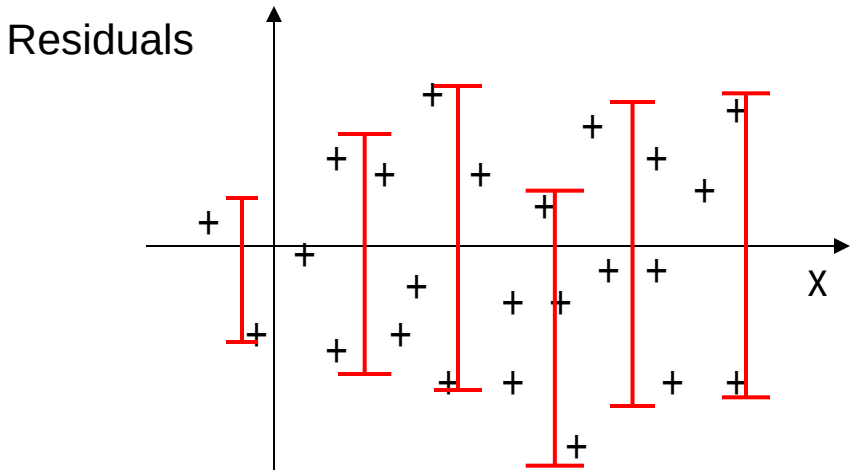
   There is no correlation between the i^{th} and j^{th} residual terms

5) **Expected value** of the residual vector, given X, is 0 ($E(\varepsilon|X) = 0$)

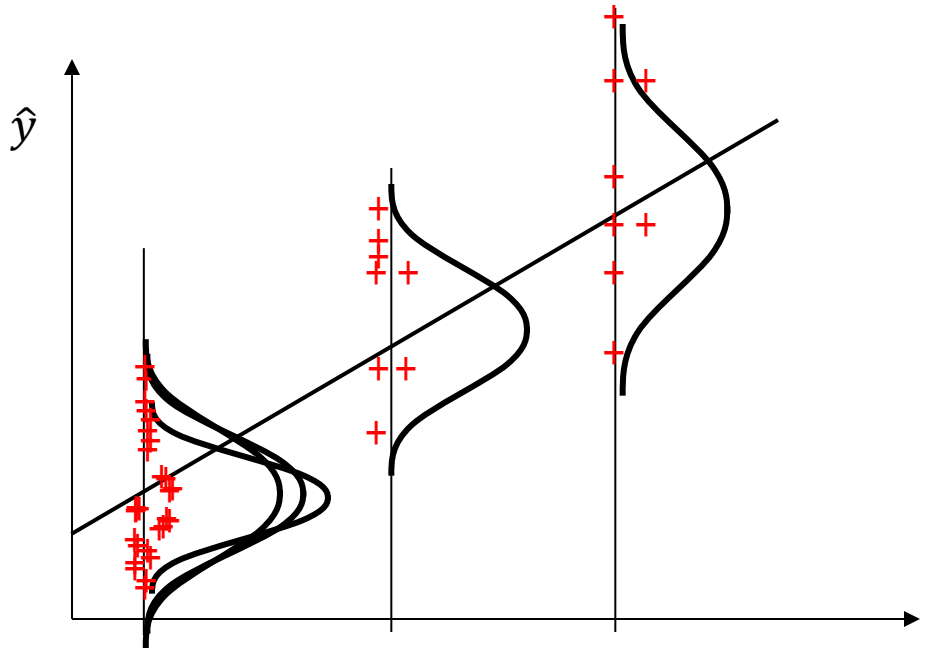
   (i.e. exogeneity ($cov(\varepsilon, X) = 0$))

Homoscedasticity

- When the requirement of **a constant variance** is not violated we have homoscedasticity.
- We also assume residuals to be **normally distributed**.
- If the data is not homoscedastic, a different estimator (e.g., generalized least squares) might be better than OLS.

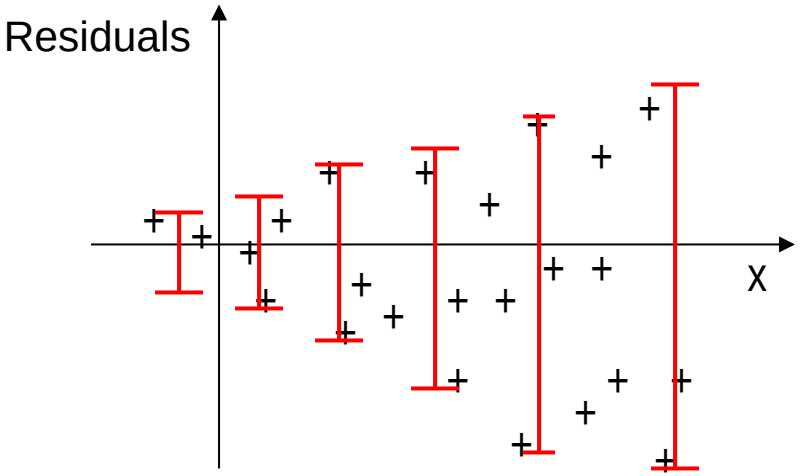


The spread of the data points does not change much.

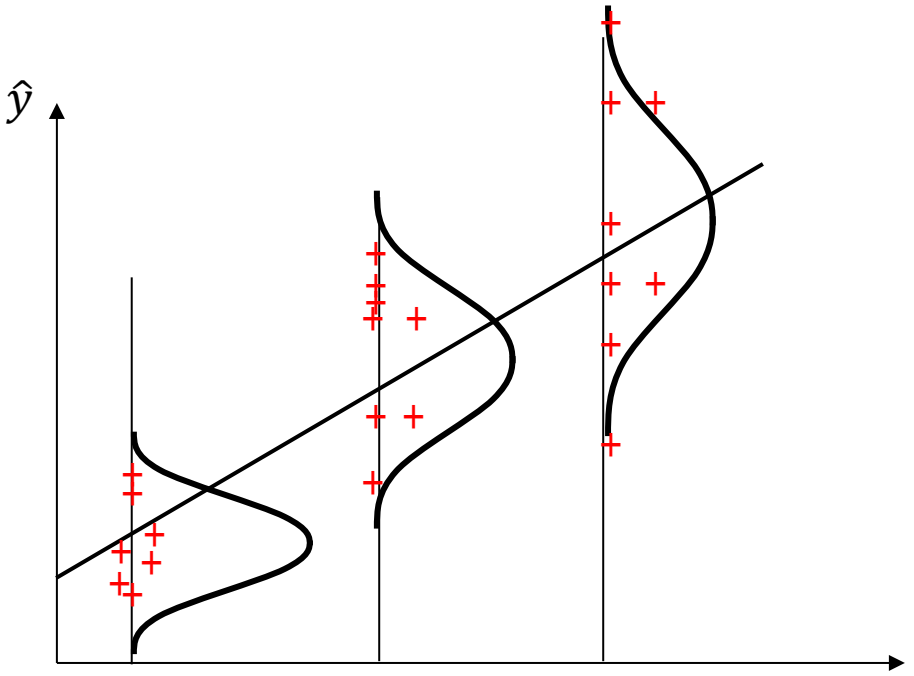


Heteroscedasticity

- When the requirement of a constant variance is violated we have heteroscedasticity ($var(\varepsilon_i|x_{1i}, \dots, x_{pi})$ not constant)
- Heteroscedasticity leads to biased error terms and p-values of significance tests.
- **Example**: annual income when predicted by age. Entry level jobs are often paid very similar, but as people grow older their income distribution increases.



The spread increases with x



Glejser Test

Apart from visual inspection, the Glejser test is a simple statistical test. It regresses the residuals on the explanatory variable that is thought to be related to the heteroscedastic variance.

1. Estimate the original regression with OLS and find the sample residuals e_i.
2. Regress the absolute value $|e_i|$ on the explanatory variable that is associated with heteroscedasticity.
   - $|e_i| = \gamma_0 + \gamma_1 X_i + v_i$
   - $|e_i| = \gamma_0 + \gamma_1 \sqrt{X_i} + v_i$
   - $|e_i| = \gamma_0 + \gamma_1 \frac{1}{X_i} + v_i$
3. Select the equation with the highest R^2 and lowest standard errors to represent heteroscedasticity.
4. Perform a t-test on γ_1. If γ_1 is statistically significant, the null hypothesis of homoscedasticity can be rejected. We test if one of the independent variables is significantly related to the variance of our residuals.

Glejser Test Implemented on Your Own

```
y <- c(37, 48, 45, 36, 25, 55, 63)
x <- c(4.5, 6.5, 3.5, 3, 2.5, 8.5, 7.5)
xinv <- 1/x
lm1 <- lm(y ~ x)
res <- residuals(lm1)
gle1 <- lm(abs(res)~ x)
summary(gle1)
gle2 <- lm(abs(res)~ (sqrt(x)))
summary(gle2)
gle3 <- lm(abs(res)~ xinv)
summary(gle3)

# None of the variables is significant, i.e. the hypothesis of
heteroscedasticity can be rejected.
# There is no relationship between the absolute value of residuals
and the explanatory variable x
```

White Test

- The White test assumes more complex relationships, and also models interaction terms. You do not have to choose a particular X and do not need normally distributed residuals.
- The test is also based on an auxiliary regression of e^2 on all the explanatory variables (X_j), their squares (X_j^2), and all their cross products

$$e^2 = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_1^2 + \beta_4 X_2^2 + \beta_5 X_1 X_2 + v$$

- With more than 2 independent variables, you need to analyze the product of each independent variable with each other independent variable.
- The test statistic is nR^2, where n is the sample size and R^2 is the unadjusted R-squared from the auxiliary OLS regression.
  - The statistic has an asymptotic chi-square (χ^2) distribution with $d.f. = p$, where p is the no. of all explanatory variables in the auxiliary model.
  - H_0: All the variances ${\sigma_i}^2$ are equal (i.e., homoscedastic)
  - Reject H_0 if $\chi^2 > {\chi_{cr}}^2$

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   (i.e. exogeneity ($cov(\varepsilon, X) = 0$))

Applications of Linear Regressions to Time Series Data

Average hours worked per week by manufacturing workers:

Period	Hours	Period	Hours	Period	Hours	Period	Hours
1	37.2	11	36.9	21	35.6	31	35.7
2	37.0	12	36.7	22	35.2	32	35.5
3	37.4	13	36.7	23	34.8	33	35.6
4	37.5	14	36.5	24	35.3	34	36.3
5	37.7	15	36.3	25	35.6	35	36.5
6	37.7	16	35.9	26	35.6		
7	37.4	17	35.8	27	35.6		
8	37.2	18	35.9	28	35.9		
9	37.3	19	36.0	29	36.0		
10	37.2	20	35.7	30	35.7		

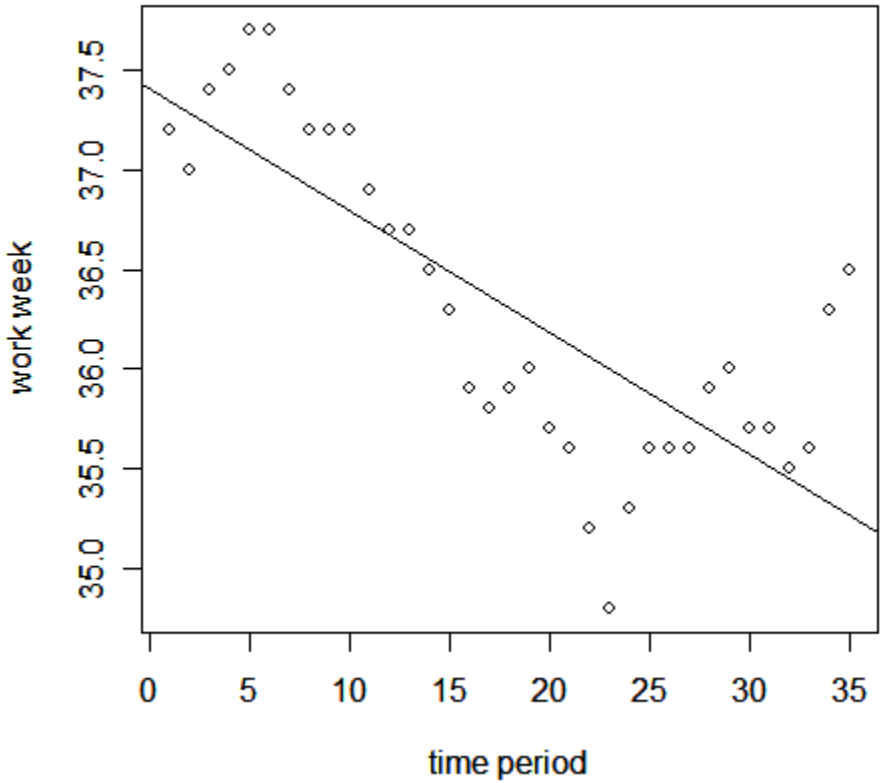
Forecasting Linear Trend using the Multiple Regression

$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$
, where: Y_i = data value for period i
$\hat{Y} = 37.416 - 0.0614 X_i$

```
Call:
lm(formula = y ~ x)

Residuals:
     Min       1Q   Median       3Q      Max
-1.20297 -0.28361  0.04711  0.30798  1.23048

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 37.40874    0.17502 213.744  < 2e-16 ***
x           -0.06112    0.00848  -7.208  2.9e-08 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.5067 on 33 degrees of freedom
Multiple R-Squared: 0.6116,     Adjusted R-squared: 0.5998
F-statistic: 51.95 on 1 and 33 DF,  p-value: 2.901e-08
```

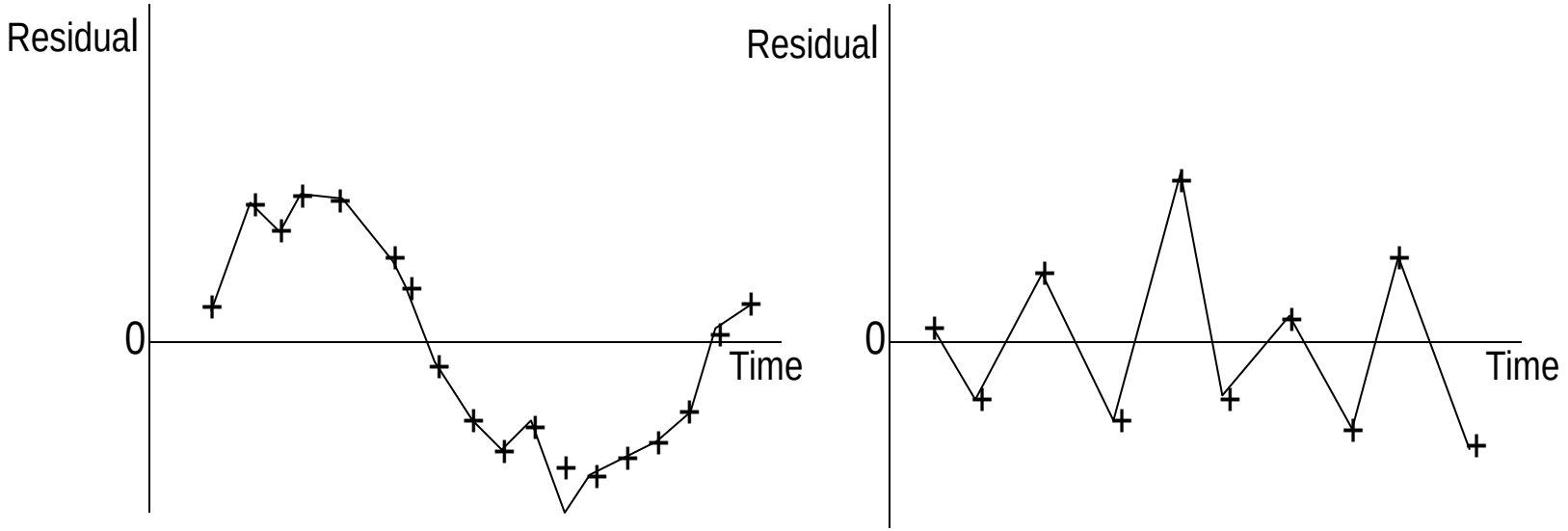
p-value for hypothesis ($\beta = 0$)

Hours Worked Data - A Linear Trend Line



Autocorrelation

- Examining the residuals over time, no pattern should be observed if the errors are independent
- Autocorrelation can be detected by graphing the residuals against time, or Durbin-Watson statistic



Autocorrelation

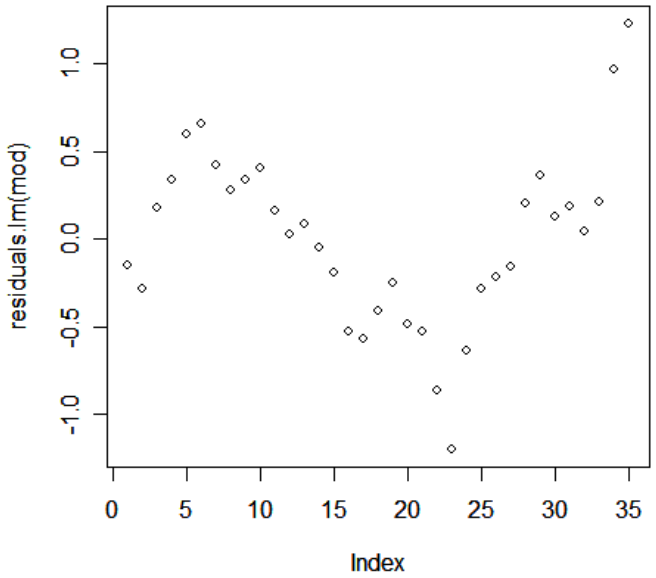
- Reasons leading to autocorrelation:
  - Omitted an important variable
  - Functional missfit
  - Measurement error in independent variable

- Use Durbin-Watson (DW) statistic to test for first order autocorrelation. DW takes values within [0, 4]. For no serial correlation, a value close to 2 (e.g., 1.5-2.5) is expected

$$DW = \frac{\sum_{i=2}^{n}(e_i - e_{i-1})^2}{\sum_{i=1}^{n} e_i^2}$$

- $DW = 2$ – no autocorrelation
- $DW = 0$ – perfect positive autocorrelation
- $DW = 4$ – perfect negative autocorrelation

Test for Autocorrelation in R



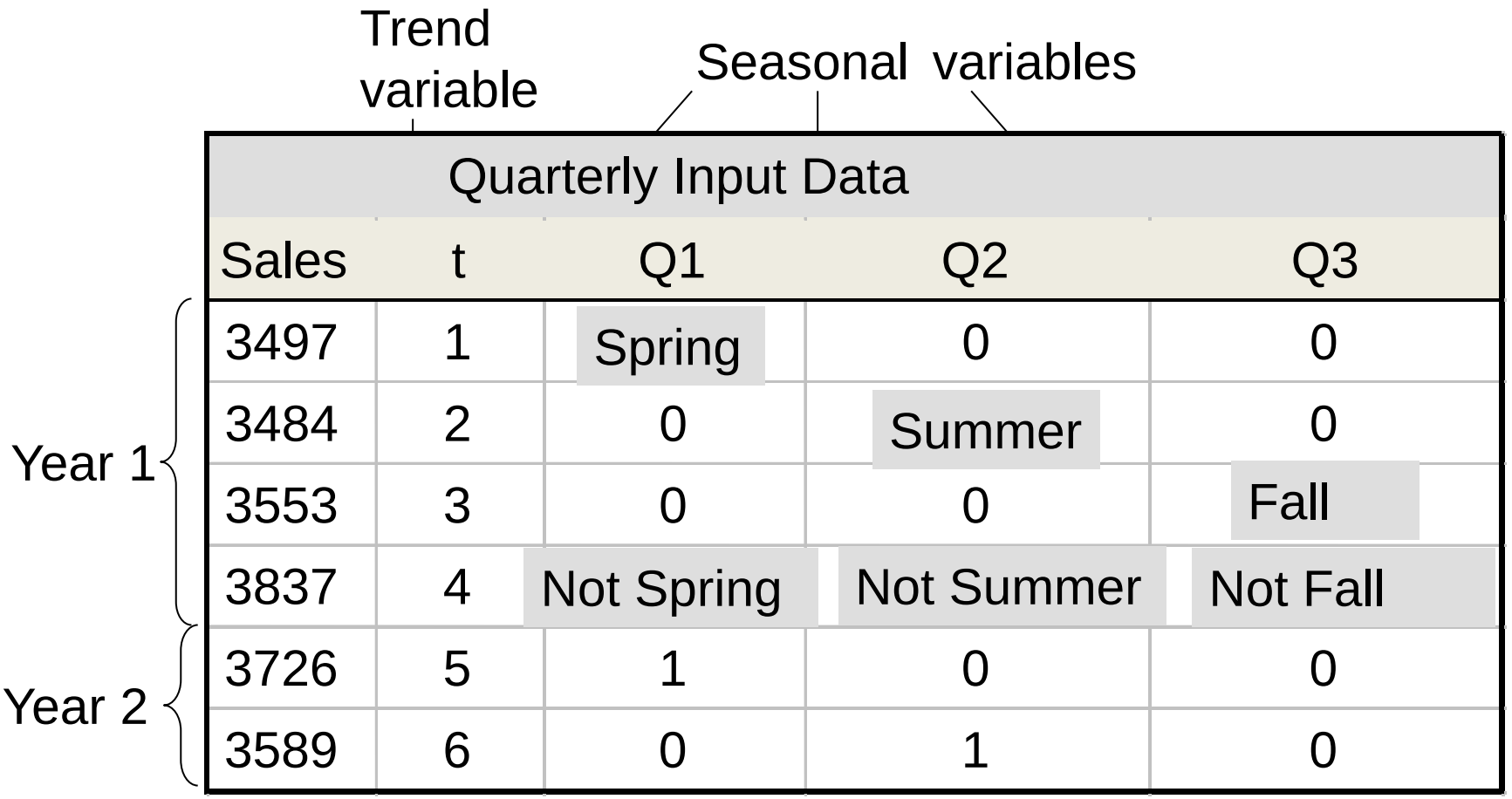

```
> plot(residuals.lm(mod))
> library(car)
> durbin.watson(mod)

lag Autocorrelation  D-W Statistic p-value
   1       0.7705505     0.2775895       0
 Alternative hypothesis: rho != 0
```

Modeling Seasonality

- A regression can estimate both the trend and additive seasonal indexes
  - Create dummy variables which indicate the season
  - Regress on time and the seasonal variables
  - Use the multiple regression model to forecast

- For any season, e.g. season 1, create a column with 1 for time periods which are season 1, and zero for other time periods (only season $-$ 1 dummy variables are required)

Dummy Variables



Modelling Seasonality

- The model which is fitted (assuming quarterly data) is

$$y = \beta_0 + \beta_1 t + \beta_2 Q_1 + \beta_3 Q_2 + \beta_4 Q_3$$

- Only 3 quarters are explicitly modelled. Otherwise:
  - $Q_1 = 1 - (Q_2 + Q_3 + Q_4)$, for all 4 quarters -> Multicollinearity

- Allows to test for seasonality

Gauss-Markov Assumptions in Detail

The OLS estimator is the best linear unbiased estimator (BLUE), iff

1) **Linearity**

   Linear relationship in parameters β

2) **No multicollinearity** of predictors

   No linear dependency between predictors

3) **Homoscedasticity**

   The residuals exhibit constant variance

4) **No autocorrelation**

   There is no correlation between the i^{th} and j^{th} residual terms

5) Expected value of the residual vector, given X, is 0 ($E(\varepsilon|X) = 0$)

   (i.e. exogeneity ($cov(\varepsilon, X) = 0$))

Endogeneity due to Omitted Variables

- Endogeneity means ($corr(\varepsilon_i, X_i) \neq 0$) => $E(\varepsilon_i | X_i) \neq 0$
  - Simple test: analyze the correlation of the residuals and an independent variable.
  - Reason for endogeneity: measurement errors, variables that affect each other, **omitted variables** (!)
- Omitted (aka. confounding) variables:
  - True model: $y_i = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + e_i$
  - Estimated model: $y_i = \beta_0 + \beta_1 x_1 + u_i$
  - Now $u_i = \beta_2 x_2 + e_i$. If x_1 and x_2 are correlated and x_2 affects y, this leads to endogeneity.

Why is this a problem?

Omitted Variables

- Consider the acceptance rates for the following groups of men and women who applied to college

Counts	**Accepted**	**Not accepted**	**Total**
Men	198	162	**360**
Women	88	112	**200**
Total	**286**	**274**	**560**

Percents	**Accepted**	**Not accepted**
Men	55%	45%
Women	44%	56%

- A higher percentage of men were accepted: Is there evidence of discrimination?

Omitted Variables

Consider the acceptance rates broken down by type of school.

Computer Science

Counts	Accepted	Not accepted	Total
Men	18	102	**120**
Women	24	96	**120**
Total	**42**	**198**	**240**

Percents	Accepted	Not accepted
Men	15%	85%
Women	**20%**	80%

School of Management

Counts	Accepted	Not accepted	Total
Men	180	60	**240**
Women	64	16	**80**
Total	**244**	**76**	**320**

Percents	Accepted	Not accepted
Men	75%	25%
Women	**80%**	20%

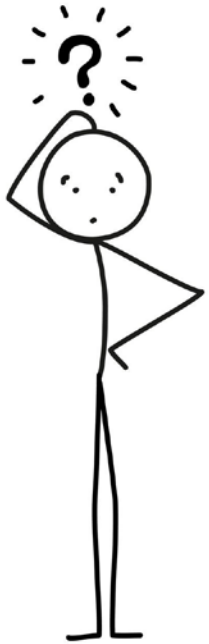
Explanations?

- Within each school a higher percentage of women were accepted!
  - There was no discrimination against women
  - Women rather applied to schools with lower acceptance rates
  - Men applied in schools with higher acceptance rates

- This is an example of **Simpson‘s paradox**:
  - When the **omitted (aka. confounding) variable** (Type of School) is ignored the data seem to suggest discrimination against women
  - However, when the type of school is considered, the association is reversed and suggests discrimination against men
  - see also https://en.wikipedia.org/wiki/Simpson%27s_paradox

- But we often do not have all relevant variables in the data ...?

Check out the Wikipedia entry for Simpson‘s paradoxon!



Omitted Variable Bias

- **Endogeneity** is given when an independent variable is correlated with the error term and the covariance is not null
  - A reason for endogeneity might be that relevant variables are omitted from the model
  - For example, enthusiasm or willingness to take risks of an individual describe unobserved heterogeneity. Can we control for these effects when estimating our regression model?

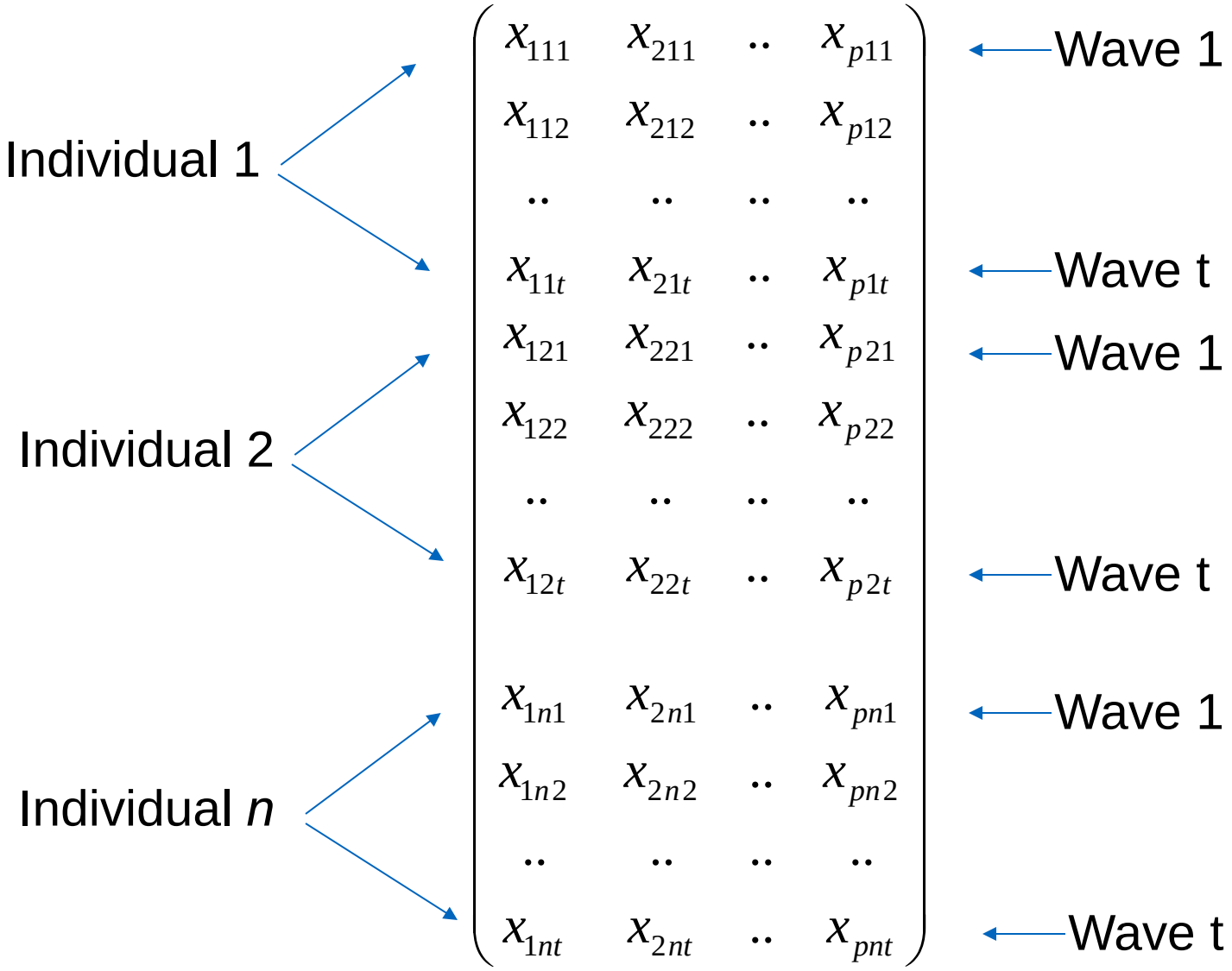
- Various techniques have been developed to address endogeneity in **panel data**
  - In panel data, the same individual is observed multiple times!
  - see https://en.wikipedia.org/wiki/Omitted-variable_bias
  - see https://en.wikipedia.org/wiki/Endogeneity_(econometrics)

Panel Data vs. Cross-Section Data

- Cross-section data collected in observational studies refers to data observing many subjects (such as individuals, firms or countries/regions) at the same point of time, or without regard to differences in time.
  - There might be omitted variables (aka. confounder) describing important characteristics of individuals, such as their risk attitudes.

- A panel data set, or longitudinal data set, is one where there are repeated observations on the same units, which makes it possible to overcome the an omitted variable bias
  - A balanced panel is one where every unit is surveyed in every time period
  - In an unbalanced panel some individuals have not been recorded in some time period

The Panel Data Structure



Treatment of Individual Effects with Panel Data

- *Fixed effect models* assume that the explanatory variable has a fixed or constant relationship with the response variable across all observations.
- *Random-effects models* assume that explanatory variables have fixed relationships with the response variable across all observations, but that these fixed effects may vary from one observation to another.

Example:

- You want to understand the probability of a vaccinated person falling sick due to Covid-19.
- You train a linear model with blood pressure, age, and the *fixed effects* gender (m/f), and country.
- One could also model the country as a *random effect*, which will incorporate the variabiliby due to picking a subset of all the countries (as only a limited number of countries were in the analysis).

- Picking the right model is not always easy, but the Hausman test can help you.

Modeling Fixed and Random Effects

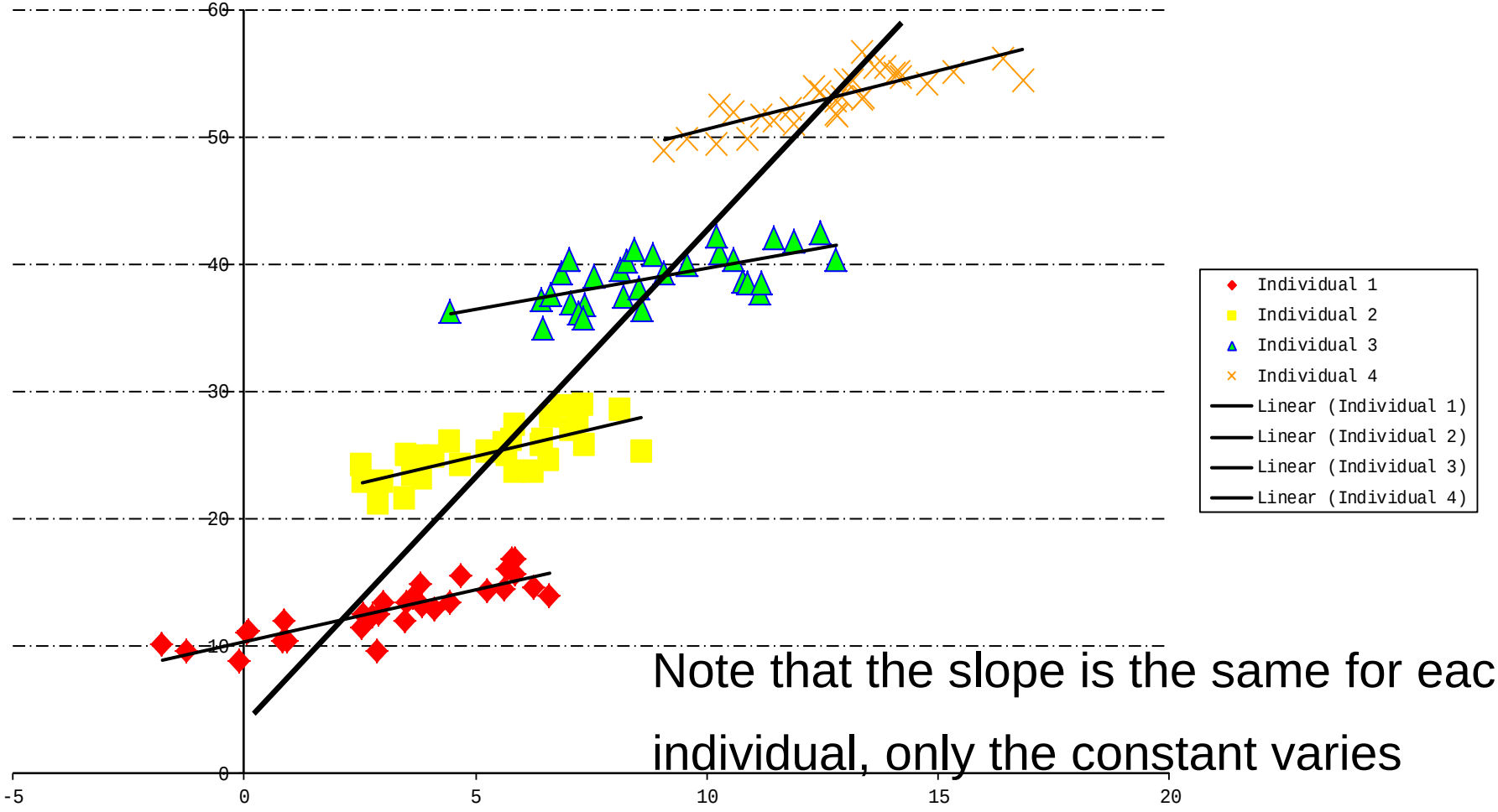
- Fixed effects
  - assume λ_i are constants (there is *endogeneity*)
    - Effects are *correlated to the other covariates**
      - see also https://en.wikipedia.org/wiki/Fixed_effects_model

- Random effects
  - assume λ_i are drawn independently from some probability distribution
    - Effects are *uncorrelated to the other covariates*
      - see also https://en.wikipedia.org/wiki/Random_effects_model

- Specific packages in R are available for fixed, random, and mixed effects models, which combine both (e.g., plm)

*We talk about a factor if it is a categorical variable and a covariate if it is continuous.

Fixed Effects Models

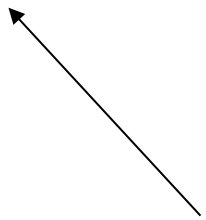


Note that the slope is the same for each individual, only the constant varies

The Fixed Effect Model

Treat λ_i (the individual-specific heterogeneity) as a constant for each individual.

$$y_{it} = \left(\beta_0 + \lambda_i\right) + \beta_1 x_{1it} + \beta_2 x_{2it} + \ldots + \beta_p x_{pit} + \varepsilon_{it}$$



λ is part of constant, but varies by individual i

Various estimators for fixed effect models: first differences, within, between, least squares dummy variable estimator

First-Differences Estimator

Eliminating unobserved heterogeneity by taking first differences

Original equation

$$y_{it} = \beta_0 + \lambda_i + \beta_1 x_{1it} + \beta_2 x_{2it} + \ldots + \beta_p x_{pit} + \varepsilon_{it}$$

Lag one period and subtract

$$\begin{aligned} y_{it} - y_{it-1} = {} & \beta_0 + \lambda_i + \beta_1 x_{1it} + \beta_2 x_{2it} + \ldots + \beta_p x_{pit} + \varepsilon_{it} \\ & - \beta_0 - \lambda_i - \beta_1 x_{1it-1} - \beta_2 x_{2it-1} - \ldots - \beta_p x_{pit-1} - \varepsilon_{it-1} \end{aligned}$$

Constant and individual effects eliminated

$$\begin{aligned} y_{it} - y_{it-1} = {} & \beta_1 \left(x_{1it} - x_{1it-1} \right) + \beta_2 \left(x_{2it} - x_{2it-1} \right) + \ldots \\ & + \beta_p \left(x_{pit} - x_{pit-1} \right) + \left(\varepsilon_{it} - \varepsilon_{it-1} \right) \end{aligned}$$

Transformed equation

$$\Delta y_{it} = \beta_1 \Delta x_{1it} + \beta_2 \Delta x_{2it} + \ldots + \beta_p \Delta x_{pit} + \Delta \varepsilon_{it}$$

How to Estimate a Model with Fixed Effects

- Least squares **dummy variable estimator**
  - uses a dummy variable for each individual (or firm, etc.), which we assume to have a fixed effect.

- **Within estimator**
  - Take deviations from individual means and apply least squares

    $y_{it} - \bar{y}_i = \beta_1(x_{1it} - \bar{x}_{1i}) + \cdots + \beta_p\left(x_{pit} - \bar{x}_{pi}\right) + (\varepsilon_{it} - \bar{\varepsilon}_i)$

    relies on variations within individuals

Random Effects Model

- The **fixed effect assumption** is that the individual specific effect is correlated with the independent variables ($cov(\lambda_i, x_{jit}) \neq 0$).

- The **random effects assumption** (in a random effects model) is that the individual specific effects are *uncorrelated* with the independent variables ($cov(\lambda_i, x_{jit}) = 0$, but λ_i might be correlated).

Original equation

$$y_{it} = \beta_0 + \beta_1 x_{1it} + \beta_2 x_{2it} + \ldots + \beta_p x_{pit} + \varepsilon_{it}$$

$$y_{it} = \beta_0 + \beta_1 x_{1it} + \beta_2 x_{2it} + \ldots + \beta_p x_{pit} + \lambda_i + u_{it}$$

λ_i is part of error term in random effects models

Fixed vs. Random Effects

- Deciding between fixed and random effects is not always easy.
- If the *individual effects are correlated with the other regressors* in the model, the fixed effect model is consistent and the random effects model is inconsistent.
- If the *individual effects are not correlated with the other regressors* in the model, *both* random and fixed effects are consistent and random effects is *efficient*.
- If the coefficients of a FE and RE model are basically the same, you should prefer the RE ones, because they have smaller standard errors.

The Hausman Test

- The Hausman test looks at the covariance matrix of the FE and RE estimators as well as the estimates and follows a chi-square (χ^2) distribution. It tests whether the errors are correlated with the regressors.

- Run a fixed effects model and save the estimates, then run a random model and save the estimates, then perform the test.

- If the p-value is significant (for example <0.05) then use fixed effects, if not use random effects.

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   (i.e. exogeneity ($cov(\varepsilon, X) = 0$))