Ergodicity of fractional stochastic differential equations and related problems

Maylis Varvenne

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INSTITUT de MATHEMATIQUES de TOULOUSE

PhD advisors : Laure Coutin & Fabien Panloup

Outline

1. Introduction
   - Fractional Brownian motion
   - Ergodicity of fractional SDEs and approximation of stationary regime

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Definition

Let $H \in (0,1)$. The d-dimensional *fractional Brownian motion* (fBm) with Hurst parameter H, denoted by $(B_t)_{t\geq 0}$, is a centered Gaussian process with covariance function given by :

$$\mathbb{E}[B_t^i B_s^j] = \frac{1}{2}\delta_{ij}\left[t^{2H} + s^{2H} - |t-s|^{2H}\right] \qquad \text{for all } t, s \geq 0.$$

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- Stationary increments :

$$\mathbb{E}\left[(B_t^i - B_s^i)(B_t^j - B_s^j)\right] = \delta_{ij}|t-s|^{2H}.$$

- Self-similarity :

$$\mathcal{L}((B_{ct})_{t\geq 0}) = \mathcal{L}(c^H (B_t)_{t\geq 0}) \quad \text{for all } c > 0.$$

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Remarks

▷ The fBm is neither a semimartingale nor a Markov process except for $H = 1/2$. In that case, B is the standard Brownian motion and has independent increments.

▷ Regularity: a.s. locally Hölder for all $\beta < H$.

Let $(W_t)_{t\in\mathbb{R}}$ be a standard Brownian motion.

Proposition (Mandelbrot Van Ness representation)

$$B_t := \int_{\mathbb{R}} (t-s)_+^{H-1/2} - (-s)_+^{H-1/2} \mathrm{d}W_s, \quad t \in \mathbb{R},$$

where $x_+ = \max(0, x)$.

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Proposition (Volterra representation)

$$B_t := \int_0^t K_H(t,s)\mathrm{d}W_s, \quad t \geqslant 0,$$

where K_H is the deterministic kernel given by

$$K_H(t,s) = c_H \left[\frac{t^{H-\frac{1}{2}}}{s^{H-\frac{1}{2}}}(t-s)^{H-\frac{1}{2}} - \left(H - \frac{1}{2}\right) \int_s^t \frac{u^{H-\frac{3}{2}}}{s^{H-\frac{1}{2}}}(u-s)^{H-\frac{1}{2}} \mathrm{d}u \right].$$

1. Introduction
   - Fractional Brownian motion
   - Ergodicity of fractional SDEs and approximation of stationary regime

SDE with additive fractional noise:

$$dY_t = b(Y_t)dt + \sigma dB_t. \qquad \textbf{(E)}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma \in \mathbb{M}_{d\times d}$

Hairer (2005):

- Homogeneous Markovian structure: $(Y_t, (W_s)_{s\le t})_{t\ge 0}$ (state space: $\mathbb{R}^d \times \mathcal{W}$).
- Existence of invariant distribution: $\mu_\star \in \mathcal{M}_1(\mathbb{R}^d \times \mathcal{W})$.
- Uniqueness of $\mu_\star$ and rate of convergence in total variation distance: $t^{-\alpha_H}$ with

$$\alpha_H = \begin{cases} H(1-2H) & \text{if } H \in (0, 1/4] \\ 1/8 & \text{if } H \in (1/4, 1) \setminus \left\{\frac{1}{2}\right\} \end{cases}.$$

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Multiplicative case:

- Fontbona, Panloup (2017): $H \in (1/2, 1)$.
- Deya, Panloup and Tindel (2019): $H \in (1/3, 1/2)$.

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Euler scheme for a fixed step $\gamma > 0$: $Z_0 = Y_0$,

$$Z_{(n+1)\gamma} = Z_{n\gamma} + \gamma b(Z_{n\gamma}) + \sigma(B_{(n+1)\gamma} - B_{n\gamma}). \qquad (\mathbf{E}_\gamma)$$

Theorem (Cohen, Panloup '11) (Cohen, Panloup, Tindel '14)

$$\lim_{\gamma\to 0} \lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} \delta_{Z_{k\gamma}} = \bar{\mu}_\star \quad a.s.$$

in the sense of weak convergence on $\mathcal{M}_1(\mathbb{R}^d)$.

2. Ergodicity of discrete-time stochastic dynamics with memory
    - Markovian structure
    - Assumptions and main result
    - Strategy of proof: coalescent coupling method

Setting

Let $X := (X_n)_{n\geqslant 0}$ be an $\mathbb{R}^d$-valued process such that

$$X_{n+1} = F(X_n, \Delta_{n+1})$$

where $(\Delta_n)_{n\in\mathbb{Z}}$ is an ergodic stationary Gaussian sequence with d-independent components and $F : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is (at least) continuous.

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Questions:

- Definition of invariant distribution in this a priori non-Markovian setting ?
- Existence and uniqueness of such measure ? Rate of convergence to equilibrium ?

Example : Euler scheme of a Gaussian SDE

Let $h > 0$ be fixed.

$$X_{n+1} = X_n + hb(X_n) + \sigma(X_n)\Delta_{n+1}$$

with $\Delta_{n+1} := Z_{(n+1)h} - Z_{nh}$ where (Z_t) is a Gaussian process with stationary increments. Then,

$$X_{n+1} = F_h(X_n, \Delta_{n+1})$$

and

$$\begin{aligned} F_h : \quad & \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \\ & (x, w) \mapsto x + hb(x) + \sigma(x)w. \end{aligned}$$

Example of noise process (Z_t)

Fractional Brownian motion with Hurst parameter $H \in (0, 1)$.

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Let $\mathcal{X} := \mathbb{R}^d$ be the state space and $\mathcal{W} := (\mathbb{R}^d)^{\mathbb{Z}^-}$ be the noise space.

Idea:

$$(X_n)_{n\in\mathbb{N}} \in \mathcal{X}^{\mathbb{N}} \dashrightarrow (X_n, (\Delta_{n+k})_{k\leqslant 0})_{n\in\mathbb{N}} \in (\mathcal{X} \times \mathcal{W})^{\mathbb{N}}$$

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Equivalent system:

$$(X_{n+1}, (\Delta_{n+1+k})_{k\leqslant 0}) = \varphi\left((X_n, (\Delta_{n+k})_{k\leqslant 0}), \Delta_{n+1}\right)$$

where

$$\begin{aligned} \varphi : (\mathcal{X} \times \mathcal{W}) \times \mathbb{R}^d &\to \mathcal{X} \times \mathcal{W} \\ ((x, w), \delta) &\mapsto (F(x, \delta), w \sqcup \delta). \end{aligned}$$

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Transition kernel: $\mathcal{Q} : \mathcal{X}\times\mathcal{W} \to \mathcal{M}_1(\mathcal{X}\times\mathcal{W})$

Definition

A measure $\mu \in \mathcal{M}_1(\mathcal{X}\times\mathcal{W})$ is said to be an **invariant distribution** for our system if it is invariant by $\mathcal{Q}$, i.e.

$$\mathcal{Q}\mu = \mu.$$

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Uniqueness: Let $\mathcal{S} : \mathcal{M}_1(\mathcal{X}\times\mathcal{W}) \to \mathcal{M}_1(\mathcal{X}^{\mathbb{N}})$ be the application which maps μ into $S\mu := \mathcal{L}((X_n^\mu)_{n\geqslant 0})$. Then

$$\mu \simeq \nu \Longleftrightarrow \mathcal{S}\mu = \mathcal{S}\nu \quad (\star)$$

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Moving average representation

Wold's decomposition theorem,

$$\forall n \in \mathbb{Z}, \quad \Delta_n = \sum_{k=0}^{+\infty} a_k \xi_{n-k}$$

with

$$\begin{cases} (a_k)_{k\geqslant 0} \in \mathbb{R}^{\mathbb{N}} \text{ such that } a_0 \neq 0 \text{ and } \sum_{k=0}^{+\infty} a_k^2 < +\infty \\ (\xi_k)_{k\in\mathbb{Z}} \text{ an i.i.d sequence such that } \xi_1 \sim \mathcal{N}(0, I_d). \end{cases}$$

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Remarks

- ▷ Without loss of generality, we assume that $a_0 = 1$. If $a_0 \neq 1$, we can come back to this case by setting $\tilde{\Delta}_n = \sum_{k=0}^{+\infty} \tilde{a}_k \xi_{n-k}$ with $\tilde{a}_k = a_k / a_0$.
- ▷ The memory induced by the noise is quantified by $(a_k)_{k\geqslant 0}$.

Preliminary tool : a Toeplitz type operator

Definition

Let $\mathbf{T_a}$ be defined on $\ell_a(\mathbb{Z}^-, \mathbb{R}^d) := \left\{ w \in (\mathbb{R}^d)^{\mathbb{Z}^-} \;\middle|\; \forall k \geqslant 0, \; \left| \sum_{l=0}^{+\infty} a_l w_{-k-l} \right| < +\infty \right\}$ by

$$\forall w \in \ell_a(\mathbb{Z}^-, \mathbb{R}^d), \quad \mathbf{T_a}(w) = \left(\sum_{l=0}^{+\infty} a_l w_{-k-l} \right)_{k \geqslant 0}.$$

Remark : This operator links $(\Delta_n)_{n\in\mathbb{Z}}$ to the underlying noise process $(\xi_n)_{n\in\mathbb{Z}}$.

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Remark : This operator links $(\Delta_n)_{n\in\mathbb{Z}}$ to the underlying noise process $(\xi_n)_{n\in\mathbb{Z}}$.

Proposition

Let $\mathbf{T_b}$ be defined on $\ell_b(\mathbb{Z}^-,\mathbb{R}^d)$ with the following sequence $(b_k)_{k\geqslant 0}$

$$b_0 = \frac{1}{a_0} \quad \text{and} \quad \forall k \geqslant 1,\ \ b_k = -\frac{1}{a_0}\sum_{l=1}^{k} a_l b_{k-l}.$$

Then, $\mathbf{T_b} = \mathbf{T_a}^{-1}$.

$(\mathbf{H_{poly}})$: The following conditions are satisfied,

- There exist $\rho, \beta > 0$ and $C_\rho, C_\beta > 0$ such that

$$\forall k \geqslant 0, \quad |a_k| \leqslant C_\rho (k+1)^{-\rho} \quad \text{and} \quad \forall k \geqslant 0, \quad |b_k| \leqslant C_\beta (k+1)^{-\beta}.$$

- There exist $\kappa \geqslant \rho + 1$ and $C_\kappa > 0$ such that

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Controlability and Lyapunov type assumptions on F.

Example : Euler scheme with step $h > 0$.

($\mathbf{H_{b,\sigma}}$): $b : \mathbb{R}^d \to \mathbb{R}^d$ is continuous, $\sigma : \mathbb{R}^d \to \mathbb{M}_{d\times d}$ is bounded, continuous and $\sigma^{-1} : x \mapsto \sigma(x)^{-1}$ is well defined and continuous. Moreover,

- $\exists C > 0$ such that $\forall x \in \mathcal{X}, \quad |b(x)| \leqslant C(1 + |x|)$
- $\exists \tilde{\beta} \in \mathbb{R}$ and $\tilde{\alpha} > 0$ such that $\forall x \in \mathcal{X}, \quad \langle x, b(x) \rangle \leqslant \tilde{\beta} - \tilde{\alpha}|x|^2$.

Theorem

Assume the two hyposthesis on the function F. Then,

(i) *There exists an invariant distribution $\mu_\star$.*

(ii) *Assume that $(\mathbf{H_{poly}})$ is true with $\rho, \beta > 1/2$ and $\rho + \beta > 3/2$. Then, uniqueness holds for $\mu_\star$. Moreover, for all initial distribution μ_0 such that $\int_{\mathcal{X}} V(x)\Pi_{\mathcal{X}}^* \mu_0(dx) < +\infty$ and for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that*

$$\|\mathcal{L}((X_{n+k}^{\mu_0})_{k\geqslant 0}) - \mathcal{S}\mu_\star\|_{TV} \leqslant C_\varepsilon \; n^{-(v(\beta,\rho)-\varepsilon)}.$$

where v is defined by

$$v(\beta,\rho) = \sup_{\alpha\in\left(\frac{1}{2}\vee\left(\frac{3}{2}-\beta\right),\rho\right)} \min\{1, 2(\rho-\alpha)\}(\min\{\alpha,\ \beta,\ \alpha+\beta-1\} - 1/2).$$

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Example: dynamics driven by fBm (with $H \in (0, 1/2)$)

Convergence to equilibrium : $n^{-(v_H-\varepsilon)}$ with

$$v_H = \begin{cases} H(1-2H) & \text{if} \quad H \in (0, 1/4] \\ 1/8 & \text{if} \quad H \in (1/4, 1/2) \end{cases}$$

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Scheme of coupling (discrete time setting) : We consider (X^1, X^2) the solution of the system :

$$\begin{cases} X^1_{n+1} = F(X^1_n, \Delta^1_{n+1}) \\ X^2_{n+1} = F(X^2_n, \Delta^2_{n+1}) \end{cases}$$

with initial conditions $(X^1_0, (\Delta^1_k)_{k\leqslant 0}) \sim \mu_0$ and $(X^2_0, (\Delta^2_k)_{k\leqslant 0}) \sim \mu_\star$.

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We have

$$\|\mathcal{L}((X^1_{n+k})_{k\geqslant 0}) - \mathcal{S}\mu_\star\|_{TV} \leqslant \mathbb{P}(\tau_\infty > n).$$

where $\tau_\infty := \inf\{n \geqslant 0 \mid X^1_k = X^2_k, \ \forall k \geqslant n\}$.

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We define the sequence of r.v. $(g_n)_{n\in\mathbb{Z}}$ by

$$\forall n \in \mathbb{Z}, \quad \xi^1_{n+1} = \xi^2_{n+1} + g_n, \qquad \text{hence} \qquad g_n = 0 \quad \forall n < 0.$$

Steps of the coupling procedure

▷ **Step 1** : Try to stick the positions at a given time with a "controlled cost".

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- ▷ **Step 2** : Try to keep the paths fastened together (specific to non-Markov process).
- ▷ **Step 3** : If Step 2 fails, impose $g_n = 0$ and wait long enough in order to allow Step 1 to be realized with a “controlled cost” and with a positive probability.

$\boxed{\text{Step 3}}$

Step 3

Compact return

$$|X^i_\tau| \leq K$$
$$\left|\sum_{k=1}^{+\infty} a_k \xi^i_{\tau+1-k}\right| \leq K \text{ for } i=1,2.$$

Step 1 (mainly)

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Memory decrease

$$\left|\sum_{k=n+1}^{+\infty} a_k g_{\tau+n-k}\right| \le (n+1)^{-\alpha} \qquad \forall n \ge 0$$

Step 2

Euler scheme (step 1)

At a given time $(\tau+1)$, we want to build $(\xi^1_{\tau+1}, \xi^2_{\tau+1})$ in order to get $X^1_{\tau+1} = X^2_{\tau+1}$, i.e.

$$X^1_\tau + hb(X^1_\tau) + \sigma(X^1_\tau)\sum_{k=0}^{+\infty} a_k \xi^1_{\tau+1-k} = X^2_\tau + hb(X^2_\tau) + \sigma(X^2_\tau)\sum_{k=0}^{+\infty} a_k \xi^2_{\tau+1-k}$$

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$$\iff \quad \xi^2_{\tau+1} = \Lambda_{\mathbf{X}}(\xi^1_{\tau+1}) \ \text{ where } \ \mathbf{X} = \left(X^1_\tau, X^2_\tau, \sum_{k=1}^{+\infty} a_k \xi^1_{\tau+1-k}, \sum_{k=1}^{+\infty} a_k \xi^2_{\tau+1-k}\right)$$

Euler scheme (step 1)

At a given time $(\tau+1)$, we want to build $(\xi^1_{\tau+1}, \xi^2_{\tau+1})$ in order to get $X^1_{\tau+1} = X^2_{\tau+1}$, i.e.

$$X^1_\tau + hb(X^1_\tau) + \sigma(X^1_\tau)\sum_{k=0}^{+\infty} a_k \xi^1_{\tau+1-k} = X^2_\tau + hb(X^2_\tau) + \sigma(X^2_\tau)\sum_{k=0}^{+\infty} a_k \xi^2_{\tau+1-k}$$

$$\iff \quad \xi^2_{\tau+1} = \Lambda_{\mathbf{X}}(\xi^1_{\tau+1}) \;\text{ where }\; \mathbf{X} = \left(X^1_\tau, X^2_\tau, \sum_{k=1}^{+\infty} a_k \xi^1_{\tau+1-k}, \sum_{k=1}^{+\infty} a_k \xi^2_{\tau+1-k} \right)$$

Coupling Lemma to build $(\xi^1_{\tau+1}, \xi^2_{\tau+1})$ such that:

- $\xi^1_{\tau+1} \sim \mathcal{N}(0, I_d)$ and $\xi^2_{\tau+1} \sim \mathcal{N}(0, I_d)$,
- ensure $\mathbb{P}\left(\xi^2_{\tau+1} = \Lambda_{\mathbf{X}}(\xi^1_{\tau+1})\right) \geq \delta_K > 0$,
- $|\xi^1_{\tau+1} - \xi^2_{\tau+1}| \leqslant M_K$ a.s.

Euler scheme (step 2)

Keep the paths fastened : $X^1_{n+1} = X^2_{n+1} \;\; \forall n \geqslant \tau + 1$, i.e.

$$X_n^1 + hb(X_n^1) + \sigma(X_n^1) \sum_{k=0}^{+\infty} a_k \xi^1_{n+1-k} = X_n^1 + hb(X_n^1) + \sigma(X_n^1) \sum_{k=0}^{+\infty} a_k \xi^2_{n+1-k}$$

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$$\iff \quad \forall n \geqslant \tau + 1, \quad \xi^1_{n+1} - \xi^2_{n+1} = g_n^{(s)} = -\sum_{k=1}^{+\infty} a_l g_{n-k}$$

$$\iff \quad \forall n \geqslant 1, \quad g^{(s)}_{\tau+n} = -\sum_{k=1}^{n} a_k g^{(s)}_{\tau+n-k} - \sum_{k=n+1}^{+\infty} a_k g_{\tau+n-k}. \tag{3.1}$$

Euler scheme (step 2)

Keep the paths fastened : $X^1_{n+1} = X^2_{n+1} \ \ \forall n \geqslant \tau + 1$, i.e.

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Coupling Lemma to build $((\xi^1_{\tau+n+1}, \xi^2_{\tau+n+1}))_{n\in[\![1,T]\!]}$ **such that**:

- ensure (3.1) with controlled positive probability,
- $\left\|(g_{\tau+n})_{n\in[\![1,T]\!]}\right\|$ a.s. controlled.

Aim : Determine for which value of $p > 0$ we can control $\mathbb{E}[\tau_\infty^p]$ since:

$$\mathbb{P}(\tau_\infty > n) \leqslant \frac{\mathbb{E}[\tau_\infty^p]}{n^p}$$

where $\tau_\infty := \inf\{n \geqslant 0 \mid X_k^1 = X_k^2,\ \forall k \geqslant n\}$.

3 Drift estimation for fractional SDEs
- Model and contruction of the estimators
- Consistency results
- Bound on the quadratic error

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   - Model and contruction of the estimators
   - Consistency results
   - Bound on the quadratic error

Let Y be an $\mathbb{R}^d$-valued process such that $Y_0 = y_0$ and

$$dY_t = b_{\vartheta_0}(Y_t)\,dt + \sigma dB_t \qquad (\mathbf{E}_{\vartheta_\mathbf{0}})$$

where $\sigma \in \mathbb{M}_{d\times d}$ is an invertible matrix, $\vartheta_0 \in \Theta$ is the unknown parameter and $\{b_\vartheta(.) \mid \vartheta \in \Theta\}$ is a known family of functions.

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$(\mathbf{C_w})$: We have $b \in \mathcal{C}^{1,1}(\mathbb{R}^d \times \Theta; \mathbb{R}^d)$ and:

(i) $\forall x, y \in \mathbb{R}^d$, $\forall \vartheta \in \Theta$,

$$\langle b_\vartheta(x) - b_\vartheta(y), x - y\rangle \leq \beta - \alpha|x-y|^2 \quad \text{and} \quad |b_\vartheta(x) - b_\vartheta(y)| \leq L|x-y|$$

(ii) $\forall x \in \mathbb{R}^d$, $\forall \vartheta \in \Theta$,

$$|\partial_\vartheta b_\vartheta(x)| \leq C\left(1 + |x|^r\right).$$

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Observations : $\{Y_{t_k} \mid 0 \leq k < n\}$ and $t_{k+1} - t_k = \kappa > 0$.

$$
\begin{aligned}
(\mathbf{C_w}) \quad & \Rightarrow (\mathbf{E}_\vartheta) \text{ admits a unique invariant distribution} \\
& \Rightarrow \nu_\vartheta \in \mathcal{M}_1(\mathbb{R}^d) \text{ (marginal invariant distribution).}
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<u>**Identifiability assumptions**</u>: denote by d a distance on $\mathcal{M}_1(\mathbb{R}^d)$.

$(\mathbf{I_w})$: We have: $\quad d(\nu_\vartheta, \nu_{\vartheta_0}) = 0 \quad \Leftrightarrow \quad \vartheta = \vartheta_0.$

$(\mathbf{I_s})$: There exists a constant $C > 0$ and a parameter $\varsigma \in (0, 1]$ such that:

$$\forall \vartheta \in \Theta, \quad d(\nu_\vartheta, \nu_{\vartheta_0}) \geq C|\vartheta - \vartheta_0|^\varsigma.$$

Approximation of ν_{ϑ_0}: $\frac{1}{n}\sum_{k=0}^{n-1}\delta_{Y_{t_k}} \Rightarrow \nu_{\vartheta_0}$.

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Approximation of ν_{ϑ}: Euler scheme associated to $(\mathbf{E}_\vartheta)$: $Z_0^\vartheta = y_0$

$$\forall k \geq 0, \quad Z_{s_{k+1}}^\vartheta = Z_{s_k}^\vartheta + (s_{k+1} - s_k)b_\vartheta(Z_{s_k}^\vartheta) + \sigma\,(B_{s_{k+1}} - B_{s_k}).$$

where $s_0 = 0$ and (s_k) is an increasing sequence such that $\lim_{k\to+\infty} s_k = +\infty$.

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$$\forall k \geq 0, \quad Z^\vartheta_{s_{k+1}} = Z^\vartheta_{s_k} + (s_{k+1} - s_k) b_\vartheta(Z^\vartheta_{s_k}) + \sigma\,(B_{s_{k+1}} - B_{s_k}).$$

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- Constant step Euler scheme $Z^{\vartheta,\gamma}$: $s_k = k\gamma$.
- Decreasing step Euler scheme Z^ϑ: $\gamma_k := s_k - s_{k-1}$ is a decreasing sequence and $\lim_{k\to+\infty}\gamma_k = 0$.

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Estimators:

$$\hat{\vartheta}_{N,n,\gamma} = \operatorname*{argmin}_{\vartheta\in\Theta} d\left(\frac{1}{n}\sum_{k=0}^{n-1}\delta_{Y_{t_k}},\ \frac{1}{N}\sum_{k=0}^{N-1}\delta_{Z^{\vartheta,\gamma}_{k\gamma}}\right),$$

$$\hat{\vartheta}_{N,n} = \operatorname*{argmin}_{\vartheta\in\Theta} d\left(\frac{1}{n}\sum_{k=0}^{n-1}\delta_{Y_{t_k}},\ \frac{1}{s_N}\sum_{k=0}^{N-1}\gamma_{k+1}\delta_{Z^{\vartheta}_{s_k}}\right).$$

where $d \in \mathcal{D}_p := \{\text{distances } d \text{ on } \mathcal{M}_1(\mathbb{R}^d);\ \exists\, c > 0, \forall \nu,\mu,\ d(\nu,\mu) \leq c\mathcal{W}_p(\nu,\mu)\}$

3. Drift estimation for fractional SDEs
    - Model and contruction of the estimators
    - Consistency results
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Theorem (Panloup, Tindel, V. '19)

Assume $(\mathbf{H_0})$, $(\mathbf{C_s})$ and $(\mathbf{I_w})$. Then, *a.s.*

$$\lim_{\gamma\to 0}\lim_{N,n\to+\infty}\hat{\vartheta}_{N,n,\gamma} = \vartheta_0 \quad \text{and} \quad \lim_{N,n\to+\infty}\hat{\vartheta}_{N,n} = \vartheta_0.$$

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Remarks

- Proof: uniform convergence of the occupation measure to the marginal invariant distribution.
- Under $(\mathbf{C_w})$, we need to discretize Θ to keep the uniform convergence.

3. Drift estimation for fractional SDEs
   - Model and contruction of the estimators
   - Consistency results
   - Bound on the quadratic error

($\mathbf{I_s}$): There exists a constant $C > 0$ and a parameter $\varsigma \in (0,1]$ such that:

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Let $\mu, \nu \in \mathcal{M}_1(\mathbb{R}^d)$. Let $d_{CF,p}$ and d_s be defined in the following way:

(*i*) let $g_p(\xi) := c_p(1 + |\xi|^2)^{-p}$ and $c_p := \left(\int_{\mathbb{R}^d}(1 + |\xi|^2)^{-p} d\xi\right)^{-1}$,

$$d_{CF,p}(\mu, \nu) := \left(\int_{\mathbb{R}^d} \left(\mu(e^{i\langle\xi,\cdot\rangle}) - \nu(e^{i\langle\xi,\cdot\rangle})\right)^2 g_p(\xi) d\xi\right)^{1/2}.$$

(*ii*) Let $\{f_i \ ; \ i \geq 1\}$ be a family of $\mathcal{C}^1_b$, supposed to be dense in the space $\mathcal{C}^0_b$.

$$d_s(\mu, \nu) := \sum_{i=0}^{+\infty} 2^{-i}(|\mu(f_i) - \nu(f_i)| \wedge 1).$$

Quadratic error (for $d_{CF,p}$ and d_s)

Theorem (Panloup, Tindel, V. '19)

Assume $(\mathbf{H_0})$, $(\mathbf{C_s})$ and $(\mathbf{I_s})$ for some given $\varsigma \in (0,1]$ hold true.

$$\mathbb{E}\left[|\hat{\vartheta}_{N,n,\gamma} - \vartheta_0|^2\right] \leq C_q \left(n^{-\frac{q}{2}(2-(2H\vee 1))} + \gamma^{qH} + (N\gamma)^{-\tilde{\eta}}\right)$$

with $q = 2/\varsigma$ and $\tilde{\eta} := \frac{q^2}{2(q+d)}(2-(2H \vee 1))$.

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$$E\left[|\hat{\vartheta}_{N,n,\gamma} - \vartheta_0|^2\right]$$

$$(\mathbf{I_s}) \qquad \Rightarrow \mathbb{E}\left[d\left(\nu_{\hat{\vartheta}_{N,n,\gamma}}, \nu_{\vartheta_0}\right)^q\right]$$

Quadratic error (for $d_{CF,p}$ and d_s)

Theorem (Panloup, Tindel, V. '19)

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Triangle inequality + def of $\hat{\vartheta}$ $$\Rightarrow d\left(\nu_{\vartheta_0}, \frac{1}{n}\sum_{k=0}^{n-1}\delta_{Y_{t_k}}\right), \quad d\left(\nu_{\vartheta}, \frac{1}{T}\int_0^T \delta_{Y_t^{\vartheta}}\, dt\right)$$

Quadratic error (for $d_{CF,p}$ and d_s)

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$\Rightarrow$ **Concentration inequalities.**

Let $Y := (Y_t)_{t \geqslant 0}$ be an $\mathbb{R}^d$-valued process such that

$$Y_t = x + \int_0^t b(Y_s)\mathrm{d}s + \sigma B_t.$$

where $x \in \mathbb{R}^d$, $b : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma \in \mathcal{M}_d(\mathbb{R})$ and B is a d-dimensional fractional Brownian motion with Hurst parameter $H \in (0, 1)$.

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Aim : For all function $f : \mathbb{R}^d \to \mathbb{R}$ Lipschitz, we wish to control :

$$* \ \mathbb{P}\left(\frac{1}{n}\sum_{k=1}^{n}(f(Y_{k\Delta}) - E[f(Y_{k\Delta})]) > r\right) \text{ for a fixed } \Delta > 0,$$

$$* \ \mathbb{P}\left(\frac{1}{T}\int_0^T (f(Y_t) - \mathbb{E}[f(Y_t)])\mathrm{d}t > r\right)$$

with respect to n and T.

4. Concentration inequalities for fractional SDEs
   - Transportation inequalities
   - Main results
   - Sketch of proof

Let (E, d) be a metric space. Given $p \geqslant 1$ and two probability measures μ and ν on E, the Wasserstein distance is defined by

$$\mathcal{W}_p(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \left(\int_{E \times E} d(x, y)^p \mathrm{d}\pi(x, y) \right)^{1/p},$$

where $\Pi(\mu, \nu) := \{\pi \in \mathcal{M}_1(E \times E)$ telles que $\pi(., E) = \mu(.)$ et $\pi(E, .) = \nu(.)\}$.

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The relative entropy of ν with respect to μ is defined by

$$\mathbf{H}(\nu|\mu) = \begin{cases} \int \log\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right) \mathrm{d}\nu, & \text{if } \nu \ll \mu, \\ +\infty & \text{otherwise.} \end{cases}$$

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The relative entropy of ν with respect to μ is defined by

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Then, we say that μ satisfies an L^p-transportation inequality with constant $C \geqslant 0$ (denoted by $\mu \in T_p(C)$) if for any probability measure ν,

$$\mathcal{W}_p(\mu,\nu) \leqslant \sqrt{2C\mathbf{H}(\nu|\mu)}.$$

Theorem (Bobkov and Götze '99)

Let (E, d) be a metric space and μ a probability measure on E. Then, $\mu \in T_1(C)$ if and only if for any μ-integrable Lipschitz function $F : (E, d) \to \mathbb{R}$ we have for all $\lambda \in \mathbb{R}$,

$$\mathbb{E}\left[e^{\lambda(F(X)-\mathbb{E}[F(X)])}\right] \leqslant \exp\left(\frac{\lambda^2}{2}C\|F\|_{\mathrm{Lip}}^2\right)$$

with $\mathcal{L}(X) = \mu$. In that case,

$$\mathbb{P}\left(F(X) - \mathbb{E}[F(X)] > r\right) \leqslant \exp\left(-\frac{r^2}{2C\|F\|_{\mathrm{Lip}}^2}\right), \quad \forall r > 0.$$

Results for our fractional SDE : $\mu = \mathcal{L}((Y_t)_{t\in[0,T]})$

- For $H > 1/2$, Saussereau shows that $\mu \in T_2(C_T)$ for two metrics on $\mathcal{C}([0,T],\mathbb{R}^d)$:

$$d_2(\gamma_1,\gamma_2) = \left(\int_0^T |\gamma_1(t) - \gamma_2(t)|^2 dt\right)^{1/2} \quad \text{and} \quad d_\infty(\gamma_1,\gamma_2) = \sup_{t\in[0,T]} |\gamma_1(t) - \gamma_2(t)|.$$

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Theorem (Saussereau '12)

Let $H > 1/2$. There exists $C > 0$ such that for all Lipschitz function $f : (\mathbb{R}^d, |\cdot|) \to (\mathbb{R}, |\cdot|)$ and for all $r \geqslant 0$,

$$\mathbb{P}\left(\frac{1}{T}\int_0^T (f(Y_t) - \mathbb{E}[f(Y_t)])\mathrm{d}t > r\right) \leqslant \exp\left(-\frac{r^2 T^{2-2H}}{4C\|f\|_{\mathrm{Lip}}^2}\right).$$

4 Concentration inequalities for fractional SDEs
- Transportation inequalities
- Main results
- Sketch of proof

Hypothesis : We assume that there exist $\alpha, L > 0$ such that: for all $x, y \in \mathbb{R}^d$,

$$\langle b(x) - b(y), x - y \rangle \leq -\alpha |x-y|^2 \quad \text{et} \quad |b(x) - b(y)| \leq L|x-y|.$$

Theorem

Let $H \in (0,1)$ and $\Delta > 0$. Let $n \in \mathbb{N}^*$ and $T \geqslant 1$. Then,

(*i*) $\mathcal{L}((Y_{k\Delta})_{1 \leq k \leq n}) \in T_1(2C_{H,\Delta} n^{2H \vee 1})$ for the metric $d_n(x,y) := \sum_{k=1}^n |x_k - y_k|$,

(*ii*) $\mathcal{L}((Y_t)_{t \in [0,T]}) \in T_1(2\tilde{C}_H T^{2H \vee 1})$ for the metric $d_T(x,y) := \int_0^T |x_t - y_t| dt$.

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Consequently,

(*i*) for all Lipschitz function $f : \left(\mathbb{R}^d, |\cdot|\right) \to (\mathbb{R}, |\cdot|)$ and for all $r \geqslant 0$,

$$\mathbb{P}\left(\frac{1}{n}\sum_{k=1}^n (f(Y_{k\Delta}) - E[f(Y_{k\Delta})]) > r\right) \leqslant \exp\left(-\frac{r^2 n^{2-(2H\vee 1)}}{4C_{H,\Delta}\|f\|_{\mathrm{Lip}}^2}\right).$$

(*ii*) for all Lipschitz function $f : \left(\mathbb{R}^d, |\cdot|\right) \to (\mathbb{R}, |\cdot|)$ and for all $r \geqslant 0$,

$$\mathbb{P}\left(\frac{1}{T}\int_0^T (f(Y_t) - \mathbb{E}[f(Y_t)])\mathrm{d}t > r\right) \leqslant \exp\left(-\frac{r^2 T^{2-(2H\vee 1)}}{4\tilde{C}_H\|f\|_{\mathrm{Lip}}^2}\right).$$

We set

$$F_Y := \frac{1}{n}\sum_{k=1}^{n} f(Y_{k\Delta}).$$

Assume $\Delta = 1$ for the sake of simplicity. We denote by $(\mathcal{F}_t)_{t\geq 0}$ the natural filtration associated to the underlying standard Brownian motion W (see the Voltera representation). We set $M_k = \mathbb{E}[F_Y|\mathcal{F}_k]$.
Then

$$F_Y - \mathbb{E}[F_Y] = M_n = \sum_{k=1}^{n} M_k - M_{k-1}.$$

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We show the existence of a deterministic sequence (u_k) such that

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$$\mathbb{E}\left[e^{\lambda(M_k - M_{k-1})} \middle| \mathcal{F}_{k-1}\right] \leq e^{\lambda^2 u_k} \implies \quad \mathbb{E}\left[e^{\lambda M_n}\right] \leq e^{\lambda^2 \sum_{k=1}^{n} u_k}$$

Since $\mathbb{E}[M_k - M_{k-1}|\mathcal{F}_{k-1}] = 0$, we are thus reduced to estimate

$$\mathbb{E}[|M_k - M_{k-1}|^p|\mathcal{F}_{k-1}] \quad , \forall p \geq 2.$$

Recall that :

$$Y_t = x + \int_0^t b(Y_s)\mathrm{d}s + \sigma \int_0^t K_H(t,s)\mathrm{d}W_s.$$

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We can see Y_t as a functional $\Phi : \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{R}^d$ depending on the time, the initial condition x and the Brownian motion :

$$\forall t \geqslant 0, \quad Y_t := \Phi_t(x, (W_s)_{s\in[0,t]}).$$

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$$\forall t \geqslant 0, \quad Y_t := \Phi_t(x, (W_s)_{s\in[0,t]}).$$

Let $k \geq 1$,

$$\begin{aligned}
&|M_k - M_{k-1}| \\
&= |\mathbb{E}[F_Y|\mathcal{F}_k] - \mathbb{E}[F_Y|\mathcal{F}_{k-1}]| = \left| \frac{1}{n}\sum_{t=k}^{n} \mathbb{E}[f(Y_t)|\mathcal{F}_k] - \mathbb{E}[f(Y_t)|\mathcal{F}_{k-1}] \right| \\
&\leqslant \frac{\|f\|_{\mathrm{Lip}}}{n} \int_\Omega \sum_{t=k}^{n} \left| \Phi_t\left(x, W_{[0,k]} \sqcup \tilde{w}_{[k,t]}\right) - \Phi_t\left(x, W_{[0,k-1]} \sqcup \tilde{w}_{[k-1,t]}\right) \right| \mathbb{P}_W(\mathrm{d}\tilde{w}) \\
&\leqslant \frac{\|f\|_{\mathrm{Lip}}}{n} \int_\Omega \sum_{u=1}^{n-k+1} \left| X_u - \tilde{X}_u \right| \mathbb{P}_W(\mathrm{d}\tilde{w})
\end{aligned}$$

Using the SDE, we get for all $u \geq 1$,

$$X_u - \tilde{X}_u = \int_0^u b(X_s) - b(\tilde{X}_s)\mathrm{d}s + \sigma \int_{k-1}^k K_H(u+k-1, s)\mathrm{d}(W - \tilde{w})_s$$

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Key elements in the proof : Assumption on the drift b + precise estimates on the kernel K_H.

Outlook

- Discrete dynamics: rate when $H > 1/2$; dependence on the step size $h > 0$ for the Euler scheme.
- Drift estimation: develop numerical aspects like optimization with gradient descent algorithms.
- Concentration: assumptions on the drift b.

(H₁): There exists $V : \mathbb{R}^d \to \mathbb{R}_+^*$ continuous such that $\lim_{|x|\to+\infty} V(x) = +\infty$ and $\exists \gamma \in (0,1)$ and $C > 0$ such that

$$\forall (x,w) \in \mathbb{R}^d \times \mathbb{R}^d, \quad V(F(x,w)) \leqslant \gamma V(x) + C(1+|w|).$$

(H₂): Let $K > 0$. We assume that there exists $\tilde{K} > 0$ such that for every $\mathbf{X} := (x, x', y, y')$ in $B(0,K)^4$, there exist $\Lambda_{\mathbf{X}} : \mathbb{R}^d \to \mathbb{R}^d$, $M_K > 0$ and $C_{\tilde{K}}$ such that the following holds

- $\Lambda_{\mathbf{X}}$ is a bijection from $\mathbb{R}^d$ to $\mathbb{R}^d$. Moreover, it is a $\mathcal{C}^1$-diffeomorphism between two open sets U and D such that $\mathbb{R}^d \backslash U$ and $\mathbb{R}^d \backslash D$ are negligible sets.
- for all $u \in B(0, \tilde{K})$,

$$F(x, u + y) = F(x', \Lambda_{\mathbf{X}}(u) + y') \tag{6.1}$$

$$\text{and} \quad |\det(J_{\Lambda_{\mathbf{X}}}(u))| \geqslant C_{\tilde{K}}. \tag{6.2}$$

- for all $u \in \mathbb{R}^d$,

$$|\Lambda_{\mathbf{X}}(u) - u| \leqslant M_K. \tag{6.3}$$

Key role of Hypothesis $\langle x-y,\ b(x)-b(y)\rangle \leq -\alpha|x-y|^2$

For the sake of simplicity, we set

$$X_u - \tilde{X}_u = \int_0^u b(X_s) - b(\tilde{X}_s)\mathrm{d}s + \tilde{B}_u^{(k)}$$

For $u > 1$,

$$\begin{aligned}
\frac{d}{du}|X_u - \tilde{X}_u|^2 &= 2\langle X_u - \tilde{X}_u,\ \frac{d}{du}(X_u - \tilde{X}_u)\rangle \\
&= 2\langle X_u - \tilde{X}_u,\ b(X_u) - b(\tilde{X}_u)\rangle + 2\langle X_u - \tilde{X}_u,\ \partial_u \tilde{B}_u^{(k)}\rangle \\
&\leq -\alpha|X_u - \tilde{X}_u|^2 + \frac{1}{\alpha}|\partial_u \tilde{B}_u^{(k)}|^2
\end{aligned}$$

Gronwall's lemma : for all $u_0 > 1$,

$$|X_u - \tilde{X}_u|^2 \leq e^{-\alpha(u-u_0)}|X_{u_0} - \tilde{X}_{u_0}|^2 + \frac{1}{\alpha}\int_{u_0}^u e^{-\alpha(u-v)}|\partial_v \tilde{B}_v^{(k)}|^2 dv.$$