CSE 544: Probability & Statistics for Data Science,
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Assignment - 4

Submitted by

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1.) (a) Let $\{X_1, X_2 --- X_n\} \sim \text{Gamma}(x, y)$

Given, mean $= E[X] = xy$

variance $= E[X^2] - (E[X])^2 = xy^2$

Step 0 $\quad k = 2 \quad (x, y)$

Step 1 $\quad \hat{\alpha}_1 = E[\hat{X}'] = \frac{1}{n} \sum_{j=1}^{n} X_j = \overline{X}$

$\hat{\alpha}_2 = E[\hat{X^2}] = \frac{1}{n} \sum_{j=1}^{n} (X_j)^2$

Step 2 $\quad \alpha_1 = E[X] = xy$

$\alpha_2 = E[X^2]$

$= \text{variance} + (E[X])^2$

$= xy^2 + (xy)^2 = xy^2 + x^2y^2$

$= xy^2(1+x)$

Step 3 $\quad$ Equating $\hat{\alpha}_1 = \alpha_1$ & $\hat{\alpha}_2 = \alpha_2$

$\Rightarrow \quad xy = \overline{X} \quad (1) \quad$ & $\quad xy^2(1+x) = \frac{\sum_{j=1}^{n} (X_j)^2}{n} \quad (2)$

From (1) $\Rightarrow x = \frac{\overline{X}}{y}$

Substituting in (2)

$\Rightarrow \frac{\overline{X}}{y} y^2 \left(1 + \frac{\overline{X}}{y}\right) = \frac{\sum_{j=1}^{n} (X_j)^2}{n}$

$$\Rightarrow \bar{x}\,y\left(\frac{y+\bar{x}}{y}\right) = \sum_{j=1}^{n}\frac{(X_j)^2}{n}$$

$$\Rightarrow \bar{x}\,y + (\bar{x})^2 = \sum_{j=1}^{n}\frac{(X_j)^2}{n}$$

$$\Rightarrow \bar{x}\,y = \sum_{j=1}^{n}\frac{(X_j)^2}{n} - (\bar{x})^2$$

$$\Rightarrow y = \frac{1}{\bar{x}}\left(\sum_{j=1}^{n}\frac{(X_j)^2}{n} - (\bar{x})^2\right)$$

$$\therefore \hat{y}_{MME} = \frac{1}{\bar{X}}\left(\sum_{j=1}^{n}\frac{(X_j)^2}{n} - (\bar{X})^2\right)$$

$$\& \quad \hat{x}_{MME} = \frac{(\bar{X})^2}{\left(\sum_{j=1}^{n}\frac{(X_j)^2}{n} - (\bar{x})^2\right)}$$

1) (b) Let $\{X_1, X_2 \dots X_n\} \sim$ Uniform (a, b)

Step 0 $\quad k = 2 \qquad (a, b)$

Step 1 $\quad \hat{\alpha}_1 = E[\hat{x}^1] = \frac{\sum_{i=1}^{n} X_i}{n} = \bar{X}$

$$\hat{\alpha}_2 = E[\hat{x}^2] = \frac{\sum_{i=1}^{n} (X_i)^2}{n}$$

Step 2 $\quad \alpha_1 = E[X] = \frac{a+b}{2}$

$$\alpha_2 = E[X^2] = Var[X] + (E[X])^2$$

$$= \frac{(b-a)^2}{12} + \left(\frac{a+b}{2}\right)^2 \qquad \left(\because \text{Variance for uniform} = \frac{(b-a)^2}{12}\right)$$

$$= \frac{b^2 + a^2 - 2ab}{12} + \frac{a^2 + b^2 + 2ab}{4}$$

$$= \frac{a^2 + b^2 + ab}{3}$$

Step 3 $\quad \hat{\alpha}_1 = \alpha_1$

$$\Rightarrow \bar{X} = \frac{a+b}{2}$$

$$\Rightarrow a = 2\bar{X} - b \quad \text{—— } (1)$$

$$\hat{\alpha}_2 = \alpha_2$$

$$\Rightarrow \frac{\sum_{i=1}^{n} (X_i)^2}{n} = \frac{a^2 + b^2 + ab}{3} \quad \text{—— } (2)$$

Substituting ① in ②

$$\Rightarrow \frac{\sum_{i=1}^{n}(X_i)^2}{n} = \frac{(2\bar{x}-b)^2 + b^2 + (2\bar{x}-b)b}{3}$$

$$\Rightarrow \frac{\sum_{i=1}^{n}(X_i)^2}{n} = \frac{4(\bar{x})^2 + b^2 - 4b\bar{x} + b^2 + 2b\bar{x} - b^2}{3}$$

$$\Rightarrow \frac{\sum_{i=1}^{n}(X_i)^2}{n} = \frac{4(\bar{x})^2 + b^2 - 2b\bar{x}}{3}$$

$$\Rightarrow \frac{\sum_{i=1}^{n}(X_i)^2}{n} = \frac{(\bar{x}-b)^2}{3} + (\bar{x})^2$$

$$\Rightarrow (\bar{x}-b)^2 = 3\left(\frac{\sum_{i=1}^{n}(X_i)^2}{n} - (\bar{x})^2\right)$$

$$\Rightarrow (\bar{x}-b)^2 = 3\bar{S}^2$$

$$\Rightarrow \bar{x} - b = \pm \bar{S}\sqrt{3}$$

Here, we will take $-\bar{S}\sqrt{3}$ only as $b > \bar{x}$ in uniform distribution so $\bar{x} - b$ will be negative

$$\therefore \bar{x} - b = -\bar{S}\sqrt{3}$$

$$\Rightarrow b = +\bar{S}\sqrt{3} + \bar{x}$$

$$\therefore \hat{b}_{MME} = \bar{x} + \sqrt{3}\,\bar{S}$$

$$\& \quad \hat{a}_{MME} = \bar{x} - \sqrt{3}\,\bar{S}$$

Q2) $x_1, x_2, \dots, x_n \sim \text{Exp}(1/\beta)$

$\therefore \text{Pdf}(x_1) \sim 1/\beta \; e^{-1/\beta \, x_1}$

$\text{Pdf}(x_2) \sim 1/\beta \; e^{-1/\beta \, x_2}$

and so on.

We know,

Likelihood $= L(p) =$ product of pdfs.

$$= \left(\frac{1}{\beta} e^{-1/\beta \, x_1}\right)\left(\frac{1}{\beta} e^{-1/\beta \, x_2}\right) \cdots \left(\frac{1}{\beta} e^{-1/\beta \, x_n}\right)$$

$$= \frac{1}{\beta^n}\left[e^{-1/\beta \, x_1} \times e^{-1/\beta \, x_2} \times \cdots \times e^{-1/\beta \, x_n}\right]$$

Since $a^x . a^y = a^{x+y}$ we have,

$$L(p) = \frac{1}{\beta^n} e^{-1/\beta \left[x_1 + x_2 + \cdots + x_n\right]} \quad \text{—— (1)}$$

Now take log on both sides, so that we can get MLE by differentiating.

So taking natural log on both sides,

$$\ln\left[L(p)\right] = \ln\left[\frac{1}{\beta^n} e^{-1/\beta\,[x_1 + x_2 + \cdots + x_n]}\right]$$

since $\ln(ab) = \ln a + \ln b$,

$$\ln\left[L(p)\right] = \ln\left[1/\beta^n\right] + \ln\left[e^{-1/\beta\,[x_1+x_2+\cdots+x_n]}\right]$$

$$= \ln\left[\beta^{-n}\right] + \left(-1/\beta\,[x_1 + x_2 + \cdots + x_n]\,(\ln e)\right)$$

Since $\ln(a^b) = b\ln a$ and $\ln e = 1$

we have,

$$\ln\left[L(p)\right] = -n\ln[\beta] - \frac{1}{\beta}\left[x_1 + x_2 + \cdots + x_n\right]$$

Now we can differentiate w.r.t β and equate to 0 to get MLE

$$\frac{d}{d\beta}\left[-n\ln\beta - \frac{1}{\beta}\left[x_1 + x_2 + \cdots + x_n\right]\right] = 0$$

$$\frac{-n}{\beta} + \frac{1}{\beta^2}\left[x_1 + x_2 + x_3 + \dots + x_n\right] = 0$$

$$\frac{n}{\cancel{\beta}} = \frac{1}{\beta^{2}}\left[x_1 + x_2 + x_3 + \dots + x_n\right]$$

$$\hat{\beta} = \frac{1}{n}\left[x_1 + x_2 + \dots + x_n\right]$$

$$\therefore \hat{\beta} = \frac{\sum_{i=1}^{n} x_i}{n} \quad \text{——} (2)$$

To show consistency we need to prove,

$\text{Bias}(\hat{\beta}) = 0$ as $n \to \infty$
$\text{Se}(\hat{\beta}) = 0$ as $n \to \infty$.

$$\text{Bias}(\hat{\beta}) = E(\hat{\beta}) - \beta$$

From (2),

$$E(\hat{\beta}) = E\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)$$

$$= \frac{1}{n} E\left(\sum_{i=1}^{n} x_i\right)$$

Since $x_1, x_2, \ldots\ldots, x_n$ are iid, and by LOE

$$= \frac{1}{n} \times \left[E(x_1) + E(x_2) + \cdots + E(x_n)\right]$$

$$= \frac{1}{\cancel{n}} \times \cancel{n} \left[E(x_1)\right]$$

$$= E(x_1)$$

Since avg / mean / E of exp is $1/\lambda$, we have,

$$E(x_1) = \frac{1}{(1/\beta)}$$

$$\therefore \quad E(\hat{\beta}) = \beta$$

$$\therefore \quad \text{Bias}(\hat{\beta}) = E(\hat{\beta}) - \beta$$

$$= \beta - \beta = 0.$$

$$\boxed{\text{Bias}(\hat{\beta}) = 0}$$

$$se(\hat{\beta}) = \sqrt{Var(\hat{\beta})}$$

$$= \sqrt{Var\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)}$$

$$= \sqrt{\frac{1}{n^2} Var\left(\sum_{i=1}^{n} x_i\right)}$$

(since $Var(ax) = a^2 Var(x)$)

By LOV, & iid,

$$= \sqrt{\frac{1}{n^2} \times n \times Var(x_1)}$$

$$= \sqrt{\frac{Var(x_1)}{n}}$$

$\because$ $Var(x_1) = 1/(1/\beta)^2 = \beta^2$

$$\therefore \quad se(\hat{\beta}) = \sqrt{\frac{\beta^2}{n}}$$

$$\boxed{se(\hat{\beta}) = \frac{\beta}{\sqrt{n}}.}$$

So as $n \to \infty$, $se(\hat{\beta}) = 0$.

Since $Bias(\hat{\beta}), se(\hat{\beta}) = 0$ when $n \to \infty$,

MLE$(\hat{\beta})$ will converge to unknown parameter $\hat{\beta}$.

Hence proved.

3.) (a) $\{X_1, X_2 \ldots X_n\} \overset{iid}{\sim} \text{Poisson}(\lambda)$

For Poisson $P_X(x) = \dfrac{e^{-\lambda} \lambda^{x}}{x!}$

Here X is discrete

$\therefore L(\lambda) = \prod_{i=1}^{n} \dfrac{e^{-\lambda} \lambda^{x_i}}{x_i!}$

$= \dfrac{e^{-n\lambda} \lambda^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}$

Taking log both sides

$l(\lambda) = \ln\left(\dfrac{e^{-n\lambda} \lambda^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!} \right)$

$l(\lambda) = \ln(e^{-n\lambda}) + \ln(\lambda^{\sum_{i=1}^{n} x_i}) - \left(\sum_{i=1}^{n} \ln(x_i) \right)$

For maxima $\dfrac{d\, l(\lambda)}{d\lambda} = 0$

$\Rightarrow$ $\therefore$ Differentiating RHS and equating to 0

$\Rightarrow -n + \dfrac{\sum_{i=1}^{n} x_i}{\lambda} - 0 = 0$

$\Rightarrow \lambda = \dfrac{\sum_{i=1}^{n} x_i}{n}$ $\Rightarrow \lambda_{MLE} = \dfrac{\sum_{i=1}^{n} x_i}{n}$

3 (b) $\{X_1, X_2 \ldots X_n\} \overset{iid}{\sim} Nor(\mu, \sigma^2)$

$$L(\vec{\theta}) = \prod_{i=1}^{n} f_X(x_i)$$

For Normal $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$\therefore$

$$L(\vec{\theta}) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$

Here μ and σ are unknown so we need to take partial derivatives with respect to them individually and equate to 0.

$$\Rightarrow L(\vec{\theta}) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{2\sigma^2}}$$

Taking log both sides

$$\Rightarrow l(\vec{\theta}) = n \ln\left(\frac{1}{\sigma\sqrt{2\pi}}\right) - \sum_{i=1}^{n}\frac{(x_i-\mu)^2}{2\sigma^2}$$

(1) Take partial derivative w.r.t μ

$$\frac{dl}{d\mu} = 0$$

$$\Rightarrow 0 - \frac{d}{d\mu}\left(\sum_{i=1}^{n}\frac{(x_i^2+\mu^2-2\mu x_i)}{2\sigma^2}\right) = 0$$

$$\Rightarrow \frac{d}{d\mu}\left(\frac{\sum_{i=1}^{n}(x_i)^2}{2\sigma^2}\right) + \frac{d}{d\mu}\left(\frac{n\mu^2}{2\sigma^2}\right) - \frac{d}{d\mu}\left(\frac{\sum_{i=1}^{n} 2\mu x_i}{2\sigma^2}\right) = 0$$

$$\Rightarrow \quad 0 + \frac{2\mu n}{2\sigma^2} - \frac{2\sum_{i=1}^{n} x_i}{2\sigma^2} = 0$$

$$\Rightarrow \quad \cancel{2}\mu n = \cancel{2}\sum_{i=1}^{n} x_i$$

$$\Rightarrow \quad \mu = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$\therefore \mu_{MLE} = \text{Sample Mean} = \frac{\sum_{i=1}^{n} x_i}{n}$$

② Taking partial derivative w.r.t σ.

$$\frac{dl}{d\sigma} = 0$$

$$\Rightarrow \frac{d}{d\sigma}\left(-n\ln(\sigma\sqrt{2\pi})\right) - \frac{d}{d\sigma}\left(\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}\right) = 0$$

$$\Rightarrow -\frac{n}{\sigma\sqrt{2\pi}}\sqrt{2\pi} - \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2}\left(-2\sigma^{-3}\right) = 0$$

$$\Rightarrow -\frac{n}{\sigma} + \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{\sigma^3} = 0$$

$$\Rightarrow \quad \sigma^2 = \frac{\sum(x_i-\mu)^2}{n}$$

$$\therefore \sigma^2_{MLE} = \begin{array}{c}\text{Uncorrected}\\ \text{Sample variance}\end{array} = \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{n}$$

3.) (c.) $\{X_1, X_2 - - - X_n\} \sim$ Normal $(\theta, 1)$

Let $\delta = E[I_{X_1 > 0}]$

$\Rightarrow \delta = Pr(X_1 > 0)$ $\quad (\because E[I_x] = Pr(x))$

$\Rightarrow \delta = 1 - Pr(X_1 \leq 0)$

$\Rightarrow \delta = 1 - F_{X_1}(0)$ $\quad$ — ①

As we need the final expression of δ_{MLE} in terms of Φ, we can use relation

$$Z = \frac{X - \mu}{\sigma}$$ where Z is the std. normal.

$\because F_{X_1}(0) = Pr(X_1 \leq 0)$

$= Pr(\mu + \sigma Z \leq 0)$

given $\rightarrow \mu = \theta$
$\sigma = 1$

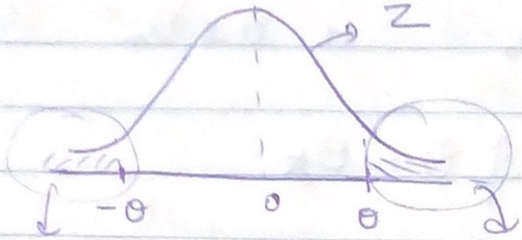
$= Pr(\theta + Z \leq 0)$

$= Pr(Z \leq -\theta)$

$= \Phi(-\theta)$

Substituting in ①

$\Rightarrow \delta = 1 - \Phi(-\theta)$



$\Phi(-\theta)$ $\qquad\qquad$ $1-\Phi(\theta)$

$$\therefore\ \delta = 1 - (1-\Phi(\theta))$$

$$= \Phi(\theta)$$

Using Equivariance property,

$$\delta_{MLE} = \Phi(\theta_{MLE})$$

$$= \Phi\left(\sum_{i=1}^{n} \frac{X_i}{n}\right)$$

Hence Proved.

4) Parametric interference with data samples :

$$X = \begin{cases} 2 & \text{with prob } \theta \\ 3 & \text{otherwise (i.e. prob } 1-\theta) \end{cases}$$

Step 0 : $k \Rightarrow 1$ unknown parameter

Step 1 : For $i = 1$, $\hat{\alpha}_1 = \frac{1}{n} \sum_{j=1}^{n} X_j = \bar{X}$

Step 2 : For $i = 1$,

$$E[X(\theta)] = 2.\theta + 3(1-\theta)$$
$$= 2\theta + 3 - 3\theta = 3-\theta$$

Step 3 : $\bar{X} = 3 - \hat{\theta}$ $\quad \boxed{\Rightarrow \hat{\theta}_{MME} = 3 - \bar{X}} \rightarrow ①$

$$\hat{\theta}_{MME} = 3 - \bar{x}$$
$$= 3 - \frac{2+3+2}{3} = 3 - \frac{7}{3} = \frac{2}{3}$$

$$\boxed{\therefore \hat{\theta}_{MME} = \frac{2}{3}}$$

(b) $se(\hat{\theta}_{MME}) = \sqrt{var(\hat{\theta}_{MME})} = \sqrt{Var(3-\bar{x})}$ (From ①)

We know : $Var(c - x) = Var(x)$

$(\because var(c + (-x)) \overset{LOV}{=} Var(c) + Var(-x))$

we know, $Var(-A) = Var(A)$

$[\because var(-A) = Var(-1.A) = (-1)^2 Var(A)]$

And $Var(c) = 0$

$\therefore \quad Var(c - \bar{x}) = Var(\bar{x})$

$\therefore \; Var(3 - \bar{x}) = Var(\bar{x})$

$$se(\hat{\theta}_{MME}) = \sqrt{Var(\bar{x})} = \sqrt{Var\left(\frac{\Sigma X_j}{n}\right)} = \sqrt{\frac{1}{n^2} Var(\Sigma X_j)}$$

By LOV, $\Rightarrow \sqrt{\frac{1}{n^2} \Sigma Var(X_j)} \xRightarrow{iid} \sqrt{\frac{1}{n^2} \cdot n \cdot Var(X_1)}$

$$se(\hat{\theta}_{MME}) = \sqrt{\frac{1}{n} \cdot Var(X_1)} = \sqrt{\frac{Var(x)}{n}}$$

$\hookrightarrow$ 2

Now,

$$Var(x) = E[x^2] - (E[x])^2$$

$$E[x^2] = \sum_{\omega} \omega^2 . p(\omega) = 4(\theta) + 9(1-\theta)$$
$$= 4\theta + 9 - 9\theta$$

$$E[x^2] = 9 - 5\theta$$

$$(E[x])^2 = (3-\theta)^2 = 9 + \theta^2 - 6\theta$$

$$Var[X] = 9 - 5\theta - 9 - \theta^2 + 6\theta = \theta - \theta^2$$

$\Rightarrow \theta(1-\theta)$

Now, From ② and

$$se(\hat{\theta}_{MME}) = \sqrt{\frac{\theta(1-\theta)}{n}}$$

As se cannot be computed directly as θ is true but unknown value, we replace θ with $\hat{\theta}_{MME}$

$$\therefore \hat{se}(\hat{\theta}) = \sqrt{\frac{\hat{\theta}_{MME}(1-\hat{\theta}_{MME})}{n}}$$

Substituting $\Rightarrow$ $\hat{se}(\hat{\theta}) = \sqrt{\frac{\frac{2}{3} \times (1-\frac{2}{3})}{3}} = \sqrt{\frac{2}{3} \times \frac{1}{3} \times \frac{1}{3}}$

$$\boxed{\therefore \hat{se}(\hat{\theta}) = \frac{1}{3}\sqrt{\frac{2}{3}}}$$

(c) $X = \begin{cases} 2 & w.p\ \theta \\ 3 & w.p\ (1-\theta) \end{cases}$

PMF in closed form,

For θ, $ax + b = 1 \Rightarrow$ $2a + b = 1$
$3a + b = 0$

For $1-\theta$, $1-(3-x) = x-2$

$\therefore P_X(x) = (1-\theta)^{x-2} \cdot \theta^{3-x}$

Now, Let $X_1, X_2 \dots X_n$ are iid data samples
Here, $n = 3$

$$L(\theta) = \prod_{i=1}^{n} (1-\theta)^{X_i-2} \cdot \theta^{3-X_i}$$

$$L(\theta) = \theta^{3n-\Sigma X_i} \, (1-\theta)^{\Sigma X_i - 2n}$$

$$l(\theta) = \ln(L(\theta)) = \ln\left(\theta^{3n-\Sigma X_i}\right) + \ln\left((1-\theta)^{\Sigma X_i - 2n}\right)$$

$$= (3n - \Sigma X_i) \ln\theta + (\Sigma X_i - 2n) \ln(1-\theta)$$

$$\frac{d(l(\theta))}{d\theta} = 0$$

$$\frac{3n - \Sigma X_i}{\theta} - \frac{\Sigma X_i - 2n}{1-\theta} = 0$$

$$\Rightarrow \quad \frac{3n - \sum x_i}{\theta} = \frac{\sum x_i - 2n}{1-\theta}$$

$$(1-\theta).(3n - \sum x_i) = \theta(\sum x_i - 2n)$$

$$3n - \sum x_i - 3n\theta + \theta \sum x_i = \theta \sum x_i - 2n\theta$$

$$3n - \sum x_i = n\theta$$

$$\hat{\theta}_{MLE} = \frac{3n - \sum x_i}{n} = 3 - \frac{\sum x_i}{n}$$

$$\hat{\theta}_{MLE} = 3 - \frac{2+3+2}{3} = 3 - \frac{7}{3} = \frac{2}{3}$$

$$\boxed{\therefore \hat{\theta}_{MLE} = \frac{2}{3}}$$

5) (a) $\{X_1, X_2 --- X_n\} \sim Exp(\lambda)$ $\hat{\lambda}_{MME} = ?$

Step 0 $k = 1$

Step 1 $\hat{\alpha}_1 = \dfrac{\sum_{i=1}^{n} X_i}{n}$

Step 2 $\alpha_1 = E[Exp(\lambda)] = \dfrac{1}{\lambda}$

Step 3 $\hat{\alpha}_1 = \alpha_1$

$\Rightarrow \dfrac{1}{\lambda} = \dfrac{\sum_{i=1}^{n} X_i}{n}$

$\therefore \boxed{\hat{\lambda}_{MME} = \dfrac{n}{\sum_{i=1}^{n} X_i}}$

(b) $\hat{\lambda}_{MLE} = ?$

$$L(\lambda) = \prod_{i=1}^{n} f_x(x_i)$$

$$= \prod_{i=1}^{n} \lambda e^{-\lambda x_i} = (\lambda)^n e^{-\lambda \sum_{i=1}^{n} x_i}$$

Taking log

$$l(\lambda) = \ln\left(\lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}\right)$$

$$= \ln(\lambda^n) - \lambda \sum_{i=1}^{n} x_i$$

$$\frac{d\,l(\lambda)}{d\lambda} = 0$$

$$\Rightarrow \frac{n}{\lambda} - \sum_{i=1}^{n} x_i = 0$$

$$\Rightarrow \frac{n}{\lambda} = \sum_{i=1}^{n} x_i$$

$$\Rightarrow \boxed{\hat{\lambda}_{MLE} = \frac{n}{\sum_{i=1}^{n} x_i}}$$

5 (c.) $\hat{\mu}_{MME} = \frac{\sum_{i=1}^{n} x_i}{n} = 15.568$

$\hat{\sigma}^2_{MME} = \frac{\sum_{i=1}^{n}(x_i)^2}{n} - \left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^2 = 7.586$

$\hat{a}_{MME} = \overline{x} - \sqrt{3}\,\overline{s} = 69.614$

$\hat{b}_{MME} = \overline{x} + \sqrt{3}\,\overline{s} = 82.406$

$\hat{\lambda}_{MME} = \frac{n}{\sum_{i=1}^{n} x_i} = 0.043$

5 (d.) $\hat{\mu}_{MLE} = \frac{\sum_{i=1}^{n} x_i}{n} = 15.568$

$\hat{\sigma}^2_{MLE} = \frac{\sum_{i=1}^{n}(x_i)^2}{n} - \left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^2 = 7.586$

$\hat{a}_{MLE} = \min(\text{data}) = 70$

$\hat{b}_{MLE} = \max(\text{date}) = 82$

$\hat{\lambda}_{MLE} = 0.043$

6)

		Test accept H_o	Test reject H_o
Ground truth	H_o true	True -ve	False +ve
	H_o false	False -ve	True +ve

$H_o \Rightarrow$ Patient is healthy
$H_1 \Rightarrow$ Patient is sick

Number of healthy patients = 100
Number of sick patients = 100

Accurate healthy reports = 98
Accurate sick reports = 99

(a) Precision :

$$\frac{\text{True positive}}{\text{True positive} + \text{False positive}}$$

positive result $\Rightarrow$ Test rejects H_o
$\hookrightarrow$ Patient given sick report

negative result $\Rightarrow$ Test accepts H_o
$\hookrightarrow$ Patient given healthy report

True positive (TP) ⇒ sick person given sick report
False positive (FP) ⇒ Healthy person given sick report

$$\text{Precision} = \frac{TP}{TP+FP} = \frac{99}{99+2} = \frac{99}{101} = \boxed{0.9801}$$

(b) Recall (coverage) $= \dfrac{\text{True positive}}{\text{False negative} + \text{True positive}}$

$$= \frac{99}{1+99} = \boxed{0.99}$$

(c) Type I error (FP) : Pr (positive test result | Ho true)
(test rejects Ho)

: Pr (patient given sick report when he is actually healthy)

= 100 − 98 ⇒ 2 people out of 100

$$= \frac{2}{100} = \boxed{0.02}$$

(d) Type 2 error: $Pr(\text{Test is accepted} \mid H_0 \text{ is false})$
(FN)

$= Pr(\text{patient given healthy report when is actually sick})$

$= 100 - 99 \Rightarrow 1$ person out of 100

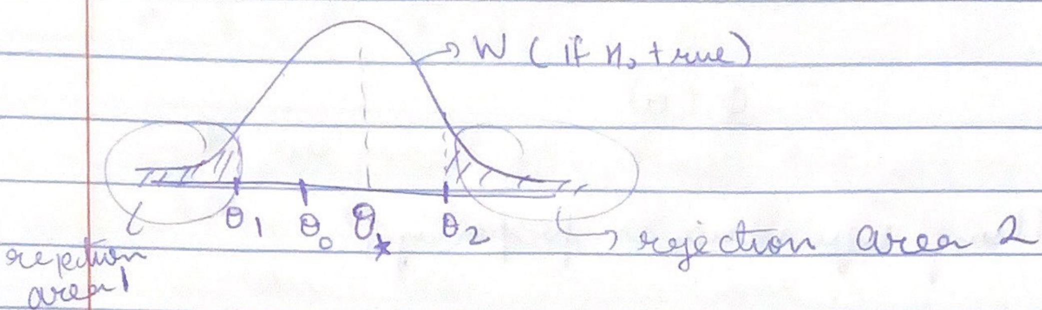
$\Rightarrow \frac{1}{100} = \boxed{0.01}$

7) (a) $H_0: \theta = \theta_0$ $H_1: \theta \neq \theta_0$

True value of $\theta = \theta_*$.

Pr(Type 2 error)

$= Pr(\text{Accept } H_0 \mid H_0 \text{ false})$



Acc. to Wald's test

$$W = \frac{\hat{\theta} - \theta_0}{\hat{se}(\hat{\theta})}$$

To accept $H_0 \rightarrow -z_{\alpha/2} \leq W \leq z_{\alpha/2}$

To reject $H_0 \rightarrow$ $W > z_{\alpha/2}$ & $W < -z_{\alpha/2}$

$$\Rightarrow \frac{\hat{\theta} - \theta_0}{\hat{se}(\theta)} > z_{\alpha/2} \quad \& \quad \frac{\hat{\theta} - \theta_0}{\hat{se}(\theta)} < -z_{\alpha/2}$$

$$\Rightarrow \hat{\theta} > z_{\alpha/2}\,\hat{se}(\theta) + \theta_0 \Rightarrow \underline{\hat{\theta} > \theta_2}$$

$$\& \quad \hat{\theta} < \theta_0 - z_{\alpha/2}\,\hat{se}(\theta)$$

$$\Rightarrow \underline{\hat{\theta} < \theta_1}$$

Now, to find $Pr(\text{Type 2 error})$

$$= Pr(\text{Accept } H_0 \mid H_0 \text{ false})$$

we can do,

$$1 - Pr\left(\hat{\theta} \text{ in rejection area 1 or 2} \mid H_0 \text{ false}\right)$$

$$\Rightarrow 1 - \left[Pr\left(\hat{\theta} \text{ in rejection area 1} \mid H_0 \text{ false}\right) + Pr\left(\hat{\theta} \text{ in rejection area 2} \mid H_0 \text{ false}\right)\right]$$

$$\Rightarrow 1 - \left[Pr\left(\hat{\theta} < \theta_0 - Z_{\alpha/2}\,\hat{se}(\hat{\theta}) \mid H_0 \text{ false}\right) + Pr\left(\hat{\theta} > \theta_0 + Z_{\alpha/2}\,\hat{se}(\hat{\theta}) \mid H_0 \text{ false}\right)\right]$$

$$\Rightarrow 1 - \left[Pr\left(\hat{\theta} < \theta_0 - Z_{\alpha/2}\,\hat{se}(\hat{\theta}) \mid \theta = \theta_*\right) + 1 - Pr\left(\hat{\theta} < \theta_0 + Z_{\alpha/2}\,\hat{se}(\hat{\theta}) \mid \theta = \theta_*\right)\right]$$

$$\cancel{\Rightarrow Pr\left(\hat{\theta} < \theta_0 + Z_{\alpha/2}\,\hat{se}(\hat{\theta}) \mid \theta = \theta_*\right) - Pr\left(\hat{\theta} < \theta_0 - Z_{\alpha/2}\,\hat{se}(\hat{\theta}) \mid \theta = \theta_*\right)}$$

$$\Rightarrow -Pr\left(\hat{\theta} < \theta_0 - Z_{\alpha/2}\,\hat{se}(\hat{\theta}) \mid \theta = \theta_*\right) + Pr\left(\hat{\theta} < \theta_0 + Z_{\alpha/2}\,\hat{se}(\hat{\theta}) \mid \theta = \theta_*\right)$$

$\Rightarrow$ So, now to get the result in the form of $\Phi(\;)$ we need to convert above term to standard normal form.

So, subtract θ_* from both sides and divide by $\hat{se}$

$$\Rightarrow -\Pr\left(\frac{\hat{\theta}-\theta_*}{\hat{se}(\hat{\theta})} < \frac{\theta_0 - \hat{se}(\hat{\theta})\, z_{\alpha/2} - \theta_*}{\hat{se}(\hat{\theta})}\right)$$

$$+ \Pr\left(\frac{\hat{\theta}-\theta_*}{\hat{se}(\hat{\theta})} < \frac{\theta_0 + z_{\alpha/2}\, \hat{se}(\hat{\theta}) - \theta_*}{\hat{se}(\hat{\theta})}\right)$$

$$\Rightarrow -\Pr\left(W < \frac{\theta_0 - \theta_*}{\hat{se}(\hat{\theta})} - z_{\alpha/2}\right)$$

$$+ \Pr\left(W < \frac{\theta_0 - \theta_*}{\hat{se}(\hat{\theta})} + z_{\alpha/2}\right)$$

Now, we know if H_0 is true and $\hat{se}$ is consistent then $W \sim Z$

$$\therefore -\Pr\left(Z < \frac{\theta_0 - \theta_*}{\hat{se}(\hat{\theta})} - z_{\alpha/2}\right)$$

$$+ \Pr\left(Z < \frac{\theta_0 - \theta_*}{\hat{se}(\hat{\theta})} + z_{\alpha/2}\right)$$

$$\Rightarrow -\Phi\left(\frac{\theta_0 - \theta_*}{\hat{se}(\hat{\theta})} - z_{\alpha/2}\right) + \Phi\left(\frac{\theta_0 - \theta_*}{\hat{se}(\hat{\theta})} + z_{\alpha/2}\right)$$

$$\Rightarrow \Phi\left(\frac{\theta_0 - \theta_*}{\hat{se}(\hat{\theta})} + z_{\alpha/2}\right) - \Phi\left(\frac{\theta_0 - \theta_*}{\hat{se}(\hat{\theta})} - z_{\alpha/2}\right)$$

7) (b.) H_0: coin unbiased

$\Rightarrow H_0: p = 0.5 \qquad H_1: p \neq 0.5$

Total trials = 100

No. of success = 46

Coin toss in a Bernoulli trial

$\hat{P}_{MLE}$ (Bern.) $= \frac{\sum x_i}{n} = \frac{46}{100} = 0.46$

$\because$ MLE is Asymptotically Normal

$\therefore$ We can apply wald's test here

$$\Rightarrow \quad W = \frac{\hat{P}_{MLE} - P_0}{\hat{se}(\hat{P}_{MLE})}$$

$$\Rightarrow \quad W = \frac{0.46 - 0.5}{\hat{se}(\hat{P}_{MLE})} \qquad \left(\because P_0 = 0.5 \text{ given}\right)$$

Denominator $\rightarrow \hat{se}(\hat{P}_{MLE})$

$$\hat{se}(\hat{P}_{MLE}) = \sqrt{Var(\hat{P}_{MLE})} = \sqrt{Var\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)}$$

$$\overset{iid}{\underset{LOV}{=}} \sqrt{\frac{Var(X_1)}{n}}$$

$$= \sqrt{\frac{\hat{P}_{MLE}(1-\hat{P}_{MLE})}{n}} \qquad \left(\because \text{Variance for Bernoulli is } P(1-P)\right)$$

$\Rightarrow \hat{se}(\hat{p}_{MLE}) = \sqrt{\dfrac{0.46(1-0.46)}{100}} = \sqrt{0.002484}$

$\therefore W = \dfrac{0.46-0.5}{\sqrt{0.002484}} = \cancel{}$

$= -0.8026$

To accept $H_0 \Rightarrow |W| < Z_{\alpha/2}$

$Z_{\alpha/2} = Z_{0.025} = 1.96$

$|W| = 0.8026$

$\therefore |W| < Z_{\alpha/2}$ so, Null can be accepted.

Case 2 $\Rightarrow H_0 : p = 0.7 \qquad H_1 \; p \neq 0.7$

$W = \dfrac{0.46-0.7}{\sqrt{0.002484}} = -4.82$

$\Rightarrow |W| = 4.82 > Z_{\alpha/2}$

$\therefore$ Null is rejected in this case.

8) (a) $D \sim Nor(\theta, \sigma^2)$

$\alpha = 0.02$

$\theta_0 = 0.5$

Acc. to Wald's test, if $|W| < Z_{\alpha/2}$ then we accept Null hypothesis else we reject.

Here, $H_0 : \theta_0 = 0.5$ $H_1 : \theta_0 \neq 0.5$

$$W = \frac{\hat{\theta} - \theta_0}{\hat{se}(\hat{\theta})}$$

From dataset $\rightarrow$ $\hat{\theta} = \frac{\sum_{i=1}^{n} X_i}{n} = 0.5409$

$$\hat{se}(\hat{\theta}) = \sqrt{Var(\hat{\theta})} = \sqrt{Var\left(\frac{\sum_{i=1}^{n} X_i}{n}\right)}$$

$$\overset{LOV}{\underset{iid}{=}} \sqrt{\frac{1}{n^2} Var\left(\sum_{i=1}^{n} X_i\right)}$$

$$= \sqrt{\frac{n \, Var(X_i)}{n^2}}$$

$$= \sqrt{\frac{Var(X_1)}{n}}$$

and corrected $Var(X_1) = \frac{\sum_{i=1}^{n} (X_i - \mu)^2}{n-1}$

this comes out to be $\Rightarrow$ 0.00649

$$\therefore \hat{se}(\hat{\theta}) = \sqrt{\frac{\cancel{1.0649}}{n}} = \sqrt{\frac{0.010649}{1000}}$$

$$= 0.00326$$

$$\therefore W \Rightarrow \frac{\hat{\theta} - \theta_0}{\hat{se}(\hat{\theta})} = 12.5476$$

For $\alpha = 0.02$

$$\Rightarrow Z_{\alpha/2} = Z_{0.01} = 2.326$$

Here $|W| > Z_{\alpha/2}$

$\therefore H_0$ is rejected.

8) b) Given

$$\left.\begin{array}{l} X = \{x_1, x_2, \cdots, x_{750}\} \overset{iid}{\sim} x(\theta_1) \\ Y = \{y_1, y_2, \cdots, y_{750}\} \overset{iid}{\sim} y(\theta_1) \end{array}\right] \text{ — (1)}$$

Let $\bar{x}$ = mean of x = Sample mean of x

$\bar{y}$ = mean of y = Sample mean of y

$$\delta = \bar{x} - \bar{y}$$

To check : $\bar{x} = \bar{y}$?

So we have,

$$H_0 : \delta = 0 \quad (vs) \quad H_1 : \delta \neq 0$$

$$W = \frac{\hat{\delta} - \delta_0}{\hat{se}(\hat{\delta})} = \frac{\hat{\delta}}{\hat{se}(\hat{\delta})} = \frac{\bar{x} - \bar{y}}{\hat{SE}(\bar{x} - \bar{y})}$$

(since $\delta_0 = 0$)

From Python we have

$$\bar{x} = 5.004734$$

$$\bar{y} = 5.845618$$

$$\bar{x} - \bar{y} = -0.840884$$

Denominator,

$$\hat{se}(\hat{\delta}) = \hat{se}(\bar{x} - \bar{y})$$

$$\hat{se}(\bar{x} - \bar{y}) = \sqrt{Var(\bar{x} - \bar{y})}$$

$$\overset{LOV}{=} \sqrt{Var(\bar{x}) + Var(\bar{y})}$$

$$= \sqrt{Var\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) + Var\left(\frac{\sum_{i=1}^{n} y_i}{n}\right)}$$

$$= \sqrt{\frac{1}{n^2}\left[n\,Var(X_1) + n\,Var(Y)\right]}$$

$$= \sqrt{\frac{Var(x) + Var(y)}{\hat{n}}}$$

From Python, we have,

$Var(x) = 2.361459$
$Var(y) = 6.472424$

$$\hat{se}(\hat{\delta}) = \sqrt{\frac{2.361459 + 6.472424}{750}}$$

$$= 0.1085288$$

W for δ : $$\frac{\bar{x} - \bar{y}}{\hat{se}(\hat{\delta})}$$

$$= 7.74802255$$

$$z_{\alpha/2} = 1.96$$

$$|W| > Z_{\alpha/2}$$

H_0 is rejected.

$\therefore$ Mean of $X \neq$ Mean of Y

For this test to be true, we assume that

$\hat{\theta}_1$, $\hat{\theta}_2$ are both asymptotically normal

from (1).

This is true for Normal distribution as any transformation to the distribution stays normal.

$\therefore$ MLE, W are also normal asymptotically.

$\therefore$ Wald's test can be applied.