CSE 544: Probability & Statistics for Data Science,
Spring 2021

Assignment - 1

Submitted by

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1.) (a) $p(\text{LAC win}) = p(\text{DEN win}) = 0.5$

After first 4 games $\Rightarrow$ $\underbrace{\underline{LAC}\ \underline{LAC}\ \underline{LAC}}_{\text{3 wins for LAC}}\ \underbrace{\underline{DEN}}_{\text{1 win for DEN}}$

This can happen in any order

$\therefore$ first choosing three games out of four $\Rightarrow {}^4C_3$

and in these three chance of winning of LAC will be

$\Rightarrow {}^4C_3 \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$

Now, for remaining 1 game DEN wins

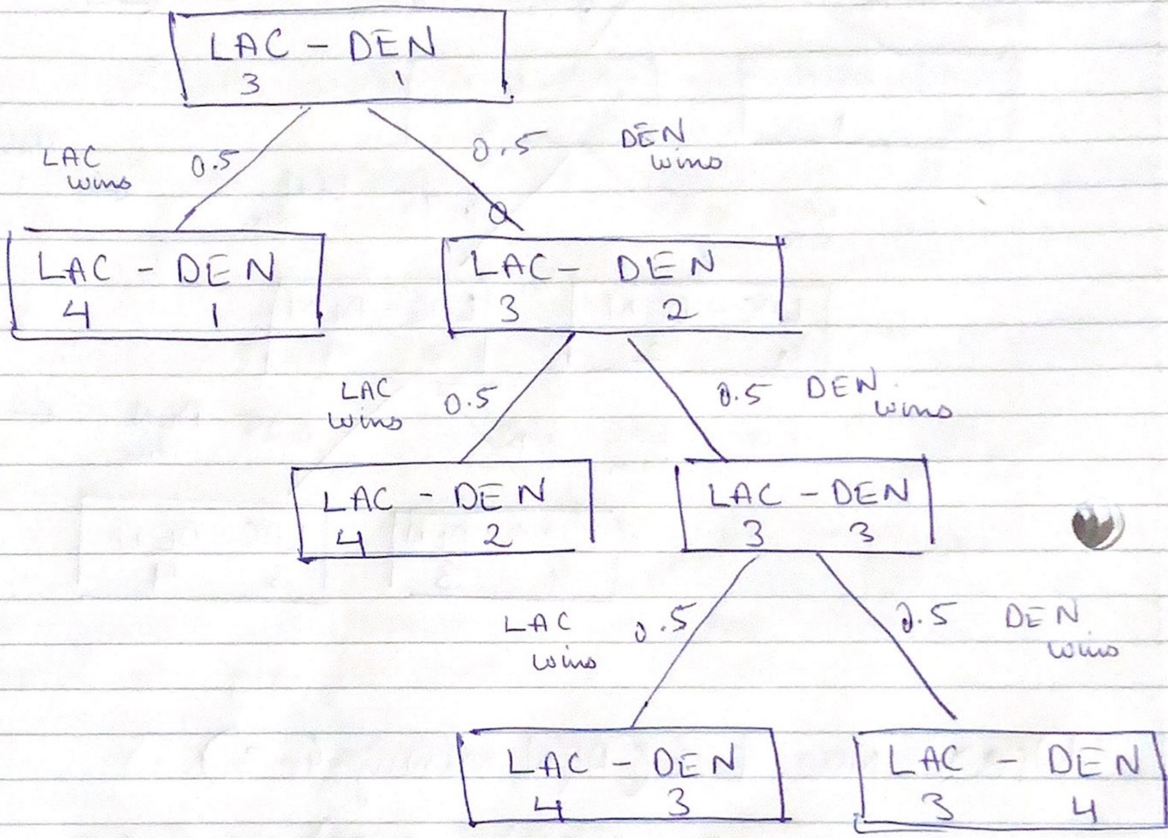
$\Rightarrow {}^4C_3 \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) \times 1 \left(\frac{1}{2}\right)$

$\Rightarrow {}^4C_3 \left(\frac{1}{2}\right)^4$

$= \frac{4!}{1!\,3!}\left(\frac{1}{2^4}\right) \Rightarrow \frac{4 \times 3!}{1 \times 3!} \times \frac{1}{16}$

$\Rightarrow 4 \times \frac{1}{16} = \frac{1}{4} = \boxed{0.25}$

1) (b) Decision tree after first 4 games :



1) (c) After (3-1), probability of DEN winning (3-4)

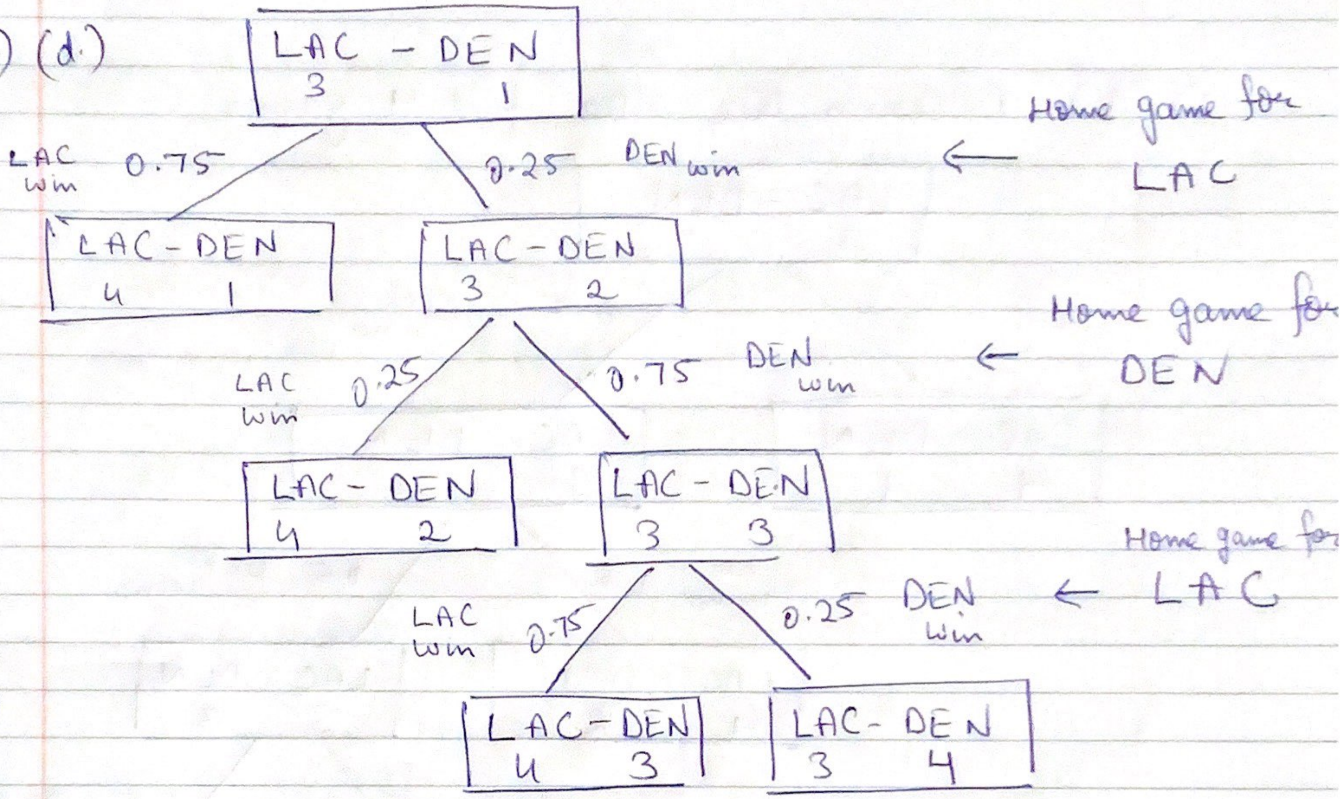
will be :

Using Decision tree

$$\left(\frac{1}{2}\right)\times\left(\frac{1}{2}\right)\times\left(\frac{1}{2}\right)=\frac{1}{8}$$

$$\boxed{=0.125}$$

1) (d.)



1)(e.) Now, Prob (DEN winning 4-3)

$= 0.25 \times 0.75 \times 0.25$

$\boxed{= 0.046875}$

Program execution results for 1.f)

Note: As N increases, the probabilities calculated are more closer or accurate to the calculated answers in 1)

Hw1 — -zsh — 131×35

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harman@harmans-MacBook-Pro Hw1 % python3 nba.py
Calculating...

FOR N = 1000, the simulated value for part (a) is 0.263
FOR N = 1000, the simulated value for part (c) is 0.12547528517110265
FOR N = 1000, the simulated value for part (e) is 0.03619909502262444


Calculating...

FOR N = 10000, the simulated value for part (a) is 0.2564
FOR N = 10000, the simulated value for part (c) is 0.11856474258970359
FOR N = 10000, the simulated value for part (e) is 0.04595744680851064


Calculating...

FOR N = 100000, the simulated value for part (a) is 0.25123
FOR N = 100000, the simulated value for part (c) is 0.1282092106834375
FOR N = 100000, the simulated value for part (e) is 0.04695910150455605


Calculating...

FOR N = 1000000, the simulated value for part (a) is 0.249416
FOR N = 1000000, the simulated value for part (c) is 0.12554928312538088
FOR N = 1000000, the simulated value for part (e) is 0.0468439602691445


Calculating...

FOR N = 10000000, the simulated value for part (a) is 0.2498361
FOR N = 10000000, the simulated value for part (c) is 0.12485185287474468
FOR N = 10000000, the simulated value for part (e) is 0.04658595203704762
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2) Let N be the number of possible arrangements on which the trial / game is to be conducted.

$\therefore$ N = { all ways of distributing n iphones across n places }

i.e, N = { all permutations of $(1, 2, \cdots, n)$ }

We know that to distribute m items in n places when $m \leq n$ = ${}^{n}P_{n}$

$$N = {}^{n}P_{n} = \frac{n!}{(n-n)!} = n!$$

Note: N is chosen this way because, we can have any 1 of these possible arrangements (n!) to start with.

All of these arrangements of iphones are equally likely for the trial / game.

We define the following events

E_i = Picking Iphone i at the i th step

E_{ij} = Picking Iphone i at the ith step and picking Iphone j at the jth step

$E_{ij} = E(E_i \cap E_j)$

Similarly $E_{ijk} = E(E_i \cap E_j \cap E_k)$

Let N_1 be the number of sequences which favor E_i

$\therefore N_1 =$ Number of sequences which has i at the ith place.

$$\_\_\quad \_\_\quad \_\_\quad \boxed{\overset{\_\_}{i}}\quad \_\_\quad \_\_$$

$\therefore N_1 = 1 \times$ No. of ways of distributing $(n-1)$ iphones to $(n-1)$ places.

$$\therefore N_1 = 1 \times (n-1)!$$

$$\boxed{N_1 = (n-1)!}$$

Similarly

$N_2 =$ Number of sequences which has i at the i th place, j at the jth place.

$\therefore N_2 = 1 \times 1 \times$ no. of ways of distributing $(n-2)$ i phones

$$\boxed{N_2 = (n-2)!}$$

similarly, $\boxed{N_3 = (n-3)!}$

As we know the outcomes favouring an event E & Total outcomes, we can calculate the following probabilities.

$$P_r(E_i) = \frac{N_1}{N}$$

$$= \frac{(n-1)!}{n!}$$

$$P_r(E_i) = \frac{1}{n}, \text{ where } i = 0, 1 \cdots, n$$

$$P_r(E_{ij}) = \frac{N_2}{N}$$

$$= \frac{(n-2)!}{n!}$$

$$P_r(E_{ij}) = \frac{1}{n(n-1)}, \text{where } 1 \leq i < j \leq n$$

Similarly, $P_r(E_{ijk}) = \frac{N_3}{N} = \frac{1}{n(n-1)(n-2)}$

where $1 \leq i < j < k \leq n$

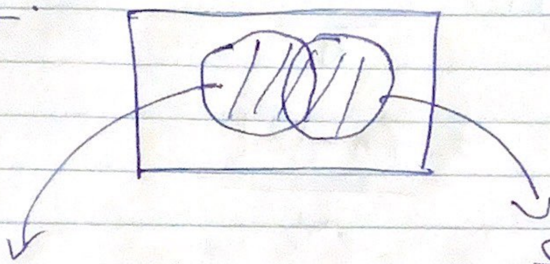
It follows that $E_{ij \cdots n}$, where $i=1, j=2$ .... (choose i th Iphone at i th step for all i) (choose Iphone with number = trial number)

$$= \frac{1}{n(n-1)(n-2)\cdots 1} = \frac{1}{n!}$$

Pr (Ending up with at least one iphone)

$$= Pr\left(\bigcup_{i=1}^{n} E_i\right)$$

<u>Why?</u>



S_1 = Sequences leading to E_1

S_2 = Sequences leading to E_2

$\ldots\ldots S_n$

Shaded = E (I have at least one Iphone)

Non shaded = E (I have no I phone)

<u>From PIE</u>

$$Pr\left(\bigcup_{i=1}^{n} E_i\right) = \sum_i Pr(E_i) - \sum_{i<j} Pr(E_i \cap E_j)$$

$$+ \sum_{i<j<k} Pr(E_i \cap E_j \cap E_k)$$

$$- \ldots\ldots + (-1)^{n+1} Pr(E_i \cap \ldots \cap E_n)$$

$$= \sum_{i} \frac{1}{n} - \sum_{i<j} \frac{1}{n(n-1)} + \sum_{i<j<k} \frac{1}{n(n-1)(n-2)} - \cdots$$

$$+ (-1)^{n+1} \frac{1}{n!}$$

$$= \frac{1}{n} \sum_{i} (1) - \frac{1}{n(n-1)} \sum_{i<j} (1) + \frac{1}{n(n-1)(n-2)} \sum_{i<j<k} (1)$$

$$- \cdots\cdots + (-1)^{n+1} \frac{1}{n!}$$

No. of terms in $\sum_{i<j}$, i.e; $1 \le i < j \le n$

$$= {}^{n}C_{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2!}$$

Similarly no. of terms in $\sum_{i<j<k}$ i.e; $1 \le i < j < k \le n$

$$= {}^{n}C_{3} = \frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{3!}$$

and so on.

$$\therefore Pr\left(\bigcup_{i=1}^{n} E_i\right) = \frac{1}{\cancel{n}} \times \cancel{n} - \frac{1}{\cancel{n(n-1)}} \times \frac{\cancel{n(n-1)}}{2!}$$

$$+ \frac{1}{\cancel{n(n-1)(n-2)}} \times \frac{\cancel{n(n-1)(n-2)}}{3!}$$

$$- \cdots\cdots + (-1)^{n+1} \frac{1}{n!}$$

$$= 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots\cdots + (-1)^{n+1} \frac{1}{n!}$$

$\therefore$ Pr $\left(\begin{array}{c}\text{Ending up with atleast} \\ \text{one iphone}\end{array}\right)$

$$= 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots\cdots + (-1)^{n+1} \frac{1}{n!}$$

Given,

3) Total number of rings $n = 10^4$,

Let Probability of Bilbo's ring being ONERING $\Rightarrow P(B_{OR})$

$$\Rightarrow P(B_{OR}) = \frac{1}{n} = \frac{1}{10^4} = 10^{-4} \quad \text{—— ①}$$

Let, event 'L' be that Owner has above average Life

$$P(L / B_{OR}) = 0.95 \quad \text{—— ②}$$

(Prob that owner has above average life given the ring is one ring)

From ② we can conclude,

$$P(\bar{L} / B_{OR}) = 1 - P(L / B_{OR})$$
$$= 1 - 0.95$$
$$= 0.05 \quad \text{—— ③}$$

where $\bar{L}$ is event that owner has below average life.

Also, given is $P(\bar{L} / \bar{B}_{OR}) = 0.75$ —— ④

$$\therefore P(L / \bar{B}_{OR}) = 1 - P(\bar{L} / \bar{B}_{OR})$$
$$= 0.25 \quad \text{—— ⑤}$$

(a) In Part a we are asked

$\boxed{P(B_{OR} / L)}$ i.e. Prob that ring is indeed ONERING given Bilbo has above average life

So, using Bayes Theorem

$$P(B_{OR}/L) = \frac{P(L/B_{OR})\,P(B_{OR})}{P(L/B_{OR})\,P(B_{OR}) + P(L/\overline{B_{OR}})\,P(\overline{B_{OR}})}$$

We have already computed all the values except $P(\overline{B_{OR}})$ → which will be :

$$P(\overline{B_{OR}}) = 1 - P(B_{OR}) = 1 - 10^{-4} \quad \text{———} (6)$$

∴ So, substituting values using (1), (2), (5) & (6)

we get

$$P(B_{OR}/L) = \frac{0.95 \times 10^{-4}}{0.95 \times 10^{-4} + 0.25\,(1 - 10^{-4})}$$

$$= 0.000379$$

$$\boxed{\therefore \text{(a)} \quad P(B_{OR}/L) = 0.000379}$$

3 (b) Now, Let event 'W' be that writing appears on the ring.

Given is $P(W/B_{OR}) = 0.9$ — (7)

$\Rightarrow P(\overline{W}/B_{OR}) = 1-0.9 = 0.1$ — (8)

Also, given is

$P(W/\overline{B_{OR}}) = 0.05$ — (9)

$\therefore P(\overline{W}/\overline{B_{OR}}) = 1-0.05 = 0.95$ — (10)

Now, we are asked

$P(B_{OR} / W \cap L)$ i.e. Prob. that ring is ONE RING given the event W & L holds.

Given is that 'W' & 'L' tests are conditionally independent on event (B_{OR}).

$\therefore$ Using Conditional Independence property that

$P(A \cap B | C) = P(A/C) \cdot P(B/C)$

(Here, A & B are conditionally independent of C)

$\therefore P(W \cap L / B_{OR}) = P(W|B_{OR})\, P(L/B_{OR})$ — (11)

Again using Bayes Theorem for part (b)

$$P(B_{OR} \mid W \cap L) = \frac{P(W \cap L \mid B_{OR})\, P(B_{OR})}{P(W \cap L \mid B_{OR}) P(B_{OR}) + P(W \cap L \mid \bar{B}_{OR}) \times P(\bar{B}_{OR})}$$

$$= \frac{P(W \mid B_{OR})\ P(L \mid B_{OR})\ P(B_{OR})}{P(W \mid B_{OR})\ P(L \mid B_{OR}) P(B_{OR}) + P(W \mid \bar{B}_{OR}) P(L \mid \bar{B}_{OR}) P(\bar{B}_{OR})}$$

Substituting all the values computed till now,

$$= \frac{0.9 \times 0.95 \times 10^{-4}}{0.9 \times 0.95 \times 10^{-4} + 0.05 \times 0.25 \times (1 - 10^{-4})}$$

$$= 0.00684$$

$$\boxed{\therefore \text{ For part (b) } P(B_{OR} / W \cap L) = 0.00684}$$

4) $X \rightarrow$ non negative Integer valued R.V

$\Rightarrow (X > 0)$

To Prove: $E[X] = \sum_{x=0}^{\infty} Pr[X > x]$

$E[x]$ is the expectation of R.V. X

which we know is,

$$E[x] = \sum_{x} x P(x)$$

here we have $x = 0$ to ∞

$\Rightarrow$ LHS $\quad \therefore E[x] = \sum_{x=0}^{\infty} x P(x) \quad —— ①$

We have value of LHS in ①, now we need to prove RHS also should be equal to ① i.e. LHS = RHS

$\Rightarrow$ RHS $\quad \sum_{x=0}^{\infty} P(X > x)$

As, X is Integer valued we can write

$$P(X > x) = P(X = x+1) + P(X = x+2) + ---- \infty$$

$$= P(x+1) + P(x+2) + P(x+3) + ---- \infty$$

So we have to calculate

$$\sum_{x=0}^{\infty} P(X > x) = \sum_{x=0}^{\infty} \left(P(x+1) + P(x+2) + P(x+3) + ---- \infty \right)$$

—— ②

Substituting different values of 'x' in (2)

$$\Rightarrow \sum_{x=0}^{\infty} \left(P(x+1) + P(x+2) + P(x+3) + \cdots \infty \right)$$

$\underline{\underline{x=0}}$

$$\Rightarrow P(0+1) + P(0+2) + P(0+3) + \cdots \infty$$
$$= P(1) + P(2) + P(3) + \cdots \infty \quad \text{——— (3)}$$

$\underline{\underline{x=1}}$

$$\Rightarrow P(1+1) + P(1+2) + P(1+3) + \cdots \infty$$
$$= P(2) + P(3) + P(4) + \cdots \infty \quad \text{——— (4)}$$

$\underline{\underline{x=2}}$

$$\Rightarrow P(2+1) + P(2+2) + P(2+3) + \cdots \infty$$
$$= P(3) + P(4) + P(5) + \cdots \infty \quad \text{——— (5)}$$

and so on up to $x = \infty$

Now (2) = (3) + (4) + (5) + $\cdots \infty$

$$\sum_{x=0}^{\infty} P(X > x) = P(1) + P(2) + P(3) + P(4) + \cdots \infty$$
$$+ \quad P(2) + P(3) + P(4) + \cdots \infty$$
$$+ \quad P(3) + P(4) + \cdots \infty$$

$$= 1 \cdot P(1) + 2P(2) + 3P(3) + 4P(4) + \cdots \infty$$

which can be written as

$$\sum_{x=0}^{\infty} x P(x)$$

$\therefore$ RHS also equals to $\sum_{x=0}^{\infty} x P(x)$

which is same as LHS

Hence Proved.

5.) (a.) $I[E] \sim$ Indicator R.V. with event E

Using PMF of Indicator R.V. introduced in Class we get

$$P_X(x) = \begin{cases} Pr(E) & \text{if } x = 1 \\ 1 - Pr(E) & \text{if } x = 0 \end{cases}$$

We know that

$$E[X] = \sum_{x} x P_X(x)$$

Here we have $x \to 0$ or $x \to 1$

$$\therefore E[X] = \sum_{0}^{1} x P_X(x)$$

$$= 0 \times P_X(0) + 1 \times P_X(1)$$

$$= 0 + 1 \times Pr(E) \qquad (\because P_X(1) = Pr(E))$$

$$= Pr(E)$$

$$\boxed{\therefore \quad E[I_E] = Pr(E)}$$

5. (b) $Var(I_E) = ?$

Let X denote I_E

we are asked $Var[X]$

which we know is $Var[X] = E[X^2] - (E[X])^2$

where $E[X^2] =$ 2nd moment of $X \Rightarrow \sum_x x^2 P(x)$

$$\Rightarrow E[X^2] = \sum_x x^2 P(x)$$

as $x \to 0$ or 1

$$\therefore E[X^2] = \sum_{x=0}^{1} x^2 P(x)$$

and $E[X] = P_r(E)$ (Already computed)

$$\therefore Var[X] = E[X^2] - (E[X])^2$$

$$= \sum_{x=0}^{1} x^2 P(x) - (P_r(E))^2$$

$$= 0\,P(0) + 1(P(1)) - (P_r(E))^2$$

$$= 0 + 1\,P_r(E) - (P_r(E))^2$$

$$= P_r(E)(1 - P_r(E))$$

$$\boxed{\therefore Var[I_E] = P_r(E)(1 - P_r(E))}$$

5.(C) $X \sim \text{Geometric}(P)$, $P < 1$

We need to calculate $E[X]$.

Also, given is $\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$ for $x < 1$

$\Rightarrow$ Using definition of Geometric RV.

$$P_X(i) = (1-P)^{i-1} (P)^1$$

(where P & i are probability of 1^{st} success at i^{th} trial.)

$$E[X] = \sum_{i=1}^{\infty} i \, P_X(i)$$

$$= \sum_{i=1}^{\infty} i \, (1-P)^{i-1} (P)$$

$$= P \sum_{i=1}^{\infty} i \, (1-P)^{i-1}$$ (Take out the constant variable)

$$\Rightarrow E[X] = P \sum_{i=1}^{\infty} i \, (1-P)^{i-1}$$

$$\Rightarrow E[X] = P \left(1 \cdot (1-P)^{1-1} + 2 \cdot (1-P)^{2-1} + 3 \cdot (1-P)^{3-1} + \cdots \infty \right)$$

$$\Rightarrow E[X] = P \left(1(1-P)^0 + 2(1-P) + 3(1-P)^2 + \cdots \infty \right)$$ — (1)

Multiply both sides of (1) by $(1-P)$

$$(1-p)\, E[x] = (1-p)\left[p\left((1-p)^0 + 2(1-p) + 3(1-p)^2 + \cdots \infty\right)\right]$$

$$(1-p)\, E[x] = p\left((1-p) + 2(1-p)^2 + 3(1-p)^3 + \cdots \infty\right) \quad ②$$

Subtracting ② from ①

$$E[x] - (1-p)E[x] = p\left((1-p)^0 + 2(1-p) + 3(1-p)^2 + 4(1-p)^3 + \cdots \infty\right) - p\left((1-p) + 2(1-p)^2 + 3(1-p)^3 + \cdots \infty\right)$$

$$(1-(1-p))\,E[x] = p\left[1 + 2(1-p) + 3(1-p)^2 + 4(1-p)^3 + \cdots \infty - \left((1-p) + 2(1-p)^2 + 3(1-p)^3 + \cdots \infty\right)\right]$$

$$\cancel{p}\, E[x] = \cancel{p}\left[1 + (1-p) + (1-p)^2 + (1-p)^3 + \cdots \infty\right]$$

$$E[x] = 1 + (1-p) + (1-p)^2 + (1-p)^3 + \cdots \infty$$

If we assume $x = 1 - p$ $(p < 1)$

We know that $1 + x + x^2 + x^3 + \cdots \infty = \frac{1}{1-x}$ for $x < 1$

as $(p < 1)$ $\therefore$ x is also < 1

$$\therefore 1 + (1-p) + (1-p)^2 + (1-p)^3 + \cdots \infty$$

$$= \frac{1}{1-(1-p)} = \frac{1}{p}$$

$$\boxed{\therefore E[x] = \frac{1}{p}}$$

5(d.) $Var(X) = E[x^2] - E[x]^2$

We know from 5(c) $\Rightarrow E[x] = \frac{1}{p}$

Also $E[x^i] = \sum_x x^i P(x)$ (where i can be 2, 3, 4 etc

$\Rightarrow E[x^2] = \sum_{x=1}^{\infty} x^2 P(x)$

For Geometric distribution we know

$$P(x) = (1-p)^{x-1}(p) \quad \text{for } 1 \leq x < \infty$$

$\therefore Var[x] = E[x^2] - E[x]^2$

$$Var[x] = \sum_{x=1}^{\infty} x^2 (1-p)^{x-1}(p) - \left(\frac{1}{p}\right)^2$$

$$\Rightarrow Var[x] = p \sum_{x=1}^{\infty} x^2 (1-p)^{x-1} - \left(\frac{1}{p}\right)^2$$

$$\Rightarrow Var[x] = p\left(1(1-p)^{1-1} + 4(1-p)^{2-1} + 9(1-p)^{3-1} + 16(1-p)^{4-1} + \cdots \infty\right) - \left(\frac{1}{p}\right)^2$$

$$\Rightarrow Var[x] = p\left((1-p)^0 + 4(1-p)^1 + 9(1-p)^2 + 16(1-p)^3 + \cdots \infty\right) - \frac{1}{p^2} \quad (1)$$

Let $t = 1-p$

Multiplying both sides by t in (1)

$$\Rightarrow t \, Var[x] = t \cdot \left((1-t)\left(t^0 + 4t^1 + 9t^2 + 16t^3 + \cdots \infty\right) - \frac{1}{(1-t)^2}\right)$$

$$\Rightarrow t \cdot Var[x] = (1-t)\left(t + 4t^2 + 9t^3 + 16t^4 + \cdots \infty\right) - \frac{t}{(1-t)^2}$$

(2.)

Subtracting (2) from (1.)

$$\Rightarrow Var[x] - t \cdot Var[x] = (1-t)\left(1 + 4t + 9t^2 + 16t^3 + \cdots \infty\right) - \frac{1}{(1-t)^2}$$
$$- \left((1-t)\left(t + 4t^2 + 9t^3 + \cdots \infty\right) - \frac{t}{(1-t)^2}\right)$$

$$\Rightarrow (1-t)\, Var[x] = (1-t)\left(1 + (4t - t) + (9t^2 - 4t^2) + (16t^3 - 9t^3) + \cdots \infty\right)$$
$$- \frac{1}{(1-t)^2} + \frac{t}{(1-t)^2}$$

$$\Rightarrow (1-t)\, Var[x] = (1-t)\left(1 + 3t + 5t^2 + 7t^3 + \cdots \infty\right) + \frac{t-1}{(1-t)^2}$$

$$\Rightarrow (1-t)\, Var[x] = (1-t)\left(1 + 3t + 5t^2 + 7t^3 + \cdots \infty\right) - \frac{1}{(1-t)}$$

$$\Rightarrow Var[x] = \left(1 + 3t + 5t^2 + 7t^3 + \cdots \infty\right) - \frac{1}{(1-t)^2}$$

(3.)

Multiply both sides of (3) by t

$$\Rightarrow t \cdot Var[x] = \left(t + 3t^2 + 5t^3 + 7t^4 + \cdots \infty\right) - \frac{t}{(1-t)^2}$$

(4.)

Subtracting (4) from (3.)

$$\Rightarrow (1-t)\,Var[X] = (1+2t+2t^2+2t^3+2t^4+\cdots\infty) + \frac{(t-1)}{(1-t)^2}$$

$$\Rightarrow (1-t)\,Var[X] = 1+2t\,(1+t+t^2+t^3+\cdots\infty) + \frac{(t-1)}{(1-t)^2}$$

We know that

$$1+t+t^2+t^3+\cdots\infty = \frac{1}{1-t} \quad \text{for } t<1$$

$$\therefore \quad (1-t)\,Var[X] = 1+2t\left(\frac{1}{1-t}\right) + \frac{(t-1)}{(1-t)^2}$$

$$\Rightarrow (1-t)\,Var[X] = 1 + \frac{2t}{(1-t)} - \frac{1}{(1-t)}$$

$$\Rightarrow Var[X] = \frac{1}{(1-t)} + \frac{1}{(1-t)}\left(\frac{2t-1}{1-t}\right)$$

Substituting back $t = 1-p$

$$\Rightarrow Var[X] = \frac{1}{p} + \frac{1}{p}\left(\frac{2(1-p)-1}{p}\right)$$

$$= \frac{1}{p} + \frac{1}{p}\left(\frac{1-2p}{p}\right)$$

$$= \frac{p+1-2p}{p^2} = \frac{1-p}{p^2}$$

$$\boxed{\therefore \; Var[X] = \frac{1-p}{p^2}}$$

6) p.m.f

$$P_X(i) = \frac{e^{-\lambda}\lambda^{i}}{i!}, \quad i \geq 0$$

a) <u>To prove</u> :

$$\sum_{i=0}^{+\infty} P_X(i) = 1$$

$$\sum_{i=0}^{+\infty} P_X(i) = \sum_{i=0}^{+\infty} \frac{e^{-\lambda}\lambda^{i}}{i!}$$

$$= e^{-\lambda} \sum_{i=0}^{+\infty} \frac{\lambda^{i}}{i!}$$

$$= e^{-\lambda}\left[\frac{\lambda^{0}}{0!} + \frac{\lambda^{1}}{1!} + \frac{\lambda^{2}}{2!} + \cdots + \frac{\lambda^{\infty}}{\infty!}\right]$$

$$= e^{-\lambda}\left[1 + \frac{\lambda}{1!} + \frac{\lambda^{2}}{2!} + \cdots + \frac{\lambda^{\infty}}{\infty!}\right]$$

From Taylor's expansion of e^{x},

$$e^{x} = 1 + x + \frac{x^{2}}{2!} + \frac{x^{3}}{3!} + \cdots$$

$$(\text{or}) \quad e^{x} = \sum_{n=0}^{\infty} \frac{x^{n}}{n!}$$

for $x \in \mathbb{R}$

$$\therefore \sum_{i=0}^{+\infty} P_x(i) = e^{-\lambda}\left[1+\frac{\lambda}{1!}+\frac{\lambda^2}{2!}+\ldots+\frac{\lambda^{\infty}}{\infty!}\right]$$

$$\boxed{1+\frac{\lambda}{1!}+\frac{\lambda^2}{2!}+\ldots = e^{\lambda}}$$

a.1

$$\therefore \sum_{i=0}^{+\infty} P_x(i) = e^{-\lambda}\left[e^{\lambda}\right]$$

$$= 1$$

$$\boxed{\therefore \sum_{i=0}^{+\infty} P_x(i) = 1}$$

b) Find $E[x]$

From definition for $E[x]$

$$E[x] = \sum_{i=0}^{+\infty} i P_x(i)$$

When $i=0$, $i P_x(i) = 0$

$$\therefore \sum_{i=0}^{+\infty} i P_x(i) = \sum_{i=1}^{+\infty} i P_x(i)$$

$$\therefore \quad E[X] = \sum_{i=1}^{+\infty} i P_X(i)$$

$$= \sum_{i=1}^{+\infty} \frac{i e^{-\lambda} \lambda^i}{i!} \quad , \quad i \geq 1$$

$$= \left(e^{-\lambda}\right) \left(\sum_{i=1}^{+\infty} \frac{i \lambda^i}{i!} \right)$$

$$= \left(e^{-\lambda}\right) \left(\sum_{i=1}^{+\infty} \frac{\lambda^i}{(i-1)!} \right)$$

We will try to change the summation to j
Where,

$$j = i-1$$

$$= \left(e^{-\lambda}\right) \left(\sum_{i=1}^{+\infty} \frac{\lambda \lambda^{i-1}}{(i-1)!} \right)$$

$$= \left(e^{-\lambda}\right)(\lambda) \left(\sum_{i=1}^{+\infty} \frac{\lambda^{i-1}}{(i-1)!} \right)$$

Using $j = i-1$,

$$E[X] = \left(e^{-\lambda}\right)(\lambda) \left(\sum_{j=0}^{+\infty} \frac{\lambda^j}{j!} \right)$$

From a.1 we know,

$$\sum_{j=0}^{+\infty} \frac{\lambda^j}{j!} = e^{\lambda}$$

$$\therefore \quad E[x] = (e^{-\lambda})(\lambda)(e^{\lambda})$$

$$= \lambda$$

$$\boxed{\therefore \quad E[x] = \lambda}$$

7)

$$f_X(x) = \alpha x^{-\alpha-1}, \quad x \geq 1, \quad 1 < \alpha < 2.$$

a) <u>To prove</u>

$$\int_1^{+\infty} f_X(x)\,dx = 1$$

$$\int_1^{+\infty} f_X(x)\,dx = \int_1^{+\infty} \alpha x^{-\alpha-1}\,dx$$

$$= \alpha \int_1^{+\infty} x^{-\alpha-1}\,dx$$

We know that,

$$\int_a^b x^n\,dx = \left[\frac{x^{n+1}}{n+1}\right]_a^b$$

$$\therefore \int_1^{+\infty} f_X(x)\,dx = \alpha \left[\frac{x^{-\alpha-1+1}}{-\alpha-1+1}\right]_1^{+\infty}$$

$$= \alpha \left[\frac{x^{-\alpha}}{-\alpha}\right]$$

$$= -\left[\frac{1}{x^{\alpha}}\right]_{1}^{\infty}$$

$$= -\left[\frac{1}{\infty^{\alpha}} - \frac{1}{1^{\alpha}}\right]$$

Since $1 < \alpha < 2$,

$$\frac{1}{\infty^{\alpha}} = 0$$

$$\frac{1}{1^{\alpha}} = \frac{1}{1} = 1$$

$$= -\left[0 - 1\right] = 1$$

$$\therefore \int_{1}^{+\infty} f_X(x)\,dx = 1$$

b) Find $E[x]$

From definition of $E[x]$ for continuous distribution,

$$E[x] = \int_{1}^{+\infty} x f_X(x)\, dx$$

$$= \int_{1}^{+\infty} x \alpha x^{-\alpha-1}\, dx$$

$$= \alpha \int_{1}^{+\infty} x^{-\alpha}\, dx.$$

$$= \alpha \left[\frac{x^{-\alpha+1}}{-\alpha+1} \right]_{1}^{+\infty}$$

$$= \frac{\alpha}{1-\alpha} \left[x^{-\alpha+1} \right]_{1}^{+\infty}$$

$$= \frac{\alpha}{1-\alpha} \left[\frac{1}{x^{\alpha-1}} \right]_{1}^{+\infty}$$

Since,

$$1 < \alpha < 2$$

$$1-1 < \alpha - 1 < 2-1$$

$$0 < \alpha - 1 < 1$$

$$E[x] = \frac{\alpha}{1-\alpha}\left[\frac{1}{\infty^{\alpha-1}} - \frac{1}{1^{\alpha-1}}\right]$$

$$= \frac{\alpha}{1-\alpha}\left[\frac{1}{\infty^{\text{+ve fraction less than 1, greater than 0}}} - \frac{1}{1^{\text{+ve Fraction less than 1, greater than 0}}}\right]$$

$$= \frac{\alpha}{1-\alpha}\left[0 - 1\right]$$

$$\boxed{E[x] = \frac{\alpha}{\alpha - 1}}$$

c) Find Var[x]

From the definition of Var[x] for Continuous random Variable,

$$Var[x] = \int_{1}^{+\infty} x^2 f_X(x)\,dx - \left(E[x]\right)^2$$

$$\int_{1}^{+\infty} x^2 f_X(x)\,dx = \int_{1}^{+\infty} x^2 \alpha x^{-\alpha-1}$$

$$= \alpha \int_{1}^{+\infty} x^{-\alpha+1}$$

$$= \alpha \left[\frac{x^{-\alpha+2}}{-\alpha+2} \right]_{1}^{+\infty}$$

$$= \frac{\alpha}{2-\alpha} \left[\frac{1}{x^{\alpha-2}} \right]_{1}^{+\infty}$$

Since $1 < \alpha < 2$,

$$1-2 < \alpha-2 < 2-2$$

$$-1 < \alpha-2 < 0$$

$$\therefore \int_{1}^{+\infty} x^2 f_X(x)\,dx = \frac{\alpha}{2-\alpha}\left[\frac{1}{x^{\alpha-2}}\right]_1^{+\infty}$$

$$= \frac{\alpha}{2-\alpha}\left[\frac{1}{\infty^{-ve\ fraction\ <0,\ >-1}} - \frac{1}{1^{-ve\ fraction\ <0,\ >1}}\right]$$

$$= \frac{\alpha}{2-\alpha}\left[\infty^{+ve\ fraction\ <1,\ >0} - 1\right]$$

$$= \frac{\alpha}{2-\alpha}\left[\infty - 1\right]$$

$$= \frac{\alpha}{2-\alpha}(\infty) = \infty.$$

$$\therefore Var[x] = \infty - (E(x))^2$$

$$= \infty - \left(\frac{\alpha}{\alpha-1}\right)^2$$

$$\boxed{Var[x] = \infty}$$

8) (a) $U \sim \text{Uniform}(0,1)$

Given that X is a R.V where

$X = F^{-1}(U)$

We have to prove CDF of X is F.

$\Rightarrow$ CDF of X is denoted as $F_X(\alpha)$ for some α

$$F_X(\alpha) = Pr(X \leq \alpha)$$

$$= Pr(F^{-1}(U) \leq \alpha) \quad (\because X = F^{-1}(U))$$

$$= Pr(F(F^{-1}(U)) \leq F(\alpha))$$

(Taking anti Inverse on both sides because it is given F^{-1} exists)

$$= Pr(U \leq F(\alpha))$$

Which is same as CDF of U

i.e. $\int_{-\infty}^{F(\alpha)} U(x)\, dx$

For uniform distribution, we know that

$$U(x) \begin{cases} 0 & x \leq a \\ \frac{1}{b-a} & a < x < b \\ 0 & x \geq b \end{cases}$$

Here we have $a = 0$, $b = 1$

$$\therefore u(x) \begin{cases} 0 & x \leq 0 \\ 1 & 0 < x < 1 \\ 0 & x \geq 1 \end{cases}$$

$$\therefore \int_{-\infty}^{F(\alpha)} u(x)\,dx$$

$$= \int_{-\infty}^{0} 0 \cdot dx + \int_{0}^{F(\alpha)} 1 \cdot dx + \cancel{\int 0 \cdot dx}$$

$$= \int_{0}^{F(\alpha)} 1 \cdot dx$$

$$= F(\alpha)$$

$$= F$$

$\therefore$ Proved that CDF of X is F.

8) (b.) Given that,
Y is a random variable with
CDF → F.

Which means that $F_Y(\alpha) = F$

for some α.

To prove → $F(Y) \sim$ Uniform (0,1)

We know that F(Y) can be the same distribution as Uniform (0,1) distribution if CDF of both are same

∴ Taking CDF of LHS first.

On LHS → CDF of F(Y)
$= Pr(F(Y) \leq \alpha)$

Taking Inverse on both sides,
$= Pr(Y \leq F^{-1}(\alpha))$ —— (1)

We know that for some α
$F_x(\alpha) = Pr(x \leq \alpha)$

∴ in (1) applying same logic

we get : $Pr(Y \leq F^{-1}(\alpha)) = F(F^{-1}(\alpha))$
$= \alpha$

$\boxed{\therefore LHS = \alpha}$

Now, taking CDF of RHS

On RHS we have $\Rightarrow$ Uniform $(0,1)$

Let $Z \sim$ Uniform $(0,1)$

So, CDF of Z

$$= F_Z(\alpha)$$

For a uniform distribution, we know that

$$F_Z(\alpha) = \frac{\alpha - a}{b - a} \quad \text{for } a < \alpha < b$$

Here we have

$$a = 0$$
$$\& \ b = 1$$

$$\therefore F_Z(\alpha) = \frac{\alpha - 0}{1 - 0} = \boxed{\alpha}$$

Since CDF of LHS and RHS are same we can say that

$$\boxed{F(Y) \sim \text{Uniform}(0,1)}$$