CSE 544: Probability & Statistics for Data Science,
Spring 2021

Assignment - 2

Submitted by

Harmanpreet Singh Khurana, SBU ID: 113262379,
Mayank Jain, SBU ID: 113263864,
Rajadorai DS, SBU ID: 113259773,
Venkatesan Ravi, SBU ID: 113263484

1) Given that

$$Cov(x, y) = E[(x - E(x))(y - E(y))] \quad \text{--- } ①$$

$$Cov(x, y) = E(xy) - E(x)E(y) \quad \text{--- } ②$$

a) Let S_1 - Event of getting heads at first toss
S_2 - Event of getting heads at second toss
S_3 - Event of getting heads at third toss

Note, S_1, S_2, S_3 are mutually independent.
i.e, $S_1 \perp S_2 \perp S_3$ --- ③

We have, $x = S_1 (or) S_2 = S_1 + S_2$
$y = S_2 (or) S_3 = S_2 + S_3$

$$\therefore Cov(x, y) = cov(S_1 + S_2, S_2 + S_3)$$

We know $P(S_i) = \frac{1}{2}$ for $\forall i \in [1, 2, 3]$
We also know that coin tosses are bernoulli distribution,

$$\therefore Var(S_i) = P(S_i)(1 - P(S_i))$$
$$= 1/2(1 - 1/2) = 1/4.$$

$$Var(S_i) = \frac{1}{4} \quad \text{--- } ④$$

$\therefore \operatorname{cov}(S_1+S_2, S_2+S_3)$

$$= E\left(\left(S_1+S_2 - E(S_1+S_2)\right)\left(S_2+S_3 - E(S_2+S_3)\right)\right)$$

using ①

We Know, $E(A+B) = E(A) + E(B)$.

$$\boxed{\begin{aligned} &\therefore E(S_1+S_2) = E(S_1) + E(S_2) \\ &E(S_2+S_3) = E(S_2) + E(S_3) \end{aligned}} \to ⑤$$

$\therefore \operatorname{Cov}(S_1+S_2, S_2+S_3)$

$$= E\left(\left(S_1+S_2 - E(S_1) - E(S_2)\right)\left(S_2+S_3 - E(S_2) - E(S_3)\right)\right)$$

using ⑤

$$= E\left[\left(\left(S_1 - E(S_1)\right) + \left(S_2 - E(S_2)\right)\right)\left(\left(S_2 - E(S_2)\right) + \left(S_3 - E(S_3)\right)\right)\right]$$

Multiplying,

$$= E\left[\left(S_1 - E(S_1)\right)\left(S_2 - E(S_2)\right) + \left(S_2 - E(S_2)\right)\left(S_2 - E(S_2)\right) + \left(S_1 - E(S_1)\right)\left(S_3 - E(S_3)\right) + \left(S_2 - E(S_2)\right)\left(S_3 - E(S_3)\right)\right]$$

$$= E\left[(S_1 - E(S_1))\ (S_2 - E(S_2))\right]$$

$$+ E\left[(S_2 - E(S_2))^2\right]$$

$$+ E\left[(S_1 - E(S_1))(S_3 - E(S_3))\right]$$

$$+ E\left[(S_2 - E(S_2))\ (S_3 - E(S_3))\right]$$

$$= \text{Cov}(S_1, S_2) + E\left[(S_2 - E(S_2))^2\right] + \text{Cov}(S_1, S_3) + \text{Cov}(S_2, S_3)$$

since,

$S_1 \perp S_2 \perp S_3$, $\text{Cov}(S_i, S_j) = 0$ $\forall\ i,j \in [1,3]$ and $i \neq j$

We also know

$$E\left[[X - E(X)]^2\right] = \text{Var}(X)$$

$$\therefore \text{Cov}(X, Y) = 0 + \text{Var}(S_2) + 0 + 0$$

from (4), $\text{Var}(S_2) = 1/4$

$$\therefore \boxed{\text{Cov}(X, Y) = 1/4}$$

b) $X =$ fair sided dice

$$\{-5, -2, 0, 2, 5\}$$

Let i be the face we get on outcome.
We have,

$$i = \{-5, -2, 0, 2, 5\}$$

We have $\sum_{i=1}^{5} i = 0$.

Also $\sum_{i=1}^{5} i^3 = (-5)^3 + (-2)^3 + (0)^3 + (2)^3 + (5)^3$

$= 0$

From (2)

$$Cov(x, y) = E(xy) - E(x)E(y)$$

$$Cov(x, x^2) = E(x^3) - E(x)E(x^2)$$

We have, $P_i = 1/5$

$P_{ii} = 1/25$

$P_{iii} = 1/125$

$$E(x) = \sum_{i=1}^{5} i\, p_i$$

$$= \sum_{i=1}^{5} i\left(\frac{1}{5}\right) = \frac{1}{5}\sum_{i=1}^{5} i$$

$$= 1/5\,(0)$$

$$= 0.$$

$$\therefore E(x)\,E(x^2) = 0 \times E(x^2) = 0$$

$$\therefore \operatorname{Cov}(x, x^2) = E(x^3)$$

$$E(x^3) = \sum_{i=1}^{5} i^3\, p_{iii}$$

$$= \sum_{i=1}^{5} i^3\left(\frac{1}{125}\right) = \frac{1}{125}\sum_{i=1}^{5} i^3$$

$$= \frac{1}{125}(0)$$

$$= 0$$

$$\therefore \operatorname{Cov}(x, x^2) = 0$$

Note that p_{iii} depends on p_i and still $\operatorname{Cov}(x, x^2) = 0$.

c) No.

$cov(x, f(x)) \neq 0$ iff $f(x)$ is linear in terms of x. Cov won't catch dependency if the two terms are not linearly dependant.

ex Let $f(x) = x^2$.

$cov(x, x^2) = 0$ from (1.b), Still x, x^2 are dependent.

$\therefore$ Covariance works only for linear dependency.

2) Inequalities:

(a) X is a non-negative RV
Mean $= \mu$
Variance $= \sigma^2$
t is a real number > 0

To prove:

$$E[X] \geq \int_t^{\infty} x \cdot f(x)\, dx$$

Solution:

$$E[X] = \int_{-\infty}^{+\infty} x \cdot f(x)\, dx$$

As $x > 0$, $E[X] = \int_0^{\infty} x \cdot f(x) \cdot dx$

t is any +ve number between 0 and ∞

$$\therefore E[X] = \underbrace{\int_0^t x \cdot f(x)\, dx}_{A} + \underbrace{\int_t^{\infty} x \cdot f(x)\, dx}_{B}$$

For A, we know $x > 0$, $f(x) > 0$ and $t > 0$

Hence, $\int_0^t x \cdot f(x)\, dx \geq 0$

$\therefore E[X] = (\geq 0) + \int_t^\infty x \cdot f(x) \cdot dx$

$\therefore E[X] \geq \int_t^\infty x \cdot f(x)\, dx$

Hence, proved

(b) To prove : $P_r(X > t) \leq \frac{E[X]}{t}$

From (a), $E[X] \geq \int_t^\infty x f(x) \cdot dx \longrightarrow (1)$

As we know $t > 0$, $x > 0$.

Minimum value of 'x' $= t$

$\therefore x \geq t$

Using (applying) $f(x)$ on both sides, (where $f(x) > 0$)

$x \cdot f(x) \geq t \cdot f(x)$

Integrating on both sides,

$$\int_t^\infty x \cdot f(x)\, dx \geq \int_t^\infty t \cdot f(x) \cdot dx$$

Hence, replacing (i) with the above explanation,

$$\Rightarrow E[x] \geq \int_t^\infty t \cdot f(x) \cdot dx$$

$$\Rightarrow E[x] \geq t \int_t^\infty f(x) \cdot dx$$

$\hookrightarrow P[x > t]$

$$\therefore E[x] \geq t\, P[x > t]$$

$$\Rightarrow \text{Hence, } P[x \geq t] \leq \frac{E[x]}{t}$$

<u>Hence, Proved.</u>

(c) To prove : $P(|X - \mu| \geq t) \leq \frac{\sigma^2}{t^2}$

Solution:

From(b), $P(X > t) \leq \frac{E[X]}{t}$

Replacing X with $(X-\mu)^2$,

$$P((X-\mu)^2 > t) \leq \frac{E[(X-\mu)^2]}{t}$$

We know, $t > 0$, hence replacing t with t^2 (inequality does not change)

$$P((X-\mu)^2 > t^2) \leq \frac{E[(X-\mu)^2]}{t^2}$$

$\hookrightarrow |X-\mu| > |t|$

(But $t > 0$, Hence,

$|X-\mu| > t$

$(\because \sigma^2 = E[(X-\mu)^2])$

$\hookrightarrow$ Hence, $P(|X-\mu| \geq t) \leq \frac{\sigma^2}{t^2}$

(As point probabilites are 0 for continuous RV, $|X-\mu| > t \Rightarrow |X-\mu| \geq t$

(3) Given,

$x_1, x_2, \cdots\cdots, x_K \Rightarrow K$ independent RVs.

Pdfs $\quad f_{x_i}(x) = \lambda_i e^{-\lambda_i x} \quad \forall \; i \in (1, 2, \cdots, K)$

Also, $\quad Z = \min(x_1, x_2, x_3, \cdots\cdots, x_K)$

$$Pr(Z \leq z) = Pr(\min(x_1, x_2, \cdots, x_K) \leq z)$$

$$= Pr(\text{Atleast one element in } (x_1, \cdots\cdots, x_K) \leq z)$$

$$= 1 - Pr(\text{No element in } (x_1 \cdots, x_2) \leq z)$$

$$= 1 - Pr(x_1 > z \text{ (and) } x_2 > z \; \cdots \; x_K > z)$$

Since, x_1, x_2 are R.V which are independent,

$$Pr(Z \leq z) = 1 - [Pr(x_1 > z) \, Pr(x_2 > z) \cdots Pr(x_K > z)]$$

— (1)

i) From definition of Pdf, CDF

$$F_{x_i}(x) = P_r(x \leq x) = CDF(x_i) \quad \text{— (2)}$$

$$F'_{x_i}(x) = f_{x_i}(x) = Pdf(x_i) \quad \text{— (3)}$$

CDF(z) from (2)

$$F_Z(z) = P_r(Z \leq z)$$

From (1)

$$F_Z(z) = 1 - \left[P_r(x_1 > z) \cdot P_r(x_2 > z) \cdots P_r(x_k > z)\right] \quad \text{— (4)}$$

We know

$$P_r(x_i > z) = 1 - P_r(x_i \leq z)$$

$$= 1 - F_{x_i}(z)$$

$$= 1 - CDF(x_i)$$

We have, $f_{X_i}(x) = \lambda_i e^{-\lambda_i x}$

$$f_{X_i}(z) = \lambda_i e^{-\lambda_i z}$$

$$F_{X_i}(z) = \int_0^z \lambda_i e^{-\lambda_i z} \, dz$$

$$= \lambda_i \int_0^z e^{-\lambda_i z} \, dz$$

$$= \lambda_i \left[-\frac{1}{\lambda_i} e^{-\lambda_i z} \right]_0^z$$

$$= -\left[e^{-\lambda_i z} \right]_0^z$$

$$= -\left[e^{-\lambda_i z} - 1 \right]$$

$$F_{X_i}(z) = P_r(X_i \le z) = 1 - e^{-\lambda_i z} \quad \text{— (5)}$$

$\therefore$ From (5) we have

$$P_r(X_i > z) = e^{-\lambda_i z} \quad \text{— (6)}$$

(6) in (4),

$$F_Z(z) = 1 - \left[e^{-\lambda_1 z} \cdot e^{-\lambda_2 z} \cdots \cdots e^{-\lambda_k z} \right]$$

$$F_Z(z) = 1 - \left[e^{-(\lambda_1 + \lambda_2 + \cdots \lambda_K) z} \right]$$

From (3) Pdf (z)

$$= F_Z'(z)$$

$$= 0 - \left[-(\lambda_1 + \lambda_2 + \cdots + \lambda_K) e^{-(\lambda_1 + \cdots + \lambda_K) z} \right]$$

$$= (\lambda_1 + \lambda_2 + \cdots + \lambda_K) e^{-(\lambda_1 + \lambda_2 + \cdots + \lambda_K) z}$$

Let $\lambda_T = \lambda_1 + \lambda_2 + \cdots + \lambda_K$

$$\boxed{\begin{array}{l} \therefore \text{Pdf}(z) = f_Z(z) = \lambda_T e^{-\lambda_T z} \\ \text{where } \lambda_T = \lambda_1 + \lambda_2 + \cdots + \lambda_K . \end{array}}$$

ii) $E[Z] = \int_0^\infty z f_Z(z) \, dz$

$$= \int_0^\infty z \lambda_T e^{-\lambda_T z} \, dz$$

$$= \lambda_T \int_0^\infty z e^{-\lambda_T z} \, dz$$

Let $u = z$, $dv = e^{-\lambda_T z} dz$

$\therefore du = dz$, $v = \frac{-1}{\lambda_T} e^{-\lambda_T z}$

$$\therefore E[z] = \lambda_T \left[uv - \int_0^\infty v\,du \right]$$

$$= \lambda_T \left[\frac{z e^{-\lambda_T z}}{-\lambda_T} \Big|_0^\infty - \int_0^\infty \frac{-1}{\lambda_T} e^{-\lambda_T z} dz \right]$$

$$= \lambda_T \left[0 + \frac{1}{\lambda_T} \int_0^\infty e^{-\lambda_T z} dz \right]$$

$$= \lambda_T \left[\frac{1}{\lambda_T} \times \frac{-1}{\lambda_T} \left[e^{-\lambda_T z} \right]_0^\infty \right]$$

$$= -\frac{1}{\lambda_T} \left[e^{-\lambda_T z} \right]_0^\infty$$

$$= -\frac{1}{\lambda_T} \left[0 - 1 \right]$$

$$= \frac{1}{\lambda_T}$$

$$\therefore E(z) = \frac{1}{\lambda_T} = \frac{1}{(\lambda_1 + \lambda_2 \cdots + \lambda_k)}$$

iii) Similarly integrating by parts twice,

we get

$E(X^2)$ for exp distribution $= \dfrac{1}{\lambda^2}$

where λ is the rate.

So $E(Z^2) = \dfrac{1}{\lambda_T^2} = \dfrac{1}{(\lambda_1 + \lambda_2 + \dots + \lambda_k)^2}$.

$\therefore Var(Z) = \dfrac{1}{(\lambda_1 + \lambda_2 + \dots + \lambda_k)^2}$.

b) $f_{xy}(x,y) = \begin{cases} 2 & 0 \le x \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$

Given $Z = XY$

$\therefore X = \frac{Z}{Y}, \ Y = \frac{Z}{X}$. —— ①

<u>To find</u>

Pdf (z)

We will use the similar approach in 3(a) to Calculate Pdf(z) by finding cdf(z)

$$cdf(z) = F_Z(z) = P_r(Z \le z)$$

$$cdf(z) = P_r(XY \le z) \text{ —— ②}$$

We have,

$Y \ge X$ (Given)

From ①

$\frac{Z}{X} > X$ (or) $Y \ge \frac{Z}{Y}$

$Y^2 \ge Z \ (\because y \ge 0)$

$\underline{y \ge \sqrt{z}}$ $(\because y \ge 0, x \ge 0, xy \ge 0)$ —— ③

From ②,

$$F_Z(z) = P_r(Z \leq z)$$

$$= 1 - P_r(Z > z)$$

$$= 1 - P_r(xy > z)$$

$P_r(xy > z)$

$$= \int_{y=b}^{1} \int_{x=a}^{y} f_{xy}(x, y)\, dx\, dy.$$

We need to find a, b.

One way is fix a and find b by lower limit of y.

From ③, we have $y \geq \sqrt{z}$ (or) $y \geq \sqrt{z}$

$(\because Z \geq z)$

Also, From ①

$$x = \frac{z}{y}.$$

$\therefore \Pr(xy > z)$

$$= \int_{y=\sqrt{z}}^{1} \int_{x=\frac{z}{y}}^{y} f_{XY}(x,y)\,dx\,dy.$$

$$= \int_{y=\sqrt{z}}^{1} \int_{x=\frac{z}{y}}^{y} 2\,dx\,dy.$$

$$= 2\left[\int_{y=\sqrt{z}}^{1} \int_{x=\frac{z}{y}}^{y} dx\,dy\right]$$

$$= 2\left[\int_{y=\sqrt{z}}^{1} \left[x\right]_{z/y}^{y} dy\right]$$

$$= 2\left[\int_{y=\sqrt{z}}^{1} \left(y - \frac{z}{y}\right) dy\right]$$

$$= 2\left[\int_{y=\sqrt{z}}^{1} y\,dy - z\int_{y=\sqrt{z}}^{1} \frac{1}{y}\,dy\right]$$

$$= 2\left[\left[\frac{y^2}{2}\right]'_{\sqrt{z}} - z\left[\log_e y\right]'_{\sqrt{z}}\right]$$

$$= 2\left[\frac{1-z}{2} - z\left(0 - \log_e \sqrt{z}\right)\right]$$

$$= 2\left[\frac{1-z}{2} + z\log_e \sqrt{z}\right]$$

$$= \left(\frac{1-z}{2}\right)2 + 2z\log_e \sqrt{z}$$

$$= (1-z) + z\log_e\left(\sqrt{z}\right)^2$$

$$\left(\because\ a\log b = \log b^a\right)$$

$$\therefore\ \Pr(Z > z) = 1 - z + z\log_e z$$

$$\therefore\ \mathrm{CDF}(z) = 1 - \left(1 - z + z\log_e z\right)$$

$$= z - z\log_e z$$

$$F'(z) = f(z) = \text{pdf}(z)$$

where $F'(z)$ is partial derivative of $F(z)$ wrt to z.

$$\therefore \text{pdf}(z) = F'(z) = 1 - \frac{\partial}{\partial z}\left(z \log_e z\right)$$

$$\frac{\partial}{\partial z}\left(z \log_e z\right) = \left(z \times \frac{1}{z}\right) + \left(\log_e z \times 1\right)$$

(by Product rule)

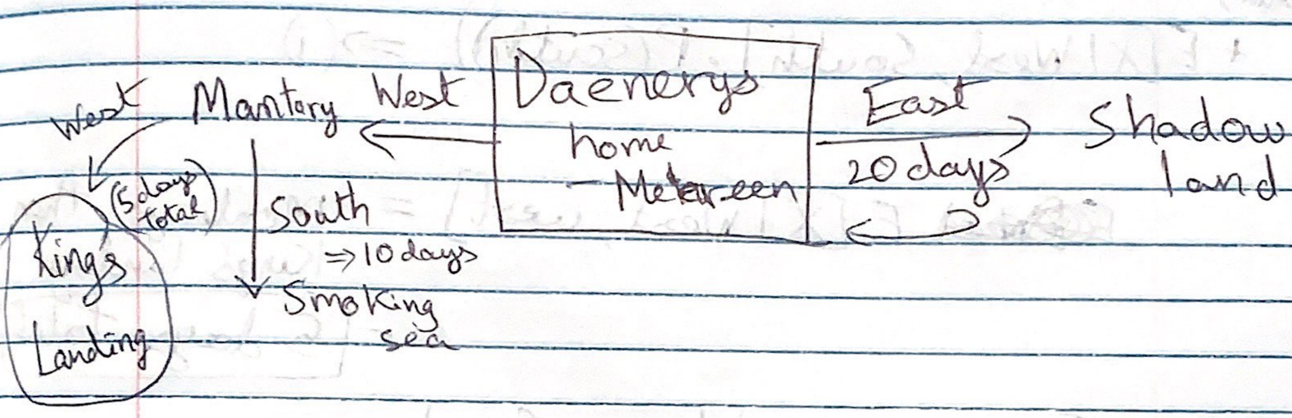
$$= 1 + \log_e z$$

$$\therefore \text{pdf}(z) = F'(z) = 1 - \left[1 + \log_e z\right]$$

$$= -\log_e z$$

$$\boxed{\therefore \text{pdf}(z) = f_z(z) = -\log_e z}$$

4)



☆ $E[X] = ?$

(where $X \Rightarrow$ time spent by Daenerys before reaching King's landing)

By law of total expectation,

$$E[X] = E[X \mid \text{east from Meeren}] \cdot Pr(\text{east from Meeren}) + E[X \mid \text{West from Meeren}] \cdot Pr(\text{west from Meeren})$$

$$Pr(\text{choosing any direction at a point in time}) = \boxed{\frac{1}{2}}$$

Further splitting the West journey,

$$E[X] = E[X \mid \text{east}] \cdot P(\text{east}) + \frac{1}{2}\Big(E[X \mid \text{West, West}] \cdot P(\text{West}) +$$

(Contd.)

$+ E[X \mid West, South] \cdot P(South)) \Rightarrow (i)$

$E[X \mid West, west]$ = Monterys then King's landing

= $\boxed{5 \text{ days total}}$

$E[X \mid West, south] \Rightarrow$ Smoking sea

↳ additional $\boxed{10 \text{ days}}$

$= \boxed{X+10}$

$E[X \mid east] \Rightarrow$ Shadow lands

↳ additional 20 days

$= \boxed{X+20}$

Substituting these values in (i),

$$= (E[X+20]) \cdot \frac{1}{2} + \frac{1}{2}\left(5 \times \frac{1}{2} + E[X+10] \cdot \frac{1}{2}\right)$$

Using Linearity of expectation,

$$\hookrightarrow E[X] = \frac{1}{2} E[X] + 10 + \frac{5}{4} + \frac{1}{4} E[X] + \frac{10}{4}$$

$$E[X] = \frac{3}{4}E[X] + 10 + \frac{15}{4}$$

(using LCM 4)

$$4E[X] - 3E[X] = \frac{55}{4}$$

$$\boxed{\therefore\ E[X] = 55}$$

(b) $Var[X] = ?$

We know $Var[X] = E[X^2] - (E[X])^2$

Obtaining $E[X^2]$ from equation (i) in part (a),

$$E[X^2] = \frac{1}{2}E[X^2 | East] + \frac{1}{2}\Big(E[X^2 | West, West] \cdot P(West) + E[X^2 | West, South] \cdot P(South)\Big)$$

$$\Rightarrow E[X^2] = \frac{1}{2}[(X+20)^2] + \frac{1}{2}\Big(\frac{1}{2} \times 25 + \frac{1}{2}E[(X+10)^2]\Big)$$

$$\Rightarrow E[X^2] = \frac{1}{2}E[400 + 40X + X^2] + \frac{1}{2}\Big(\frac{1}{2} \cdot 25 + \frac{1}{2}E[X^2 + 100 + 20X]\Big)$$

Using LOE,

$$E[X^2] = \frac{925}{4} + 25 + \frac{E[X^2]}{4} + 5E[X] + 20E[X] + 200$$

$$E[X^2] = \frac{3E[X^2]}{4} + 25E[X] + \frac{925}{4}$$

(using LCM 4)

$$4E[X^2] - 3E[X^2] = 100E[X] + 925$$

(Using (a), $E[X] = 55$)

$$= (100 \times 55) + 925$$

$\therefore$

$$E[X^2] = 6425$$

$$Var(X) = E[X^2] - (E[X])^2$$

$$= 6425 - (55)^2$$

$$= \boxed{3400}$$

Given ↓

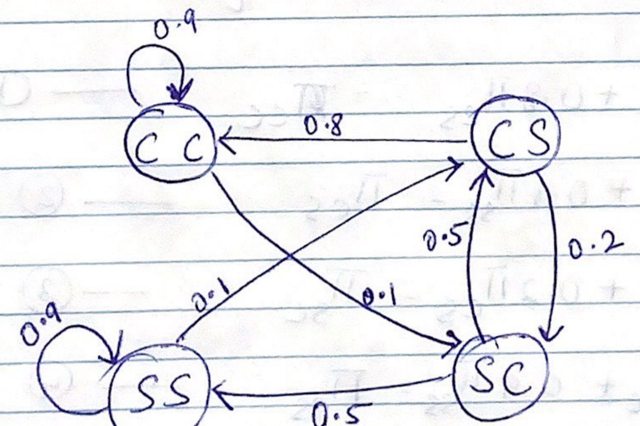
5.) $Pr[C|CC] = 0.9 \Rightarrow Pr[S|CC] = 1-0.9 = 0.1$

$Pr[C|CS] = 0.8 \Rightarrow Pr[S|CS] = 1-0.8 = 0.2$

$Pr[C|SC] = 0.5 \Rightarrow Pr[S|SC] = 1-0.5 = 0.5$

$Pr[C|SS] = 0.1 \Rightarrow Pr[S|SS] = 1-0.1 = 0.9$

(a.) Markov Chain :



Probability matrix →

$$(P) = \begin{array}{c|cccc} & CC & CS & SC & SS \\ \hline CC & 0.9 & 0 & 0.1 & 0 \\ CS & 0.8 & 0 & 0.2 & 0 \\ SC & 0 & 0.5 & 0 & 0.5 \\ SS & 0 & 0.1 & 0 & 0.9 \end{array}$$

Using Global Balance

$\rightarrow P^{n-1} \cdot P = P^n$

$P^{n-1} \approx P^n \quad (\text{as } n \rightarrow \infty)$

$\rightarrow P^{n-1} \cdot P = P^{n}$

$$[\pi_{cc}\ \pi_{cs}\ \pi_{sc}\ \pi_{ss}] \begin{bmatrix} 0.9 & 0 & 0.1 & 0 \\ 0.8 & 0 & 0.2 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0.1 & 0 & 0.9 \end{bmatrix}$$

$$= [\pi_{cc}\ \pi_{cs}\ \pi_{sc}\ \pi_{ss}]$$

$\Rightarrow$ $0.9\,\pi_{cc} + 0.8\,\pi_{cs} = \pi_{cc}$ —— ①

$0.5\,\pi_{sc} + 0.1\,\pi_{ss} = \pi_{cs}$ —— ②

$0.1\,\pi_{cc} + 0.2\,\pi_{cs} = \pi_{sc}$ —— ③

$0.5\,\pi_{sc} + 0.9\,\pi_{ss} = \pi_{ss}$ —— ④

Also, $\sum P_{rows} = 1$

$\Rightarrow$ $\pi_{cc} + \pi_{cs} + \pi_{sc} + \pi_{ss} = 1$ —— ⑤

Solving equations ①, ②, ③, ④ and ⑤

$$\left\{ \begin{aligned} \pi_{sc} &= \frac{1}{15} \\ \pi_{ss} &= \frac{5}{15} = \frac{1}{3} \\ \pi_{cc} &= \frac{8}{15} \\ \pi_{cs} &= \frac{1}{15} \end{aligned} \right\}$$

(b.) Pr. (Snowy 3 days from today) = ?

TP rule

$$= \pi_{CC} \times P(S \mid CC) + \pi_{CS} \times P(S \mid CS) + \pi_{SC}\, P(S \mid SC) + \pi_{SS} \times P(S \mid SS)$$

$$= \frac{8}{15} \times 0.1 + \frac{1}{15} \times 0.2 + \frac{1}{15} \times (0.5) + \frac{1}{3}(0.9)$$

$$= \frac{0.8}{15} + \frac{0.2}{15} + \frac{0.5}{15} + \frac{4.5}{15}$$

$$= \frac{6}{15}$$

$$\boxed{= 0.4}$$

5(c) Steady_State : Power iteration >>

$$\begin{array}{cccc} \pi_{cc} & \pi_{cs} & \pi_{sc} & \pi_{ss} \end{array}$$
$$\begin{bmatrix} 0.533333 & 0.066667 & 0.066667 & 0.333333 \end{bmatrix}$$

Complete Transition matrix with $k = 2000$ after python simulation

$$\begin{array}{c|c} & \begin{array}{cccc} CC & CS & SC & SS \end{array} \\ \begin{array}{c} CC \\ CS \\ SC \\ SS \end{array} & \begin{bmatrix} 0.533333 & 0.066667 & 0.066667 & 0.333333 \\ 0.533333 & 0.066667 & 0.066667 & 0.333333 \\ 0.533333 & 0.066667 & 0.066667 & 0.333333 \\ 0.533333 & 0.066667 & 0.066667 & 0.333333 \end{bmatrix} \end{array}$$

Using the steady state that we got from simulation to calculate part (b)

→ Pr (Snowy 3 days from today) = ?

$$\overset{\text{TP rule}}{=} \pi_{cc} P(S|CC) + \pi_{cs} P(S|CS) + \pi_{sc} P(S|SC) + \pi_{ss} P(S|SS)$$

$$= 0.533333 \times 0.1 + 0.066667 \times 0.2 + 0.066667 \times 0.5 + 0.333333 \times 0.9$$

$= 0.399999 \approx 0.4$ as calculated via our mathematical formulas in part a & b.

Code execution output for (5.c)

Hw2 — -zsh — 91×24

```
harman@harmans-MacBook-Pro Hw2 % python3 a2_5.py
Transition Matrix
[[0.533333 0.066667 0.066667 0.333333]
 [0.533333 0.066667 0.066667 0.333333]
 [0.533333 0.066667 0.066667 0.333333]
 [0.533333 0.066667 0.066667 0.333333]]

Steady State:
[0.533333 0.066667 0.066667 0.333333]

CC:  0.533333
CS:  0.066667
SC:  0.066667
SS:  0.333333

5(b): Steady State Probability for Snow third day from today: 0.39999989999999996
harman@harmans-MacBook-Pro Hw2 %
```

6.) (a.) Given ⇒ $X = (X_1, X_2, \cdots X_k)$ is a multivariate normal

∴ $t_1 X_1 + t_2 X_2 + \cdots t_k X_k$ is a linear Combination having Normal distribution.

$\Rightarrow t_1 X_1 + t_2 X_2 + t_3 X_3 + \cdots t_j X_j + \cdots t_k X_k$

Suppose in this combination all t_i's $= 0$ except t_j

$\therefore \quad 0 + 0 + 0 + \cdots t_j X_j + \cdots 0 \sim$ Normal

$\Rightarrow \quad t_j X_j \sim$ Normal

$$\boxed{\therefore \quad X_j \sim \text{Normal}}$$

(b.) $X = \text{Normal}(0, 1)$

$$S = \begin{cases} 1 & P = 1/2 \\ -1 & P = 1/2 \end{cases}$$

Also, $Y = SX$ is a Normal R.V.

$\therefore Y = SX$

$$\therefore Y \begin{cases} X & \text{with } P = 1/2 \\ -X & \text{with } P = 1/2 \end{cases}$$

Let $Z = (X, Y)$

To show Z is not a multivariate Normal we can show that some Linear combination of X & Y is not Normally distributed

Let linear combination be : $X + Y$

Now,
$$X+Y \begin{cases} X + (-X) & \text{with } P = 1/2 \\ X + (X) & \text{with } P = 1/2 \end{cases}$$

$$\Rightarrow X+Y \begin{cases} 0 & \text{with } P = 1/2 \\ 2X & \text{with } P = 1/2 \end{cases}$$

As we know if a distribution is Normal then the point probabilities are 0 because Normal distribution is Continuous and Point Probabilities in Continuous distribution are zero.

But here $X+Y \begin{cases} 0 & \text{with } P=1/2 \\ 2X & \text{with } P=1/2 \end{cases}$

$P(X+Y=0) > 0$

$\therefore$ $X+Y$ is not normally distributed

Similarly $X-Y$ will also not be normally distributed

$\therefore$ $Z=(X,Y)$ will not be a multivariate Normal

Because Linear combination of X & Y are not normal.

(c) (i) $Z, W \sim \text{Normal}(0,1)$

Also Z, W are iid RV

Let $X = (Z, W)$ be a R.V.

Linear combination of $Z, W \Rightarrow t_1 Z + t_2 W$

$\Rightarrow t_1 Z + t_2 W$ is also Normally distributed using Weighted sum of Independent Normals property

Because Z & W are given iid.

So, for any t_1, t_2 $t_1 Z + t_2 W$ will have a ~~[illegible]~~ Normal distribution

$\therefore$ Using defintion of Multivariate Normals

$X = (Z, W)$ will be a multivariate Normal distribution.

(ii) $(Z + 2W, 3Z + 5W)$

Let $X = (Z + 2W, 3Z + 5W)$

Linear combination $\Rightarrow$ $t_1(Z + 2W) + t_2(3Z + 5W)$

$\Rightarrow t_1 Z + 2t_1 W + 3t_2 Z + 5t_2 W$

$\Rightarrow (t_1 + 3t_2) Z + (2t_1 + 5t_2) W$

Let $t_1 + 3t_2 = t_3$
& $2t_1 + 5t_2 = t_4$

$\therefore \Rightarrow t_3 Z + t_4 W$

Now Z & W are given i.i.d & normals
$\therefore$ Using weighted sum of Independent Normals

$t_3 Z + t_4 W$ will be Normally distributed

$\therefore X \Rightarrow (Z + 2W, 3Z + 5W)$
is a multivariate Normal.

d.) $X = (X_1, X_2, \dots X_n)$
$Y = (Y_1, Y_2, \dots Y_m)$ are
multivariate normals & X, Y are independent

$\Rightarrow W = (X_1, X_2 \dots X_n, Y_1, Y_2 \dots Y_m)$
$\Rightarrow W = (X, Y)$

Linear Combination $\rightarrow t_1 X + t_2 Y$

$\because$ It is given X, Y are Independent & Normally distributed.

$\therefore$ $t_1 X + t_2 Y$ will also be Normally distributed using Weighted sum of Independent Normals

$\Rightarrow \therefore W = (X, Y)$ will be a multivariate Normal distribution.

(e.) X, Y $\Rightarrow$ iid & Normal

Let $Z = (X+Y, X-Y)$

Using Fact 2 (Prop. of Covariance)

$$\text{Covariance}(Z) \Rightarrow \text{Covariance}(X+Y, X-Y)$$

$$= 1.1\ Cov(X,X) + 1.(-1)\ Cov(X,Y) + 1.1\ Cov(Y,X) + 1.(-1)\ Cov(Y,Y)$$

$$= 1.Cov(X,X) - 1.Cov(X,Y) + 1.Cov(Y,X) - 1.Cov(Y,Y)$$

$$= 1.Cov(X,X) - 1.Cov(Y,Y)$$

$$= \text{Variance}(X) - \text{Variance}(Y)$$

Given that X & Y are iid $\therefore Var(X) - Var(Y) = 0$

$\therefore$ Covariance $(X+Y, X-Y) = 0$

$\therefore$ $(X+Y, X-Y)$ are uncorrelated

Using Fact 1 (Uncorrelated implies independence)

$\Rightarrow$ $(X+Y, X-Y)$ are Independent

$\therefore$ $Z = (X+Y, X-Y)$

is Multivariate normal using Fact 1 & Fact 2

NOTE :- Since $(X+Y), (X-Y)$ are linearly related and their covariance is zero we can say that they are uncorrelated or Independent.

7) X = no. of days needed to complete the goal

Let P_i $(P_1, P_2, \dots P_n)$ be the days to capture Pokemon 'i' after capturing 'i-1' Pokemon

Then, $X = P_1 + P_2 + P_3 + \dots P_n$

P_i can be thought of as Random variable with Geometric distribution as Geometric distribution indicates time to first success.

In our case, success can be said to be equivalent to capturing a Pokemon of type 'i' after some no. of days.

Also, $P_1, P_2 \dots P_n$ are Independent as we are calculating P_2 after P_1 has already occurred.

a) $E[X] = ?$

$\because X = P_1 + P_2 + P_3 + \dots P_n$

$\therefore E[X] = E[P_1 + P_2 + \dots P_n]$

Using Linearity of Expectation,

$\Rightarrow E[X] = E[P_1] + E[P_2] + E[P_3] + \dots E[P_n]$

$\because$ P_i has a Geometric Distribution

$\therefore$ Expectation of P_i $E[P_i] = \dfrac{1}{\text{Probability}}$

(Already calculated in Assignment 1
question - 5 (C))

Now, probability of event 1
Lets say $Pr(P_1)$ we need to find.

$Pr(P_1) = \dfrac{n}{n}$ (Using counting method
i.e. for first type of
Pokemon we have
'n' choices to capture
out of Total 'n' Pokemons)

Similarly,

$Pr(P_2) = \dfrac{n-1}{n}$ (After capturing Pokemon
of Type 1, 'n-1' are
left)

$Pr(P_3) = \dfrac{n-2}{n}$

$\vdots$

$Pr(P_n) = \dfrac{n-(n-1)}{n} = \dfrac{1}{n}$

$\therefore E[X] = E[P_1] + E[P_2] + \cdots E[P_n]$

$= \frac{1}{Pr(P_1)} + \frac{1}{Pr(P_2)} + \frac{1}{Pr(P_3)} + \cdots \frac{1}{Pr(P_n)}$

$= \frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \cdots \frac{n}{1}$

$= n\left(\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \cdots \frac{1}{2} + 1\right)$

$$\boxed{\therefore E[x] = n\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n-1} + \frac{1}{n}\right)}$$

(b) $Var[X]$

$\Rightarrow Var[X] = Var[P_1 + P_2 + P_3 + \cdots P_n]$

Using Linearity of Variance, as P_i's are Independent

$\Rightarrow Var[X] = var[P_1] + Var[P_2] + Var[P_3] + \cdots Var[P_n]$

As, P_i are Geometrically distributed,

$\therefore Var[P_i] = \frac{1 - Pr(P_i)}{(Pr(P_i))^2}$ (Already calculated in Assignment 1 Q 5(d))

$$\therefore \text{Var}[P_1] = \frac{1 - Pr(P_1)}{(Pr(P_1))^2} = \frac{1 - (n/n)}{(n/n)^2} = 0$$

$$\text{Var}[P_2] = \frac{1 - (n-1/n)}{\left(\frac{n-1}{n}\right)^2} = n^2 \left(\frac{1/n}{(n-1)^2}\right)$$

$$\text{Var}[P_3] = \frac{1 - (n-2/n)}{\left(\frac{n-2}{n}\right)^2} = n^2 \left(\frac{2/n}{(n-2)^2}\right)$$

$$\vdots$$

$$\text{Var}[P_n] = \frac{1 - (1/n)}{\left(\frac{1}{n}\right)^2} = n^2 \left((n-1)/n\right)$$

$$\Rightarrow \text{Var}[X] = \text{Var}[P_1] + \text{Var}[P_2] + \text{Var}[P_3] + \cdots \text{Var}[P_n]$$

$$= 0 + n^2\left(\frac{1/n}{(n-1)^2}\right) + n^2\left(\frac{2/n}{(n-2)^2}\right) + \cdots n^2\left(\frac{(n-1)/n}{1}\right)$$

$$= n\left[\frac{1}{(n-1)^2} + \frac{2}{(n-2)^2} + \cdots \frac{(n-1)}{1}\right]$$

$$\Rightarrow \boxed{\text{Var}[X] = n\left(\frac{1}{(n-1)^2} + \frac{2}{(n-2)^2} + \cdots (n-1)\right)}$$