CO-2 – Probability
Session-10:
Session Topic: **Expectation of a Function of a Random Variable**

Probability, Statistics and Queueing Theory
(Course code: 21MT2103RA)

(DEEMED TO BE UNIVERSITY)

COMPUTER SCIENCE & ENGINEERING

CO#2
(Probability)

- Continuous Random Variables: Uniform, Exponential and Normal Random Variables
- Expectation of a Random Variable : Discrete and Continuous Case
- Expectation of a Function of a Random Variable
- Higher Order Moments, Variance, Standard Deviation
- Jointly Distributed Random Variables
- Joint Distribution Functions, Independent Random Variables

Functions of a Random Variable

If X is a random variable and Y=g(X), then Y itself is a random variable.

Examples of functions of random variable are

aX where a is a constant

X +b where b is a constant

aX + b

$X*X = X^2$

$(X-\boldsymbol{\mu})^2$

X^4

Let us learn to compute the expected value of Y, a function of X.

Expectation of a Function of a Discrete Random Variable

Let X be a discrete random variable with PMF $P_X(x)$ and let Y=g(X). Then Y itself is a discrete random variable.

The expectation of a function of X, g(X), is given by the following formula:

$$E[g(X)] = \sum_{x_k \in R_X} g(x_k) P_X(x_k)$$

Expectation of a Function of a Discrete Random Variable: example

Example question:
Let X be a random variable showing the value of the outcome of throwing a fair dice. Compute E[Y] where Y=X*X.

Solution:
The expectation of a function of X, g(X), is given by the following formula:

$$E[g(X)] = \sum_{x_k \in R_X} g(x_k) P_X(x_k)$$

Since all 6 outcomes are equally likely, the PMF of X is

P(x) = ⅙ for x=1,2,3,4,5,6.

So, E[Y] = E[X*X] = Σ x*x*P(x)

= ⅙ [1x1 + 2x2 + 3x3 + 4x4 + 5x5 + 6x6]
= ⅙ [91]
= 91/6.

Expectation of a Function of a Continuous Random Variable

Let X be a continuous random variable with pdf f(x).
Then the expectation of a function of X, g(X), is given by the following

Formula:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f(x)\, dx.$$

Expectation of a Function of a Continuous Random Variable: example

Let X denote the time a person waits for an elevator to arrive. Compute E{X^2] when the pdf of X is given by

$$f(x) = \begin{cases} x, & \text{for } 0 \le x \le 1 \\ 2 - x, & \text{for } 1 < x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

.

We will use the following formula: $$E[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f(x)\, dx.$$

$$\mathrm{E}[X^2] = \int_0^1 x^2 \cdot x\, dx + \int_1^2 x^2 \cdot (2 - x)\, dx$$

$$= \int_0^1 x^3\, dx + \int_1^2 (2x^2 - x^3)\, dx = \frac{1}{4} + \frac{11}{12} = \frac{7}{6}.$$

Linear property of Expectation

Statement: If a and b are constants, then E[a X + b] = a E[X] + b

Proof: Let g(X) = a X + b where X is a continuous random variable.

$$\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx.$$

E[a X + b] $$= \int_{-\infty}^{\infty} (ax + b)\, f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} ax\, f_X(x)\, dx + \int_{-\infty}^{\infty} b\, f_X(x)\, dx$$

$$= a \int_{-\infty}^{\infty} x\, f_X(x)\, dx + b \int_{-\infty}^{\infty} f_X(x)\, dx$$

= a E[X] + b

Linear property of Expectation

- If a and b are constants, then **E[a + b X] = a + b E[X]**.

- For any random variables X and Y, **E[X + Y] = E[X] + E[Y]**.

The above two properties show that expectation is a linear operator.

From Discrete to Continuous

	Discrete	Continuous
PMF/PDF	$p_X(x) = P(X = x)$	$f_X(x) \neq P(X = x) = 0$
CDF	$F_X(x) = \sum_{t \leq x} p_X(t)$	$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$
Normalization	$\sum_x p_X(x) = 1$	$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$
Expectation	$\mathbb{E}[g(X)] = \sum_x g(x) p_X(x)$	$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$

source: http://www.alextsun.com/files/Prob_Stat_for_CS_Book.pdf

References

* Section 3.2, 4.1,4.2 of TS1: Alex Tsun, Probability & Statistics with Applications to Computing (Available at: http://www.alextsun.com/files/Prob_Stat_for_CS_Book.pdf)

* https://www.probabilitycourse.com/chapter4/4_1_0_continuous_random_vars_distributions.php

* https://www.probabilitycourse.com/chapter3/3_2_3_functions_random_var.php

* Chapter IX.2 of TP1: William Feller, An Introduction to Probability Theory and Its Applications: Volume 1, Third Edition, 1968 by John Wiley & Sons,Inc.

* Video: