CO-1 – Queueing Theory
Session-5:
Session Topic: Random Variables, Probability Distribution Function

Probability, Statistics and Queueing Theory
(Course code: 21MT2103RA)

(DEEMED TO BE UNIVERSITY)
COMPUTER SCIENCE & ENGINEERING

CO-1
(Probability)

- **Introduction to Probability: Sample Space and Events**
- **Probabilities defined on Events, Conditional Probabilities**
- **Independent Events, Bayes Formula.**
- **Random Variables, Probability Distribution Function**
- **Cumulative Distribution Function**
- **Discrete Random Variables: Bernoulli, Binomial, and Poisson.**

Probability and Statistics

Probability is the chance of an outcome in an experiment (also called event).

Event: Tossing a fair coin
Outcome: Head, Tail

Probability deals with **predicting** the likelihood of **future** events.

Statistics involves the **analysis** of the **frequency** of **past** events

In other words:
In probability, we are given a model and asked what kind of data we are likely to see.

In statistics, we are given data and asked what kind of model is likely to have generated it.

RANDOM VARIABLES

2.1 Introduction

2.2 Discrete probability distribution

2.3 Continuous probability distribution

2.4 Cumulative distribution function

2.5 Expected value, variance and standard deviation

Introduction

- In an experiment of chance, outcomes occur randomly. We often summarize the outcome from a random experiment by a simple number.

Variable	• is a symbol such as X or Y that assumes values for different elements. If the variable can assume only one value, it is called a constant.
Random variable	• A function that assigns a real number to each outcome in the sample space of a random experiment. • Denote by an uppercase letter. i.e: *X* • After experiment, denoted as lowercase letter. i.e: $x=70$ miliampere

Example

A balanced coin is tossed two times. List the elements of the sample space, the corresponding probabilities and the corresponding values ***X***, where ***X*** is the number of getting head.

Solution

Random Variable, X : the number of getting head

Elements of sample space	**Probability**	**X**
HH	¼	2
HT	¼	1
TH	¼	1
TT	¼	0

Exercise

- The time to recharge the flash is tested in three cell phone cameras. The probability a camera passes the test is 0.8 and the camera perform independently. List the elements of the sample space, the corresponding probabilities and the corresponding values ***X***, where ***X*** denotes the number of camera passes the test.

Solution

X : the number of cameras that pass the test

Camera 1	Camera 2	Camera 3	Probability	X
Pass	Pass	Pass	0.512	3
Pass	Pass	Fail	0.128	2
Pass	Fail	Pass	0.128	2
Pass	Fail	Fail	0.032	1
Fail	Pass	Pass	0.128	2
Fail	Pass	Fail	0.032	1
Fail	Fail	Pass	0.032	1
Fail	Fail	Fail	0.008	0

TWO TYPES OF RANDOM VARIABLES

Discrete Random Variables

- A random variable is **discrete** if its set of possible values consist of **discrete points** on the number line.

Continuous Random Variables

- A random variable is **continuous** if its set of possible values consist of an **entire interval** on the number line.

EXAMPLES

Examples of discrete random variables:

- number of scratches on a surface
- number of defective parts among 1000 tested
- number of transmitted bits received error

Examples of continuous random variables:

- electrical current
- length
- Time
- Temperature
- weight

DISCRETE PROBABILITY DISTRIBUTIONS

Definition 2.3:

- If X is a discrete random variable, the function given by ***f(x)=P(X=x)*** for each x within the range of X is called the **probability distribution** of *X.*

- Requirements for a discrete probability distribution:

1) The probability of each value of the discrete random variable is between 0 and 1, inclusive. That is, $0 \le P(X = x) \le 1$

2) The sum of all the probabilities is 1. That is, $\sum_{x \in S} P(X = x) = 1$

Example

Check whether the distribution is a probability distribution.

X	**0**	**1**	**2**	**3**	**4**
P(X=x)	0.125	0.375	0.025	0.375	0.125

$$\sum_{0}^{4} P(X=x) = P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)$$

$$= 0.125+0.375+0.025+0.375+0.125$$

$$= 1.025$$

$$\neq 1$$

Since the summation of all probabilities is not equal to 1, so the distribution is not a probability distribution

Example

- Check whether the given function can serve as the probability distribution random variable

$$\boxed{f(x) = \frac{x+2}{25} \text{ for } x\text{=1,2,3,4,5}}$$

Solution

$$\sum_{1}^{5} f(x) = \sum_{1}^{5} \frac{x+2}{25}$$

$$= f(1) + f(2) + f(3) + f(4) + f(5)$$

$$= \frac{1+2}{25} + \frac{2+2}{25} + \frac{3+2}{25} + \frac{4+2}{25} + \frac{5+2}{25}$$

$$= \frac{3}{25} + \frac{4}{25} + \frac{5}{25} + \frac{6}{25} + \frac{7}{25}$$

$$= \frac{25}{25}$$

$$= 1$$

 # so the given function is a probability distribution of a discrete random variable.

CONTINUOUS PROBABILITY DISTRIBUTIONS

Definition

- A function with values ***f(x),*** defined over the set of all numbers, is called a **probability density function** of the continuous random variable *X* if and only if

$$P(a \le X \le b) = \int_a^b f(x)\, dx$$

for any real constant a and b with $a \le b$

- Requirements for a probability density function of a continuous random variable *X*:

1) $f(x) \geq 0$ for $-\infty \leq x \leq \infty$

2) $\int_{-\infty}^{\infty} f(x)\, dx = 1$. That is the total area under graph is 1.

Example

Let *X* be a continuous random variable with the following

$$f(x) = \begin{cases} \frac{3}{4}(x^2+1), & 0 \le x \le 1 \\ 0, & otherwise \end{cases}$$

a) Verify whether this distribution is a probability density function

b) Find $P(0 \le X \le 0.5)$

c) Find $P(0.5 \le X \le 2)$

Answer

a) The distribution is probability density function if it fulfill the following requirements,

1) All $f(x) \geq 0$

2) If $\int_{-\infty}^{\infty} f(x)dx = 1$

In this problem,

1) First requirement

   $f(0) = 3/4 \geq 0$,

   $f(1) = 3/2 \geq 0$,

   $f(x) = 0$, otherwise

   - **All $f(x) \geq 0$**

Must write the conclusion so that we know the first requirement is fulfill!

2) Second requirement

$$\int_{-\infty}^{0} 0dx + \int_{0}^{1} \frac{3}{4}(x^2+1)dx + \int_{1}^{\infty} 0dx$$

$$= \frac{3}{4}[\frac{x^3}{3} + x]_0^1$$

$$= \frac{3}{4}[\frac{4}{3}]$$

$$= 1$$

$$\therefore \int_{-\infty}^{\infty} f(x)dx = 1$$

Must write the conclusion so that we know the second requirement is fulfill!

Write last conclusion to answer the question!

Since all requirements are fulfill, the distribution is probability density function.

$$b)P(0 \le X \le 0.5)$$

$$= \int_0^{0.5} \frac{3}{4}(x^2+1)dx$$

$$= \frac{3}{4}\int_0^{0.5} (x^2+1)dx$$

$$= \frac{3}{4}\left[\frac{x^3}{3}+x\right]_0^{0.5}$$

$$= \frac{3}{4}\left[(\frac{0.5^3}{3}+0.5)-0\right]$$

$$= 0.4063$$

$c) P(0.5 \le X \le 2)$

$$= \int_{0.5}^{1} \frac{3}{4}(x^2+1)dx + \int_{1}^{2} 0dx$$

$$= \frac{3}{4}\int_{0.5}^{1}(x^2+1)dx + 0$$

$$= \frac{3}{4}\left[\frac{x^3}{3}+x\right]_{0.5}^{1} = \frac{3}{4}\left[\left(\frac{1^3}{3}+1\right)-\left(\frac{0.5^3}{3}+0.5\right)\right]$$

$$= 0.5938$$

Example

Let X be a continuous random variable with the following probability density function

$$f(x) = \begin{cases} c(2x^3 + 5) & , \quad -1 \le x \le 1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

1) Evaluate c

2) Find $P(0 \le X \le 1)$

Solution

$$\int_{-\infty}^{-1} 0dx + \int_{-1}^{1} c(2x^3 + 5)dx + \int_{1}^{\infty} 0dx = 1$$

$$c\int_{-1}^{1} (2x^3 + 5)dx = 1$$

$$c(\frac{2x^4}{4} + 5x)\Big|_{-1}^{1} = 1$$

$$c\left[\left(\frac{2(1)^4}{4} + 5(1)\right) - \left(\frac{2(-1)^4}{4} + 5(-1)\right)\right] = 1$$

$$c\left[\left(\frac{11}{2}\right) - \left(-\frac{9}{2}\right)\right] = 1$$

$$c(10) = 1$$

$$c = \frac{1}{10}$$

b)

$$P(0 \le X \le 1) = \int_0^1 \frac{1}{10}\left(2x^3 + 5\right)dx$$

$$= \frac{1}{10}\left(\frac{2x^4}{4} + 5x\right)\Bigg|_0^1$$

$$= \frac{1}{10}\left[\left(\frac{2(1)^4}{4} + 5(1)\right) - \left(\frac{2(0)^4}{4} + 5(0)\right)\right]$$

$$= \frac{1}{10}\left(\frac{11}{2}\right)$$

$$= \frac{11}{20}$$

$$= 0.55$$

EXERCISE

1. A random variable X can assume 0,1,2,3,4. A probability distribution is shown here:

X	0	1	2	3	4
P(X)	0.1	0.3	0.3	?	0.1

(b) Find $P(X = 3)$

(c) Find $P(X \geq 2)$

2. Let

$$f(x)=\begin{cases}12.5x-1.25 & , \quad 0.1\le x\le 0.5\\ 0 & , \quad \text{otherwise}\end{cases}$$

Find $P(0.2\le X\le 0.3)$

3. Let

$$f(x)=\begin{cases}\mathrm{e}^{-x+6} & , \quad x>6\\ 0 & , \quad \text{otherwise}\end{cases}$$

(a)Find $P(X>6)$

(b) Find $P(6\le \mathrm{X}<8)$

CUMULATIVE DISTRIBUTION FUNCTION

- The cumulative distribution function of a **discrete random variable** ***X*** , denoted as ***F(x)***, is

$$\boxed{F(x) = P(X \le x) = \sum_{t \le x} f(t) \quad \text{for } -\infty < x < \infty}$$

- For a discrete random variable *X*, *F(x)* satisfies the following properties:

$$\boxed{\begin{array}{l} 1)\ 0 \le F(x) \le 1 \\ 2)\ \text{If } \ x \le y, \text{ then } F(x) \le F(y) \end{array}}$$