CO-2 – Probability
Session-09:

Session Topic: **Expectation of a Random Variable : Discrete and Continuous Case**

Probability, Statistics and Queueing Theory
(Course code: 21MT2103RA)

(DEEMED TO BE UNIVERSITY) | COMPUTER SCIENCE & ENGINEERING

CO#2

(Probability)

- Continuous Random Variables: Uniform, Exponential and Normal Random Variables

- Expectation of a Random Variable : Discrete and Continuous Case

- Expectation of a Function of a Random Variable

- Higher Order Moments, Variance, Standard Deviation

- Jointly Distributed Random Variables

- Joint Distribution Functions, Independent Random Variables

Expectation of a random variable: introduction

- One can think of a random variable as a random quantity whose value we do not know yet. You might want to know what you might expect it to equal on average.
- For example, X could be the random variable which represents the number of babies born in Vijayawada per day.
  - On average, X might be equal to 25, and we would write that its average/mean/expectation/expected value is E[X] = 25.

Let us consider a **discrete random variable**.

- The expected value of a random variable with a finite number of outcomes is a weighted average of all possible outcomes.
- Here, the weights are the probabilities associated with the outcomes. These probabilities are given by the Probability Mass Function (PMF) of the discrete random variable.

Expectations of a Random Variable: introduction

Expectations of Random Variables

1. The **expected value** of a random variable is denoted by $E[X]$. The expected value can be thought of as the "average" value attained by the random variable; in fact, the expected value of a random variable is also called its **mean**, in which case we use the notation μ_X. (μ is the Greek letter mu.)

2. The formula for the expected value of a discrete random variable is this:

$$E[X] = \sum_{\text{all possible } x} xP(X = x).$$

   In words, the expected value is the sum, over all possible values x, of x times its probability $P(X = x)$.

3. Example: The expected value of the roll of a die is

$$1(\frac{1}{6}) + 2(\frac{1}{6}) + 3(\frac{1}{6}) + \cdots + 6(\frac{1}{6}) = 21/6 = 3.5.$$

   Notice that the expected value is not one of the possible outcomes: you can't roll a 3.5. However, if you average the outcomes of a large number of rolls, the result approaches 3.5.

Expectations of a Discrete Random Variable: Definition

Let X be a discrete random variable with range $R_X=\{x_1,x_2, x_3,...\}$ (finite or countably infinite).

The expected value of X, denoted by E[X] is defined as

$$E[X] = \sum_{x_k \in R_X} x_k P(X = x_k) = \sum_{x_k \in R_X} x_k P_X(x_k).$$

The intuition behind E[X] is that if you repeat the random experiment independently N times and take the average of the observed data, the average gets closer and closer to E[X] as N gets larger and larger.

We sometimes denote E[X] by $\boldsymbol{\mu}_X$

Expectation: an example

What is the *expected* number of heads in 2 flips of a fair coin? **1**
Since X is the # of heads in 2 flips of a fair coin, we denote this $E[X]$.

How did we get this number? Imagine we repeated this experiment 4 times. Then, we would expect to get HH, HT, TH, TT each once. We would divide by the number of times (4), to get 1 on average.

$$\frac{2+1+1+0}{4} = 2 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} = 1$$

$$\text{E[X]} \quad = 2 \cdot \frac{1}{4} + 1 \cdot \frac{2}{4} + 0 \cdot \frac{1}{4} = 1 = \sum_{x_k \in R_X} x_k P_X(x_k).$$

Explanation of the last line:
Here, the random variable, X, denotes the number of heads observed in repeating the '2 flips of a fair coin' experiment 4 times. The different values of X would be {0,1,2}. The corresponding probabilities are ¼, ½, ¼ respectively. E[X] is the weighted average of all possible outcomes, where the weights are the probabilities associated with the outcomes.

Source: http://www.alextsun.com/files/Prob_Stat_for_CS_Book.pdf

Expectation: Bernoulli RV

Let X~Bernoulli(p), possible pdf is given below

$$P_X(x) = \begin{cases} p & \text{for } x = 1 \\ 1-p & \text{for } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

Find E[X] (mean or expected value of a Bernoulli distribution)

For the Bernoulli distribution, the range of X is = {0,1}.

P(0) = 1-p and P(1) = p

E[X] = $\sum_{x_k \in R_X} x_k P_X(x_k).$

E[X] = 0. (1-p) + 1. p

= p

Expectation: Poisson RV

Let X~Poisson(λ). Find E[X]

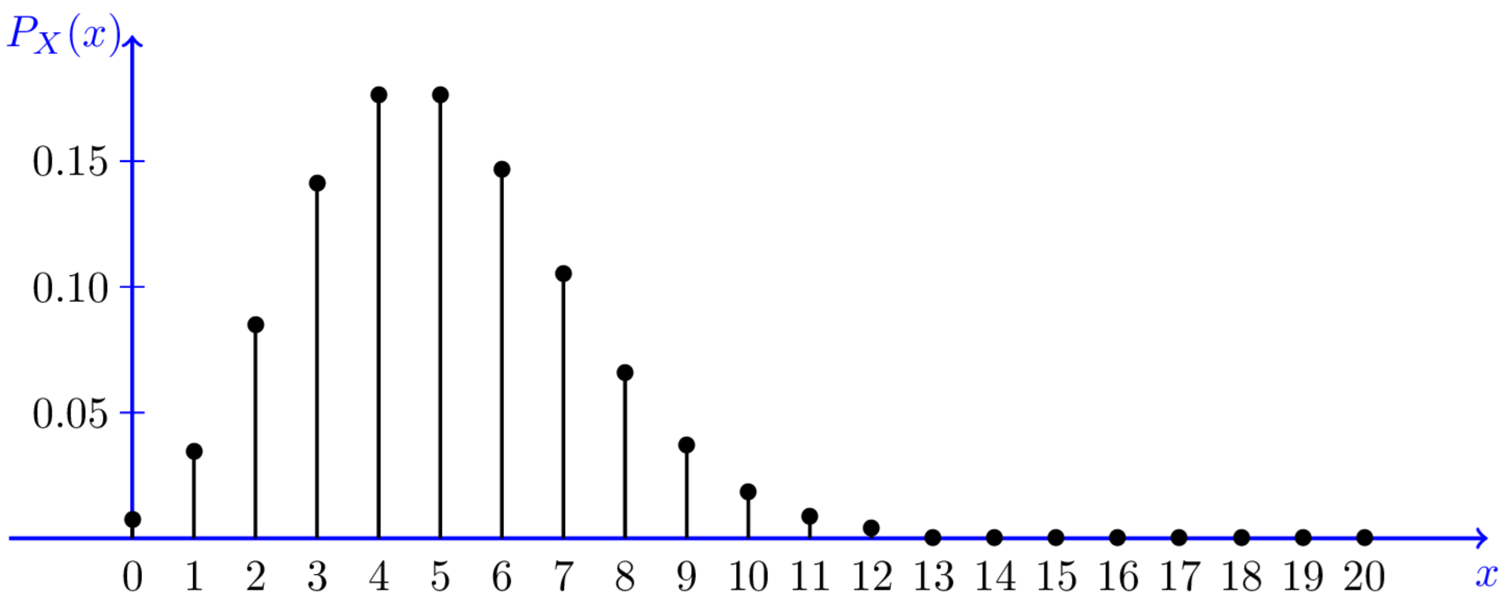
$$P_X(k) = \begin{cases} \frac{e^{-\lambda}\lambda^k}{k!} & \text{for } k \in R_X \\ 0 & \text{otherwise} \end{cases}$$

The PMF is shown on the right.

The range is {0,1,2,3,...}.

Notice the location of the peak (mode) at around λ

$X \sim Poisson(\lambda = 5)$



Expectation: Poisson RV

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The PMF is shown on the right.

The range is {0,1,2,3,...}.

$$
\begin{aligned}
E[X] &= \sum_{x_k \in R_X} x_k P_X(x_k) \\
&= \sum_{k=0}^{\infty} k \frac{e^{-\lambda}\lambda^k}{k!} \\
&= e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} \\
&= e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^{(j+1)}}{j!} \\
&= \lambda e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} \\
&= \lambda e^{-\lambda} e^{\lambda} \\
&= \lambda.
\end{aligned}
$$

Let j = k-1

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

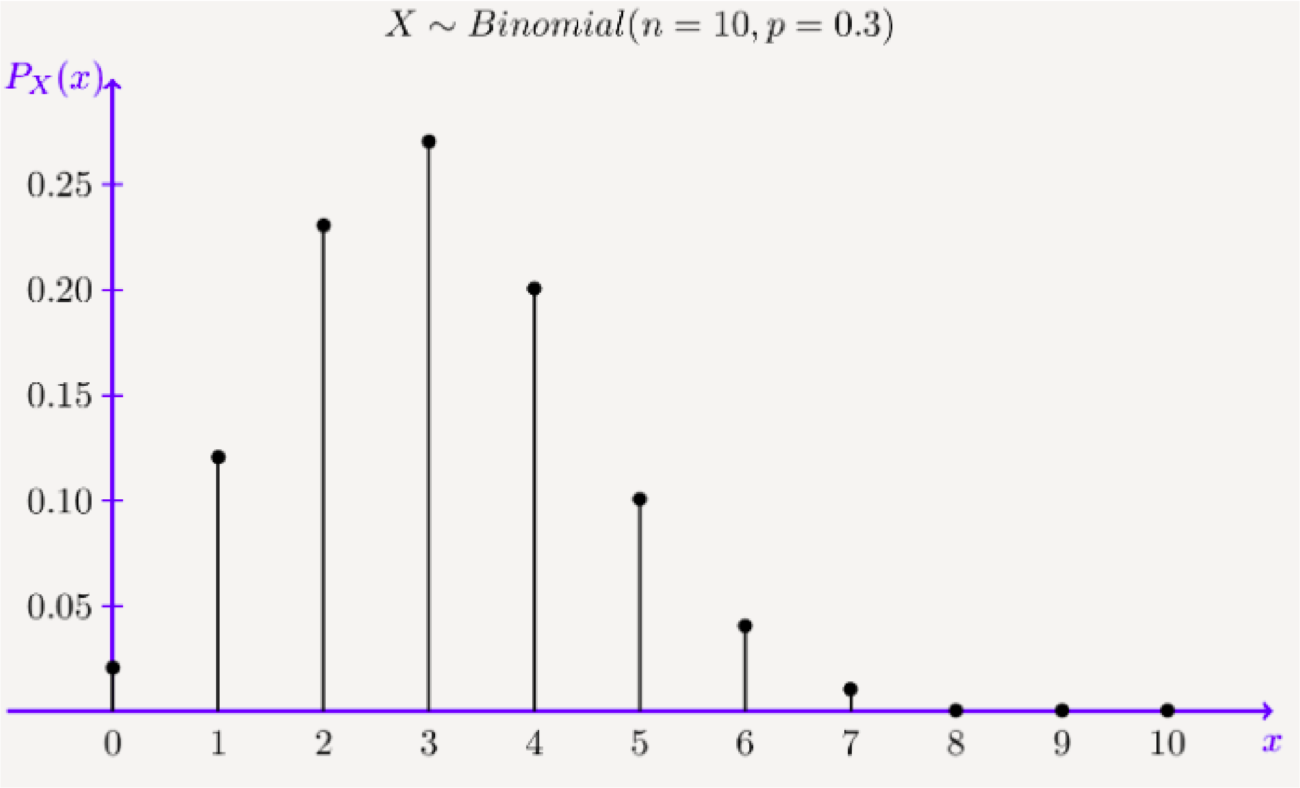
Maclaurin series

(finding the approximate value of the function)

Expectation: Binomial RV

Let X~Binomial(n,p), if its PMF is $P(X=r)=\binom{n}{r}p^r(1-p)^{n-r},\ r=0,\ldots,n$ where 0 < p < 1.

Notice that the peak of the distribution (mode) is around n*p = 3



Expectation: Binomial RV

Let X~Binomial(n,p), if its PMF is $P(X=r)=\binom{n}{r}p^r(1-p)^{n-r},\ r=0,\ldots,n$ where 0 < p < 1.

$$
\begin{aligned}
E[X] &= \sum_{r=0}^{n} r p^r (1-p)^{n-r} \binom{n}{r} \\
&= \sum_{r=0}^{n} r \frac{n!}{r!(n-r)!} p^r (1-p)^{n-r} \\
&= np \sum_{r=1}^{n} \frac{(n-1)!}{(r-1)!(n-r)!} p^{r-1} (1-p)^{n-r} \\
&= np \sum_{r=0}^{n-1} \frac{(n-1)!}{r!(n-1-r)!} p^r (1-p)^{n-1-r} \\
&= np \sum_{r=0}^{n-1} \binom{n-1}{r} p^r (1-p)^{n-1-r} \\
&= np.
\end{aligned}
$$

refer to the graph in the previous slide: n=10, p=0.3

Linear property of Expectation

*) If a and b are constants then $E[a + bX] = a + bE[X]$.

*) For any random variables X, Y then $E[X + Y] = E[X] + E[Y]$.

The above two properties show that expectation is a linear operator.

Expected Value of a Continuous Random Variable

Definition

If X is a continuous random variable with pdf $f(x)$, then the ***expected value*** (or ***mean***) of X is given by

$$\mu = \mu_X = \mathrm{E}[X] = \int_{-\infty}^{\infty} x \cdot f(x)\, dx.$$

The formula for the expected value of a continuous random variable is the continuous analog of the expected value of a discrete random variable, where instead of *summing* over all possible values we *integrate*

Example

Let the random variable X denote the time a person waits for an elevator to arrive. Suppose the longest one would need to wait for the elevator is 2 minutes, so that the possible values of X (in minutes) are given by the interval $[0,2]$. A possible pdf for X is given by

$$f(x) = \begin{cases} x, & \text{for } 0 \leq x \leq 1 \\ 2-x, & \text{for } 1 < x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

Applying Definition · we compute the expected value of X:

$$\mathrm{E}[X] = \int_0^1 x \cdot x \, dx + \int_1^2 x \cdot (2-x) \, dx = \int_0^1 x^2 \, dx + \int_1^2 (2x - x^2) \, dx = \frac{1}{3} + \frac{2}{3} = 1.$$

Thus, we expect a person will wait 1 minute for the elevator on average. Figure 1 demonstrates the graphical representation of the expected value as the center of mass of the pdf.

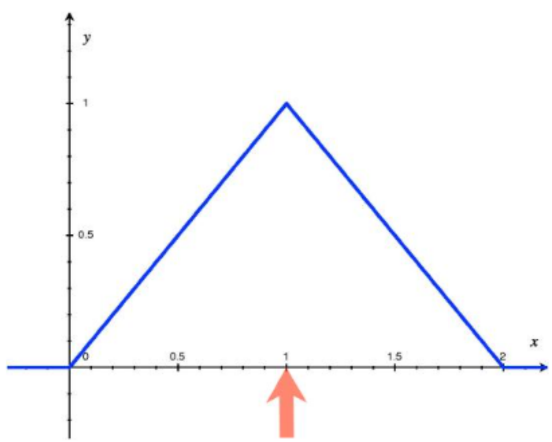


Figure 1: The red arrow represents the center of mass, or the expected value, of X.

References

* Section 3.1.3, of TS1: Alex Tsun, Probability & Statistics with Applications to Computing (Available at: http://www.alextsun.com/files/Prob_Stat_for_CS_Book.pdf)

* https://www.probabilitycourse.com/chapter3/3_2_2_expectation.php

* Chapter IX.2 of TP1: William Feller, An Introduction to Probability Theory and Its Applications: Volume 1, Third Edition, 1968 by John Wiley & Sons,Inc.

* Section 7.3, 17,2 of Richard Weber's course on Probability, http://www.statslab.cam.ac.uk/~rrw1/prob/prob-weber.pdf

* Video: https://www.probabilitycourse.com/videos/chapter3/video3_6.php

Expectation: tutorial?

Prob) Let X be the value of single roll of a fair six-sided dice.

What is the range R_X, the PMF $P_X(k)$, and the expectation E[X]?

Solution:

The range is $R_X = \{1, 2, 3, 4, 5, 6\}$.

The PMF is $p_X(k) = \frac{1}{6}$, $k \in R_X$

The expectation is

$$\mathbb{E}[X] = \sum_{k \in R_X} k \cdot p_X(k) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \cdots + 6 \cdot \frac{1}{6} = \frac{1}{6}(1 + 2 + \cdots + 6) = 3.5$$