CO-1 – Probability
Session-03 & 04:
Session Topic: **Introduction to probability: Sample space and Events**

Probability, Statistics and Queueing Theory

(Course code: 21MT2103RA)

CO-1

(Probability)

- Introduction to Probability: Sample Space and Events

- Probabilities defined on Events, Conditional Probabilities

- Independent Events, Bayes Formula.

- Random Variables, Probability Distribution Function

- Cumulative Distribution Function

- Discrete Random Variables: Bernoulli, Binomial, and Poisson.

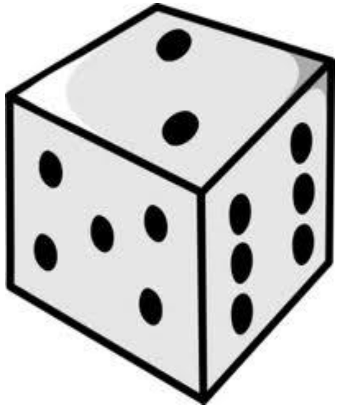
Agenda

- Probabilities defined on Events
- Independent Events
- Conditional Probabilities
- Bayes Formula.

Event: A single result of an experiment

Example Events:

1. Getting a Tail when tossing a coin is an event
2. Rolling a "5" is an event.
3. An event can include one or more possible outcomes:
4. Choosing a "King" from a deck of cards (any of the 4) is an event
5. Rolling an "even number" (2, 4 or 6) is also an event



Dependent and Independent events

An event is deemed dependent if it provides information about another event. An event is deemed independent if it offers no information about other events.

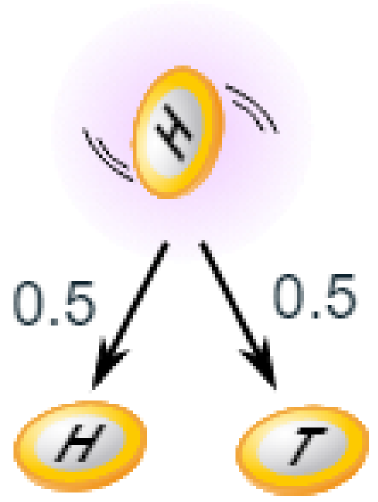
Independent events do not affect one another and do not increase or decrease the probability of another event happening.

Dependent events influence the probability of other events – or their probability of occurring is affected by other events.

Independent Events

Independent Events are **not affected** by previous events.

A coin does not "know" it came up heads before.
And each toss of a coin is a perfect isolated thing.



Probability of an event happening = *Number of ways it can happen*/**Total number of outcomes**

Example: You toss a coin and it comes up "Heads" three times ... what is the chance that the next toss will also be a "Head"?

The chance is simply ½ (or 0.5) just like ANY toss of the coin.

What it did in the past will not affect the current toss!

Some people think "it is overdue for a Tail", but really truly the next toss of the coin is totally independent of any previous tosses.

Independent Events...

Example:

What is the probability of getting a "4" or "6" when rolling a die?

Solution:

Number of ways it can happen: 2 ("4" and "6")
Total number of outcomes: 6 ("1", "2", "3", "4", "5" and "6")

So the probability = *2*/**6** = *1*/**3** = 0.333...

Independent Events…

Other examples of pairs of independent events include:

- ✓ Taking an Uber ride and getting a free meal at your favorite restaurant
- ✓ Winning a card game and running out of bread
- ✓ Growing the perfect tomato and owning a cat

Note: Independent events don't influence one another or have any effect on how probable another event is.

Dependent Events?

Two events are dependent when the outcome of the first event influences the outcome of the second event.

Examples

1.Boarding a plane first and finding a good seat

2.Getting into a traffic accident is dependent upon driving or riding in a vehicle.

3.If you park your vehicle illegally, you're more likely to get a parking ticket.

4.You must buy a lottery ticket to have a chance at winning; your odds of winning are increased if you buy more than one ticket.

5.Committing a serious crime – such as breaking into someone's home – increases your odds of getting caught and going to jail.

6.Not paying your power bill on time and having your power cut off.

Method to Identify Independent Events

Before applying probability formulas, one needs to identify an independent event. Few steps for checking whether the probability belongs to a **dependent or independent events**:

Step 1: Check if it possible for the events to happen in any order? If yes, go to Step 2, or else go to Step 3
Step 2: Check if one event affects the outcome of the other event? If yes, go to step 4, or else go to Step 3
Step 3: The event is independent. Use the formula of independent events and get the answer.
Step 4: The event is dependent. Use the formula of dependent event and get the answer.

Ref: https://www.cuemath.com/data/independent-events/

Probabilities defined on Events

Event : An event is the basic element to which probability can be applied.

Define probability of an event?

The probability of an event refers to the likelihood that the event will occur.

Mathematically, the probability that an event will occur is expressed as a number between 0 and 1.

Notionally, the probability of event A is represented by P(A).

If the probability of an event is 1, then it is an absolutely certain event.

The probability of an event can never be greater than 1 and always greater than or equal to zero.$0 \leq P \leq 1$

Probabilities of Dependent and Independent Events

Independent Events

The outcome of one event does not affect the outcome of the other.

If A and B are independent events then the probability of both occurring is

$$P(A \text{ and } B) = P(A) \times P(B)$$

Dependent Events

The outcome of one event affects the outcome of the other.

If A and B are dependent events then the probability of both occurring is

$$P(A \text{ and } B) = P(A) \times P(B|A)$$

Probability of B given A

Independence…

One of the objectives of calculating conditional probability is to determine whether two events are related.

In particular, we would like to know whether they are ***independent***, that is, if the probability of one event is ***not affected*** by the occurrence of the other event.

Two events A and B are said to be ***independent*** if

$$P(A|B) = P(A)$$

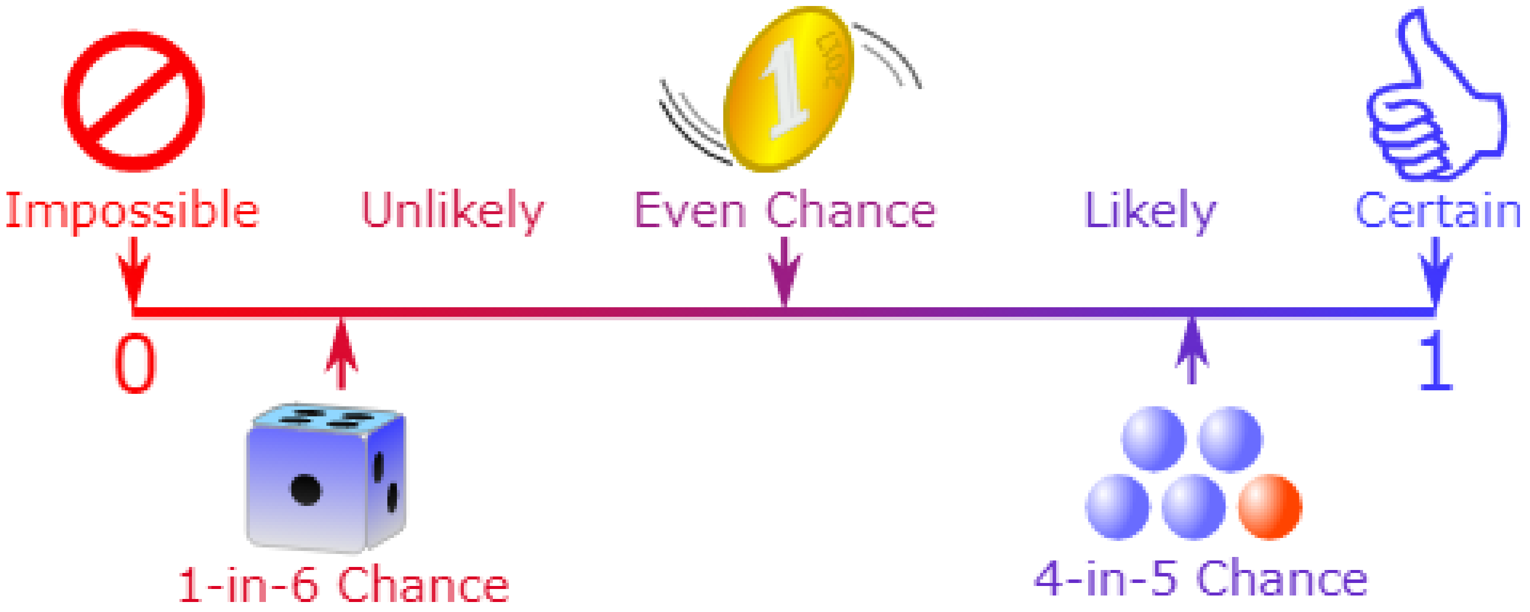
or

$$P(B|A) = P(B)$$

- Let k be an event and n(k) be the number of elements of event k. Now, total probability is 1
- Then P(k)=No. of elements in sample space/ n(k) ≤1
- And we know that, probability of an event can't be negative. So, P(k)≥0
- Hence, 0≤P(k)≤1.

Ways of Showing Probability

- Probability goes from **0** (impossible) to **1** (certain):



Conditional probability

In probability theory, conditional probability is a measure of the likelihood of an event occurring, given that another event (by assumption, presumption, assertion or evidence) has already occurred.

If the event of interest is *A* and the event *B* is known or assumed to have occurred, "the conditional probability of *A* given *B*", or "the probability of *A* under the condition *B*", is usually written as P(*A*|*B*) . This can also be understood as the fraction of probability B that intersects with A:

The symbol "|" represent the word "given". This means the event after "|" is already occurred

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

Conditional probability can be contrasted with unconditional probability. Unconditional probability refers to the likelihood that an event will take place irrespective of whether any other events have taken place or any other conditions are present.

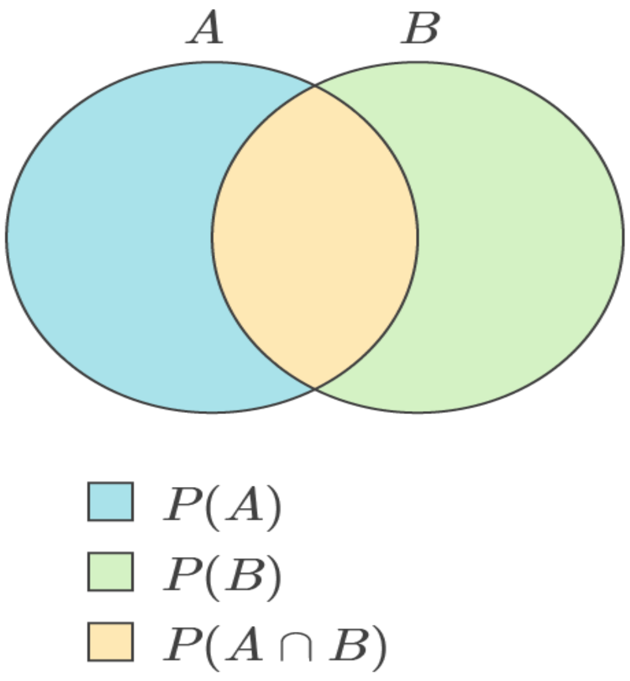
Conditional Probabilities

Notations

Event: A, B

A∩B → Both (A and B)

A ∪ B → Either (A or B)



Conditional Probability Formula

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Probability that A occurs given that B has already occured

P(A ∩ B) = P(A and B) =P(A)+P(B)-P(A ∪ B)

Example : In a group of 100 computer buyers, 40 bought CPU, 30 purchased monitor, and 20 purchased CPU and monitors. If a computer buyer chose at random and bought a CPU, what is the probability they also bought a Monitor?

Solution: As per the first event, 40 out of 100 bought CPU,

So, $P(A) = 40\%$ or 0.4

Now, according to the question, 20 buyers purchased both CPU and monitors. So, this is the intersection of the happening of two events. Hence,

$P(A \cap B) = 20\%$ or 0.2

By the formula of conditional probability we know;

$P(B|A) = P(A \cap B)/P(B)$

$P(B|A) = 0.2/0.4 = 2/4 = \frac{1}{2} = 0.5$

The probability that a buyer bought a monitor, given that they purchased a CPU, is 50%.

Example:

In a batch, there are 80% C programmers, and 40% are Java and C programmers. What is the probability that a C programmer is also Java programmer?

Let A --> Event that a student is Java programmer

B --> Event that a student is C programmer

$P(A|B) = P(A \cap B) / P(B)$

$= (0.4) / (0.8)$

$= 0.5$

So there are 50% chances that student that knows C also knows Java

Bayes Theorem

Bayes theorem, determines the conditional probability of an event A given that event B has already occurred. Bayes theorem is also known as the Bayes Rule or Bayes Law.

Bayes theorem states that the conditional probability of an event A, given the occurrence of another event B, is equal to the product of the likelihood of B, given A and the probability of A. It is given as:

Formula For Bayes' Theorem

$$P(A|B) = \frac{P(A \bigcap B)}{P(B)} = \frac{P(A) \cdot P(B|A)}{P(B)}$$

where:

$P(A) =$ The probability of A occurring

$P(B) =$ The probability of B occurring

$P(A|B) =$ The probability of A given B

$P(B|A) =$ The probability of B given A

$P\left(A \bigcap B\right)) =$ The probability of both A and B occurri

Bayes Theorem

LIKELIHOOD
The probability of “B” being True, given “A” is True

PRIOR
The probability “A” being True. This is the knowledge.

$$P(A|B) = \frac{P(B|A).P(A)}{P(B)}$$

POSTERIOR
The probability of “A” being True, given “B” is True

MARGINALIZATION
The probability “B” being True.

Bayes Theorem Derivation

$P(A|B) = P(A \cap B) / P(B)$, if $P(B) \neq 0$,

$P(B|A) = P(B \cap A) / P(A)$, if $P(A) \neq 0$,

$P(A \cap B) = P(A|B).P(B) = P(B|A).P(A)$,

Here, the joint probability $P(A \cap B)$ of both events A and B being true such that,
$P(B \cap A) = P(A \cap B)$

$P(A|B) = P(B|A).P(A)/P(B)$, if $P(B) \neq 0$,

https://byjus.com/maths/bayes-theorem/

Bayes Theorem Statement

Let $E_1, E_2, \ldots, E_n$ be a set of events associated with a sample space S, where all the events $E_1, E_2, \ldots, E_n$ have nonzero probability of occurrence and they form a partition of S. Let A be any event associated with S, then according to Bayes theorem:

$$P(E_i \mid A) = \frac{P(E_i)P(A \mid E_i)}{\sum_{k=1}^{n} P(E_k)P(A \mid E_k)}$$

for any k = 1, 2, 3, …., n

Bayes Theorem Proof

According to the conditional probability formula,

$$P(E_i \mid A) = \frac{P(E_i \cap A)}{P(A)} \quad \cdots\cdots\cdots\cdots(1)$$

Using the multiplication rule of probability,

$$P(E_i \cap A) = P(E_i)P(A \mid E_i) \quad \cdots\cdots\cdots\cdots(2)$$

Using total probability theorem,

$$P(A) = \sum_{k=1}^{n} P(E_k)P(A \mid E_k) \quad \cdots\cdots\cdots\cdots(3)$$

Putting the values from equations (2) and (3) in equation 1, we get

$$P(E_i \mid A) = \frac{P(E_i)P(A \mid E_i)}{\sum_{k=1}^{n} P(E_k)P(A \mid E_k)}$$

Hypotheses: The events $E_1, E_2, \ldots E_n$ is called the hypotheses
Priori Probability: The probability $P(E_i)$ is considered as the priori probability of hypothesis E_i
Posteriori Probability: The probability $P(E_i|A)$ is considered as the posteriori probability of hypothesis E_i
Bayes' theorem is also called the formula for the probability of "causes". Since the E_i's are a partition of the sample space S, one and only one of the events E_i occurs (i.e. one of the events E_i must occur and the only one can occur). Hence, the above formula gives us the probability of a particular E_i (i.e. a "Cause"), given that the event A has occurred.

Difference Between Conditional Probability and Bayes Theorem

Conditional Probability	Bayes Theorem
Conditional Probability is the probability of an event A that is based on the occurrence of another event B.	Bayes Theorem is derived using the definition of conditional probability. The Bayes theorem formula includes two conditional probabilities.
Formula: $P(A\|B) = \frac{P(A \cap B)}{P(B)}$	Formula: $P(A\|B) = \frac{P(B\|A)P(A)}{P(B)}$

Is Bayes Theorem for Independent Events?
If two events A and B are independent, then $P(A|B) = P(A)$ and $P(B|A) = P(B)$, therefore Bayes theorem cannot be used here to determine the conditional probability as we need to determine the total probability and there is no dependency of events.

Bayes theorem Applications

One of the many applications of Bayes' theorem is Bayesian inference, a particular approach to statistical inference. Bayesian inference has found application in various activities, including medicine, science, philosophy, engineering, sports, law, etc. For example, we can use Bayes' theorem to define the accuracy of medical test results by considering how likely any given person is to have a disease and the test's overall accuracy. Bayes' theorem relies on consolidating prior probability distributions to generate posterior probabilities. In Bayesian statistical inference, prior probability is the probability of an event before new data is collected.

Example of Bayes Theorem

- Given:
  - A doctor knows that meningitis causes stiff neck 50% of the time
  - Prior probability of any patient having meningitis is 1/50,000
  - Prior probability of any patient having stiff neck is 1/20

- If a patient has stiff neck, what's the probability he/she has meningitis?

$$P(M \mid S) = \frac{P(S \mid M)P(M)}{P(S)} = \frac{0.5 \times 1/50000}{1/20} = 0.0002$$

Bayes' rule formula - tests

The Bayes' theorem can be extended to two or more cases of event A. This can be useful when testing for false positives and false negatives. The probability of event B is then defined:

$P(B) = P(A) * P(B|A) + P(\text{not } A) * P(B|\text{not } A)$,

where $P(\text{not } A)$ is the probability of the event A not occurring.
The following equation is true:

$P(\text{not } A) + P(A) = 1$ as either event A occurs or it does not.

The extended Bayes' rule formula would then be:

$P(A|B) = [P(B|A) * P(A)] / [P(A) * P(B|A) + P(\text{not } A) * P(B|\text{not } A)]$,

Q. Given the following statistics, what is the probability that a woman has cancer if she has a positive mammogram result?

1. One percent of women over 50 have breast cancer.
2. 90% of women who have breast cancer test positive on mammograms.
3. 8 percent of women will have false positives.

Step 1: Assign events to A or X. You want to know what a woman's probability of having cancer is, given a positive mammogram. For this problem, actually having cancer is A and a positive test result is X.
Step 2: List out the parts of the equation (this makes it easier to work the actual equation):
$P(A)=0.01$
$P(\sim A)=0.99$
$P(X|A)=0.9$
$P(X|\sim A)=0.08$
Step 3: Insert the parts into the equation and solve. Note that as this is a medical test, we're using the form of the equation from example #2:
$(0.9 * 0.01) / ((0.9 * 0.01) + (0.08 * 0.99) = 0.10.$
The probability of a woman having cancer, given a positive test result, is 10%.

Questions?

1. How do you find conditional probability?
2. What is the difference between probability and conditional probability?
3. Why do we need conditional probability?
4. What does given mean in probability?
5. What is an example of conditional probability?
6. How do you find conditional probability?
7. What is the difference between probability and conditional probability?
8. Let $P(E)$ denote the probability of the event E. Given $P(A) = 1$, $P(B) = 1/2$, the values of $P(A \mid B)$ and $P(B \mid A)$ respectively are:
   (A) 1/4, 1/2
   (B) 1/2, 1/14
   (C) 1/2, 1
   (D) 1, 1/2