# CO-2 – Probability
## Session-08:
## Session Topic: Continuous Random Variables: Uniform, Exponential and Normal Random Variables

## Probability, Statistics and Queueing Theory
# (Course code: 21MT2103RA)

(DEEMED TO BE UNIVERSITY) KL | CSE COMPUTER SCIENCE & ENGINEERING

**CO#2**

**(Probability)**

**- Continuous Random Variables: Uniform, Exponential and Normal Random Variables**

**- Expectation of a Random Variable : Discrete and Continuous Case**

**- Expectation of a Function of a Random Variable**

**- Higher Order Moments, Variance, Standard Deviation**

**- Jointly Distributed Random Variables**

**- Joint Distribution Functions, Independent Random Variables**

# Continuous Random Variables

X is a **continuous** random variable if X can have values in an **interval** on the real number line.

Examples of continuous random variables

| Description | Range of X |
|---|---|
| C, the temperature in Celsius of water | (0,100) |
| B, the amount of time I wait for the next bus (in hours) | $(0,\infty)$ |

| | | |
|---|---|---|
| Notation: | X | name of a random variable |
| | x | the value of X after a random experiment |

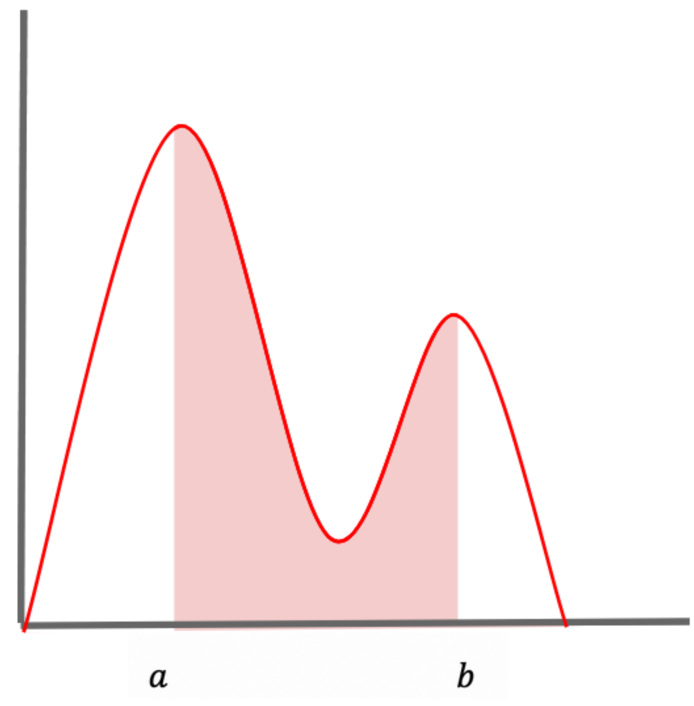
# **Probability Density Function**: definition

The **probability density function** (pdf) gives the probability of a continuous random variable X having values in a short **interval** around a given value x.

If f(x) is the probability density function of a continuous random variable X, then the probability of X having values in an interval [a,b] is given by

$$P(a \le X \le b) = \int_a^b f(x)\,dx$$

f(x)




source: http://www.alextsun.com/files/Prob_Stat_for_CS_Book.pdf

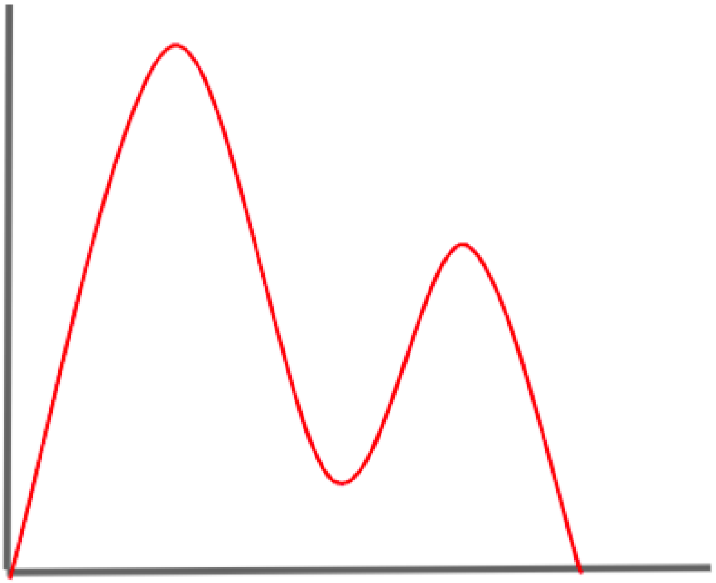
x ⟶

# **Probability Density Function**: properties

Properties of a **probability density function**:

1. $$f(x) \geq 0$$

2. f(x) $$\int_{-\infty}^{\infty} f(x)\, \mathrm{d}x = 1$$



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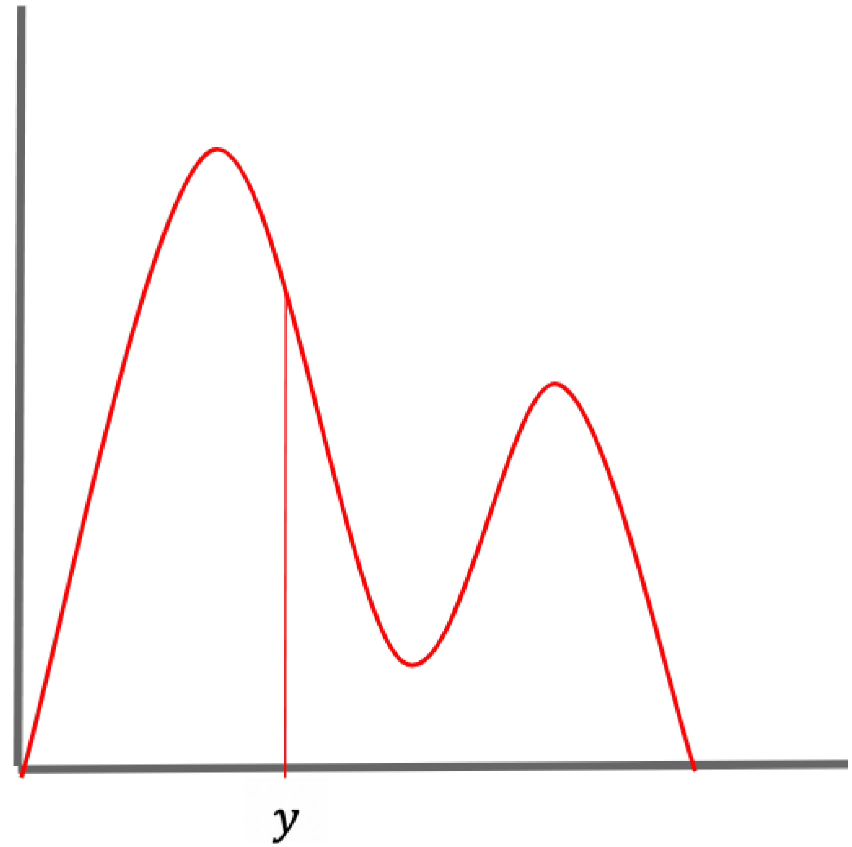
# Probability Density Function (pdf)




$$P(X = y) \ = \ P(y \ \leq \ X \ \leq \ y) \ = \ \int_{y}^{y} f(x)\, dx \ = \ 0$$

The probability of a continuous random variable X taking a **particular** value y is 0, even though the value of the probability density function is nonzero at y.

# Cumulative Distribution Function (CDF)

The cumulative distribution function (CDF) of a continuous random variable is defined as

$$F(x) = \int_{-\infty}^{x} f(x)\, dx$$

where f(x) is the probability density function of the continuous random variable.

It can be shown that

$$\mathrm{P}(\mathrm{a} \leq \mathrm{X} \leq \mathrm{b}) = F(b) - F(a)$$

Notation:

| | |
|---|---|
| f(x) | probability density function of a continuous random variable |
| F(x) | cumulative distribution function |

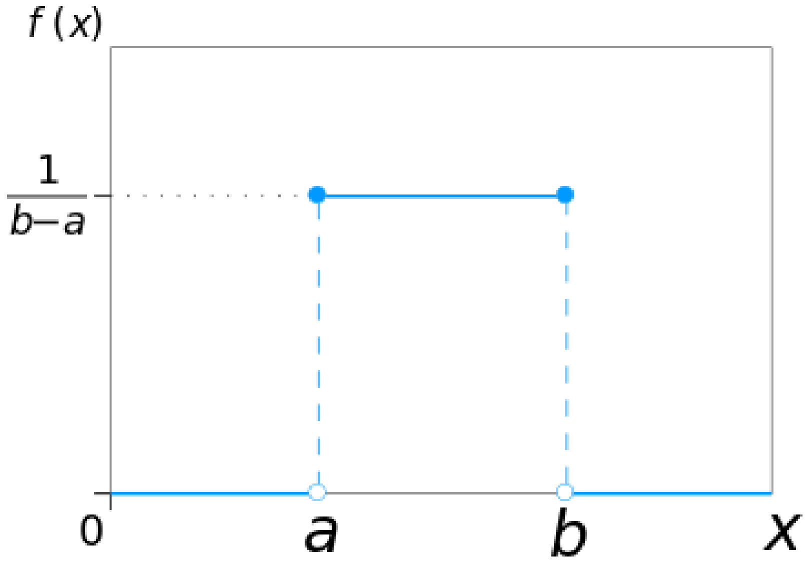
# Continuous Random Variables: Uniform, Exponential and Normal Random Variables

# 1. Uniform distribution: pdf

A continuous random variable X is said to have a Uniform distribution over the interval [a,b], shown as X~U[a,b], if its pdf, f(x), is as follows:

$$\begin{cases} \frac{1}{b-a} & \text{for } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

X is equally likely to be take on any value in [a, b]. The graph of f(x) is shown below.



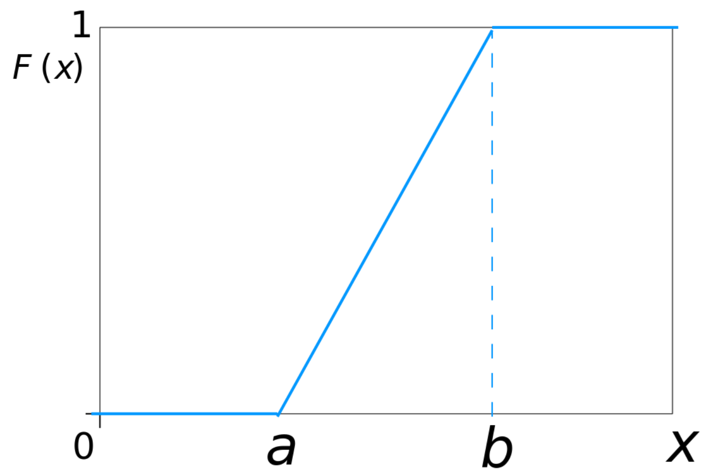

source: https://en.wikipedia.org/wiki/Continuous_uniform_distribution

# Uniform distribution: CDF

The CDF, F(x), of a uniform random variable is shown below:

$$\begin{cases} 0 & \text{for } x < a \\ \frac{x-a}{b-a} & \text{for } x \in [a,b] \\ 1 & \text{for } x > b \end{cases}$$

The graph of F(x) is shown below:

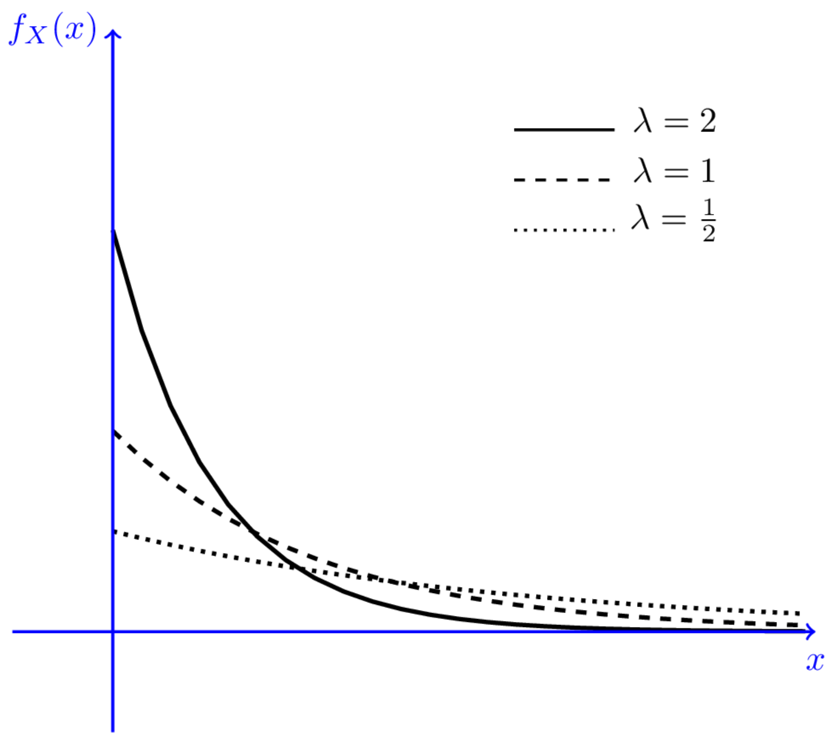



source: https://en.wikipedia.org/wiki/Continuous_uniform_distribution

# 2. Exponential distribution: pdf

A continuous random variable X is said to have a said to have an exponential distribution with parameter λ>0, shown as X~Exponential(λ), if its pdf is given by

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

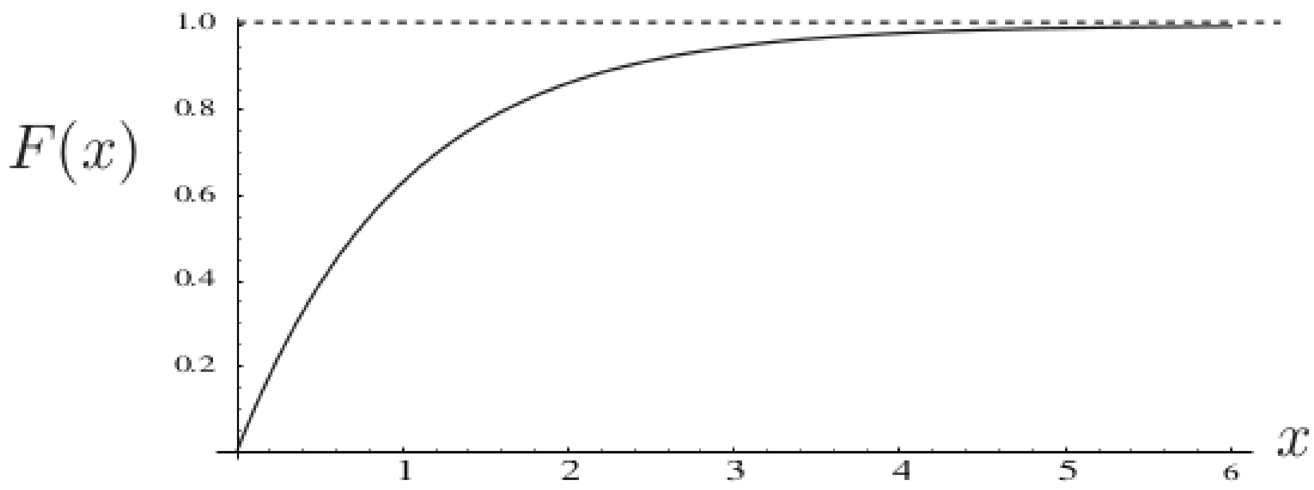



Graphs of pdfs of exponential distributions with different parameters.

# Exponential distribution: CDF

The CDF of exponential random variable (for x > 0) is given by

$$F_X(x) = \int_0^x \lambda e^{-\lambda t} dt = 1 - e^{-\lambda x}.$$




c.d.f. of $F(x) = 1 - e^{-x}$ for exponential r.v. $\mathscr{E}(1)$

source: http://www.statslab.cam.ac.uk/~rrw1/prob/prob-weber.pdf

## **Exponential distribution**: memoryless property

Exponential distribution is often used to model the time elapsed between events.

An important fact about the exponential distribution it that it has the

$$P\left(X \geq x+z \mid X \geq z\right)=\frac{P\left(X \geq x+z\right)}{P\left(X \geq z\right)}=\frac{e^{-\lambda(x+z)}}{e^{-\lambda z}}=e^{-\lambda x}=P\left(X \geq x\right).$$

If X were the life of something, such as ‘how long this phone call to my mother will last’, the memoryless property says that after we have been talking for 5 minutes, the distribution of the remaining duration of the call is just the same as it was at the start. This is close to what happens in real life.

# 3. Normal distribution

The **normal** distribution (or **Gaussian** distribution) with parameters **μ** and $\boldsymbol{\sigma^2}$ has the following probability density function:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \qquad -\infty < x < \infty.$$

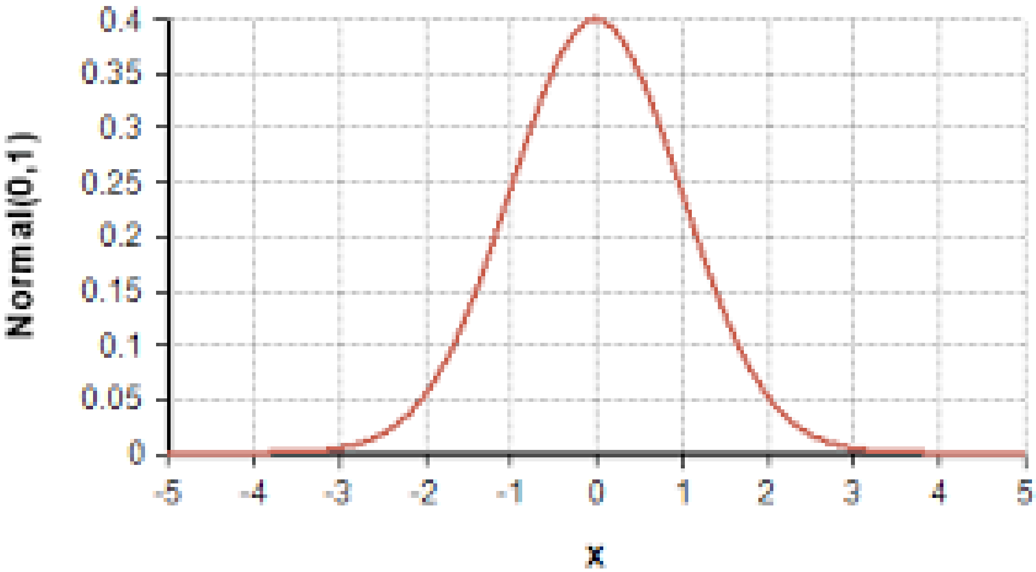
To indicate that X has this distribution we write $X \sim N(\mu, \sigma^2)$.

The parameter **μ** is the **mean** of the distribution, and $\boldsymbol{\sigma^2}$ is the **variance**. The positive square root of variance, σ, is called the **standard deviation**. σ is a measure of the spread of data around the mean.

# Standard Normal distribution

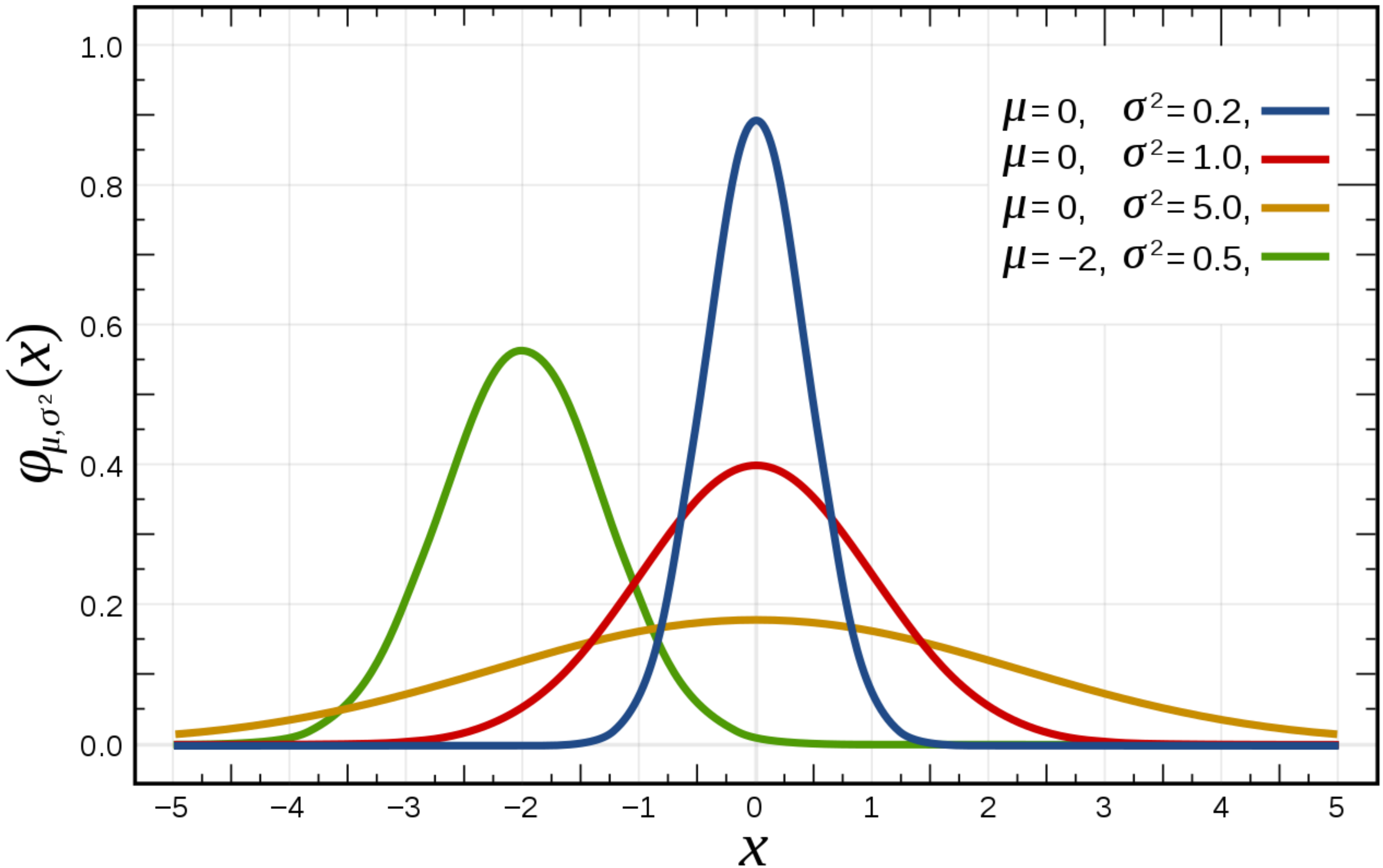
$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \qquad -\infty < x < \infty.$$

is the probability density function of $X \sim N(\mu, \sigma^2)$. When $\mu = 0$, and $\sigma = 1$, the distribution is called the **standard normal distribution**. The graph of the probability density function of $X \sim N(0, 1)$ is shown below:




source: https://wiki.analytica.com/index.php?title=Normal_distribution

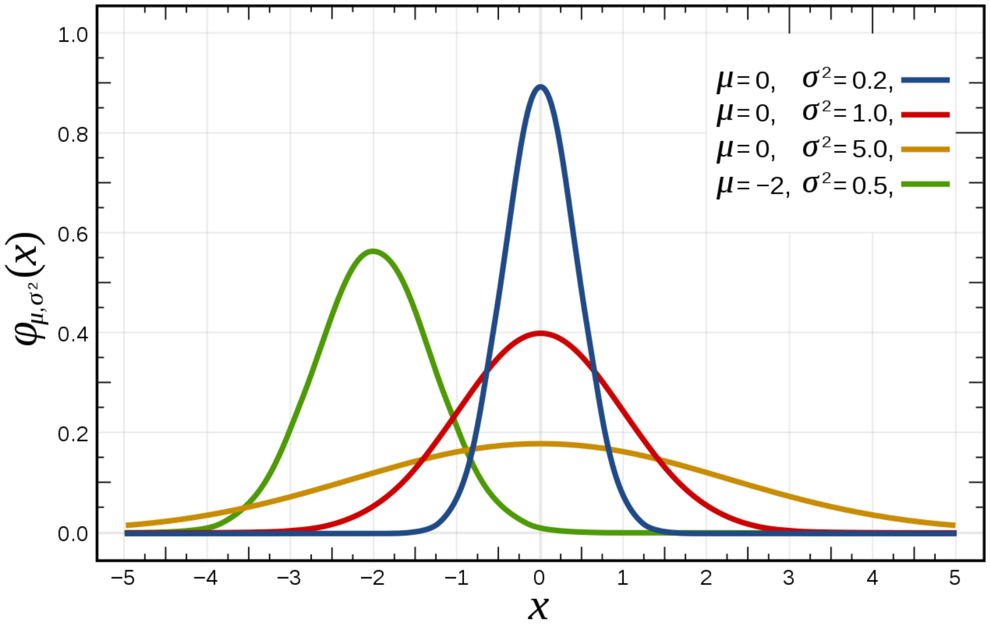
# **Normal distribution:** pdfs with varying parameter values



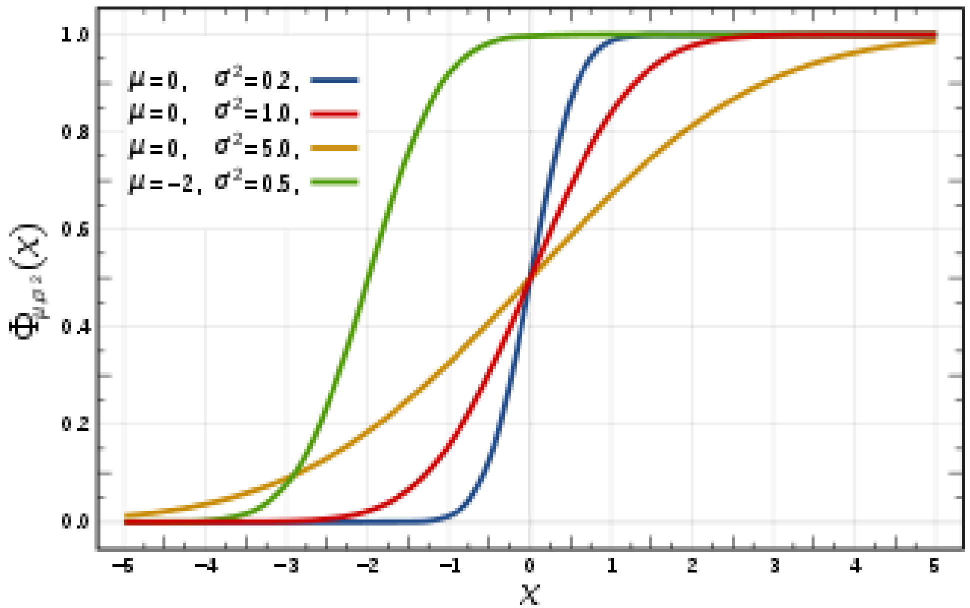

source: https://en.wikipedia.org/wiki/Normal_distribution

# Normal distribution:

pdfs




CDFs




Note: CDF ($x = \mu$) = 0.5

source: https://en.wikipedia.org/wiki/Normal_distribution

# References

* Section 4.1, 4.2 of TS1: Alex Tsun, Probability & Statistics with Applications to Computing (Available at: http://www.alextsun.com/files/Prob_Stat_for_CS_Book.pdf)

* https://www.probabilitycourse.com/chapter4/4_1_0_continuous_random_vars_distributions.php

* Chapter VII of TP1: William Feller, An Introduction to Probability Theory and Its Applications: Volume 1, Third Edition, 1968 by John Wiley & Sons,Inc.

* Richard Weber's course on Probability, http://www.statslab.cam.ac.uk/~rrw1/prob/prob-weber.pdf

Video:

https://www.youtube.com/watch?v=-5sOBWV0qH8&list=PLeB45KifGiuHesi4PALNZSYZFhViVGQJK&index=19