# CO-1 – Probability
## Session-07:
## Session Topic: Discrete Random Variables: Bernoulli, Binomial, and Poisson

# Probability, Statistics and Queueing Theory
# (Course code: 21MT2103RA)

## CO#1
## (Probability)

- Introduction to Probability: Sample Space and Events
- Probabilities defined on Events, Conditional Probabilities
- Independent Events, Bayes Formula.
- Random Variables, Probability Distribution Function
- Cumulative Distribution Function
- **Discrete Random Variables: Bernoulli, Binomial, and Poisson.**

# Discrete and Continuous Random Variables

In this course, we will study two important classes of random variables:

- discrete random variables
- continuous random variables.

**Discrete random variable:**

Suppose we conduct an experiment with sample space S. A random variable is a numeric function of the outcome, $X : S \to R$. That is, X maps outcomes ($s \in S$) to numbers: $s \to X(s)$. The set of possible values X can take on is its range. If the range is finite or countably infinite (typically integers or a subset), X is a **discrete random variable**.

**X is a discrete random variable, if its range is countable**

Examples of discrete random variables

| Description | Range |
|---|---|
| X, the # of heads in **n** flips of a fair coin. | {0,1,2, …, **n**} |
| Y, the # of people born this year. | {0,1,2, …} |

# Continuous Random Variable

If the range of a random variable is uncountably large (an interval on the real number line), X is a continuous random variable.

Examples of continuous random variables

| Description | Range of X |
|---|---|
| C, the temperature in Celsius of water | (0,100) |
| B, the amount of time I wait for the next bus in seconds | $(0,\infty)$ |

**Notation**: random variable **X** numerical value **x**

# **Probability Mass Function:** introduction

Consider a random experiment of tossing two fair coins.


The sample space is

$$S = \{TT, HT, TH, HH\}$$

Let X denote the number of heads observed.

$$X(TT) = 0 \qquad X(HT) = 1 \qquad X(TH) = 1 \qquad X(HH) = 2$$

So, the random variable X can take 3 distinct values: 0, 1, 2.

# Probability Mass Function: introduction


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So, the random variable X can take 3 distinct values: 0, 1, 2.

Let us compute the probability of these 3 values of X using relative frequency approach by assuming that all 4 outcomes of the sample space are equally likely.

Then,

$$p(0) = P(X=0) = \tfrac{1}{4}$$

$$p(1) = P(X=1) = 2/4 = \tfrac{1}{2}$$

$$p(2) = P(X=2) = \tfrac{1}{4}$$

Such a listing of probabilities of all possible values a **discrete** random variable can take is called the **Probability Mass Function** of X.

# Probability Mass Function (PMF)

**Definition**:

Let $x_1$, $x_2$, $x_3$, … be the possible values a discrete random variable X can take.

The function

$$P_X(x_k) = P(X = x_k), \text{ for } k = 1, 2, 3, \ldots,$$

is called the **probability mass function** (PMF) of X.

In other words, the probability mass function (PMF) of a discrete random variable X assigns probabilities to the possible values of the random variable.

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**Properties of PMF**:

- $0 \le P_X(x) \le 1$ for all $x$, and
- $\sum_{x \in R_X} P_X(x) = 1$.

The cumulative distribution function (CDF) of a discrete random variable X can be computed in terms of PMF of X as follows:

$$F_X(x) = \sum_{x_k \le x} P_X(x_k).$$

# Bernoulli Random Variable

A **Bernoulli random variable** is a random variable that can only take two possible values, usually 0 and 1. This random variable models random experiments that have two possible outcomes, sometimes referred to as "success" and "failure."

Some examples:

- You take a pass-fail exam. You either pass (resulting in X=1) or fail (resulting in X=0).
- The outcome of flipping a coin is either H or T.
- A child is born. The gender is either male or female.

| **Values of a Bernoulli random variable** | |
|---|---|
| 1 | 0 |
| success | failure |
| H | T |

If probability of "success" is **p**, the probability of "failure" is **1-p**.

# Bernoulli Random Variable

**Definition**: A random variable X is said to be a **Bernoulli random variable** with parameter p, shown as X~Bernoulli(p), if its PMF is given by

$$P_X(x) = \begin{cases} p & \text{for } x = 1 \\ 1-p & \text{for } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

where 0 < p < 1.

Bernoulli Distribution Graph

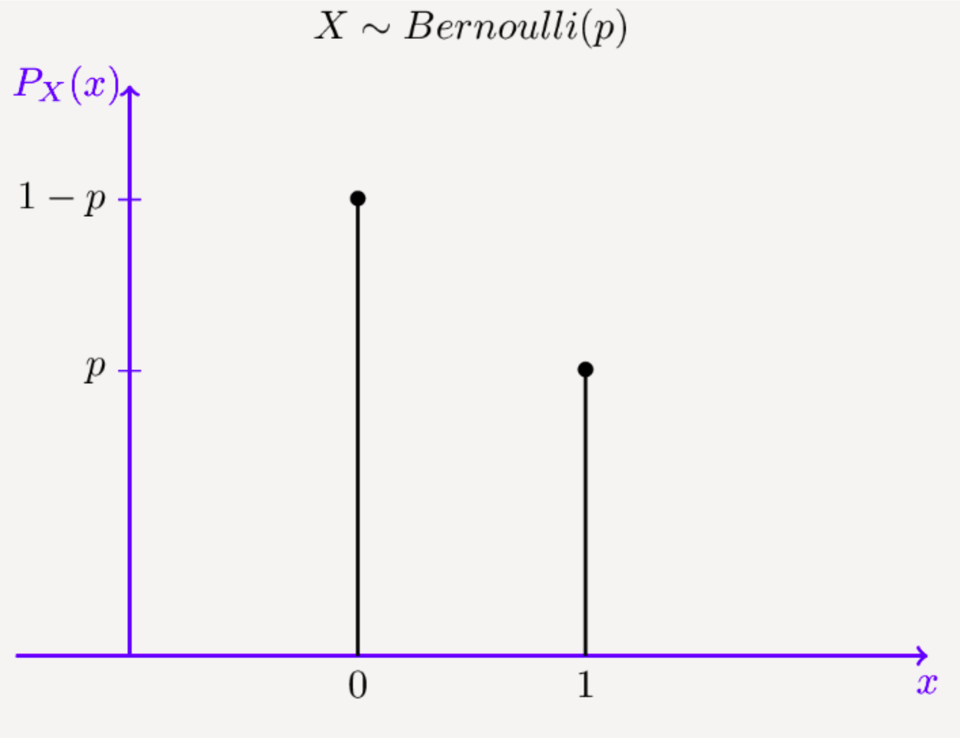


Fig.3.2 - PMF of a $Bernoulli(p)$ random variable.

# Example

**1.** A basketball player can shoot a ball into the basket with a probability of 0.6. What is the probability that he misses the shot?

**Solution:**

We know that success probability $P(X = 1) = p = 0.6$

Thus, probability of failure is $P(X = 0) = 1 - p = 1 - 0.6 = 0.4$

**Answer:**

The probability of failure of the Bernoulli distribution is 0.4

# Binomial random variable

A binomial random variable is random variable that represents the number of successes in n successive independent trials of a Bernoulli experiment

If X is a Binomial random variable, we denote this X ~ Bin(n, p), where p is the probability of success in a given trial.

The binomial distribution formula is for any random variable X, given by;

$$P(x) = \binom{n}{x} p^x q^{n-x} = \frac{n!}{(n-x)!x!} p^x q^{n-x}$$

where

$n$ = the number of trials (or the number being sampled)

$x$ = the number of successes desired

$p$ = probability of getting a success in one trial

$q$ = 1 - p = the probability of getting a failure in one trial

# Binomial random variable

Suppose that I have a coin with P(H)=p. I toss the coin n times and define X to be the total number of heads that I observe. Then X is binomial with parameters n and p, and we write X~Binomial(n,p). The range of X in this case is

$R_X=\{0,1,2,...,n\}$.

Let n=5, and let us compute the probability of getting exactly 4 heads, P (X = 4). The probability of one sample sequence (HTHHH) is given by

$$\mathbb{P}(HTHHH) = p \cdot (1-p) \cdot p \cdot p \cdot p = p^4 (1-p)^{5-4}$$

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There are $\binom{5}{4}$ such sequences of coin flips that give exactly 4 heads.
So, we need to add the probability of $\binom{5}{4}$ cases.

So, we get
$$\mathbb{P}(X=4) = \binom{5}{4} p^4(1-p)^{5-1}$$

We can generalize this to get the PMF of a binomial random variable.

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

# Binomial distribution

**Definition**: A random variable X is said to be a **binomial random variable** with parameters n and p, shown as X~Binomial(n,p), if its PMF is given by

$$P_X(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & \text{for } k = 0, 1, 2, \cdots, n \\ 0 & \text{otherwise} \end{cases}$$

where $0 < p < 1$.

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where 0 < p < 1.

Since $P_X(k)$ is a PMF, the sum of all probabilities should add upto 1.

$$\begin{aligned} \sum_{k=0}^{n} p_X(k) &= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \\ &= (p + (1-p))^n \\ &= 1^n = 1 \end{aligned}$$

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Binomial theorem: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$

**Example 1: If a coin is tossed 5 times, using binomial distribution find the probability of:**
**(a) Exactly 2 heads**
**(b) At least 4 heads.**

**Solution:**
**(a)** The repeated tossing of the coin is an example of a Bernoulli trial.
According to the problem:
Number of trials: $n=5$
Probability of head: $p= 1/2$ and hence the probability of tail, $q =1/2$
For exactly two heads:
$x=2$
$P(x=2) = {}^5C2\ p^2\ q^{5-2} = 5!\ /\ 2!\ 3! \times (½)^2 \times (½)^3$
$P(x=2) = 5/16$

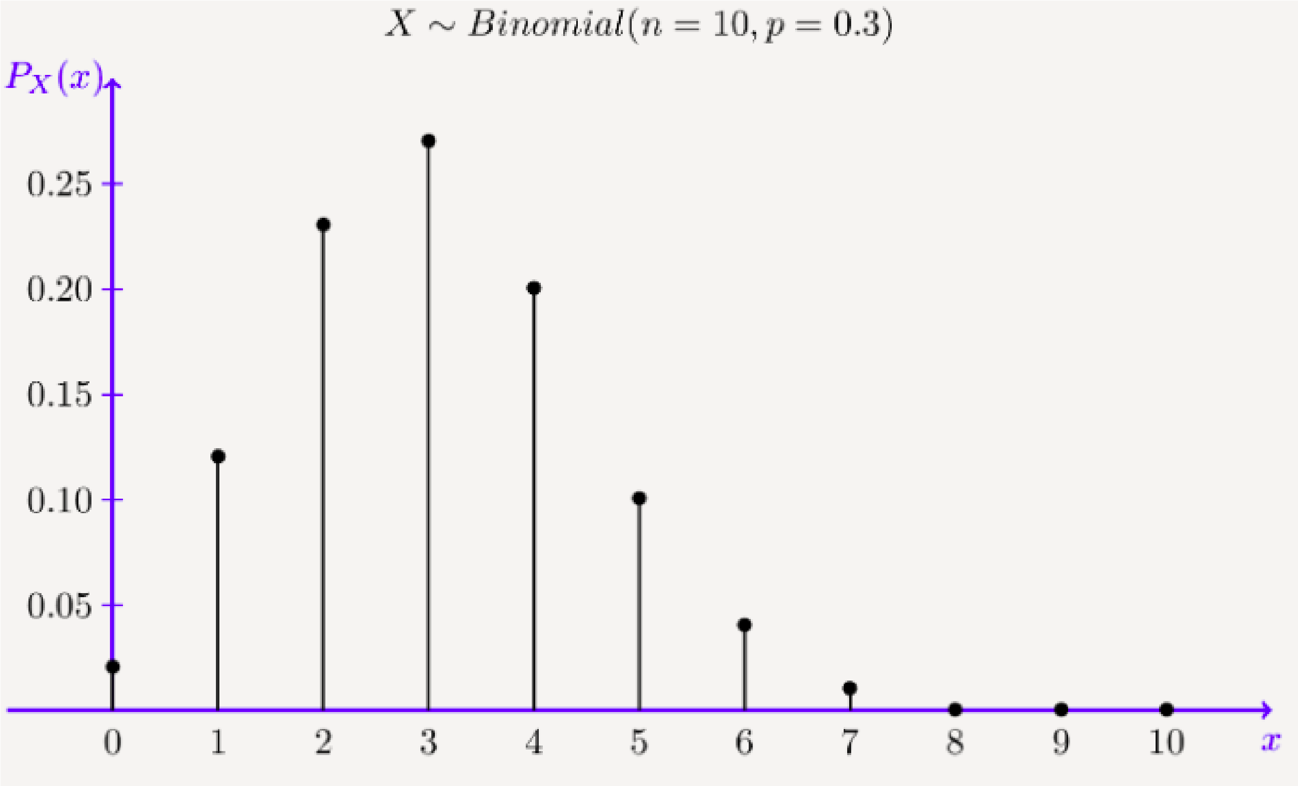
**(b)** For at least four heads,
$x \geq 4$, $P(x \geq 4) = P(x = 4) + P(x=5)$
Hence,
$P(x = 4) = {}^5C4\ p^4\ q^{5-4} = 5!/4!\ 1! \times (½)^4 \times (½)^1 = 5/32$
$P(x = 5) = {}^5C5\ p^5\ q^{5-5} = (½)^5 = 1/32$
**Answer: Therefore, $P(x \geq 4) = 5/32 + 1/32 = 6/32 = 3/16$**

# Binomial distribution

$$P_X(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & \text{for } k = 0, 1, 2, \cdots, n \\ 0 & \text{otherwise} \end{cases}$$

**Example PMF**:

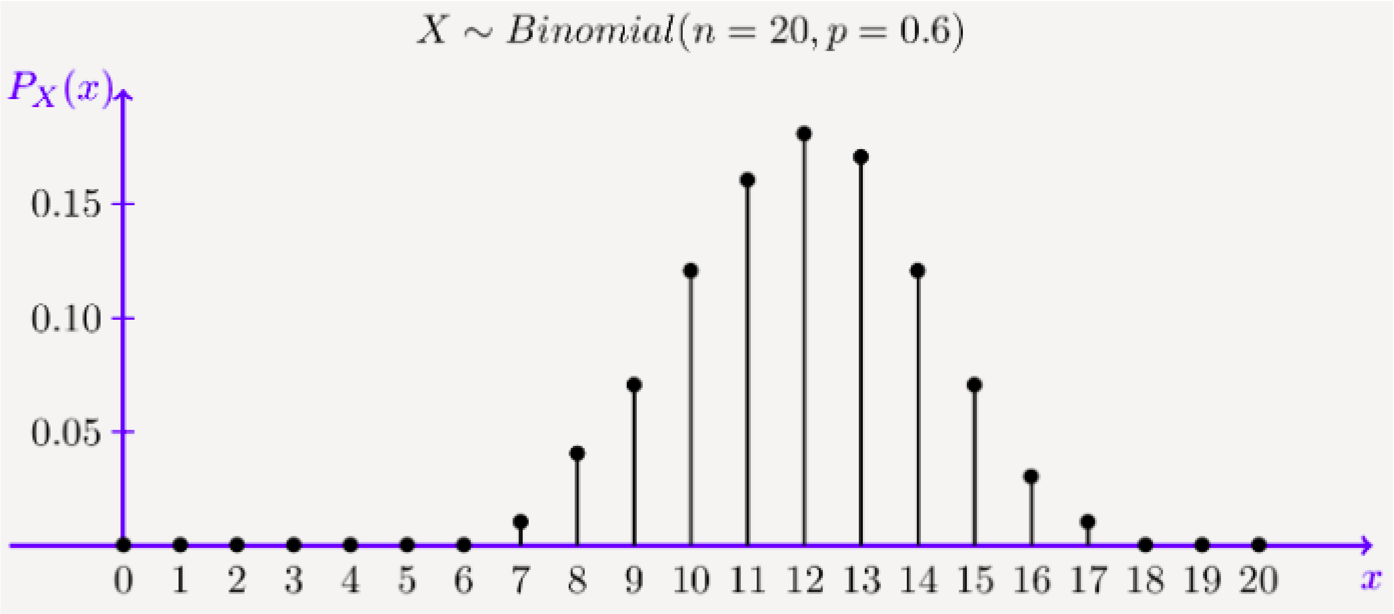
PMF of a Binomial(10,0.3) random variable




PMF of a Binomial(10,0.3) random variable

# Binomial distribution

Another example of Binomial distribution with n=20 and p=0.6
Since p > 0.5 the mode (peak) of the distribution occurs to the right of n/2 (20/2 = 10).




PMF of a Binomial(20,0.6) random variable

# Binomial distribution

**Business Applications**

- Banks and other financial institutions use Binomial Distribution to determine the likelihood of borrowers defaulting, and apply the number towards pricing insurance, and figuring out how much money to keep in reserve, or how much to loan.

| **Bernoulli Distribution** | **Binomial Distribtuion** |
|---|---|
| Bernoulli distribution is used when we want to model the outcome of a single trial of an event. | If we want to model the outcome of multiple trials of an event, Binomial distribution is used. |
| It is represented as X ~~ Bernoulli (p). Here, p is the probability of success. | It is denoted as X ~~ Binomial (n, p). Where n is the number of trials. |
| Mean, E[X] = p | Mean, E[X] = np |
| Variance, Var[X] = p(1-p) | Variance, Var[X]= np(1-p) |
| Example:<br>Suppose the probability of passing an exam is 80% and failing is 20%. Then the Bernoulli distribution can be used to model the passing or failing in such an exam. | Example:<br>Suppose the probability of passing an exam is 80% and failing is 20%. Then if we want to find the probability that a student will pass in exactly 4 out of 5 exams, we use the Binomial Distribution. |

# The Poisson random variable: motivation

The Poisson distribution is one of the most widely used probability distributions. It is usually used in scenarios **where we are counting the occurrences of certain events in an interval of time or space**. In practice, it is often an approximation of a real-life random variable. A Poisson random variable can be used to answer real-life questions such as

- How many babies born in the next minute?
- How many car crashes happen per hour?
- How many customers visit a store in an hour?

Suppose that we are counting the number of customers who visit a certain store from 1pm to 2pm. Based on data from previous days, we know that on average $\lambda=15$ customers visit the store. Of course, there will be more customers some days and fewer on others. Here, we may model the random variable X showing the number customers as a **Poisson random variable** with parameter $\lambda=15$.

## Business Applications

- Predicting customer sales on particular days/times of the year.
- Supply and demand estimations to help with stocking products.
- Service industries can prepare for an influx of customers, hire temporary help, order additional supplies, and make alternative plans to reroute customers if needed.

The Poisson distribution

# The Poisson distribution

A random variable X is said to be a Poisson random variable with parameter λ, shown as X~Poisson(λ), if its range is $R_X$={0,1,2,3,...}, and its PMF is given by

$$P_X(k) = \begin{cases} \frac{e^{-\lambda}\lambda^k}{k!} & \text{for } k \in R_X \\ 0 & \text{otherwise} \end{cases}$$

As λ increases, exp(-λ) decreases, but $\lambda^k$ increases; a peak is formed.

$$\begin{aligned} \sum_{k\in R_X} P_X(k) &= \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} \\ &= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \\ &= e^{-\lambda} e^{\lambda} \\ &= 1. \end{aligned}$$

# The Poisson distribution: examples

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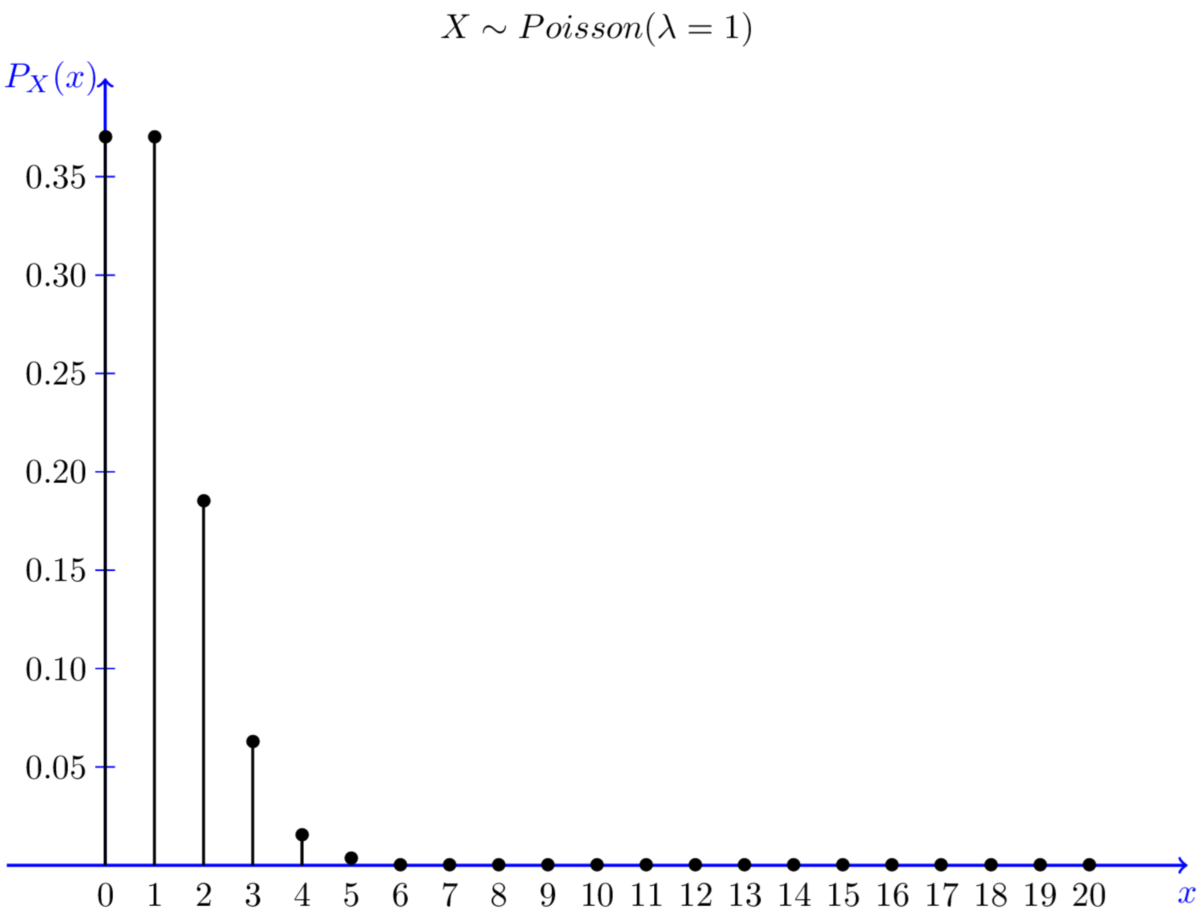
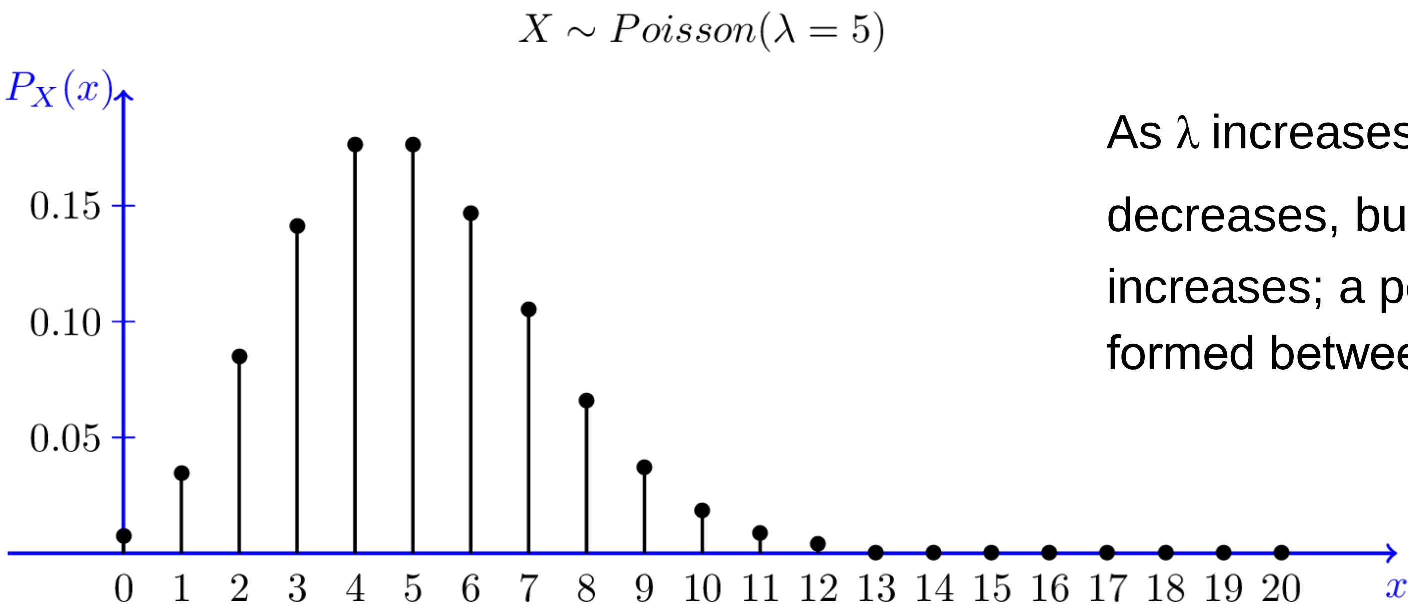



Note: the peak of the distribution is between λ and λ-1.

Asymmetric distribution

# **Poisson distribution**: examples

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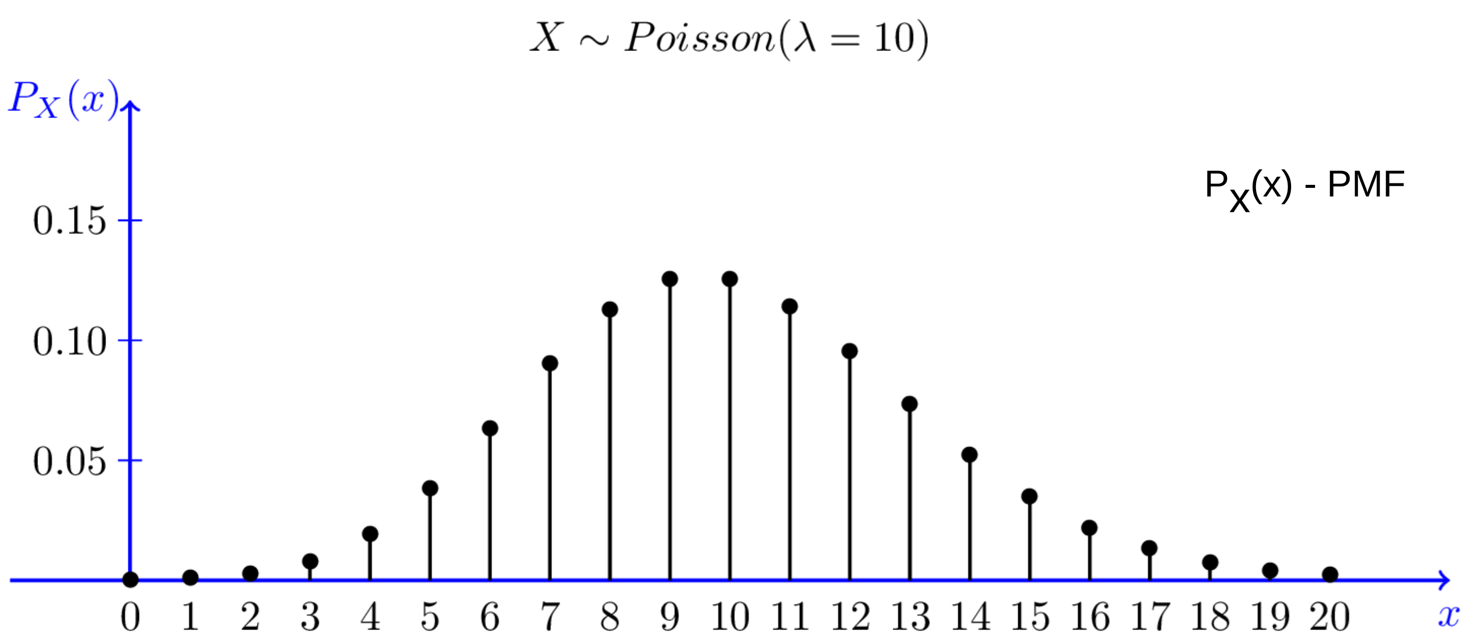


$P_X(x)$ - PMF

EXAMPLE:

You are a manager of car dealership and the average number of cars sold is 2 cars per day. What is the probability that exactly 5 cars will be sold tomorrow?

Solution

- Since we have 2 cars sold per day, $\mu = 2$.
- Since we want to know the likelihood that 5 cars being sold tomorrow, $x = 5$.
- We know that $e = 2.71828$ (constant).

Substitute in Poisson formula as follows:

$$P(x;\ \mu) = (e^{-\mu})\,(\mu^{x})\,/\,x!$$
$$P(5;\ 2) = (2.71828^{-2})\,(2^{5})\,/\,5!$$
$$P(5;\ 2) = 0.036$$

Thus, the probability of selling 5 cars tomorrow is 0.036.

# **Poisson distribution**: Application areas

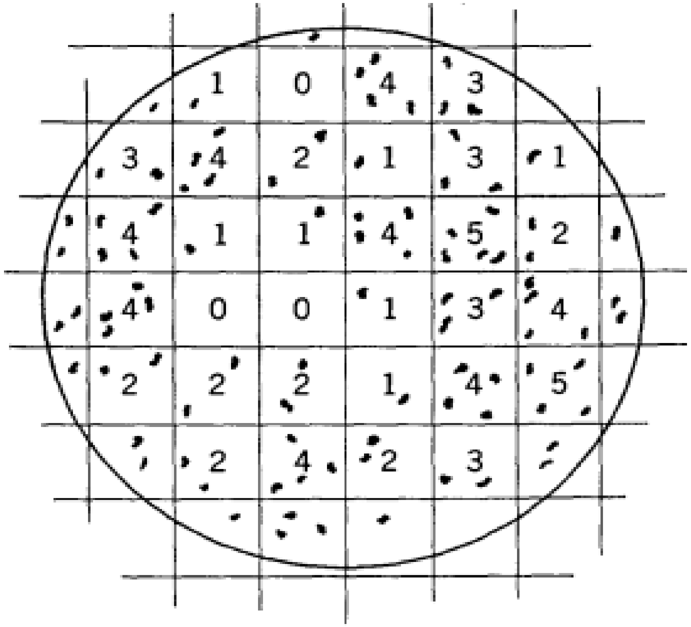

Figure 1. Bacteria on a Petri plate.

Figure 1 reproduces a photograph of a Petri plate with **bacterial colonies**, which are visible under the microscope as dark spots. The plate is divided into small squares. The observed numbers of squares with exactly k dark spots follow a Poisson distribution.

**Chromosome interchanges in cells**: Irradiation by X-rays produces certain processes in organic cells, called chromosome interchanges. According to theory, the numbers N, of cells with exactly k interchanges should follow a Poisson distribution.

# References


- Section 3.1.1 of TS1:  Alex Tsun, Probability & Statistics with Applications to Computing (Available at: http://www.alextsun.com/files/Prob_Stat_for_CS_Book.pdf)
- https://www.probabilitycourse.com/chapter3/3_1_2_discrete_random_var.php
- Chapter VI of TP1: William Feller, An Introduction to Probability Theory and Its Applications: Volume 1, Third Edition, 1968 by John Wiley & Sons,Inc.