Задание к занятию №9

№1

$z = z(x;y);\quad x = x(t);\quad y = y(t);\quad \frac{dz}{dt} - ?$

(1.1) $z = x^2 + y^2 + xy,\quad x = a\sin t,\quad y = a\cos t.$

$z = a^2\sin^2 t + a^2\cos^2 t + a^2\sin t\cos t =$

$= a^2 + \frac{a^2}{2}\sin 2t.$

$\frac{dz}{dt} = 0 + \frac{a^2}{2}\cos 2t \cdot 2 = a^2\cos 2t.$

(1.2) $z = x^2y^3u,\quad x = t;\quad y = t^2,\quad u = \sin t.$

$\frac{\partial z}{\partial x} = 2xy^3u;\quad \frac{\partial z}{\partial y} = 3x^2y^2u;\quad \frac{\partial z}{\partial u} = x^2y^3;$

$\frac{\partial x}{\partial t} = 1;\quad \frac{\partial y}{\partial t} = 2t;\quad \frac{\partial u}{\partial t} = \cos t.$

$\frac{dz}{dt} = 2xy^3u + 3x^2y^2u \cdot 2t + x^2y^3 \cdot \cos t =$

$= 2 \cdot t^7 \cdot \sin t + 6t^7\sin t + t^8\cos t =$

$= 8t^7\sin t + t^8\cos t;$

№2

$z = f(x;y),\quad x = x(u;v);\quad y = y(u;v);\quad \frac{\partial z}{\partial u} - ?;\quad \frac{\partial z}{\partial v} - ?$

(2.1) $z = x^3 + y^3;\quad x = uv;\quad y = \frac{u}{v};$

$\frac{\partial z}{\partial x} = 3x^2;\quad \frac{\partial z}{\partial y} = 3y^2;$

$\frac{\partial x}{\partial u} = v;\quad \frac{\partial x}{\partial v} = u;\quad \frac{\partial y}{\partial u} = \frac{1}{v};\quad \frac{\partial y}{\partial v} = -\frac{u}{v^2};$

$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = 3x^2\cdot v + \frac{3y^2}{v};$

$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} = 3x^2\cdot u - \frac{3y^2\cdot v^2}{u}$

2.2 $z = \cos xy$; $x = ue^{v}$; $y = v \ln u$

$\frac{\partial z}{\partial x} = -\sin(xy)\cdot y$; $\frac{\partial z}{\partial y} = -\sin(xy)\cdot x$;

$\frac{\partial x}{\partial u} = e^{v}$; $\frac{\partial x}{\partial v} = ue^{v}$; $\frac{\partial y}{\partial u} = \frac{v}{u}$; $\frac{\partial y}{\partial v} = \ln u$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = -y\sin(xy)\cdot e^{v} - \frac{x\sin(xy)\cdot v}{u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} = -y\sin(x;y)u\cdot e^{v} - x\sin(x;y)\cdot\ln u$$

№3

$y'(x)$ - ? $xe^{2y} - y\ln x = 8$.

$F(x;y) = xe^{2y} - y\ln x - 8$

$F'_x = e^{2y} - \frac{y}{x}$

$F'_y = xe^{2y}\cdot 2 - \ln x$

$$y' = -\frac{F'_x(x;y)}{F'_y(x;y)} = -\frac{e^{2y} - \frac{y}{x}}{2xe^{2y} - \ln x}$$

№4

$F(x;y) = x^3 y - y^3 x - 6$; $M_0(2;1)$

$F(2;1) = 8 - 2 - 6 = 0$.

$F'_x = 3x^2 y - y^3$; $F'_y = x^3 - 3y^2 x$

$F'_x(2;1) = 12 - 1 = 11$; $F'_y(2;1) = 8 - 6 = 2$; $\neq 0$!

$$k = y'(1) = -\frac{F'_x(2;1)}{F'_y(2;1)} = -\frac{11}{2};$$

(t): $y - y_0 = k(x - x_0)$ (n): $y - y_0 = -\frac{1}{k}(x - x_0)$

$y - 2 = -\frac{11}{2}(x - 1)$; $y - 2 = \frac{2}{11}(x - 1)$

№3

5.1 $z = \sin x \sin y$, $d^2 z$ - ?

$$\frac{\partial z}{\partial x} = \cos x \sin y; \qquad \frac{\partial z}{\partial y} = \sin x \cos y$$

$$\frac{\partial^2 z}{\partial x^2} = -\sin x \sin y; \qquad \frac{\partial^2 z}{\partial y^2} = -\sin x \sin y$$

$$\frac{\partial^2 z}{\partial x \partial y} = \cos x \cos y;$$

$$d^2 z = \frac{\partial^2 z}{\partial x^2} dx^2 + 2 \frac{\partial^2 z}{\partial x \partial y} dx dy + \frac{\partial^2 z}{\partial y^2} dy^2 =$$

$$= -\sin x \sin y \, dx^2 + 2 \cos x \cos y \, dx \, dy - \sin x \sin y \, dy^2 =$$

$$= -\sin x \sin y (dx^2 + dy^2) + 2 \cos x \cos y \, dx \, dy$$

5.2 $z = xy + \sin(x+y)$, $\frac{\partial^2 z}{\partial x^2}$ - ?

$$\frac{\partial z}{\partial x} = y + \cos(x+y) \cdot 1$$

$$\frac{\partial^2 z}{\partial x^2} = 0 - \sin(x+y) \cdot 1 = -\sin(x+y)$$

5.3 $z = \operatorname{arctg} \frac{x+y}{1-xy}$; $\frac{\partial^2 z}{\partial x \partial y}$ - ?

$$\frac{\partial z}{\partial x} = \frac{1}{1 + \frac{(x+y)^2}{(1-xy)^2}} \cdot \frac{1 \cdot (1-xy) - (x+y) \cdot (0-y)}{(1-xy)^2} =$$

$$= \frac{1 - xy + y(x+y)}{(1-xy)^2 + \frac{(x+y)^2 \cdot (1-xy)^2}{(1-xy)^2}} = \frac{1 - xy + xy - y^2}{(1-xy)^2 + (x+y)^2} =$$

$$= \frac{1-y^2}{(1-xy)^2 + (x+y)^2} = \frac{1-y^2}{1 - 2xy + x^2y^2 + x^2 + 2xy + y^2} = \frac{1-y^2}{x^2 + y^2 + x^2y^2 + 1}$$

$$\frac{\partial z}{\partial x} = \frac{1-y^2}{x^2+y^2+x^2y^2+1}$$

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{-2y(x^2+y^2+x^2y^2+1)-(1-y^2)(2y+2x^2y)}{(x^2+y^2+x^2y^2+1)^2}$$

№6

$x^2 - xy - 2y^2 + x - y = 1$

$y(0) = 1;$

$y' - ?;\ y'' - ?;\ y''' - ?$

- $2x - y - xy' - 4yy' + 1 - y' = 0$

- $2 - y' - y' - xy'' - 4(y')^2 - 4yy'' - y'' = 0$

  $2 - 2y' - xy'' - 4(y')^2 - 4yy'' - y'' = 0$

- $-2y'' - y'' - xy''' - 8y'y'' - 4y'y'' - 4yy''' - y''' = 0$

- $y(0) = 1$

$y'(0):\quad 2\cdot 0 - 1 - 0\cdot y' - 4\cdot 1\cdot y' + 1 - y' = 0$

$-1 - 5y' = 0$

$y'(0) = -\frac{1}{5};$

$y''(0):\quad 2 + 2\cdot\frac{1}{5} - 0\cdot y'' - 4\left(-\frac{1}{5}\right)^2 - 4\cdot 1y'' - y'' = 0$

$2 + \frac{2}{5} - \frac{4}{25} - 5y'' = 0$

$y'' = \frac{56/25}{5} = \frac{56}{125};$

$y'''(0):\quad -2\cdot\frac{56}{125} - \frac{56}{125} - 0\cdot\frac{56}{125} - 8\cdot\left(-\frac{1}{5}\right)\cdot\frac{56}{125} - 4\left(-\frac{1}{5}\right)\cdot\frac{56}{125} - 4y''' - y'''$

$\frac{56}{125}\left(-2 - 1 + \frac{8}{5} + \frac{4}{5}\right) = 5y'''$

$$-\frac{56}{125}\cdot\frac{3}{5}=5y^{14}$$

$$y^{14}=-\frac{168}{3125}$$

№7

$$z=\operatorname{arctg}\frac{y}{x};$$

построить линии уровне и градиента сравнить направление в точках $A(1;1)$; $B(1;-1)$

$$z'_x=\frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{-y}{x^2}=\frac{-y}{x^2+y^2};$$

$$z'_y=\frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x}=\frac{1}{x+\frac{y^2}{x}}=\frac{x}{x^2+y^2}$$

$$\vec{AB}=\vec{l}=(1-1;\ -1-1)=(0;-2);$$

- для точки A: $z'_x(1;1)=\frac{-1}{1+1}=-\frac{1}{2}$;

$$z'_y(1;1)=\frac{1}{1+1}=\frac{1}{2};$$

$$\vec{grad}\,z(1;1)=\left(-\frac{1}{2};\ \frac{1}{2}\right)$$

$$|\vec{grad}\,z(1;1)|=\sqrt{\left(-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}=\sqrt{\frac{1}{2}};$$

- для точки B: $z'_x(1;-1)=\frac{1}{1+1}=\frac{1}{2}$;

$$z'_y(1;-1)=\frac{1}{1+1}=\frac{1}{2};$$

$$\vec{grad}\,z(1;-1)=\left(\frac{1}{2};\ \frac{1}{2}\right)$$

$$|\vec{grad}\,z(1;-1)|=\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}=\sqrt{\frac{1}{2}};$$

- угол α между градиентами в точках A и B:

$$\cos\alpha=\frac{\vec{grad}\,z(1;1)\cdot\vec{grad}\,z(1;-1)}{|\vec{grad}\,z(1;1)|\cdot|\vec{grad}\,z(1;-1)|}=$$

$$=\frac{-\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}}{\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}}}=0\ \Rightarrow\ \alpha=90^\circ;$$

Построение

$z = \operatorname{arctg}\left(\frac{y}{x}\right)$

$\frac{y}{x} = \operatorname{tg} z \quad \Rightarrow \quad y = x \cdot \operatorname{tg} z$

- $z = 0$: $\quad y = x \operatorname{tg} 0 = 0$
- $z = \frac{\pi}{6}$: $\quad y = x \operatorname{tg} \frac{\pi}{6} = \frac{1}{\sqrt{3}} x$

  $z = -\frac{\pi}{6}$: $\quad y = -\frac{1}{\sqrt{3}} x$;
- $z = \frac{\pi}{4}$: $\quad y = x \operatorname{tg} \frac{\pi}{4} = x$.

  $z = -\frac{\pi}{4}$: $\quad y = -x$;
- $z = \frac{\pi}{3}$: $\quad y = x \operatorname{tg} \frac{\pi}{3} \Rightarrow y = \sqrt{3} x$.

  $z = -\frac{\pi}{3}$: $\quad \Rightarrow \quad y = -\sqrt{3} x$.