Exercises of chapter 1

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1 Approximation

Determine experimentally in which situations it is convenient to resort to approximations of the factorial like Equation 2 and Equation 3. That is to say, how much computational time is saved, and at what precision cost. Additionally, what happens if the values ??of e are also approximates. (For example, with Equation 2 or 3 or some other of the various forms that are known to approximate their value) and of π (for which there are numerous approximations) instead of assuming they were constants of fixed precision.

Factorial numbers can be used in combinational probability, to calculate combinations and permutations. Through this, factorials are also often used to calculate probabilities, such as binomial coefficient where ! marks the factorial number, and it is defined as all integer number k $\in \mathbb{Z}$, such as

$$k! = k * (k-1) * (k-2) * ... * 2 * 1 \tag{1}$$

A very useful approximation of the factorial is the Stirling approximation:

$$k! \approx k^k e^{-k} \sqrt{2\pi k} \tag{2}$$

and the improved version by Gosper:

$$k! \approx \sqrt{\pi(2k + \frac{1}{3})} k^k e^{-k} \tag{3}$$

This approximations are used when the factorial number is too big. However, this approximations come with various degrees of error. Equation 4 shows the formula to get the percentage error.

$$E = \mid \frac{Tv - Av}{Tv} \mid * 100\% \tag{4}$$

Where:

E: Percentage error.
Tv: True value
Av: Approximation value

In Python the library math has already a fixed parameter for the value π and e.

```
In [27]:
import math
from math import pi, e
from timeit import timeit

print (pi, e)
```

```
3.141592653589793 2.718281828459045
```

First we begin our experiment by taking those parameters of π and e from the math library, and using the Stirling approximation. [1] In Table 1 we can se the results. Column N is the number we used to take the value of the factorial. N! is the answer provided by python form the same math library, and the column Stirling is the approximation. Time is the processing time it took to compile, and the error was calculated with Equation 4.

Table 1: Results of the Stirling approximation

N	**N!**	**Stirling**	**Time**	**Error**
1	1	0.922137009		0.077862991
5	120	118.941305	0.000825167	0.008822459
10	3628800	3598814.56	0.001646757	0.008263183
25	1.55112E+25	1.54596E+25	0.001966	0.003327607
50	3.04141E+64	3.03634E+64	0.002142906	0.001665256
75	2.4809E+109	2.4782E+109	0.002707958	0.001110487
100	9.3326E+157	9.3248E+157	0.002886772	0.000832983
125	1.8827E+209	1.8814E+209	0.003069878	0.000666443
130	6.4669E+219	6.4627E+219	0.003262997	0.000640819

Then in Table 2 we have the results of the Gosper Equation 3.
We see that the bigger the value of N, the approximation gets similar to the true value.

As the next step on our experimentation, we wanted to explore a little more inside the formulas of Stirling and Gosper, since both of them use the π and e values, and they are approximations as well.

[1] All experimentation mentioned we used a code in python, and it can be consulted here:

Table 2: Results of the Gosper approximation.

N	**N!**	**Gosper**	**Time**	**Error**
1	1	0.996021807		0.003978193
5	120	120.966052	0.012824059	-0.008050433
10	3628800	3628681.791	0.013508797	3.25753E-05
25	1.55112E+25	1.5511E+25	0.013875008	1.08842E-05
50	3.04141E+64	3.0414E+64	0.014073133	2.7494E-06
75	2.4809E+109	2.4809E+109	0.014258862	1.22616E-06
100	9.3326E+157	9.3326E+157	0.016538858	6.90896E-07
125	1.8827E+209	1.8827E+209	0.016858816	4.42627E-07
130	6.4669E+219	6.4669E+219	0.017882824	4.09269E-07

Both of this values have formulas to get a better approximation to its real value. For example, Equation 5 for the π value called the Leibniz formula, and Equation 6 for the e value. In Table 3 the column of Value for π and e is the value of n in the equations.

$$\pi = 4 * \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+2} = \frac{\pi}{4} \tag{5}$$

$$e = (1 + \frac{1}{n})^n \tag{6}$$

Table 3: Description of the distribution of the values for π and e.

Value for π and e	π	**e**
10	3.041839619	2.59374246
100	3.131592904	2.704813829
1000	3.140592654	2.716923932
10000	3.141492654	2.718145927
100000	3.141582654	2.718268237
1000000	3.141591654	2.718280469
10000000	3.141592554	2.718281694
100000000	3.141592644	2.718281798

Now we show the results of the experimentation in Table 4 and 5. Here is where the computational time shows some significance, because in the larger ones, it took more than a minute to compile and give us an answer.

Table 4: Results of the Stirling approximation using the different values of π and e

N	N!	Stirling	Time	Error
1	1	0.922232625		0.077767375
5	120	147.7417932	0.002756834	-0.23118161
10	3628800	3776077.14	0.004431009	-0.040585632
25	1.55112E+25	1.56514E+25	0.009263992	-0.009039953
50	3.04141E+64	3.0439E+64	0.04496026	-0.000817607
75	2.4809E+109	2.4791E+109	0.343361855	0.000737427
100	9.3326E+157	9.3253E+157	2.778558016	0.000783175
125	1.8827E+209	1.8814E+209	25.75114012	0.000660286
130	6.4669E+219	6.4627E+219	251.0401418	0.000639381

Table 5: Results of the Gosper approximation using the different values of π and e.

N	N!	Gosper	Time	Error
1	1	1.527525232	0	-0.527525232
5	120	150.7740198	0.00718689	-0.256450165
10	3628800	3807416.223	0.005645037	-0.049221843
25	1.55112E+25	1.57035E+25	0.012120962	-0.012397832
50	3.04141E+64	3.04896E+64	0.056209087	-0.002484249
75	2.4809E+109	2.4818E+109	0.380222797	-0.000372249
100	9.3326E+157	9.3331E+157	3.328314781	-4.91594E-05
125	1.8827E+209	1.8827E+209	30.81744933	-5.71849E-06
130	6.4669E+219	6.4669E+219	299.2242897	-1.02921E-06

2 Conclusions

At the end of the experimentation, we noticed that the second part of the experiment, when we began to change the parameters of π and e, the computational time started to grow, and comparing the first results where we used the default ones given by python, there is not much difference in accuracy of the approximation on the bigger numbers.

On the smaller numbers, the difference between the first and the second experiment is quite noticeable. This being because the accuracy of the π and e values are not very good.

With this we conclude that the accuracy of the π and e values are very significant on the result of the Stirling and Gosper approximation formulas. Also, the Gosper formula proves to be the more accurate of the two on the bigger numbers.