香港科技大學
THE HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY

COMP 5212
Machine Learning
Lecture 13

Dimensionality Reduction

Junxian He
Oct 22, 2024

Midterm Exam

Thursday

Tomorrow (Oct 24), 1:20pm-2:40pm, one A4-size double-sided cheetsheet is allowe
(either printing or handwriting is fine)

We have two rooms for the exam for sparse seat plans:

1. For SIS ID ending with an even digit: Room 2303
2. For SIS ID ending with an odd digit: Room 2504

next Tuesday

High-Dimensional Data

- High-Dimensions = Lot of Features

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Document classification

Features per document =

thousands of words/unigrams

millions of bigrams, contextual information

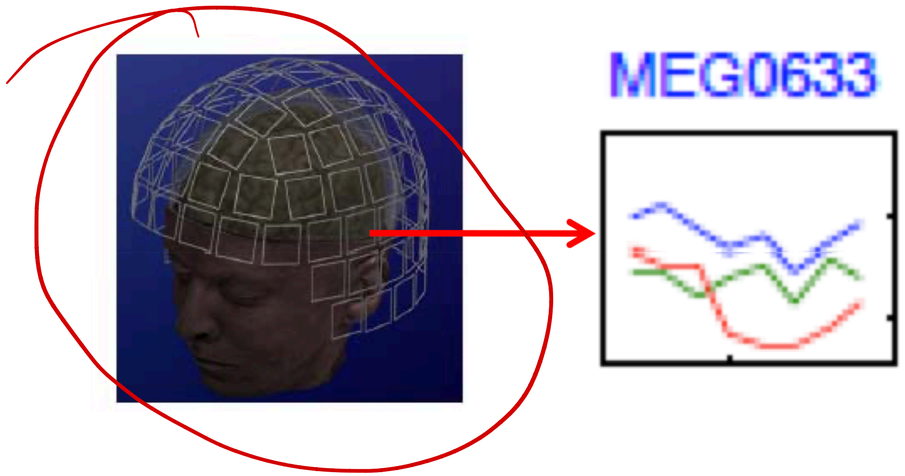
High-Dimensional Data

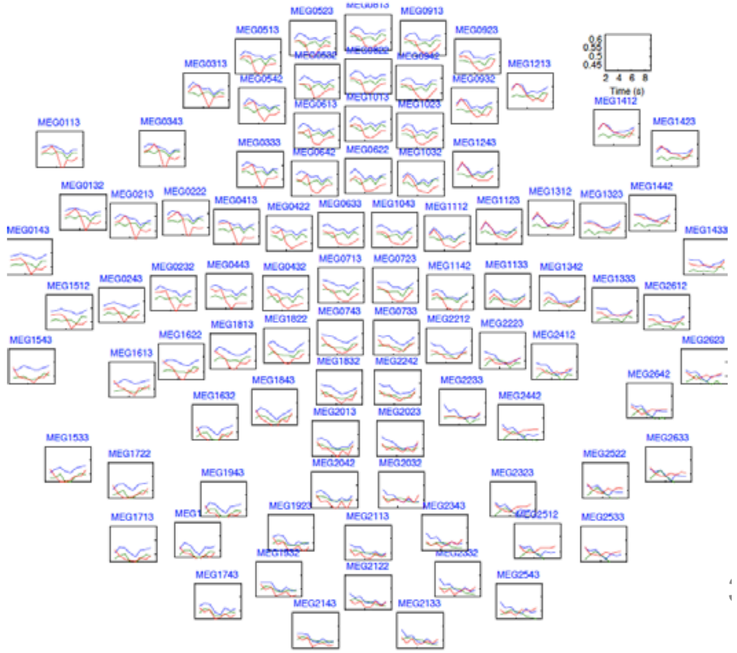
- High-Dimensions = Lot of Features

MEG Brain Imaging

120 locations x 500 time points

x 20 objects





Curse of Dimensionality

- Why are more features bad?

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  - Redundant features (not all words are useful to classify a document) more noise added than signal

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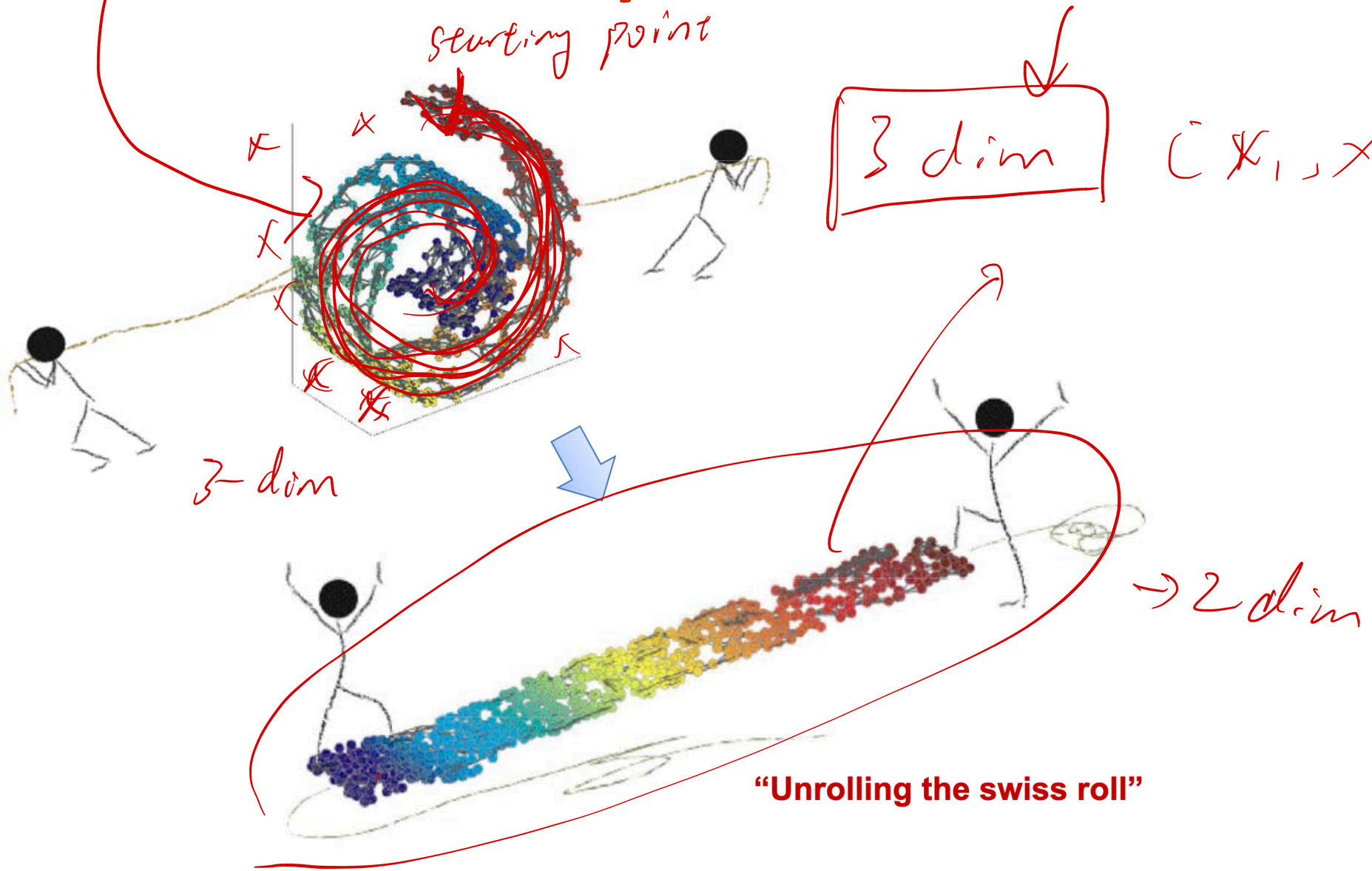
Curse of Dimensionality

- Why are more features bad?
  - Redundant features (not all words are useful to classify a document) more noise added than signal
  - Hard to store and process data (computationally challenging)
  - Hard to interpret and visualize
  - Complexity of decision rule tends to grow with # features

Dimensionality Reduction

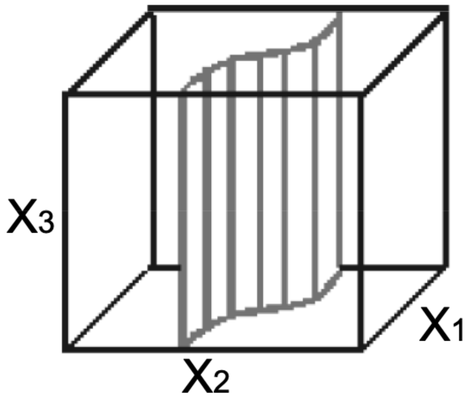


"Unrolling the swiss roll"

Dimensionality Reduction

Dimensionality Reduction

- Feature Selection – Only a few features are relevant to the learning task

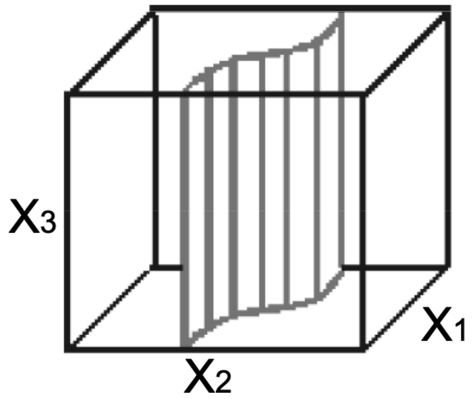


X3 - Irrelevant

supervised learning

Dimensionality Reduction

- Feature Selection – Only a few features are relevant to the learning task



X_3 - Irrelevant

- Latent features – Some linear/nonlinear combination of features provides a more efficient representation than observed features

Dimensionality Reduction

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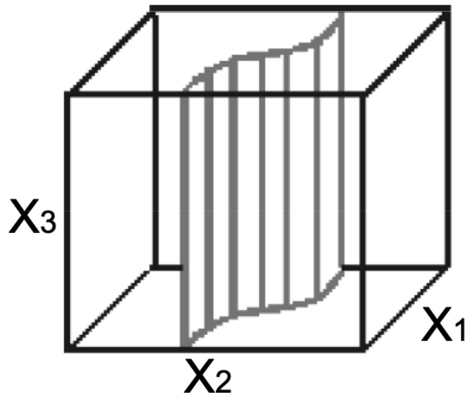

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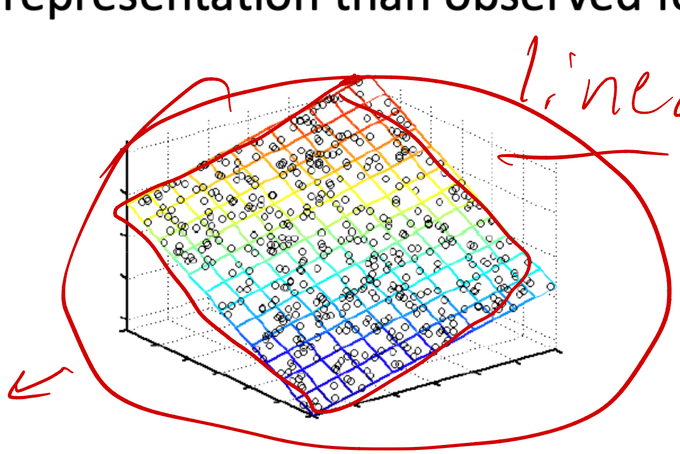

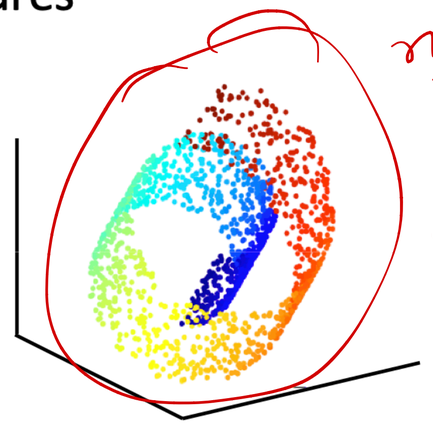

unsupervised

Latent Feature Extraction

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Combinations of observed features provide more efficient representation, and capture underlying relations that govern the data

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Topics (sports, science, news, etc.) instead of documents

Latent Feature Extraction

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E.g. Ego, personality and intelligence are hidden attributes that characterize human behavior instead of survey questions

Topics (sports, science, news, etc.) instead of documents

- Linear
  - Principal Component Analysis (PCA)
  - Factor Analysis
  - Independent Component Analysis (ICA)
- Nonlinear
  - ISOMAP
  - Local Linear Embedding (LLE)
  - Laplacian Eigenmaps

Principal Component Analysis (PCA)

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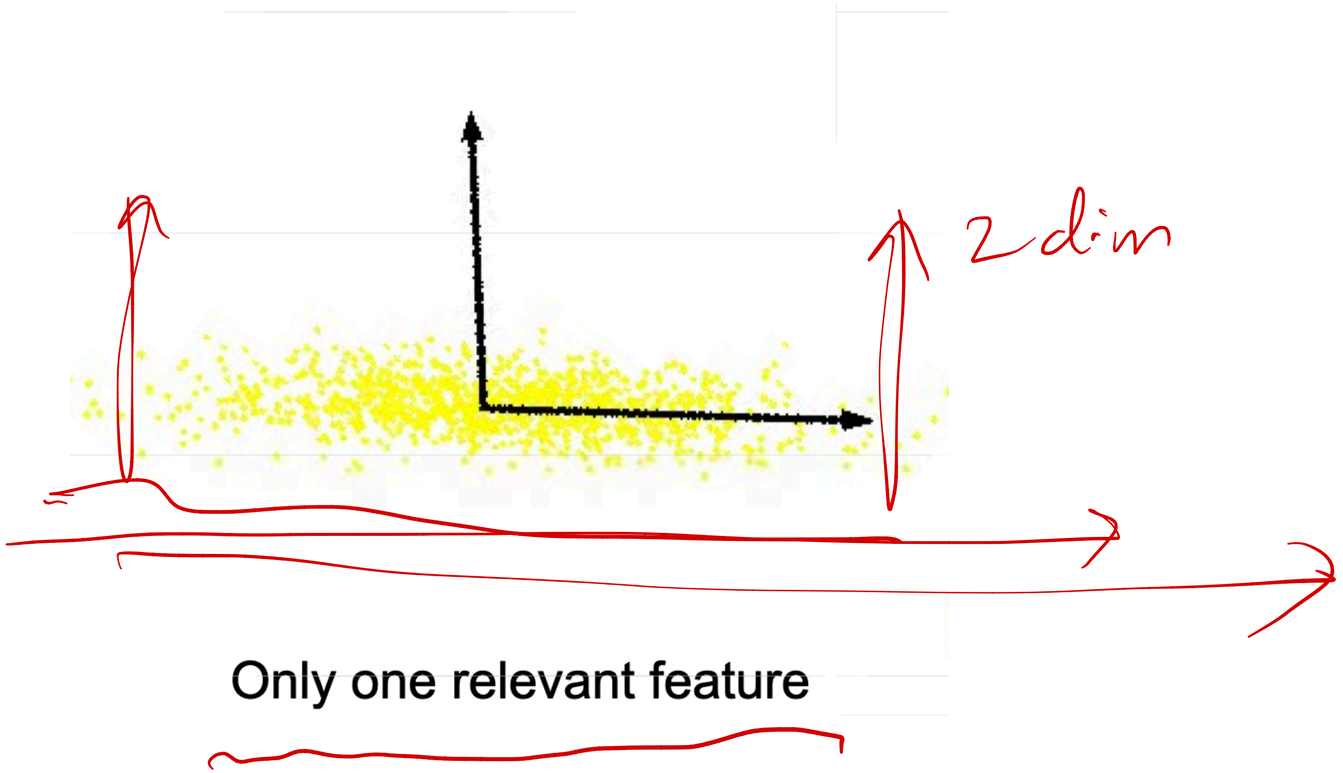
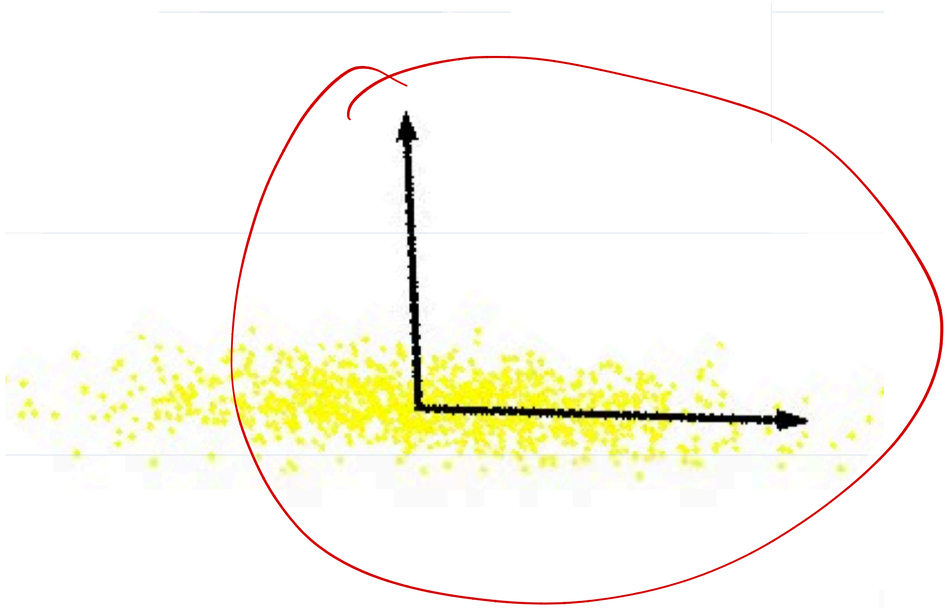
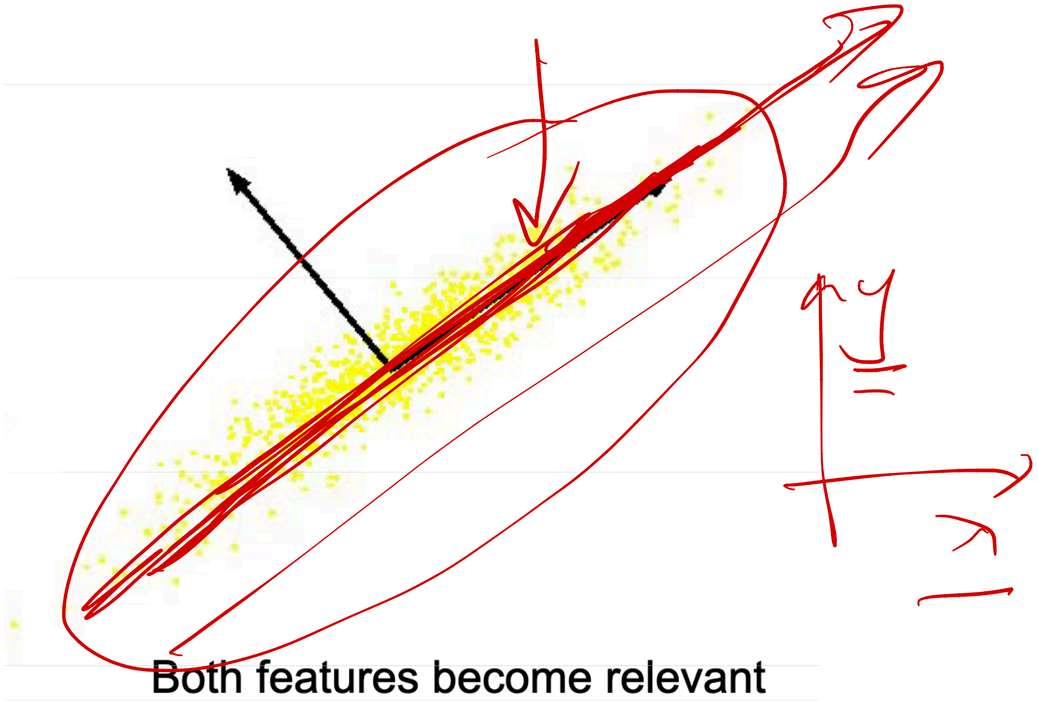


Principal Component Analysis (PCA)

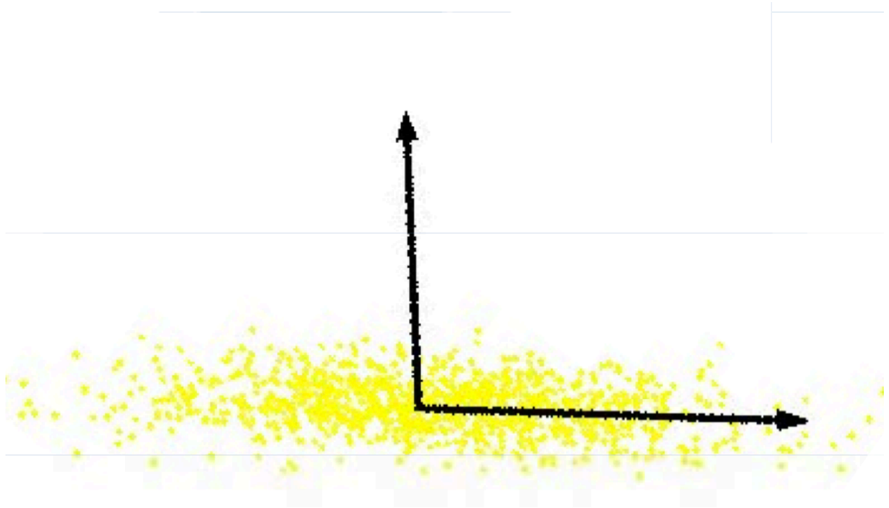


Only one relevant feature

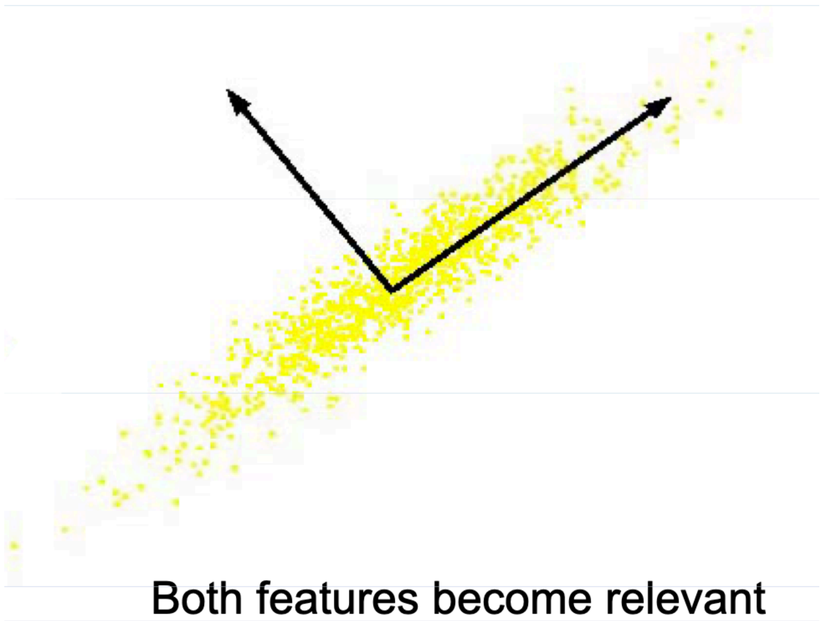


Both features become relevant

Principal Component Analysis (PCA)



Only one relevant feature



Both features become relevant

Can we transform the features so that we only need to preserve one latent feature? Find linear projection so that projected data is uncorrelated.

Principal Component Analysis (PCA)

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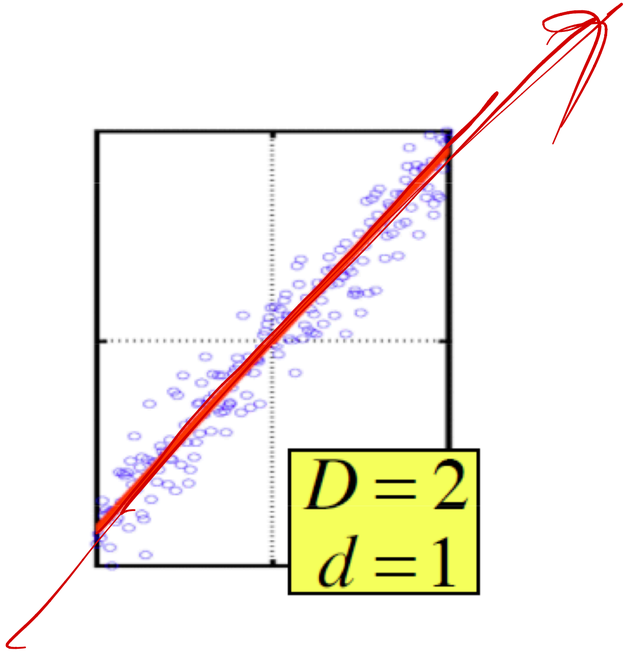
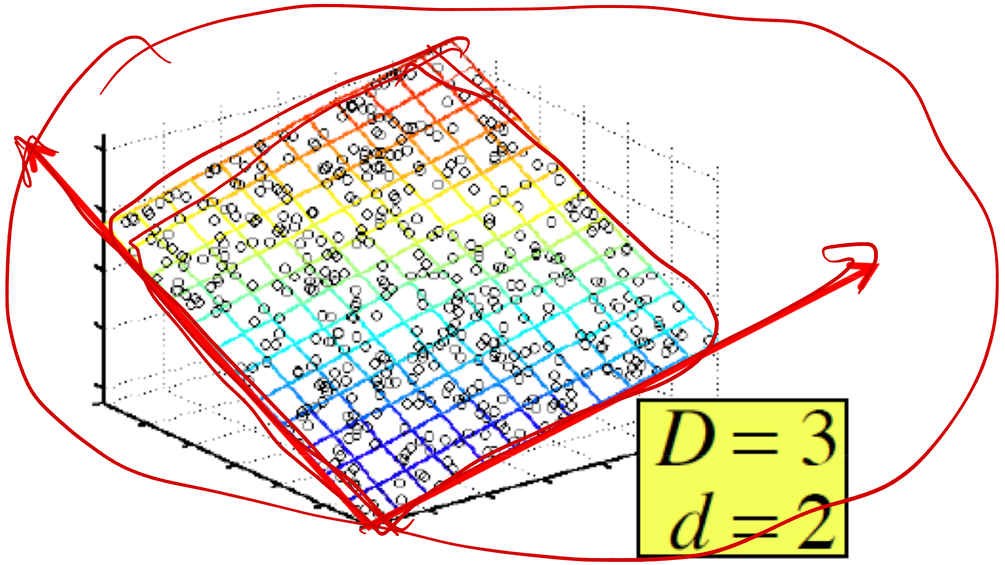
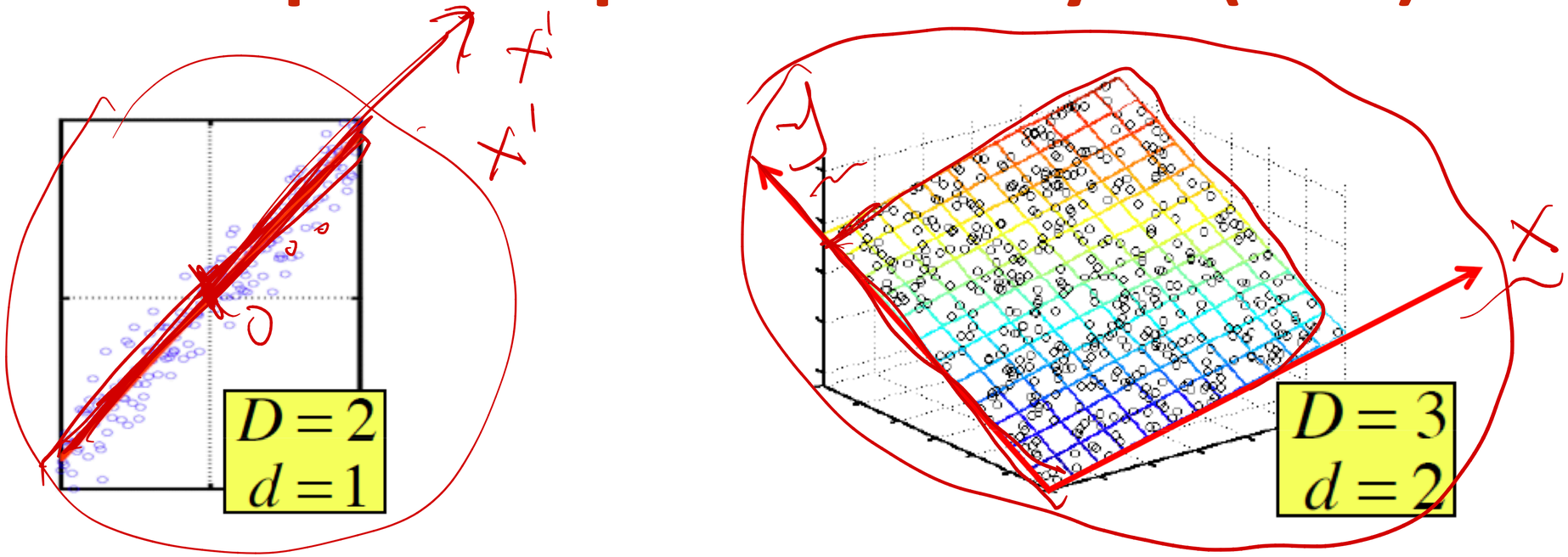




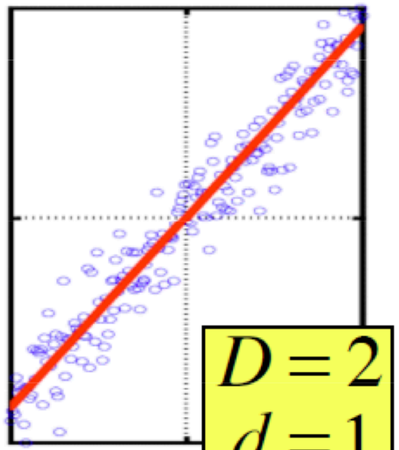
Principal Component Analysis (PCA)

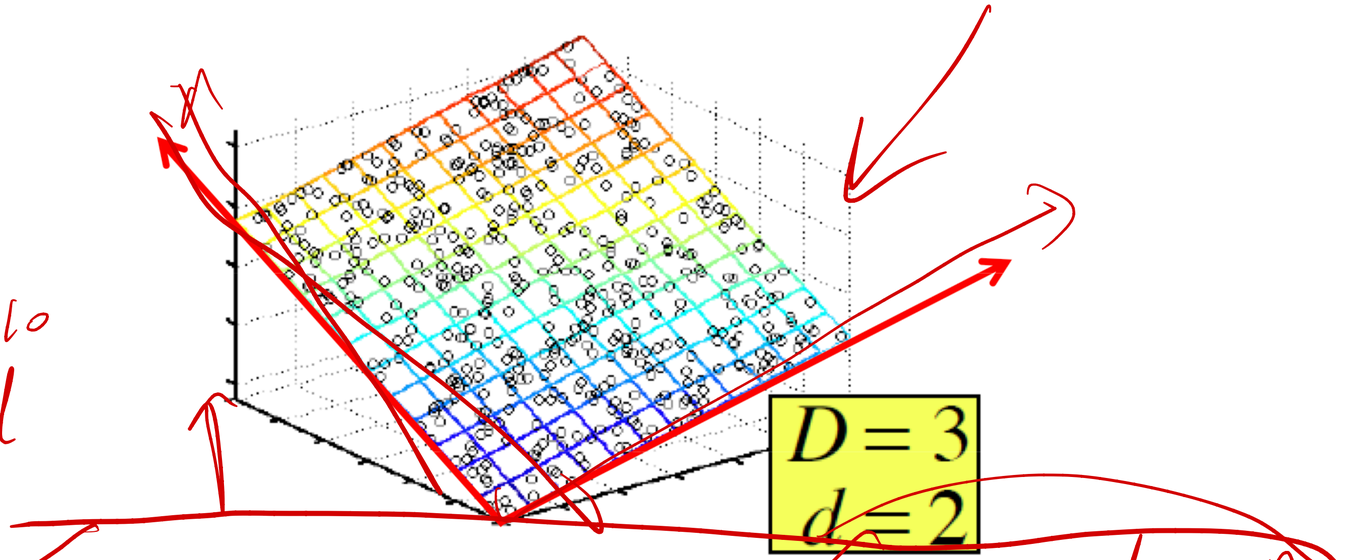


Assumption: Data lies on or near a low d-dimensional linear subspace.

Axes of this subspace are an effective representation of the data

Principal Component Analysis (PCA)





Assumption: Data lies on or near a low d-dimensional linear subspace

Axes of this subspace are an effective representation of the data

Identifying the axes is known as Principal Components Analysis, and can be obtained by Eigen or Singular value decomposition

Principal Component Analysis (PCA)

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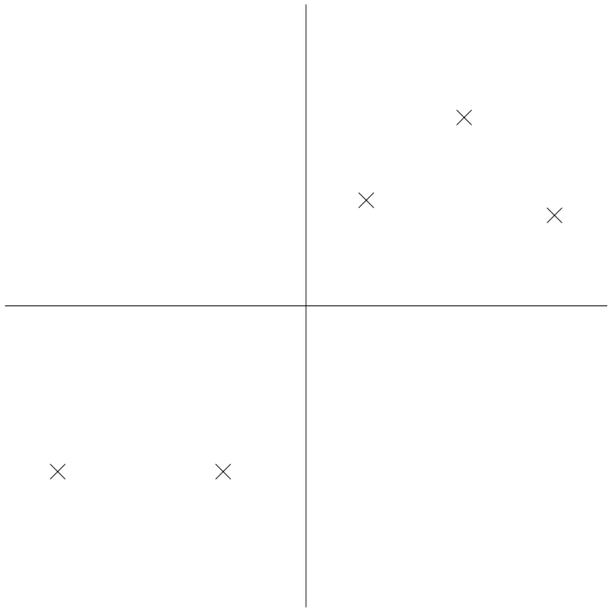

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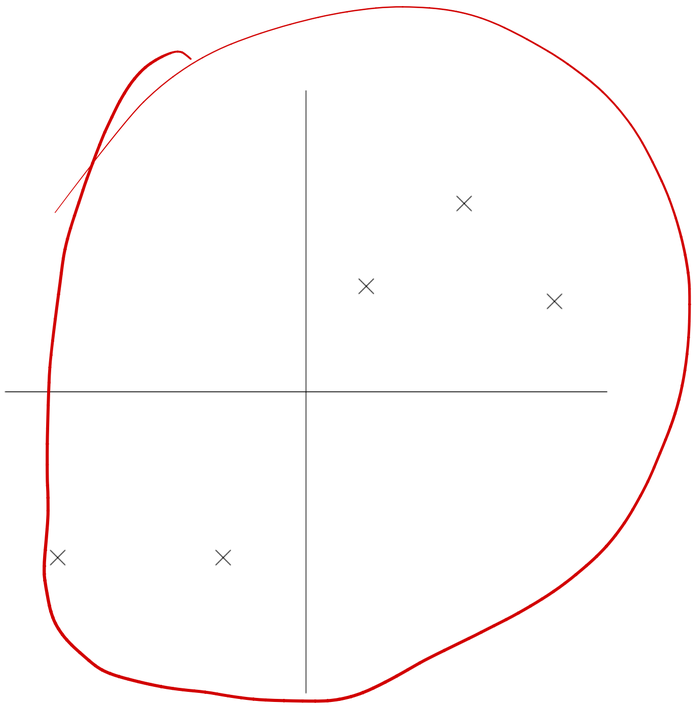
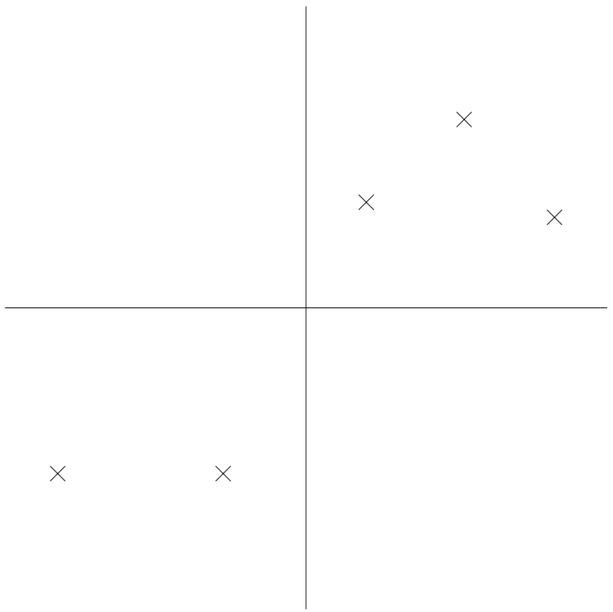
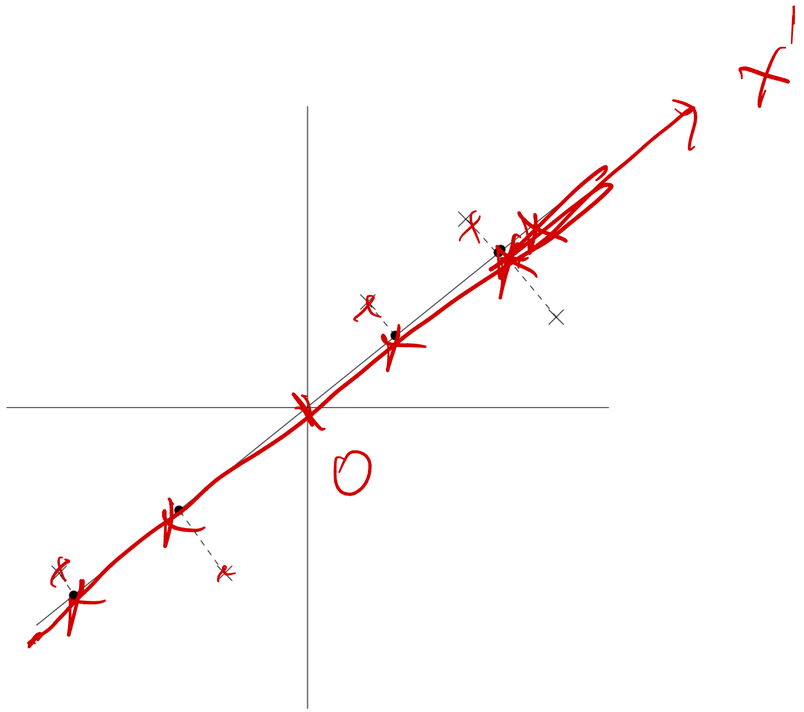


Project the data onto different directions

Principal Component Analysis (PCA)





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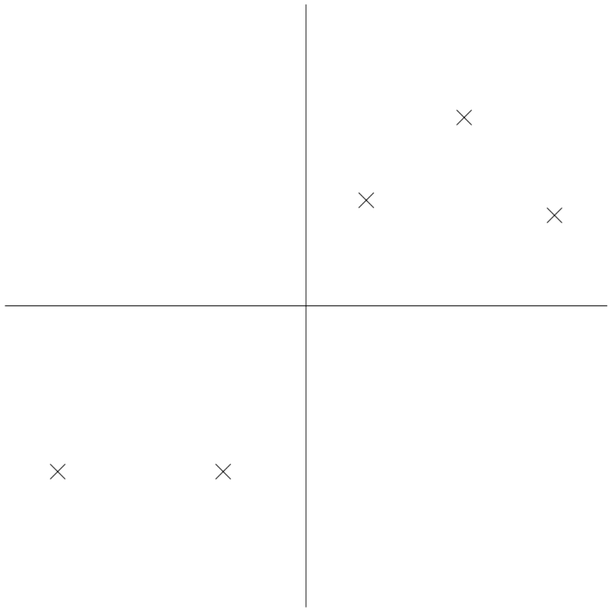

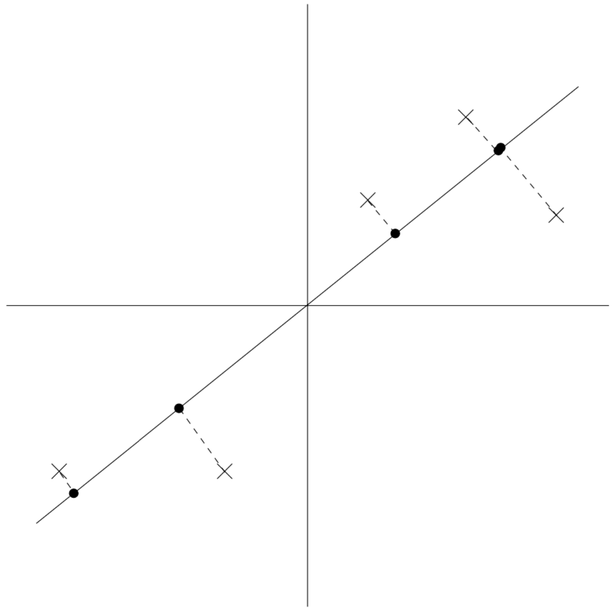

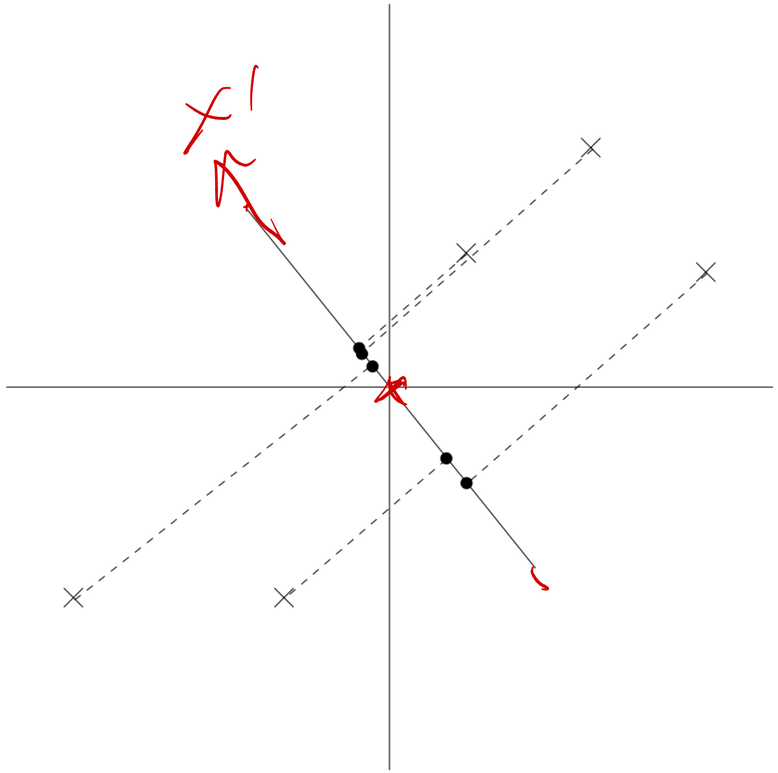

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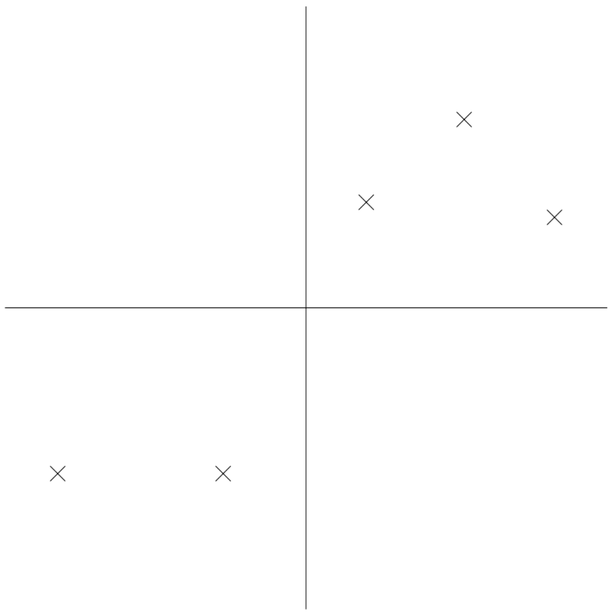
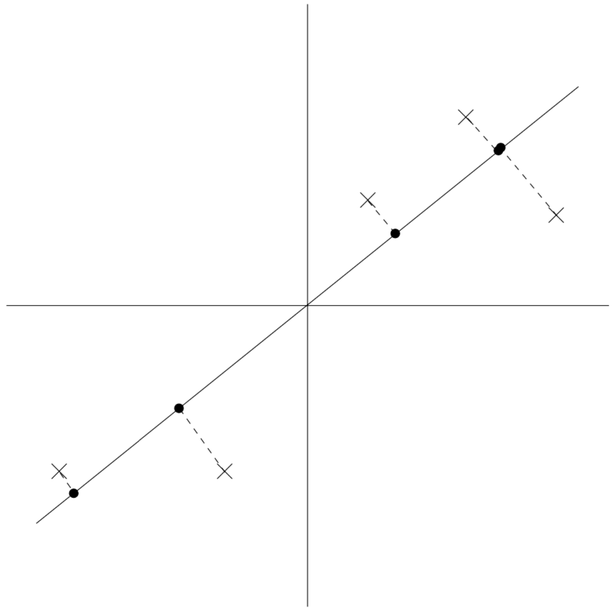
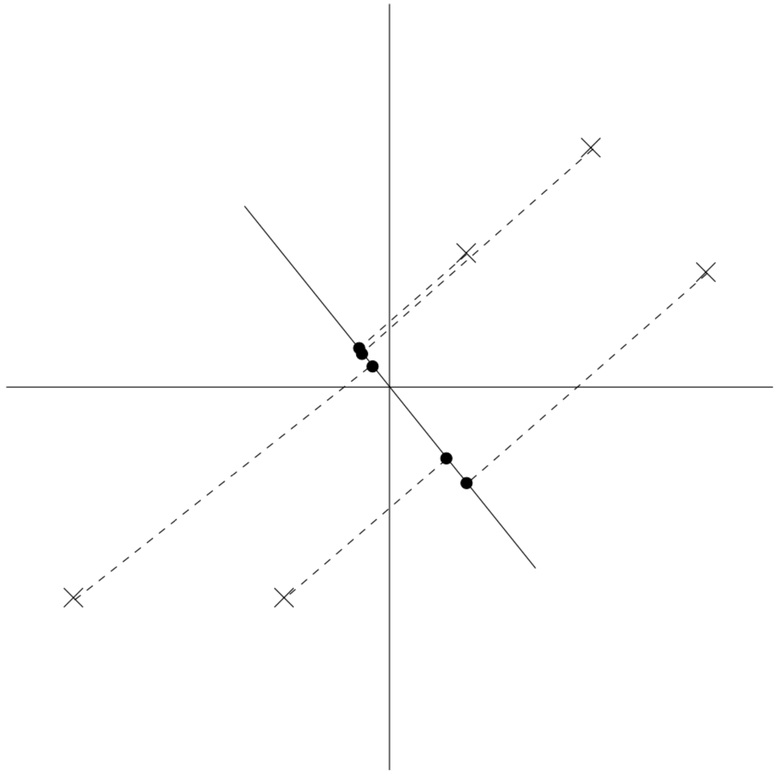


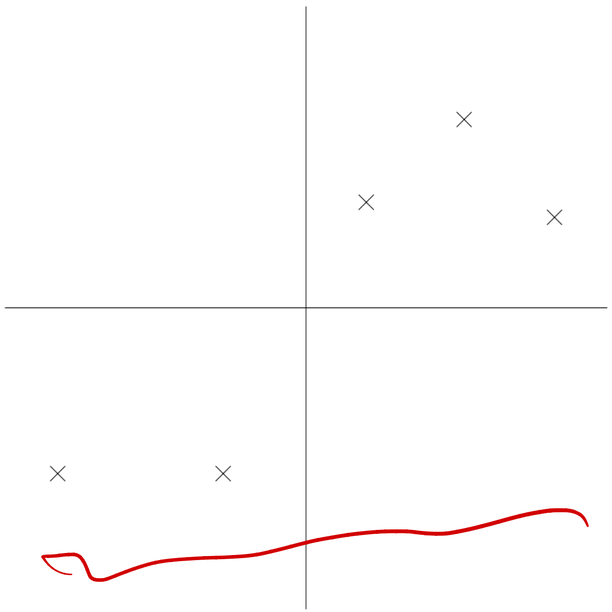


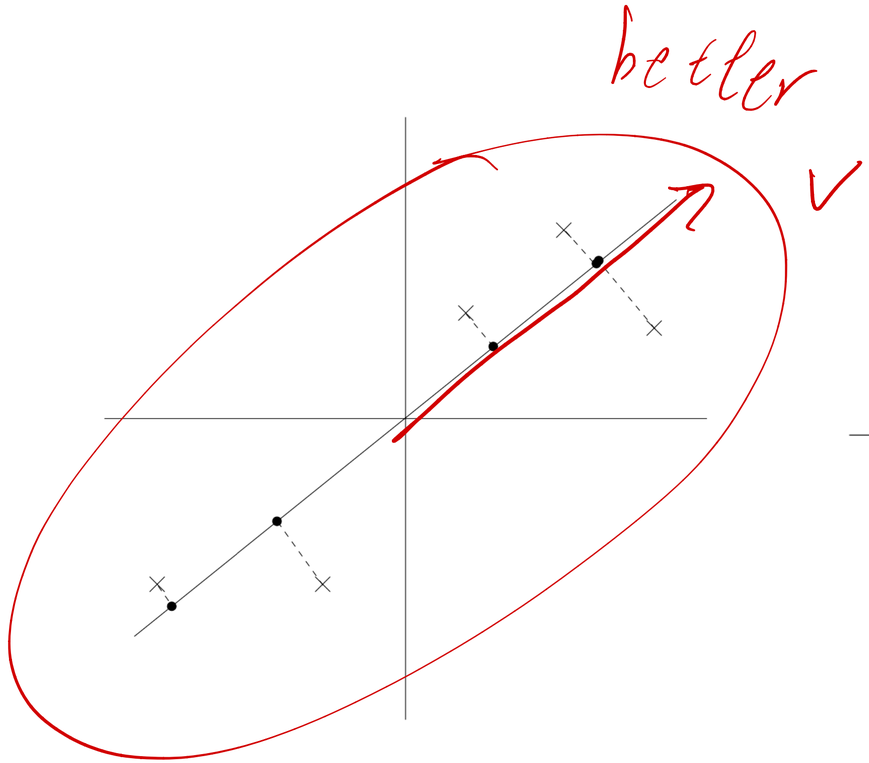


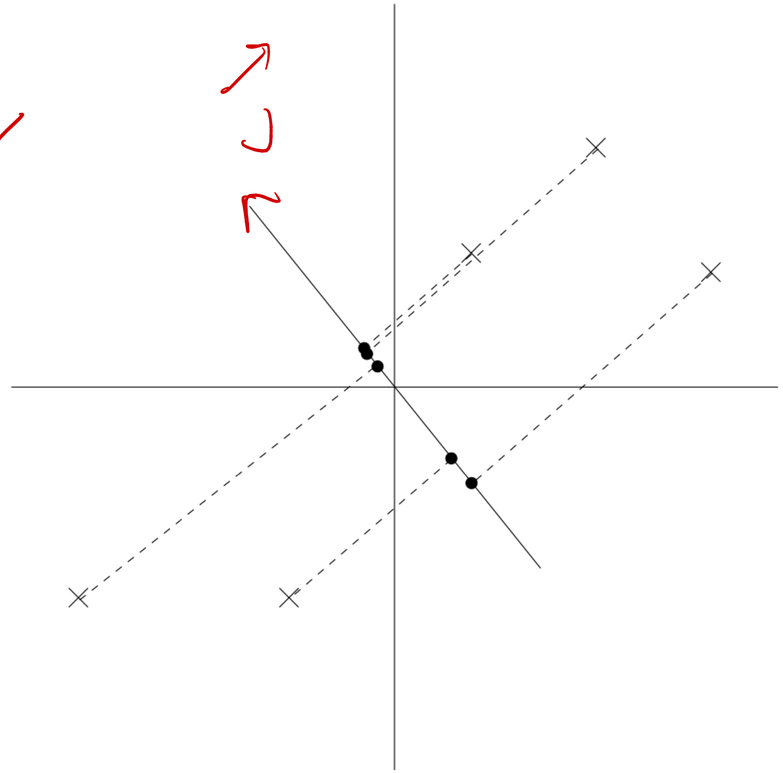
Project the data onto different directions

Which projection is better?

Principal Component Analysis (PCA)







Project the data onto different directions

Which projection is better?

We want the low-dim features that can discriminate the data the most

Normalizing Data

$$x_j^{(i)} \leftarrow \frac{x_j^{(i)} - \mu_j}{\sigma_j}$$

$$\mu_j = \frac{1}{n}\sum_{i=1}^{n} x_j^{(i)} \qquad \sigma_j^2 = \frac{1}{n}\sum_{i=1}^{n}(x_j^{(i)} - \mu_j)^2$$

Different features may have different scales

After normalization, each feature has 0 mean and variance 1

Principal Component Analysis (PCA)

Let v be the principal component

unit vector

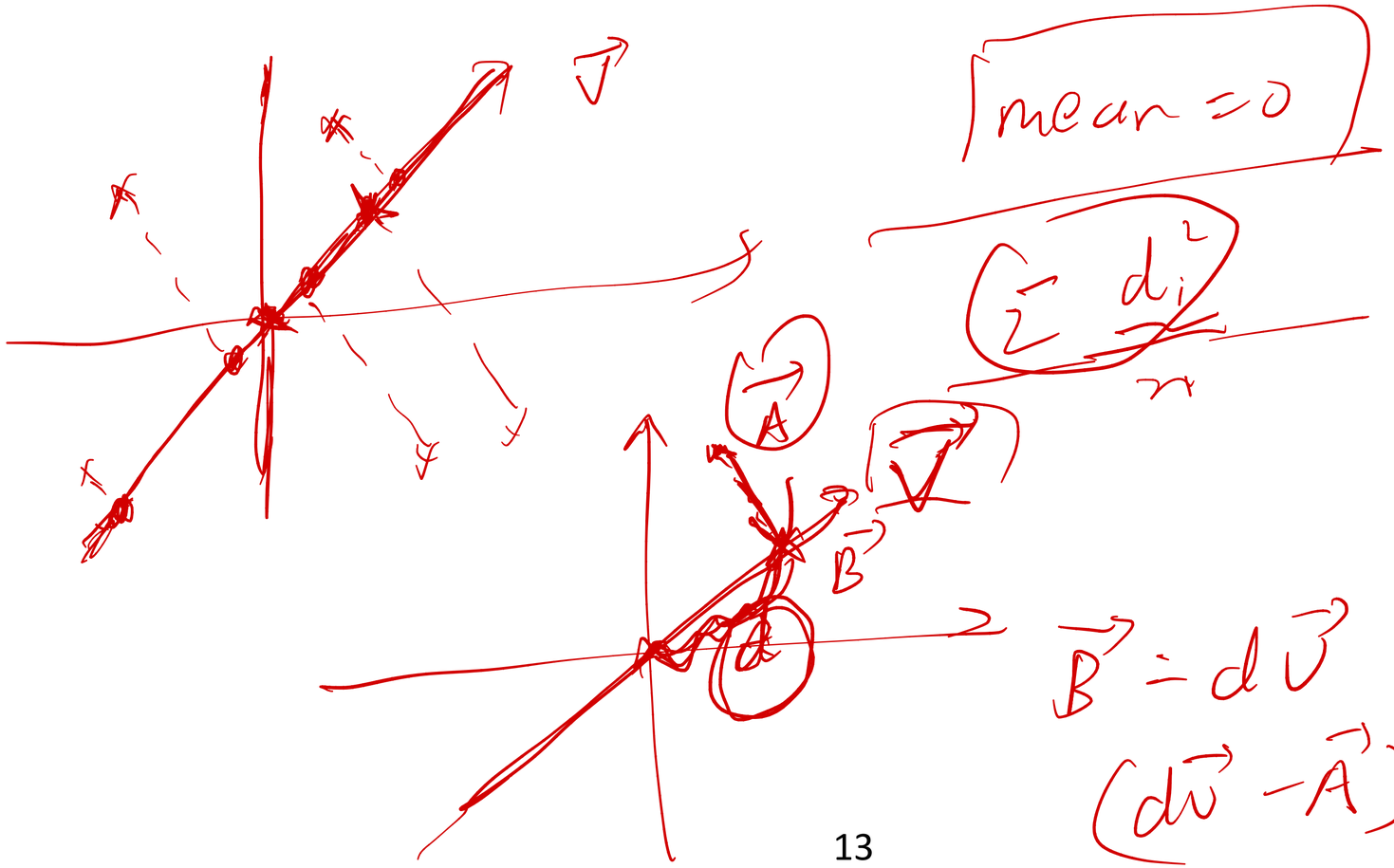
$\|v\| = 1$

x_j mean $(x_j) = 0$ Var $(x_j) = 1$

Principal Component Analysis (PCA)

Let v be the principal component

Find vector that maximizes sample variance of projection



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Find vector that maximizes sample variance of projection

$$\frac{1}{n}\sum_{i=1}^{n}(\mathrm{v}^T\mathrm{x}_i)^2 = \frac{1}{n}\mathrm{v}^T X X^T \mathrm{v}$$

max

d_i^2

V?

$[V^T x_i]$?

$X = \begin{bmatrix} x_1^T \\ x_2^T \\ x_3^T \\ \vdots \end{bmatrix}$

Principal Component Analysis (PCA)

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$$\max_{\mathbf{v}} \ \mathbf{v}^T\mathbf{X}\mathbf{X}^T\mathbf{v} \quad \text{s.t.} \quad \mathbf{v}^T\mathbf{v} = 1$$

Lagrangian: $\max_{\mathrm{v}} \mathrm{v}^T X X^T \mathrm{v} - \lambda(\mathrm{v}^T\mathrm{v} - 1)$

$$\partial/\partial\mathbf{v} = 0 \qquad (\mathbf{X}\mathbf{X}^T - \lambda\mathbf{I})\mathbf{v} = 0$$

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Lagrangian: $\max_{\mathrm{v}} \mathrm{v}^T XX^T\mathrm{v} - \lambda(\mathrm{v}^T\mathrm{v} - 1)$

$$\partial/\partial\mathbf{v} = 0 \qquad (\mathbf{X}\mathbf{X}^T - \lambda\mathbf{I})\mathbf{v} = 0 \qquad \Rightarrow \boxed{(\mathbf{X}\mathbf{X}^T)\mathbf{v} = \lambda\mathbf{v}}$$

Definition of eigenvectors

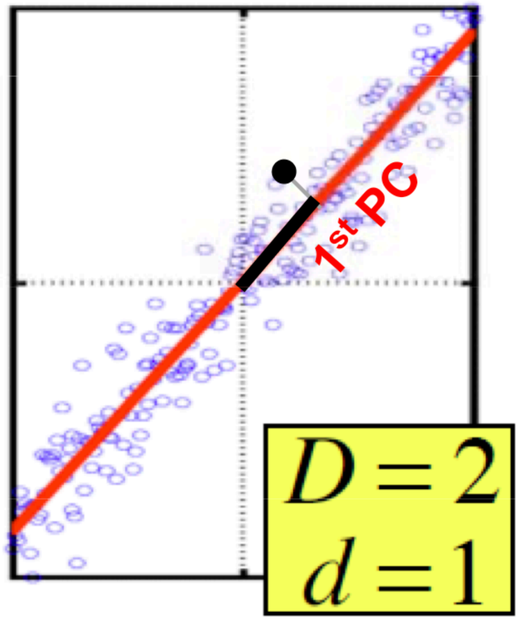
K-dimensional Cases

If we project our data into a k-dimensional subspace (k<d), we should choose $v_1, v_2, \ldots, v_k$ to be the top k eigenvectors of XX^T

why? largest λ

For symmetric matrices, eigenvectors for distinct eigenvalues are orthogonal

Principal Component Analysis (PCA)



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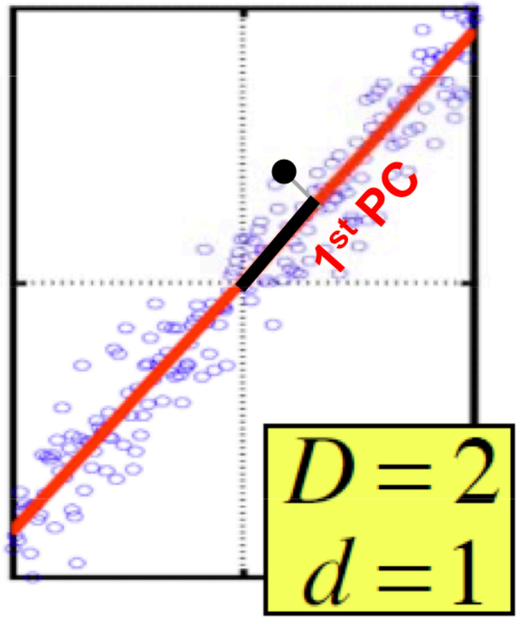
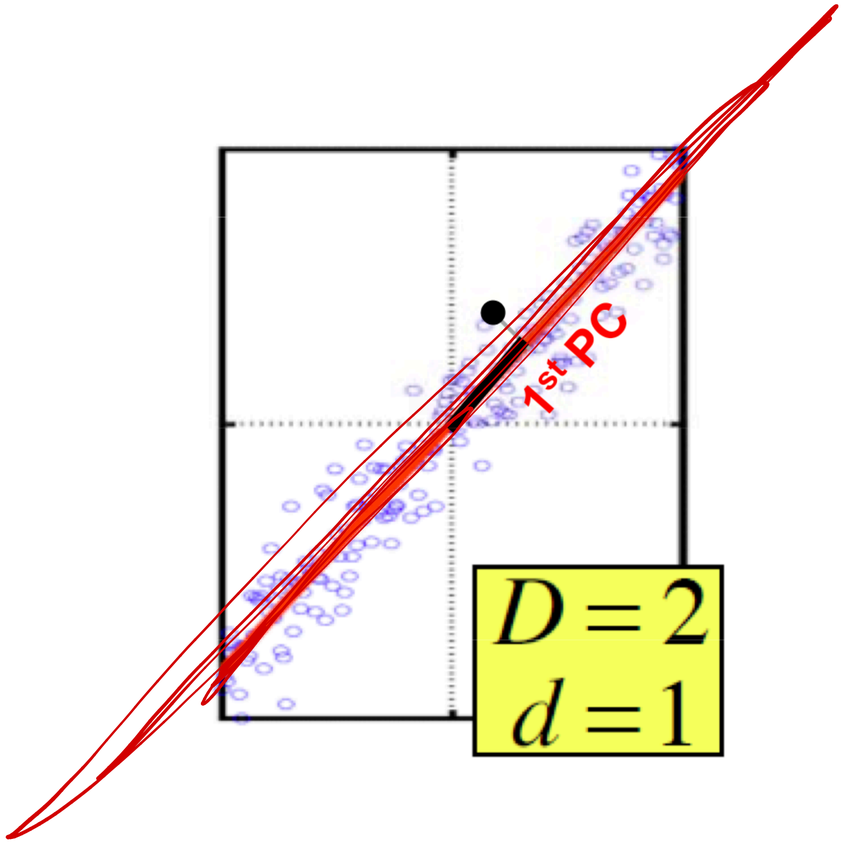


Principal Components (PC) are orthogonal directions that capture most of the variance in the data

Principal Component Analysis (PCA)



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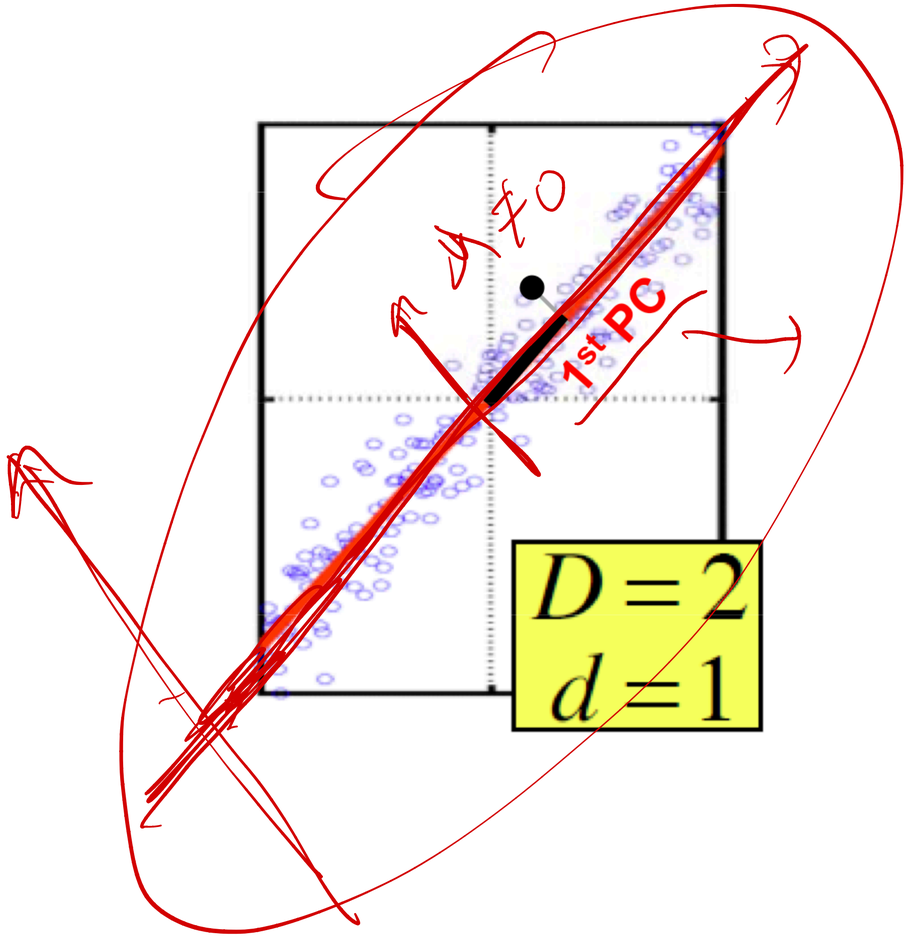
1st PC – direction of greatest variability in data

Principal Component Analysis (PCA)



Principal Components (PC) are orthogonal directions that capture most of the variance in the data

1st PC – direction of greatest variability in data

Projection of data points along 1st PC discriminate the data most along any one direction

Principal Component Analysis (PCA)

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Sample variance of projection $= \mathbf{v}^T \mathbf{X}\mathbf{X}^T \mathbf{v} = \lambda \mathbf{v}^T \mathbf{v} = \lambda$

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Thus, the eigenvalue λ denotes the amount of variability captured along that dimension.

Principal Component Analysis (PCA)

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Thus, the eigenvalue λ denotes the amount of variability captured along that dimension.

$x \in R^d$

The 1st Principal component v_1 is the eigenvector of the sample covariance matrix XX^T associated with the largest eigenvalue λ_1

The 2nd Principal component v_2 is the eigenvector of the sample covariance matrix XX^T associated with the second largest eigenvalue λ_2

And so on …

d

d^{th}

Computing the Principal Components (PCs)

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Eigenvectors are solutions of the following equation:

$$(\mathbf{X}\mathbf{X}^T)\mathbf{v} = \lambda\mathbf{v} \qquad (\mathbf{X}\mathbf{X}^T - \lambda\mathbf{I})\mathbf{v} = 0$$

Computing the Principal Components (PCs)

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Non-zero solution $\mathbf{v} \neq 0$ possible only if

$$\det(\mathbf{X}\mathbf{X}^T - \lambda\mathbf{I}) = 0$$

Computing the Principal Components (PCs)

Eigenvectors are solutions of the following equation:

$$(\mathbf{X}\mathbf{X}^T)\mathbf{v} = \lambda\mathbf{v} \qquad (\mathbf{X}\mathbf{X}^T - \lambda\mathbf{I})\mathbf{v} = 0$$

Non-zero solution v ≠ 0 possible only if

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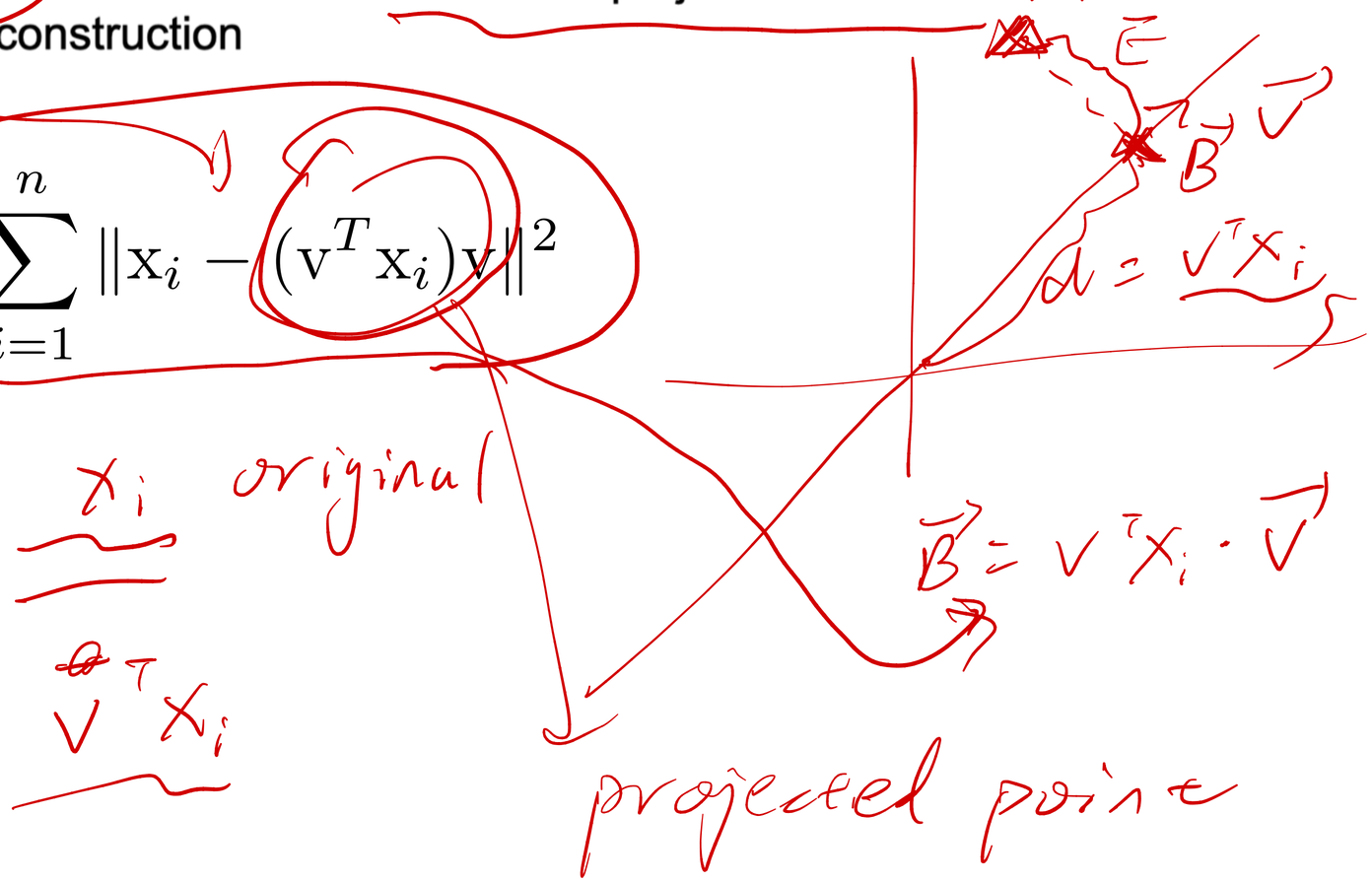
Once eigenvalues are computed, solve for eigenvectors (Principal Components) using

$$(\mathbf{X}\mathbf{X}^T - \lambda\mathbf{I})\mathbf{v} = 0$$

Another Interpretation

Minimum Reconstruction Error: PCA finds vectors v such that projection on to the vectors yields minimum MSE reconstruction

$$\frac{1}{n}\sum_{i=1}^{n}\|\mathrm{x}_i - (\mathrm{v}^T\mathrm{x}_i)\mathrm{v}\|^2$$



Dimensionality Reduction using PCA

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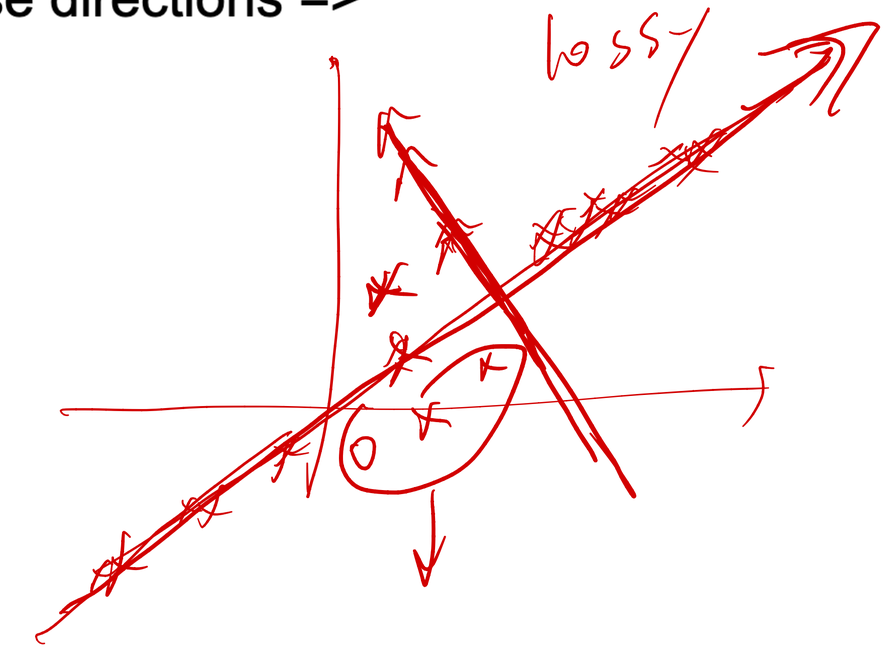
The eigenvalue λ denotes the amount of variability captured along that dimension.

Zero eigenvalues indicate no variability along those directions => data lies exactly on a linear subspace



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Only keep data projections onto principal components with non-zero eigenvalues, say $v_1, \ldots, v_d$ where $d = \text{rank}(XX^T)$

lossless dim Reduction

$D \rightarrow d = \text{rank}(XX^T)$

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Transformed representation

projections

$[v_1^T x_i, v_2^T x_i, \ldots v_d^T x_i]$

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Usually data lies near a linear subspace, as noise introduces small variability

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Can *ignore* the components of lesser significance.

small eigenvalue

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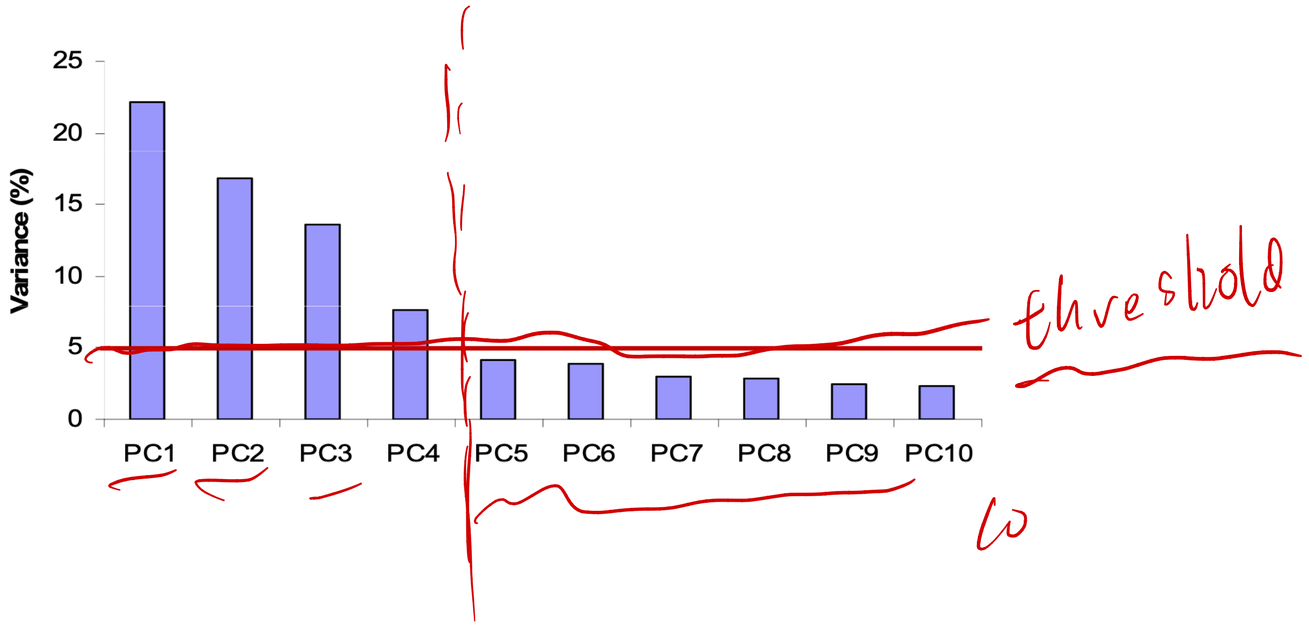

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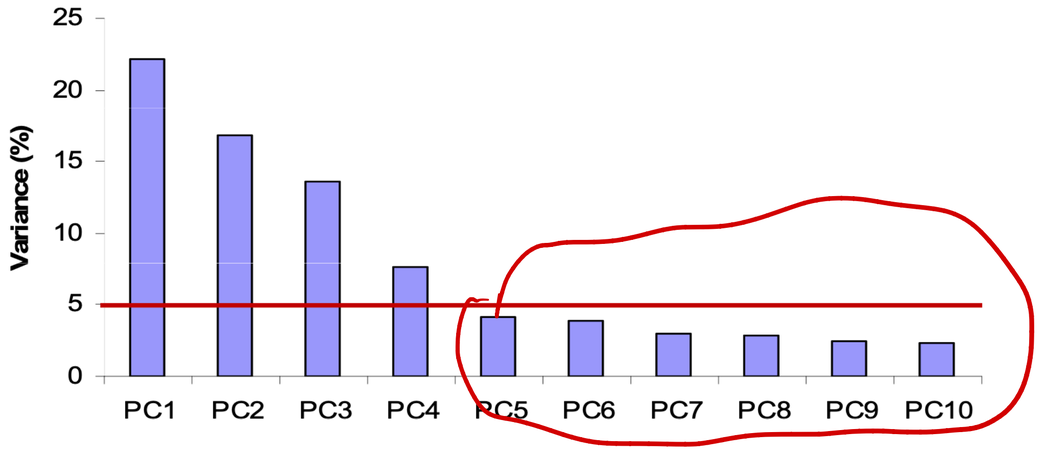

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lossy

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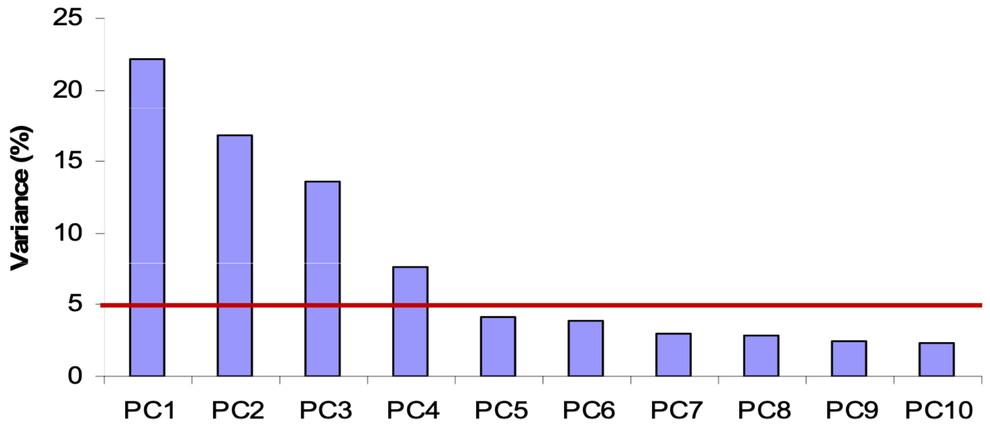

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It is not lossless compression

Example: faces

Example: faces

Eigenfaces from 7562 images:

top left image is linear combination of rest.

Sirovich & Kirby (1987)
Turk & Pentland (1991)

10000 images

A ⟶ B

Source coding theorem

[200 x 200]

10000 x [200 x 200]

Original

compression is intelligent

PCA

15 eigen images

A ⟶ B

10000 images ⟶

PCA

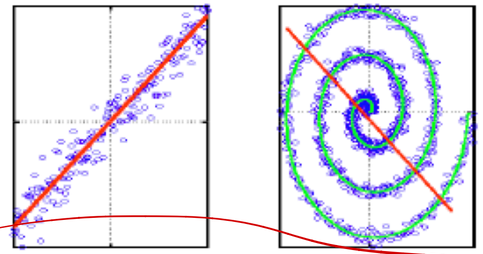
15 x [200 x 200]

+

10000 x 15

Properties of PCA

- **Strengths**
  - Eigenvector method
  - No tuning parameters
  - Non-iterative
  - No local optima
- **Weaknesses**
  - Limited to second order statistics
  - Limited to linear projections



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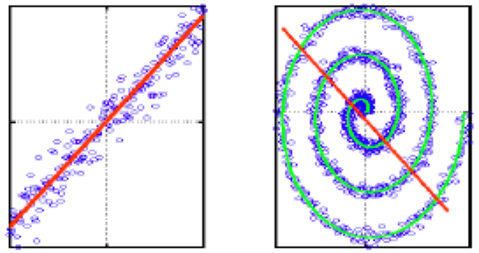
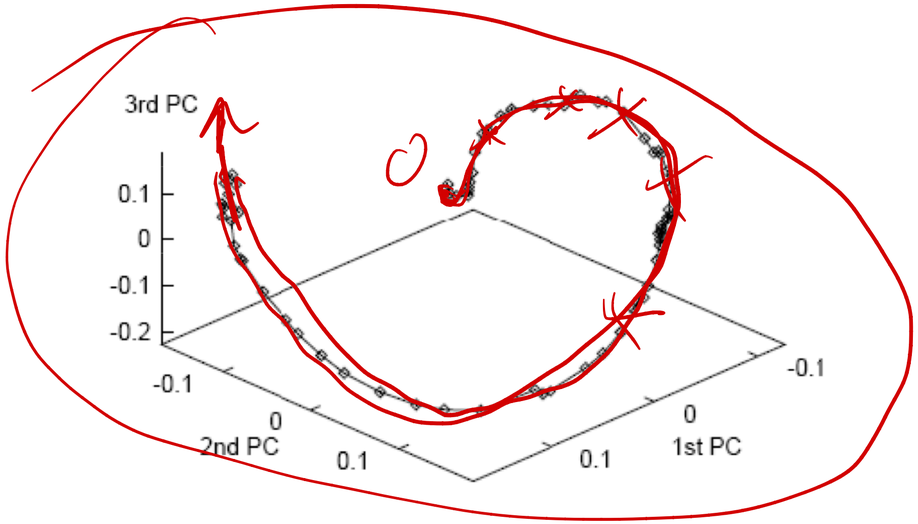


Nonlinear example

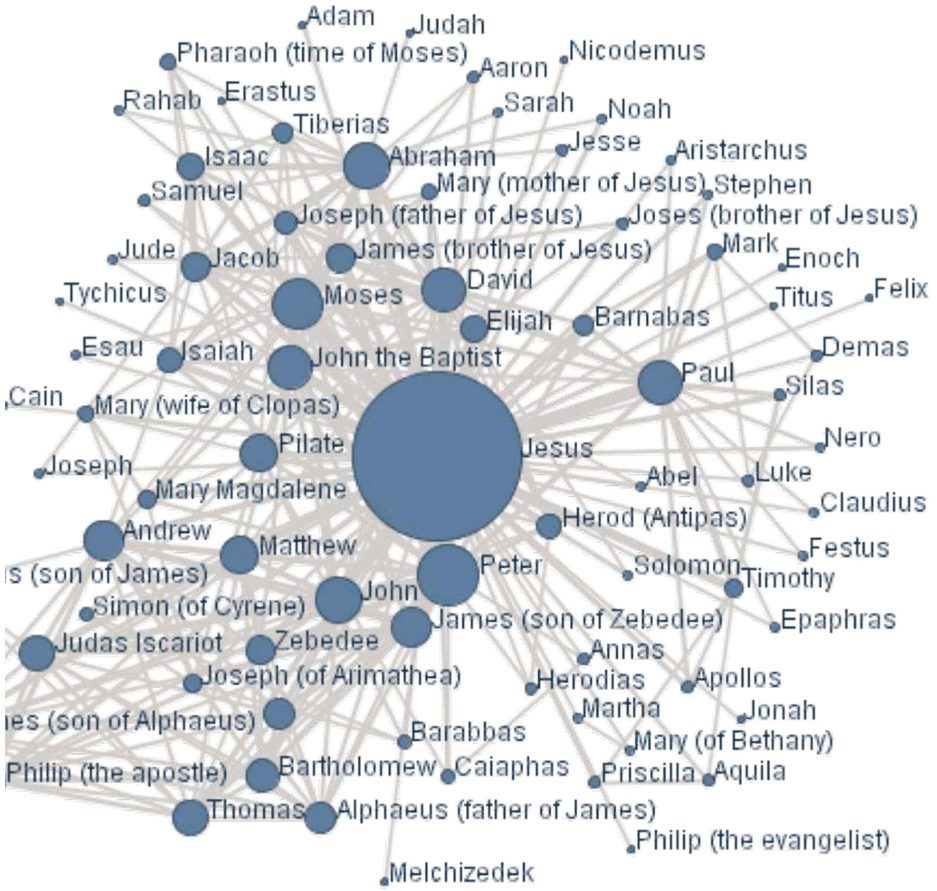


香港科技大學
THE HONG KONG
UNIVERSITY OF SCIENCE
AND TECHNOLOGY

COMP 5212
Machine Learning
Lecture 14

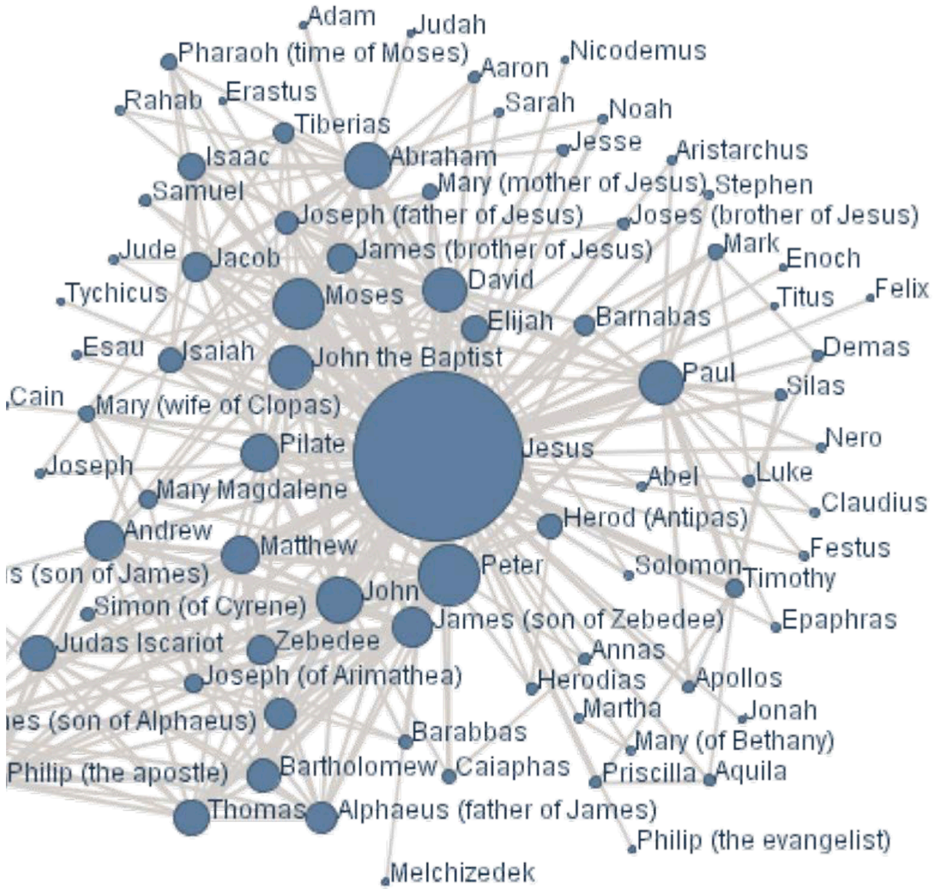
Probabilistic Graphical Models

What Are Graphical Models?



What Are Graphical Models?

- Informally, a GM is just a graph representing relationship among random variables
  - Nodes: random variables (features, not examples)
  - Edges (or absence of edges): relationship



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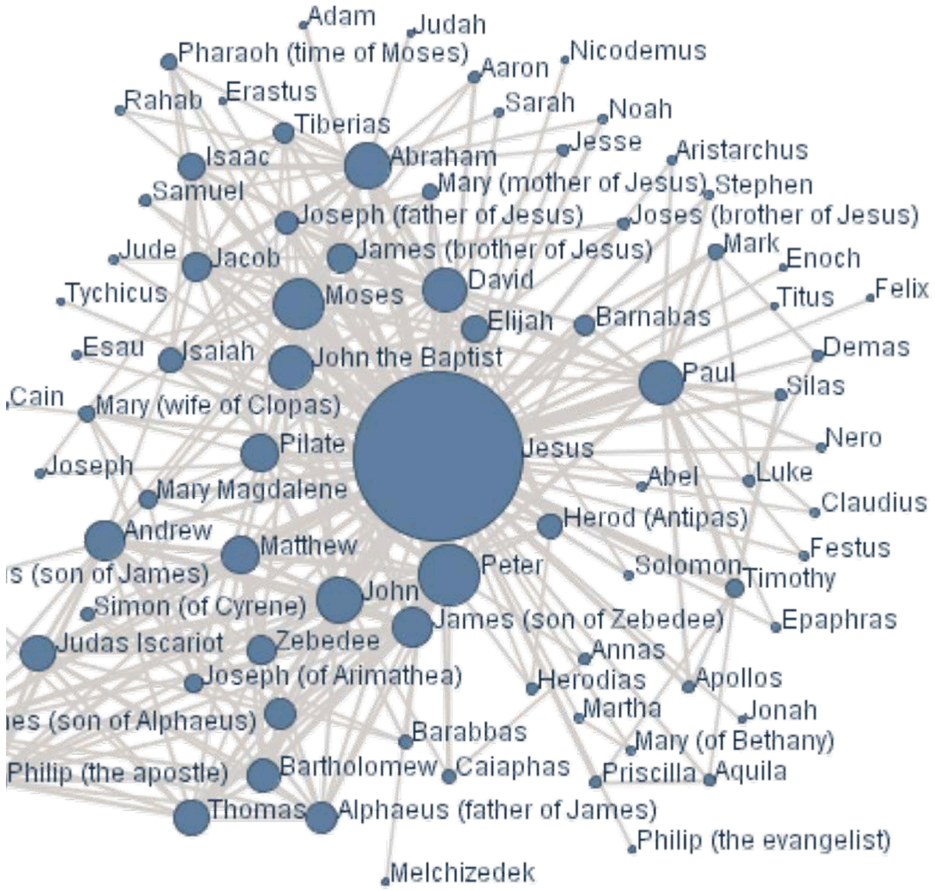
- Looks simple!
  - But detail matters, as always.
  - What exactly do we mean by relationship?



Relationship between two random variables

- Many types of relationships exist:
  - X and Y are correlated
  - X and Y are dependent
  - X and Y are independent
  - X and Y are partially correlated given Z
  - X and Y are conditionally dependent given Z
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  - X causes Y
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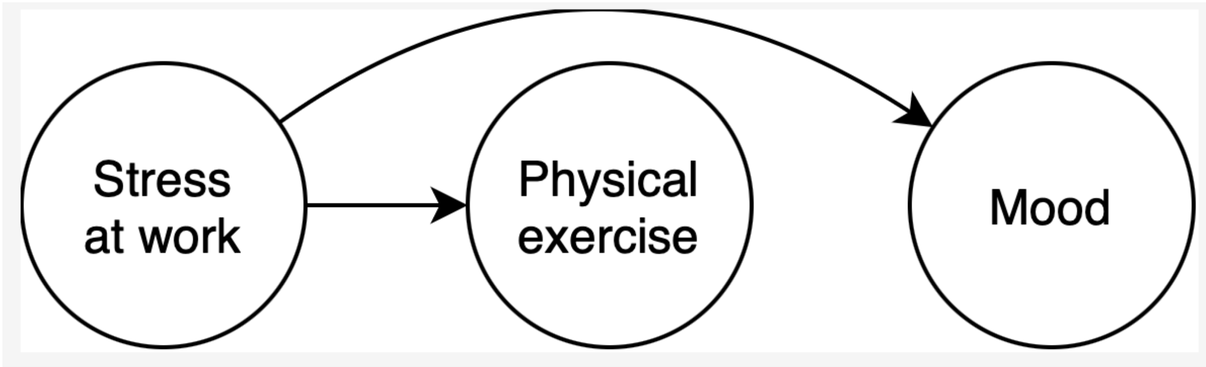

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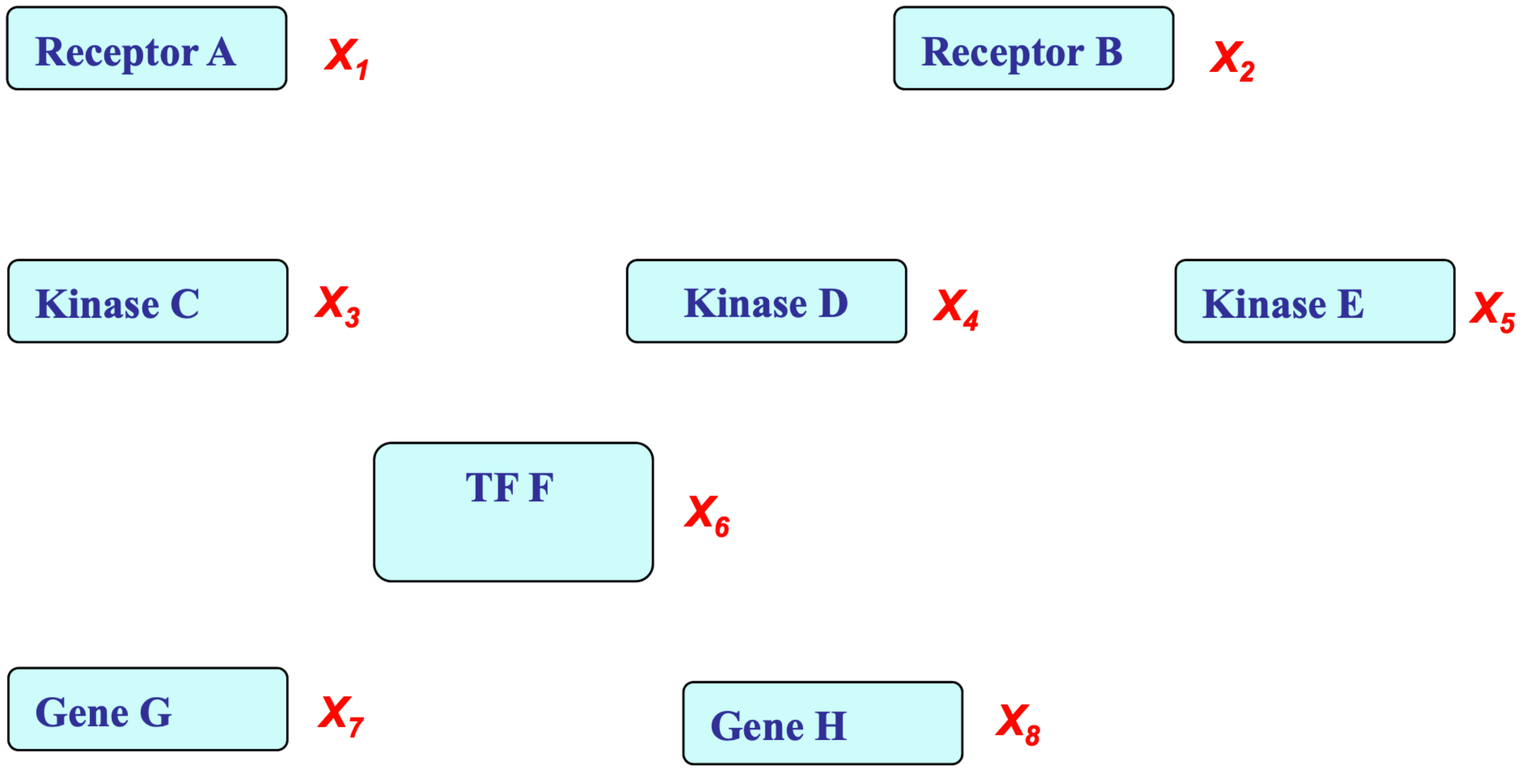
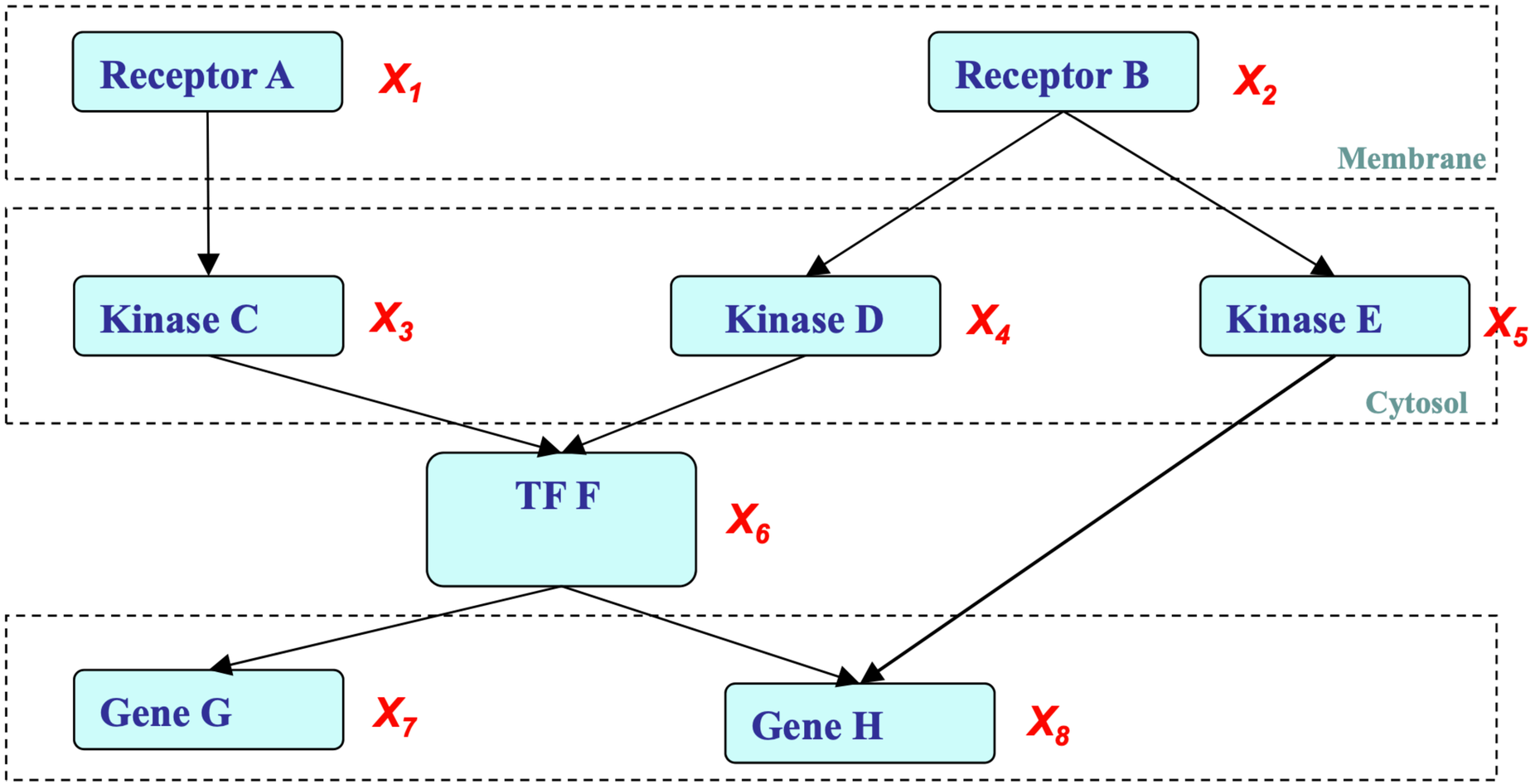


Structure Simplifies Representation

Dependencies among variables



Probabilistic Graphical Models

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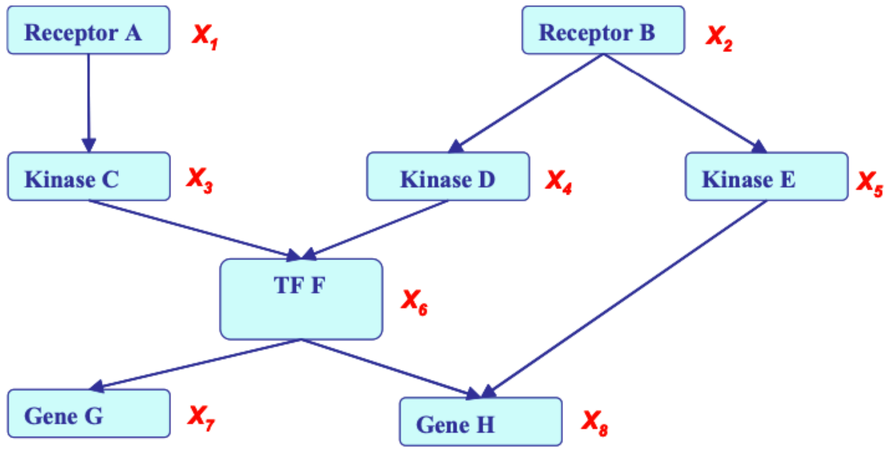
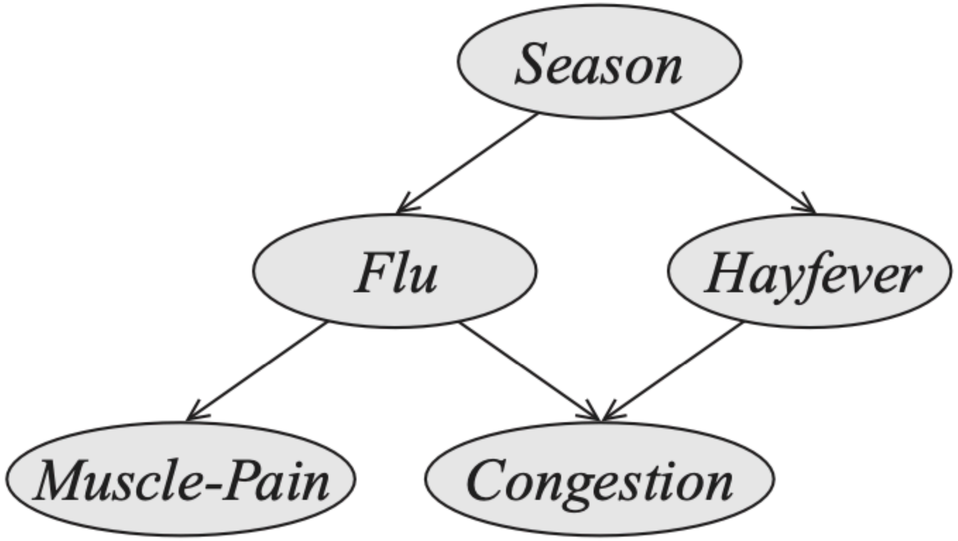


$$P(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8)$$

$$= P(X_1)\, P(X_2)\, P(X_3|\, X_1)\, P(X_4|\, X_2)\, P(X_5|\, X_2)$$
$$P(X_6|\, X_3, X_4)\, P(X_7|\, X_6)\, P(X_8|\, X_5, X_6)$$

Stay tune for what are these independencies!

Another Example



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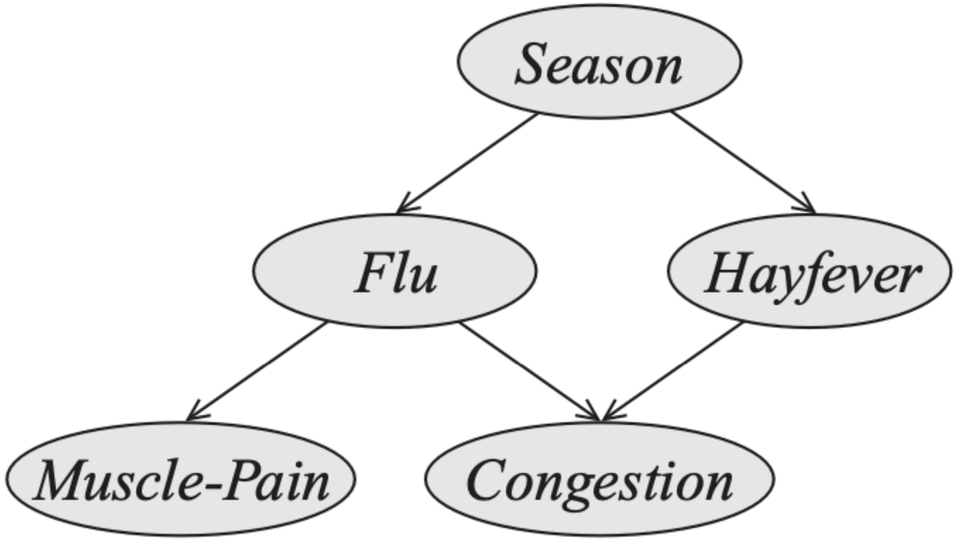

$$P(Congestion \mid Flu, Hayfever, Season) = P(Congestion \mid Flu, Hayfever);$$

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It is a smart way to write/specify/compose/design exponentially-large probability distributions without paying an exponential cost, and at the same time endow the distributions with ***structured semantics***

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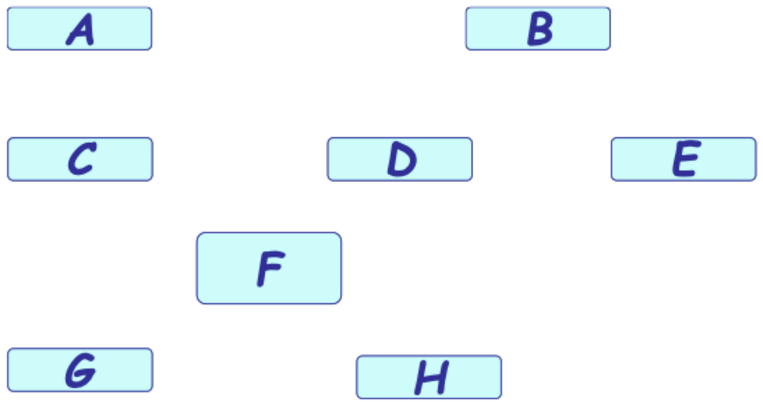

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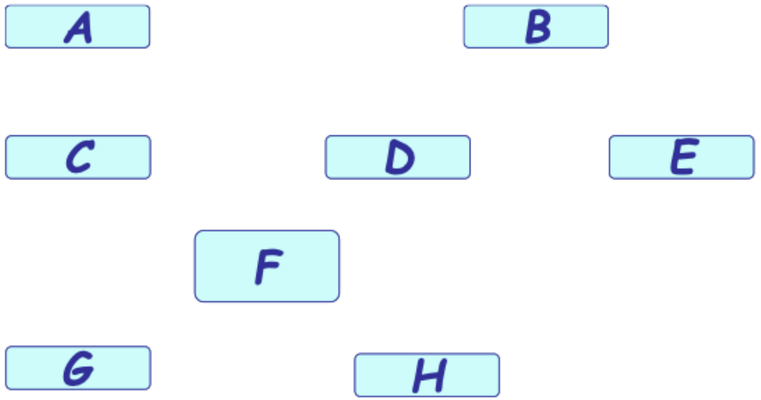

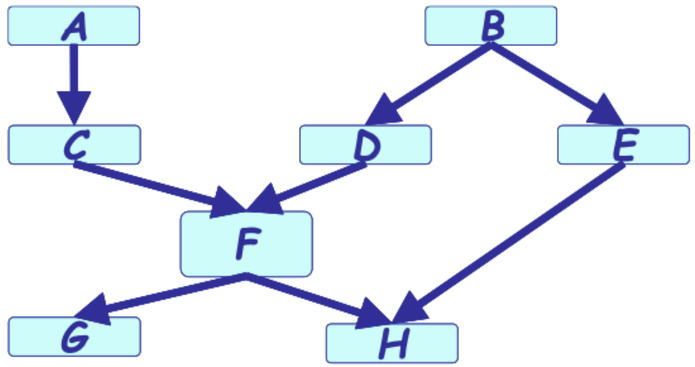

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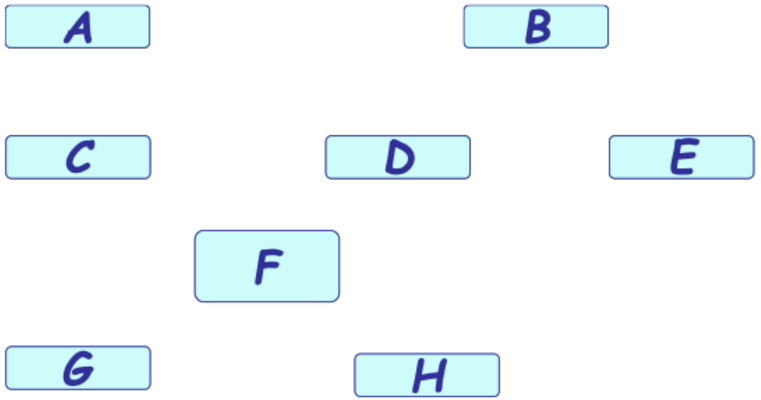

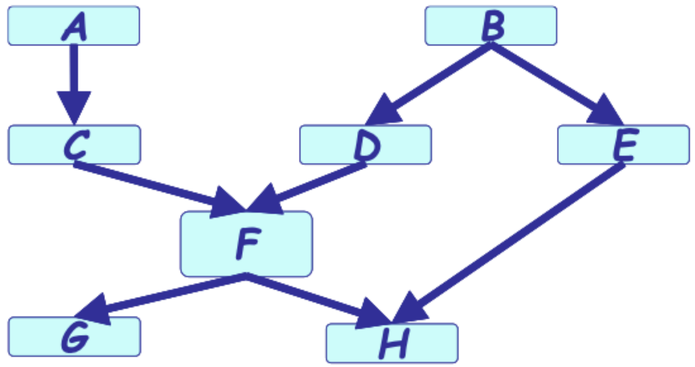

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More formal definition:

It refers to a family of distributions on a set of random variables that are compatible with all the probabilistic independence propositions encoded by a graph that connects these variables

Probabilistic Graphical Model is a graphical language to express conditional independence

Thank You!
Q & A