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Thermal Spreading Resistance of Eccentric Heat Sources on Rectangular Flux Channels

A general solution, based on the separation of variables method for thermal spreading resistances of eccentric heat sources on a rectangular flux channel is presented. Solutions are obtained for both isotropic and compound flux channels. The general solution can also be used to model any number of discrete heat sources on a compound or isotropic flux channel using superposition. Several special cases involving single and multiple heat sources are presented. [DOI: 10.1115/1.1568125]

Keywords: *Conduction, Electronics Cooling, Heat Spreaders, Heat Sinks, Spreading Resistance*

Introduction

Thermal spreading resistance results from discrete heat sources in many engineering systems. Typical applications include cooling of electronic devices both at the package and system level, and cooling of power semi-conductors using heat sinks. In these applications, heat dissipated by electronic devices is conducted through electronic packages into printed circuit boards or heat sink baseplates which are convectively cooled. In heat sink applications the convective film resistance is replaced by an effective extended surface film coefficient.

Analytical results for an eccentric heat source on a finite rectangular flux channel are obtained for isotropic and compound systems. These solutions may be used to model single or multi-source systems by means of superposition. The present approach differs from other methods [1–3], in that the heat source specification is incorporated in the definition of the thermal boundary conditions rather than in the governing partial differential equation. This results in analytical expressions which can be easily manipulated in most advanced mathematical software packages [4–7].

The general solution for the spreading resistance of a single constant flux eccentric heat source with convective or conductive cooling at one boundary will be presented, see Fig. 1. A review of the literature reveals that this particular configuration has not yet been analyzed [8]. The two-dimensional eccentric strip heat source for a semi-infinite flux channel was obtained by Veziroglu and Chandra [9]. While the solution for a centrally located heat source on an isotropic rectangular plate was obtained by Krane [10]. Recently, Yovanovich et al. [11] obtained a solution for a centrally located heat source on a compound rectangular flux channel.

The general solution will depend on several dimensionless geometric and thermal parameters. In general, the total resistance is given by

$$R_T = R_{1D} + R_s \tag{1}$$

where R_s is the spreading resistance and R_{1D} is the one dimensional resistance of the system given by

$$R_{1D} = \frac{t_1}{k_1 A_b} + \frac{1}{h A_b} \tag{2}$$

where $A_b = ab$, and h is a heat transfer coefficient which may be a contact conductance or effective fin conductance.

The total thermal resistance is defined as

$$R_T = \frac{\overline{T_s} - T_f}{Q} \tag{3}$$

where $\overline{T_s}$ is the average source temperature given by

$$\overline{T_s} = \frac{1}{A_s} \int \int_{A_s} T(x,y,0)\, dA_s \tag{4}$$

The thermal resistance for the configuration shown in Fig. 1 is a function of

$$R = f(a,b,c,d,X_c,Y_c,t_1,h,k_1) \tag{5}$$

Problem Statement

The governing equation for the system shown in Fig. 1 is Laplace's equation

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{6}$$

which is subjected to a uniform flux distribution

$$\left.\frac{\partial T}{\partial z}\right|_{z=0} = -\frac{(Q/A_s)}{k_1} \tag{7}$$

within the heat source area, $A_s = cd$, and

$$\left.\frac{\partial T}{\partial z}\right|_{z=0} = 0 \tag{8}$$

outside the heat source area, and a convective or mixed boundary condition on the bottom surface

$$\left.\frac{\partial T}{\partial z}\right|_{z=t_1} = -\frac{h}{k_1}[T(x,y,t_1) - T_f] \tag{9}$$

In extended surface applications such as heat sinks, the value of h is replaced by an effective value which accounts for both the heat transfer coefficient on the fin surface and the increased surface area

$$h_{\text{eff}} = \frac{1}{A_b R_{\text{fins}}} \tag{10}$$

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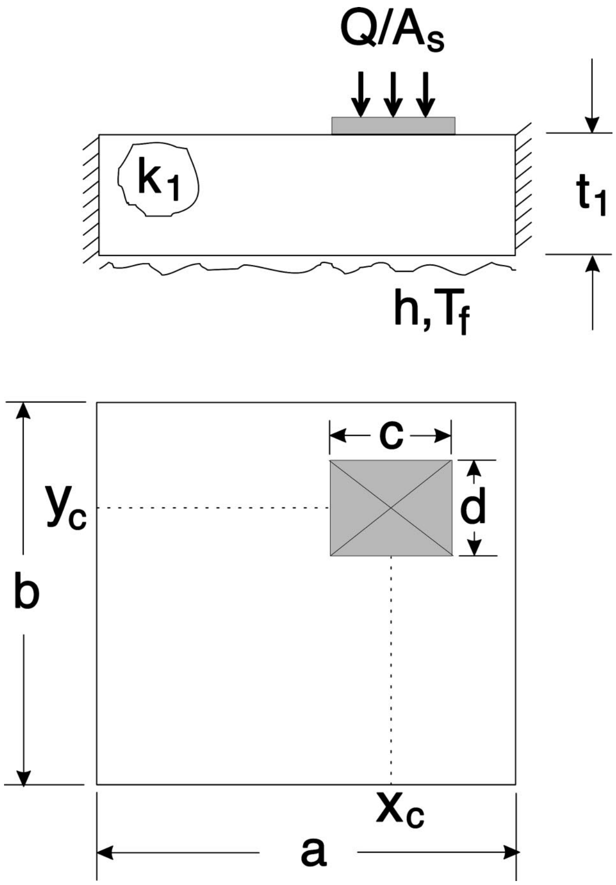

Fig. 1 Isotropic plate with eccentric heat source

Along the edges of the plate, the following conditions are also required:

$$\left.\frac{\partial T}{\partial x}\right|_{x=0,a}=0 \tag{11}$$

and

$$\left.\frac{\partial T}{\partial y}\right|_{y=0,b}=0 \tag{12}$$

The general solution for the total thermal resistance and temperature distribution will be obtained for the system shown in Fig. 1. In a later section, the solution will be extended to compound systems as shown in Fig. 2.

In a compound system Laplace's equation must be solved in each layer. In addition to the boundary conditions specified for the isotropic system, the following conditions with perfect contact at the interface are required:

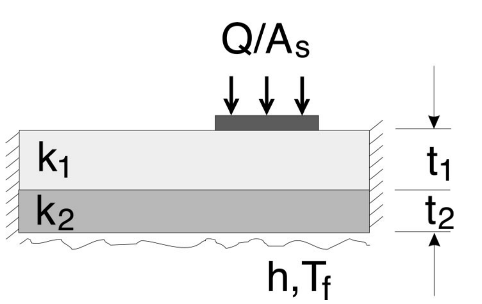

Fig. 2 Compound plate with eccentric heat source

$$T_1(x,y,t_1)=T_2(x,y,t_1) \tag{13}$$

and

$$k_1\left.\frac{\partial T_1}{\partial z}\right|_{z=t_1}=k_2\left.\frac{\partial T_2}{\partial z}\right|_{z=t_1} \tag{14}$$

while along the bottom surface $z=t_1+t_2$, the boundary condition to be satisfied becomes

$$\left.\frac{\partial T_2}{\partial z}\right|_{z=t_1+t_2}=-\frac{h}{k_2}[T_2(x,y,t_1+t_2)-T_f] \tag{15}$$

The full solution is obtained for the isotropic case, however, it may easily be applied to the compound system with only slight modification, using the results of Yovanovich et al. [11].

General Solution

The solution for the isotropic plate may be obtained by means of separation of variables [12–15]. The solution is assumed to have the form $\theta(x,y,z)=X(x)*Y(y)*Z(z)$, where $\theta(x,y,z)=T(x,y,z)-T_f$. Applying the method of separation of variables yields the following general solution for the temperature excess in the plate which satisfy the thermal boundary conditions along ($x=0$, $x=a$) and ($y=0$, $y=b$)

$$\begin{aligned}\theta(x,y,z)=A_0+B_0z&+\sum_{m=1}^{\infty}\cos(\lambda x)[A_1\cosh(\lambda z)+B_1\sinh(\lambda z)]\\&+\sum_{n=1}^{\infty}\cos(\delta y)[A_2\cosh(\delta z)+B_2\sinh(\delta z)]\\&+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\cos(\lambda x)\cos(\delta y)[A_3\cosh(\beta z)\\&+B_3\sinh(\beta z)]\end{aligned} \tag{16}$$

where $\lambda=m\pi/a$, $\delta=n\pi/b$, and $\beta=\sqrt{\lambda^2+\delta^2}$.

The solution contains four components, a uniform flow solution and three spreading (or constriction) solutions which vanish when the heat flux is distributed uniformly over the entire surface $z=0$. Since the solution is a linear superposition of each component, they may be dealt with separately. Application of the boundary conditions in the z direction will yield solutions for the unknown constants.

Application of the thermal boundary condition at $z=t_1$ for an isotropic rectangular plate yields the following result for the Fourier coefficients:

$$B_i=-\phi(\zeta)A_i \quad i=1,2,3 \tag{17}$$

where

$$\phi(\zeta)=\frac{\zeta\sinh(\zeta t_1)+h/k_1\cosh(\zeta t_1)}{\zeta\cosh(\zeta t_1)+h/k_1\sinh(\zeta t_1)} \tag{18}$$

and ζ is replaced by λ, δ, or β, accordingly.

The final Fourier coefficients A_m, A_n, and A_{mn} are obtained by taking Fourier series expansions of the boundary condition at the surface $z=0$. This results in

$$A_1=\frac{Q}{bck_1\lambda_m\phi(\lambda_m)}\frac{\int_{X_c-c/2}^{X_c+c/2}\cos(\lambda_m x)dx}{\int_0^a\cos^2(\lambda_m x)dx} \tag{19}$$

or

$$A_m=\frac{2Q\left[\sin\left(\frac{(2X_c+c)}{2}\lambda_m\right)-\sin\left(\frac{(2X_c-c)}{2}\lambda_m\right)\right]}{abck_1\lambda_m^2\phi(\lambda_m)} \tag{20}$$

and

$$A_2=\frac{Q}{adk_1\delta_n\phi(\delta_n)}\frac{\int_{Y_c-d/2}^{Y_c+d/2}\cos(\delta_n y)dy}{\int_0^b\cos^2(\delta_n y)dy} \tag{21}$$

or

$$A_n=\frac{2Q\left[\sin\left(\frac{(2Y_c+d)}{2}\delta_n\right)-\sin\left(\frac{(2Y_c-d)}{2}\delta_n\right)\right]}{abdk_1\delta_n^2\phi(\delta_n)} \tag{22}$$

and

$$A_3=\frac{Q}{cdk_1\beta_{m,n}\phi(\beta_{m,n})}\cdot\frac{\int_{Y_c-d/2}^{Y_c+d/2}\int_{X_c-c/2}^{X_c+c/2}\cos(\lambda_m x)\cos(\delta_n y)dxdy}{\int_0^b\int_0^a\cos^2(\lambda_m x)\cos^2(\delta_n y)dxdy} \tag{23}$$

or

$$A_{mn}=\frac{16Q\cos(\lambda_m X_c)\sin\left(\frac{1}{2}\lambda_m c\right)\cos(\delta_n Y_c)\sin\left(\frac{1}{2}\delta_n d\right)}{abcdk_1\beta_{m,n}\lambda_m\delta_n\phi(\beta_{m,n})} \tag{24}$$

Finally, values for the coefficients in the uniform flow solution are given by

$$A_0=\frac{Q}{ab}\left(\frac{t_1}{k_1}+\frac{1}{h}\right) \tag{25}$$

and

$$B_0=-\frac{Q}{k_1ab} \tag{26}$$

Mean Temperature Excess. The general solution for the mean temperature excess of a single heat source may be obtained by integrating Eq. (16) over the heat source area. Carrying out the necessary integrations leads to the following expression for the mean source temperature:

$$\begin{aligned}\overline{\theta}=\overline{\theta}_{1D}&+2\sum_{m=1}^{\infty}A_m\frac{\cos(\lambda_m X_c)\sin\left(\frac{1}{2}\lambda_m c\right)}{\lambda_m c}\\&+2\sum_{n=1}^{\infty}A_n\frac{\cos(\delta_n Y_c)\sin\left(\frac{1}{2}\delta_n d\right)}{\delta_n d}\\&+4\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\frac{\cos(\delta_n Y_c)\sin\left(\frac{1}{2}\delta_n d\right)\cos(\lambda_m X_c)\sin\left(\frac{1}{2}\lambda_m c\right)}{\lambda_m c\,\delta_n d}\end{aligned} \tag{27}$$

where

$$\overline{\theta}_{1D}=\frac{Q}{ab}\left(\frac{t_1}{k_1}+\frac{1}{h}\right) \tag{28}$$

for an isotropic system.

Thermal Spreading Resistance. The thermal spreading resistance may now be computed using the mean temperature excess. In general, the total resistance as defined by Eq. (3), results in

$$R_T=\frac{\overline{\theta}}{Q}=R_{1D}+R_s \tag{29}$$

where R_{1D} is the one-dimensional thermal resistance and R_s is the thermal spreading resistance. The thermal spreading resistance component is defined by the three series solution terms in Eq. (27).

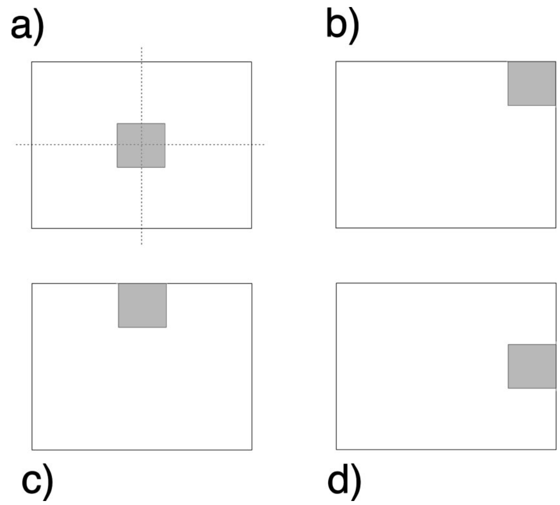

Fig. 3 Special cases of a plate with eccentric heat source

Compound Systems. In many applications an interface material may be added to reduce thermal contact conductance and/or promote thermal spreading. The solution obtained for the isotropic rectangular flux channel may be used for a compound flux channel with only minor modifications. In Yovanovich et al. [11] the authors obtained a solution for a compound rectangular flux channel. It is not difficult to show that ϕ in Eqs. (20), (22), (24) can be replaced by

$$\phi(\zeta)=\frac{(\alpha e^{4\zeta t_1}-e^{2\zeta t_1})+\varrho(e^{2\zeta(2t_1+t_2)}-\alpha e^{2\zeta(t_1+t_2)})}{(\alpha e^{4\zeta t_1}+e^{2\zeta t_1})+\varrho(e^{2\zeta(2t_1+t_2)}+\alpha e^{2\zeta(t_1+t_2)})} \tag{30}$$

where

$$\varrho=\frac{\zeta+h/k_2}{\zeta-h/k_2}\quad\text{and}\quad\alpha=\frac{1-\kappa}{1+\kappa}$$

with $\kappa=k_2/k_1$, and ζ is replaced by λ, δ, or β, accordingly. This simple extension is possible, since the effect of the additional layer results from solving for the unknown coefficients by application of the boundary conditions in the z direction. The general solution for all but one of the Fourier coefficients is identical to the case for a central source. Since the spreading resistance is based upon the mean source temperature at the *surface* of the flux channel, it is not necessary to resolve the complete system of equations. However, complete solution for all coefficients is required for calculating the temperature within the solid.

Additionally,

$$\overline{\theta}_{1D}=\frac{Q}{ab}\left(\frac{t_1}{k_1}+\frac{t_2}{k_2}+\frac{1}{h}\right) \tag{31}$$

for a compound system. Application of the above results is only valid for computing the temperature distribution at the *surface* of the baseplate and to compute the spreading resistance. Equation (30) is merely the reciprocal of a similar expression reported in Yovanovich et al. [11].

Special Cases

Several special cases of an eccentric heat source may be obtained from the general solution. These are shown in Fig. 3. Solution for a central heat source on an isotropic plate was obtained by Krane [10]. However, the results are only presented for the thermal resistance based upon the maximum or centroidal temperature difference. Recently, Yovanovich et al. [11] obtained the solution for the spreading resistance of a centrally located heat source on a compound plate. A special case of the solution of Yovanovich et al. [11] is that for an isotropic plate; see Fig. 4.

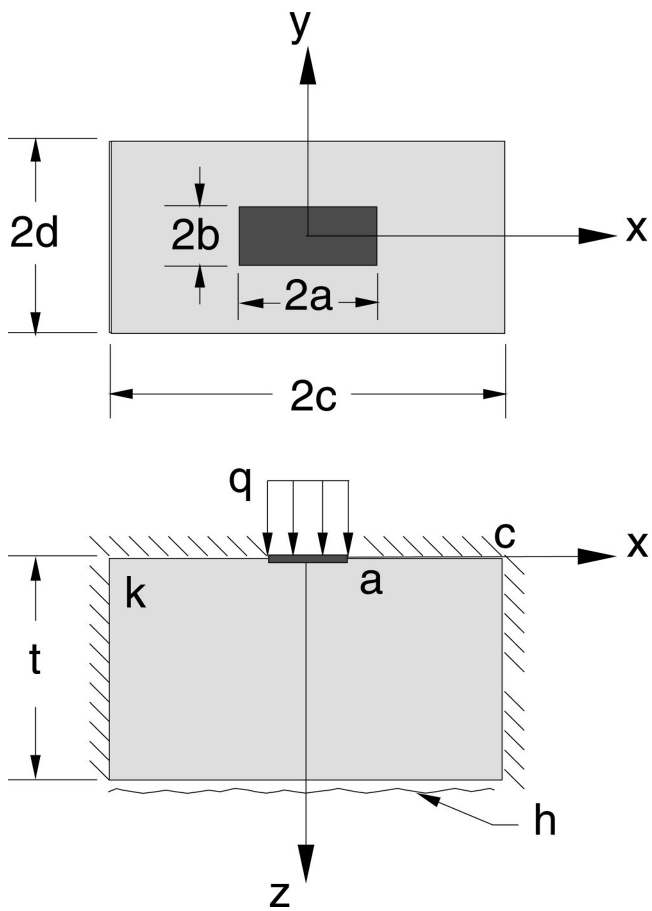

Fig. 4 Isotropic plate with central heat source

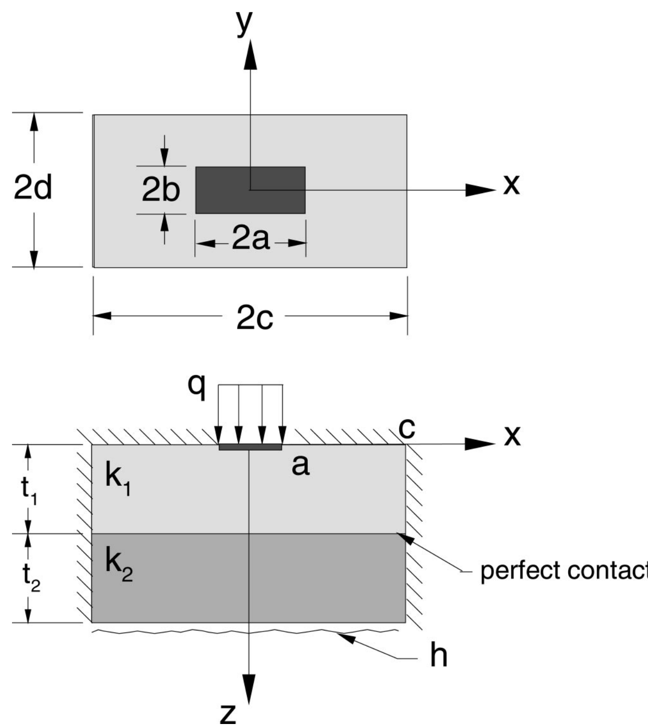

Fig. 5 Compound plate with central heat source

Additionally, the remaining cases in Fig. 3 may also be obtained from the solution for a central heat source using the method of images.

Central Heat Source. The spreading resistance of Yovanovich et al. [11] is obtained from the following general expression which shows the explicit relationships with the geometric and thermal parameters of the system according to the notation in Figs. 4 and 5:

$$R_s=\frac{1}{2a^2cdk_1}\sum_{m=1}^{\infty}\frac{\sin^2(a\,\delta_m)}{\delta_m^3}\cdot\varphi(\delta_m)+\frac{1}{2b^2cdk_1}\sum_{n=1}^{\infty}\frac{\sin^2(b\lambda_n)}{\lambda_n^3}\cdot\varphi(\lambda_n)+\frac{1}{a^2b^2cdk_1}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin^2(a\,\delta_m)\sin^2(b\lambda_n)}{\delta_m^2\lambda_n^2\beta_{m,n}}\cdot\varphi(\beta_{m,n}) \tag{32}$$

where

$$\varphi(\zeta)=\frac{(e^{2\zeta t}+1)\zeta-(1-e^{2\zeta t})h/k_1}{(e^{2\zeta t}-1)\zeta+(1+e^{2\zeta t})h/k_1} \tag{33}$$

If the system is composed of two layers, then

$$\varphi(\zeta)=\frac{(\alpha e^{4\zeta t_1}+e^{2\zeta t_1})+\varrho(e^{2\zeta(2t_1+t_2)}+\alpha e^{2\zeta(t_1+t_2)})}{(\alpha e^{4\zeta t_1}-e^{2\zeta t_1})+\varrho(e^{2\zeta(2t_1+t_2)}-\alpha e^{2\zeta(t_1+t_2)})} \tag{34}$$

where

$$\varrho=\frac{\zeta+h/k_2}{\zeta-h/k_2}\quad\text{and}\quad\alpha=\frac{1-\kappa}{1+\kappa}$$

with $\kappa=k_2/k_1$. The eigenvalues for these solutions are: $\delta_m=m\pi/c$, $\lambda_n=n\pi/d$ and $\beta_{m,n}=\sqrt{\delta_m^2+\lambda_n^2}$.

Edge and Corner Heat Sources. The solution obtained by Yovanovich et al. [11] may also be used to model the three additional special cases shown in Fig. 3. By means of symmetry, the solution for the spreading resistance may be obtained by considering that each of the special cases represents an element of the system with a centrally located heat source. For an edge source, the resistance is given by $R_s=2R$, and for a corner heat source the total resistance is given by $R_s=4R$, where R is the resistance of the system composed by mirroring the image(s) of the edge or corner heat sources to obtain a system with a central heat source.

Semi-Infinite Isotropic Flux Channel. Additional results may be obtained for semi-infinite flux channels for the case where $t_1\rightarrow\infty$ and the effect of the conductance h is no longer a factor. The solution for a semi-infinite flux channel is obtained when the parameter

$$\phi(\zeta)=1 \tag{35}$$

Semi-Infinite Compound Flux Channel. The general expression for $\phi(\zeta)$ reduces to a simpler expression when $t_2\rightarrow\infty$, (see Fig. 2). The solution for this particular case arises from Eq. (30) with

$$\phi(\zeta)=\frac{(e^{2\zeta t_1}+1)\kappa+(e^{2\zeta t_1}-1)}{(e^{2\zeta t_1}-1)\kappa+(e^{2\zeta t_1}+1)} \tag{36}$$

where the influence of the convective conductance has vanished, but the influence of the substrate remains.

Eccentric Strip Solutions. Finally, solutions for both isotropic and compound eccentric strips may be obtained from the general solution. If the dimensions of the heat source extend to two of the boundaries to form a continuous strip, the general solution simplifies considerably. In Eq. (27), the general solution consists of four terms, a uniform flow component, two strip solutions (single summations) and a rectangular source solution (double summation). For the case of an eccentric strip, one need only

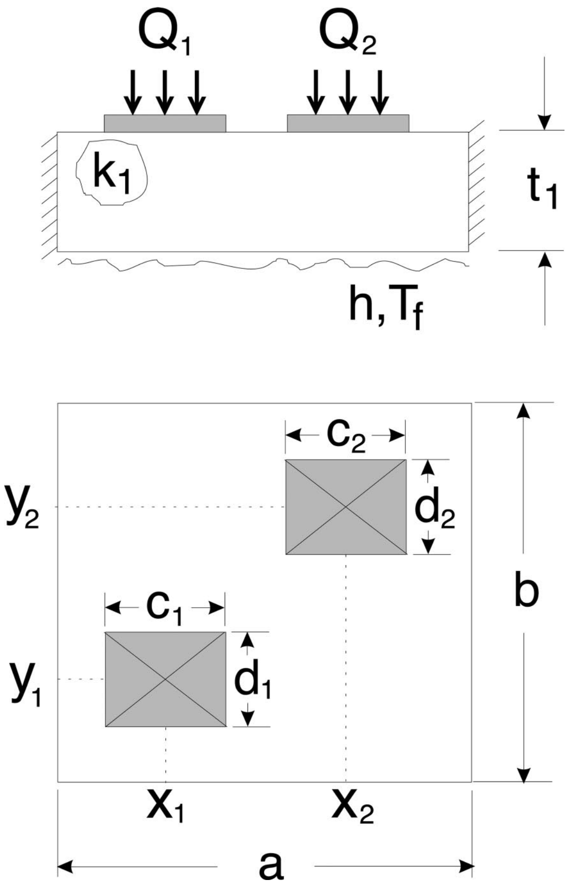

Fig. 6 Plate with multiple heat sources

consider the appropriate strip solution and the uniform flow solution. The remaining summations fall out of the solution when $b = d$ or $c = a$. In the case of semi-infinite eccentric strip solutions, Eqs. (35) and (36) also hold.

Multiple Heat Sources

If more than one heat source is present (see Fig. 6), the solution for the temperature distribution on the *surface* of the circuit board or heat sink may be obtained using superposition. For N discrete heat sources, the *surface* temperature distribution is given by

$$T(x,y,0)-T_f=\sum_{i=1}^{N}\theta_i(x,y,0) \tag{37}$$

where θ_i is the temperature excess for each heat source by itself. The temperature excess of each heat source may be computed using Eq. (16) evaluated at the surface

$$\theta_i(x,y,0)=A_0^i+\sum_{m=1}^{\infty}A_m^i\cos(\lambda x)+\sum_{n=1}^{\infty}A_n^i\cos(\delta y) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}^i\cos(\lambda x)\cos(\delta y) \tag{38}$$

with ϕ defined by Eq. (18) or (30) and $A_0=\overline{\theta_{1D}}$ given by Eq. (28) or (31).

The mean temperature of an arbitrary rectangular patch of dimensions c_j and d_j, located at $X_{c,j}$ and $Y_{c,j}$ may be computed by integrating Eq. (37) over the region $A_j=c_jd_j$

$$\overline{\theta_j}=\frac{1}{A_j}\int\int_{A_j}\theta dA_j=\frac{1}{A_j}\int\int_{A_j}\sum_{i=1}^{N}\theta_i(x,y,0)dA_j \tag{39}$$

Table 1 Results for case 1(a)

	$\hat{T}_1$	$\overline{T}_1$	$\hat{T}_2$	$\overline{T}_2$	$\overline{T}_b$
Present	84.97	80.56	108.43	101.81	53.06
Culham [16]	85.95	80.20	109.92	101.26	53.06

which may be written

$$\overline{\theta_j}=\sum_{i=1}^{N}\frac{1}{A_j}\int\int_{A_j}\theta_i(x,y,0)dA_j=\sum_{i=1}^{N}\overline{\theta_i} \tag{40}$$

Using Eqs. (27) and (40) results in the following expression for the mean temperature of the jth heat source:

$$\overline{T_j}-T_f=\sum_{i=1}^{N}\overline{\theta_i} \tag{41}$$

where

$$\overline{\theta_i}=A_o^i+2\sum_{m=1}^{\infty}A_m^i\frac{\cos(\lambda_m X_{c,j})\sin\left(\frac{1}{2}\lambda_m c_j\right)}{\lambda_m c_j} +2\sum_{n=1}^{\infty}A_n^i\frac{\cos(\delta_n Y_{c,j})\sin\left(\frac{1}{2}\delta_n d_j\right)}{\delta_n d_j}+4\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}^i \times\frac{\cos(\delta_n Y_{c,j})\sin\left(\frac{1}{2}\delta_n d_j\right)\cos(\lambda_m X_{c,j})\sin\left(\frac{1}{2}\lambda_m c_j\right)}{\lambda_m c_j\delta_n d_j}$$

Equation (41) represents the sum of the effects of all sources over an arbitrary location. Equation (41) is evaluated over the region of interest c_j, d_j located at $X_{c,j}$, $Y_{c,j}$, with the coefficients A_0^i, A_m^i, A_n^i and A_{mn}^i evaluated at each of the ith source parameters.

The coordinate system is based on the origin placed at the lower left corner of the plate, and each source is located using the coordinates of the centroid.

Application of Results

To demonstrate the usefulness of the present approach, several examples of systems containing multiple sources are presented. First, an example is given which shows the effect of a heat spreader on a low conductivity substrate. Next, the method is applied to model a heat sink containing a number of discrete heat sources uniformly located on the baseplate. Finally, a uniform flux heat source of complex shape is analyzed.

Case 1. In the first case, the dimensions of a plate or circuit board are: $a=300$ mm, $b=300$ mm, $t_1=10$ mm, $h=10$ W/m^{2}K and $k=10$ W/mK, with $T_f=25$°C. Two heat sources having dimensions $c=25$ mm and $d=25$ mm each. The first source with a power of $Q=10$ W is located at $X_c=Y_c=90$ mm and the second having a power of $Q=15$ W at $X_c=Y_c=210$ mm. The basic equations may be programmed into any symbolic or numerical mathematics software package. For the present calculations, the symbolic mathematics program Maple V [4] was employed. A total of 50 terms were used in each of the single and double summations.

The results for the mean $\overline{T}$ and centroidal $\hat{T}$ temperatures of the first case are presented in Table 1. In this example the centroidal temperatures of each heat source were found to be 84.97°C and 108.43°C. A three-dimensional plot of the surface temperature profile is given in Fig. 7. Comparisons with a generalized Fourier series approach of Culham and Yovanovich [16] yields 85.95°C and 109.92°C. The primary difference between the present ap-

Table 2 Results for case 1(b)

	$\hat{T}_1$	$\overline{T}_1$	$\hat{T}_2$	$\overline{T}_2$	$\overline{T}_b$
Present	56.55	56.05	59.94	59.19	53.06
Culham [16]	56.63	56.03	60.08	59.16	53.06

proach and that of Culham and Yovanovich [16] is that the present work yields simplified expressions which may be easily programmed in any Mathematics or Spreadsheet software, whereas the latter uses a numerical least squares approximation to solve for a mixed boundary value problem.

For the same configuration, a thin, $t=2$ mm, highly conductive layer $k=350$ W/mK, is added to the original substrate and the problem reanalyzed. The results are summarized in Table 2. In this example the centroidal temperatures of each heat source were found to be 56.55 and 59.94°C. Comparisons with a generalized Fourier series approach of Culham and Yovanovich [16] yields 56.63 and 60.08°C. A three-dimensional plot of the surface temperature profile is given in Fig. 8. It is clearly seen that adding a spreader has reduced the maximum source temperatures considerably and equalized the temperature.

Case 2. In this example, the baseplate of a heat sink is to be analyzed. The dimensions of the baseplate are $a=50$ mm, $b=50$ mm, $k=150$ W/mK, $t=10$ mm, and an effective extended surface heat transfer coefficient $h=400$ W/m^{2}K. Four sources having the characteristics summarized in Table 3, are attached to the baseplate assuming negligible contact or interface resistance. The temperature results are summarized in Table 4 and in Fig. 9.

Case 3. In the final example, a heat source with a complex shape is analyzed by the present approach. The source is composed of five square heat sources each having dimensions $c=20$ mm by $d=20$ mm and dissipating 5 W. The heat sources are arranged in the form of a cross in the center of a plate having dimensions $a=100$ mm by $b=100$ mm, thickness of $t=10$ mm. This results in an irregular shaped isoflux heat source which cannot be solved using conventional approaches. The thermal conductivity of the plate is $k=50$ W/mK, while $h=50$ W/m^{2}K, and $T_f=25$°C. The value of the maximum temperature at the center of

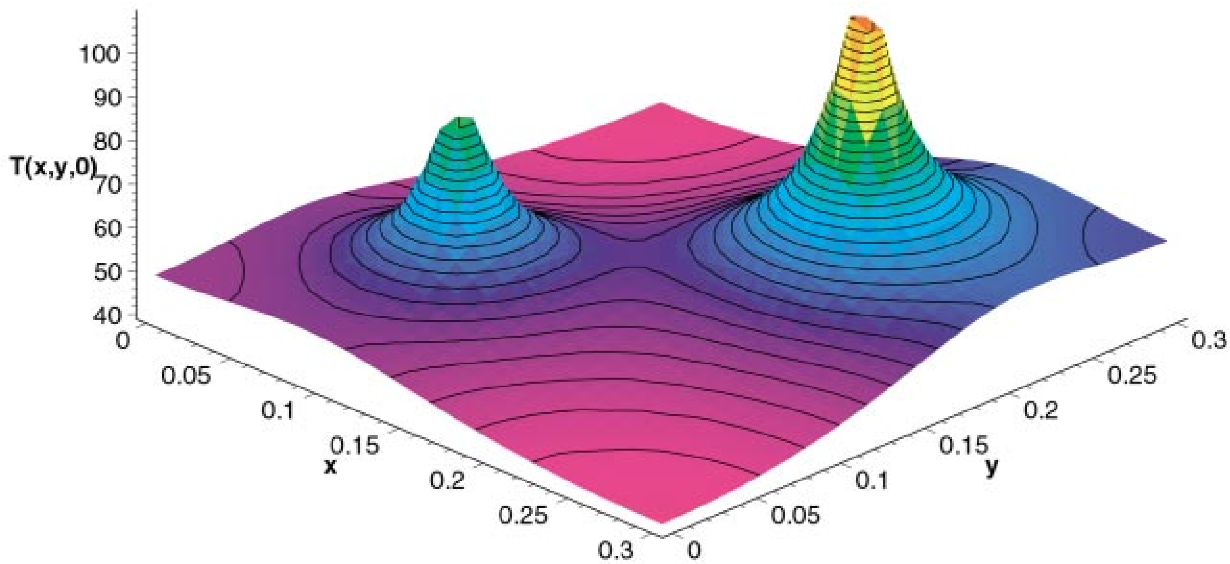

Fig. 7 Surface temperature for an isotropic plate with two heat sources

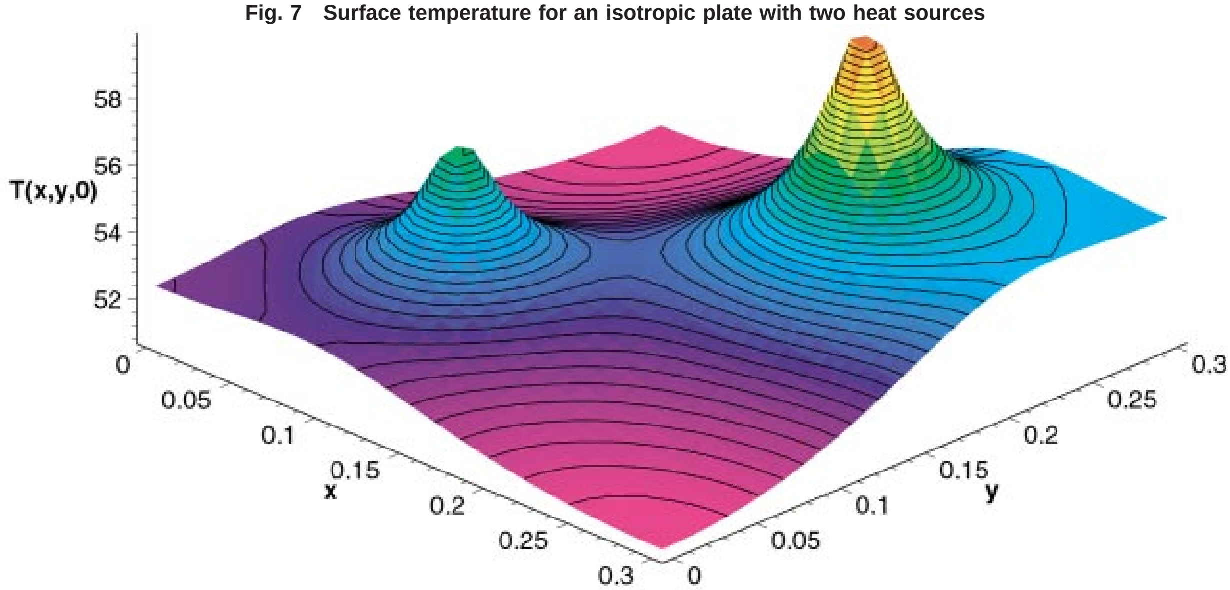

Fig. 8 Surface temperature for a compound plate with two heat sources

Table 3 Source characteristics for case 2

	Q W	c mm	d mm	X_c mm	Yc mm
Source 1	10	10	10	12.5	12.5
Source 2	25	10	10	37.5	12.5
Source 3	20	10	10	12.5	37.5
Source 4	15	10	10	37.5	37.5

Table 4 Source temperatures for case 2

	$\hat{T}$	$\overline{T}$
Source 1	97.40	96.85
Source 2	103.82	102.33
Source 3	101.28	100.10
Source 4	99.94	99.07

the plate is found to be 80.61°C. The area weighted mean source temperature is found to be 78.87°C. The thermal contour plot is given in Fig. 10.

Discussion

The general solution obtained may easily be coded in a number of ways. The simplest approach is through the use of mathematical programs [4–7]. These packages allow for symbolic and numerical computation of mathematical expressions. They also provide graphical functions for generating three-dimensional plots such as those presented earlier. One advantage of these packages is the minimal effort required to enter the basic equations. Computation time varies among packages and is also dependent upon the number of sources specified. The present results were obtained using Maple V6 [4] with 50 terms in each of the summations. A single source calculation typically required a few seconds. Reasonable convergence with 50 terms is obtained for most problems.

Another method of computation which was assessed is the use of computer languages such as C or Basic. Computation time is much faster with a compiled code, however, a considerable amount of code is required to achieve the same results as those produced using mathematical software. The method is also amenable to spreadsheet calculations with or without the use of macros. However, the use of macros allows for easier development.

In addition to providing details of the surface temperature distribution and centroidal and mean source temperatures, the effective thermal resistance may also be computed for each source. This approach was not taken in the present work, since no unique value of thermal resistance is definable when more than one source is present. The location and strength of each additional heat source will affect the value of thermal resistance for a particular source of interest.

Finally, if the ambient temperature increases as a result of heat transfer due to the film coefficient h, a wake effect may be approximated in the final solution by defining an ambient temperature which is a function of flow position

$$T_f = T_i + \frac{Q}{\dot{m} C_p} \frac{x}{L} \tag{42}$$

where T_i is the inlet temperature, $\dot{m}$ is the mass flow through the system, Q is the sum of all heat sources, and x/L the local position in the flow direction.

Summary and Conclusions

General expressions for determining the spreading resistance of an eccentric isoflux rectangular source on the surface of finite isotropic and compound rectangular flux channels were presented. The solution for the temperature at the surface of a rectangular flux channel was also presented. It was shown that this solution may be used to predict the centroidal temperatures for any number of heat sources using superposition. Additional special cases of spreading resistance from single eccentric heat sources on isotropic, compound, finite, and semi-infinite flux channels were also presented. Finally, it was shown that the solution for a central heat source may be used to compute the spreading resistance for corner and edge heat sources using the method of images. Several applications of the general solution to systems with multiple heat sources were also given.

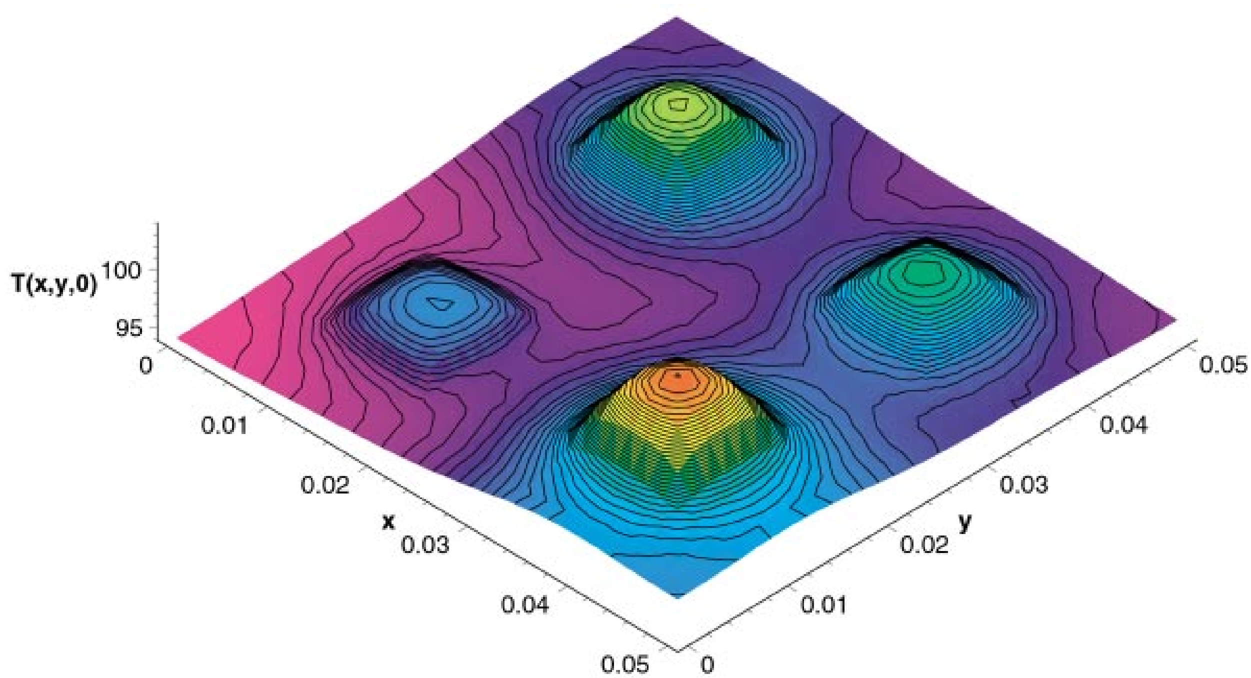

Fig. 9 Surface temperature for a heat sink with four heat sources

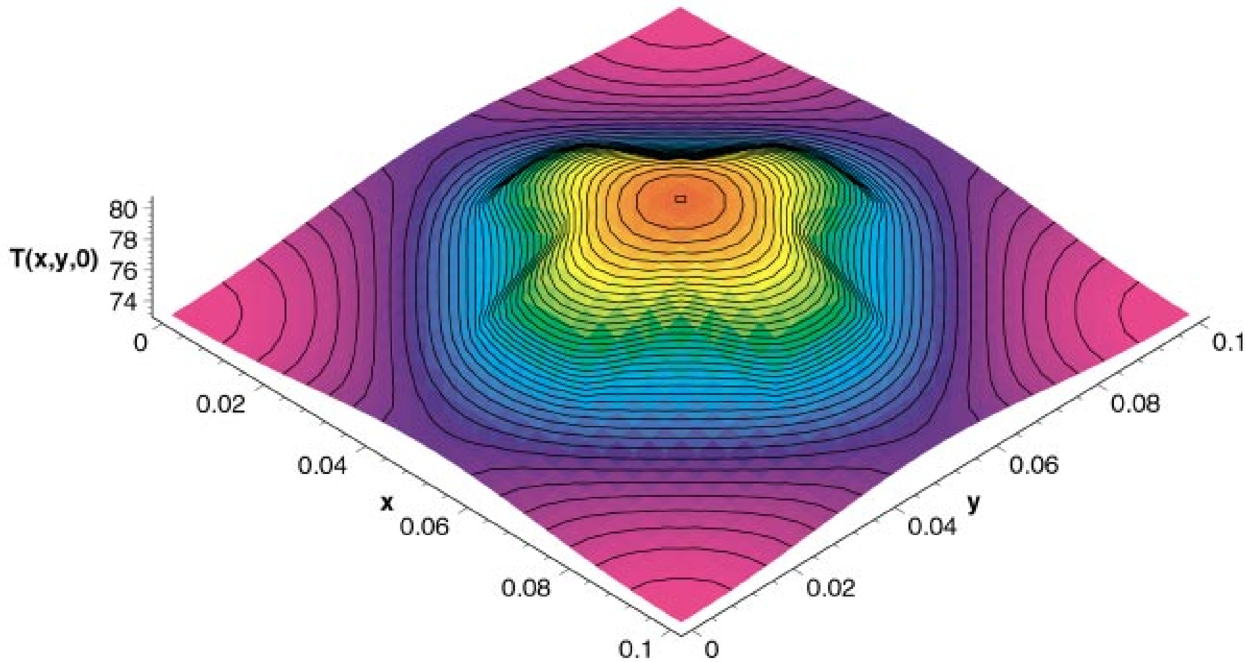

Fig. 10 Surface temperature for a complex heat source

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Nomenclature

a, b, c, d = linear dimensions, m
A_b = baseplate area, m^2
A_s = heat source area, m^2
A_0, A_m, A_n, A_{mn} = Fourier coefficients
Bi = Biot no., $h\mathcal{L}/k$
C_p = heat capacity, J/Kg·K
h = contact conductance or film coefficient, $W/m^2 \cdot K$
i = index denoting layers 1 and 2
k, k_1, k_2 = thermal conductivities, W/m·K
$\mathcal{L}$ = arbitrary length scale, m
$\dot{m}$ = mass flow rate, kg/s
m, n = indices for summations
N = no. of heat sources
Q = heat flow rate, W
q = heat flux, W/m^2
R = thermal resistance, K/W
R^* = dimensionless thermal resistance, $\equiv kR\mathcal{L}$
R_{1D} = one-dimensional resistance, K/W
R_s = spreading resistance, K/W
R_T = total resistance, K/W
t, t_1, t_2 = total and layer thicknesses, m
T_1, T_2 = layer temperatures, K
$\overline{T}_s$ = mean source temperature, K
T_f = sink temperature, K
T_i = inlet or initial temperature, K
Xc, Yc = heat source centroid, m
$\beta_{m,n}$ = eigenvalues, $\equiv \sqrt{\lambda_m^2 + \delta_n^2}$
δ_n = eigenvalues, $(n\pi/b)$
ϵ = relative contact size, $\equiv b/a$
θ = temperature excess, $\equiv T - T_f$, K
$\overline{\theta}$ = mean temperature excess, $\equiv \overline{T} - T_f$, K
$\hat{\theta}$ = centroidal temperature excess, $\equiv \hat{T} - T_f$, K
$\overline{\theta}_b$ = mean base temperature excess, $\equiv \overline{T}_b - T_f$, K
κ = relative conductivity, k_2/k_1
λ_m = eigenvalues, $(m\pi/a)$
ϕ = spreading function
τ = relative thickness, $\equiv t/\mathcal{L}$
ζ = dummy variable, m^{-1}

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