Solving Higher Order ODEs

Solving Higher Order ODEs

Second Order Differential Equations

In general, if we wish to solve an ODE of the form

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$$

with initial conditions $y(x = x_0) = y_0$ and $y'(x = x_0) = y_0'$, we can transform these into a system of coupled first order equations by introducing the variable:

$$v = \frac{dy}{dx}$$

which gives us the equations:

$$\begin{aligned}\frac{dy}{dx} &= v \\ \frac{dv}{dx} &= f(x, y, v)\end{aligned}$$

with the initial conditions

As the ODE for y depends on v and the ODE for v depends on y, these equations need to be integrated simultaneously.

Worked Example

Consider second order ODE:

$$\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 100y = 100|\sin(t)|$$

which we wish to solve for the initial conditions $y = 0.1$, $dy/dx = -0.5$ at $t = 0$.

Firstly let's rearrange the equation to make y'' the subject:

$$\frac{d^2y}{dt^2} = 100|\sin(t)| - 10\frac{dy}{dt} - 100y$$

We start by introducing the variables:

$$v_0 = y$$
$$v_1 = \frac{dy}{dt} = \frac{dv_0}{dt}$$

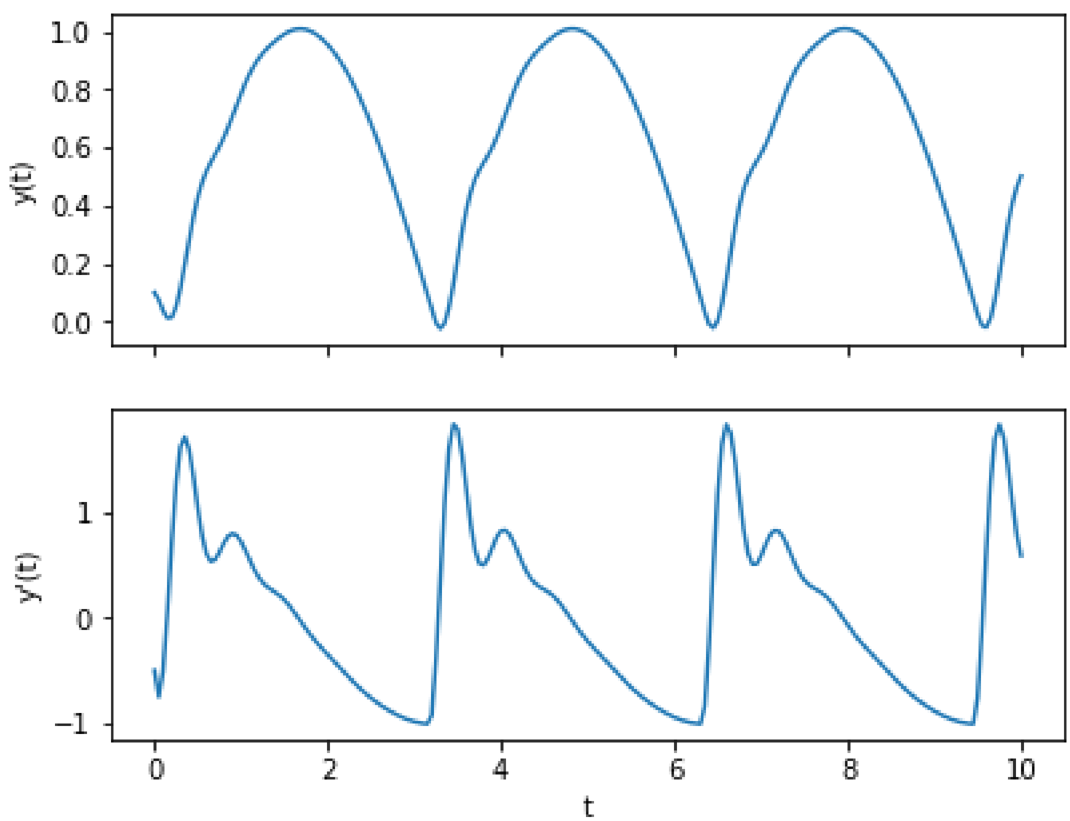
in order to reduce the second order ODE to a coupled system of two first order ODEs:

$$\frac{dv_0}{dt} = v_1$$
$$\frac{dv_1}{dt} = 100|sin(t)| - 10v_1 - 100v_0$$

[8]:

```
import numpy as np
import matplotlib.pyplot as plt

t0, y0, v0 = 0, 0.1, -0.5 #initial conditions
h = 0.05 #step size
t_end = 10


#The ODE function
def f1(t, y, v):
    return v

def f2(t, y, v):
    return 100*np.abs(np.sin(t)) - 10 * v - 100 * y


#Constructing the arrays:
t_arr = np.arange(t0, t_end + h, h) #make sure it goes up to and including x_end

y_arr = np.zeros(t_arr.shape)
v_arr = np.zeros(t_arr.shape)

#Setting the initial conditions
y_arr[0] = y0
v_arr[0] = v0
```

```
#Performing the Euler method, note we don't use the last x value in the update
↪calculations
for i,t in enumerate(t_arr[:-1]):
    y_arr[i + 1] = y_arr[i] + h * f1(t, y_arr[i], v_arr[i])
    v_arr[i + 1] = v_arr[i] + h * f2(t, y_arr[i], v_arr[i])

##Plotting both curves
fig, ax = plt.subplots(2,1, sharex = True, figsize = (6.4, 5))

ax[0].plot(t_arr, y_arr)

ax[0].set_ylabel('y(t)')

ax[1].plot(t_arr, v_arr)

ax[1].set_xlabel('t')
ax[1].set_ylabel("y'(t)")

plt.show()
```



In the solution above we used separate variables to store the values for $y(x)$ and $v(x)$. In the example below, we shall see that it is more practical to store these values in a single 2D array.

Higher Order Differential Equations

We can extend this technique of creating a system of coupled first order equations to an ODE of arbitrary order:

$$\frac{d^n y}{dx^n} = f\left(x, \frac{dy}{dx}, \frac{d^2 y}{dx^2}, \frac{d^3 y}{dx^3}, \ldots, \frac{d^{n-1} y}{dx^{n-1}}\right)$$

with initial conditions

$$y(x = x_0) = y_0 \qquad \frac{dy}{dx}(x = x_0) = y_0' \qquad \frac{d^2 y}{dx^2}(x = x_0) = y_0'' \qquad \ldots \qquad \frac{d^{n-1} y}{dx^{n-1}}(x = x_0) = y_0^{(n-1)}$$

We start by introducing the variables:

$$v_0 = y \qquad v_1 = \frac{dy}{dx} \qquad v_2 = \frac{d^2 y}{dx^2} \qquad \ldots \qquad v_{n-1} = \frac{d^{n-1} y}{dx^{n-1}}$$

we can transform the order n ODE to a set of n first order coupled differential equations:

$$\begin{aligned}
\frac{dv_0}{dx} &= v_1 \\
\frac{dv_1}{dx} &= v_2 \\
\frac{dv_2}{dx} &= v_3 \\
&\vdots \\
\frac{dv_{n-2}}{dx} &= v_{n-1} \\
\frac{dv_{n-1}}{dx} &= f(x, v_0, v_1, v_2, v_3, \ldots, v_{n-2}, v_{n-1})
\end{aligned}$$

As the subscripts suggest, it is practical to store the v_i values in a vector.

These equations can be integrated simultaneously, and the solution for y given by v_0.

Worked Example

Consider the order 3 ODE:

$$\frac{d^3 y}{dx^3} + y\frac{d^2 y}{dx^2} = 0$$

with the initial conditions $y = 1$, $\frac{d}{dx}y = 0.5$ and $\frac{d^2}{dx^2}y = 0.7$ at $x = 0$.

To solve this we introduce the variables:

$$y_0 = y$$
$$y_1 = \frac{dy}{dx}$$
$$y_2 = \frac{d^2y}{dx^2}$$

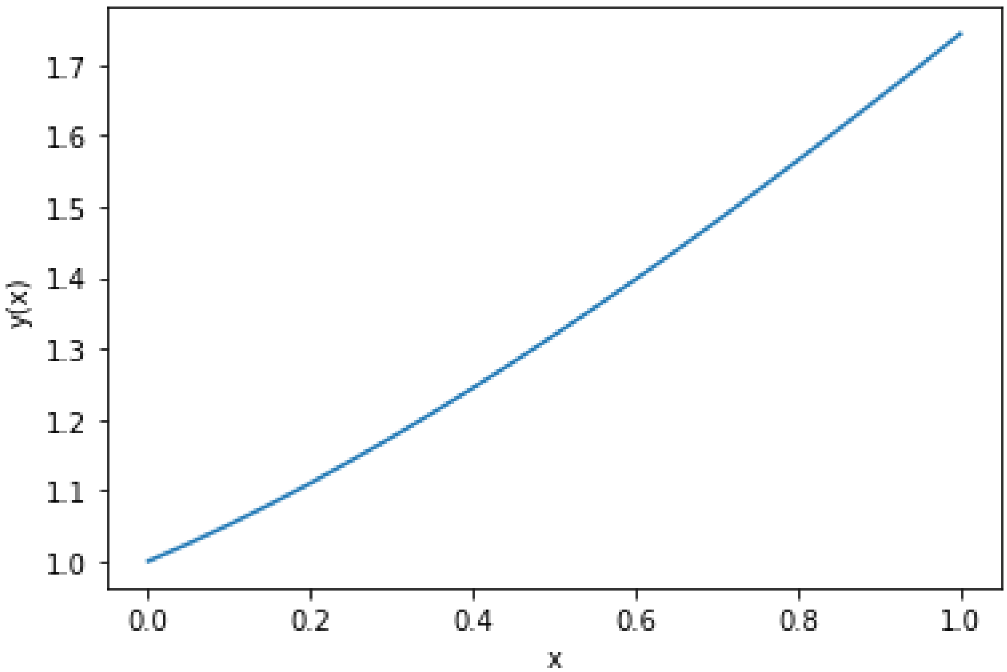
This gives us the system of equations:

$$\frac{dy_0}{dx} = y_1$$
$$\frac{dy_1}{dx} = y_2$$
$$\frac{dy_2}{dx} = -y_0 y_2$$

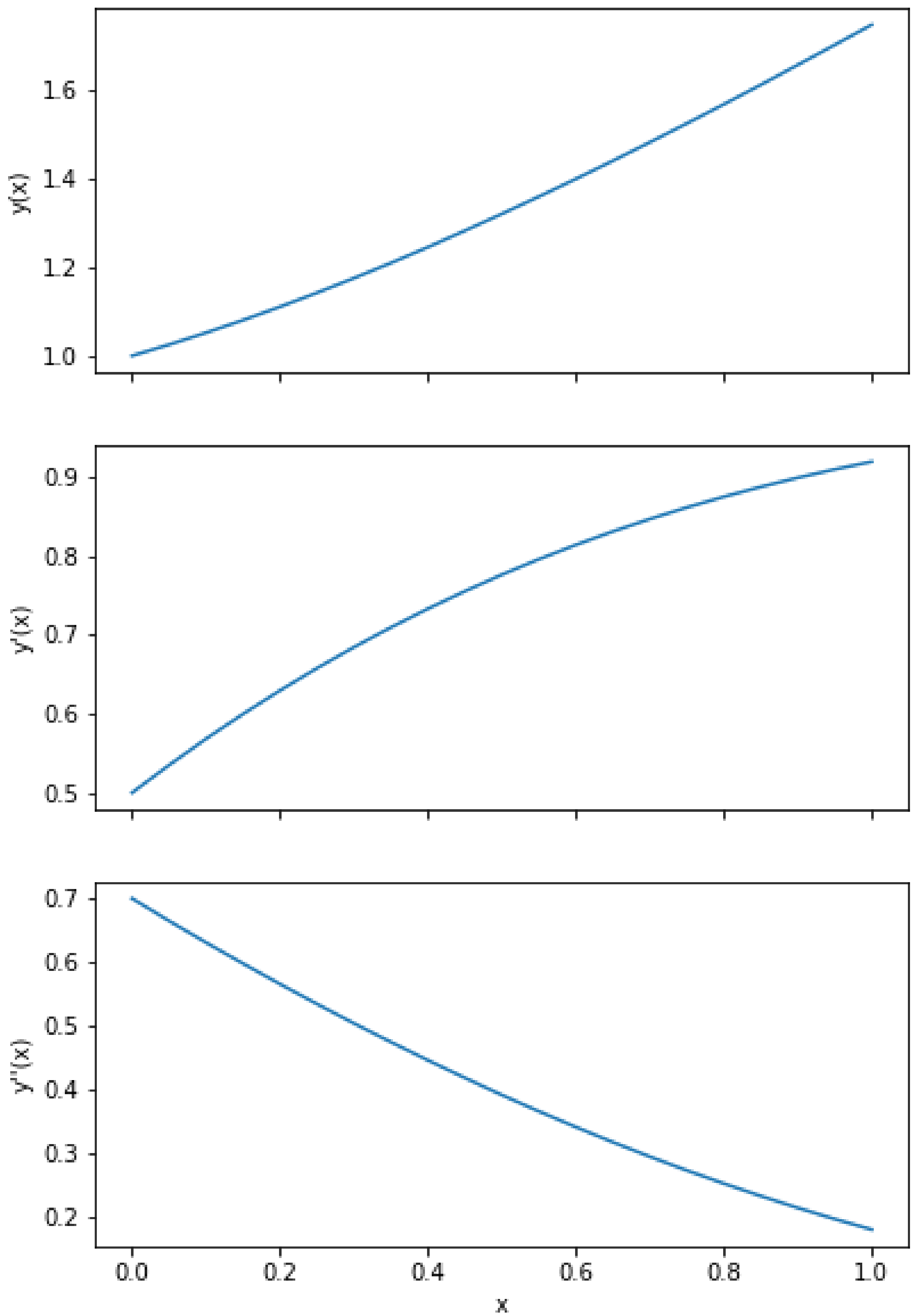
The solution in Python is:

[20]:
```
import numpy as np
import matplotlib.pyplot as plt

x0, y0 = 0, [1, 0.5, 0.7] #initial conditions
h = 0.05
x_end = 1

##The ODE function (takes y as an array and returns an array of values)

def f(x, y):
    return np.array([
        y[1],
        y[2],
        - y[0] * y[2],
    ])


#Contructing the arrays
x_arr = np.arange(x0, x_end + h, h)

y_arr = np.zeros((x_arr.size, len(y0))) #Using y instead of v as there is no␣
↪ambiguity
y_arr[0, :] = y0 #setting the initial conditions
```

```python
#Performing the Euler method, note we don't use the last x value in the update calculations
for i,x in enumerate(x_arr[:-1]):
    y_arr[i+1,:] = y_arr[i, :] + h*f(x, y_arr[i, :])


##Plotting the solution for y(x) only
fig1, ax1 = plt.subplots()

ax1.plot(x_arr, y_arr[:, 0])

ax1.set_xlabel('x')
ax1.set_ylabel('y(x)')

plt.show()

##Plotting the solutions to the derivatives
fig2, ax2 = plt.subplots(len(y0),1, sharex = True, figsize = (6.4, 10))


for i in range(len(y0)):
    ax2[i].plot(x_arr, y_arr[:, i])

    ax2[i].set_ylabel('y{}(x)'.format("'"*i))

ax2[-1].set_xlabel('x')

plt.show()
```



1.7
1.6
1.5
1.4
1.3
1.2
1.1
1.0
y(x)
0.0
0.2
0.4
0.6
0.8
1.0
x



y(x)
1.0
1.2
1.4
1.6
y'(x)
0.5
0.6
0.7
0.8
0.9
y''(x)
0.2
0.3
0.4
0.5
0.6
0.7
0.0
0.2
0.4
0.6
0.8
1.0
x